Math2411 - Calculus II
Guided Lecture Notes
Power Series

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Power Series Introduction:

Our objective is to study a class of infinite series called ***power series***.

Definition

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \tag{6.1}$$

is a power series centered at $x = 0$. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots \tag{6.2}$$

is a power series centered at $x = a$.

Question: What do you notice that is different about a power series compared to the series we have previously studied?

Answer: The terms of a power series contain a variable quantity x and so can be thought of as functions $f(x)$. One of the most important questions we ask about a function f is the following: "What is the domain of f?" In other words, what are the x-values where our function $f(x)$ is defined (or makes sense)? Since we are dealing with an infinite series we could pose this question as follows: For which x-values does the power series converge? We can summarize the answer to this question with the following theorem.

Theorem 6.1: Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.

ii. The series converges for all real numbers x.

iii. There exists a real number $R > 0$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. At the values x where $|x - a| = R$, the series may converge or diverge.

Look at the following diagram to get a better sense of the statement of the theorem.

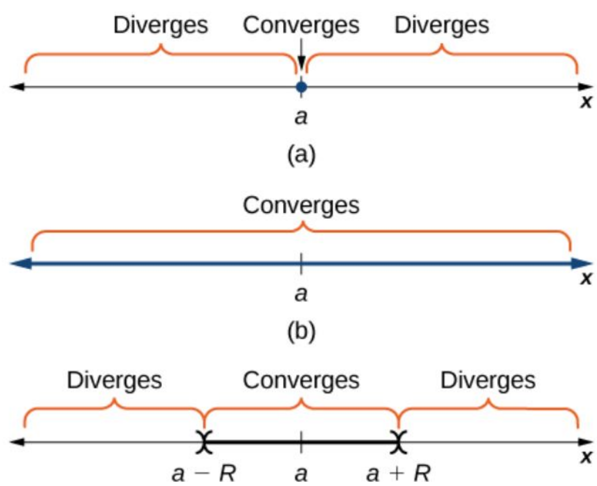

Figure 1: Three Convergence Possibilities for a Power Series

Here is some useful terminology that we will be using.

Definition

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$

Let's work some examples and discover the process.

Power Series Examples:

Example 1. Find the interval of convergence and the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Workspace:

Solution:

Observe that the power series is centered at $a = 0$. We will use the Ratio Test and so compute

$$\begin{aligned} r &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \\ &= |x| \lim_{n\to\infty} \frac{1}{n+1} \\ &= 0 < 1 \quad \text{for all } x\text{-values.} \end{aligned}$$

So the power series converges for all $x \in \mathbb{R}$ and the interval of convergence is $(-\infty, \infty)$ and radius of convergence is $R = \infty$.

Let's try another example.

Example 2. Find the interval of convergence and the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}.$$

Workspace:

Workspace Continued:

Solution:

Observe that the power series is centered at $a = 2$. We will use the Ratio Test and so compute

$$\begin{aligned} r &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}\cdot\frac{(n+1)3^n}{(x-2)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)(n+1)}{3(n+2)}\right| \\ &= |x-2|\lim_{n\to\infty}\frac{n+1}{3(n+2)} \\ &= \frac{1}{3}\cdot|x-2| \end{aligned}$$

So we solve $r = \frac{1}{3}|x-2| < 1$ for x.

$$\frac{1}{3}\cdot|x-2| < 1 \quad\Leftrightarrow\quad \underbrace{|x-2| < 3}_{R=3} \quad\Leftrightarrow\quad -3 < x-2 < 3 \quad\Leftrightarrow\quad -1 < x < 5$$

So we see that the radius of convergence is $R = 3$ and by the Ratio Test the power series converges absolutely when $-1 < x < 5$ and diverges when $x < -1$ and when $x > 5$. However, the Ratio Test fails at $x = -1$ and $x = 5$ because at those points we have $r = 1$. So we must check for convergence at those endpoint x-values using some other convergence test.

