Math2411 - Calculus II
Guided Lecture Notes
The Comparison Tests

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The Comparison Tests Introduction:

Our objective is to study the comparison tests for convergence of infinite series. Let's motivate with an example.

$$\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$$

We might consider the Integral Test since $f(x) = \frac{1}{x^3+3x+1}$ satisfies all three properties for the Integral Test: f is continuous for all $x|ge1$; $f(x) > 0$ for all $x > 1$; and f is decreasing for all $x > 1$. However, the integral

$$\int_{x=1}^{\infty} \frac{1}{x^3+3x+1}\,dx$$

is a difficult integral. The obvious technique of choice would be partial fractions, but the denominator x^3+3x+1 is difficult to factor. In fact, it factors into the product of a linear factor and an irreducible quadratic where the one real zero is

$$x = -\left(\frac{2}{\sqrt{5}-1}\right)^{1/3} + \left(\frac{\sqrt{5}-1}{2}\right)^{1/3}.$$

Since we can rarely determine the exact value of an infinite series we are often satisfied by simply determining convergence or divergence. And if the series converges, we can estimate the value of the sum with partial sums.

Direct Comparison Test:

Theorem 5.11: Comparison Test

i. Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Suppose there exists an integer N such that $a_n \ge b_n \ge 0$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Direct Comparison Test Examples:

Example 1. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges or diverges.

Workspace:

Solution:

We compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ which is a p-series with $p = 3 > 1$. Then we have

$$\frac{1}{n^3+3n+1} < \frac{1}{n^3}$$

for every positive integer n. Therefore, we can conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges.

Here is another example.

Example 2. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges or diverges.

Workspace:

Solution:

We compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ which is a convergent geometric series with $r = 1/2$. Then we have

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

for every positive integer n. Therefore, we see that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

Here is another example.

Example 3. Determine whether $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ converges or diverges.

Workspace:

Solution:

We compare $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent Harmonic series. Then we have

$$\frac{1}{\ln(n)} > \frac{1}{n}$$

for every integer $n \geq 2$ and $\sum_{n=2}^{\infty} 1/n$ diverges, we have that $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverge

Limit Comparison Test:

Let's consider the following example.

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 3}$$

Our obvious direct comparison series is $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which is a convergent p-series with $p = 2$. However,

$$\frac{1}{n^2 - 3} > \frac{1}{n^2}$$

for all $n \geq 2$ and so our direct comparison is inconclusive. But it would be convenient to be able to use our knowledge about the convergence of $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Fortunately, there is another comparison test.

Theorem 5.12: Limit Comparison Test

Let $a_n,\ b_n \geq 0$ for all $n \geq 1$.

i. If $\lim_{n \to \infty} a_n / b_n = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

ii. If $\lim_{n \to \infty} a_n / b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

iii. If $\lim_{n \to \infty} a_n / b_n = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

In our current example we have $a_n = 1/(n^2 - 2)$ and $b_n = 1/n^2$. Then we have

$$L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1/(n^2 - 2)}{1/n^2} = \lim_{n \to \infty} \frac{n^2}{n^2 - 2} = 1.$$

Since $L > 0$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent p-series we conclude that $\sum_{n=2}^{\infty} \frac{1}{n^2 - 3}$ converges by the Limit Comparison Test.

Limit Comparison Test Examples:

Example 4. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 1}$ converges or diverges.

Workspace:

Workspace Cont.:

Solution:

We compare $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is the divergent p-series. But Direct comparison is inconclusive since for all positive n we have

$$\frac{1}{\sqrt{n}+1} < \frac{1}{\sqrt{n}}.$$

Let's use the Limit Comparison Test with $a_n = 1/(\sqrt{n}+1)$ and $b_n = 1/\sqrt{n}$ and set up $\lim_{n\to\infty} \frac{a_n}{b_n}$.

$$\lim_{n\to\infty} \frac{1/(\sqrt{n}+1)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

By the limit comparison test, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges.

Example 5. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges or diverges.

Workspace:

Solution:

We compare $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ with $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ which is a convergent geometric series with $r = 2/3$. But Direct comparison is inconclusive since for all positive n we have

$$\frac{2^n+1}{3^n} > \left(\frac{2}{3}\right)^n.$$

Let's use the Limit Comparison Test with $a_n = (2^n+1)/3^n$ and $b_n =$ and set up $\lim_{n\to\infty} \frac{a_n}{b_n}$.

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = \lim_{n\to\infty} \frac{2^n+1}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} \left[1 + \left(\frac{1}{2}\right)^n\right] = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges.

Example 6. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges or diverges.

Workspace:

Solution:

Noticing that $n > \ln(n)$ for sufficiently large n, we compare $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent Harmonic series. But Direct comparison is inconclusive since for all positive n we have

$$\frac{\ln(n)}{n^2} > \frac{n}{n^2} = \frac{1}{n}.$$

Let's use the Limit Comparison Test with $a_n = \ln(n)/n^2$ and $b_n =$ and set up $\lim_{n\to\infty} \frac{a_n}{b_n}$.

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\ln n}{n}.$$

In order to evaluate $\lim_{n\to\infty} \ln n/n$, evaluate the limit as $x \to \infty$ of the real-valued function $\ln(x)/x$. These two limits are equal, and making this change allows us to use L'Hôpital's rule. We obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore, $\lim_{n\to\infty} \ln n/n = 0$, and, consequently,

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n} = 0.$$

Since the limit is 0 but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the limit comparison test does not provide any information.

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ instead. In this case,

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n^2} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \ln n = \infty.$$

Since the limit is ∞ but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the test still does not provide any information.

So now we try a series between the two we already tried. Choosing the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, we see that

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}.$$

As above, in order to evaluate $\lim_{n\to\infty} \ln n/\sqrt{n}$, evaluate the limit as $x \to \infty$ of the real-valued function $\ln x/\sqrt{x}$. Using L'Hôpital's rule,

....Solution Continued

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0.$$

Since the limit is 0 and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we can conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges

We can see from this last example that sometime the comparison series is not obvious at first glance and some experimentation is required to find a series where comparison will be conclusive. This ends the notes on the Direct Comparison Test and the Limit Comparison Test.

Please let me know if you have any questions, comments, or corrections!