Math2411 - Calculus II
Guided Lecture Notes
Properties of Power Series

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Properties of Power Series Introduction:

One of the best things about working with power series is they are very easy to manipulate to create other series. For example, we can add, subtract, scale, and multiply power series in the most natural ways.

Theorem 6.2: Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.

i. The power series $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.

ii. For any integer $m \geq 0$ and any real number b, the power series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ on I.

iii. For any integer $m \geq 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.

Theorem 6.3: Multiplying Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let

$$\begin{aligned} e_n &= c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1} d_1 + c_n d_0 \\ &= \sum_{k=0}^{n} c_k d_{n-k}. \end{aligned}$$

Then

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

and

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x) \cdot g(x) \text{ on } I.$$

The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the Cauchy product of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.

We can also make substitutions. Lets consider a few examples with substitutions.

Power Series Examples:

Example 1. We can show that the function $f(x) = \frac{1}{1-x}$ can be represented as the following power series with interval of convergence $(-1, 1)$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Can we use this series to find a power series representation of the function $\frac{1}{1+2x}$?

Workspace:

Let's try another example.

Example 2. We can show that the function $f(x) = \frac{1}{1-x}$ can be represented as the following power series with interval of convergence $(-1, 1)$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Can we use this series to find a power series representation of the function $\frac{1}{1+x^2}$?

<u>Workspace:</u>

Let's try another example.

Example 3. We can show that the function $f(x) = \frac{1}{1+x^2}$ can be represented as the following power series with interval of convergence $(-1, 1)$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

Can we use this series to find a power series representation of the function $\tan^{-1}(x)$? Yes.

Theorem 6.4: Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on the interval $(a-R, a+R)$ for some $R > 0$. Let f be the function defined by the series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \end{aligned}$$

for $|x-a| < R$. Then f is differentiable on the interval $(a-R, a+R)$ and we can find f' by differentiating the series term-by-term:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} nc_n (x-a)^{n-1} \\ &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \end{aligned}$$

for $|x-a| < R$. Also, to find $\int f(x)dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R, a+R)$, and we have

$$\begin{aligned} \int f(x)dx &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \\ &= C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots \end{aligned}$$

for $|x-a| < R$.

Workspace:

Workspace Cont.:

Let's try another example.

Example 4. We can show that the function $f(x) = \sin(x)$ can be represented as the following power series with interval of convergence $(-\infty, \infty)$.

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

Can we use this series to find a power series representation of the function $\cos(x)$? Yes.

Workspace:

Example 5. Use the power series for $\sin(x)$ centered at $a = 0$ to evaluate the integral $\displaystyle\int_{x=0}^{x=1} \frac{\sin(x)}{x}\, dx$.

Workspace: