Math2411 - Calculus II
Guided Lecture Notes
Trigonometric Integrals

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

Trigonometric Integrals Introduction:

Our objective is to integrate function involving of products and powers of $\sin(x)$ and $\cos(x)$, or products and powers of $\sec(x)$ and $\tan(x)$

We will need a few basic trig identities.

$$\sin^2(x) + \cos^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \tan^2(x) = 1 - \sec^2(x)$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

Integrals Involving $\sin^k(x)$ and $\cos^j(x)$:

Let's consider a simple example together.

Integrating $\int \cos^j x \sin x\, dx$

Evaluate $\int \cos^3 x \sin x\, dx$.

Solution

Use u-substitution and let $u = \cos x$. In this case, $du = -\sin x\, dx$. Thus,

$$\begin{aligned} \int \cos^3 x \sin x\, dx &= -\int u^3\, du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4 x + C. \end{aligned}$$

Notice this is just a basic u-substitution problem. Let's consider another more complicated example.

Example 1. Integrating $\int \cos^j(x)\sin^k(x)\,dx$ when j or k is odd. As a concrete example, evaluate the integral

$$\int \cos^2(x)\sin^3(x)\,dx$$

Evaluate $\int \cos^2 x \sin^3 x\,dx$.

Solution

To convert this integral to integrals of the form $\int \cos^j x \sin x\,dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$. Thus,

$$\begin{aligned}
\int \cos^2 x \sin^3 x\,dx &= \int \cos^2 x(1-\cos^2 x)\sin x\,dx \quad \text{Let } u = \cos x; \text{ then } du = -\sin x\,dx. \\
&= -\int u^2\left(1-u^2\right)du \\
&= \int \left(u^4 - u^2\right)du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.
\end{aligned}$$

Notice after the first step the $\sin(x)\,dx$ looks like du in a u-substitution, so this leads us to write all other trig functions in terms of $\cos(x)$.

$$\int \cos^2(x)\sin^2(x)\ \overset{\substack{\text{Wants to be our } du \\ \text{term in a } u\text{-sub} \\ \downarrow}}{\left[\sin(x)\,dx\right]} \ \xRightarrow[\text{for our } u\text{-sub}]{\text{Natural choice}}\ u = \cos(x) \ \xRightarrow[\text{perfect match}]{\text{We have a}}\ -du = \sin(x)\,dx$$

Observation: If the exponent over $\sin(x)$ is odd we can "attach" one of the $\sin(x)$ terms to the differential dx and prepare for a u-substitution. Let's set up another example.

Example 2. Evaluate the following integral.

$$\int \cos^3(x)\sin^5(x)\,dx$$

Workspace:

Solution:

$$
\begin{aligned}
\int \cos^3(x)\sin^5(x)\,dx &= \int \cos^3(x) \quad \overset{\overset{\text{This is } \left(\sin^2(x)\right)^2}{\downarrow}}{\sin^4(x)} \quad \overset{\overset{\text{Wants to be our } du \text{ term in a } u\text{-sub}}{\downarrow}}{\left[\sin(x)\,dx\right]} \\
&= \int \cos^3(x)\Big(1-\cos^2(x)\Big)^2 \sin(x)\,dx \\
&= -\int u^3\Big(1-u^2\Big)^2\,du \qquad \text{Letting } u=\cos(x) \implies -du = \sin(x)\,dx \\
&= -\int u^3 - 2u^5 + u^7\,du \\
&= -\frac{u^4}{4} + \frac{2u^6}{6} - \frac{u^8}{8} + C \\
&= -\frac{\cos^4(x)}{4} + \frac{2\cos^6(x)}{6} - \frac{\cos^8(x)}{8} + C
\end{aligned}
$$

Let's have you try another example on your own.

