

Math2411 - Calculus II
 Guided Lecture Notes
 Trigonometric Substitution

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Trigonometric Substitution Introduction:

Our objective is to integrate function involving square roots of differences and sums of squares.

$$\sqrt{a^2 - x^2} \quad \sqrt{x^2 - a^2} \quad \sqrt{x^2 + a^2}$$

We will need a few basic trig identities.

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1 & \sin(2x) = \sin^2(x) &= \frac{1 - \cos(2x)}{2} \\ \cos^2(x) &= \frac{1 + \cos(2x)}{2} & \tan^2(x) &= \sec^2(x) - 1\end{aligned}$$

Integrals involving $\sqrt{a^2 - x^2}$:

Let's consider an example together. The general strategy is to make a substitution $x = a \sin(\theta)$.

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a \sin \theta)^2} && \text{Let } x = a \sin \theta \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \text{ Simplify.} \\ &= \sqrt{a^2 - a^2 \sin^2 \theta} && \text{Factor out } a^2. \\ &= \sqrt{a^2(1 - \sin^2 \theta)} && \text{Substitute } 1 - \sin^2 x = \cos^2 x. \\ &= \sqrt{a^2 \cos^2 \theta} && \text{Take the square root.} \\ &= |a \cos \theta| \\ &= a \cos \theta.\end{aligned}$$

Then our square root quantity is converted into a simple trig function. Here is a right triangle for reference.

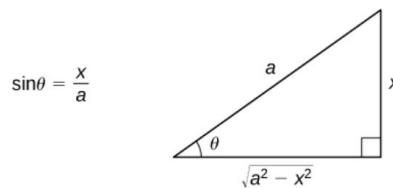


Figure 1: Reference triangle for $\sqrt{a^2 - x^2}$.

Example 1. Evaluate $\int \sqrt{9 - x^2} dx$.

Workspace:

Solution:

We let $x = 3 \sin(\theta)$ and construct our reference triangle.

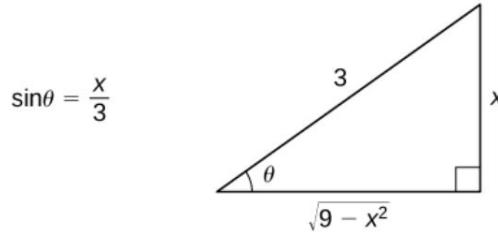


Figure 2: Reference triangle for $\sqrt{9 - x^2}$.

Then we have

$$\begin{aligned}
 \int \sqrt{9 - x^2} dx &= \int \sqrt{9 - (3 \sin\theta)^2} 3 \cos\theta d\theta && \text{Substitute } x = 3 \sin\theta \text{ and } dx = 3 \cos\theta d\theta. \\
 &= \int \sqrt{9(1 - \sin^2\theta)} 3 \cos\theta d\theta && \text{Simplify.} \\
 &= \int \sqrt{9 \cos^2\theta} 3 \cos\theta d\theta && \text{Substitute } \cos^2\theta = 1 - \sin^2\theta. \\
 &= \int 3|\cos\theta| 3 \cos\theta d\theta && \text{Take the square root.} \\
 &= \int 9 \cos^2\theta d\theta && \text{Simplify. Since } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \cos\theta \geq 0 \text{ and} \\
 &&& |\cos\theta| = \cos\theta. \\
 &= \int 9 \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta && \text{Use the strategy for integrating an even power} \\
 &= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C && \text{of } \cos\theta. \\
 &= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C && \text{Evaluate the integral.} \\
 &= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2} \cdot \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} + C && \text{Substitute } \sin(2\theta) = 2\sin\theta\cos\theta. \\
 &= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9 - x^2}}{2} + C. && \text{Substitute } \sin^{-1}\left(\frac{x}{3}\right) = \theta \text{ and } \sin\theta = \frac{x}{3}. \text{ Use} \\
 &&& \text{the reference triangle to see that} \\
 &&& \cos\theta = \frac{\sqrt{9 - x^2}}{3} \text{ and make this substitution.} \\
 &&& \text{Simplify.}
 \end{aligned}$$

Let's now have you work an example.

Example 2. Evaluate the integral $\int x^3 \sqrt{1 - x^2} dx$.

Workspace:

Solution:

Let $x = \sin\theta$. In this case, $dx = \cos\theta d\theta$. Using this substitution, we have

$$\begin{aligned}
 \int x^3 \sqrt{1-x^2} dx &= \int \sin^3 \theta \cos^2 \theta d\theta \\
 &= \int (1-\cos^2 \theta) \cos^2 \theta \sin \theta d\theta && \text{Let } u = \cos \theta. \text{ Thus, } du = -\sin \theta d\theta. \\
 &= \int (u^4 - u^2) du \\
 &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C && \text{Substitute } \cos \theta = u. \\
 &= \frac{1}{5}\cos^5 \theta - \frac{1}{3}\cos^3 \theta + C && \text{Use a reference triangle to see that} \\
 &= \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C. && \cos \theta = \sqrt{1-x^2}.
 \end{aligned}$$

We can also solve this using u -sub. See if you can solve this using u -sub.

