Math2411 - Calculus II
Guided Lecture Notes
Arclength

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Arclength Introduction

Suppose that we want to determine the arclength along a 2D curve $y = f(x)$ where $a \leq x \leq b$. We start with an estimation using straight lines.
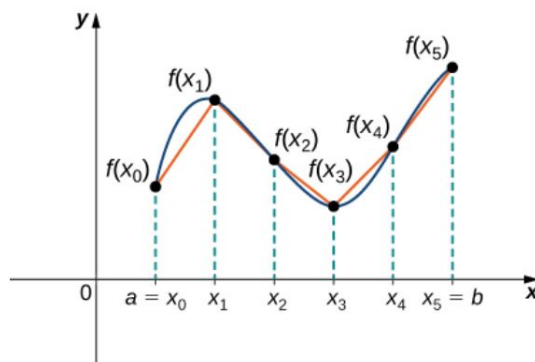

Figure 1: Estimating arclength using straight line segments

The total arclength will be approximated as the sum of the lengths of all line segments. If we write $\triangle s_k$ for the actual length along the kth segment of the curve and l_k the length of the kth line segment we have
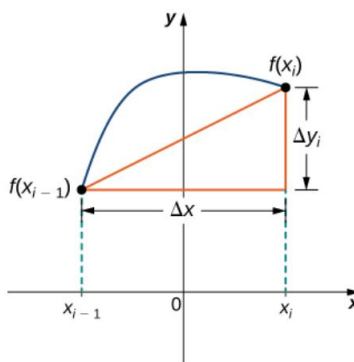
$$Arclength \approx \sum_{k=1}^{n} l_k$$

Figure 2: Length of the ith line segment

Our approximation can be written as follows.

$$\begin{aligned}
\triangle s_k &\approx \sqrt{\left(\triangle x_k\right)^2 + \left(\triangle y_k\right)^2} \\
&= \sqrt{1 + \left(\frac{\triangle y_k}{\triangle x_k}\right)^2} \triangle x_k \\
&= \sqrt{1 + \left(f'(x_k^*)\right)^2} \triangle x_k \qquad \text{by Mean Value Theorem}
\end{aligned}$$

So we have

$$\text{Total Length} \approx \sum_{k=1}^{n} \sqrt{1 + \left(f'(x_k^*)\right)^2} \triangle x_k \quad \xRightarrow[\text{Let } \triangle x \to 0]{\substack{\text{Approximation} \\ \text{Improves}}} \quad \text{Total Length} = \int_{x=a}^{x=b} \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

As $\triangle x \to 0$ we have $l_k \to \triangle s_k$ and $\triangle s_k \to 0$ and an improved approximation. So we can write the differential

$$ds = \sqrt{1 + \left(f'(x_k^*)\right)^2}\, dx$$

and the integral can be written in two different forms as

$$s = \int_{x=a}^{x=b} \sqrt{1 + \left[f'(x)\right]^2}\, dx \quad \text{or} \quad s = \int_{\mathcal{C}} ds.$$

Start with a concrete example.

Example 1. Find the arclength on the curve $y = 2x^{3/2}$ when $0 \le x \le 1$.

__Workspace:__

Now try setting up an example on your own.

Example 2. Find the arclength on the curve $y = \sin(3x)$ when $0 \leq x \leq \pi$.

Workspace: