Math2411 - Calculus II
Guided Lecture Notes
Integration by Parts

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Integration by Parts Introduction:

$$\int xe^{x^2}\,dx \qquad \text{versus} \qquad \int xe^{x}\,dx$$

We need a technique to evaluate integrals of products, where u-sub does not work. Something like a "product rule" for integration.

$$\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x)+u(x)v'(x) \quad \xrightarrow[\text{with respect to x}]{\text{Integrate both sides}} \quad u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

We can rewrite this as

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

or in differential form (by suppressing the variable x).

$$\int \underbrace{u(x)}_{\text{Write as } u} \underbrace{v'(x)\,dx}_{\text{Write as } dv} = \underbrace{u(x)v(x)}_{\text{Write as } uv} - \int \underbrace{v(x)}_{\text{Write as } v} \underbrace{u'(x)\,dx}_{\text{Write as } dv} = \int u\,dv = uv - \int v\,du.$$

Integration by Parts Examples:

Let's work an example together.

Example 1. Evaluate the integral $\int xe^{x}\,dx$

Workspace:

Solution:

Choose $u(x) = x$ and $v'(x) = e^x$ so that $u'(x) = \frac{d}{dx}[x] = 1$ and then we have $v(x) = \int e^x\, dx = e^x$

So we have that

$$\int xe^x\, dx = xe^x - \int e^x \cdot 1\, dx = xe^x - e^x + C = e^x(x-1) + C.$$

So

$$\int xe^x\, dx = xe^x - e^x + C = e^x(x-1) + C.$$

Suppose we had decided to choose $u(x) = e^x$ and $v'(x) = x$. Then $u'(x) = e^x$ and $v(x) = x^2/2$. Check the solution using integration by parts. What do you notice?

Workspace:

Solution:

If we choose $u(x) = e^x$ and $v'(x) = x$. Then $u'(x) = e^x$ and $v(x) = x^2/2$ and we have

$$\int xe^x\,dx = \frac{1}{2}x^2e^x - \frac{1}{2}\int x^2e^x\,dx.$$

This expression is correct, but the new integral produced by parts is more complicated.

Question: Can you think of a criteria for choosing $u(x)$ and $v'(x)$?

- We want our new integral to be simpler (or at the least not more complicated)! As a general rule, if there is a polynomial term such as a x^k, we will choose $u(x) = x^k$ because its derivative is simpler:

$$\frac{d}{dx}\left[x^k\right] = kx^{k-1}.$$

Later, we will see examples where we don't make this choice. But first, try this next example on your own.

Example 2. Evaluate the integral $\int 2x\sin x\,dx$

Workspace:

Solution:

Choose $u(x) = 2x$, and $v'(x) = \sin x$. So that $u'(x) = \frac{d}{dx}[2x] = 2$ and $v(x) = \int \sin x\, dx = -\cos x$

Then we have that:

$$\begin{aligned}\int x \sin x\, dx &= 2x\big(-\cos(x)\big) - \int -2\cos x\, dx \\ &= -2x\cos x + 2\int \cos x\, dx \\ &= -2x\cos x + 2\sin x + C\end{aligned}$$

So the integral evaluates to:

$$\int x \sin x\, dx = -2x\cos x + 2\sin x + C$$

Let's try another example on your own.

Example 3. Evaluate the integral using integration by parts $\displaystyle\int x\sqrt{x+1}\, dx$.

Workspace:

Solution:

You may notice that you can solve this integral using u-sub from calculus I. You can also use integration by parts. So for the sake of practice, let's use integration by parts to solve this integral.

Choose $u(x) = x$ and $v'(x) = \sqrt{x+1}$. So that $u'(x) = \frac{d}{dx}[x] = 1$ and $v(x) = \int \sqrt{x+1}\,dx = \frac{2}{3}(x+1)^{\frac{3}{2}}$

So we have that that

$$\begin{aligned} \int x\sqrt{x+1}\,dx &= x \cdot \frac{2}{3}(x+1)^{\frac{3}{2}} - \int \frac{2}{3}(x+1)^{\frac{3}{2}} \cdot 1\,dx \\ &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3} \cdot \frac{2}{5}(x+1)^{\frac{5}{2}} \end{aligned}$$

So the integral evaluates to

$$\int x\sqrt{x+1}\,dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C.$$

Let's have you try another example on your own.

