

**STSCI 4780/5780**  
**Multivariate relationships, 2:**  
**The bivariate normal distribution and**  
**joint/conditional/marginal relationships**

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# Recap

Building models for multivariate data via joint/conditional/marginal distribution relationships:

- Types of studies of multivariate data:
  - ▶ Correlation/dependence: Study joint,  $p(x, y)$
  - ▶ Regression: Study conditional,  $p(y|x)$
- BT as posterior = (joint for everything)/(marginal for knowns)
- Directed acyclic graphs (DAGs)
- Conditional independence
- Example: binomial prediction:  $n_2 \perp\!\!\!\perp n_1 | \alpha$

# Agenda

- ① Bivariate normal distribution & regression
- ② Joint from conditionals

# Agenda

① Bivariate normal distribution & regression

② Joint from conditionals

# Regression perspective on the bivariate normal distribution

Outline of BVN development:

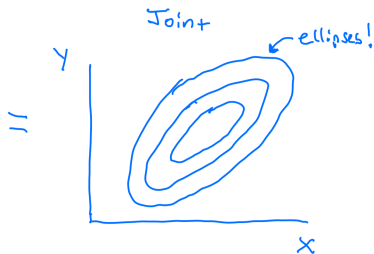
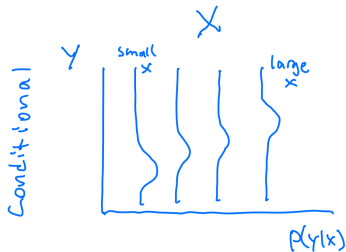
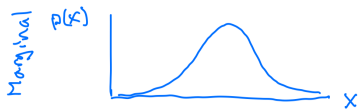
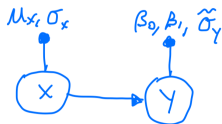
- Write a joint dist'n for two variables  $(x, y)$  factored as [marginal for  $x$ ]  $\times$  [cond'l for  $y$  given  $x$ ]:
  - ▶ Normal marginal for  $x$ ; mean  $\mu_x$ , variance  $\sigma_x^2$
  - ▶ Normal cond'l for  $y$ ; conditional mean  $\tilde{\mu}_y(x)$ , conditional variance  $\tilde{\sigma}_y^2$  (constant wrt  $x$  — homoskedastic)
  - ▶ Simplest nontrivial regression function:

$$\tilde{\mu}_y(x) \equiv \mathbb{E}(y|x) = \beta_0 + \beta_1 x$$

$$\Rightarrow p(x, y) = \text{Norm}(x|\mu_x, \sigma_x) \times \text{Norm}(y|\beta_0 + \beta_1 x, \tilde{\sigma}_y)$$

Five parameters:  $\mu_x, \sigma_x; \beta_0, \beta_1, \tilde{\sigma}_y$

Visualizing  $p(x, y) = \text{Norm}(x|\mu_x, \sigma_x) \times \text{Norm}(y|\beta_0 + \beta_1 x, \tilde{\sigma}_y)$ :



$$\begin{aligned}
 p(x, y) &= p(x) p(y|x) \\
 &\propto \exp \left[ -\frac{(x - \mu_x)^2}{2\sigma_x^2} - \frac{[y - (\beta_0 + \beta_1 x)]^2}{2\tilde{\sigma}_y^2} \right] \\
 &= \exp \left[ -\frac{Q(x, y)}{2} \right]
 \end{aligned}$$

with

$$Q(x, y) = \frac{(x - \mu_x)^2}{\sigma_x^2} - \frac{[y - (\beta_0 + \beta_1 x)]^2}{\tilde{\sigma}_y^2}$$

