Math2411 - Calculus II
Guided Lecture Notes
Trigonometric Substitution

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Trigonometric Substitution Introduction:

Our objective is to integrate function involving square roots of differences and sums of squares.

$$\sqrt{a^2 - x^2} \qquad \sqrt{x^2 - a^2} \qquad \sqrt{x^2 + a^2}$$

We will need a few basic trig identities.

$$\sin^2(x) + \cos^2(x) = 1 \qquad \sin(2x) = \sin^2(x) = \frac{1 - \cos(2x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \qquad \tan^2(x) = \sec^2(x) - 1$$

Integrals involving $\sqrt{a^2 - x^2}$:

Let's consider an example together. The general strategy is to make a substitution $x = a\sin(\theta)$.

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin\theta)^2} && \text{Let } x = a\sin\theta \text{ where } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\text{. Simplify.}\\
&= \sqrt{a^2 - a^2\sin^2\theta} && \text{Factor out } a^2.\\
&= \sqrt{a^2(1 - \sin^2\theta)} && \text{Substitute } 1 - \sin^2 x = \cos^2 x.\\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.}\\
&= |a\cos\theta| &&\\
&= a\cos\theta. &&
\end{aligned}$$

Then our square root quantity is converted into a simple trig function. Here is a right triangle for reference.

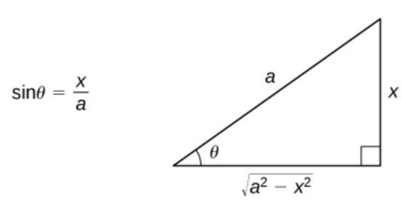

Figure 1: Reference triangle for $\sqrt{a^2 - x^2}$.

Example 1. Evaluate $\int \sqrt{9-x^2}\,dx$.

Workspace:

Let's now have you work an example.

Example 2. Evaluate the integral $\int x^3\sqrt{1-x^2}\,dx$.

Workspace:

We can also solve his using u-sub. See if you can solve this using u-sub.

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Integrals involving $\sqrt{x^2+a^2}$:

Let's consider integrals with a term $\sqrt{x^2+a^2}$. We let $x = a\tan(\theta)$ and build a reference triangle.

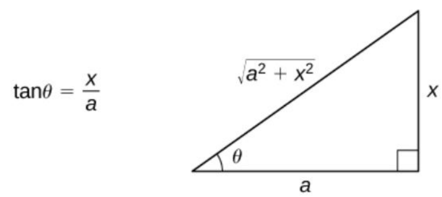

Figure 2: Reference triangle for $\sqrt{x^2+a^2}$

We can use the following problem solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = |a\sec\theta| = a\sec\theta$. (Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from **Figure 3.7** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (*Note*: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

Example 3. Calculate the length of the curve $y = x^2$ on the interval $[0, 1/2]$. Our arclength formula gives us

$$s = \int_{x=a}^{x=b} \sqrt{1+\left[f'(x)\right]^2}\,dx = \int_{x=0}^{x=1/2} \sqrt{1+4x^2}\,dx.$$

This looks like a tricky integral since there is no obvious u-substitution. We try a trig-sub. We let $x = \frac{1}{2}\tan(\theta)$ so that $dx = \frac{1}{2}\sec^2(\theta)\,d\theta$. Now continue on your own.

***Note*:** Notice that the quantity $\sqrt{1+4x^2}$ suggests a reference triangle with leg lengths of 1 and $2x$ giving a hypotenuse with length of $\sqrt{1+4x^2}$. Build your own reference triangle.

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Example 4. Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$.

Start by letting $x = \tan(\theta)$ and forming the reference triangle.

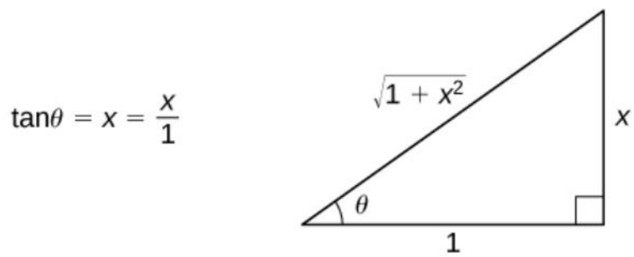

Figure 3: Reference triangle for $\sqrt{1+x^2}$

Now solve the integral by yourself.

Workspace:

Integrals involving $\sqrt{x^2 - a^2}$:

Example 5. Evaluate the integral $\displaystyle\int_{x=3}^{x=5} \sqrt{x^2 - 9}\, dx$.

The geometry suggests we let $x = 3\sec(\theta)$ and so then have $dx = 3\sec(\theta)\tan(\theta)\, d\theta$.

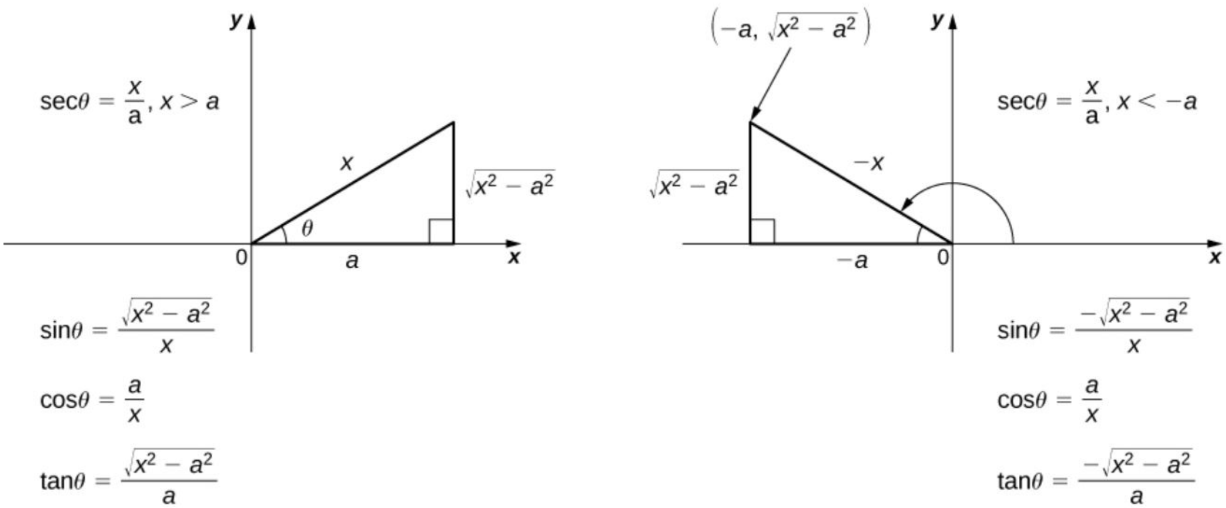

Figure 4: Reference triangle for $\sqrt{x^2 - a^2}$

Notice there are different reference triangles depending on whether the x-values are positive (so we have $x > a$) or negative (so we have $x < -a$). The main consequences is when x is positive we have $\sqrt{x^2 - a^2} = a\tan(\theta)$. When x is negative we have $\sqrt{x^2 - a^2} = -a\tan(\theta)$.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$. This substitution yields

$$\sqrt{x^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta + 1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|.$$

   For $x \geq a$, $|a\tan\theta| = a\tan\theta$ and for $x \leq -a$, $|a\tan\theta| = -a\tan\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from **Figure 3.9** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (*Note*: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

Let's try an example on your own.

Example 6. Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$, assuming that $x < -2$.

***Note*:** The domain of the integrand $f(x) = 1/\sqrt{x^2-4}$ is $\big(-\infty, -2\big) \cup \big(2, \infty\big)$.

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