Example 4. Evaluate the integral using integration by parts $\int x\ln(x)\,dx$.

Workspace:

Solution:

Suppose we choose $u(x) = x$ and $v'(x) = \ln(x)$. Then $u'(x) = \frac{d}{dx}[x] = 1$. But we don't know $\int \ln(x)\,dx$. So it looks like we have no choice but to try $u(x) = \ln(x)$ and $v'(x) = x$. Then $u'(x) = \frac{d}{dx}[\ln(x)] = 1/x$ and $v(x) = \int x\,dx = x^2/2$. Then we have

$$\begin{aligned}\int x\ln(x)\,dx &= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int \frac{1}{x}\cdot x\,dx \\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int 1\,dx\end{aligned}$$

So the integral evaluates to

$$\int x\ln(x)\,dx = \frac{1}{2}x^2\ln(x) - \frac{x}{2} + C.$$

So if we our unable to integrate our choice of $v'(x)$ then we have to make another choice. Some people prefer a helpful mnemonic device to make these choice. Here is a popular one for choosing $u(x)$.

Choosing a $u(x)$***:*** We begin integration by parts problems by choosing a u. A helpful mnemonic device to help you choose a u is "LIATE". This does not always work, but is a good way to start a problem if you're stuck. Choose $u(x)$ by which terms comes first:

L: Logarithmic functions
I: Inverse trigonometric functions
A: Algebraic functions (things like x^2, x^3)
T: Trigonometric functions (such as $\sin x, \cos x, \tan x$)
E: Exponential functions (such as $e^x, 3^x$)

Let's have you try another example on your own.

Example 5. Evaluate the integral $\displaystyle\int x^2 \sin(10x)\,dx$.

Workspace:

Workspace Cont:

Solution:

Choose $u(x) = x^2$ and $v'(x) = \sin(10x)$. So that $u'(x) = \frac{d}{dx}[x^2] = 2x$ and $v(x) = \int \sin(10x)\,dx = -\frac{1}{10}\cos(10x)$.

It follows that

$$\begin{aligned}\int x^2 \sin(10x)\,dx &= x^2\Big(-\frac{1}{10}\cos(10x)\Big) - \int -\frac{1}{10}\cos(10x)\cdot 2x\,dx. \\ &= -\frac{x^2}{10}\cos(10x) + \frac{1}{5}\int x\cos(10x)\,dx\end{aligned}$$

Observation: Our new integral produced by parts is indeed simpler. (How?) So we now choose a strategy to evaluate the second integral. Here it looks like integration by parts is needed to solve $\int x\cos(10x)dx$.

To solve this integral choose $u(x) = x$, and $v'(x) = \cos(10x)$. So that $u'(x) = 1$ and $v(x) = \frac{1}{10}\sin(10x)$. It follows that

$$\int x\cos(10x)\,dx = \frac{1}{10}x\sin(10x) - \frac{1}{10}\int \sin(10x)\,dx = \frac{x}{10}\sin(10x) + \frac{1}{10}\cos(10x).$$

Putting it altogether we have that

$$\begin{aligned}\int x^2\sin(10x)\,dx &= -\frac{x^2}{10}\cos(10x) + \frac{1}{5}\left(\frac{x}{10}\sin(10x) + \frac{1}{10}\cos(10x)\right) + C \\ &= -\frac{x^2}{10}\cos(10x) + \frac{x}{50}\sin(10x) + \frac{1}{500}\cos(10x) + C\end{aligned}$$

The Tabular Method

Example 6. Evaluate the integral $\int x^2 e^{3x}\,dx$

Workspace:

Workspace Cont:

Solution:

We have

$$\begin{aligned}
\int x^2 e^{3x}\,dx &= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\,dx \\
&= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx\right) \\
&= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x} + C
\end{aligned}$$

We will keep track of the previous steps using a table.

