

# Math2411 - Calculus II

## Guided Lecture Notes

### Partial Fractions

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

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#### Partial Fractions Introduction:

Our objective is to integrate rational functions where  $u$ -sub or algebraic manipulations are not helpful or clear. Let's look at some examples.

$$\int \frac{x^2 + 4x - 1}{x^3 + 6x^2 - 3x + 7} dx \quad \int \frac{2}{x^2 + 4x + 8} dx$$

The first integral can be solved using  $u$ -substitution letting  $u = x^3 + 6x^2 - 3x + 7$  so that  $du = (3x^2 + 12x - 3) dx$ . We then have

$$\begin{aligned} \int \frac{x^2 + 4x - 1}{x^3 + 6x^2 - 3x + 7} dx &= \frac{1}{3} \int \frac{3(x^2 + 4x - 1)}{x^3 + 6x^2 - 3x + 7} dx \\ &= \frac{1}{3} \int \frac{3x^2 + 12x - 3}{x^3 + 6x^2 - 3x + 7} dx \\ &= \frac{1}{3} \int \frac{du}{u} \\ &= \frac{1}{3} \ln |u| + C \\ &= \frac{1}{3} \ln |x^3 + 6x^2 - 3x + 7| + C \end{aligned}$$

The second integral can be solved by algebraic manipulations, a  $u$ -sub, and a known antiderivative.

$$\begin{aligned} \int \frac{2}{x^2 + 4x + 8} dx &= \int \frac{2}{x^2 + 4x + 4 + 4} dx \\ &= \int \frac{2}{(x+2)^2 + 4} dx \\ &= \int \frac{2}{u^2 + 4} du \quad (\text{Letting } u = x+2) \\ &= \tan^{-1}(u/2) + C \\ &= \tan^{-1}\left(\frac{x+2}{2}\right) + C \end{aligned}$$

But some integrals of rational expressions resist any  $u$ -sub or algebraic manipulations. For example,

$$\int \frac{2x+5}{x^2+4x-5} dx.$$

We can not turn this into a log-form by  $u$ -substitution since with  $u = x^2 + 4x - 5$  we have  $du = (2x + 4) dx$  and the numerator is not a scalar multiple of the denominator.

So we split up the integral.

$$\int \frac{2x+5}{x^2+4x-5} dx = \int \frac{2x+4}{x^2+4x-5} dx + \int \frac{1}{x^2+4x-5} dx$$

The first integral is a simple  $u$ -substitution. But what about the second integral. We could try to complete the square as follows.

$$\begin{aligned} \int \frac{1}{x^2+4x-5} dx &= \int \frac{1}{x^2+4x-4-1} dx \\ &= \int \frac{1}{(x-2)^2-1} dx \\ &= \int \frac{1}{u^2-1} du \quad (\text{Letting } u = x-2) \end{aligned}$$

But the new integral is not in a familiar form. So it looks like we need a new strategy. We will use the following algebraic fact. It is a good exercise to check this equality.

$$\frac{2x+5}{x^2+4x-5} = \frac{7}{6(x-1)} + \frac{5}{6(x+5)}$$

Now we have the following.

$$\begin{aligned} \int \frac{2x+5}{x^2+4x-5} dx &= \int \frac{7}{6(x-1)} + \frac{5}{6(x+5)} dx \\ &= \frac{7}{6} \int \frac{1}{x-1} dx + \frac{5}{6} \int \frac{1}{x+5} dx \\ &= \frac{7}{6} \ln|x-1| + \frac{5}{6} \ln|x+5| + C \end{aligned}$$

So if we can break apart a complicated rational expression into a sum of simpler rational expressions, often the new expressions will be easier to handle as far as integration.

## Partial Fractions Method

Let's see how the previous fraction was decomposed into simpler fractions.

**Step #1:** Factor the denominator.

$$x^2 + 4x - 5 = (x-1)(x+5)$$

Do you notice anything about these factors?

**Step #2:** Try to break up the fraction so that the parts correspond to the factors.

$$\frac{2x+5}{x^2+4x-5} = \frac{2x+5}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}$$

**Step #3:** Solve for  $A$  and  $B$ . That is, find values of  $A$  and  $B$  that satisfy the previous equality. This is the crux of the process. We start by combining terms on the RHS and then setting numerators equal.

$$\frac{2x+5}{(x-1)(x+5)} = \frac{A(x+5) + B(x-1)}{(x-1)(x+5)} \quad \xrightarrow{\text{Set Numerators Equal}} \quad 2x+5 = A(x+5) + B(x-1)$$

Methods to solve for  $A$  and  $B$ :

- **Method of Strategic Substitution:** Set  $x = 1$  and substitute to get

$$7 = 6A \implies A = 7/6.$$

Next set  $x = -5$  and substitute to get

$$-5 = -6B \implies B = 5/6.$$

So we have

$$\frac{2x+5}{x^2+4x-5} = \frac{A}{x-1} + \frac{B}{x+5} = \frac{7}{6(x-1)} + \frac{5}{6(x+5)}.$$

- **Method of Equating Coefficients:** Expand both sides and equate coefficients.

$$2x+5 = (A+B)x + (5B-A)$$

We see that

$$\begin{aligned} A+B &= 2 \\ 5A-B &= 1 \end{aligned}$$

Solving this system of equations gives us  $A = 7/6$  and  $B = 5/6$ . So we have

$$\frac{2x+5}{x^2+4x-5} = \frac{A}{x-1} + \frac{B}{x+5} = \frac{7}{6(x-1)} + \frac{5}{6(x+5)}.$$

The partial fractions decomposition is an algebraic technique to simplify an integral. There is no calculus involved in the partial fractions.

Let's work an example on your own.

## Partial Fractions Examples

**Example 1.** Evaluate  $\int \frac{x+3}{x^3-x^2-2x} dx$

Workspace:

Let's work another example together.

**Example 2.** Evaluate  $\int \frac{x-2}{(2x-1)^2(x-1)} dx$

Workspace:

**Example 3.** Evaluate  $\int \frac{2x - 3}{x^3 + x} dx$

Workspace:

Let's have you work another example independently.

**Example 4.** Evaluate  $\int \frac{1}{x^3 - 8} dx$

**Hint:** Since  $x = 2$  is a zero of  $x^3 - 8$ , in order to factor the denominator divide  $x - 2$  into  $x^3 - 8$ .

Workspace:

**Workspace Continued:**

## What if Numerator Degree is Too Large?

For partial fractions to work the degree of the numerator MUST BE smaller than the degree of the denominator!

**Example 5.** Evaluate  $\int \frac{x^2 + 3x + 1}{x^2 - 4} dx$

Workspace: