STSCI 4780/5780
Hierarchical/graphical models for measurement error:
Regression

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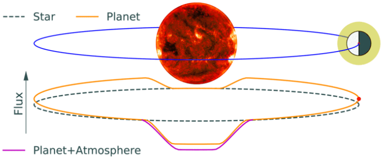
Agenda

1. **Lec21: Density estimation with measurement error**

2. **Regression with measurement error**

Exoplanet detection and characterization

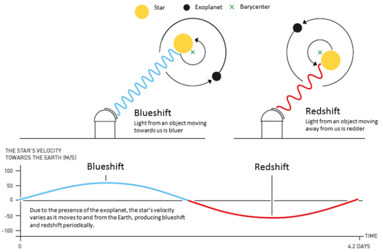
Transit method



Banjac 2019

Measures period, radius, orbit geometry, atmospheric properties

Doppler radial velocity method

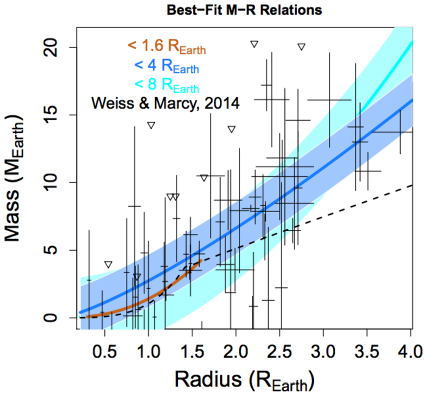


Cardenas 2019

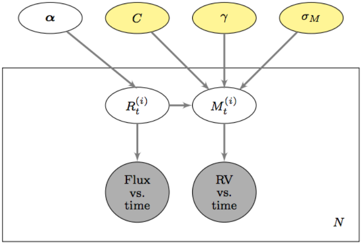
Measures period, mass, orbit geometry

Exoplanets: Regression with measurement error

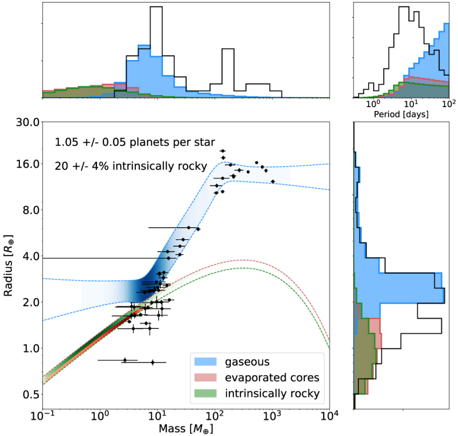
Exoplanet mass vs. radius → avg. density → composition



Wolfgang[+] 2014



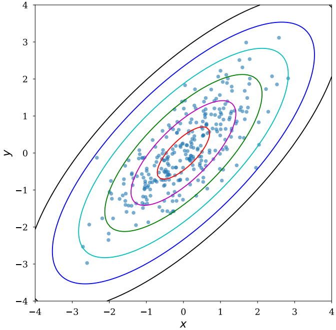
Exoplanet mass/radius/period distribution — Probes exoplanet composition/structure (Neil & Rogers 2020)

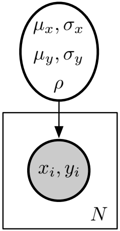


"To fit our models to the data, we use the Python implementation of the Stan statistical software package. Stan uses the No-U-Turn Sampler (NUTS) Markov Chain Monte Carlo (MCMC) algorithm, a method of numerically evaluating hierarchical Bayesian models."

