

# Math2411 - Calculus II

## Guided Lecture Notes

### Trigonometric Integrals

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### Trigonometric Integrals Introduction:

Our objective is to integrate function involving of products and powers of  $\sin(x)$  and  $\cos(x)$ , or products and powers of  $\sec(x)$  and  $\tan(x)$

We will need a few basic trig identities.

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1 & \sin(2x) &= 2 \sin(x) \cos(x) \\ \cos^2(x) &= \frac{1 + \cos(2x)}{2} & \tan^2(x) &= \sec^2(x) - 1 \\ \sin^2(x) &= \frac{1 - \cos(2x)}{2}\end{aligned}$$

### Integrals Involving $\sin^k(x)$ and $\cos^j(x)$ :

Let's consider a simple example together.

**Integrating**  $\int \cos^j x \sin x \, dx$

Evaluate  $\int \cos^3 x \sin x \, dx$ .

#### Solution

Use  $u$ -substitution and let  $u = \cos x$ . In this case,  $du = -\sin x \, dx$ . Thus,

$$\begin{aligned}\int \cos^3 x \sin x \, dx &= - \int u^3 \, du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4 x + C.\end{aligned}$$

Notice this is just a basic  $u$ -substitution problem. Let's consider another more complicated example.

**Example 1.** Integrating  $\int \cos^j(x) \sin^k(x) dx$  when  $j$  or  $k$  is odd. As a concrete example, evaluate the integral

$$\int \cos^2(x) \sin^3(x) dx$$

Evaluate  $\int \cos^2 x \sin^3 x dx$ .

### Solution

To convert this integral to integrals of the form  $\int \cos^j x \sin x dx$ , rewrite  $\sin^3 x = \sin^2 x \sin x$  and make the substitution  $\sin^2 x = 1 - \cos^2 x$ . Thus,

$$\begin{aligned} \int \cos^2 x \sin^3 x dx &= \int \cos^2 x (1 - \cos^2 x) \sin x dx \text{ Let } u = \cos x; \text{ then } du = -\sin x dx. \\ &= - \int u^2 (1 - u^2) du \\ &= \int (u^4 - u^2) du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C. \end{aligned}$$

Notice after the first step the  $\sin(x) dx$  looks like  $du$  in a  $u$ -substitution, so this leads us to write all other trig functions in terms of  $\cos(x)$ .

$$\int \cos^2(x) \sin^2(x) \left[ \sin(x) dx \right] \xrightarrow{\substack{\text{Wants to be our } du \\ \text{term in a } u\text{-sub}}} u = \cos(x) \xrightarrow{\substack{\text{Natural choice} \\ \text{for our } u\text{-sub}}} -du = \sin(x) dx \xrightarrow{\substack{\text{We have a} \\ \text{perfect match}}}$$

**Observation:** If the exponent over  $\sin(x)$  is odd we can “attach” one of the  $\sin(x)$  terms to the differential  $dx$  and prepare for a  $u$ -substitution. Let’s set up another example.

**Example 2.** Evaluate the following integral.

$$\int \cos^3(x) \sin^5(x) dx$$

Workspace:

Let's have you try another example on your own.

**Example 3.** Evaluate the integral  $\int \cos^3(x) \sin^6(x) dx$

*Workspace:*

Let's have you try another example on your own.

**Example 4.** Evaluate the integral  $\int \cos^2(x) \sin^2(x) dx$ .

**Question:** What do you notice about the exponents?

**Workspace:**

Let's try another example together.

**Example 5.** Evaluate the integral  $\int \tan^6(x) \sec^4(x) dx$ .

Workspace:

Let's consider another well-known example.

**Example 6.** Evaluate the integral  $\int \sec^3(x) dx$ .

Below are some general strategies.

**Problem-Solving Strategy: Integrating Products and Powers of  $\sin x$  and  $\cos x$** 

To integrate  $\int \cos^j x \sin^k x dx$  use the following strategies:

1. If  $k$  is odd, rewrite  $\sin^k x = \sin^{k-1} x \sin x$  and use the identity  $\sin^2 x = 1 - \cos^2 x$  to rewrite  $\sin^{k-1} x$  in terms of  $\cos x$ . Integrate using the substitution  $u = \cos x$ . This substitution makes  $du = -\sin x dx$ .
2. If  $j$  is odd, rewrite  $\cos^j x = \cos^{j-1} x \cos x$  and use the identity  $\cos^2 x = 1 - \sin^2 x$  to rewrite  $\cos^{j-1} x$  in terms of  $\sin x$ . Integrate using the substitution  $u = \sin x$ . This substitution makes  $du = \cos x dx$ . (Note: If both  $j$  and  $k$  are odd, either strategy 1 or strategy 2 may be used.)
3. If both  $j$  and  $k$  are even, use  $\sin^2 x = (1/2) - (1/2)\cos(2x)$  and  $\cos^2 x = (1/2) + (1/2)\cos(2x)$ . After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

**Problem-Solving Strategy: Integrating  $\int \tan^k x \sec^j x dx$** 

To integrate  $\int \tan^k x \sec^j x dx$ , use the following strategies:

1. If  $j$  is even and  $j \geq 2$ , rewrite  $\sec^j x = \sec^{j-2} x \sec^2 x$  and use  $\sec^2 x = \tan^2 x + 1$  to rewrite  $\sec^{j-2} x$  in terms of  $\tan x$ . Let  $u = \tan x$  and  $du = \sec^2 x$ .
2. If  $k$  is odd and  $j \geq 1$ , rewrite  $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$  and use  $\tan^2 x = \sec^2 x - 1$  to rewrite  $\tan^{k-1} x$  in terms of  $\sec x$ . Let  $u = \sec x$  and  $du = \sec x \tan x dx$ . (Note: If  $j$  is even and  $k$  is odd, then either strategy 1 or strategy 2 may be used.)
3. If  $k$  is odd and  $j \geq 3$  and  $j = 0$ , rewrite  $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x (\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$ . It may be necessary to repeat this process on the  $\tan^{k-2} x$  term.
4. If  $k$  is even and  $j$  is odd, then use  $\tan^2 x = \sec^2 x - 1$  to express  $\tan^k x$  in terms of  $\sec x$ . Use integration by parts to integrate odd powers of  $\sec x$ .