- Rewrite it as [marginal for  $y$ ]  $\times$  [cond'l for  $x$  given  $y$ ]:
  - ▶ Find the marginal for  $y$  by completing the square in  $x \rightarrow$  normal integral (after a surprising amount of algebra!)
  - ▶ Find the cond'l for  $x$  via  $p(x, y) = p(y)p(x|y)$

$$\rightarrow p(x|y) = \frac{p(x, y)}{p(y)}$$

$$\Rightarrow p(x, y) = \text{Norm}(y|\mu_y, \sigma_y) \times \text{Norm}(x|\alpha_0 + \alpha_1 y, \tilde{\sigma}_y)$$

with

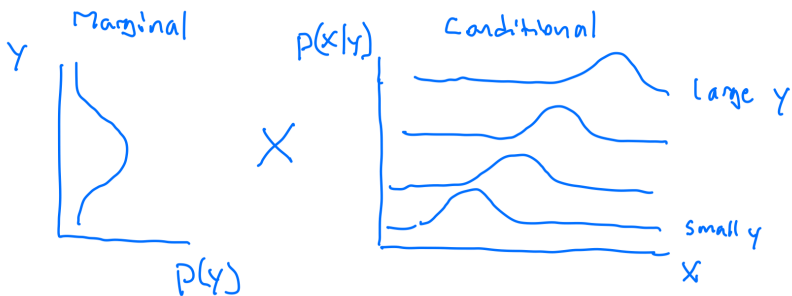
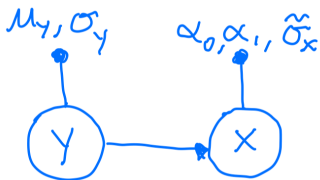
$$\mu_y = \beta_0 + \beta_1 \mu_x, \quad \sigma_y^2 = \tilde{\sigma}_y^2 + \beta_1^2 \sigma_x^2;$$

$$\alpha_0 = \mu_x - \beta_1 \frac{\sigma_x^2}{\sigma_y^2} \mu_y, \quad \alpha_1 = \beta_1 \frac{\sigma_x^2}{\sigma_y^2};$$

$$\tilde{\sigma}_x^2 = \sigma_x^2 \frac{\tilde{\sigma}_y^2}{\sigma_y^2}$$



Visualizing  $p(x, y) = \text{Norm}(y|\mu_y, \sigma_y) \times \text{Norm}(x|\alpha_0 + \alpha_1 y, \tilde{\sigma}_x)$ :



- Write the joint *symmetrically*, using the four marginal parameters  $(\mu_x, \sigma_x, \mu_y, \sigma_y)$  and the *correlation coefficient*:

$$\rho = \beta_1 \frac{\sigma_x}{\sigma_y} = \alpha_1 \frac{\sigma_y}{\sigma_x}$$

$$\begin{aligned} \Rightarrow p(x, y) &= C \exp \left[ -\frac{Q(x, y)}{2} \right] \\ &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp \left[ -\frac{Q'(x, y)}{2(1-\rho^2)} \right] \end{aligned}$$

$$Q'(x, y) = \left( \frac{x - \mu_x}{\sigma_x} \right)^2 + \left( \frac{y - \mu_y}{\sigma_y} \right)^2 - 2\rho \frac{x - \mu_x}{\sigma_x} \frac{y - \mu_y}{\sigma_y}$$

with  $|\rho| \leq 1$

- Regression functions in terms of joint parameters:

$$\begin{aligned}\mathbb{E}(y|x) &= \mu_y + \beta_1(x - \mu_x) \\ &= \mu_y + \rho\sigma_y \left( \frac{x - \mu_x}{\sigma_x} \right)\end{aligned}$$

$$\begin{aligned}\mathbb{E}(x|y) &= \mu_x + \alpha_1(y - \mu_y) \\ &= \mu_x + \rho\sigma_x \left( \frac{y - \mu_y}{\sigma_y} \right)\end{aligned}$$

The regression lines track the conditional mean, which is the (conditional) mode for normal distributions; thus they intersect contours of  $p(x, y)$  where they are vertical (for  $y|x$ ) or horizontal (for  $x|y$ )

- Matrix forms (inverse covariance matrix vs. inverse correlation matrix):

