Math2411 - Calculus II
Guided Lecture Notes
Power Series

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Power Series Introduction:

Our objective is to study a class of infinite series called ***power series***.

Definition

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \tag{6.1}$$

is a power series centered at $x = 0$. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots \tag{6.2}$$

is a power series centered at $x = a$.

Question: What do you notice that is different about a power series compared to the series we have previously studied?

Answer: The terms of a power series contain a variable quantity x and so can be thought of as functions $f(x)$. One of the most important questions we ask about a function f is the following: "What is the domain of f?" In other words, what are the x-values where our function $f(x)$ is defined (or makes sense)? Since we are dealing with an infinite series we could pose this question as follows: For which x-values does the power series converge? We can summarize the answer to this question with the following theorem.

Theorem 6.1: Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.

ii. The series converges for all real numbers x.

iii. There exists a real number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. At the values x where $|x-a| = R$, the series may converge or diverge.

Look at the following diagram to get a better sense of the statement of the theorem.

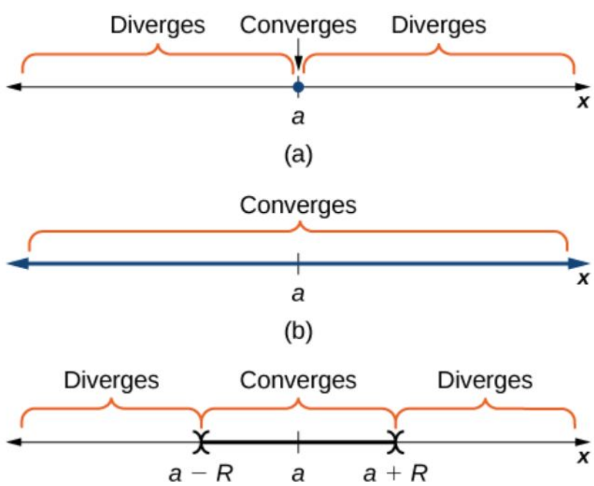

Figure 1: Three Convergence Possibilities for a Power Series

Here is some useful terminology that we will be using.

Definition

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$

Let's work some examples and discover the process.

Power Series Examples:

Example 1. Find the interval of convergence and the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Workspace:

Let's try another example.

Example 2. Find the interval of convergence and the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}.$$

Workspace:

Let's try another example.

Example 3. Find the interval of convergence and the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}}.$$

Workspace:

Let's try another example.

Example 4. Find the interval of convergence and the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{n(x+1)^n}{e^n}.$$

Workspace: