Math2411 - Calculus II
Guided Lecture Notes
Parametric Equations

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Parametric Equations Introduction:

Our objective is to examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path. One example could be describing the orbit of the Earth around the sun.

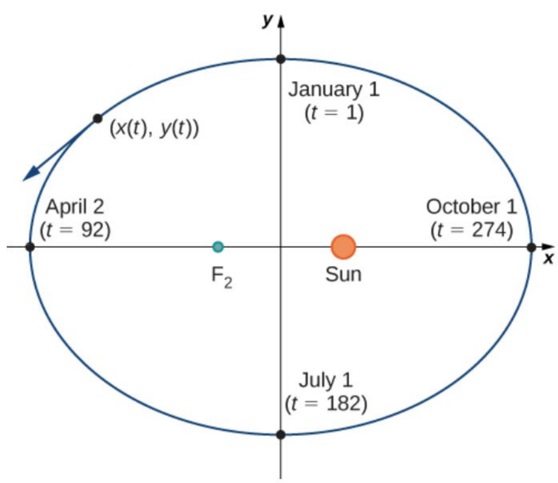

Figure 1: Coordinate Axes Superimposed on the Orbit of Earth

Definition

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t) \text{ and } y = y(t)$$

are called parametric equations and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a **parametric curve** or *plane curve*, and is denoted by C.

Parametric Curve Examples:

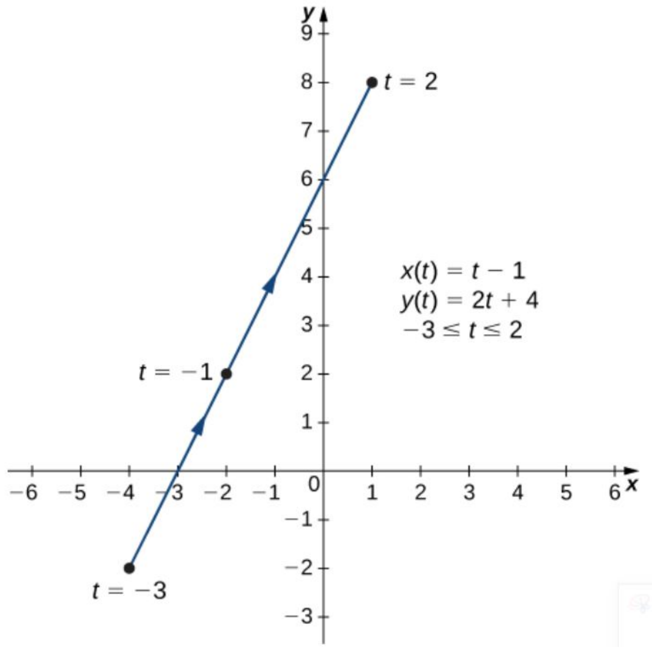

Figure 2: Graph of Parametric Curve with $x(t) = t - 1$ and $y(t) = 2t + 4$ where $-3 \leq t \leq 2$

