# Math2411 - Calculus II
# Section 001 Fall 2024
# Introduction to Sequences and Series

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## Sequences and Series Introduction:

Our objective is to study the basics of ***infinite sequences*** and ***infinite series***. So to get started, what is an infinite sequence?

**Definition**

An **infinite sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$a_1, a_2, \ldots, a_n, \ldots .$$

The subscript $n$ is called the **index variable** of the sequence. Each number $a_n$ is a **term** of the sequence. Sometimes sequences are defined by **explicit formulas**, in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

## Sequence Examples:

**Example 1.** Consider the sequence $\{a_n\}$ where $a_n = 2^n$ for natural numbers $n$.

$$\{2^n\}_{n\in\mathbb{N}} = \{2, 4, 8, 16, 32, \ldots\}$$

Notice that we will use curly set brackets for sequence notation. And as above we can describe a sequence either by some explicit rule, or by listing enough of the sequence elements to understand the pattern (if there is indeed a pattern). Here is a graph of the sequence.

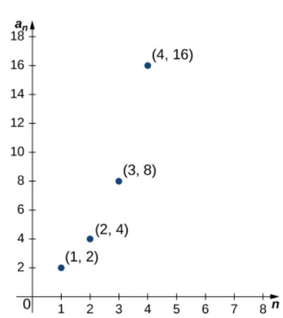


Figure 1: Graph of the sequence where $a_n = 2^n$.

Here is another example.

**Example 2.** The Fibonnaci sequence is one of the most famous sequences in all of mathematics. This is defined recursively as follows.

$$a_n = a_{n-1} + a_{n-2}, \quad \text{where } a_0 = a_1 = 1$$

Notice that the sequence is not defined as an explicit formula involving $n$. That is, we are not given a rule $a_n = f(n)$. Rather, the $n^{th}$ term in the sequence is determined by the previous two terms. Here is the sequence.

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$$

Here is another simpler example.

**Example 3.** Define a sequence as follows so that $a_n = \dfrac{n}{n^2+1}$.

$$\{a_n\}_{n\in\mathbb{N}} = \left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \dots\right\}$$

Here is another example.

**Example 4.** Here is is one of the most important sequences in all of mathematics. Let $k$ be any real number and define

$$a_n = \left(1 + \frac{k}{n}\right)^n$$

So we have

$$\{a_n\}_{n=1}^{\infty} = \left\{1 + k, \left(1 + \frac{k}{2}\right)^2, \left(1 + \frac{k}{3}\right)^3, \left(1 + \frac{k}{4}\right)^4, \dots\right\}$$

## Limiting Values of Sequences:

The main question we will ask about sequences is whether of not the sequence converges to some value or diverges.

**Definition**

Given a sequence $\{a_n\}$, if the terms $a_n$ become arbitrarily close to a finite number $L$ as $n$ becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and $L$ is the **limit of the sequence**. In this case, we write

$$\lim_{n\to\infty} a_n = L.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence**.

There is a nice graphic that reflects this idea.

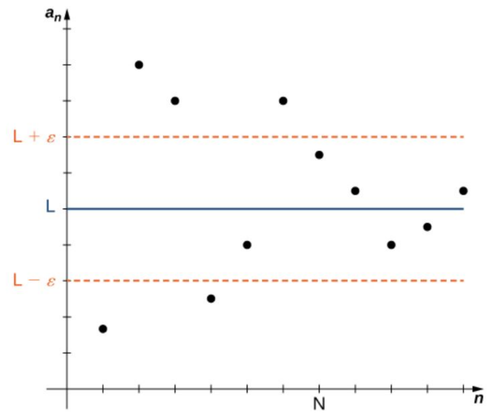


Figure 2: Graphic Representation of a Sequence $\{a_n\}$ Converging to $L$.

The above definition is a somewhat informal definition that we will use in this class. However, we should realize that this definition can be formalized to be perfectly precise as follows.

**Definition**

A sequence $\{a_n\}$ converges to a real number $L$ if for all $\varepsilon > 0$, there exists an integer $N$ such that $|a_n - L| < \varepsilon$ if $n \geq N$. The number $L$ is the limit of the sequence and we write

$$\lim_{n \to \infty} a_n = L \, or \, a_n \to L.$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We will not be working with this formal definition in this class. So how will we determine if a sequence converges or diverges? We will be exclusively evaluating explicit sequences and the following theorem will be our main tool for determining convergence or divergence.

**Theorem 5.1: Limit of a Sequence Defined by a Function**

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number $L$ such that

$$\lim_{x \to \infty} f(x) = L,$$

then $\{a_n\}$ converges and

$$\lim_{n \to \infty} a_n = L.$$

This theorem tells us that we can use all of our previous knowledge of function limits to evaluate sequence limits.

