Math2411 - Calculus II
Guided Lecture Notes
Volumes - The Slicing Method

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Volumes - The Slicing Method Introduction

Suppose that we want to determine the volume of a 3D region/solid with a varying cross-section such as shown below.

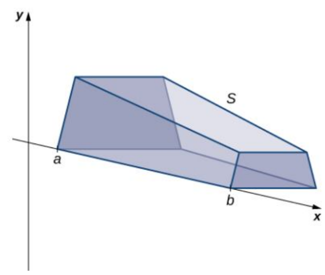

Figure 1: 3D region/solid with a varying cross-section

In this section we will start by dividing the solid into slices along the x-axis and then analyze the volume of the slices.

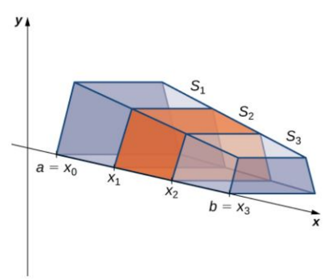

Figure 2: 3D region/solid divided into three slices

The volume of a slice located at x_k^* can be estimated as $Vol\Big(slice_k\Big) = A(x_k^*)\triangle x$ where $A(x_k^*)$ is the area of a cross-section at point x_k^*. Then, if there are n total slices we have

$$\text{Total Volume} \approx \sum_{k=1}^{n} Vol\Big(slice_k\Big) = \sum_{k=1}^{n} A(x_k^*)\triangle x.$$

As the number of slices increases, the width of the slices decreases and we can calculate the total volume as an integral.

$$Volume = \overset{\text{Sum}}{\overset{\downarrow}{\int_{x=a}^{x=b}}} \underbrace{\overset{\text{Cross-Section Area}}{\overset{\downarrow}{A(x)}} \qquad \overset{\text{Width of Slice}}{\overset{\downarrow}{dx}}}_{\text{Volume of Slice}}$$

We will focus on solids formed by revolving regions in the (x, y)-plane about a coordinate axis.

Solids of Revolution

Start with a region below the graph $y = f(x)$ and revolve around the x-axis.

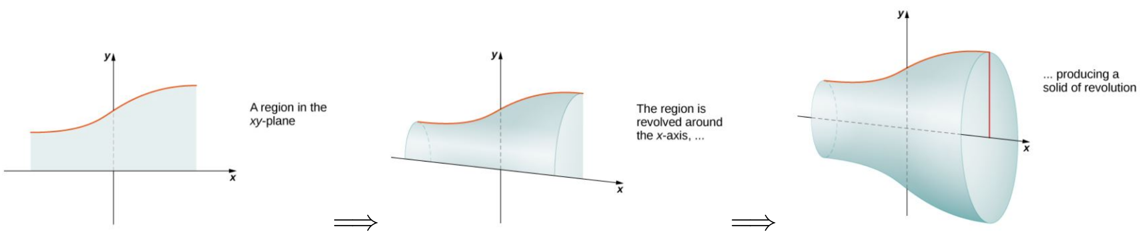

Figure 3: Creating a solid of revolution

The Disc Method:

Start with a concrete example.

Example 1. Find the volume of the solid of revolution formed by rotating the region below $y = (x-1)^2 + 1$ on the interval $[-1, 3]$ about the x-axis.

Start by graphing the region.

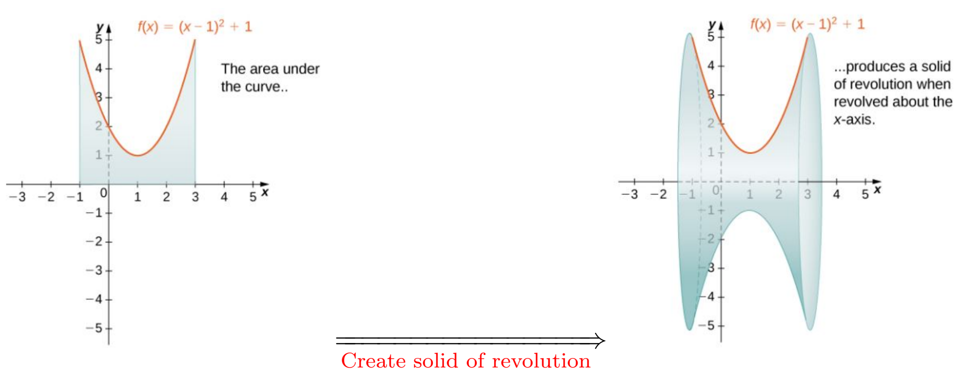

Figure 4: Solid of revolution created from a region below $y = (x-1)^2 + 1$

We now analyze the volume of a slice of the region.

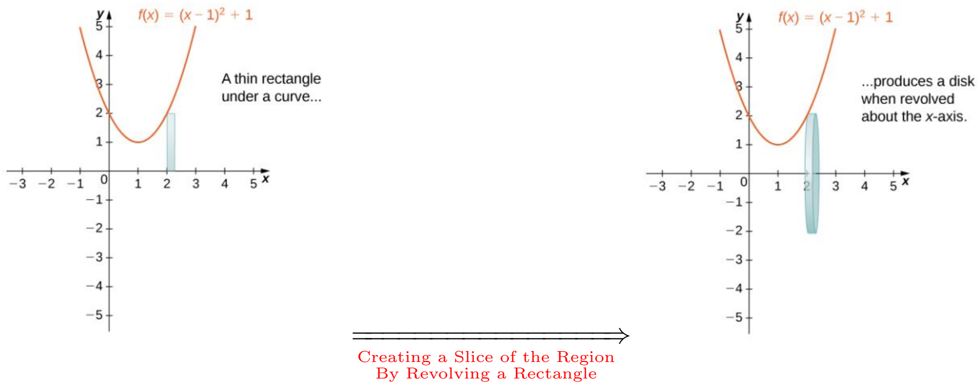

Figure 5: The slices of the 3D solid are circular discs

The cross-sectional area of a circular disc located at x is $A(x) = \pi\big[f(x)\big]^2$. So we evaluate the following integral.

$$\begin{aligned}
Volume &= \pi \int_{x=a}^{x=b} \big[f(x)\big]^2\,dx \\
&= \pi \int_{x=-1}^{x=3} \big[(x-1)^2+1\big]^2\,dx \\
&= \pi \int_{x=-1}^{x=3} (x-1)^4 + 2(x-1)^2 + 1\,dx \\
&= \pi \left[\frac{(x-1)^5}{5} + \frac{2}{3}(x-1)^3 + x\right]_{x=-1}^{x=3} \\
&= \pi \left[\left(\frac{2^5}{5} + \frac{2}{3}\cdot 2^3 + 3\right) - \left(\frac{(-2)^5}{5} + \frac{2}{3}\cdot(-2)^3 - 1\right)\right] \\
&= \frac{412}{15}\pi
\end{aligned}$$

Now try an example on your own.

Example 2. Find the volume of the solid of revolution formed by rotating the region below $y = \sqrt{x}$ on the interval $[1, 4]$ about the x-axis.

Workspace:

Solution:

Start by graphing the region:

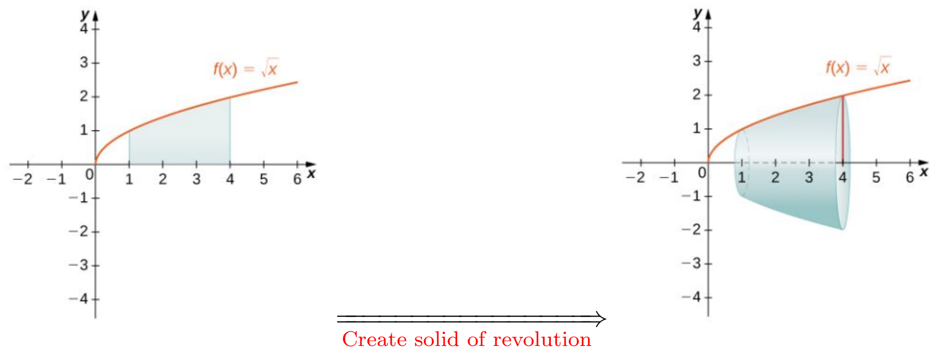

Figure 6: Solid of revolution created from the region below $y = \sqrt{x}$ on the interval $[1, 4]$.