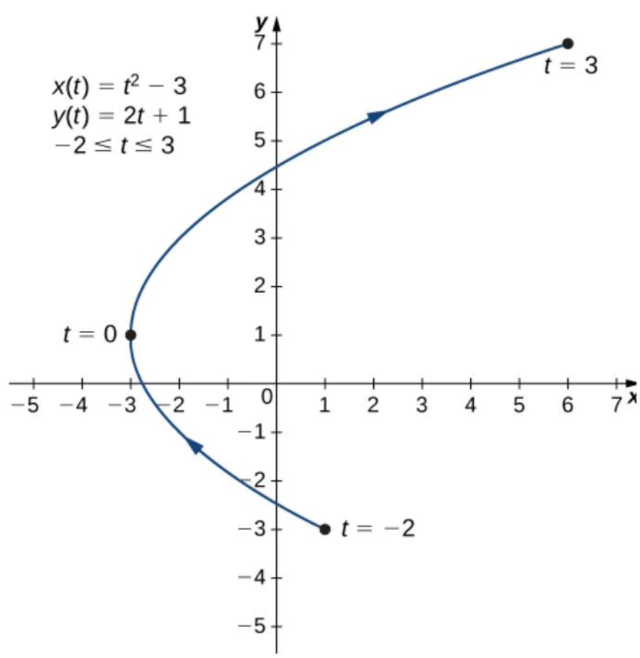

Figure 3: Graph of Parametric Curve with $x(t) = t^2 - 3$ and $y(t) = 2t + 1$ where $-2 \leq t \leq 3$

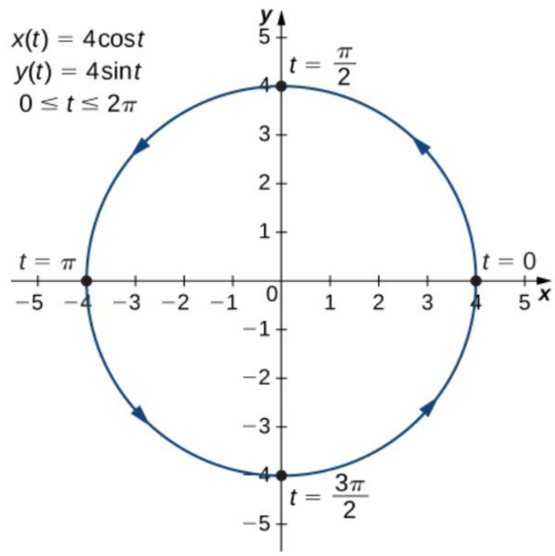

Figure 4: Graph of Parametric Curve with $x(t) = 4\cos(t)$ and $y(t) = 4\sin(t)$ where $0 \le t \le 2\pi$

Eliminating the Parameter:

Suppose we have a parametric curve described as $x(t) = t - 1$ and $y(t) = 2t + 4$ where $-3 \le t \le 2$. Can we write this in the form $y = f(x)$? This is called eliminating the parameter.

If possible, solve $x = x(t)$ for t. $x = t - 1$ gives us $t = x + 1$. Then substitute back into $y(t) = 2t + 4$ to get

$$y = 2t + 4 = 2(x + 1) + 4 = 2x + 6.$$

Because $-3 \le t \le 2$ we have $-3 \le x + 1 \le 2$ or $-4 \le x \le 1$. So the parametric curve is the same graph as the straight line given by $y = 2x + 6$ on the interval $[-4, 1]$. This agrees with our earlier sketch of the parametric curve.

Let's have you work an example independently.

Example 1. Eliminate the parameter for the parametric curve given by $x(t) = 4\cos(t)$ and $y(t) = 4\sin(t)$ where $0 \le t \le 2\pi$.

Workspace:

Example 2. Eliminate the parameter for the parametric curve given by $x(t) = t^2 - 3$ and $y(t) = 2t + 1$ where $-2 \leq t \leq 3$.

Workspace:

Let's work another example.

Example 3. Eliminate the parameter for the parametric curve given by $x(t) = \sqrt{2t+4}$ and $y(t) = 2t+1$ where $-2 \leq t \leq 6$.

Workspace:

Parameterize a Curve:

Suppose we have $y = 2x^2 - 3$ for all x-values. The simplest way to parameterize the curve is to let $x(t) = t$ and $y(t) = 2t^2 - 3$. But this is not the only choice. We could choose any $x(t)$ that takes on all x-values. For example, we could let $x(t) = 3t - 2$ for all t-values. Then we have

$$y(t) = 2x^2 - 3 = 2\big(3t - 2\big)^2 - 3 = 18t^2 - 24t + 6.$$

We have the parametric curve $x(t) = 3t - 2$ and $y(t) = 18t^2 - 24t + 6$ for all t-values.

Please let me know if you have any questions, comments, or corrections!