Math2411 - Calculus II
Guided Lecture Notes
The Ratio and Root Tests

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The Ratio and Root Tests Introduction:

Many important infinite series can not be evaluated using the techniques we have studied so far. For example, the following series does not satisfy the criteria of any of our current convergence tests.

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

Our objective is to develop tests to handle this type of series. The first test is called the ***Ratio Test***.

The Ratio Test

Theorem 5.16: Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

Let's work a few examples.

Ratio Test Examples:

Example 1. Determine whether $\sum_{n=o}^{\infty} \frac{2^n}{n!}$ converges or diverges.

Workspace:

Solution:

We use the Ratio Test and let

$$\begin{aligned} r &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right| \\ &= \lim_{n\to\infty}\frac{2}{n+1} \\ &= 0 \end{aligned}$$

Since $r = 0 < 1$ the series converges absolutely by the Ratio Test.

Here is another example.

Example 2. Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$ converges or diverges.

Workspace:

Solution:

We use the Ratio Test and let

$$\begin{aligned} r &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\right| \\ &= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \\ &= e \end{aligned}$$

Since $r = e > 1$ the series diverges by the Ratio Test.

Here is another example.

Example 3. Determine whether $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(n!)^2}{(2n)!}$ converges or diverges.

Workspace:

Solution:

We use the Ratio Test and let

$$\begin{aligned} r &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|(-1)^{n+1}\frac{\big((n+1)!\big)^2}{(2n+2)!}\cdot(-1)^n\frac{(2n)!}{(n!)^2}\right| \\ &= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)} \\ &= \frac{1}{4} \end{aligned}$$

Since $r = 1/4 < 1$ the series converges absolutely by the Ratio Test.

Discussion:

You may ask yourself why we didn't use the Alternating Series Test? In fact, we have the tools to use the Alternating Series Test, but it will only determine convergence. What it does not tell us is whether we have absolute convergence or conditional convergence. One strength of the Ratio Test is that it tells us that we have absolute convergence, which is a much stronger form of convergence. Additionally, in this example it is much easier computationally to use the Ratio Test and so in my opinion it is the superior test for this example. I will provide the details for the Alternating Series Test as an optional section below for those who are curious about those details.

Optional Section: Let's try the Alternating Series Test and first show that the non-alternating component of the series terms is positive and decreasing. That is, show that $0 \le b_{n+1} \le b_n$. Of course all terms are positive so we only need to check the right inequality. The following statements are all equivalent.

$$\begin{aligned} b_{n+1} &\le b_n \\ \frac{b_{n+1}}{b_n} &\le 1 \\ \frac{\Big[(n+1)!\Big]^2\cdot(2n)!}{(2n+2)!\cdot\big(n!\big)^2} &\le 1 \\ \frac{(n+1)(n+1)}{(2n+2)(2n+1)} &\le 1 \\ \underbrace{\left(\frac{n+1}{2n+2}\right)}_{<1}\underbrace{\left(\frac{n+1}{2n+1}\right)}_{<1} &\le 1 \end{aligned}$$

Since the last inequality is certainly true we have shown that $0 \leq b_{n+1} \leq b_n$ and the non-alternating components of the series terms are positive and decreasing.

Next we show that $\lim_{n\to\infty} b_n = 0$. This can be done as follows using Stirling's Approximation which states that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1.$$

Then we have

$$\lim_{n\to\infty} \frac{\left(n!\right)^2}{(2n)!} = \lim_{n\to\infty} \frac{2\pi n\left(\frac{n}{e}\right)^{2n}}{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}} = \lim_{n\to\infty} \frac{\sqrt{\pi n}}{4^n} = 0.$$

Hopefully you're convinced that although the Alternating Series Test does apply in this example, the Ratio Test is the much preferred test for numerous reasons.

Here is another example.

Example 4. Determine whether the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges or diverges.

Workspace:

Solution:

We use the Ratio Test and let

$$\begin{aligned} r &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n^p}{(n+1)^p}\right| \\ &= \lim_{n\to\infty}\frac{n^p}{(n+1)^p} \\ &= 1 \end{aligned}$$

Since $r = 1$ the Ratio Test is inconclusive. This makes sense because there are convergent p-series and divergent p-series. The main point here is that the Ratio Test will not be useful every infinite series. There are many series where the test will be inconclusive.

We have another test which can be useful when the Ratio Test is difficult to use. It is called the Root Test.

The Root Test:

Consider the infinite series

$$\sum_{n=1}^{\infty}\frac{\left(n^2+3n\right)^n}{(4n^2+5)^n}.$$

The Ratio Test will require significant simplification to evaluate the limit. Fortunately, there is another test that is well-suited for this type of series. It is called the ***Root Test***.

Theorem 5.17: Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

Root Test Examples:

Let's work through an example.

Example 5. Determine if the infinite series $\sum_{n=1}^{\infty} \frac{\left(n^2+3n\right)^n}{(4n^2+5)^n}$ converges or diverges.

Workspace:

Solution:

We try the Root Test and compute the following.

$$\begin{aligned}\rho &= \lim_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \lim_{n\to\infty} \sqrt[n]{\left|\frac{\left(n^2+3n\right)^n}{(4n^2+5)^n}\right|} \\ &= \lim_{n\to\infty} \frac{n^2+3n}{4n^2+5} \\ &= \frac{1}{4}\end{aligned}$$

Since $\rho = 1/4 < 1$ the series converges absolutely by the Root Test.

Example 6. Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{(-12)^n}{n}$ converges or diverges.

Workspace:

Solution:

We try the Root Test and compute the following.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \sqrt[n]{|a_n|} \\
&= \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-12)^n}{n}\right|} \\
&= \lim_{n\to\infty} \frac{12}{n^{1/n}} \\
&= \frac{12}{1} \\
&= 12
\end{aligned}$$

Since $\rho = 12 > 1$ the series diverges by the Root Test.

Example 7. Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{1+2n}}{2^{5n-3}}$ converges or diverges.

Workspace:

Solution:

We try the Root Test and compute the following.

$$\begin{aligned}\rho &= \lim_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-5)^{1+2n}}{2^{5n-3}}\right|} \\ &= \lim_{n\to\infty} \left|\frac{(-5)^{\frac{1}{n}+2}}{2^{5-\frac{3}{n}}}\right| \\ &= \left|\frac{(-5)^2}{2^5}\right| \\ &= \frac{25}{32}\end{aligned}$$

Since $\rho = 25/32 < 1$ the series diverges by the Root Test.

Let's consider an example from the section on Alternating Series Test

Example 8. Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{n+1}\right)^{n^2}$ converges or diverges.

Workspace:

Solution:

We try the Root Test and compute the following.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \sqrt[n]{|a_n|} \\
&= \lim_{n\to\infty} \sqrt[n]{\left|(-1)^{n+1}\left(\frac{n}{n+1}\right)^{n^2}\right|} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} \\
&= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^{n}} \\
&= \frac{1}{e}
\end{aligned}$$

Since $\rho = 1/e < 1$ the series converges absolutely by the Root Test.

Choosing a Convergence Test

Problem-Solving Strategy: Choosing a Convergence Test for a Series

Consider a series $\sum_{n=1}^{\infty} a_n$. In the steps below, we outline a strategy for determining whether the series converges.

1. Is $\sum_{n=1}^{\infty} a_n$ a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a $p-$series or geometric series? If so, check the power p or the ratio r to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If we are interested in absolute convergence, proceed to step $3,$ considering the series of absolute values $\sum_{n=1}^{\infty} |a_n|$.
3. Is the series similar to a $p-$series or geometric series? If so, try the comparison test or limit comparison test.
4. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = b_n^n,$ try the root test first. Otherwise, try the ratio test first.
5. Use the divergence test. If this test does not provide any information, try the integral test.

Please let me know if you have any questions, comments, or corrections!