

# Math2411 - Calculus II

## Guided Lecture Notes

### Physics Applications

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

## Physics Applications Introduction

Suppose that we want to determine the work done when a force moves an object. Recall that in physics we define work as

$$\text{Work} = \text{Force} \times \text{Distance}.$$

For example, compute the work done by compressing or stretching a spring.

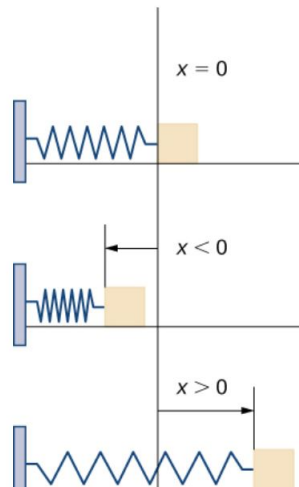


Figure 1: Block attached to a spring at equilibrium, compressed, and stretched.

Hooke's Law states that the force applied from a spring at position  $x$  from equilibrium can be described as  $F(x) = kx$  for some positive constant  $k$ . We want to move a block from  $x = a \rightarrow x = b$ . Over a short interval  $[x_{i-1}, x_i]$  we can estimate the work done moving the block from  $x_{i-1} \rightarrow x_i$ .

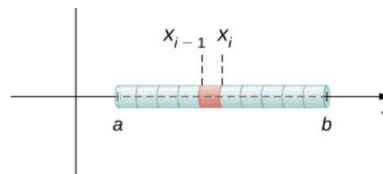


Figure 2: The interval from  $x_{i-1} \rightarrow x_i$ .

$$W_i \approx F(x_i)(x_i - x_{i-1}) \quad \xRightarrow[\text{Add up each work component}]{} \quad \text{Total Work} \approx \sum_{i=1}^n F(x_i)(x_i - x_{i-1}) = \sum_{i=1}^n kx(x_i - x_{i-1})$$

Then our approximation improves as we let  $\Delta x_i = x_i - x_{i-1} \rightarrow 0$  and so we have

$$\text{Total Work} = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n F(x_i) \cdot \Delta x = \int_{x=a}^{x=b} F(x) dx = \int_{x=a}^{x=b} kx dx.$$

We can interpret this integral as adding up all work done from  $x = a \rightarrow x = b$ .

$$\text{Total Work} = \int_{x=a}^{x=b} \underbrace{kx \, dx}_{\text{Work Quantity}}$$

Sum  
 ↓  
 Spring Force at  
 Position  $x$   
 ↓  
 Small Distance  
 ↓

Continue with a concrete example.

**Example 1.** Suppose it takes a force of  $10 \, \text{N}$  (in the negative direction) to compress a spring  $0.2 \, \text{m}$  from the equilibrium position. How much work is done to stretch the spring  $0.5 \, \text{m}$  from the equilibrium position?

**Workspace:**

Now try another example.

**Example 2.** Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

**Workspace:**

Before we try another example we can write down a problem solving strategy.

### Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.
6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as  $n \rightarrow \infty$  and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

**Example 3.** Assume a tank in the shape of an inverted cone, with height 12m and base radius 4m. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4m. How much work is required to pump out that amount of water?

**Workspace:**