

Math2411 - Calculus II

Guided Lecture Notes

The Alternating Series Test

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The Alternating Series Test Introduction:

Our objective is to study *alternating series* whose terms alternating between positive and negative values.

Definition

Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \dots \quad (5.13)$$

or

$$\sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \dots \quad (5.14)$$

Where $b_n \geq 0$ for all positive integers n .

A famous example of an alternating series is the alternating harmonic series which converges to the value $\ln(2)$.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln(2).$$

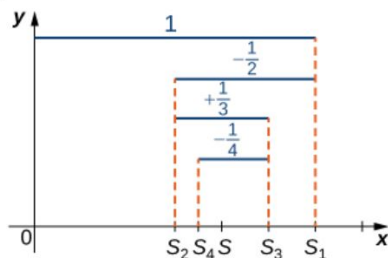


Figure 1: Partial Sums of the Alternating Harmonic Series

The pattern of partial sums will hold in general for alternating series as shown in the following diagram. Notice how the sequence of partial sums is oscillating and seems to be converging to some value S .

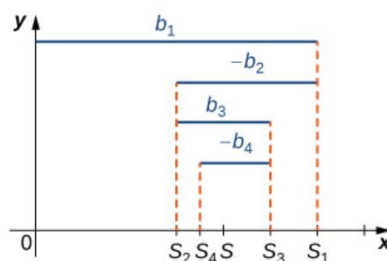


Figure 2: Partial Sums of an Alternating Series

It turns out that determining the convergence or divergence of an alternating series is based on a very simple test.

Theorem 5.13: Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ or } \sum_{n=1}^{\infty} (-1)^n b_n$$

converges if

- i. $0 \leq b_{n+1} \leq b_n$ for all $n \geq 1$ and
- ii. $\lim_{n \rightarrow \infty} b_n = 0$.

This is known as the **alternating series test**.

Let's work a couple of examples.

Alternating Series Test Examples:

Example 1. Determine whether $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converges or diverges.

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Here is another example.

Example 2. Determine whether $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n-1}$ converges or diverges.

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Here is another example.

Example 3. Determine whether $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2(1/n)$ converges or diverges.

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Here is a challenging example.

Example 4. Determine whether $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{n+1} \right)^{n^2}$ converges or diverges.

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Alternating Series Remainder:

For most series we can not determine an actual value of the series and are forced to make approximations using partial sums. A nice property of alternating series is that it is easy to analyze the error in an approximation.

Theorem 5.14: Remainders in Alternating Series

Consider an alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ or } \sum_{n=1}^{\infty} (-1)^n b_n$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the N th partial sum. For any integer $N \geq 1$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leq b_{N+1}.$$

Let's work through an example.

Example 5. Estimate $\sin(1)$ to within five decimal places.

Note: We use the fact that

$$\sin(1) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots.$$

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Absolute and Conditional Convergence:

When a particular series, such as an alternating series, has both positive and negative terms we consider two kinds of convergence: **absolute convergence** and **conditional convergence**.

Definition

A series $\sum_{n=1}^{\infty} a_n$ exhibits **absolute convergence** if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ exhibits **conditional convergence** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem 5.15: Absolute Convergence Implies Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Examples:

Example 6. Determine whether the following series converge conditionally, converge absolutely, or diverge.

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

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2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$

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3. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n+4}} = \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} + \cdots$

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Example 7. Determine whether $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^{3/2}} = \cos(1) + \frac{\cos(2)}{2^{3/2}} + \frac{\cos(3)}{3^{3/2}} + \cdots$ converges.

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