The slices of the region are circular discs and so we simply evaluate the following integral.

$$
\begin{aligned}
Volume &= \pi \int_{x=a}^{x=b} \left[f(x)\right]^2 dx \\
&= \pi \int_{x=1}^{x=4} \left[\sqrt{x}\right]^2 dx \\
&= \pi \int_{x=1}^{x=4} x \, dx \\
&= \pi \left[\frac{x^2}{2}\right]_{x=1}^{x=4} \\
&= \pi \left[\frac{4^2}{2} - \frac{1^2}{2}\right] \\
&= \frac{15}{2}\pi
\end{aligned}
$$

The Washer Method:

Sometimes the slices will not be discs. Consider the following example where we rotate the region between $y = \sqrt{x}$ and $y = 1$ on the interval $[1, 4]$ about the x-axis.

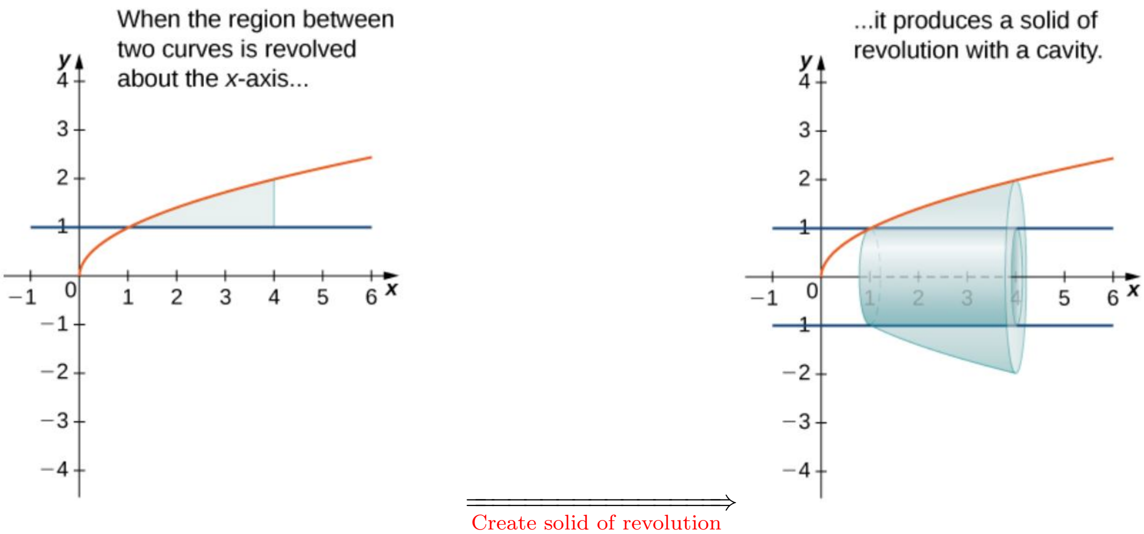

Figure 7: Rotate the region between $y = \sqrt{x}$ and $y = 1$ on the interval $[1, 4]$.

Look at the slices of the solid. They are not discs.

