Math2411 - Calculus II
Guided Lecture Notes
Partial Fractions

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Partial Fractions Introduction:

Our objective is to integrate rational functions where u-sub or algebraic manipulations are not helpful or clear. Let's look at some examples.

$$\int \frac{x^2+4x-1}{x^3+6x^2-3x+7}\,dx \qquad \int \frac{2}{x^2+4x+8}\,dx$$

The first integral can be solved using u-substitution letting $u = x^3+6x^2-3x+7$ so that $du = \big(3x^2+12x-3\big)\,dx$. We then have

$$\begin{aligned}
\int \frac{x^2+4x-1}{x^3+6x^2-3x+7}\,dx &= \frac{1}{3}\int \frac{3\big(x^2+4x-1\big)}{x^3+6x^2-3x+7}\,dx \\
&= \frac{1}{3}\int \frac{3x^2+12x-3}{x^3+6x^2-3x+7}\,dx \\
&= \frac{1}{3}\int \frac{du}{u} \\
&= \frac{1}{3}\ln|u|+C \\
&= \frac{1}{3}\ln\left|x^3+6x^2-3x+7\right|+C
\end{aligned}$$

The second integral can be solved by algebraic manipulations, a u-sub, and a known antiderivative.

$$\begin{aligned}
\int \frac{2}{x^2+4x+8}\,dx &= \int \frac{2}{x^2+4x+4+4}\,dx \\
&= \int \frac{2}{(x+2)^2+4}\,dx \\
&= \int \frac{2}{u^2+4}\,du \qquad \text{(Letting } u = x+2\text{)} \\
&= \tan^{-1}(u/2)+C \\
&= \tan^{-1}\left(\frac{x+2}{2}\right)+C
\end{aligned}$$

But some integrals of rational expressions resist any u-sub or algebraic manipulations. For example,

$$\int \frac{2x+5}{x^2+4x-5}\,dx.$$

We can not turn this into a log-form by u-substitution since with $u = x^2 + 4x - 5$ we have $du = (2x + 4)\ dx$ and the numerator is not a scalar multiple of the denominator.

So we split up the integral.

$$\int \frac{2x+5}{x^2+4x-5}\,dx = \int \frac{2x+4}{x^2+4x-5}\,dx + \int \frac{1}{x^2+4x-5}\,dx$$

The first integral is a simple u-substitution. But what about the second integral. We could try to complete the square as follows.

$$\begin{aligned}
\int \frac{1}{x^2+4x-5}\,dx &= \int \frac{1}{x^2+4x-4-1}\,dx \\
&= \int \frac{1}{(x-2)^2-1}\,dx \\
&= \int \frac{1}{u^2-1}\,du \qquad \text{(Letting } u = x-2\text{)}
\end{aligned}$$

But the new integral is not in a familiar form. So it looks like we need a new strategy. We will use the following algebraic fact. It is a good exercise to check this equality.

$$\frac{2x+5}{x^2+4x-5} = \frac{7}{6(x-1)} + \frac{5}{6(x+5)}$$

Now we have the following.

$$\begin{aligned}
\int \frac{2x+5}{x^2+4x-5}\,dx &= \int \frac{7}{6(x-1)} + \frac{5}{6(x+5)}\,dx \\
&= \frac{7}{6}\int \frac{1}{x-1}\,dx + \frac{5}{6}\int \frac{1}{x+5}\,dx \\
&= \frac{7}{6}\ln|x-1| + \frac{5}{6}\ln|x+5| + C
\end{aligned}$$

So if we can break apart a complicated rational expression into a sum of simpler rational expressions, often the new expressions will be easier to handle as far as integration.

Partial Fractions Method

Let's see how the previous fraction was decomposed into simpler fractions.

Step #1: Factor the denominator.

$$x^2 + 4x - 5 = (x-1)(x+5)$$

Do you notice anything about these factors?

***Step #2*:** Try to break up the fraction so that the parts correspond to the factors.

$$\frac{2x+5}{x^2+4x-5} = \frac{2x+5}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}$$

***Step #3*:** Solve for A and B. That is, find values of A and B that satisfy the previous equality. This is the crux of the process. We start by combining terms on the RHS and then seeting numerators equal.

$$\frac{2x+5}{(x-1)(x+5)} = \frac{A(x+5)+B(x-1)}{(x-1)(x+5)} \xRightarrow[\text{Equal}]{\text{Set Numerators}} 2x+5 = A(x+5)+B(x-1)$$

Methods to solve for A and B:

- ***Method of Strategic Substitution:*** Set $x = 1$ and substitute to get

  $$7 = 6A \quad \Longrightarrow \quad A = 7/6.$$

  Next set $x = -5$ and substitute to get

  $$-5 = -6B \quad \Longrightarrow \quad B = 5/6.$$

  So we have

  $$\frac{2x+5}{x^2+4x-5} = \frac{A}{x-1} + \frac{B}{x+5} = \frac{7}{6(x-1)} + \frac{5}{6(x+5)}.$$

- ***Method of Equating Coefficients:*** Expand both sides and equate coefficients.

  $$2x+5 = (A+B)x + (5B-A)$$

  We see that

  $$\begin{aligned} A+B &= 2 \\ 5A-B &= 1 \end{aligned}$$

  Solving this system of equations gives us $A = 7/6$ and $B = 5/6$. So we have

  $$\frac{2x+5}{x^2+4x-5} = \frac{A}{x-1} + \frac{B}{x+5} = \frac{7}{6(x-1)} + \frac{5}{6(x+5)}.$$

The partial fractions decomposition is an algebraic technique to simplify an integral. There is no calculus involved in the partial fractions.