- $x = -1$: We have the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n}{(n+1)3^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ which is the convergent alternating Harmonic Series. The original power series converges at $x = -1$.

- $x = 5$: We have the series $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{(n+1)3^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ which is the divergent Harmonic Series. The original power series diverges at $x = 5$.

So we have the interval of convergence is $[-1, 5)$. Observe that the center of the interval is $a = 2$.

Let's try another example.

Example 3. Find the interval of convergence and the radius of convergence for the power series

$$\sum_{n=0}^{\infty}\frac{x^{2n+2}}{4^{n+1}}.$$

Workspace:

Solution:

Observe that the power series is centered at $a = 0$. We will use the Ratio Test and so compute

$$\begin{aligned} r &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^{2n+4}}{4^{n+2}} \cdot \frac{4^{n+1}}{x^{2n+2}} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^2}{4} \right| \\ &= \frac{x^2}{4} \end{aligned}$$

So we solve $r = x^2/4 < 1$ for x.

$$\frac{x^2}{4} < 1 \quad \Leftrightarrow \quad x^2 < 4 \quad \Leftrightarrow \quad \underbrace{|x| < 2}_{R=2} \quad \Leftrightarrow \quad -2 < x < 2$$

So we see that the radius of convergence is $R = 2$ and by the Ratio Test the power series converges absolutely when $-2 < x < 2$ and diverges when $x < -2$ and when $x > 2$. However, the Ratio Test fails at $x = -2$ and $x = 2$ because at those points we have $r = 1$. So we must check for convergence at those endpoint x-values using some other convergence test.

- $x = -2$: We have the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{2n+2}}{4^{n+1}} = \sum_{n=0}^{\infty} 1$ which is a divergent series. The original power series diverges at $x = -2$.

- $x = 2$: We have the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+2}}{4^{n+1}} = \sum_{n=0}^{\infty} 1$ which is a divergent series. The original power series diverges at $x = 2$.

So we have the interval of convergence is $(-2, 2)$. Observe that the center of the interval is $a = 0$.

Let's try another example.

Example 4. Find the interval of convergence and the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{n(x+1)^n}{e^n}.$$

Workspace:

Solution:

Observe that the power series is centered at $a = 0$. We will use the Ratio Test and so compute

$$\begin{aligned} r &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)(x+1)^{n+1}}{e^{n+1}} \cdot \frac{e^n}{n(x+1)^n} \right| \\ &= \frac{|x+1|}{e} \lim_{n\to\infty} \frac{n+1}{n} \\ &= \frac{|x+1|}{e} \end{aligned}$$

So we solve $r = |x+1|/e < 1$ for x.

$$\frac{|x+1|}{e} < 1 \quad \Leftrightarrow \quad \underbrace{|x+1| < e}_{R=e} \quad \Leftrightarrow \quad -e < x+1 < e \quad \Leftrightarrow \quad -1-e < x < -1+e$$

So we see that the radius of convergence is $R = e$ and by the Ratio Test the power series converges absolutely when $-1-e < x < -1+e$ and diverges when $x < -1-e$ and when $x > -1+e$. However, the Ratio Test fails at $x = -1-e$ and $x = -1+e$ because at those points we have $r = 1$. So we must check for convergence at those endpoint x-values using some other convergence test.

- $x = -1-e$: We have the series $\displaystyle\sum_{n=0}^{\infty} \frac{n\big(-e\big)^n}{e^n} = \sum_{n=0}^{\infty} (-1)^n \cdot n$ which is a divergent series. The original power series diverges at $x = -1-e$.

- $x = -1+e$: We have the series $\displaystyle\sum_{n=0}^{\infty} \frac{ne^n}{e^n} = \sum_{n=0}^{\infty} n$ which is a divergent series. The original power series diverges at $x = -1+e$.

So we have the interval of convergence is $(-1-e, -1+e)$. Observe that the center of the interval is $a = -1$.

Please let me know if you have any questions, comments, or corrections!