Math2411 - Calculus II
Guided Lecture Notes
Taylor Series

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Taylor Series Introduction:

Our objective is to extend a Taylor polynomial for a function f centered at a point a to an infinite series called a ***Taylor series***. First recall our definition of a Taylor polynomial.

Definition

If f has n derivatives at $x = a$, then the nth Taylor polynomial for f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth Taylor polynomial for f at 0 is known as the nth Maclaurin polynomial for f.

Then a Taylor series is a power series with the same coefficients as a Taylor polynomial.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Since a Taylor series is just a power series all of our knowledge of power series applies to Taylor series. But with a Taylor series for a function $f(x)$ we have to make sure that the Taylor series actually converges to the correct function value $f(x)$. Fortunately we can determine this using the remainder formula. Recall the following.

Theorem 6.7: Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n+1$ times on an interval I containing the real number a. Let p_n be the nth Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x)$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

If there exists a real number M such that $\left|f^{(n+1)}(x)\right| \le M$ for all $x \in I$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x in I.

Then we have the following theorem.

Theorem 6.8: Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n \to \infty} R_n(x) = 0$$

for all x in I.

Let's work an example.

Example 1. Show that the Taylor series for $f(x) = \sin(x)$ centered at the point $a = 0$ converges to $\sin(x)$ for all x-values.

***Note*:** Recall the Taylor series for $f(x) = \sin(x)$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

Workspace:

Solution:

We could use the Ratio Test to check for convergence, but we want to know if the series converges to the actual function values. So we will use Taylor's Remainder Theorem which states that for all x-values and an n^{th} partial sum $S_n(x)$ we have

$$\big|R_n(x)\big| = \big|\sin(x) - S_n(x)\big| = \frac{\big|f^{(n+1)}(c)\big|}{(n+1)!} \cdot |x|^{n+1}$$

for some c in between x and a. But the derivatives of the sine function are bounded. That is, the absolute value of the k^{th} derivative evaluated at some value c always satisfies $\big|f^{(k)}(c)\big| \le 1$. So for all x-values we have

$$0 \le |R_n| \le \frac{1}{(n+1)!} \cdot |x|^{n+1} \quad \Longrightarrow \quad 0 \le \lim_{n\to\infty} |R_n| \le \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot |x|^{n+1} = 0$$

and so $\lim_{n\to\infty} R_n = 0$ for all x-values and the Taylor series converges to $f(x) = \sin(x)$ for all x-values.

Discussion:

Fortunately, all of the Taylor series for the elementary functions we will be working with actually converge to the correct function values. But it is worth noting that there are functions whose Taylor series does not converge to the correct function values. For example, the Taylor series for

$$f(x) = \begin{cases} e^{-1/x^2} & : \quad x \neq 0 \\ 0 & : \quad x = 0 \end{cases}$$

is the zero Taylor series (since all of the derivatives at $a = 0$ equal zero) and so does not converge to the actual function values when $x \neq 0$. Showing the details is beyond the scope of this course. But looking at the graph of the function we can see that the graph function appears very "flat" around $x = 0$.

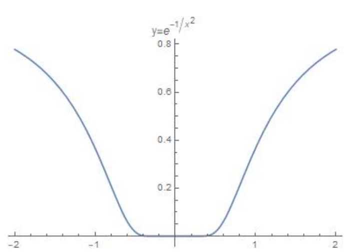

Figure 1: Graph of $y = e^{-1/x^2}$

Since we have already done so much work with power series in previous sections, for these notes we will simply provide a list of the most common and important Taylor series and show a few examples. Following this we will work one concrete example for the binomial series.

List of Taylor Series:

Function	Maclaurin Series	Interval of Convergence
$f(x) = \frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$f(x) = e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$f(x) = \ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$-1 < x \leq 1$
$f(x) = \tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 < x \leq 1$
$f(x) = (1+x)^r$	$\sum_{n=0}^{\infty} \binom{r}{n} x^n$	$-1 < x < 1$

So we have the following expressions for some important numbers.

$$
\begin{aligned}
e &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \\
\pi &= 4\tan^{-1}(1) = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right) \\
\ln(2) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{1}{6} + \cdots
\end{aligned}
$$

Let's now spend a little time with the binomial series.