Let's work an example on your own.

Partial Fractions Examples

Example 1. Evaluate $\displaystyle\int \frac{x+3}{x^3 - x^2 - 2x}\,dx$

Workspace:

Solution:

We start by factoring the denominator and breaking apart to get

$$\frac{x+3}{x^3-x^2-2x} = \frac{x+3}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1} = \frac{A(x+1)(x-2)+Bx(x+1)+Cx(x-2)}{x(x-2)(x+1)}$$

After putting the pieces back together we get

$$\xRightarrow[\text{Equal}]{\text{Set Numerators}} \quad x+3 = A(x+1)(x-2)+Bx(x+1)+Cx(x-2) \qquad \text{(Eqn. 3.8)}$$

We now solve equation 3.8 for A, B, and C using one of our described methods.

Rule: Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of A, B, and C that satisfy **Equation 3.8** for *all* values of x. That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find A, B, and C easily. For example, if we substitute $x = 0$, the equation reduces to $2 = A(-2)(1)$. Solving for A yields $A = -1$. Next, by substituting $x = 2$, the equation reduces to $8 = B(2)(3)$, or equivalently $B = 4/3$. Last, we substitute $x = -1$ into the equation and obtain $-1 = C(-1)(-3)$. Solving, we have $C = -\frac{1}{3}$.

Rule: Method of Equating Coefficients

Rewrite **Equation 3.8** in the form

$$3x+2 = (A+B+C)x^2 + (-A+B-2C)x + (-2A).$$

Equating coefficients produces the system of equations

$$\begin{aligned} A+B+C &= 0 \\ -A+B-2C &= 3 \\ -2A &= 2. \end{aligned}$$

To solve this system, we first observe that $-2A = 2 \Rightarrow A = -1$. Substituting this value into the first two equations gives us the system

$$\begin{aligned} B+C &= 1 \\ B-2C &= 2. \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces

$$-3C = 1,$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B+C=1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1$, $B = \frac{4}{3}$, and $C = -\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

....Solution Continued

$$\vdots$$

Now we're ready for the integral.

$$\begin{aligned}\int \frac{x+3}{x^3-x^2-2x}\,dx &= -\int \frac{1}{x}\,dx + \frac{4}{3}\int \frac{1}{x-2}\,dx - \frac{1}{3}\int \frac{1}{x+1}\,dx \\ &= -\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C\end{aligned}$$

Let's work another example together.

Example 2. Evaluate $\displaystyle\int \frac{x-2}{(2x-1)^2(x-1)}\,dx$

Workspace:

Solution:

The denominator is already factored. Then we notice that the factor $(2x-1)$ is a repeated factor. In fact it's repeated twice and so this factor gives us two pieces, one for each power, as follows.

$$\frac{x-2}{(2x-1)^2(x-1)} = \underbrace{\frac{A}{2x-1} + \frac{B}{(2x-1)^2}}_{\text{Repeated factors require multiple components}} + \frac{C}{x-1} = \frac{A(2x-1)(x-1)+B(x-1)+C(2x-1)^2}{(2x-1)^2(x-1)}$$

After putting the pieces back together we can solve an equation.

$$\xRightarrow[\text{Equal}]{\text{Set Numerators}} \quad x-2 = A(2x-1)(x-1)+B(x-1)+C(2x-1)^2$$

We now solve the equation for A, B, and C using one of our described methods. Equating coefficients we get $A=2$, $B=3$, $C=-1$. Then we have

$$\begin{aligned}\int \frac{x-2}{(2x-1)^2(x-1)}\,dx &= 2\int \frac{1}{2x-1}\,dx + 3\int \frac{1}{(2x-1)^2}\,dx - \int \frac{1}{x-1}\,dx \\ &= \ln|2x-1| - \frac{3}{2(2x-1)} - \ln|x-1| + C\end{aligned}$$

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that $\text{degree}(P(x)) < \text{degree}(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $\deg(P(x)) < \deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.
   a. If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying
   $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$
   b. If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain
   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$
   c. For each irreducible quadratic factor ax^2+bx+c that $Q(x)$ contains, the decomposition must include
   $$\frac{Ax+B}{ax^2+bx+c}.$$
   d. For each repeated irreducible quadratic factor $\left(ax^2+bx+c\right)^n$, the decomposition must include
   $$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

e. After the appropriate decomposition is determined, solve for the constants.

f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Example 3. Evaluate $\displaystyle\int \frac{2x-3}{x^3+x}\,dx$

Workspace:

Solution:

The denominator needs to be factored.

$$x^3 + x = x(x^2 + 1)$$

Then we notice that the factor $(x^2 + 1)$ is a quadratic factor. Looking at the previous table, item c., we break apart as follows.

$$\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)} = \underbrace{\frac{Ax+B}{x^2+1}}_{\substack{\text{Quadratic factors require} \\ \text{Linear Numerator}}} + \frac{C}{x} = \frac{(Ax+B)x + C(x^2+1)}{x(x^2+1)}$$

After putting the pieces back together we can solve an equation.

$$\xRightarrow[\substack{\text{Set Numerators} \\ \text{Equal}}]{} \quad 2x - 3 = (Ax + B)x + C(x^2 + 1)$$

We now solve the equation for A, B, and C using one of our described methods. Equating coefficients we get $A = 3$, $B = 2$, $C = -3$. Then we have

$$\begin{aligned} \int \frac{2x-3}{x^3+x}\,dx &= \int \frac{3x+2}{x^2+1}\,dx - 3\int \frac{1}{x}\,dx \\ &= \frac{3}{2}\underbrace{\int \frac{2x}{x^2+1}\,dx}_{\substack{\text{This is a} \\ \text{Log-Form}}} + 2\underbrace{\int \frac{1}{x^2+1}\,dx}_{\substack{\text{Arctangent} \\ \text{Form}}} - 3\underbrace{\int \frac{1}{x}\,dx}_{\substack{\text{Log} \\ \text{Form}}} \\ &= \frac{3}{2}\ln(x^2+1) + 2\tan^{-1}(x) - 3\ln|x| + C \end{aligned}$$

Let's have you work another example independently.

Example 4. Evaluate $\displaystyle\int \frac{1}{x^3 - 8}\,dx$

***Hint*:** Since $x = 2$ is a zero of $x^3 - 8$, in order to factor the denominator divide $x - 2$ into $x^3 - 8$.

Workspace:

Workspace Continued:

Solution:

We can start by factoring $x^3 - 8 = (x-2)(x^2+2x+4)$. We see that the quadratic factor $x^2 + 2x$ · irreducible since $2^2 - 4(1)(4) = -12 < 0$. Using the decomposition described in the problem-solving st we get

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

After obtaining a common denominator and equating the numerators, this becomes

$$1 = A\left(x^2+2x+4\right) + (Bx+C)(x-2).$$

Applying either method, we get $A = \frac{1}{12}$, $B = -\frac{1}{12}$, and $C = -\frac{1}{3}$.

Rewriting $\int \frac{dx}{x^3-8}$, we have

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{1}{x-2}dx - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}dx.$$

We can see that

$\int \frac{1}{x-2}dx = \ln|x-2| + C$, but $\int \frac{x+4}{x^2+2x+4}dx$ requires a bit more effort. Let's begin by completi square on x^2+2x+4 to obtain

$$x^2+2x+4 = (x+1)^2 + 3.$$

By letting $u = x+1$ and consequently $du = dx$, we see that

$\int \frac{x+4}{x^2+2x+4}dx$	$= \int \frac{x+4}{(x+1)^2+3}dx$	Complete the square on tl denominator.
	$= \int \frac{u+3}{u^2+3}du$	Substitute $u = x+1$, $x =$ and $du = dx$.
	$= \int \frac{u}{u^2+3}du + \int \frac{3}{u^2+3}du$	Split the numerator apart.
	$= \frac{1}{2}\ln\left\|u^2+3\right\| + \frac{3}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + C$	Evaluate each integral.
	$= \frac{1}{2}\ln\left\|x^2+2x+4\right\| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$	Rewrite in terms of x and simplify.

Substituting back into the original integral and simplifying gives

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\ln|x-2| - \frac{1}{24}\ln\left|x^2+2x+4\right| - \frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2+2x+4 > 0$ for all x.

What if Numerator Degree is Too Large?

For partial fractions to work the degree of the numerator MUST BE smaller than the degree of the denominator!

Example 5. Evaluate $\int \frac{x^2+3x+1}{x^2-4}\,dx$

Workspace:

Solution:

Since degree$(x^2+3x+1) \geq$ degree(x^2-4), we must perform long division of polynomials. This results in

$$\frac{x^2+3x+1}{x^2-4} = 1 + \frac{3x+5}{x^2-4}.$$

Next, we perform partial fraction decomposition on $\dfrac{3x+5}{x^2-4} = \dfrac{3x+5}{(x+2)(x-2)}$. We have

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Thus,

$$3x+5 = A(x+2) + B(x-2).$$

Solving for A and B using either method, we obtain $A = 11/4$ and $B = 1/4$.

Rewriting the original integral, we have

$$\int \frac{x^2+3x+1}{x^2-4}dx = \int \left(1 + \frac{11}{4} \cdot \frac{1}{x-2} + \frac{1}{4} \cdot \frac{1}{x+2}\right) dx.$$

Evaluating the integral produces

$$\int \frac{x^2+3x+1}{x^2-4}dx = x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C.$$

Please let me know if you have any questions, comments, or corrections!