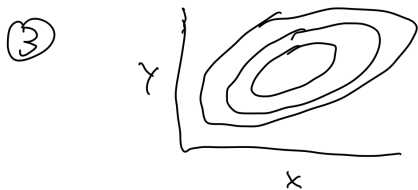
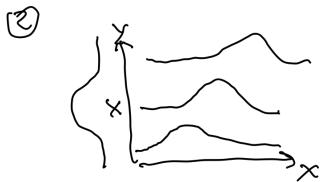
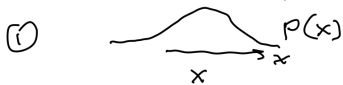
$$\begin{aligned}
 Q'(x, y) &= \left( \frac{x - \mu_x}{\sigma_x} \right)^2 + \left( \frac{y - \mu_y}{\sigma_y} \right)^2 - 2\rho \frac{x - \mu_x}{\sigma_x} \frac{y - \mu_y}{\sigma_y} \\
 &= \begin{bmatrix} (x - \mu_x) & (y - \mu_y) \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_x^2} & -\frac{\rho}{\sigma_x \sigma_y} \\ -\frac{\rho}{\sigma_x \sigma_y} & \frac{1}{\sigma_y^2} \end{bmatrix} \begin{bmatrix} x - \mu_x \\ y - \mu_y \end{bmatrix} \\
 &= \begin{bmatrix} \frac{x - \mu_x}{\sigma_x} & \frac{y - \mu_y}{\sigma_y} \end{bmatrix} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} \frac{x - \mu_x}{\sigma_x} \\ \frac{y - \mu_y}{\sigma_y} \end{bmatrix}
 \end{aligned}$$

- Covariance and correlation matrices:

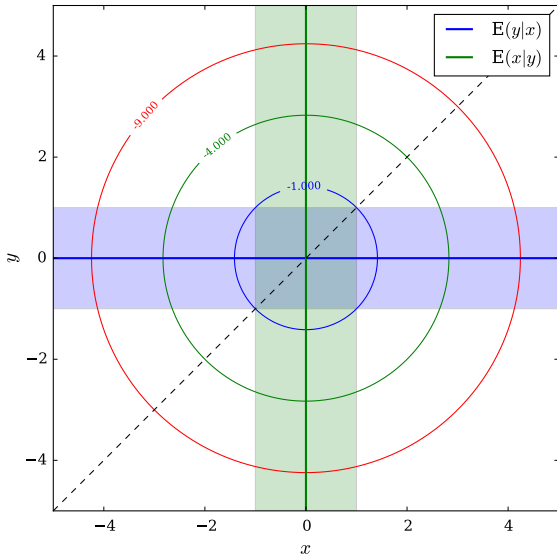
$$\text{Covar} = \begin{bmatrix} \sigma_x^2 & \rho \sigma_x \sigma_y \\ \rho \sigma_x \sigma_y & \sigma_y^2 \end{bmatrix}, \quad \text{Corr} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Visualizing the three descriptions:

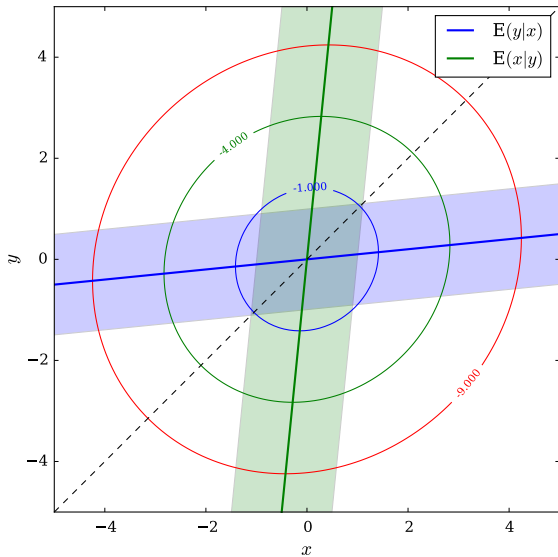
3 descriptions of  $p(x, y)$



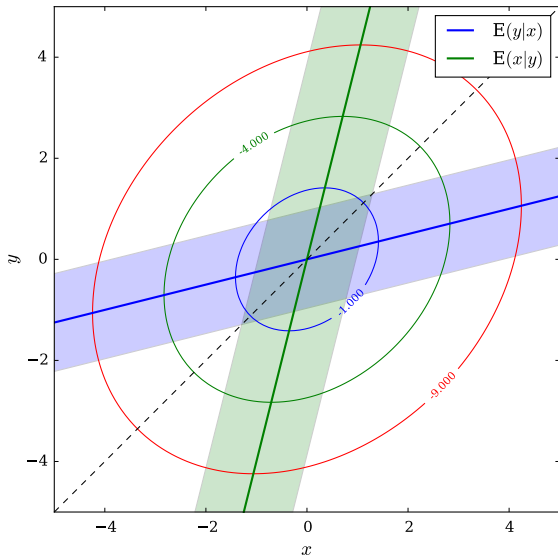
$$\rho = 0, \mu_x = \mu_y = 0, \sigma_x = \sigma_y = 1$$



$$\rho = 0.1$$

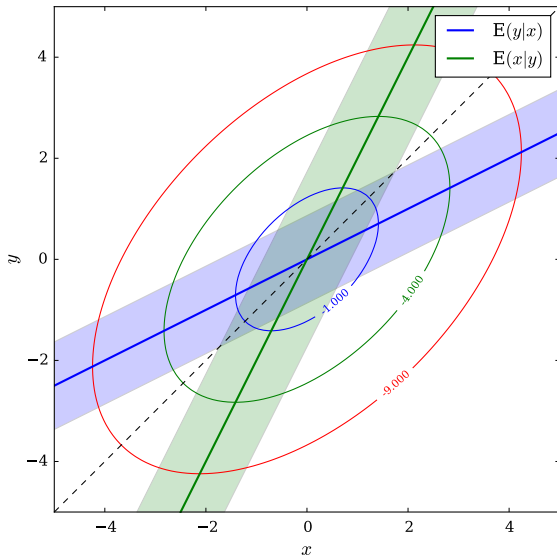


$$\rho = 0.25$$

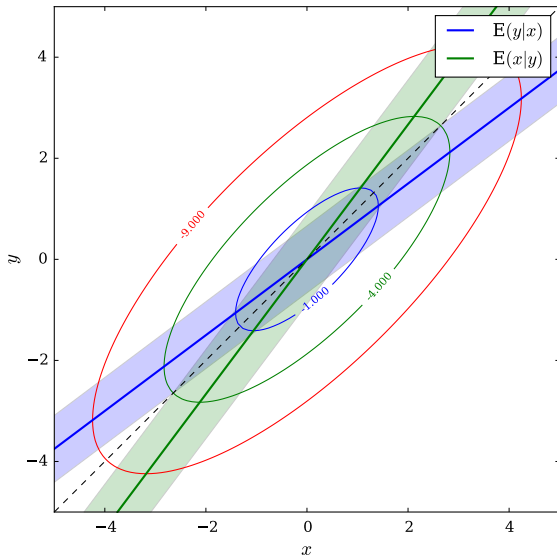




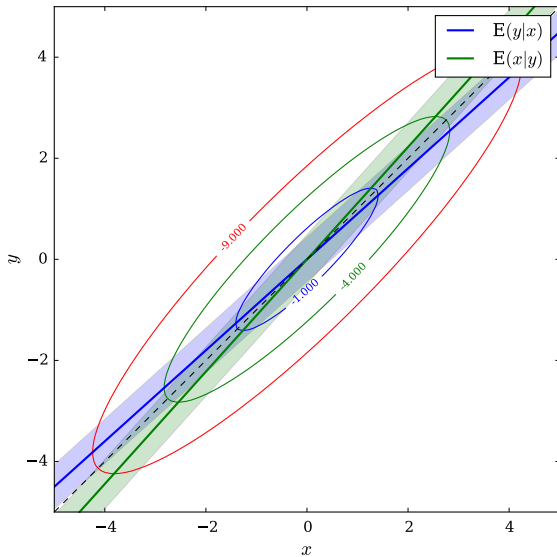
$$\rho = 0.5$$



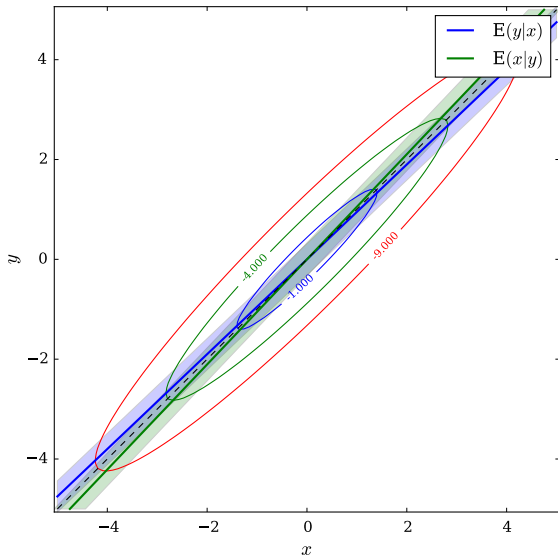
$$\rho = 0.75$$



$$\rho = 0.9$$



$$\rho = 0.95$$



## Features to note

- The regression lines track the conditional mean, which is the (conditional) mode for normal distributions; thus they intersect contours where they are vertical (for  $y|x$ ) or horizontal (for  $x|y$ )
- The regression lines for  $y|x$  and  $x|y$  are not the same; i.e., to find  $\mathbb{E}(x|y)$  (a value of  $x$  as a function of  $y$ ) we *do not* take the (blue) line:

$$y(x) = \mathbb{E}(y|x) = \beta_0 + \beta_1 x$$

and solve for  $x(y) = -\beta_0 + \frac{1}{\beta_1} y$

In fact, the slopes of *both* regression lines are  $\propto \beta_1$

- Neither regression line is the symmetry axis of the elliptical countours, except when  $\rho = \pm 1$ , when the ellipse collapses to a line
- The ellipse *always* has slope 1 (or  $-1$ ), regardless of the value of  $\rho$  (or  $\beta_1$ ); more generally its slope is  $\sigma_y/\sigma_x$

## Whence “regression”?

Recall the regression function for  $y$  given  $x$

$$\mathbb{E}(y) = \mu_y + \rho \sigma_y \frac{x - \mu_x}{\sigma_x}$$

Compute *relative shifts from means*:

$$\rightarrow \frac{\mathbb{E}(y - \mu_y)}{\sigma_y} = \rho \frac{x - \mu_x}{\sigma_x}$$

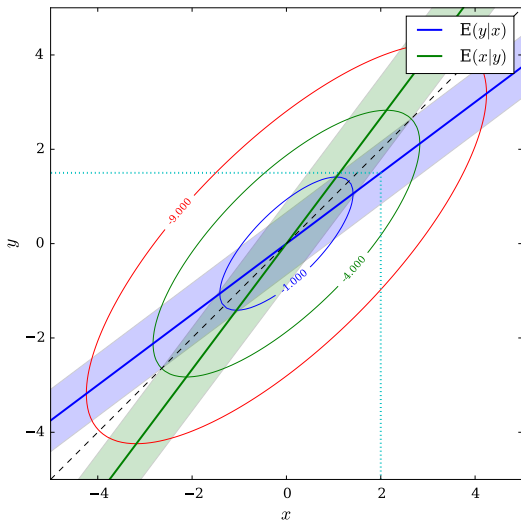
Since  $|\rho| \leq 1$ , so long as  $\rho \neq 0$ ,

For a given (observed)  $x$ ,  
we expect  $y$  to deviate from its mean  
by a smaller relative amount than  $x$  does from its mean

Galton, in a study of inheritance of stature, referred to this effect as “regression to mediocrity”

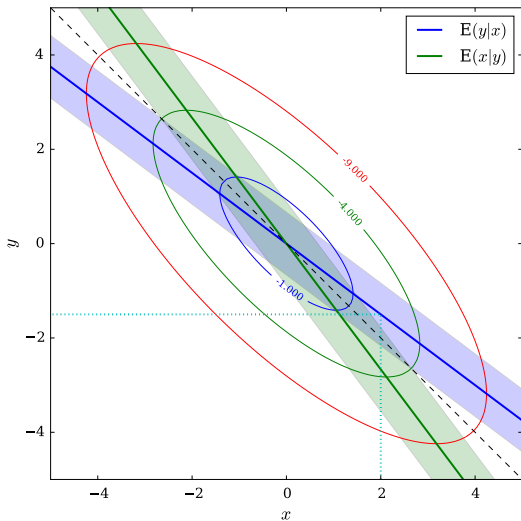
$$\rho = 0.75, \mu_x = \mu_y = 0, \sigma_x = \sigma_y = 1$$

$$\mathbb{E}(y - \mu_y | x) < x - \mu_x \quad \text{when } x > \mu_x \quad (\text{else reverse})$$



Negative correlation:  $\rho = -0.75$

$$|\mathbb{E}(y - \mu_y|x)| < |x - \mu_x| \quad \text{when } x > \mu_x$$





# Agenda

① Bivariate normal distribution & regression

② Joint from conditionals

## Joint distribution from conditionals?

The symmetric parameterization of the BVN has 5 parameters:

- Marginal means:  $\mu_x, \mu_y$
- Marginal standard deviations:  $\sigma_x, \sigma_y$
- Correlation coefficient:  $\rho$

If we fix  $(\mu_x, \sigma_x, \mu_y, \sigma_y)$  and vary  $\rho$ , we generate a family of distributions with *identical marginals but different joint distributions*

*Specifying marginals does not uniquely determine the joint*

Specifying one marginal and its associated conditional does give the joint:

$$\begin{aligned} p(x, y) &= p(x) p(y|x) \\ &= p(y) p(x|y) \end{aligned}$$

What about *specifying the two conditionals*?

## Hammersley-Clifford theorem

We'll be evaluating joint, marginal, and conditional distributions for multiple choices of  $(x, y)$ , so we introduce notation distinguishing the various functions (instead of using  $p()$  for everything):

$$f(x, y) \equiv p(x, y)$$

$$m_1(x) \equiv p(x) = \int dy \, p(x, y)$$

$$m_2(y) \equiv p(y) = \int dx \, p(x, y)$$

$$c_{12}(x; y) \equiv p(x|y)$$

$$c_{21}(y; x) \equiv p(y|x)$$

From the product rule, for any choice of  $a, b$ ,

$$\begin{aligned} f(a, b) &= m_1(a) c_{21}(b; a) \\ \rightarrow m_1(a) &= \frac{f(a, b)}{c_{21}(b; a)}, \text{ for any } b \end{aligned}$$

$$\text{similarly } m_2(b) = \frac{f(a, b)}{c_{12}(a; b)}, \text{ for any } a$$

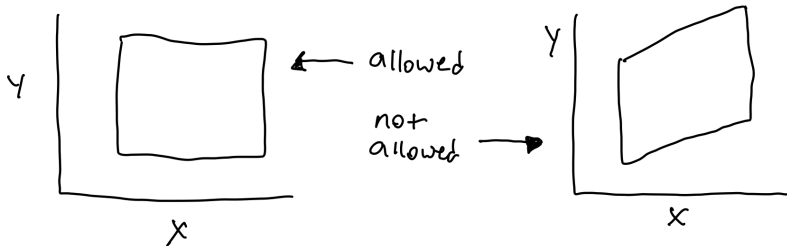
Now use the product rule for  $p(x, y)$ , replacing marginals:

$$\begin{aligned} f(x, y) &= m_1(x) c_{21}(y; x) \\ &= \frac{f(x, b)}{c_{21}(b; x)} c_{21}(y; x), \text{ for any } b \\ &= \frac{m_2(b) c_{12}(x; b)}{c_{21}(b; x)} c_{21}(y; x) \\ &= f(a, b) \frac{c_{12}(x; b)}{c_{12}(a; b)} \frac{c_{21}(y; x)}{c_{21}(b; x)} \end{aligned}$$

for any choice  $(a, b)$  (requires a *positivity condition*: support of joint = cartesian product of supports of marginals)

About the H-C positivity condition:

Box  $\approx$  support, where  $p(x,y) > 0$



$$f(x, y) = f(a, b) \frac{c_{12}(x; b)}{c_{12}(a; b)} \frac{c_{21}(y; x)}{c_{21}(b; x)}$$

Here  $f(a, b)$  is independent of  $(x, y)$ , playing the role of a normalization constant for the remaining  $(x, y)$ -dependent factors

Knowing all the conditionals  
uniquely determines the joint

Uses of this result (and its generalizations):

- Complex graphical models—Markov random fields
- Pseudo-likelihood methods
- *Gibbs sampling*: Using conditionals to build a MH proposal distribution