Math2411 - Calculus II
Guided Lecture Notes
Parametric Equations

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Parametric Equations Introduction:

Our objective is to examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path. One example could be describing the orbit of the Earth around the sun.

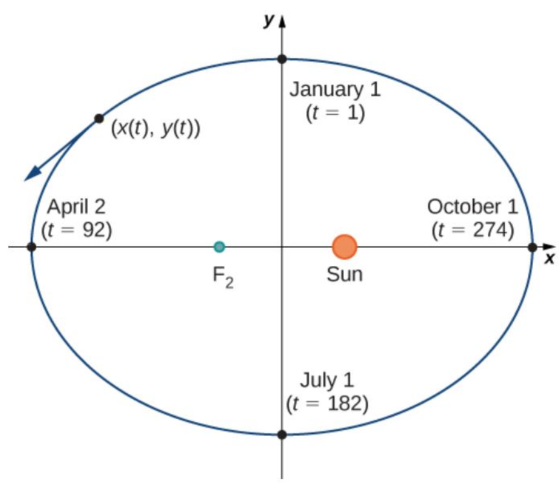

Figure 1: Coordinate Axes Superimposed on the Orbit of Earth

Definition

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t) \text{ and } y = y(t)$$

are called parametric equations and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a **parametric curve** or *plane curve*, and is denoted by C.

Parametric Curve Examples:

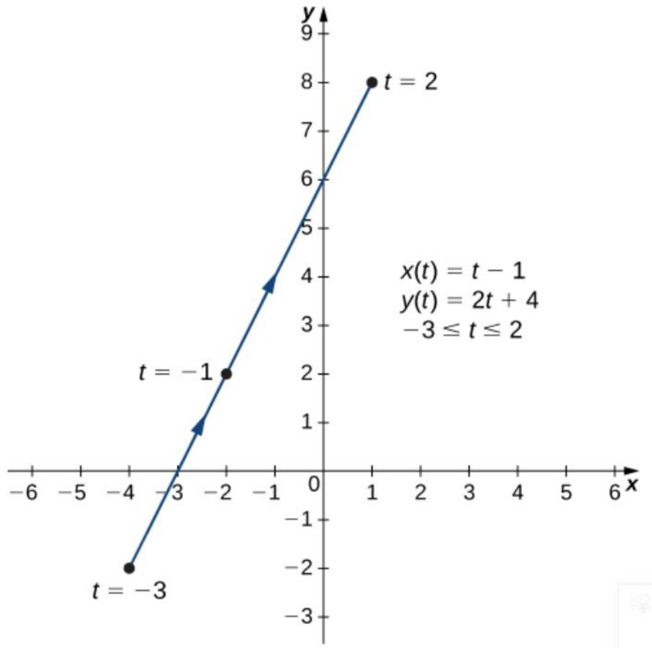

Figure 2: Graph of Parametric Curve with $x(t) = t - 1$ and $y(t) = 2t + 4$ where $-3 \leq t \leq 2$

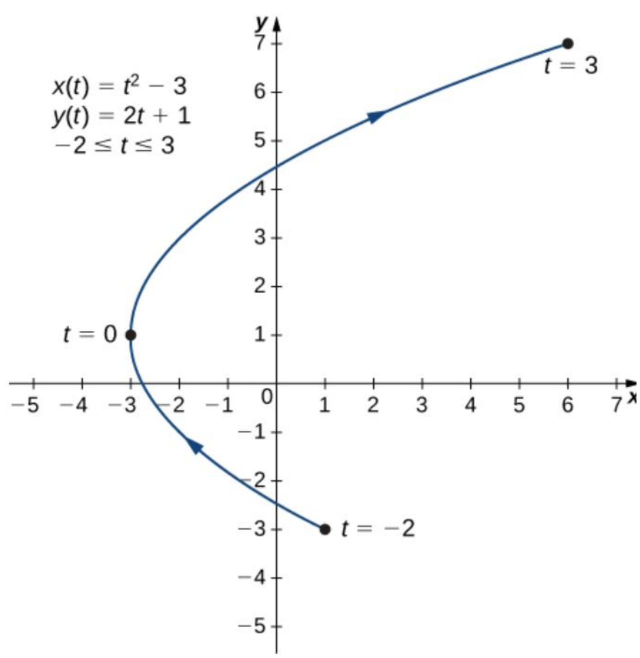

Figure 3: Graph of Parametric Curve with $x(t) = t^2 - 3$ and $y(t) = 2t + 1$ where $-2 \leq t \leq 3$

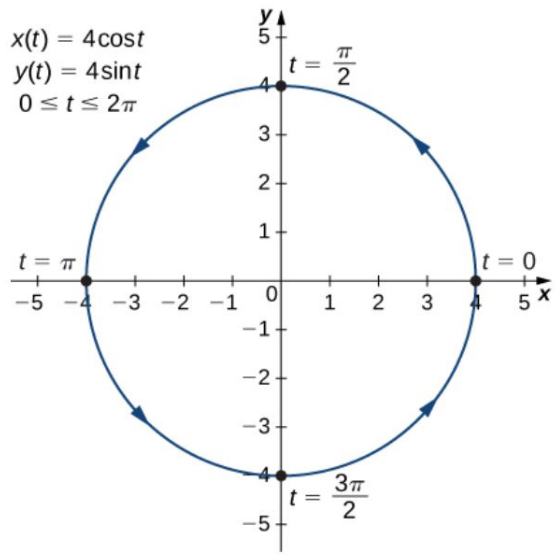

Figure 4: Graph of Parametric Curve with $x(t) = 4\cos(t)$ and $y(t) = 4\sin(t)$ where $0 \le t \le 2\pi$

Eliminating the Parameter:

Suppose we have a parametric curve described as $x(t) = t - 1$ and $y(t) = 2t + 4$ where $-3 \le t \le 2$. Can we write this in the form $y = f(x)$? This is called eliminating the parameter.

If possible, solve $x = x(t)$ for t. $x = t - 1$ gives us $t = x + 1$. Then substitute back into $y(t) = 2t + 4$ to get

$$y = 2t + 4 = 2(x + 1) + 4 = 2x + 6.$$

Because $-3 \le t \le 2$ we have $-3 \le x + 1 \le 2$ or $-4 \le x \le 1$. So the parametric curve is the same graph as the straight line given by $y = 2x + 6$ on the interval $[-4, 1]$. This agrees with our earlier sketch of the parametric curve.

Let's have you work an example independently.

Example 1. Eliminate the parameter for the parametric curve given by $x(t) = 4\cos(t)$ and $y(t) = 4\sin(t)$ where $0 \le t \le 2\pi$.

Workspace:

Solution:

We could try to solve $x(t) = 4\cos(t)$ for t arriving at $t = \cos^{-1}(x/4)$ so that

$$y = \sin\left(\cos^{-1}\left(\frac{x}{4}\right)\right)$$

where $0 \leq t \leq 2\pi$ is equivalent to $-4 \leq x \leq 4$. Some basic trigonometry allows us to write

$$y = \pm\sqrt{16 - x^2}$$

with $-4 \leq x \leq 4$ which is the equation of a circle of radius 4. Maybe there is an easier approach?

Fortunately, there is! We can also proceed as follows: $x(t) = 4\cos(t)$ and $y(t) = 4\sin(t)$ implies

$$x^2 + y^2 = 16\cos^2(t) + 16\sin^2(t) = 16 \quad \Longrightarrow \quad y = \pm\sqrt{16 - x^2}.$$

Example 2. Eliminate the parameter for the parametric curve given by $x(t) = t^2 - 3$ and $y(t) = 2t + 1$ where $-2 \leq t \leq 3$.

Workspace:

Solution:

Let's solve $y(t) = 2t + 1$ for t getting $t = (y - 1)/2$. Now a substitution gives us

$$x = t^2 - 3 = \left(\frac{y-1}{2}\right)^2 - 3 = \frac{1}{4}\big(y-1\big)^2 - 3.$$

Moreover, $-2 \le t \le 3$ is gives us $-2 \le (y-1)/2 \le 3$ which is equivalent to $-3 \le y \le 7$. So the parametric curve is the as the graph of

$$x = \frac{1}{4}\big(y-1\big)^2 - 3$$

which is a scaled parabola whose vertex is shifted up one unit and left 3 units on the interval $-3 \le y \le 7$. This agrees with the previous graph.

Let's work another example.

Example 3. Eliminate the parameter for the parametric curve given by $x(t) = \sqrt{2t+4}$ and $y(t) = 2t + 1$ where $-2 \le t \le 6$.

Workspace:

Solution:

Let's solve $x(t) = \sqrt{2t+4}$ for t getting $t = (x^2-4)/2$. Now a substitution gives us

$$y = 2t + 1 = 2\left(\frac{x^2-4}{2}\right) + 1 = x^2 - 3.$$

Moreover, $-2 \le t \le 6$ is gives us $-2 \le (x^2-4)/2 \le 6$ which is equivalent to $0 \le x \le 4$ since $\sqrt{2t+4}$ in increasing. So the parametric curve is the as the graph of

$$y = x^2 - 3$$

on the interval $0 \le x \le 4$.

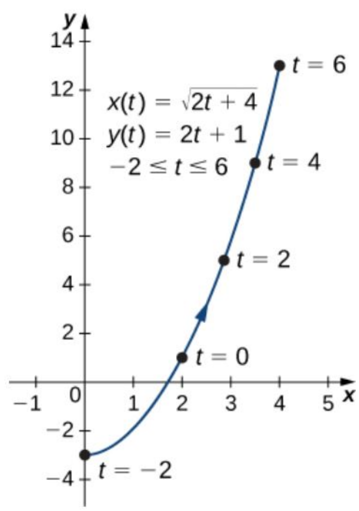

Figure 5: Parametric Curve with $x(t) = \sqrt{2t+4}$ and $y(t) = 2t+1$ where $-2 \le t \le 6$

Parameterize a Curve:

Suppose we have $y = 2x^2 - 3$ for all x-values. The simplest way to parameterize the curve is to let $x(t) = t$ and $y(t) = 2t^2 - 3$. But this is not the only choice. We could choose any $x(t)$ that takes on all x-values. For example, we could let $x(t) = 3t - 2$ for all t-values. Then we have

$$y(t) = 2x^2 - 3 = 2\big(3t-2\big)^2 - 3 = 18t^2 - 24t + 6.$$

We have the parametric curve $x(t) = 3t - 2$ and $y(t) = 18t^2 - 24t + 6$ for all t-values.

An Interesting Example - The Cycloid:

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a ***cycloid***. A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a\big(t - \sin t\big) \qquad y(t) = a\big(1 - \cos t\big).$$

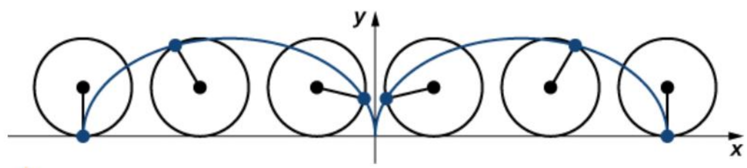

Figure 6: Graph of Cycloid Curve with $x(t) = a\big(t - \sin t\big)$ and $y(t) = a\big(1 - \cos t\big)$

There are many interesting variations and applications of cycloid curves. Anyone who is interested can easily find many resources about cycloid curves.

Please let me know if you have any questions, comments, or corrections!