**Example 5.** Determine if the sequence $\{a_n\}$, where $a_n = n/(n^2 + 1)$, converges or diverges.

***Workspace:***

***Solution:*** We see that $a_n = f(n)$ for the function $f(x) = x/(x^2+1)$. Then our knowledge of growth rates tells us that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+1} = 0.$$

Therefore, the sequence $\{a_n\}$ converges to zero.

Alternatively we could apply L'Hopital's rule.

$$\begin{array}{rcl} \displaystyle\lim_{x\to\infty} \frac{x}{x^2+1} & = & \displaystyle\frac{\infty}{\infty} \quad \text{form} \\ & \downarrow & \text{by L'Hopital's Rule} \\ \displaystyle\lim_{x\to\infty} \frac{x}{x^2+1} & = & \displaystyle\lim_{x\to\infty} \frac{1}{2x} = \frac{1}{\infty} \quad \text{form} \end{array}$$

So $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0.$

Let's consider another example.

**Example 6.** Determine if the sequence $\{a_n\}$, where $a_n = 2n^3/(5n^3 - n^2 + 1)$, converges or diverges.

***Workspace:***

***Solution:*** We see that $a_n = f(n)$ for the function $f(x) = 2x^3/(5x^3 - x^2 + 1)$. Then our knowledge of growth rates tells us that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+1} = \frac{2}{5}.$$

Therefore, the sequence $\{a_n\}$ converges to $2/5$.

Alternatively we could apply algebraic techniques.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^3}{5x^3 - x^2 + 1} &= \lim_{x\to\infty} \frac{x^3}{x^3} \cdot \left( \frac{2}{5 - \frac{1}{x} + \frac{1}{x^3}} \right) \\
&= \lim_{x\to\infty} \frac{2}{5 - \frac{1}{x} + \frac{1}{x^3}} \\
&= \frac{2}{5 - 0 + 0} \\
&= \frac{2}{5}
\end{aligned}$$

So $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^3}{5n^3 - n^2 + 1} = \frac{2}{5}$.

Let's try another example.

**Example 7.** Determine if the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = 7n/\ln(n+1)$, converges or diverges.

***Workspace:***

***Solution:*** We see that $a_n = f(n)$ for the function $f(x) = 7x/\ln(x+1)$. Then our knowledge of growth rates tells us that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{7x}{\ln(x+1)} = \infty.$$

So the sequence $\{a_n\}$ diverges to $\infty$.

Alternatively we could apply L'Hopital's rule.

$$\begin{array}{rcll} \displaystyle\lim_{x\to\infty} \frac{7x}{\ln(x+1)} & = & \dfrac{\infty}{\infty} & \text{form} \\ & \downarrow & \text{by L'Hopital's Rule} & \\ \displaystyle\lim_{x\to\infty} 7(x+1) & = & \infty & \end{array}$$

So $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7n}{\ln(n+1)} = \infty$ and the sequence diverges.

Let's try another extremely important example.

**Example 8.** Determine if the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \left(1 + \dfrac{k}{n}\right)^n$, converges or diverges.

***<u>Workspace:</u>***

***Solution:*** We see that $a_n = f(n)$ for the function $f(x) = \left(1 + \frac{k}{x}\right)^x$. Then our knowledge of limits from Calculus I tells us that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(1 + \frac{k}{x}\right)^x = e^k.$$

So the sequence $\{a_n\}$ converges to the exponential $e^k$.

Alternatively we could apply L'Hopital's rule to verify the above limit, but from this point on we will define

$$\lim \left(1 + \frac{k}{n}\right)^n := e^k.$$

And now one more example that will be important for us in this course.

**Example 9.** Determine if the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = cr^n$ for some real numbers $r$ and $c$, converges or diverges. This sequence is called a ***geometric sequence***.

***Workspace:***

***Solution:*** The sequence is listed as

$$\{a_n\}_{n=0}^{\infty} = \left\{c, cr, cr^2, cr^3, cr^4, \dots\right\}.$$

The defining property of this sequence is the common ration between consecutive terms.

$$\dots, cr^{k-1}, \underbrace{cr^k, cr^{k+1}}_{a_{k+1}=ra_k}, \dots$$

The limiting value of this sequence depends on the value of $r$.

- If $-1 < r < 1$, then $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} cr^k = 0$.
- If $r = 1$, then $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} c1^k = c$.
- If $r > 1$, then $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} cr^k = \infty$.
- If $r \le -1$, then $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} cr^k$ does not exist.

