Math2411 - Calculus II
Guided Lecture Notes
Separable Differential Equations

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

Separable Differential Equations Introduction:

Our objective is to solve ***separable differential equations***. But what is a differential equation? It is an equation that describes a relation between a dependent variable, the independent variable, and one or more derivatives of the dependent variable.

> **Definition**
>
> A **differential equation** is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Here are some examples of differential equations and solutions.

Equation	Solution
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$
$y'' - 3y' + 2y = 24e^{-2x}$	$y = 3e^x - 4e^{2x} + 2e^{-2x}$

Figure 1: Differential Equations and Solutions

We will be dealing exclusively with ***first order differential equations***.

> **Definition**
>
> The **order of a differential equation** is the highest order of any derivative of the unknown function that appears in the equation.

In general, for first order equations we will be able to write $y' = F(x, y)$ to describe a known relationship between the quantities y', y and x.

Here are a few examples.

$$\begin{aligned}
y' &= \left(x^2 - 4\right)(3y + 2) \\
y' &= 6x^2 + 4x \\
y' &= \sec y + \tan y \\
y' &= xy + 3x - 2y - 6.
\end{aligned}$$

Figure 2: Examples of Differential Equations in the Form $y' = F(x, y)$.

Separable Differential Equations:

We will be studying ***separable differential equations***. As the name suggests, these are equations in which the quantities involving the independent variable can be separated from quantities involving the dependent variable. That is, a ***separable differential equation*** is a differential equation that can be written in the form

$$n(y) \cdot y' = m(x).$$

We are hoping for a solution algorithm so that we can avoid guessing and checking. To understand the algorithm we should remember implicit differentiation. Suppose that $y = y(x)$. Then, by the chain rule, the following calculus relationships hold.

$$N(y) = M(x) + C \quad \overset{\frac{d}{dx}}{\underset{\int \quad \frac{d}{dx}}{\overset{\Longrightarrow}{\Longleftarrow}}} \quad \frac{dN}{dy}\frac{dy}{dx} = \frac{dM}{dx} \quad \text{or we can write as} \quad n(y) \cdot y' = m(x)$$

This means the following:

- If we differentiate (with respect to x) both sides of the equation

  $$N(y) = M(x) + C$$

  we end up with the relationship

  $$n(y) \cdot y' = m(x) + C.$$

  This is the familiar implicit differentiation.

- If we integrate (with respect to x) both sides of the equation

  $$n(y) \cdot y' = m(x) + C$$

  we end up with the relationship

  $$N(y) = M(x) + C$$

  In some sense we are simply reversing implicit differentiation.

We are starting with the equation $n(y) \cdot y' = m(x) + C$. The above tells us that the desired relationship that describes our solution is

$$N(y) = M(x) + C \qquad \iff \qquad \int n(y)\,dy = \int m(x)\,dx + C$$

Let's work an example together.

Example 1. Solve the differential equation $y' = 5y^2x^3$.

Workspace:

Question: How do we determine the constant C? That is, which of the above graphs will be our desired solution?

Answer: An initial condition or starting state.

Suppose we knew that $y(0) = 2$. We can use this information to solve for C. Why don't you give a try on your own.

Workspace:

Let's try another example.

Example 2. Solve the initial value problem $y' = ky$ where $y(0) = y_0$ and $k \neq 0$. This is a basic model for population growth or decay problems.

Workspace:

Let's try another example.

Example 3. Solve the initial value problem $y' = (2x+3)(y^2-4)$ where $y(0) = -1$.

Workspace:

Let's try another example.

Example 4. Solve the initial value problem $y' = ky(M - y)$ where $y(0) = y_0$, $y \geq 0$, and $k > 0$. This is a logistic model for population growth.

Workspace:

Workspace Cont: