Math2411 - Calculus II
Section 001 Fall 2024
Separable Differential Equations

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Separable Differential Equations Introduction:

Our objective is to solve ***separable differential equations***. But what is a differential equation? It is an equation that describes a relation between a dependent variable, the independent variable, and one or more derivatives of the dependent variable.

> **Definition**
>
> A **differential equation** is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Here are some examples of differential equations and solutions.

Equation	Solution
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$
$y'' - 3y' + 2y = 24e^{-2x}$	$y = 3e^x - 4e^{2x} + 2e^{-2x}$

Figure 1: Differential Equations and Solutions

We will be dealing exclusively with ***first order differential equations***.

> **Definition**
>
> The **order of a differential equation** is the highest order of any derivative of the unknown function that appears in the equation.

In general, for first order equations we will be able to write $y' = F(x, y)$ to describe a known relationship between the quantities y', y and x.

Here are a few examples.

$$
\begin{aligned}
y' &= \left(x^2-4\right)(3y+2) \\
y' &= 6x^2+4x \\
y' &= \sec y + \tan y \\
y' &= xy + 3x - 2y - 6.
\end{aligned}
$$

Figure 2: Examples of Differential Equations in the Form $y' = F(x, y)$.

Separable Differential Equations:

We will be studying ***separable differential equations***. As the name suggests, these are equations in which the quantities involving the independent variable can be separated from quantities involving the dependent variable. That is, a ***separable differential equation*** is a differential equation that can be written in the form

$$n(y) \cdot y' = m(x).$$

We are hoping for a solution algorithm so that we can avoid guessing and checking. To understand the algorithm we should remember implicit differentiation. Suppose that $y = y(x)$. Then, by the chain rule, the following calculus relationships hold.

$$N(y) = M(x) + C \quad \underset{\int \quad \frac{d}{dx}}{\overset{\frac{d}{dx}}{\Longleftrightarrow}} \quad \frac{dN}{dy}\frac{dy}{dx} = \frac{dM}{dx} \quad \text{or we can write as} \quad n(y) \cdot y' = m(x)$$

This means the following:

- If we differentiate (with respect to x) both sides of the equation
$$N(y) = M(x) + C$$
we end up with the relationship
$$n(y) \cdot y' = m(x) + C.$$
This is the familiar implicit differentiation.

- If we integrate (with respect to x) both sides of the equation
$$n(y) \cdot y' = m(x) + C$$
we end up with the relationship
$$N(y) = M(x) + C$$
In some sense we are simply reversing implicit differentiation.

We are starting with the equation $n(y) \cdot y' = m(x) + C$. The above tells us that the desired relationship that describes our solution is

$$N(y) = M(x) + C \qquad \Longleftrightarrow \qquad \int n(y)\, dy = \int m(x)\, dx + C$$

Let's work an example together.

Example 1. Solve the differential equation $y' = 5y^2x^3$.

Workspace:

Solution: We first must separate the equation as $\left(\dfrac{1}{y^2}\right) \cdot y' = 5x^3$. We identify $n(y) = 1/y^2$ and $m(x) = 5x^3$. So our solution is given by the following.

$$\begin{aligned} \int \frac{1}{y^2}\,dy &= \int 5x^3\,dx + C \\ -\frac{1}{y} &= \frac{5}{4}x^4 + C \\ &\downarrow \\ y(x) &= \frac{1}{C - \frac{5}{4}x^4} = \frac{4}{C - 5x^4} \end{aligned}$$

This is the ***general solution***. Then different values of the constant C give different solution behavior.

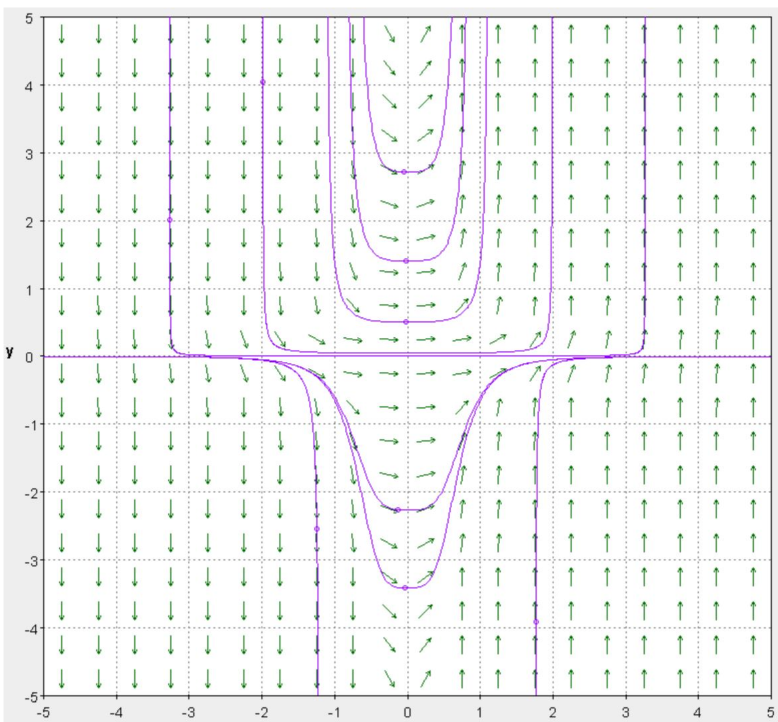

Figure 3: Family of Solutions to $y' = 5y^2x^3$.

Question: How do we determine the constant C? That is, which of the above graphs will be our desired solution?

Answer: An initial condition or starting state.

Suppose we knew that $y(0) = 2$. We can use this information to solve for C. Why don't you give a try on your own.

Workspace:

Solution: Setting $y = 2$ and $x = 0$ we get

$$2 = \frac{4}{C - 5 \cdot 0^2} \quad \Longrightarrow \quad C = \frac{1}{2}$$

So we have an ***explicit solution***.

$$y(x) = \frac{4}{\frac{1}{2} - 5x^2} = \frac{8}{1 - 10x^2}$$

Next Question: Over what interval is the solution valid?

Well we must satisfy $1 - 10x^2 \neq 0$ and so we satisfy $x \neq \pm\sqrt[4]{1/10}$. Then since our initial condition with $x = 0$ is between $-\sqrt[4]{1/10}$ and $\sqrt[4]{1/10}$ our interval of validity is $\left(-\sqrt[4]{1/10}, \sqrt[4]{1/10}\right)$. We can see the graph in the following diagram.

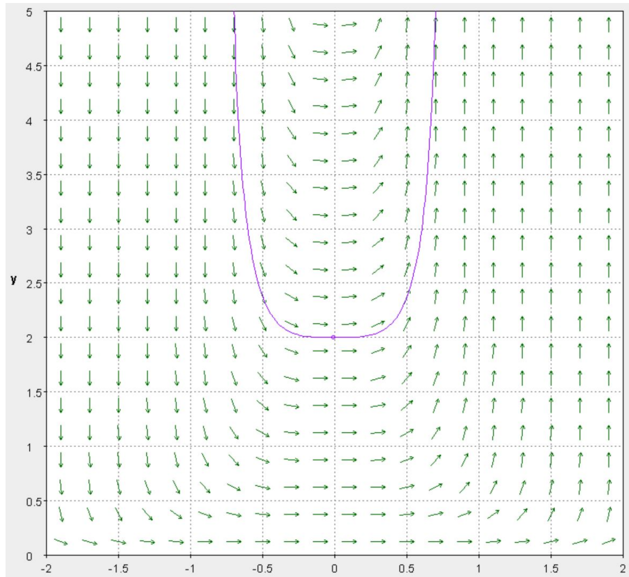

Figure 4: Explicit Solution to $y' = 5y^2x^3$ when $y(0) = 2$.

Let's try another example.

Example 2. Solve the initial value problem $y' = ky$ where $y(0) = y_0$ and $k \neq 0$. This is a basic model for population growth or decay problems.

Workspace:

Solution: We first must separate the equation as $\left(\frac{1}{y}\right) \cdot y' = ky$. Note that we are implicitly assuming $y \neq 0$ in this step. We identify $n(y) = 1/y$ and $m(x) = k$. So our solution is given by the following.

$$\int \frac{1}{y}\,dy = \int k\,dx + C.$$

$$\begin{aligned} \int \frac{1}{y}\,dy = \int k\,dx + C \\ \ln|y| &= kx + C \\ |y| &= e^{kx+C} \\ y(x) = Ce^{kx} \end{aligned}$$

This is our general solution. Then substituting $y(0) = y_0$ we get

$$y_0 = Ce^0 \qquad \Longrightarrow \qquad y(x) = y_0 e^{kx}.$$

Clearly, the interval of validity is $(-\infty, 0)$.

Let's try another example.

Example 3. Solve the initial value problem $y' = (2x+3)(y^2-4)$ where $y(0) = -1$.

<u>Workspace:</u>

Workspace Continued:

Solution: We first must separate the equation as $\left(\frac{1}{y^2}\right) \cdot y' = 5x^3$. We identify $n(y) = 1/y^2$ and $m(x) = 5x^3$. So our solution is given by the following.

$$\int \frac{1}{y^2-4}\, dy = \int 2x + 3\, dx + C.$$

To evaluate the left-hand side, use the method of partial fraction decomposition. This leads to the identity

$$\frac{1}{y^2-4} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right).$$

Next we have the following:

$$\begin{aligned} \frac{1}{4}\int\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy &= \int (2x+3)\,dx \\ \frac{1}{4}(\ln|y-2| - \ln|y+2|) &= x^2 + 3x + C. \end{aligned}$$

This is an implicit description of our solution. It is possible to solve for y by using properties of logarithms and then exponentiating both sides.

$$\left|\frac{y-2}{y+2}\right| = e^{4x^2+12x+C} = Ce^{4x^2+12x}$$

We can remove the absolute values since our constant on the right-hand side of the equation can absorb the negative 1. We next rearrange to get

$$y - 2 = C(y+2)e^{4x^2+12x} \quad \Longrightarrow \quad y\left(1 - Ce^{4x^2+12x}\right) = 2\left(1 + Ce^{4x^2+12x}\right)$$

Finally we have the general solution.

$$y(x) = \frac{2\left(1 + Ce^{4x^2+12x}\right)}{1 - Ce^{4x^2+12x}}$$

We next need to solve for C using $y(0) = -3$. Substituting into our solution we get

$$-1 = \frac{2\left(1 + Ce^{4\cdot 0^2+12\cdot 0}\right)}{1 - Ce^{4\cdot 0^2+12\cdot 0}} \quad \Longrightarrow \quad -1 = \frac{2(1+C)}{1-C} \quad \Longrightarrow \quad C = -3.$$

So our explicit solution is given as

$$y(x) = \frac{2\left(1 - 3e^{4x^2+12x}\right)}{1 + 3e^{4x^2+12x}}$$

Since the denominator is never zero the interval of validity for this solution is $(-\infty, \infty)$.

