Math2411 - Calculus II
Guided Lecture Notes
Area Between Curves

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Area Between Curves Introduction

Suppose that we want to determine the area between two curves $y = f(x)$ and $y = g(x)$ on an interval $[a, b]$.

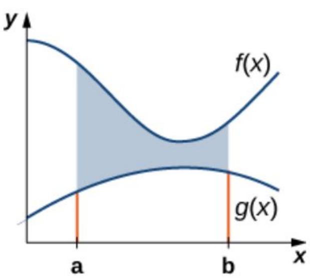

Figure 1: Region between graphs $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$

It is very natural to approach the problem by subtracting the area below the lower graph (in this example $y = g(x)$) from the area under the upper graph (in this example $y = f(x)$) to have

$$\text{Area} = \int_{x=a}^{x=b} f(x)\,dx - \int_{x=a}^{x=b} g(x)\,dx \xRightarrow[\text{of Integrals}]{\text{By Properties}} \text{Area} = \int_{x=a}^{x=b} f(x) - g(x)\,dx.$$

Another more general approach is to create a familiar type estimate of the area using rectangles.

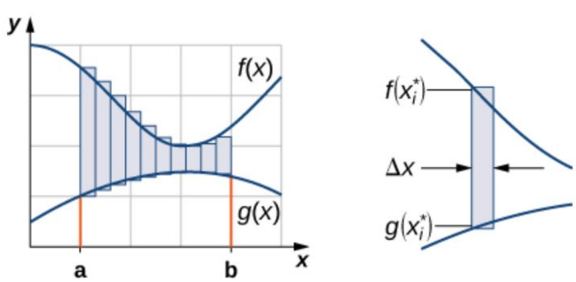

Figure 2: Estimated area between graphs $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$.

If we improve the estimate by letting the number of rectangles increase we have the same formula that we just saw above.

Theorem 2.1: Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b [f(x) - g(x)]dx. \tag{2.1}$$

If we label the upper graph as $y = upper(x)$ and the lower graph as $y = lower(x)$, we can rewrite the formula 2.1 and also interpret the integral geometrically (as we've seen before) as a sum of the areas of "very, very, thin" rectangles.

$$\int_{x=a}^{x=b} [upper(x) - lower(x)]\, dx \xRightarrow{\text{Can Be Interpreted As}} \overset{\text{Sum}}{\overset{\downarrow}{\int_{x=a}^{x=b}}} \underbrace{\overset{\text{Rectangle Height}}{\overset{\downarrow}{[upper(x) - lower(x)]}} \quad \overset{\text{Rectangle Width}}{\overset{\downarrow}{dx}}}_{\text{Rectangle Area}}$$

Area Between Curves Examples:

Let's work some examples.

Example 1. Find the area between the curves $y = 9 - x^2/4$ and $y = 6 - x$.

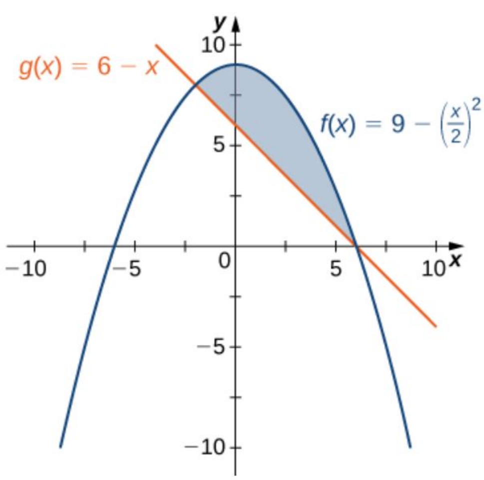

Figure 3: Region between graphs $y = 9 - x^2/4$ and $y = 6 - x$.

Start by finding the interval. At which x-values do the curves intersect?