Step #	u		v'		Results
1	x^2		e^{3x}		
		$\searrow \times$		$\xrightarrow[\text{Result after one step}]{}$	$\frac{1}{3}x^2e^{3x} - \int 2x \cdot \frac{1}{3}e^{3x}\,dx$
2	$2x$	$\xleftrightarrow[\text{Integrate}]{-}$	$\frac{1}{3}e^{3x}$		
		$\searrow - \times$		$\xrightarrow[\text{Result after two steps}]{}$	$\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \int 2 \cdot \frac{1}{9}e^{3x}\,dx$
3	2	$\xleftrightarrow[\text{Integrate}]{}$	$\frac{1}{9}e^{3x}$		
		$\searrow \times$		$\xrightarrow[\text{Result after three steps}]{}$	$\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$
	0	$\xleftrightarrow[\text{Integrate}]{-}$	$\frac{1}{27}e^{3x}$	$\int 0\,dx = C$	

Let's put this in a more compact and readable table.

$$\vdots$$

....Solution Continued

$$\vdots$$

$$\begin{array}{ccc} u & v' & +/- \\ \hline x^2 & e^{3x} & + \\ & \searrow + & \\ 2x & \frac{1}{3}e^{3x} & - \\ & \searrow - & \\ 2 & \frac{1}{9}e^{3x} & + \\ & \searrow + & \\ 0 & \frac{1}{27}e^{3x} & - \end{array}$$

We build the table by differentiating down the u-column until we reach zero, and integrating down the v' column. Then reading the products by the arrows and changing signs every other term we arrive at the answer.

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

Example 7. Evaluate the integral $\int x^4 \cos(5x)\,dx$ using the tabular method.

Workspace:

Workspace Cont:

Solution:

We choose $u(x) = x^4$ and $v'(x) = \cos(5x)$

Now complete the table.

u		v'	$+/-$
x^4		$\cos(5x)$	$+$
	$\searrow +$		
$4x^3$		$\frac{1}{5}\sin(5x)$	$-$
	$\searrow -$		
$12x^2$		$-\frac{1}{25}\cos(5x)$	$+$
	$\searrow +$		
$24x$		$-\frac{1}{125}\sin(5x)$	$-$
	$\searrow -$		
24		$\frac{1}{625}\cos(5x)$	$+$
	$\searrow +$		
0		$\frac{1}{3125}\sin(5x)$	$-$

We can now read off the answer.

$$\int x^4\cos(5x)\,dx = \frac{1}{5}x^4\sin(5x)+\frac{4}{25}x^3\cos(5x)-\frac{12}{125}x^2\sin(5x)-\frac{24}{625}x\cos(5x)+\frac{24}{3125}\sin(5x)+C$$

Definite Integrals Using Integration By Parts

We have the following formula.

$$\int_{x=a}^{x=b} u(x)v'(x)\,dx = u(x)v(x)\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v(x)u'(x)\,dx$$

The notation in this formula often leads to some confusion. So let's consider an example.

Example 8. Evaluate the integral $\displaystyle\int_{x=0}^{x=\pi/3} 2x \sin x \, dx$

Workspace:

Solution:

We have already found the antiderivative earlier. We'll use this.

$$\int x \sin x \, dx = -2x \cos x + 2 \sin x + C$$

Then the definite integral evaluates to:

$$\begin{aligned}
\int_{x=0}^{x=\pi/3} 2x \sin x \, dx &= -2x \cos x + 2 \sin x \Big|_{x=0}^{x=\pi/3} \\
&= \Big(-2 \cdot \frac{\pi}{3} \cdot \cos(\pi/3) + 2 \sin(\pi/3)\Big) - \Big(0 + 0\Big) \\
&= -\frac{\pi}{3} + \sqrt{3}
\end{aligned}$$

Let's try another example.

