Math2411 - Calculus II
Guided Lecture Notes
Trigonometric Substitution

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

Trigonometric Substitution Introduction:

Our objective is to integrate function involving square roots of differences and sums of squares.

$$\sqrt{a^2 - x^2} \qquad \sqrt{x^2 - a^2} \qquad \sqrt{x^2 + a^2}$$

We will need a few basic trig identities.

$$\sin^2(x) + \cos^2(x) = 1 \qquad \sin(2x) = \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \qquad \tan^2(x) = 1 - \sec^2(x)$$

Integrals involving $\sqrt{a^2 - x^2}$:

Let's consider an example together. The general strategy is to make a substitution $x = a\sin(\theta)$.

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin\theta)^2} && \text{Let } x = a\sin\theta \text{ where } -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}. \text{ Simplify.} \\
&= \sqrt{a^2 - a^2\sin^2\theta} && \text{Factor out } a^2. \\
&= \sqrt{a^2(1 - \sin^2\theta)} && \text{Substitute } 1 - \sin^2 x = \cos^2 x. \\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.} \\
&= |a\cos\theta| \\
&= a\cos\theta.
\end{aligned}$$

Then our square root quantity is converted into a simple trig function. Here is a right triangle for reference.

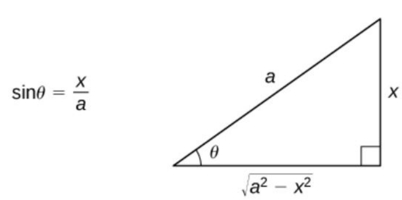

Figure 1: Reference triangle for $\sqrt{a^2 - x^2}$.

Example 1. Evaluate $\int \sqrt{9-x^2}\,dx$.

Workspace:

Solution:

We let $x = 3\sin(\theta)$ and construct our reference triangle.

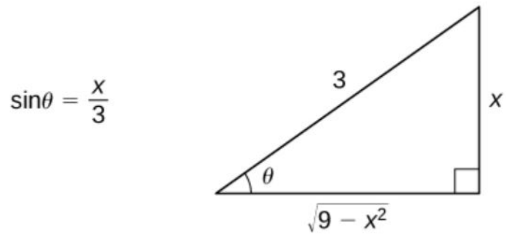

Figure 2: Reference triangle for $\sqrt{9-x^2}$.

Then we have

$$\begin{aligned}
\int \sqrt{9-x^2}\,dx &= \int \sqrt{9-(3\sin\theta)^2}\,3\cos\theta\, d\theta && \text{Substitute } x = 3\sin\theta \text{ and } dx = 3\cos\theta\, d\theta. \\
&= \int \sqrt{9(1-\sin^2\theta)}\,3\cos\theta\, d\theta && \text{Simplify.} \\
&= \int \sqrt{9\cos^2\theta}\,3\cos\theta\, d\theta && \text{Substitute } \cos^2\theta = 1-\sin^2\theta. \\
&= \int 3|\cos\theta|\,3\cos\theta\, d\theta && \text{Take the square root.} \\
&= \int 9\cos^2\theta\, d\theta && \text{Simplify. Since } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2},\ \cos\theta \ge 0 \text{ and } |\cos\theta| = \cos\theta. \\
&= \int 9\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta && \text{Use the strategy for integrating an even power of } \cos\theta. \\
&= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C && \text{Evaluate the integral.} \\
&= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C && \text{Substitute } \sin(2\theta) = 2\sin\theta\cos\theta. \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C && \text{Substitute } \sin^{-1}\left(\tfrac{x}{3}\right) = \theta \text{ and } \sin\theta = \tfrac{x}{3}. \text{ Use the reference triangle to see that } \cos\theta = \tfrac{\sqrt{9-x^2}}{3} \text{ and make this substitution.} \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C. && \text{Simplify.}
\end{aligned}$$

Let's now have you work an example.

Example 2. Evaluate the integral $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

Workspace:

Solution:

Let $x = \sin\theta$. In this case, $dx = \cos\theta d\theta$. Using this substitution, we have

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3\theta\cos^2\theta d\theta \\
&= \int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta d\theta && \text{Let } u=\cos\theta. \text{ Thus, } du=-\sin\theta d\theta. \\
&= \int \left(u^4-u^2\right)du \\
&= \frac{1}{5}u^5-\frac{1}{3}u^3+C && \text{Substitute } \cos\theta=u. \\
&= \frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta+C && \text{Use a reference triangle to see that } \cos\theta=\sqrt{1-x^2}. \\
&= \frac{1}{5}\left(1-x^2\right)^{5/2}-\frac{1}{3}\left(1-x^2\right)^{3/2}+C.
\end{aligned}$$

We can also solve his using u-sub. See if you can solve this using u-sub.

Workspace:

Solution:

We let $u = 1 - x^2$ so that $du = 2x\,dx$.

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) && \text{Make the substitution.}\\
&= -\frac{1}{2}\int (1-u)\sqrt{u}\,du && \text{Expand the expression.}\\
&= -\frac{1}{2}\int \left(u^{1/2} - u^{3/2}\right)du && \text{Evaluate the integral.}\\
&= -\frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C && \text{Rewrite in terms of } x.\\
&= -\frac{1}{3}\left(1-x^2\right)^{3/2} + \frac{1}{5}\left(1-x^2\right)^{5/2} + C.
\end{aligned}$$

Integrals involving $\sqrt{x^2+a^2}$:

Let's consider integrals with a term $\sqrt{x^2+a^2}$. We let $x = a\tan(\theta)$ and build a reference triangle.

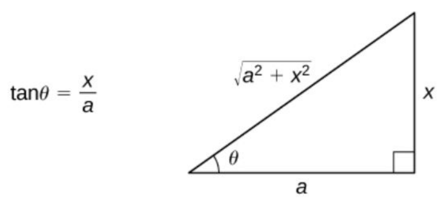

Figure 3: Reference triangle for $\sqrt{x^2+a^2}$

We can use the following problem solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = |a\sec\theta| = a\sec\theta$. (Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from **Figure 3.7** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (*Note*: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \le 0$.)

Example 3. Calculate the length of the curve $y = x^2$ on the interval $[0, 1/2]$. Our arclength formula gives us

$$s = \int_{x=a}^{x=b} \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_{x=0}^{x=1/2} \sqrt{1+4x^2}\, dx.$$

This looks like a tricky integral since there is no obvious u-substitution. We try a trig-sub. We let $x = \frac{1}{2}\tan(\theta)$ so that $dx = \frac{1}{2}\sec^2(\theta)\, d\theta$. Now continue on your own.

***Note*:** Notice that the quantity $\sqrt{1+4x^2}$ suggests a reference triangle with leg lengths of 1 and $2x$ giving a hypotenuse with length of $\sqrt{1+4x^2}$. Build your own reference triangle.

Workspace:

Solution:

Now we have

$\displaystyle\int_0^{1/2} \sqrt{1+4x^2}\,dx = \int_0^{\pi/4} \sqrt{1+\tan^2\theta}\,\frac{1}{2}\sec^2\theta\,d\theta$	After substitution, $\sqrt{1+4x^2} = \tan\theta$. Substitute $1+\tan^2\theta = \sec^2\theta$ and simplify.
$\displaystyle= \frac{1}{2}\int_0^{\pi/4} \sec^3\theta\,d\theta$	We derived this integral in the previous section.
$\displaystyle= \frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta + \ln\lvert\sec\theta+\tan\theta\rvert\right)\Big\rvert_0^{\pi/4}$	Evaluate and simplify.
$\displaystyle= \frac{1}{4}(\sqrt{2} + \ln(\sqrt{2}+1)).$	

Notice that even in trig-sub we ***change our limits of integration*** after the change of variables. Let's try another example

Example 4. Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$.

Start by letting $x = \tan(\theta)$ and forming the reference triangle.

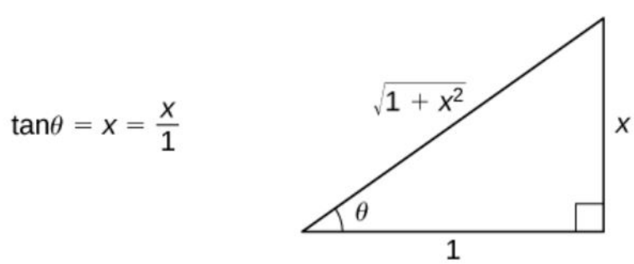

Figure 4: Reference triangle for $\sqrt{1+x^2}$

Now solve the integral by yourself.

Workspace:

Solution:

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec\theta} d\theta$$ Substitute $x = \tan\theta$ and $dx = \sec^2\theta d\theta$. This substitution makes $\sqrt{1+x^2} = \sec\theta$. Simplify.

$$= \int \sec\theta d\theta$$ Evaluate the integral.

$$= \ln|\sec\theta + \tan\theta| + C$$ Use the reference triangle to express the result in terms of x.

$$= \ln\left|\sqrt{1+x^2} + x\right| + C.$$

Integrals involving $\sqrt{x^2 - a^2}$:

Example 5. Evaluate the integral $\displaystyle\int_{x=3}^{x=5} \sqrt{x^2 - 9}\, dx$.

The geometry suggests we let $x = 3\sec(\theta)$ and so then have $dx = 3\sec(\theta)\tan(\theta)\, d\theta$.

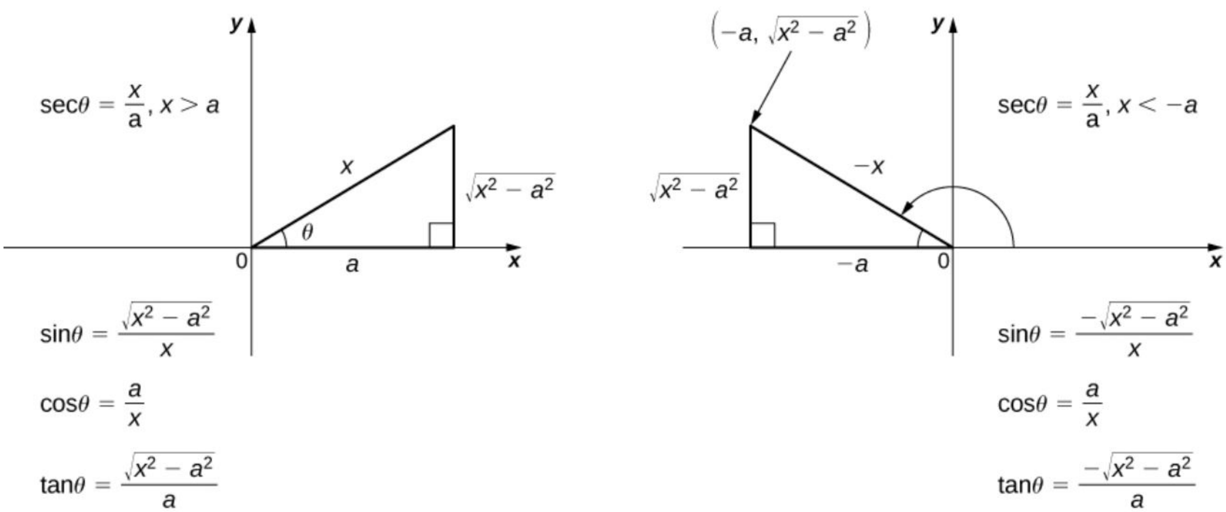

Figure 5: Reference triangle for $\sqrt{x^2 - a^2}$

Notice there are different reference triangles depending on whether the x-values are positive (so we have $x > a$) or negative (so we have $x < -a$). The main consequences is when x is positive we have $\sqrt{x^2 - a^2} = a\tan(\theta)$. When x is negative we have $\sqrt{x^2 - a^2} = -a\tan(\theta)$.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$. This substitution yields

$$\sqrt{x^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta + 1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|.$$

   For $x \geq a$, $|a\tan\theta| = a\tan\theta$ and for $x \leq -a$, $|a\tan\theta| = -a\tan\theta$.

3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from **Figure 3.9** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (*Note*: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

Solution:

$$= \int_3^5 \sqrt{x^2 - 9}\,dx$$

$$= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta\sec\theta d\theta \qquad \text{Use } \tan^2\theta = 1 - \sec^2\theta.$$

$$= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta - 1)\sec\theta d\theta \qquad \text{Expand.}$$

$$= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta - \sec\theta) d\theta \qquad \text{Evaluate the integral.}$$

$$= \left(\frac{9}{2}\ln|\sec\theta + \tan\theta| + \frac{9}{2}\sec\theta\tan\theta\right) - 9\ln|\sec\theta + \tan\theta|\Big|_0^{\sec^{-1}(5/3)} \qquad \text{Simplify.}$$

$$= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta|\Big|_0^{\sec^{-1}(5/3)} \qquad \text{Evaluate. Use } \sec\left(\sec^{-1}\frac{5}{3}\right) = \frac{5}{3} \text{ and } \tan\left(\sec^{-1}\frac{5}{3}\right) = \frac{4}{3}.$$

$$= \frac{9}{2}\cdot\frac{5}{3}\cdot\frac{4}{3} - \frac{9}{2}\ln\left|\frac{5}{3} + \frac{4}{3}\right| - \left(\frac{9}{2}\cdot 1\cdot 0 - \frac{9}{2}\ln|1 + 0|\right)$$

$$= 10 - \frac{9}{2}\ln 3.$$

Let's try an example on your own.

Example 6. Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{x^2 - 4}}\,dx$, assuming that $x < -2$.

***Note*:** The domain of the integrand $f(x) = 1/\sqrt{x^2 - 4}$ is $\big(-\infty, -2\big) \cup \big(2, \infty\big)$.

Workspace:

Please let me know if you have any questions, comments, or corrections!