Math2411 - Calculus II
Guided Lecture Notes
Arclength

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

Arclength Introduction

Suppose that we want to determine the arclength along a 2D curve $y = f(x)$ where $a \leq x \leq b$. We start with an estimation using straight lines.

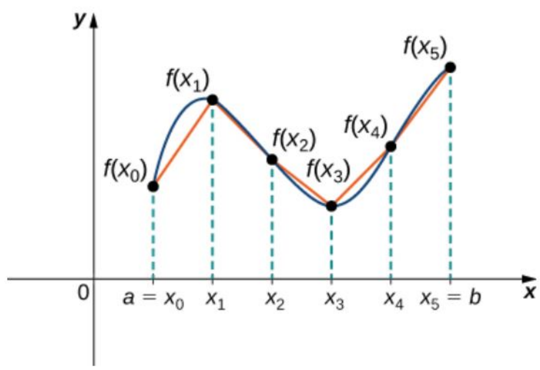

Figure 1: Estimating arclength using straight line segments

The total arclength will be approximated as the sum of the lengths of all line segments. If we write $\triangle s_k$ for the actual length along the k^{th} segment of the curve and l_k the length of the k^{th} line segment we have

$$Arclength \approx \sum_{k=1}^{n} l_k$$

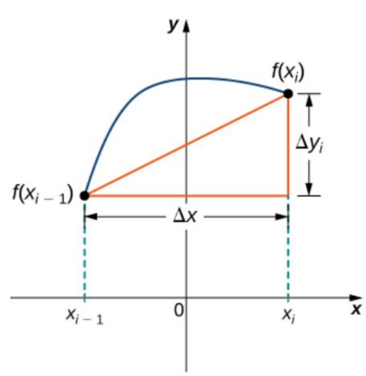

Figure 2: Length of the i^{th} line segment

Our approximation can be written as follows.

$$\begin{aligned}
\triangle s_k &\approx \sqrt{\left(\triangle x_k\right)^2 + \left(\triangle y_k\right)^2} \\
&= \sqrt{1 + \left(\frac{\triangle y_k}{\triangle x_k}\right)^2}\,\triangle x_k \\
&= \sqrt{1 + \left(f'(x_k^*\right)^2}\,\triangle x_k \qquad \text{by Mean Value Theorem}
\end{aligned}$$

So we have

$$\text{Total Length} \approx \sum_{k=1}^{n} \sqrt{1 + \left(f'(x_k^*\right)^2}\,\triangle x_k \quad \xrightarrow[\text{Let } \triangle x \to 0]{\text{Approximation Improves}} \quad \text{Total Length} = \int_{x=a}^{x=b} \sqrt{1 + \left[f'(x)\right]^2}\,dx.$$

As $\triangle x \to 0$ we have $l_k \to \triangle s_k$ and $\triangle s_k \to 0$ and an improved approximation. So we can write the differential

$$ds = \sqrt{1 + \left(f'(x_k^*\right)^2}\,dx$$

and the integral can be written in two different forms as

$$s = \int_{x=a}^{x=b} \sqrt{1 + \left[f'(x)\right]^2}\,dx \quad \text{or} \quad s = \int_{\mathcal{C}} ds.$$

Start with a concrete example.

Example 1. Find the arclength on the curve $y = 2x^{3/2}$ when $0 \le x \le 1$.

Workspace:

Solution:

While not necessary, we can start by graphing the curve.

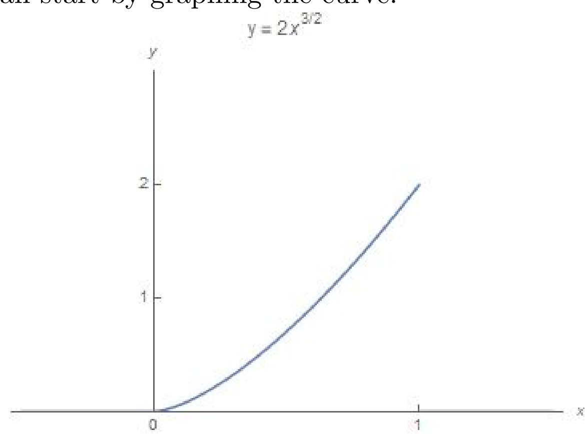

Figure 3: Graph of $y = 2x^{3/2}$ on interval $[0, 1]$.

We can now set up the integral.

$$
\begin{aligned}
s &= \int_{x=a}^{x=b} \sqrt{1 + \left[f'(x)\right]^2}\, dx \\
&= \int_{x=0}^{x=1} \sqrt{1 + \left[3x^{1/2}\right]^2}\, dx \\
&= \int_{x=0}^{x=1} \sqrt{1 + 9x}\, dx \\
&= \frac{1}{9}\int_{u=1}^{u=10} \sqrt{u}\, du \quad \big(\text{Letting } u = 1 + 9x\big) \\
&= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{u=1}^{u=10} \\
&= \frac{2}{27}\left[\, 10\sqrt{10} - 1 \,\right]
\end{aligned}
$$

Now try setting up an example on your own.

Example 2. Find the arclength on the curve $y = \sin(3x)$ when $0 \le x \le \pi$.

Workspace:

Workspace Cont:

Solution:

We set up the integral.

$$s = \int_{x=0}^{x=\pi} \sqrt{1 + \left[3\cos(3x)\right]^2}\, dx = \int_{x=0}^{x=\pi} \sqrt{1 + 9\cos^2(3x)}\, dx \approx 6.9872.$$

Usually the integral will be too difficult to evaluate by hand and so we will simply set up the integral and allow a computer to estimate the value of the integral.

Please let me know if you have any questions, comments, or corrections!