We solve the following equation.

$$\begin{aligned} 9 - x^2/4 &= 6 - x \\ x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0 \end{aligned}$$

So the interval will be $[-2, 6]$. On this interval the upper graph is $y = 9 - x^2/4$ and the lower graph is $y = 6 - x$ and so the integral is

$$\text{Area} = \int_{x=a}^{x=b} \left[upper(x) - lower(x) \right] dx = \int_{x=-2}^{x=6} \left[\left(9 - \frac{x^2}{4}\right) - \left(6 - x\right) \right] dx$$

We evaluate the integral.

$$\begin{aligned} \int_{x=-2}^{x=6} \left[\left(9 - \frac{x^2}{4}\right) - \left(6 - x\right) \right] dx &= \int_{x=-2}^{x=6} \left[-\frac{x^2}{4} + x + 3 \right] dx \\ &= -\frac{x^3}{12} + \frac{x^2}{2} + 3x \Big|_{x=-2}^{x=6} \\ &= \left(-\frac{6^3}{12} + \frac{6^2}{2} + 3 \cdot 6 \right) - \left(-\frac{(-2)^3}{12} + \frac{(-2)^2}{2} + 3 \cdot (-2) \right) \\ &= \frac{64}{3} \end{aligned}$$

So the area between the two graphs is Area=64=3.

Let's have you try an example on your own.

Example 2. Find the area between the curves $y = x + 4$ and $y = 3 - x/2$ on the interval $[1, 4]$.

Workspace:

Solution:

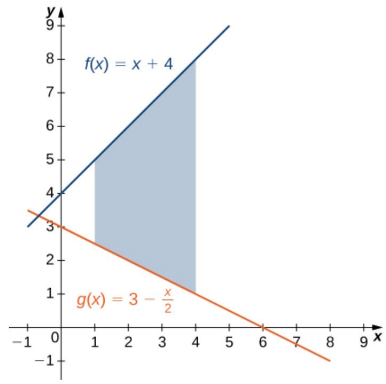

Figure 4: Region between graphs $y = x + 4$ and $y = 3 - x/2$ on interval $[1, 4]$.

Since the interval $[1, 4]$ is given we need to identify the upper graph and lower graph. Then we have to solve the integral

$$\begin{aligned}
\int_{x=a}^{x=b} \left[upper(x) - lower(x)\right] dx &= \int_{x=1}^{x=4} \left[\left(3 - \frac{x}{2}\right) - \left(x+4\right)\right] dx \\
&= \int_{x=1}^{x=4} \left[1 + \frac{3x}{2}\right] dx \\
&= x + \frac{3x^2}{4}\Big|_{x=1}^{x=4} \\
&= \left(4 + \frac{3 \cdot 4^2}{4}\right) - \left(1 + \frac{3 \cdot 1^2}{4}\right) \\
&= \frac{57}{4}
\end{aligned}$$

So the area between the two graphs is Area$= 57/4$. Let's have you try another example on your own.

Example 3. Find the area between the curves $y = \sin(x)$ and $y = \cos(x)$ on the interval $[0, \pi]$.

Workspace:

Solution:

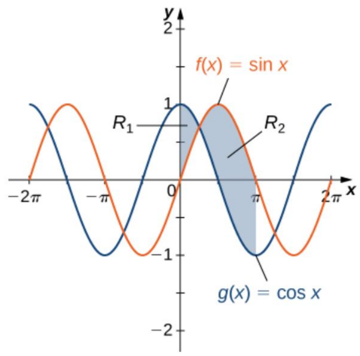

Figure 5: Region between graphs $y = \sin(x)$ and $y = \cos(x)$ on interval $[0, \pi]$.

Since the interval $[0, \pi]$ is given we need to identify the upper graph and lower graph. But since the graphs cross we consider two intervals. Where do the graphs cross? We solve $\sin(x) = \cos(x)$ when $0 \leq x \leq \pi$. They functions are equal at $x = \pi/4$ and so we have two intervals $[0, \pi/4]$ and $[\pi/4, \pi]$. On $[0, \pi/4]$ we have the upper function as $y = \cos(x)$ and the lower function as $y = \sin(x)$. On the second interval $[\pi/4, \pi]$ we have the upper function as $y = \sin(x)$ and the lower function as $y = \cos(x)$. We will need two integrals.