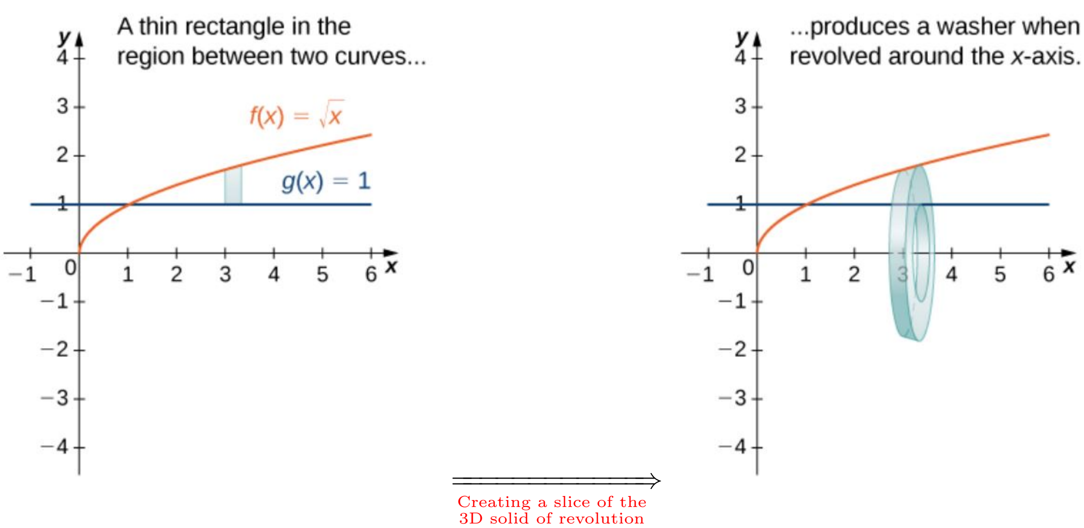

Figure 8: Slices after rotating the region between $y = \sqrt{x}$ and $y = 1$ on the interval $[1, 4]$.

The slices are "washers." Fortunately, the cross-sectional area of a washer is easy.

$$A(x) = \pi\big[R(x)\big]^2 - \pi\big[r(x)\big]^2 = \pi\left(\big[R(x)\big]^2 - \big[r(x)\big]^2\right),$$

where $R(x)$ is the larger radius to the outside edge of the washer, and $r(x)$ is the smaller radius to the inside edge of the washer. And so we simply evaluate the following integral.

$$Volume = \pi\int_{x=a}^{x=b} \big[R(x)\big]^2 - \big[r(x)\big]^2\,dx$$

Now try another example on your own. Be careful with the cross-sectional area.

Example 3. Find the volume of the solid of revolution formed by rotating the region between $y = x$ and $y = 1/x$ on the interval $[1, 4]$ about the x-axis.

Workspace:

Solution:

Start by graphing the region:

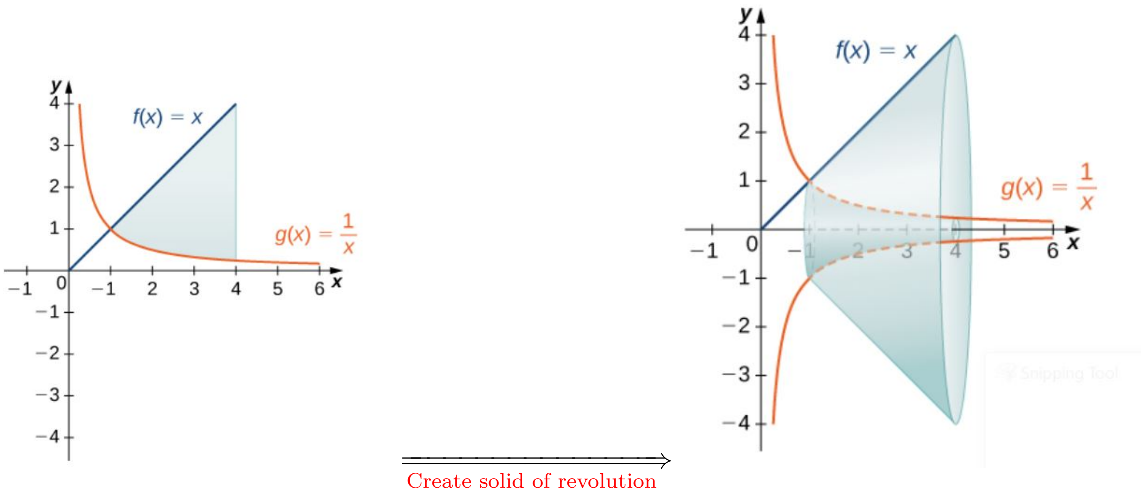

Figure 9: Rotate the region between $y = x$ and $y = 1/x$ on the interval $[1, 4]$.

