# Math2411 - Calculus II
# Guided Lecture Notes
# Volumes - The Shell Method

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## Volumes - The Shell Method  Introduction

Suppose that we want to determine the volume of a 3D region/solid with a varying cross-section such as shown below.

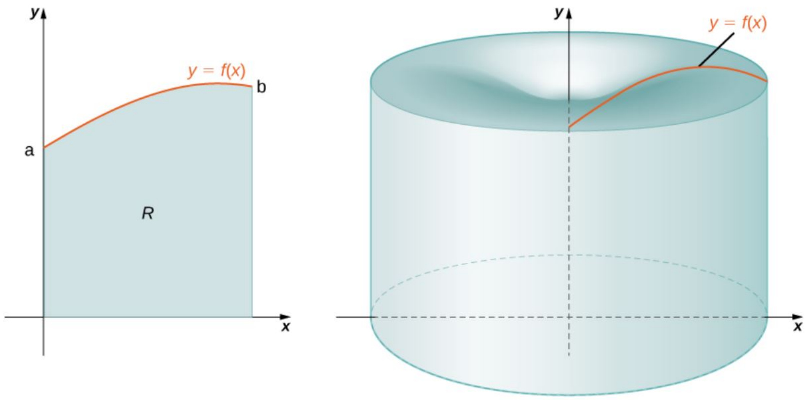


Figure 1: 3D region/solid with a varying cross-section

In this section, instead of slicing and building 3D discs, we will build cylindrical shells to approximate the area of the solid of interest. To do this we will use rectangles that are parallel to the axis of revolution.

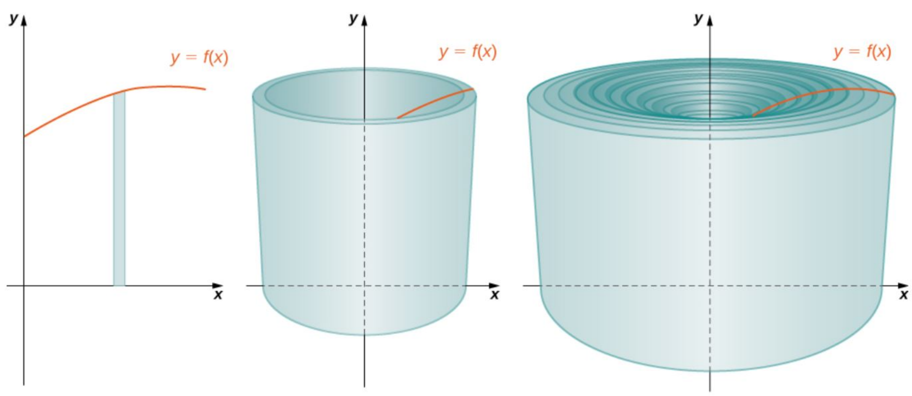


Figure 2: Creating Cylindrical Shells

Now we need to determine the volume of a shell to complete our approximation.

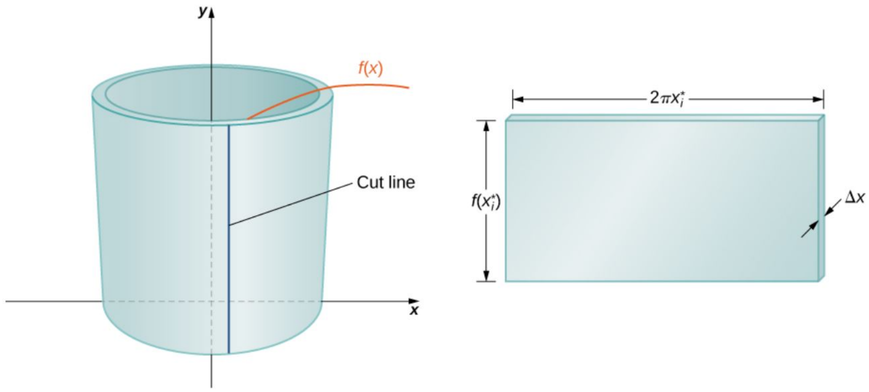


Figure 3: Determining volume of a representative shell.

Multiplying base, height, and thickness of the rectangular slab we get

$$Vol_{Shell} \approx 2\pi x^* \cdot f(x^*) \cdot \triangle x.$$

We now have a estimation formula which leads to an integral as we let the width of the rectangles approach zero.

$$\text{Total Volume} \approx \sum_{i=1}^{n} 2\pi x_i^* \cdot f(x_i^*) \cdot \triangle x_i \quad \xRightarrow[\text{Let } \triangle x \to 0]{} \quad \text{Total Volume} = \int_{x=a}^{x=b} 2\pi x f(x)\, dx$$

Let's compute some concrete examples.

## Shell Method Examples:

**Example 1.** Define $R$ as the region bounded above by the graph of $f(x) = 1/x$ and below by the $x$-axis over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y$-axis

***Workspace:***

*Solution:*

Start by graphing the region:

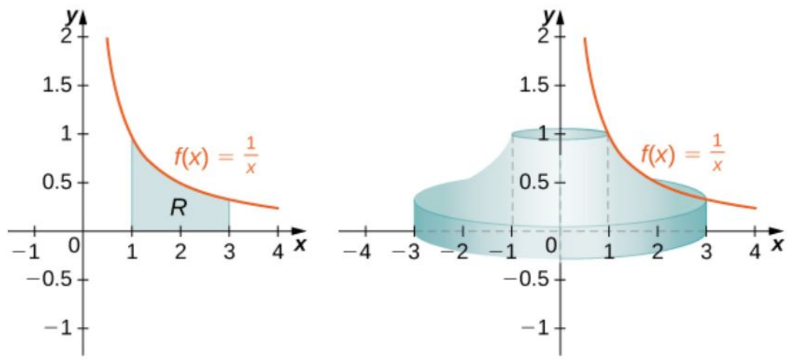


Figure 4: The region $R$ and the solid of revolution.

Then the volume of the solid is given by

$$\begin{aligned} \int_{x=a}^{x=b} 2\pi x f(x)\,dx &= 2\pi \int_{x=1}^{x=3} x\frac{1}{x}\,dx \\ &= 2\pi \int_{x=1}^{x=3} dx \\ &= 2\pi\Big(3-1\Big) \\ &= 4\pi \end{aligned}$$

Now try another example on your own.

**Example 2.** Define $R$ as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y$-axis.

***Workspace:***

*Solution:*

Start by graphing the region:

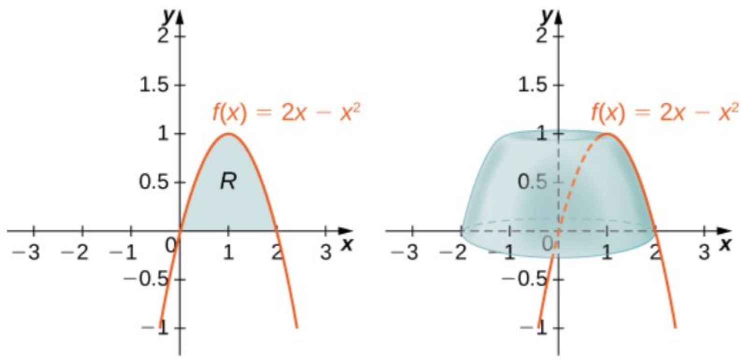


Figure 5: The region $R$ and the solid of revolution.

Then the volume of the solid is given by

$$\begin{aligned}
\int_{x=a}^{x=b} 2\pi x f(x)\, dx &= 2\pi \int_{x=0}^{x=2} x\Big(2x - x^2\Big)\, dx \\
&= 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{x=0}^{x=2} \\
&= 2\pi \left[\left(\frac{2}{3}\cdot 8 - \frac{1}{4}\cdot 16\right) - \Big(0-0\Big)\right] \\
&= \frac{8\pi}{3}
\end{aligned}$$

Let's consider another example.

**Example 3.** Define $Q$ as the region bounded on the right by the graph of $f(x) = x^2/4$, the line $y = 4$, and on the left by the $y$-axis. The solid of revolution will be formed by revolving $Q$ around the $x$-axis. If we use the slicing method we end up with washers. If we want to use shells we can rewrite everything in terms of the variable $y$. So we have the reqion bounded by $g(y) = 2\sqrt{y}$, the line $y = 4$, and on the left by the $y$-axis. Find the volume of the solid of revolution formed by revolving $Q$ around the $x$-axis

***Workspace:***

*Solution:*

Start by graphing the region:

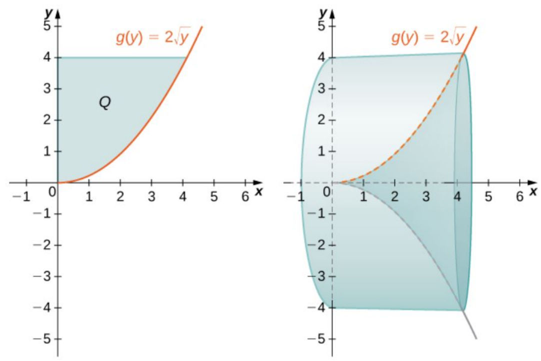


Figure 6: The region $Q$ and the solid of revolution.

Then the volume of the solid is given by

$$\begin{aligned}
\int_{y=c}^{y=d} 2\pi y g(y)\, dy &= 2\pi \int_{y=0}^{y=4} y\left(2\sqrt{y}\right) dy \\
&= 4\pi \int_{y=0}^{y=4} y^{3/2}\, dy \\
&= 4\pi \left[\frac{2}{5} y^{5/2}\right]_{y=0}^{y=4} \\
&= \frac{256\pi}{5}
\end{aligned}$$

We consider one more final example where the region is bounded between two curves.

**Example 4.** Define $R$ as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1/x$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving $R$ around the $y$-axis.

***Workspace:***

*Solution:*

Start by graphing the region:

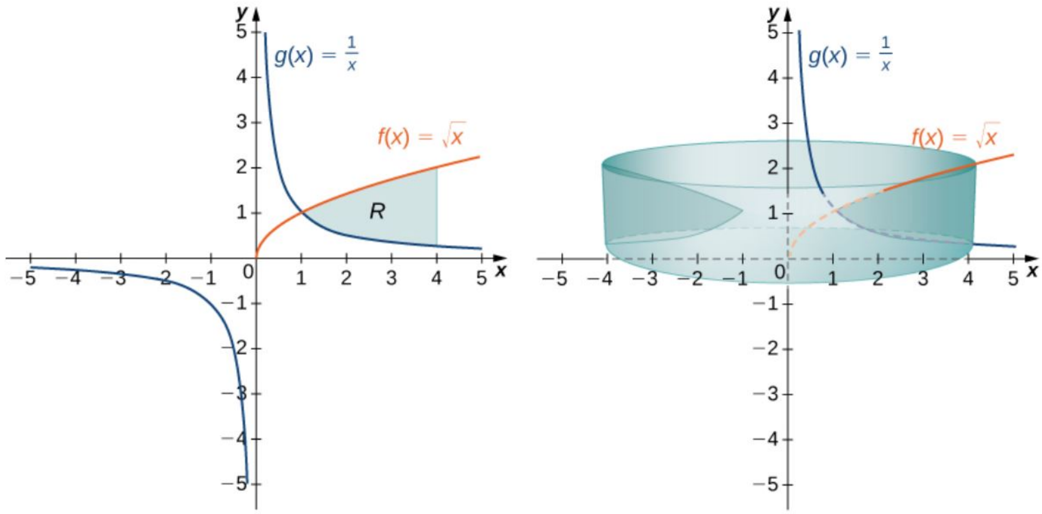


Figure 7: The region $R$ and the solid of revolution.

Then the volume of the solid is easily calculated. Note that the axis of revolution is the $y$-axis, so the radius of a shell is given simply by $x$. We don't need to make any adjustments to the $x$-term of our integrand. The height of a shell, though, is given by $f(x) - g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^4 (2\pi x(f(x) - g(x)))dx \\ &= \int_1^4 \left(2\pi x\left(\sqrt{x} - \frac{1}{x}\right)\right)dx = 2\pi \int_1^4 \left(x^{3/2} - 1\right)dx \\ &= 2\pi\left[\frac{2x^{5/2}}{5} - x\right]\Bigg|_1^4 = \frac{94\pi}{5}\text{ units}^3. \end{aligned}$$

**Please let me know if you have any questions, comments, or corrections!**