## Infinite Series:

Next we will be studying ***infinite series***. So what is an infinite series? It is simply a non-terminating discrete sum. We can think of it as an infinite sum of the terms in an infinite sequence.

### Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

For each positive integer $k$, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the $k$th **partial sum** of the infinite series. The partial sums form a sequence $\{S_k\}$. If the sequence of partial sums converges to a real number $S$, the infinite series converges. If we can describe the **convergence of a series** to $S$, we call $S$ the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we have the **divergence of a series**.

## Infinite Series Examples:

**Example 10.** Determine if the following infinite series converges.

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots$$

***Workspace:***

***Solution:*** Recognizing this series as corresponding to the decimal expansion of $1/3$ we can see that the series will converge to $1/3$. But let's be more formal and use the given definition.

$$s_k = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k}.$$

Multiplying both sides by $1/10$ we have

$$s_k/10 = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots + \frac{3}{10^{k+1}}.$$

It follows that

$$s_k - \frac{s_k}{10} = \frac{9s_k}{10} = \frac{3}{10} - \frac{3}{10^{k+1}} \quad \Longrightarrow \quad s_k = \frac{10}{9}\left(\frac{3}{10} - \frac{3}{10^{k+1}}\right) = \frac{1}{3} - \frac{1}{3 \cdot 10^k}.$$

To keep things as concrete as possible let's see what this means. If we have added $k = 5$ terms then the $5^{th}$ partial sum is

$$s_5 = \frac{1}{3} - \frac{1}{3 \cdot 10^5} = 0.33333.$$

If we have added $k = 10$ terms then the $10^{th}$ partial sum is

$$s_{10} = \frac{1}{3} - \frac{1}{3 \cdot 10^{10}} = 0.3333333333.$$

If we have added $k = 100$ terms then the $100^{th}$ partial sum is

$$s_{100} = \frac{1}{3} - \frac{1}{3 \cdot 10^{100}} = 0.\underbrace{33333\ldots3333}_{\text{One Hundred 3's}}.$$

It should be clear that as we add more and more terms we are getting arbitrarily close to the value $1/3$.

More formally,

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1}{3} - \frac{1}{3 \cdot 10^k} = \frac{1}{3} - 0 = \frac{1}{3}.$$

So the infinite series converges to $s = 1/3$.

This example is a specific case of what's known as a geometric series. Let's consider the general case for a geometric series (one of the most important sequences).

**Example 11.** Determine if the following infinite series converges.

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + cr^4 + \cdots$$

***<u>Workspace</u>:***

***Workspace Cont.:***

***Solution:*** Let's be very careful and use the formal definition.

$$s_k = c + cr + cr^2 + cr^3 + cr^4 + \cdots + cr^k.$$

Multiplying both sides by 1/10 we have

$$rs_k = cr + cr^2 + cr^3 + cr^4 + \cdots + cr^{k+1}.$$

It follows that

$$s_k - \frac{s_k}{10} = (1-r)s_k = c - cr^{k+1} \quad \Longrightarrow \quad s_k = c\left(\frac{1-r^{k+1}}{1-r}\right).$$

Now formally,

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} c\left(\frac{1-r^{k+1}}{1-r}\right) = c\left(\frac{1}{1-r}\right) \quad \text{if} \quad -1 < r < 1.$$

Otherwise the limit does not exist. So the geometric series converges to $s = \dfrac{c}{1-r}$ only when $|r| < 1$. In this case, we write

$$\sum_{n=0}^{\infty} cr^k = \frac{c}{1-r}.$$

Otherwise, the series diverges. This is a fact that should be memorized from this point onwards.

Let's try another important example.

**Example 12.** Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

***Note*:** This is an important infinite series known as the ***Harmonic Series***.

***Workspace:***

***Solution:*** Let's group terms of the sum as follows.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 1/2} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{\geq 1/2} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq 1/2} + \underbrace{\frac{1}{17} + \cdots + \frac{1}{32}}_{\geq 1/2} + \cdots$$

It is easy to see that the sequence of partial sums $\{s_k\}$ is increasing and not bounded above. Therefore we have

$$\lim_{k\to\infty} s_k = \infty$$

and the infinite series diverges to $\infty$.

We will discover that for most infinite series it may be practically impossible to discover a formula for the $k^{th}$ partial sum. So we will need to spend a good deal of time developing tests for convergence and divergence of infinite series that don't require knowledge of any formula for the partial sums. This content will occupy a good deal of time in the next part of the course. See you there.

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**Please let me know if you have any questions, comments, or corrections!**