The slices of the region are circular washers. The cross-section area of a washer located at point x is $\pi\big[R(x)\big]^2 - \pi\big[r(x)\big]^2$, where $R(x)$ is the larger radius to the outside edge of the washer, and $r(x)$ is the smaller radius to the inside edge of the washer. And so we simply evaluate the following integral.

$$\begin{aligned}
Volume &= \pi \int_{x=a}^{x=b} \big[R(x)\big]^2 - \big[r(x)\big]^2 \, dx \\
&= \pi \int_{x=1}^{x=4} x^2 - \left(\frac{1}{x}\right)^2 \, dx \\
&= \pi \int_{x=1}^{x=4} x^2 - \frac{1}{x^2} \, dx \\
&= \pi \left[\frac{x^3}{3} + \frac{1}{x}\right]_{x=1}^{x=4} \\
&= \pi \left[\left(\frac{4^3}{3} + \frac{1}{4}\right) - \left(\frac{1^3}{3} + \frac{1}{1}\right)\right] \\
&= \frac{81}{4}\pi
\end{aligned}$$

Example 4. We can rotate a regions about the y-axis as well. Suppose we have the region under $y = 4 - x^2$ on the interval $[0, 2]$ about the y-axis. Since we rotate about the y-axis we should probably switch variables to have $x = \sqrt{4 - y}$.

Workspace:

Solution:

Start by graphing the region:

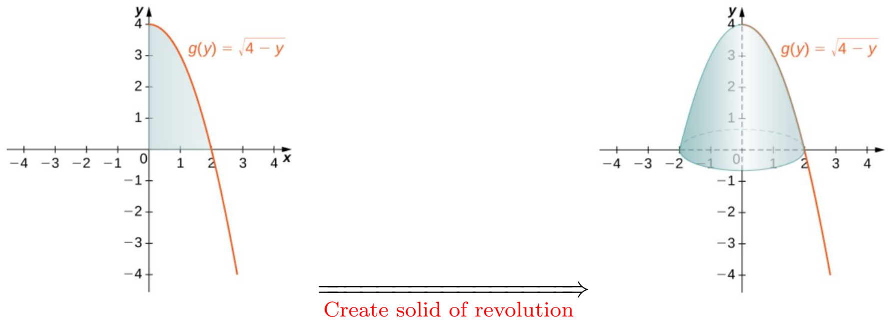

Figure 10: Rotate the region $x = \sqrt{4-y}$ on the interval $[0,4]$.

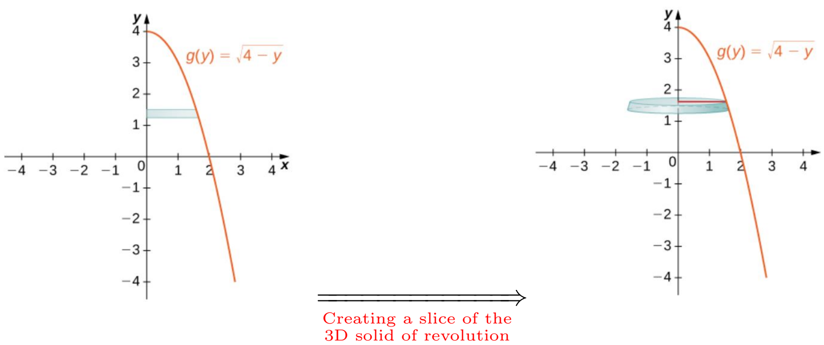

Figure 11: Slices after rotating the region under $x = \sqrt{4-y}$ on the interval $[0,4]$.

The slices are discs and so $A(y) = \pi\big[r(y)\big]^2 = \pi\big[\sqrt{4-y}\big]^2$. And so we simply evaluate the following integral.

$$\begin{aligned}
Volume &= \pi \int_{y=0}^{y=4} \big[\sqrt{4-y}\big]^2\, dy \\
&= \pi \int_{y=0}^{y=4} 4 - y\, dy \\
&= \pi \left[4y - \frac{y^2}{2}\right]_{y=0}^{y=4} \\
&= \pi \left[\left(16 - \frac{4^2}{2}\right) - \Big(0 - 0\Big)\right] \\
&= 8\pi
\end{aligned}$$

Please let me know if you have any questions, comments, or corrections!