Math2411 - Calculus II
Guided Lecture Notes
The Alternating Series Test

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The Alternating Series Test Introduction:

Our objective is to study ***alternating series*** whose terms alternating between positive and negative values.

Definition

Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots \qquad (5.13)$$

or

$$\sum_{n-1}^{\infty} (-1)^{n} b_n = -b_1 + b_2 - b_3 + b_4 - \cdots \qquad (5.14)$$

Where $b_n \geq 0$ for all positive integers n.

A famous example of an alternating series is the alternating harmonic series which converges to the value $\ln(2)$.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2).$$

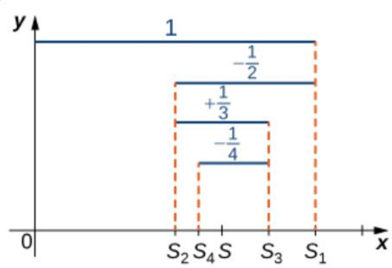

Figure 1: Partial Sums of the Alternating Harmonic Series

The pattern of partial sums will hold in general for alternating series as shown in the following diagram. Notice how the sequence of partial sums is oscillating and seems to be converging to some value S.

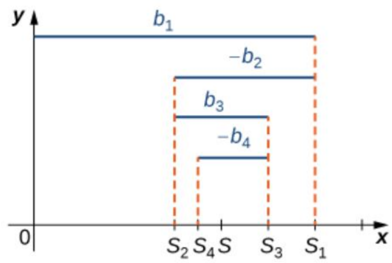

Figure 2: Partial Sums of an Alternating Series

It turns out that determining the convergence or divergence of an alternating series is based on a very simple test.

Theorem 5.13: Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ or } \sum_{n=1}^{\infty} (-1)^n b_n$$

converges if

i. $0 \leq b_{n+1} \leq b_n$ for all $n \geq 1$ and

ii. $\lim_{n \to \infty} b_n = 0.$

This is known as the **alternating series test**.

Let's work a couple of examples.

Alternating Series Test Examples:

Example 1. Determine whether $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ converges or diverges.

Workspace:

Solution:

We see that the sequence $a_n = 1/n^2$ is decreasing since for all $n \geq 1$ we have

$$\frac{1}{(n+1)^2} \leq \frac{1}{n^2}.$$

Moreover, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^2} = 0$ and so the series converges by the Alternating Series Test.

Here is another example.

Example 2. Determine whether $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n-1}$ converges or diverges.

Workspace:

Solution:

We immediately see that the $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n/(2n-1) = 1/2 \neq 0$. So the Alternating Series test does not apply. (Notice that given the knowledge of this limit we do not need to verify that the corresponding positive terms are decreasing.) Moving on, we now know that

$$\lim_{n\to\infty}(-1)^{n+1}\frac{n}{2n-1} \quad \text{Does Not Exist.}$$

So the limit does not equal zero and the series diverges by the Divergence Test.

Here is another example.

Example 3. Determine whether $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$ converges or diverges.

Workspace:

Solution:

We can see that

$$\lim_{n\to\infty} \sin^2(1/n) = \sin^2(0) = 0.$$

So if the non-alternating portion of the sequence is decreasing the Alternating Series Test applies. We use the fact that on the interval $[0,1]$ the function $f(x) = \sin^2(x)$ is an increasing function.

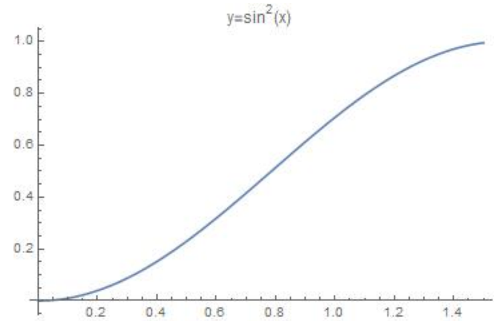

Figure 3: $f(x) = \sin^2(x)$ is Increasing on the Interval $[0,1]$.