Workspace:

Solution:

We let $u = 1 - x^2$ so that $du = 2x dx$.

$$\begin{aligned}
 \int x^3 \sqrt{1-x^2} dx &= -\frac{1}{2} \int x^2 \sqrt{1-x^2} (-2x dx) && \text{Make the substitution.} \\
 &= -\frac{1}{2} \int (1-u) \sqrt{u} du && \text{Expand the expression.} \\
 &= -\frac{1}{2} \int (u^{1/2} - u^{3/2}) du && \text{Evaluate the integral.} \\
 &= -\frac{1}{2} \left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right) + C && \text{Rewrite in terms of } x. \\
 &= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C.
 \end{aligned}$$

Integrals involving $\sqrt{x^2 + a^2}$:

Let's consider integrals with a term $\sqrt{x^2 + a^2}$. We let $x = a \tan(\theta)$ and build a reference triangle.

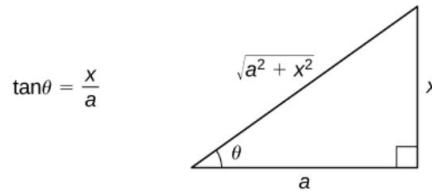


Figure 3: Reference triangle for $\sqrt{x^2 + a^2}$

We can use the following problem solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a \tan \theta$ and $dx = a \sec^2 \theta d\theta$. This substitution yields

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a \tan \theta)^2} = \sqrt{a^2(1 + \tan^2 \theta)} = \sqrt{a^2 \sec^2 \theta} = |a \sec \theta| = a \sec \theta.$$

(Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec \theta > 0$ over this interval, $|a \sec \theta| = a \sec \theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from **Figure 3.7** to rewrite the result in terms of x . You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}(\frac{x}{a})$. (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

Example 3. Calculate the length of the curve $y = x^2$ on the interval $[0, 1/2]$. Our arclength formula gives us

$$s = \int_{x=a}^{x=b} \sqrt{1 + [f'(x)]^2} dx = \int_{x=0}^{x=1/2} \sqrt{1 + 4x^2} dx.$$

This looks like a tricky integral since there is no obvious u -substitution. We try a trig-sub. We let $x = \frac{1}{2} \tan(\theta)$ so that $dx = \frac{1}{2} \sec^2(\theta) d\theta$. Now continue on your own.

Note: Notice that the quantity $\sqrt{1 + 4x^2}$ suggests a reference triangle with leg lengths of 1 and $2x$ giving a hypotenuse with length of $\sqrt{1 + 4x^2}$. Build your own reference triangle.

Workspace:

Solution:

Now we have

$$\begin{aligned}
 \int_0^{1/2} \sqrt{1+4x^2} dx &= \int_0^{\pi/4} \sqrt{1+\tan^2 \theta} \frac{1}{2} \sec^2 \theta d\theta \\
 &= \frac{1}{2} \int_0^{\pi/4} \sec^3 \theta d\theta \\
 &= \frac{1}{2} \left(\frac{1}{2} \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \Big|_0^{\pi/4} \\
 &= \frac{1}{4} (\sqrt{2} + \ln(\sqrt{2} + 1)).
 \end{aligned}$$

After substitution,
 $\sqrt{1+4x^2} = \tan \theta$. Substitute
 $1 + \tan^2 \theta = \sec^2 \theta$ and simplify.
 We derived this integral in the
 previous section.
 Evaluate and simplify.

Notice that even in trig-sub we *change our limits of integration* after the change of variables. Let's try another example

Example 4. Evaluate the integral $\int \frac{1}{\sqrt{1+x^2}} dx$.

Start by letting $x = \tan(\theta)$ and forming the reference triangle.

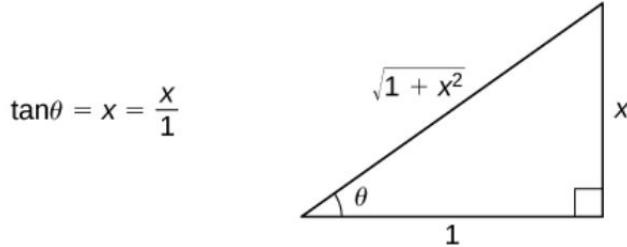


Figure 4: Reference triangle for $\sqrt{1+x^2}$

Now solve the integral by yourself.

Workspace:

Solution:

$$\begin{aligned}
 \int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\sec^2 \theta}{\sec \theta} d\theta \\
 &= \int \sec \theta d\theta \\
 &= \ln|\sec \theta + \tan \theta| + C \\
 &= \ln|\sqrt{1+x^2} + x| + C.
 \end{aligned}$$

Substitute $x = \tan \theta$ and $dx = \sec^2 \theta d\theta$. This substitution makes $\sqrt{1+x^2} = \sec \theta$. Simplify.

Evaluate the integral.

Use the reference triangle to express the result in terms of x .

Integrals involving $\sqrt{x^2 - a^2}$:

Example 5. Evaluate the integral $\int_{x=3}^{x=5} \sqrt{x^2 - 9} dx$.

The geometry suggests we let $x = 3 \sec(\theta)$ and so then have $dx = 3 \sec(\theta) \tan(\theta) d\theta$.

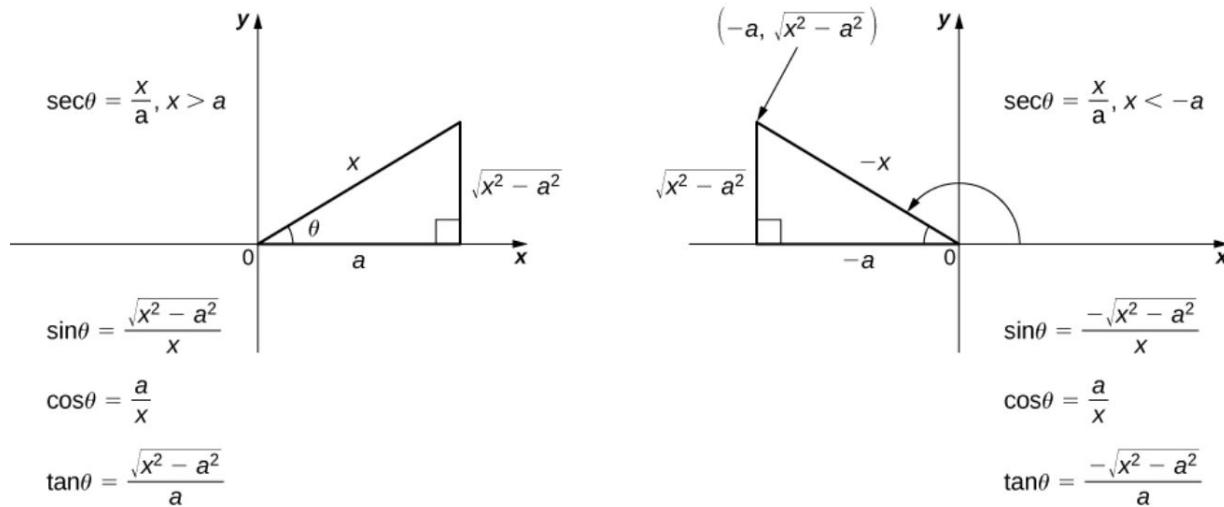


Figure 5: Reference triangle for $\sqrt{x^2 - a^2}$

Notice there are different reference triangles depending on whether the x -values are positive (so we have $x > a$) or negative (so we have $x < -a$). The main consequence is when x is positive we have $\sqrt{x^2 - a^2} = a \tan(\theta)$. When x is negative we have $\sqrt{x^2 - a^2} = -a \tan(\theta)$.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a \sec \theta$ and $dx = a \sec \theta \tan \theta d\theta$. This substitution yields

$$\sqrt{x^2 - a^2} = \sqrt{(a \sec \theta)^2 - a^2} = \sqrt{a^2(\sec^2 \theta + 1)} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta|.$$

For $x \geq a$, $|a \tan \theta| = a \tan \theta$ and for $x \leq -a$, $|a \tan \theta| = -a \tan \theta$.

3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from **Figure 3.9** to rewrite the result in terms of x . You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

Solution:

$$\begin{aligned}
 &= \int_3^5 \sqrt{x^2 - 9} dx \\
 &= \int_0^{\sec^{-1}(5/3)} 9 \tan^2 \theta \sec \theta d\theta && \text{Use } \tan^2 \theta = 1 - \sec^2 \theta. \\
 &= \int_0^{\sec^{-1}(5/3)} 9(\sec^2 \theta - 1) \sec \theta d\theta && \text{Expand.} \\
 &= \int_0^{\sec^{-1}(5/3)} 9(\sec^3 \theta - \sec \theta) d\theta && \text{Evaluate the integral.} \\
 &= \left(\frac{9}{2} \ln |\sec \theta + \tan \theta| + \frac{9}{2} \sec \theta \tan \theta \right) \Big|_0^{\sec^{-1}(5/3)} && \text{Simplify.} \\
 &= \frac{9}{2} \sec \theta \tan \theta - \frac{9}{2} \ln |\sec \theta + \tan \theta| \Big|_0^{\sec^{-1}(5/3)} && \text{Evaluate. Use } \sec\left(\sec^{-1}\frac{5}{3}\right) = \frac{5}{3} \\
 &= \frac{9}{2} \cdot \frac{5}{3} \cdot \frac{4}{3} - \frac{9}{2} \ln \left| \frac{5}{3} + \frac{4}{3} \right| - \left(\frac{9}{2} \cdot 1 \cdot 0 - \frac{9}{2} \ln |1 + 0| \right) && \text{and } \tan\left(\sec^{-1}\frac{5}{3}\right) = \frac{4}{3}. \\
 &= 10 - \frac{9}{2} \ln 3.
 \end{aligned}$$

Let's try an example on your own.

Example 6. Evaluate the integral $\int \frac{1}{\sqrt{x^2 - 4}} dx$, assuming that $x < -2$.

Note: The domain of the integrand $f(x) = 1/\sqrt{x^2 - 4}$ is $(-\infty, -2) \cup (2, \infty)$.

Workspace:

Please let me know if you have any questions, comments, or corrections!