Example 2. Apply the Taylor series for $f(x) = (1 + x)^r$ centered at the point $a = 0$ to estimate the value of $\sqrt{1.5}$.

Workspace:

Solution:

We have the following result (which we will not compute in these notes). The Taylor series for $f(x) = (1+x)^r$ is given as

$$\sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$$

and has an interval of convergence $(-1, 1)$. So a Taylor series for $\sqrt{1+x}$ is given by

$$1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \frac{(1/2)(-1/2)(-3/2)(-5/2)}{4!}x^4 + \cdots$$

If we use a degree= 3 Taylor polynomial we get

$$\sqrt{1.5} = \sqrt{1+1/2} \approx 1 + \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{8}\left(\frac{1}{2}\right)^2 + \frac{1}{16}\left(\frac{1}{2}\right)^3 \approx 1.2266$$

A calculator gives a value of $\sqrt{1.5} = 1.2247448714$ and so with just a few terms we are within two decimal places accuracy. Or using Taylor Remainder Theorem we could find that

$$\left|R_3(0.5)\right| \leq \frac{15}{4! \cdot 2^4}(0.5)^4 = 0.00244.$$

You are encourage to work out the details for yourself. You can also see that $p_3(x)$ appears to be a good approximating function when $x = 1/2$.

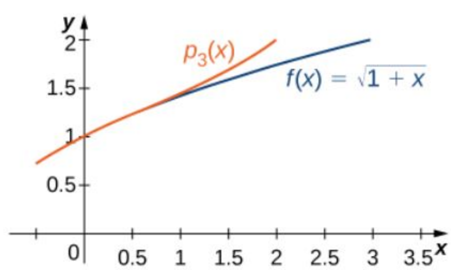

Figure 2: Graph of $y = \sqrt{1+x}$ versus $p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$.

Example 3. Let's look at another application of the binomial series. In particular, we want to solve the equation for the period of an oscillating pendulum. If T is the period, L is the length of the pendulum, g is acceleration due to gravity, θ_{max} is the maximum angle of swing, and $k = \sin(\theta_{max}/2)$ we have

$$T = 4\sqrt{\frac{L}{g}} \int_{\theta=0}^{\theta=\pi/2} \frac{1}{\sqrt{1-k^2\sin^2(\theta)}}\, d\theta.$$

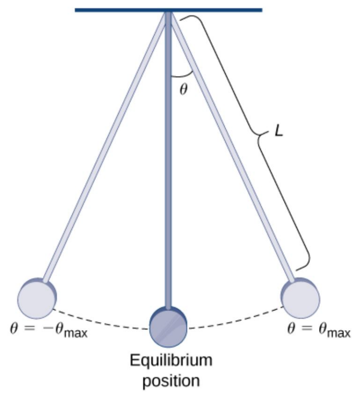

Figure 3: Oscillating Undamped Pendulum

Workspace:

Solution:

We use the binomial series substituting $x = -k^2 \sin^2(\theta)$ and $r = -1/2$ to get

$$T = 4\sqrt{\frac{L}{g}} \int_{\theta=0}^{\theta=\pi/2} 1 + \frac{1}{2}k^2 \sin^2(\theta) + \frac{1 \cdot 3}{2! \cdot 2^2} k^4 \sin^4(\theta) + \cdots \, d\theta.$$

It turns out that if θ_{max} is small then we get a good approximation using only the first term of the infinite series (because the k-terms are very "small"). This gives us the well-known formula

$$T \approx 4\sqrt{\frac{L}{g}} \cdot \frac{\pi}{2} = 2\pi\sqrt{\frac{L}{g}}$$

Otherwise, we can use the first two terms in the infinite series.

$$T \approx 4\sqrt{\frac{L}{g}} \int_{\theta=0}^{\theta=\pi/2} 1 + \frac{1}{2}k^2 \sin^2(\theta) \, d\theta.$$

This is a good exercise to evaluate this trigonometric integral. After evaluation we get

$$T \approx 2\pi\sqrt{\frac{L}{g}} \left(1 + \frac{k^2}{4}\right).$$

Hopefully we now understand how powerful we have become with our new knowledge of power series. Additionally, power series are fairly easy to manipulate to discover power series for certain desired functions. Let's consider one more example along with an application.

Example 4. We can show that the function $f(x) = e^x$ can be represented as the following power series with interval of convergence $(-\infty, \infty)$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Can we use this series to find a Taylor series representation of the function e^{-x^2}? Can we then use this Taylor series to evaluate the integral $\displaystyle\int_{x=0}^{x=1} e^{-x^2} \, dx$? Yes to both.

Workspace:

Workspace Continued:

Solution:

We have the power series representation

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots$$

for all x-values. We can use this to solve the difficult (and important) problem of evaluating the following integral.

$$\int_{x=0}^{x=1} e^{-x^2}\, dx$$

The difficulty is that there is no elementary antiderivative for the function e^{-x^2} Fortunately, we can integrate power series and so we have

$$\begin{aligned}
\int_{x=0}^{x=1} e^{-x^2}\, dx &= \int_{x=0}^{x=1} \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \right) dx \\
&= \int_{x=0}^{x=1} 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots\, dx \\
&= x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{11}}{11 \cdot 5!} + \cdots \Bigg|_{x=0}^{x=1} \\
&= \left(1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} - \frac{1}{11 \cdot 5!} + \cdots \right) - \Big(0 - 0 + 0 - 0 + 0 - 0 + \cdots \Big) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)n!}
\end{aligned}$$

This is an alternating series and so we can easily approximate the value of the integral and have an easy analysis of the error in our approximation. For example, consider the 6^{th} partial sum.

$$S_6 = \sum_{n=0}^{6} (-1)^n \frac{1}{(2n+1)n!} = 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} - \frac{1}{11 \cdot 5!} + \frac{1}{13 \cdot 6!} = 0.746836$$

We know that the error can be analyzed as follows.

$$|R_6| = \left|S - S_6\right| < \frac{1}{15 \cdot 7!} = 0.0000132275.$$

So we have

$$0.746823 < \int_{x=0}^{x=1} e^{-x^2}\, dx < 0.746849$$

and our estimate is accurate to four decimal places.

Discussion:

Where this integral shows up is in probability and statistics applications. The normal distribution is one of the most important probability distributions and to calculate probabilities we integrate the following density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

See the following diagram.

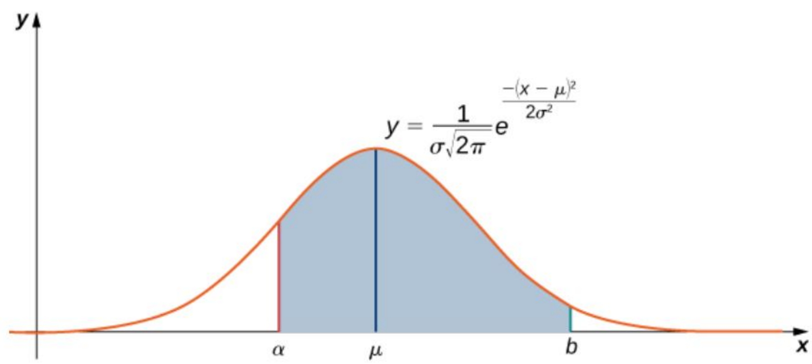

Figure 4: Probabilites For a Normal Distribution as Area

We can see that our knowledge of Taylor series makes us pretty powerful when it comes to solving certain difficult problems. There are many other applications that we do not have the time to cover in this course. Anyone interested is encouraged to research the various applications.

Please let me know if you have any questions, comments, or corrections!