We evaluate as follows.

$$\begin{aligned}
\text{Area} &= \int_{x=0}^{x=\pi/4} \Big[\cos(x) - \sin(x)\Big]\,dx + \int_{x=\pi/4}^{x=\pi} \Big[\sin(x) - \cos(x)\Big]\,dx \\
&= \Big[\sin(x) + \cos(x)\Big]_{x=0}^{x=\pi/4} + \Big[-\cos(x) - \sin(x)\Big]_{x=\pi/4}^{x=\pi} \\
&= \Big[\Big(\sin\left(\frac{\pi}{4}\right) + \left(\frac{\pi}{4}\right)\Big) - \Big(\sin(0) + \cos(0)\Big)\Big] + \Big[\Big(-\cos(\pi) - \sin(\pi)\Big) - \Big(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\Big)\Big] \\
&= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \Big(0 + 1\Big)\right] + \left[\Big(-(-1) - 0\Big) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] \\
&= \Big[\sqrt{2} - 1\Big] + \Big[1 + \sqrt{2}\Big] \\
&= 2\sqrt{2}
\end{aligned}$$

So the area between the two graphs is Area$= 2\sqrt{2}$.

Let's have you try another example on your own.

Example 4. Find the area between the curves $y = \sqrt{x}$ and $y = 3/2 - x/2$ and $y = 0$.

Workspace:

Workspace Cont.:

Solution:

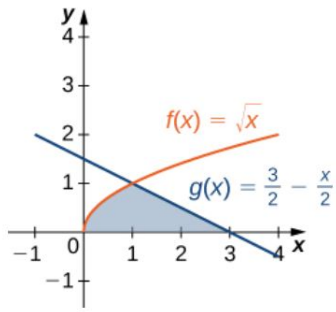

Figure 6: Region between graphs $y = \sqrt{x}$ and $y = 3/2 - x/2$ and $y = 0$.

You might notice that integrating with respect to x will require two integrals because the graphs cross. How about using the variable y. Look at the following diagram.

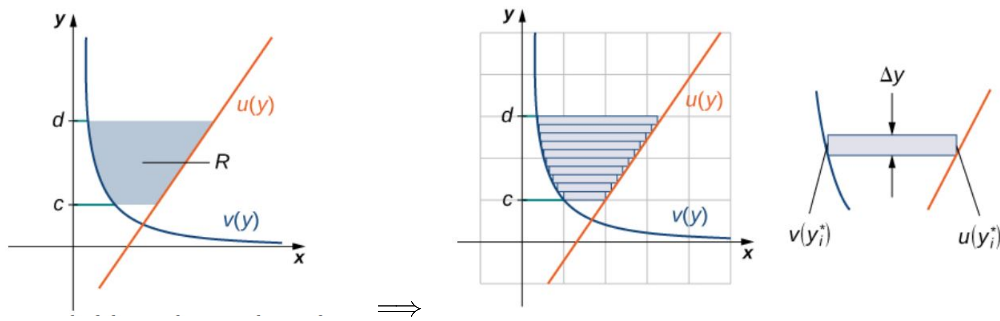

Figure 7: Estimating the area of a region using the variable y.

If we write the equations in terms of y we have $y = \sqrt{x} \Leftrightarrow x = y^2$ and $y = 3/2 - x/2 \Leftrightarrow x = 3 - 2y$. The intersection point of the two graphs is $(x, y) = (1, 1)$ we can now solve the single integral.

$$\begin{aligned}
\text{Area} &= \int_{y=c}^{y=d} \big[upper(y) - lower(y)\big]\, dy \\
&= \int_{y=0}^{y=1} \Big[\Big(3 - 2y\Big) - \Big(y^2\Big)\Big]\, dy \\
&= 3y - y^2 - \frac{y^3}{3}\Bigg|_{y=0}^{y=1} \\
&= \frac{5}{3}
\end{aligned}$$

So we have Area$= 5/3$.

Please let me know if you have any questions, comments, or corrections!