Math2411 - Calculus II
Guided Lecture Notes
Sequences and Series

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

Sequences and Series Introduction:

Our objective is to study the basics of ***infinite sequences*** and ***infinite series***. So to get started, what is an infinite sequence?

Definition

An **infinite sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$a_1, a_2, \ldots, a_n, \ldots .$$

The subscript n is called the **index variable** of the sequence. Each number a_n is a **term** of the sequence. Sometimes sequences are defined by **explicit formulas**, in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Sequence Examples:

Example 1. Consider the sequence $\{a_n\}$ where $a_n = 2^n$ for natural numbers n.

$$\{2^n\}_{n\in\mathbb{N}} = \{2, 4, 8, 16, 32, \ldots\}$$

Notice that we will use curly set brackets for sequence notation. And as above we can describe a sequence either by some explicit rule, or by listing enough of the sequence elements to understand the pattern (if there is indeed a pattern). Here is a graph of the sequence.

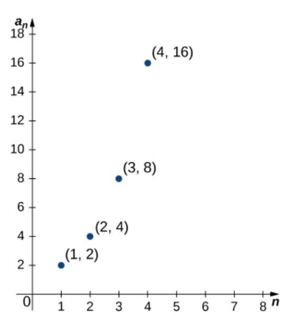

Figure 1: Graph of the sequence where $a_n = 2^n$.

Here is another example.

Example 2. The Fibonnaci sequence is one of the most famous sequences in all of mathematics. This is defined recursively as follows.

$$a_n = a_{n-1} + a_{n-2}, \quad \text{where } a_0 = a_1 = 1$$

Notice that the sequence is not defined as an explicit formula involving n. That is, we are not given a rule $a_n = f(n)$. Rather, the n^{th} term in the sequence is determined by the previous two terms. Here is the sequence.

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$$

Here is another simpler example.

Example 3. Define a sequence as follows so that $a_n = \dfrac{n}{n^2+1}$.

$$\{a_n\}_{n\in\mathbb{N}} = \left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \dots\right\}$$

Here is another example.

Example 4. Here is is one of the most important sequences in all of mathematics. Let k be any real number and define

$$a_n = \left(1 + \frac{k}{n}\right)^n$$

So we have

$$\{a_n\}_{n=1}^{\infty} = \left\{1 + k, \left(1 + \frac{k}{2}\right)^2, \left(1 + \frac{k}{3}\right)^3, \left(1 + \frac{k}{4}\right)^4, \dots\right\}$$

Limiting Values of Sequences:

The main question we will ask about sequences is whether of not the sequence converges to some value or diverges.

Definition

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence**. In this case, we write

$$\lim_{n\to\infty} a_n = L.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence**.

There is a nice graphic that reflects this idea.

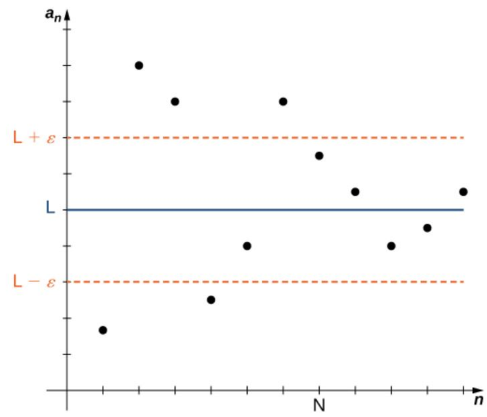

Figure 2: Graphic Representation of a Sequence $\{a_n\}$ Converging to L.

The above definition is a somewhat informal definition that we will use in this class. However, we should realize that this definition can be formalized to be perfectly precise as follows.

Definition

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0$, there exists an integer N such that $|a_n - L| < \varepsilon$ if $n \geq N$. The number L is the limit of the sequence and we write

$$\lim_{n \to \infty} a_n = L \text{ or } a_n \to L.$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We will not be working with this formal definition in this class. So how will we determine if a sequence converges or diverges? We will be exclusively evaluating explicit sequences and the following theorem will be our main tool for determining convergence or divergence.

Theorem 5.1: Limit of a Sequence Defined by a Function

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x \to \infty} f(x) = L,$$

then $\{a_n\}$ converges and

$$\lim_{n \to \infty} a_n = L.$$

This theorem tells us that we can use all of our previous knowledge of function limits to evaluate sequence limits.

Example 5. Determine if the sequence $\{a_n\}$, where $a_n = n/(n^2 + 1)$, converges or diverges.

Workspace:

Let's consider another example.

Example 6. Determine if the sequence $\{a_n\}$, where $a_n = 2n^3/(5n^3 - n^2 + 1)$, converges or diverges.

Workspace:

Let's try another example.

Example 7. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = 7n/\ln(n+1)$, converges or diverges.

Workspace:

Let's try another extremely important example.

Example 8. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \left(1 + \dfrac{k}{n}\right)^n$, converges or diverges.

Workspace:

And now one more example that will be important for us in this course.

Example 9. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = cr^n$ for some real numbers r and c, converges or diverges. This sequence is called a ***geometric sequence***.

Workspace:

Infinite Series:

Next we will be studying ***infinite series***. So what is an infinite series? It is simply a non-terminating discrete sum. We can think of it as an infinite sum of the terms in an infinite sequence.

Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the kth **partial sum** of the infinite series. The partial sums form a sequence $\{S_k\}$. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we have the **divergence of a series**.

Infinite Series Examples:

Example 10. Determine if the following infinite series converges.

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots$$

Workspace:

This example is a specific case of what's known as a geometric series. Let's consider the general case for a geometric series (one of the most important sequences).

Example 11. Determine if the following infinite series converges.

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + cr^4 + \cdots$$

Workspace:

Let's try another important example.

Example 12. Determine if the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

Note*:** This is an important infinite series known as the ***Harmonic Series.

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