

# Math2411 - Calculus II

## Guided Lecture Notes

### Trigonometric Substitution

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## Trigonometric Substitution Introduction:

Our objective is to integrate function involving square roots of differences and sums of squares.

$$\sqrt{a^2 - x^2} \quad \sqrt{x^2 - a^2} \quad \sqrt{x^2 + a^2}$$

We will need a few basic trig identities.

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 & \sin(2x) &= 2\sin(x)\cos(x) \\ \cos^2(x) &= \frac{1 + \cos(2x)}{2} & \tan^2(x) &= \sec^2(x) - 1 \end{aligned}$$

## Integrals involving $\sqrt{a^2 - x^2}$ :

Let's consider an example together. The general strategy is to make a substitution  $x = a \sin(\theta)$ .

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - (a \sin \theta)^2} && \text{Let } x = a \sin \theta \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \text{ Simplify.} \\ &= \sqrt{a^2 - a^2 \sin^2 \theta} && \text{Factor out } a^2. \\ &= \sqrt{a^2(1 - \sin^2 \theta)} && \text{Substitute } 1 - \sin^2 \theta = \cos^2 \theta. \\ &= \sqrt{a^2 \cos^2 \theta} && \text{Take the square root.} \\ &= |a \cos \theta| \\ &= a \cos \theta. \end{aligned}$$

Then our square root quantity is converted into a simple trig function. Here is a right triangle for reference.

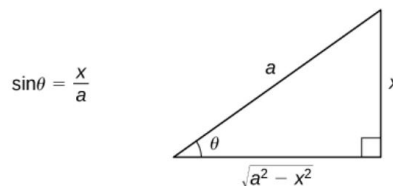


Figure 1: Reference triangle for  $\sqrt{a^2 - x^2}$ .

**Example 1.** Evaluate  $\int \sqrt{9 - x^2} \, dx$ .

**Workspace:**

Let's now have you work an example.

**Example 2.** Evaluate the integral  $\int x^3 \sqrt{1-x^2} \, dx$ .

**Workspace:**

We can also solve this using  $u$ -sub. See if you can solve this using  $u$ -sub.

**Workspace:**

## Integrals involving $\sqrt{x^2 + a^2}$ :

Let's consider integrals with a term  $\sqrt{x^2 + a^2}$ . We let  $x = a \tan(\theta)$  and build a reference triangle.

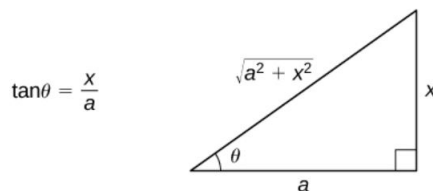


Figure 2: Reference triangle for  $\sqrt{x^2 + a^2}$

We can use the following problem solving strategy.

### Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute  $x = a \tan \theta$  and  $dx = a \sec^2 \theta d\theta$ . This substitution yields  

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a \tan \theta)^2} = \sqrt{a^2(1 + \tan^2 \theta)} = \sqrt{a^2 \sec^2 \theta} = |a \sec \theta| = a \sec \theta.$$
 (Since  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$  and  $\sec \theta > 0$  over this interval,  $|a \sec \theta| = a \sec \theta$ .)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from **Figure 3.7** to rewrite the result in terms of  $x$ . You may also need to use some trigonometric identities and the relationship  $\theta = \tan^{-1}(\frac{x}{a})$ . (Note: The reference triangle is based on the assumption that  $x > 0$ ; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which  $x \leq 0$ .)

**Example 3.** Calculate the length of the curve  $y = x^2$  on the interval  $[0, 1/2]$ . Our arclength formula gives us

$$s = \int_{x=a}^{x=b} \sqrt{1 + [f'(x)]^2} dx = \int_{x=0}^{x=1/2} \sqrt{1 + 4x^2} dx.$$

This looks like a tricky integral since there is no obvious  $u$ -substitution. We try a trig-sub. We let  $x = \frac{1}{2} \tan(\theta)$  so that  $dx = \frac{1}{2} \sec^2(\theta) d\theta$ . Now continue on your own.

**Note:** Notice that the quantity  $\sqrt{1 + 4x^2}$  suggests a reference triangle with leg lengths of 1 and  $2x$  giving a hypotenuse with length of  $\sqrt{1 + 4x^2}$ . Build your own reference triangle.

**Workspace:**

**Workspace Cont.:**

**Example 4.** Evaluate the integral  $\int \frac{1}{\sqrt{1+x^2}} dx$ .

Start by letting  $x = \tan(\theta)$  and forming the reference triangle.