Lec18: Regression with precise predictor data

BVN density estimation

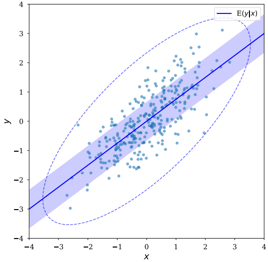


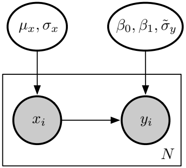


$$\text{Joint:} \quad p(\mu_x, \sigma_x, \mu_y, \sigma_y, \rho) \prod_{i=1}^{N} p(x_i, y_i | \mu_x, \sigma_x, \mu_y, \sigma_y, \rho)$$

$$\text{Inference:} \quad p(\mu_x, \sigma_x, \mu_y, \sigma_y, \rho | \{x_i, y_i\}) \propto \text{Joint}$$

BVN regression





Regression focuses on $p(y|x)$:

$$\text{Joint:} \quad p(\mu_x, \sigma_x)\, p(\beta_0, \beta_1, \tilde{\sigma}_y) \prod_{i=1}^{N} p(x_i|\mu_x, \sigma_x)\, p(y_i|x_i, \beta_0, \beta_1, \tilde{\sigma}_y)$$

$$\text{Inference:} \quad p(\beta_0, \beta_1, \tilde{\sigma}_y|\{x_i, y_i\}) = \int d\mu_x \int d\sigma_x\, p(\mu_x, \sigma_x, \beta_0, \beta_1, \tilde{\sigma}_y|\{x_i, y_i\})$$

$$\propto p(\beta_0, \beta_1, \tilde{\sigma}_y) \prod_{i=1}^{N} p(y_i|x_i, \beta_0, \beta_1, \tilde{\sigma}_y)$$

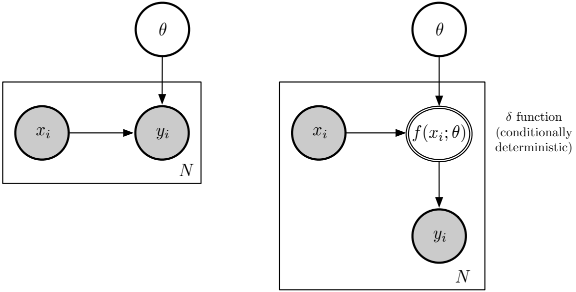
Note that $p(x_i|\cdots)$ drops out

Parametric regression

Infer θ determining the *conditional expectation*

$$\mathbb{E}(y_i|x_i,\theta) = f(x_i;\theta)$$

Regression function may be implicit (conditional expectation of y dist'n) or explicit ("true + error" or "typical + scatter")



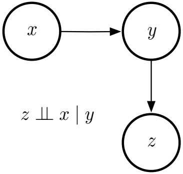
Often natural to express this via an additive error model:

$$y_i = f(x_i;\theta) + \epsilon_i; \qquad \mathbb{E}(\epsilon_i) = 0$$

Lec16: DAGs with missing edges

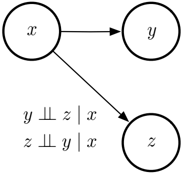
Conditional independence

$p(x)\,p(y|x)\,p(z|y)$



"Causal chain"

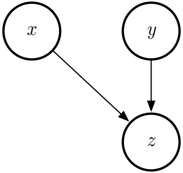
$p(x)\,p(y|x)\,p(z|x)$



"Common cause"

Conditional dependence

$p(x)\,p(y)\,p(z|x,y)$

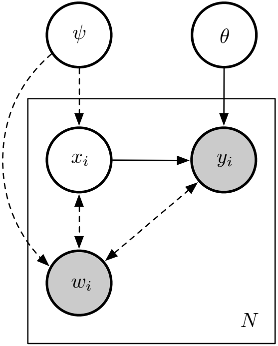


"Multiple causes
or common effect"

Regression with measurement error

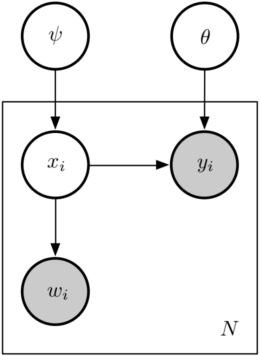
Suppose x_i is not measured precisely—perhaps a surrogate or related quantity, w_i, is measured/specified, or more complex data are available that are linked to x_i in a probabilistic way

There are many possible variations!



See: *Measurement error in nonlinear models* (Carroll, Ruppert, Stefanski, Crainiceanu 2006) CRC Press

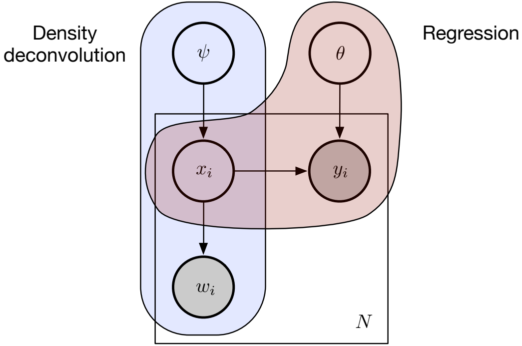
Classical measurement error



Prototype: Noisy measurement of a predictor

$$
\begin{aligned}
x_i &\sim p(x_i|\psi) \\
w_i &= x_i + \delta_i; \quad \delta_i \sim \mathcal{N}(0, \sigma_w^2) \\
y_i &= f(x_i; \theta) + \epsilon_i; \quad \epsilon_i \sim \mathcal{N}(0, \sigma_y^2)
\end{aligned}
$$

May have $\sigma_y \in \theta$ and/or $\sigma_x \in \psi$



Suppose we just want to learn θ, and ψ is *known*, so $p(x_i)$ is a fixed dist'n:

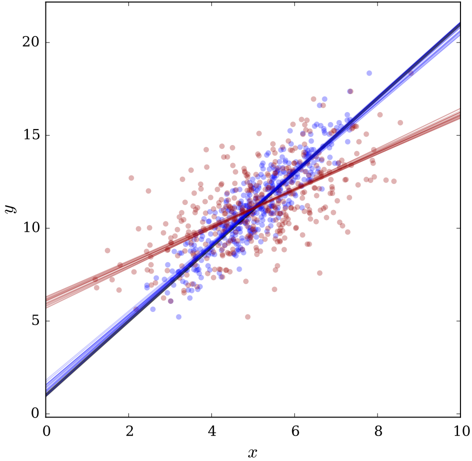
$$p(\theta|\{w_i, y_i\}) \propto \pi(\theta) \prod_i \int dx_i \, p(x_i) \, p(w_i|x_i) \, p(y_i|x_i, \theta)$$

Here *$p(x_i)$ does* not *drop out*; we must know or infer $p(x_i)$, and regression inference is now affected by the x_i marginal

Attenuation

Classical measurement error makes the noisy predictor measurement more dispersed than the true predictor values $\rightarrow$ tend to *under*estimate slopes

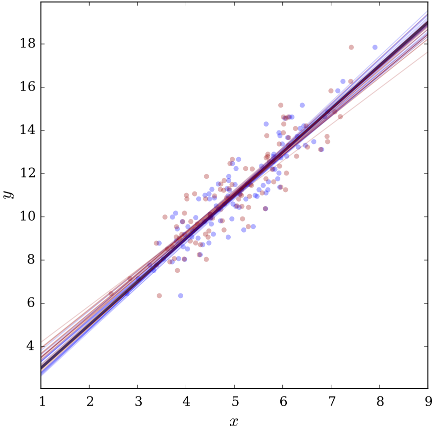
Example: $\sigma_x = 1$, $\sigma_w = 1 \quad \Rightarrow \quad \sqrt{\sigma_x^2 + \sigma_w^2} = \sigma_x \sqrt(2)$



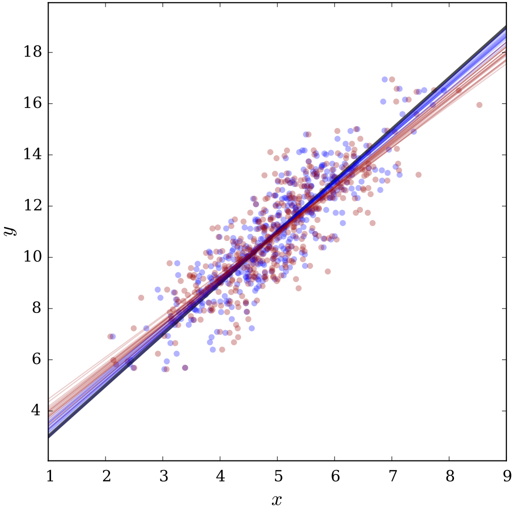
Measurement error does not "average out"

Small measurement error may produce a small effect on estimates, but the effect does not diminish as sample size grows $\rightarrow$ Ignoring measurement error often produces *inconsistent estimates*

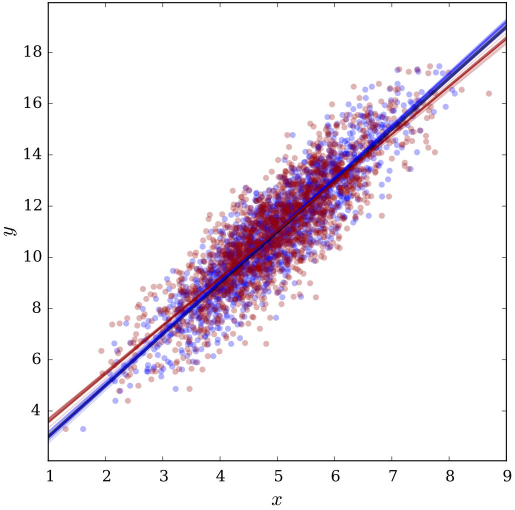
Example: $\sigma_x = 1$, $\sigma_w = 0.286 \quad \Rightarrow \quad \sqrt{\sigma_x^2 + \sigma_w^2} = 1.04\sigma_x$
$N = 100$; Lines are posterior using x_is (blue) or w_is (red)



$N = 400$



$N = 1600$

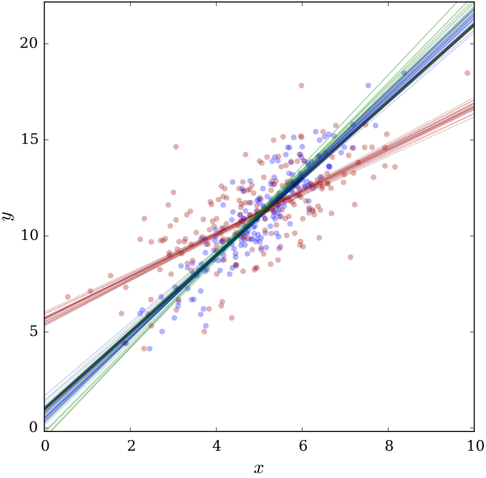


Latent variable model in Stan

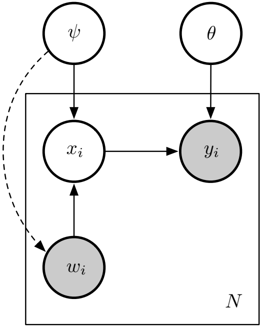
```
merr_code = """
data {
    int<lower=0> n; // number of samples
    real w[n]; // samples
    real y[n]; // samples
}

parameters {
    real beta_0;
    real beta_1;
    real x[n]; // latents
}

model {
    beta_0 ~ normal(0, 10.);  // prior is a wide normal
    beta_1 ~ normal(0, 10.);
    for (i in 1:n) {
        x[i] ~ normal(5., 1.);  // normal marginal for x
        w[i] ~ normal(x[i], %f);
        y[i] ~ normal(beta_0 + beta_1*x[i], 1.);
    }
}
""" % sig_err
```

Example with large measurement error: $\sigma_x = 1$, $\sigma_w = 1$

- Blue: Standard regression using (non-noisy) x_is
- Red: Standard regression using (noisy) w_is
- Green: Classical measurement error model using w_is



Berkson measurement error

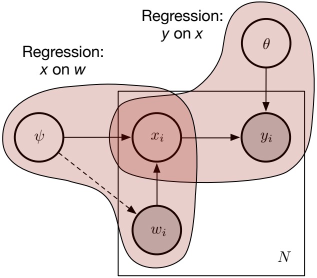


Prototype: Dose-response model

$$w_i \sim p(w_i|\psi) \quad \text{(dose)}$$
$$x_i \sim p(x_i|w_i, \psi)$$
$$x_i = g(w_i; \psi) + \delta_i; \quad \delta_i \sim \mathcal{N}(0, \sigma_x^2)$$
$$y_i = f(x_i; \theta) + \epsilon_i; \quad \epsilon_i \sim \mathcal{N}(0, \sigma_y^2)$$

Simplest: $x_i = w_i + \delta_i$, e.g., received dose is not exactly administered dose

May have $\sigma_y \in \theta$ and/or $\sigma_x \in \psi$



Suppose we just want to learn θ, and ψ is *known* (or absent), so $p(w_i)$ and $p(x_i|w_i)$ are fixed (fully specified) dist'ns:

$$
\begin{aligned}
p(\theta|\{w_i, y_i\}) &\propto \pi(\theta) \prod_i \int dx_i \; p(w_i)\, p(x_i|w_i)\, p(y_i|x_i, \theta) \\
&\propto \pi(\theta) \left(\prod_i p(w_i)\right) \prod_i \int dx_i \; p(x_i|w_i)\, p(y_i|x_i, \theta)
\end{aligned}
$$

Here $p(x_i|w_i)$ *does* not *drop out*; we must know or infer it

But $p(w_i)$ **does** drop out (as long as $w_i \perp\!\!\!\perp \psi$)

Classical vs. Berkson measurement error

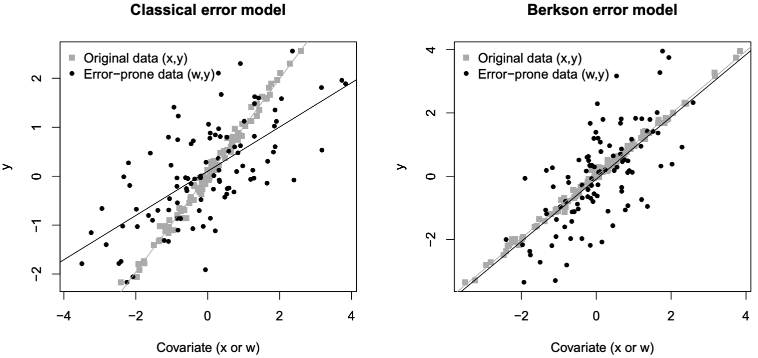


Fig. 3. Effect of ME in the linear model. Left: Classical ME. Two effects can be seen: 1) The absolute value of the covariate estimate is biased (attenuated); 2) The variability around the regression line in the data with ME (black circles) is much larger than in the case of the truly observed data (grey squares). Right: Berkson ME. The absolute value of the covariate estimate is unbiased in the linear model, while the variability around the regression line is larger for the data with ME.

Muff et al. 2013

Clustered data and mixed effects models

Examples of clustered data

- Subpopulations: Country $\rightarrow$ States $\rightarrow$ Counties
- Repeated measurements of related items
- Longitudinal studies of populations
- Meta-analysis—combining results of multiple studies

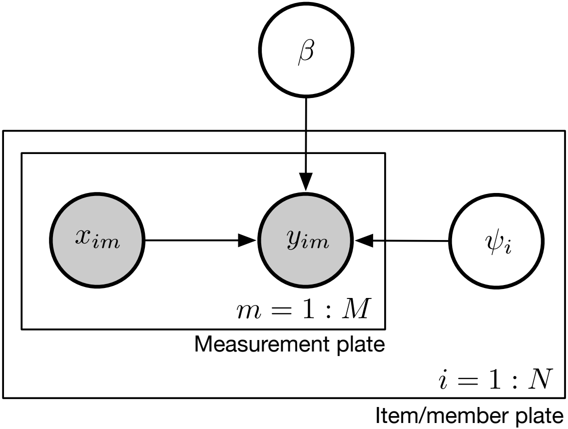
Mixed effects models

Model for measurement m (replication/time/study) on item i:

$$y_{im} = f(x_{im}; \beta) + g(x_{im}; \psi_i) + \epsilon_{im}$$

- β: *fixed (shared/bulk) effect*
  $f(x_{im})$ is like a shared *template*
- ψ_i: *random (individual/specific/peculiar) effect*
  $g(x_{im})$ is the individual departure from the template

DAG for mixed effects models



Mixed effects — Explicit shared/specific effects