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Math2411 - Calculus II
Guided Lecture Notes
Physics Applications

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

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Physics Applications Introduction

Suppose that we want to determine the work done when a force moves an object. Recall that in physics we define work as

$$Work = Force \times Distance.$$

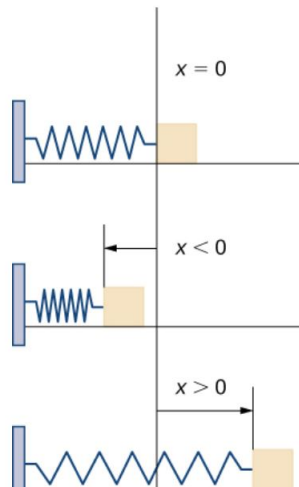For example, compute the work done by compressing or stretching a spring.

Figure 1: Block attached to a spring at equilibrium, compressed, and stretched.

Hooke's Law states that the force applied from a spring at position x from equilibrium can be described as $F(x) = kx$ for some positive constant k. We want to move a block from $x = a \to x = b$. Over a short interval $[x_{i-1}, x_i]$ we can estimate the work done moving the block from $x_{i-1} \to x_i$.
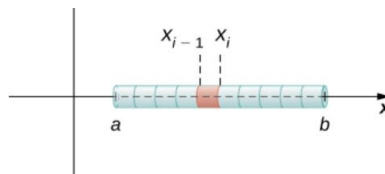

Figure 2: The interval from $x_{i-1} \to x_i$.

$$W_i \approx F(x_i)\big(x_i - x_{i-1}\big) \xrightarrow[\substack{\text{Add up each} \\ \text{work component}}]{} \text{Total Work} \approx \sum_{i=1}^{n} F(x_i)\big(x_i - x_{i-1}\big) = \sum_{i=1}^{n} kx\big(x_i - x_{i-1}\big)$$

Then our approximation improves as we let $\triangle x_i = x_i - x_{i-1} \to 0$ and so we have

$$\text{Total Work} = \lim_{\triangle x \to 0} \sum_{i=1}^{n} F(x_i) \cdot \triangle x = \int_{x=a}^{x=b} F(x)\, dx = \int_{x=a}^{x=b} kx\, dx.$$

We can interpret this integral as adding up all work done from $x = a \to x = b$.

$$\text{Total Work} = \int_{x=a}^{x=b} \underbrace{\overset{\overset{\text{Sum}}{\downarrow}}{} \overset{\overset{\text{Spring Force at Position x}}{\downarrow}}{kx} \quad \overset{\overset{\text{Small Distance}}{\downarrow}}{dx}}_{\text{Work Quantity}}$$

Continue with a concrete example.

Example 1. Suppose it takes a force of $10\,N$ (in the negative direction) to compress a spring $0.2m$ from the equilibrium position. How much work is done to stretch the spring $0.5m$ from the equilibrium position?

Workspace:

Solution:

We first determine the spring constant k.

$$
\begin{aligned}
F(x) &= kx \\
-10 &= -0.2k \\
k &= 50
\end{aligned}
$$

We can now set up the integral.

$$
\begin{aligned}
\text{Work} &= \int_{x=a}^{x=b} F(x)\, dx \\
&= \int_{x=0}^{x=0.5} 50x\, dx \\
&= 25x^2 \Big|_{x=0}^{x=0.5} \\
&= 25(0.5)^2 - 25(0)^2 \\
&= 6.25
\end{aligned}
$$

So the total work is 6.25 J (or Newton-Meters).

Now try another example.

Example 2. Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

Workspace:

Solution:

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes. Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let x represent the vertical distance above the bottom of the tank. That is, we orient the x-axis vertically, with the origin at the bottom of the tank and the upward direction being positive.
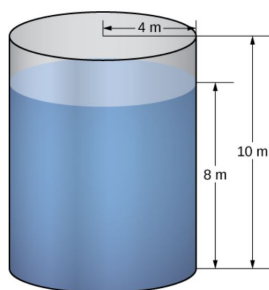

Figure 3: Cylindrical Tank of water.

Using this coordinate system, the water extends from $x = 0$ to $x = 8$. Therefore, we partition the interval $[0, 8]$ and look at the work required to lift each individual "layer" of water. For $i = 0, 1, 2, \ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$.
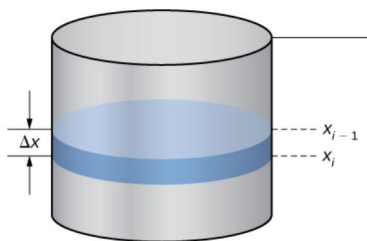

Figure 4: Representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is 9800 N/m^3 calculating the volume of each layer gives us the weight.

$$Volume = \pi r^2 \triangle x = 16\pi \triangle x \quad \xrightarrow[\substack{\text{Multipy Volume} \\ \text{by Density}}]{} \quad Weight = 9800 \cdot 16\pi \triangle x = 156,800\pi \triangle x.$$

....*Solution Continued*

Then the distance this representative layer or water must move is $10 - x$ (the distance from height x to the top of the tank). Therefore the work required to move a representative layer of water at height x_i^* to the top of the tank is approximately

$$W_i \approx 156,800\pi\left(10 - x_i^*\right)\triangle x$$

Therefore, we have

$$\text{Total Work} \approx \sum_{i=1}^{n} 156,800\pi\left(10 - x_i^*\right)\triangle x \quad \xRightarrow[\substack{\text{As } \triangle x \to 0 \text{ we} \\ \text{have an integral}}]{} \quad \text{Total Work} = 156,800\pi\int_{x=0}^{x=8} 10 - x\, dx$$

So we have

$$
\begin{aligned}
\text{Total Work} \;&=\; 156,800\pi\int_{x=0}^{x=8} 10 - x\, dx \\
&=\; 156,800\pi\left[10x - \frac{x^2}{2}\right]_{x=0}^{x=8} \\
&=\; 156,800\pi\left[\left(10\cdot 8 - \frac{8^2}{2}\right) - \left(0 - 0\right)\right] \\
&=\; 7,526,400\pi
\end{aligned}
$$

So it requires approximately 23,644,883 J to pump all the water out the top of the tank.

Before we try another example we can write down a problem solving strategy.

Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.
6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

Example 3. Assume a tank in the shape of an inverted cone, with height 12m and base radius 4m. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4m. How much work is required to pump out that amount of water?

Workspace:

Solution:

First sketch the tank, determine your coordinate system (with bottom of the tank at $x = 0$, and sketch a representative "layer" of water.
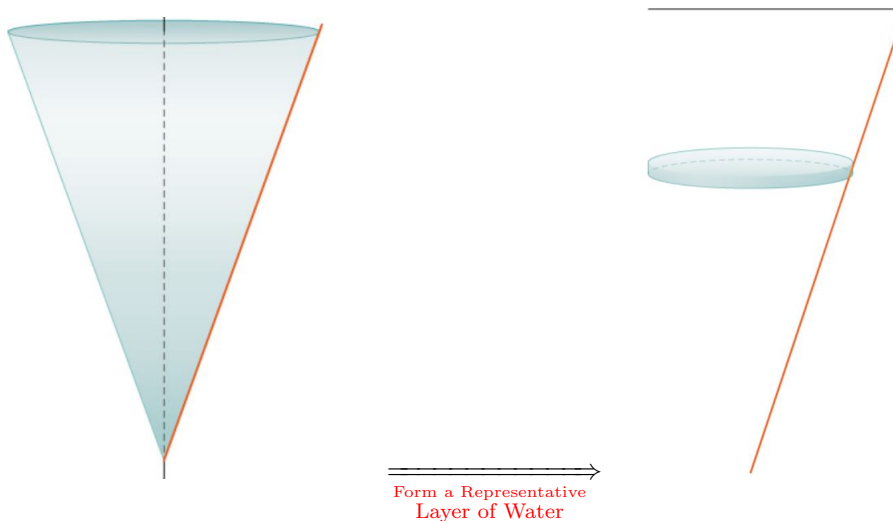

Form a Representative
Layer of Water

Figure 5: Inverted Conical Tank of water and representative layer of water.

Find the weight of the representative layer of water at height x_i^* from the bottom of the tank.

$$Volume = \pi \left(\frac{4x}{12}\right)^2 \triangle x \xrightarrow[\text{by Density}]{\text{Multipy Volume}} Weight = 9800 \cdot \pi \left(\frac{4x}{12}\right)^2 \triangle x$$

So we have

$$
\begin{aligned}
\text{Total Work} &= 9800\pi \int_{x=4}^{x=12} \frac{x^2}{9}\left(12 - x\right) dx \\
&= 9800\pi \int_{x=4}^{x=12} \frac{4x^2}{3} - \frac{x^3}{9} \, dx \\
&= 9800\pi \left[\frac{4x^3}{9} - \frac{x^4}{36}\right]_{x=4}^{x=12} \\
&= 9800\pi \left[\left(\frac{4(12)^3}{9} - \frac{(12)^4}{36}\right) - \left(\frac{4(4)^3}{9} - \frac{(4)^4}{36}\right)\right] \\
&= \frac{16,307,200}{9}\pi
\end{aligned}
$$

So it requires approximately 5,692,118.76 J to pump all the water out the top of the tank.

Please let me know if you have any questions, comments, or corrections!