

Math2411 - Calculus II

Guided Lecture Notes

Integration by Parts

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Integration by Parts Introduction:

$$\int x e^{x^2} dx \quad \text{versus} \quad \int x e^x dx$$

We need a technique to evaluate integrals of products, where u -sub does not work. Something like a “product rule” for integration.

$$\frac{d}{dx} [u(x)v(x)] = u'(x)v(x) + u(x)v'(x) \quad \xrightarrow[\text{Integrate both sides with respect to } x]{} \quad u(x)v(x) = \int u'(x)v(x) dx + \int u(x)v'(x) dx$$

We can rewrite this as

$$\int u(x)v'(x) dx = u(x)v(x) - \int v(x)u'(x) dx$$

or in differential form (by suppressing the variable x).

$$\int \underbrace{u(x)}_{\text{Write as } u} \underbrace{v'(x) dx}_{\text{Write as } dv} = \underbrace{u(x)v(x)}_{\text{Write as } uv} - \int \underbrace{v(x)}_{\text{Write as } v} \underbrace{u'(x) dx}_{\text{Write as } dv} = \int u dv = uv - \int v du.$$

Integration by Parts Examples:

Let's work an example together.

Example 1. Evaluate the integral $\int x e^x dx$

Workspace:

Suppose we had decided to choose $u(x) = e^x$ and $v'(x) = x$. Then $u'(x) = e^x$ and $v(x) = x^2/2$. Check the solution using integration by parts. What do you notice?

Workspace:

Later, we will see examples where we don't make this choice. But first, try this next example on your own.

Example 2. Evaluate the integral $\int 2x \sin x \, dx$

Workspace:

Let's try another example on your own.

Example 3. Evaluate the integral using integration by parts $\int x\sqrt{x+1} \, dx$.

Workspace:

Let's have you try another example on your own.

Example 4. Evaluate the integral using integration by parts $\int x \ln(x) dx$.

Workspace:

So if we are unable to integrate our choice of $v'(x)$ then we have to make another choice. Some people prefer a helpful mnemonic device to make these choices. Here is a popular one for choosing $u(x)$.

Choosing a $u(x)$: We begin integration by parts problems by choosing a u . A helpful mnemonic device to help you choose a u is "LIATE". This does not always work, but is a good way to start a problem if you're stuck. Choose $u(x)$ by which term comes first:

L: Logarithmic functions

I: Inverse trigonometric functions

A: Algebraic functions (things like x^2, x^3)

T: Trigonometric functions (such as $\sin x, \cos x, \tan x$)

E: Exponential functions (such as $e^x, 3^x$)

Let's have you try another example on your own.

Example 5. Evaluate the integral $\int x^2 \sin(10x) dx$.

Workspace:

The Tabular Method

Example 6. Evaluate the integral $\int x^2 e^{3x} dx$

Workspace:

Example 7. Evaluate the integral $\int x^4 \cos(5x) dx$ using the tabular method.

Workspace:

Definite Integrals Using Integration By Parts

We have the following formula.

$$\int_{x=a}^{x=b} u(x)v'(x) dx = u(x)v(x) \Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v(x)u'(x) dx$$

The notation in this formula often leads to some confusion. So let's consider an example.

Example 8. Evaluate the integral $\int_{x=0}^{x=\pi/3} 2x \sin x \, dx$

Workspace:

Let's try another example.

Example 9. Evaluate the integral $\int_{x=0}^{x=\pi/12} \frac{x^2}{\sec(4x)} dx$

Hint: Can you rewrite $1/\sec(4x)$?

Workspace:

Some Interesting and Important Examples

Example 10. Evaluate the integral $\int e^x \sin x \, dx$

Workspace:

Example 11. Evaluate the integral $\int \ln(x) dx$.

Workspace:

Now see if you can use the same idea on another example.

Example 12. Evaluate the integral $\int \tan^{-1}(x) dx$.

Workspace:

The idea is if we are integrating a function where we know its derivative but not its antiderivative sometimes multiplying the function by 1 and then using integration by parts will be helpful. But not always. The following antiderivative does NOT have an elementary solution. None can be found no matter the technique tried.

$$\int e^{-x^2} dx.$$

It should be enlightening to realize that not all mathematical problems have elementary solution techniques. But integration by parts allows to integrate many more functions than we could before.