Math2411 - Calculus II
Guided Lecture Notes
A Nice Limit Result

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In our guided lecture notes we considered the infinite series

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(n!)^2}{(2n)!}.$$

We showed the series converges by the Ratio Test but we also wanted to try the Alternating Series Test. The question is how to evaluate the limit

$$\lim_{n\to\infty}\frac{(n!)^2}{(2n!)}$$

without using the heavy machinery of Stirling's approximation $n! \approx \sqrt{2\pi n}\left(\frac{e}{n}\right)^n$?

Instead we will use Bernoulli's Inequality.

Theorem 1 (Bernoulli)**.** *Let $x \geq -1$ be a real number and r be any non-negative real number. Then we satisfy the following inequalities.*

1. *If $r \geq 1$, then $(1+x)^r \geq 1 + rx$.*
2. *If $0 \leq r \leq 1$, then $(1+x)^r \leq 1 + rx$.*

To evaluate the limit we will consider the sequence $a_n = (2n)!/(n!)^2$ and first observe that

$$\frac{(2n)!}{(n!)^2} = \frac{(2n)(2n-1)(2n-2)!}{n^2\cdot\left[(n-1)!\right]^2} = 4\left(1-\frac{1}{2n}\right)\frac{\left[2(n-1)\right]!}{\left[(n-1)!\right])^2}$$

From this and an induction argument we see that

$$\begin{aligned}
\frac{(2n)!}{(n!)^2} &= 4^n\prod_{k=0}^{n-1}\left(1-\frac{1}{2k+2}\right)\\
&= \frac{4^n}{2}\prod_{k=1}^{n-1}\left(1-\frac{1}{2k+2}\right)\\
&= \frac{4^n}{2}\prod_{k=1}^{n-1}\left(1-\frac{1}{2}\left(\frac{1}{k+1}\right)\right)\\
&\geq \frac{4^n}{2}\prod_{k=1}^{n-1}\left(1-\frac{1}{k+1}\right)^{1/2} \qquad \text{By Bernoulli's Inequality}\\
&= \frac{4^n}{2\sqrt{n}}
\end{aligned}$$

So we have the following inequalities.

$$\frac{(2n)!}{\big(n!\big)^2} \geq \frac{4^n}{2\sqrt{n}} \quad \Longrightarrow \quad 0 \leq \frac{\big(n!\big)^2}{(2n)!} \leq \frac{2\sqrt{n}}{4^n}.$$

Now since $\lim\limits_{n\to\infty} \dfrac{2\sqrt{n}}{4^n} = 0$, we have by the Squeeze Theorem that $\lim\limits_{n\to\infty} \dfrac{\big(n!\big)^2}{(2n)!} = 0$.

Please let me know if you have any questions, comments, or corrections!