Example 3. Evaluate the integral $\int \cos^3(x)\sin^6(x)\,dx$

Workspace:

Solution:

$$\begin{aligned}
\int \cos^3(x)\sin^6(x)\,dx &= \int \cos^2(x)\sin^6(x)\quad \big[\overset{\substack{\text{Wants to be our } du \\ \text{term in a } u\text{-sub} \\ \downarrow}}{\cos(x)\,dx}\big] \\
&= \int \Big(1-\sin^2(x)\Big)\sin^6(x)\cos(x)\,dx \\
&= \int \Big(1-u^2\Big)u^6\,du \qquad \text{Letting } u=\sin(x) \implies du=\cos(x)\,dx \\
&= \int u^6-u^8\,du \\
&= \frac{u^7}{7}-\frac{u^9}{9}+C \\
&= \frac{\sin^7(x)}{7}-\frac{\sin^9(x)}{9}+C
\end{aligned}$$

Let's have you try another example on your own.

Example 4. Evaluate the integral $\int \cos^2(x)\sin^2(x)\,dx$.

Question: What do you notice about the exponents?

Workspace:

Solution:

$$\begin{aligned}
\int \cos^2(x)\sin^2(x)\,dx &= \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)dx \\
&= \frac{1}{4}\int 1-\cos^2(2x)\,dx \qquad \left[\text{Can also write the integrand as } \sin^2(2x)\right] \\
&= \frac{1}{4}\int 1-\left(\frac{1+\cos(4x)}{2}\right)dx \\
&= \frac{1}{8}\int 1-\cos(4x)\,dx \\
&= \frac{1}{8}x-\frac{1}{32}\sin(4x)+C
\end{aligned}$$

Another possible solution method is

$$\begin{aligned}
\int \cos^2(x)\sin^2(x)\,dx &= \frac{1}{4}\int \sin^2(2x)\,dx \\
&= \frac{1}{8}\int 1-\cos(4x)\,dx \\
&= \frac{1}{8}x-\frac{1}{32}\sin(4x)+C
\end{aligned}$$

Let's try another example together.

Example 5. Evaluate the integral $\int \tan^6(x)\sec^4(x)\,dx$.

Workspace:

Solution:

Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the firs $\sec^2 x$ in terms of $\tan x$. Thus,

$$\begin{aligned}
\int \tan^6 x \sec^4 x\, dx &= \int \tan^6 x\left(\tan^2 x + 1\right)\sec^2 x\, dx && \text{Let } u = \tan x \text{ and } du = \sec^2 x. \\
&= \int u^6\left(u^2+1\right)du && \text{Expand.} \\
&= \int (u^8 + u^6)du && \text{Evaluate the integral.} \\
&= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C && \text{Substitute } \tan x = u. \\
&= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C.
\end{aligned}$$

Let's consider another well-known example.

Example 6. Evaluate the integral $\displaystyle\int \sec^3(x)\, dx$.

Solution:

This integral requires integration by parts. To begin, let $u = \sec x$ and $dv = \sec^2 x$. These choices make $du = \sec x \tan x$ and $v = \tan x$. Thus,

$$\begin{aligned}
\int \sec^3 x\, dx &= \sec x \tan x - \int \tan x \sec x \tan x\, dx \\
&= \sec x \tan x - \int \tan^2 x \sec x\, dx && \text{Simplify.} \\
&= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x\, dx && \text{Substitute } \tan^2 x = \sec^2 x - 1. \\
&= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx && \text{Rewrite.} \\
&= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\, dx. && \text{Evaluate} \int \sec x\, dx.
\end{aligned}$$

We now have

$$\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\, dx.$$

Since the integral $\int \sec^3 x\, dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3 x\, dx$ by adding it to both sides. In doing so, we obtain

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Dividing by 2, we arrive at

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Below are some general strategies.

Problem-Solving Strategy: Integrating Products and Powers of sin *x* and cos *x*

To integrate $\int \cos^j x \sin^k x\, dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x\, dx$.
2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x\, dx$. (*Note*: If both j and k are odd, either strategy 1 or strategy 2 may be used.)
3. If both j and k are even, use $\sin^2 x = (1/2) - (1/2)\cos(2x)$ and $\cos^2 x = (1/2) + (1/2)\cos(2x)$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x\, dx$

To integrate $\int \tan^k x \sec^j x\, dx$, use the following strategies:

1. If j is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x$.
2. If k is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x\, dx$. (*Note*: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)
3. If k is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x(\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.
4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.

Please let me know if you have any questions, comments, or corrections!