

Math2411 - Calculus II

Guided Lecture Notes

Trigonometric Integrals

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Trigonometric Integrals Introduction:

Our objective is to integrate function involving of products and powers of $\sin(x)$ and $\cos(x)$, or products and powers of $\sec(x)$ and $\tan(x)$

We will need a few basic trig identities.

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1 & \sin(2x) &= 2 \sin(x) \cos(x) \\ \cos^2(x) &= \frac{1 + \cos(2x)}{2} & \tan^2(x) &= \sec^2(x) - 1 \\ \sin^2(x) &= \frac{1 - \cos(2x)}{2}\end{aligned}$$

Integrals Involving $\sin^k(x)$ and $\cos^j(x)$:

Let's consider a simple example together.

Integrating $\int \cos^j x \sin x \, dx$

Evaluate $\int \cos^3 x \sin x \, dx$.

Solution

Use u -substitution and let $u = \cos x$. In this case, $du = -\sin x \, dx$. Thus,

$$\begin{aligned}\int \cos^3 x \sin x \, dx &= -\int u^3 \, du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4 x + C.\end{aligned}$$

Notice this is just a basic u -substitution problem. Let's consider another more complicated example.

Example 1. Integrating $\int \cos^j(x) \sin^k(x) dx$ when j or k is odd. As a concrete example, evaluate the integral

$$\int \cos^2(x) \sin^3(x) dx$$

Evaluate $\int \cos^2 x \sin^3 x dx$.

Solution

To convert this integral to integrals of the form $\int \cos^j x \sin x dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$. Thus,

$$\begin{aligned} \int \cos^2 x \sin^3 x dx &= \int \cos^2 x (1 - \cos^2 x) \sin x dx \quad \text{Let } u = \cos x; \text{ then } du = -\sin x dx. \\ &= -\int u^2 (1 - u^2) du \\ &= \int (u^4 - u^2) du \\ &= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\ &= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C. \end{aligned}$$

Notice after the first step the $\sin(x) dx$ looks like du in a u -substitution, so this leads us to write all other trig functions in terms of $\cos(x)$.

$$\int \cos^2(x) \sin^2(x) \quad \begin{array}{c} \text{Wants to be our } du \\ \text{term in a } u\text{-sub} \\ \downarrow \\ [\sin(x) dx] \end{array} \quad \begin{array}{c} \xRightarrow{\text{Natural choice}} \\ \text{for our } u\text{-sub} \end{array} \quad u = \cos(x) \quad \begin{array}{c} \xRightarrow{\text{We have a}} \\ \text{perfect match} \end{array} \quad -du = \sin(x) dx$$

Observation: If the exponent over $\sin(x)$ is odd we can “attach” one of the $\sin(x)$ terms to the differential dx and prepare for a u -substitution. Let’s set up another example.

Example 2. Evaluate the following integral.

$$\int \cos^3(x) \sin^5(x) dx$$

Workspace:

Let's have you try another example on your own.

Example 3. Evaluate the integral $\int \cos^3(x) \sin^6(x) dx$

Workspace:

Let's have you try another example on your own.

Example 4. Evaluate the integral $\int \cos^2(x) \sin^2(x) dx$.

Question: What do you notice about the exponents?

Workspace:

Let's try another example together.

Example 5. Evaluate the integral $\int \tan^6(x) \sec^4(x) dx$.

Workspace:

Let's consider another well-known example.

Example 6. Evaluate the integral $\int \sec^3(x) dx$.

Below are some general strategies.

Problem-Solving Strategy: Integrating Products and Powers of $\sin x$ and $\cos x$

To integrate $\int \cos^j x \sin^k x dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x dx$.
2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x dx$. (Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)
3. If both j and k are even, use $\sin^2 x = (1/2) - (1/2)\cos(2x)$ and $\cos^2 x = (1/2) + (1/2)\cos(2x)$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x dx$

To integrate $\int \tan^k x \sec^j x dx$, use the following strategies:

1. If j is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x dx$.
2. If k is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x dx$. (Note: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)
3. If k is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x (\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.
4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.