That is,

$$x_1 \le x_2 \quad \Longrightarrow \quad \sin^2(x_1) \le \sin^2(x_2.)$$

Then the composition of an increasing function with a decreasing function is a decreasing function.

$$\frac{1}{n+1} \le \frac{1}{n} \quad \Longrightarrow \quad \sin^2\left(\frac{1}{n+1}\right) \le \sin^2\left(\frac{1}{n}\right).$$

So our non-alternating component of the series terms is a decreasing sequence and the Alternating Series Test applies. So we conclude that the series converges by the Alternating Series Test.

Here is another method to determine whether the sequence terms $\sin^2(1/n)$ are decreasing. Consider the function $f(x) = \sin^2(1/x)$ on the interval $(0,1]$. Then

$$f'(x) = \underbrace{2\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)}_{>0} \cdot \underbrace{\left(\frac{-1}{x^2}\right)}_{<0} = -\frac{2}{x}\cdot\sin\left(\frac{1}{x}\right)\cdot\cos\left(\frac{1}{x}\right) = \underbrace{-\frac{1}{x}}_{<0}\cdot\underbrace{\sin\left(\frac{2}{x}\right)}_{>0} < 0$$

Since $f'(x) < 0$ when $0 < x \le 1$ we have the same conclusion that the sequence terms $\sin^2(1/n)$ are decreasing.

Here is a challenging example.

Example 4. Determine whether $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{n+1}\right)^{n^2}$ converges or diverges.

Workspace:

Solution:

We first check if the Alternating Series Test applies. Let's determine if the non-alternating component of the series terms gives a decreasing sequence. That is, do we satisfy the following?

$$0 \le \left(\frac{n+1}{n+2}\right)^{(n+1)^2} \le \left(\frac{n}{n+1}\right)^{n^2}$$

This will require a bit of work. We will use the fact that $\ln(1+x) < x$ when $x > -1$ and then compute a derivative. To see why the stated inequality is true we let $g(x) = x - \ln(1+x)$. Then

$$g'(x) = 1 - \frac{1}{1+x} \quad \text{and} \quad g''(x) = \frac{1}{(1+x)^2} > 0.$$

Then there is a critical point when $g'(x) = 0$ which occurs at $x = 0$. Moreover, since $g''(0) = 1 > 0$ we see that there is a minimum at $x = 0$. Then $g(0) = 0$ and $g(0)$ being a minimum is equivalent to $g(x) = x - \ln(1+x) \ge 0$ when $x > -1$. Equivalently, $\ln(1+x) \le x$ whenever $x > -1$. Now we take the following derivative assuming $x \ge 1$.

$$\begin{aligned}
\frac{d}{dx}\left[\left(\frac{x}{x+1}\right)^{x^2}\right] &= \frac{d}{dx}\left[e^{x^2 \ln\left(\frac{x}{x+1}\right)}\right] \\
&= e^{x^2 \ln\left(\frac{x}{x+1}\right)} \cdot \left(2x \ln\left(1 - \frac{1}{x+1}\right) + \frac{x}{x+1}\right) \\
&< e^{x^2 \ln\left(\frac{x}{x+1}\right)} \cdot \left(-2x\left(\frac{1}{x+1}\right) + \frac{x}{x+1}\right) \\
&= e^{x^2 \ln\left(\frac{x}{x+1}\right)} \cdot \left(-\frac{x}{x+1}\right) \\
&< 0
\end{aligned}$$

And the non-alternating terms of the series form a decreasing sequence as is required by part(i) of the Alternating Series Test.