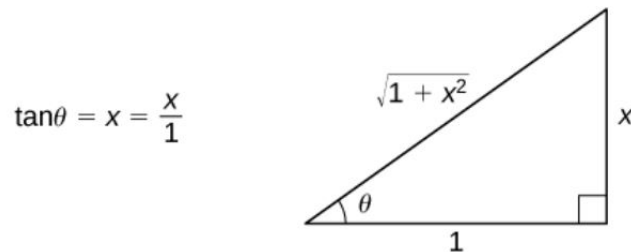


Figure 3: Reference triangle for  $\sqrt{1+x^2}$

Now solve the integral by yourself.

**Workspace:**

## Integrals involving $\sqrt{x^2 - a^2}$ :

**Example 5.** Evaluate the integral  $\int_{x=3}^{x=5} \sqrt{x^2 - 9} dx$ .

The geometry suggests we let  $x = 3 \sec(\theta)$  and so then have  $dx = 3 \sec(\theta) \tan(\theta) d\theta$ .

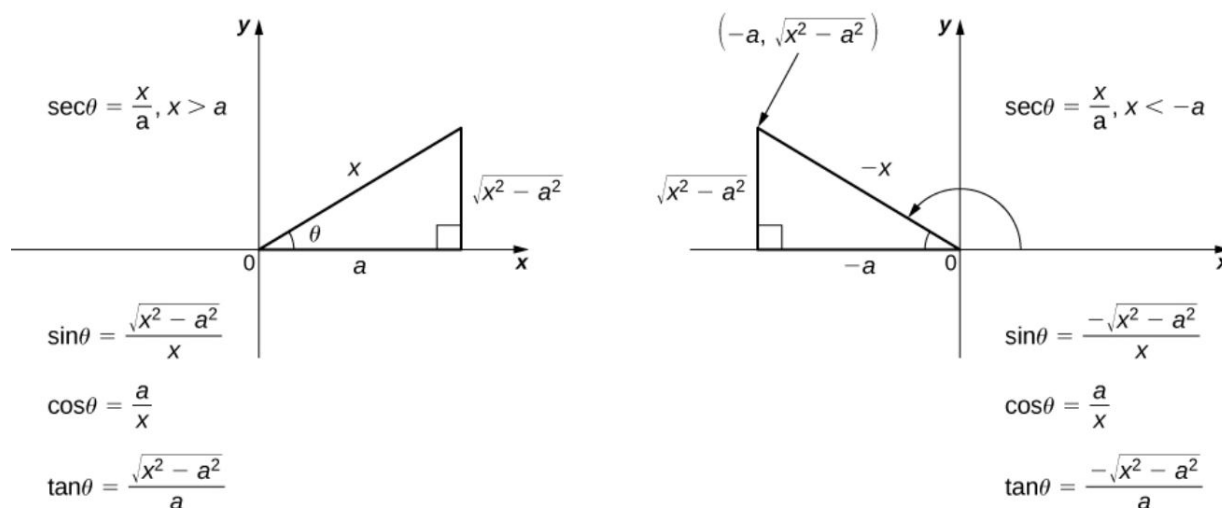


Figure 4: Reference triangle for  $\sqrt{x^2 - a^2}$

Notice there are different reference triangles depending on whether the  $x$ -values are positive (so we have  $x > a$ ) or negative (so we have  $x < -a$ ). The main consequence is when  $x$  is positive we have  $\sqrt{x^2 - a^2} = a \tan(\theta)$ . When  $x$  is negative we have  $\sqrt{x^2 - a^2} = -a \tan(\theta)$ .

### Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute  $x = a \sec \theta$  and  $dx = a \sec \theta \tan \theta d\theta$ . This substitution yields

$$\sqrt{x^2 - a^2} = \sqrt{(a \sec \theta)^2 - a^2} = \sqrt{a^2(\sec^2 \theta + 1)} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta|.$$

For  $x \geq a$ ,  $|a \tan \theta| = a \tan \theta$  and for  $x \leq -a$ ,  $|a \tan \theta| = -a \tan \theta$ .

3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from **Figure 3.9** to rewrite the result in terms of  $x$ . You may also need to use some trigonometric identities and the relationship  $\theta = \sec^{-1}(\frac{x}{a})$ . (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether  $x > a$  or  $x < -a$ .)

Let's try an example on your own.



**Example 6.** Evaluate the integral  $\int \frac{1}{\sqrt{x^2 - 4}} dx$ , assuming that  $x < -2$ .

**Note:** The domain of the integrand  $f(x) = 1/\sqrt{x^2 - 4}$  is  $(-\infty, -2) \cup (2, \infty)$ .

**Workspace:**