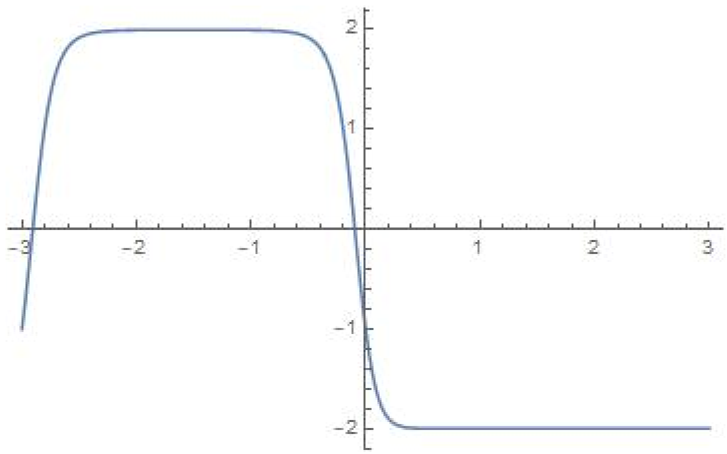

Figure 5: Explicit Solution to $y' = (2x+3)(y^2-4)$ where $y(0) = -1$.

Let's try another example.
Let's try another example.

Example 4. Solve the initial value problem $y' = ky(M - y)$ where $y(0) = y_0$, $y \geq 0$, and $k > 0$. This is a logistic model for population growth.

Workspace:

Solution: We first must separate the equation as $\left(\frac{1}{y(M-y)}\right) \cdot y' = ky$. Note that we are implicitly assuming $y \neq 0$ in this step. We identify $n(y) = 1/y(M-y)$ and $m(x) = k$. So our solution is given by the following.

$$\int \frac{1}{y(M-y)}\, dy = \int k\, dx + C.$$

Our first step is partial fraction decomposition of the left side.

$$\frac{1}{y(M-y)} = \frac{A}{y} + \frac{B}{M-y} = \frac{1}{M}\left(\frac{1}{y} + \frac{1}{M-y}\right).$$

$$\begin{aligned}
\int \frac{1}{y(M-y)}\, dy = \int k\, dx + C \\
\frac{1}{M}\int \frac{1}{y} + \frac{1}{M-y}\, dy &= \int k\, dx + C \\
\frac{1}{M}\Big(\ln|y| - \ln|M-y|\Big) &= kx + C \\
\frac{1}{M}\ln\left|\frac{y}{M-y}\right| &= kx + C \\
\ln\left|\frac{y}{M-y}\right| &= Mkx + C \\
\frac{y}{M-y} &= Ce^{Mkx} \\
&\vdots \\
y(x) &= \frac{M}{1 + Ce^{-Mkx}}
\end{aligned}$$

This is our general solution. If we let $y(0) = y_0$ then we have $C = (M - y_0)/y_0$. Clearly, the interval of validity is $(-\infty, \infty)$.

Question: Think about why this population growth model might be an improvement over our exponential growth model from before. You can see the plot of the family of solution functions on the next page where we have $k = 2$ and $M = 10$.

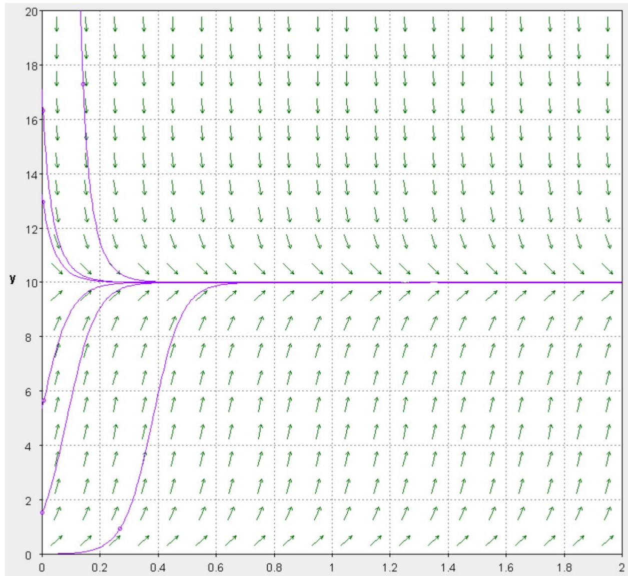

Figure 6: Family of Solutions to $y' = 2y(10 - y)$.

Please let me know if you have any questions, comments, or corrections!