Math2411 - Calculus II
Guided Lecture Notes
The Comparison Tests

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The Comparison Tests Introduction:

Our objective is to study the comparison tests for convergence of infinite series. Let's motivate with an example.

$$\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$$

We might consider the Integral Test since $f(x) = \frac{1}{x^3+3x+1}$ satisfies all three properties for the Integral Test: f is continuous for all $x|ge1$; $f(x) > 0$ for all $x > 1$; and f is decreasing for all $x > 1$. However, the integral

$$\int_{x=1}^{\infty} \frac{1}{x^3+3x+1}\,dx$$

is a difficult integral. The obvious technique of choice would be partial fractions, but the denominator x^3+3x+1 is difficult to factor. In fact, it factors into the product of a linear factor and an irreducible quadratic where the one real zero is

$$x = -\left(\frac{2}{\sqrt{5}-1}\right)^{1/3} + \left(\frac{\sqrt{5}-1}{2}\right)^{1/3}.$$

Since we can rarely determine the exact value of an infinite series we are often satisfied by simply determining convergence or divergence. And if the series converges, we can estimate the value of the sum with partial sums.

Direct Comparison Test:

Theorem 5.11: Comparison Test

i. Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Suppose there exists an integer N such that $a_n \ge b_n \ge 0$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Direct Comparison Test Examples:

Example 1. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges or diverges.

Workspace:

Here is another example.

Example 2. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ converges or diverges.

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Here is another example.

Example 3. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ converges or diverges.

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Limit Comparison Test:

Let's consider the following example.

$$\sum_{n=2}^{\infty} \frac{1}{n^2-3}$$

Our obvious direct comparison series is $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which is a convergent p-series with $p = 2$. However,

$$\frac{1}{n^2-3} > \frac{1}{n^2}$$

for all $n \geq 2$ and so our direct comparison is inconclusive. But it would be convenient to be able to use our knowledge about the convergence of $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Fortunately, there is another comparison test.

Theorem 5.12: Limit Comparison Test

Let $a_n,\ b_n \geq 0$ for all $n \geq 1$.

i. If $\lim_{n\to\infty} a_n/b_n = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

ii. If $\lim_{n\to\infty} a_n/b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

iii. If $\lim_{n\to\infty} a_n/b_n = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

In our current example we have $a_n = 1/(n^2-2)$ and $b_n = 1/n^2$. Then we have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^2-2)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-2} = 1.$$

Since $L > 0$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent p-series we conclude that $\sum_{n=2}^{\infty} \frac{1}{n^2-3}$ converges by the Limit Comparison Test.

Limit Comparison Test Examples:

Example 4. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ converges or diverges.

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Example 5. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges or diverges.

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Example 6. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges or diverges.

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