

Math2411 - Calculus II
 Section H01 - Spring 2025
 Arclength

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics - Dr. Rostermundt

Arclength Introduction

Suppose that we want to determine the arclength along a 2D curve $y = f(x)$ where $a \leq x \leq b$. We start with an estimation using straight lines.

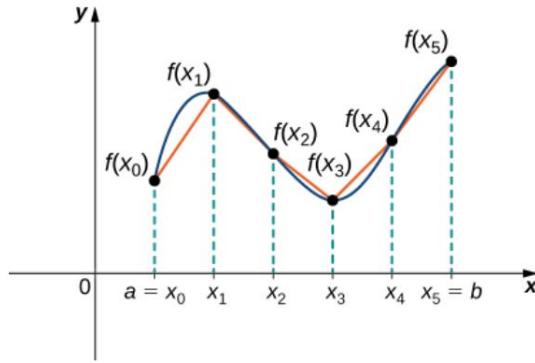


Figure 1: Estimating arclength using straight line segments

The total arclength will be approximated as the sum of the lengths of all line segments. If we write Δs_k for the actual length along the k^{th} segment of the curve and l_k the length of the k^{th} line segment we have

$$\text{Arc length} \approx \sum_{k=1}^n l_k$$

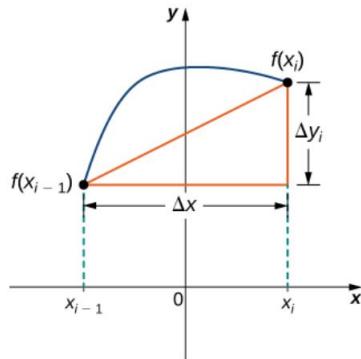


Figure 2: Length of the i^{th} line segment

Our approximation can be written as follows.

$$\begin{aligned}
 \Delta s_k &\approx \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\
 &= \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \Delta x_k \\
 &= \sqrt{1 + (f'(x_k^*))^2} \Delta x_k \quad \text{by Mean Value Theorem}
 \end{aligned}$$

So we have

$$\text{Total Length} \approx \sum_{k=1}^n \sqrt{1 + (f'(x_k^*))^2} \Delta x_k \xrightarrow[\text{Let } \Delta x \rightarrow 0]{\substack{\text{Approximation} \\ \text{Improves}}} \text{Total Length} = \int_{x=a}^{x=b} \sqrt{1 + [f'(x)]^2} dx.$$

As $\Delta x \rightarrow 0$ we have $l_k \rightarrow \Delta s_k$ and $\Delta s_k \rightarrow 0$ and an improved approximation. So we can write the differential

$$ds = \sqrt{1 + (f'(x_k^*))^2} dx$$

and the integral can be written in two different forms as

$$s = \int_{x=a}^{x=b} \sqrt{1 + [f'(x)]^2} dx \quad \text{or} \quad s = \int_C ds.$$

Start with a concrete example.

Example 1. Find the arclength on the curve $y = 2x^{3/2}$ when $0 \leq x \leq 1$.

Workspace:

Solution:

While not necessary, we can start by graphing the curve.

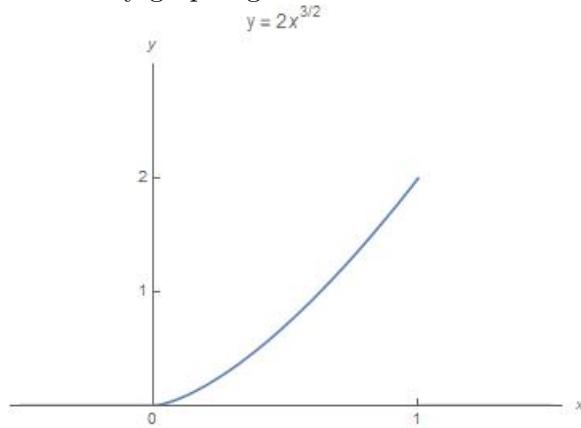


Figure 3: Graph of $y = 2x^{3/2}$ on interval $[0, 1]$.

We can now set up the integral.

$$\begin{aligned}
 s &= \int_{x=a}^{x=b} \sqrt{1 + [f'(x)]^2} dx \\
 &= \int_{x=0}^{x=1} \sqrt{1 + [3x^{1/2}]^2} dx \\
 &= \int_{x=0}^{x=1} \sqrt{1 + 9x} dx \\
 &= \frac{1}{9} \int_{u=1}^{u=10} \sqrt{u} du \quad (\text{Letting } u = 1 + 9x) \\
 &= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{u=1}^{u=10} \\
 &= \frac{2}{27} [10\sqrt{10} - 1]
 \end{aligned}$$

Now try setting up an example on your own.

Example 2. Find the arclength on the curve $y = \sin(3x)$ when $0 \leq x \leq \pi$.

Workspace:

Workspace Cont:

Solution:

We set up the integral.

$$s = \int_{x=0}^{x=\pi} \sqrt{1 + [3 \cos(3x)]^2} dx = \int_{x=0}^{x=\pi} \sqrt{1 + 9 \cos^2(3x)} dx \approx 6.9872.$$

Usually the integral will be too difficult to evaluate by hand and so we will simply set up the integral and allow a computer to estimate the value of the integral.

Please let me know if you have any questions, comments, or corrections!