We next need to evaluate

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2}.$$

We calculate as follows. First observe that

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2} = \lim_{n\to\infty} e^{\ln\left[\left(\frac{n}{n+1}\right)^{n^2}\right]} = e^{\lim_{n\to\infty} \ln\left[\left(\frac{n}{n+1}\right)^{n^2}\right]}$$

Then we compute the following.

$$\lim_{n\to\infty} \ln\left[\left(\frac{n}{n+1}\right)^{n^2}\right]$$

$$\vdots$$

....Solution Continued

$$\vdots$$

$$\begin{aligned}
\lim_{n\to\infty} \ln\left[\left(\frac{n}{n+1}\right)^{n^2}\right] &= \lim_{n\to\infty} n^2 \cdot \ln\left(\frac{n}{n+1}\right) \\
&= \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{1/n^2} = \frac{0}{0} \quad \text{form} \\
&= \lim_{n\to\infty} \frac{\left(\frac{n+1}{n}\right)\left(\frac{1}{(n+1)^2}\right)}{\left(-\frac{2}{n^3}\right)} \quad \text{by L'Hopital's Rule} \\
&= \lim_{n\to\infty} \frac{-n^3}{2n(n+1)} \\
&= -\infty \quad \text{by known growth rates}
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2} = e^{-\infty} \quad \text{form}$$

and so $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2} = 0$ and the series converges by the Alternating Series Test.

Optional Section: For those that like to dive into some problem solving, here is another argument that the sequence $\left\{\left(\frac{n}{n+1}\right)^{n^2}\right\}$ is decreasing.

The first part of our argument is that

$$\left\{\left(\frac{n}{n+1}\right)^{n^2}\right\} \quad \text{is decreasing} \quad \Leftrightarrow \quad \left\{\left(\frac{n+1}{n}\right)^{n^2}\right\} \quad \text{is increasing.}$$

We will show that the sequence terms on the right above are increasing. We can start by rewriting those sequence terms as

$$\left(\frac{n+1}{n}\right)^{n^2} = \left[\left(\frac{n+1}{n}\right)^{n}\right]^{n}.$$

Since the composition of increasing functions is increasing, and $b_n = c^n$ is an increasing function when $c > 1$, it is sufficient to show that the terms $\left(\frac{n+1}{n}\right)^{n}$ are increasing when $n \geq 1$.

So we need to show that for all $n \geq 1$ we have

$$\left(\frac{n+1}{n}\right)^n \leq \left(\frac{n+2}{n+1}\right)^{n+1}.$$

The AM-GM inequality states the following for non-negative real numbers $x_1, x_2, \ldots, x_{n+1}$:

$$\sqrt[n+1]{x_1 \cdot x_2 \cdots x_{n+1}} \leq \frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1}$$

Let $x1 = x_2 = \cdots x_n = 1 + \frac{1}{n}$ and $x_{n+1} = 1$. Then

$$\left[\left(1+\frac{1}{n}\right)^n \cdot 1\right]^{1/n+1} \leq \frac{n\left(1+\frac{1}{n}\right)+1}{n+1} \implies \left(\frac{n+1}{n}\right)^n \leq \left(\frac{n+2}{n+1}\right)^{n+1}$$

and we have shown the desired inequality. So the sequence terms $\left(\frac{n+1}{n}\right)^{n^2}$ are increasing and so the terms $\left(\frac{n}{n+1}\right)^{n^2}$ are decreasing as desired.

Alternating Series Remainder:

For most series we can not determine an actual value of the series and are forced to make approximations using partial sums. A nice property of alternating series is that it is easy to analyze the error in an approximation.

Theorem 5.14: Remainders in Alternating Series

Consider an alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ or } \sum_{n=1}^{\infty} (-1)^n b_n$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the Nth partial sum. For any integer $N \geq 1$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leq b_{N+1}.$$

Let's work through an example.

Example 5. Estimate $\sin(1)$ to within five decimal places.

***Note*:** We use the fact that

$$\sin(1) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots.$$

Workspace:

Solution:

Since sin(1) can be represented as an alternating series we will use the Alternating Series Remainder Theorem.

$$|\, R_k \,| = \left| \sin(1) - \sum_{n=0}^{k} (-1)^n \underbrace{\frac{1}{(2n+1)!}}_{b_n} \right| \leq \underbrace{\frac{1}{(2k+3)!}}_{b_{k+1}}$$

So if we satisfy $1/(2k+3)! < 10^{-5}$ we will have enough terms to approximate within our desired accuracy.

$1! = 1 \quad 2! = 2 \quad 3! = 6 \quad 4! = 24 \quad 5! = 120 \quad 6! = 720 \quad 7! = 5040 \quad 8! = 40,320 \quad 9! = 352,880$

$$\frac{1}{(2k+3)!} < \frac{1}{10^5} \quad \Longrightarrow \quad (2k+3)! > 10^5 \quad \Longrightarrow \quad 2k+3 > 9 \quad \Longrightarrow k > 3$$

That means $|R_4| = \left|\sin(1) - S_4\right| < 10^{-5}$. We calculate

$$S_4 = \sum_{n=0}^{4} (-1)^n \frac{1}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} = 0.8414710878.$$

The true value is $\sin(1) = 0.84147098$ and our approximation is accurate to 5 decimal places.

Absolute and Conditional Convergence:

When a particular series, such as an alternating series, has both positive and negative terms we consider two kinds of convergence: ***absolute convergence*** and ***conditional convergence***.

Definition

A series $\sum_{n=1}^{\infty} a_n$ exhibits **absolute convergence** if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ exhibits **conditional convergence** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem 5.15: Absolute Convergence Implies Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Examples:

Example 6. Determine whether the following series converge conditionally, converge absolutely, or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

   Workspace:

Solution:

The series converges by the Alternating Series Test. But

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

which is the divergent Harmonic series. It follows that the original series converges conditionally.

2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$

Workspace:

Solution:

The series converges by the Alternating Series Test. And

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

is a convergent p-series with $p = 2 > 1$. It follows that the original series converges absolutely.

3. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{\sqrt{n+4}} = \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} + \cdots$

Workspace:

Solution:

The series converges by the Alternating Series Test. And we evaluate

$$\sum_{n=0}^{\infty}\left|(-1)^n\frac{1}{\sqrt{n+4}}\right| = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+4}} = \sum_{n=4}^{\infty}\frac{1}{\sqrt{n}}.$$

This is a divergent p-series with $p = 1/2 \leq 1$. It follows that the original series converges conditionally.

Example 7. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^{3/2}} = \cos(1) + \frac{\cos(2)}{2^{3/2}} + \frac{\cos(3)}{3^{3/2}} + \cdots$ converges.

Workspace:

Solution:

We look at the series approximating the cosine values.

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^{3/2}} \approx 0.54 - \frac{0.42}{2^{3/2}} - \frac{0.98}{3^{3/2}} + \cdots$$

The series is not an alternating series since the signs do not alternate $+ - + - + - + \cdots$. Additionally the positive components of the series terms do ot form a decreasing sequence. So the Alternating Series Test does not apply. Let's check for absolute convergence and so evaluate the series

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^{3/2}} \right| = \sum_{n=1}^{\infty} \frac{\left|\cos(n)\right|}{n^{3/2}}.$$

Since $0 \leq \left|\cos(n)\right| \leq 1$ we will compare with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

$$0 \leq \frac{\left|\cos(n)\right|}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

and so the series $\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^{3/2}} \right|$ converges by the Direct Comparison Test which implies that the original series converges absolutely. Every absolutely convergent series is convergent and so we conclude that the original series $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^{3/2}}$ converges.

Discussion:

One of the consequences of absolute convergence is that we may safely rearrange or regroup the terms in the series without changing the value of the sum. But if a series converges conditionally, a rearrangement or regrouping of the series terms may change the value of the series. Consider the following example

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots .$$

This is a conditionally convergent alternating series. It turns out that if we rearrange the terms we can show

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3\ln(2)}{2}.$$

The point is we need to be careful when we rearrange the terms of a conditionally convergent series. Later in the course we will need the fact that the terms of an absolutely convergent series can be rearranged and regrouped without changing the value of the series.

Please let me know if you have any questions, comments, or corrections!