Math2411 - Calculus II
Guided Lecture Notes
Taylor Polynomials

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Taylor Polynomials Introduction:

Our objective is to approximate general functions using polynomials. Many of the functions that we work with in everyday applications are quite difficult or impossible to evaluate exactly. Examples could be $\sin(x)$, $\cos(x)$, $\tan^{-1}(x)$, e^x, $\ln(x)$, etc. Now sometimes we can easily handle these functions. Suppose you were asked to evaluate $\sin(\pi/6)$. This is a familiar angle measure for the sine function and we immediately know that $\sin(\pi/6) = 1/2$. Even some unfamiliar angle measures can be substituted in the sine function and be evaluated. For example, to evaluate $\sin(\pi/12)$ we could use a half angle identity.

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \quad \Longrightarrow \quad \sin(\pi/12) = \sqrt{\frac{1-\cos(\pi/6)}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

But what if we were asked to evaluate $\sin(1)$ and write as a decimal, fraction, or an expression involving radicals? The angle 1 radians is not a familiar input for the sine function and so must be approximated. Similarly, suppose you are asked to write $\ln(e)$ as a decimal. This is an easy question as we know $\ln(e) = 1$. Similarly, we could easily state that $\ln\left(e^7\right) = 7\ln(e) = 7$. But what about the exact value of $\ln(4)$ as a decimal, fraction, or expression involving radicals? We might try to write $\ln(4) = 2\ln(2)$. But now we are stuck and need an approximation of $\ln(2)$. Our focus in this section is developing a formal method to make these approximations. Before we develop the method let's look at some examples using the sine function. We are going to look at the graph $y = \sin(x)$ compared to the graphs of various polynomial functions.

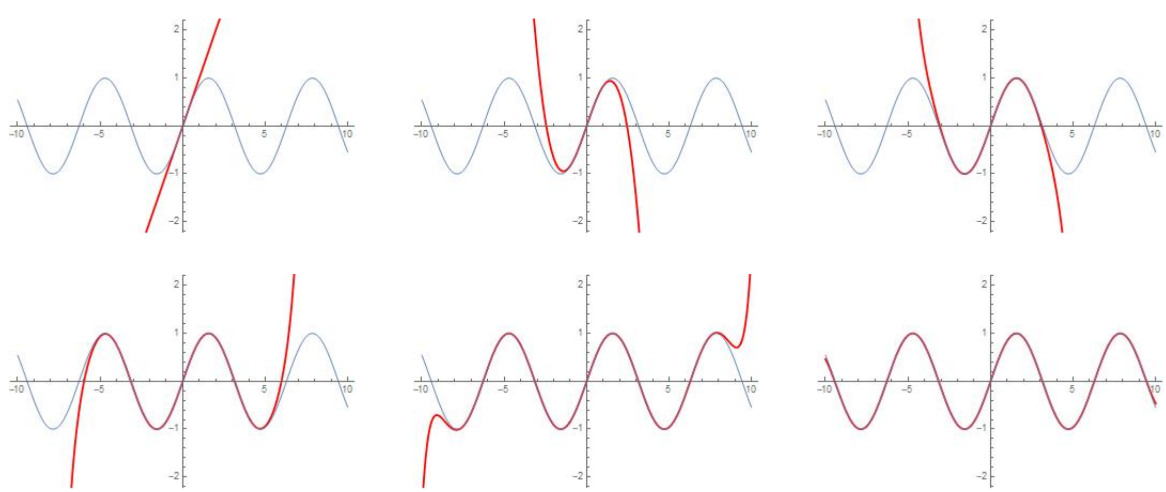

Figure 1: Various Polynomial Approximations of the Sine Function

The previous Figure 1. shows approximating polynomials of varying degrees, from a first degree linear polynomial approximation to a degree 25 polynomial approximation. Before we move on, let's look at the graph $y = e^x$ compared to the graphs of various polynomial functions.

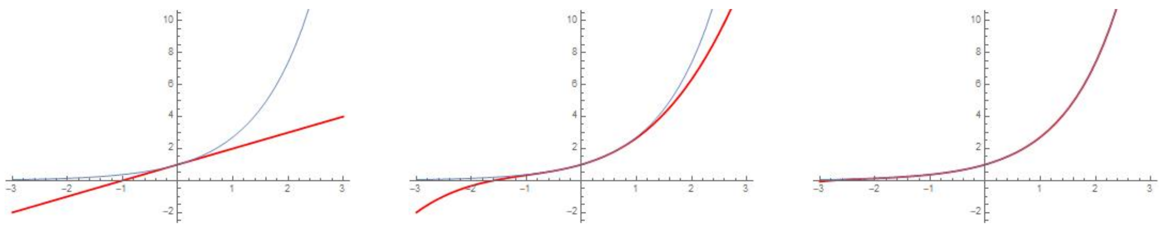

Figure 2: Various Polynomial Approximations of the Exponential Function

The previous figure shows approximating polynomials of varying degrees, from a first degree linear polynomial approximation to a degree 7 polynomial approximation. For example, the degree 7 polynomial is given by

$$p_7(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040}.$$

Then, for certain x-values, we could approximate $e^x \approx p_7(x)$. So

$$e = e^1 \approx p(1) = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = 2.718253968$$

The above polynomials are called ***Taylor polynomials***. Each of these polynomials is ***centered*** at some point $x = a$. This will be a point where we have lots of information about the function, especially information about derivatives. The idea is to match derivatives for the function $f(x)$ and the polynomial function. The more derivatives we match, our intuition suggests that we should have a better approximation. This is because derivatives control the shape of the graph. Our general polynomial will have the form

$$p_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n.$$

Then we want

$$\begin{aligned} f(a) &= p(a) \\ f'(a) &= p'(a) \\ f''(a) &= p''(a) \\ &\vdots \\ f^{(n)}(a) &= p^{(n)}(a) \end{aligned}$$

Taking derivatives and solving for the coefficients $c_0, c_1, \ldots, c_n$ gives us the following formula for the n^{th} Taylor polynomial centered at $x = a$.

Definition

If f has n derivatives at $x = a$, then the nth Taylor polynomial for f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth Taylor polynomial for f at 0 is known as the nth Maclaurin polynomial for f.

***Note*:** For those unfamiliar with the factorial function, we define $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. And for various reasons we also set $0! = 1$.

Let's try an example together.

Taylor Polynomial Examples:

Example 1. Find a degree 3 Taylor polynomial for $f(x) = e^x$ centered at $a = 0$. Then find a general formula for an n^{th} degree Taylor polynomial.

Workspace:

Let's try another example.

Example 2. Find a Taylor polynomial formula for $f(x) = \sin(x)$ centered at $a = 0$.

<u>Workspace</u>:

Let's try another example.

Example 3. Find a Taylor polynomial formula for $f(x) = \ln(x)$ centered at $a = 1$.

Workspace:

Let's try another example.

Example 4. Find a Taylor polynomial formula for $f(x) = \tan^{-1}(x)$ centered at $a = 0$.

Workspace:

Let's try another example.

Example 5. Use the fact that $\cos(x) = \dfrac{d}{dx}\big[\sin(x)\big]$ to find the formula for a Taylor polynomial for $f(x) = \cos(x)$.

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