Example 9. Evaluate the integral $\displaystyle\int_{x=0}^{x=\pi/12} \frac{x^2}{\sec(4x)} \, dx$

***Hint*:** Can you rewrite $1/\sec(4x)$?

Workspace:

Solution:

We find the antiderivative by parts first seeing that $1/\sec(4x) = \cos(4x)$.

u	v'	$+/-$
x^2	$\cos(4x)$	$+$
$2x$	$\frac{1}{4}\sin(4x)$	$-$
2	$-\frac{1}{16}\cos(4x)$	$+$
0	$-\frac{1}{64}\sin(4x)$	$-$

Now we have

$$\begin{aligned}
\int_{x=0}^{x=\pi/12} \frac{x^2}{\sec(4x)}\,dx &= \frac{x^2}{4}\sin(4x) + \frac{x}{8}\cos(4x) - \frac{1}{32}\sin(4x)\Big|_{x=0}^{x=\pi/12} \\
&= \left[\frac{\pi^2}{576}\cdot\sin\left(\frac{\pi}{3}\right) + \frac{\pi}{96}\cdot\cos\left(\frac{\pi}{3}\right) - \frac{1}{32}\sin\left(\frac{\pi}{3}\right)\right] - \left[0 + 0 - \frac{1}{32}\sin(0)\right] \\
&= \left(\frac{\pi^2}{576}\right)\cdot\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\pi}{96}\right)\cdot\left(\frac{1}{2}\right) - \left(\frac{1}{32}\right)\cdot\left(\frac{\sqrt{3}}{2}\right) \\
&= \frac{\pi^2\sqrt{3}}{1152} + \frac{\pi}{192} - \frac{\sqrt{3}}{64} \approx 0.004
\end{aligned}$$

Some Interesting and Important Examples

Example 10. Evaluate the integral $\int e^x \sin x\,dx$

Workspace:

Workspace Cont.:

Solution:

Choose $u(x) = e^x$, and $v'(x) = \sin x$. So that $u'(x) = e^x$ and $v(x) = -\cos x$. Then we have

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x cos(x) \, dx.$$

We will need to use integration by parts to solve $\int e^x \cos x \, dx$. So choose $u(x) = e^x$ and $v'(x) = \cos x$. So that $u'(x) = e^x$ and $v(x) = \sin x$. It follows that

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

Putting it together we have

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx.$$

Or if you prefer using the tabular method:

u		v'	$+/-$
e^x		$\sin(x)$	$+$
	$\searrow +$		
e^x		$-\cos(x)$	$-$
	$\searrow -$		
e^x	$\xleftrightarrow[\text{Integrate}]{+}$	$-\sin(x)$	$+$

Remember we can exit the tabular algorithm whenever we want (such as the third row) using integration. Then reading off the table we have

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx.$$

It may appear that we have simply gone in a circle. But we can add $\int e^x \sin x \, dx$ on each side of the equation to obtain:

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + C.$$

Therefore

$$\int e^x \sin x \, dx = \frac{e^x \sin x - e^x \cos x}{2} + C.$$

Example 11. Evaluate the integral $\int \ln(x)\,dx$.

Workspace:

Solution:

We can write $\ln(x) = 1 \cdot \ln(x)$ and then choose $u(x) = \ln(x)$ and $v'(x) = 1$ giving us $u'(x) = 1/x$ and $v(x) = x$.

$$\int \ln(x)\, dx = x\ln(x) - \int 1\, dx = x\ln(x) - x + C.$$

Now see if you can use the same idea on another example.

Example 12. Evaluate the integral $\int \tan^{-1}(x)\, dx$.

Workspace:

Solution:

We can write $\tan^{-1}(x) = 1 \cdot \tan^{-1}(x)$ and then choose $u(x) = \tan^{-1}(x)$ and $v'(x) = 1$ giving us $u'(x) = 1/(x^2+1)$ and $v(x) = x$.

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx = x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) + C.$$

The idea is if we are integrating a function where we know its derivative but not its antiderivative sometimes mutliplying the function by 1 and then using integration by parts will be helpful. But not always. The following antiderivative does NOT have an elementary solution. None can be found no matter the technique tried.

$$\int e^{-x^2}\,dx.$$

It should be enlightening to realize that not all mathematical problems have elementary solution techniques. But integration by parts allows to integrate many more functions than we could before.

Please let me know if you have any questions, comments, or corrections!