Math2411 - Calculus II
Guided Lecture Notes
Properties of Power Series

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

Properties of Power Series Introduction:

One of the best things about working with power series is they are very easy to manipulate to create other series. For example, we can add, subtract, scale, and multiply power series in the most natural ways.

Theorem 6.2: Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.

i. The power series $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.

ii. For any integer $m \geq 0$ and any real number b, the power series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ on I.

iii. For any integer $m \geq 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.

Theorem 6.3: Multiplying Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I.

Let

$$\begin{aligned} e_n &= c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1} d_1 + c_n d_0 \\ &= \sum_{k=0}^{n} c_k d_{n-k}. \end{aligned}$$

Then

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

and

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x) \cdot g(x) \text{ on } I.$$

The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the Cauchy product of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.

We can also make substitutions. Lets consider a few examples with substitutions.

Power Series Examples:

Example 1. We can show that the function $f(x) = \frac{1}{1-x}$ can be represented as the following power series with interval of convergence $(-1, 1)$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Can we use this series to find a power series representation of the function $\frac{1}{1+2x}$?

Workspace:

Solution:

We see that $\frac{1}{1+2x} = \frac{1}{1-(-2x)} = f(-2x)$. So we have the power series representation

$$\frac{1}{1+2x} = \sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^n = 1 - 2x + 4x^2 - 8x^3 + 16x^4 + \cdots$$

when $-1 < -2x < 1$, which is equivalent to $-1/2 < x < 1/2$. The radius of convergence for the new series is $(-1/2, 1/2)$ and the radius of convergence as $R = 1/2$. Notice that with this substitution we do not need a Ratio Test to determine the radius of convergence and interval of convergence. We simply use the known radius of convergence and interval of convergence for the original series.

Let's try another example.

Example 2. We can show that the function $f(x) = \frac{1}{1-x}$ can be represented as the following power series with interval of convergence $(-1, 1)$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Can we use this series to find a power series representation of the function $\frac{1}{1+x^2}$?

Workspace:

Solution:

We see that $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = f\left(-x^2\right)$. So we have the power series representation

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 + x^{10} + \cdots$$

when $-1 < -x^2 < 1$, which is equivalent to $-1 < x < 1$. The radius of convergence for the new series is $(-1, 1)$ and the radius of convergence as $R = 1$. Notice that with this substitution we do not need a Ratio Test to determine the radius of convergence and interval of convergence. We simply use the known radius of convergence and interval of convergence for the original series.

Let's try another example.

Example 3. We can show that the function $f(x) = \frac{1}{1+x^2}$ can be represented as the following power series with interval of convergence $(-1, 1)$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

Can we use this series to find a power series representation of the function $\tan^{-1}(x)$? Yes.

Theorem 6.4: Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges on the interval $(a-R,\ a+R)$ for some $R > 0$. Let f be the function defined by the series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n(x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \end{aligned}$$

for $|x-a| < R$. Then f is differentiable on the interval $(a-R,\ a+R)$ and we can find f' by differentiating the series term-by-term:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \\ &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \end{aligned}$$

for $|x-a| < R$. Also, to find $\int f(x)dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R,\ a+R)$, and we have

$$\begin{aligned} \int f(x)dx &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \\ &= C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots \end{aligned}$$

for $|x-a| < R$.

Workspace:

Solution:

We know that $\tan^{-1}(x) = \int_{t=0}^{t=x} \frac{1}{1+t^2}\,dt$. It follows that the power series for $\tan^{-1}(x)$ can be found by integrating the series for $\frac{1}{1+x^2}$.

$$\begin{aligned}
\int_{t=0}^{t=x} (-1)^n t^{2n}\,dt &= \int_{t=0}^{t=x} 1 - t^2 + t^4 - t^6 + t^8 - t^{10} + \cdots\,dt \\
&= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \frac{t^{11}}{11} + \cdots \Bigg|_{t=0}^{t=x} \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots\right) - \left(0 - 0 + 0 - 0 + 0 - 0 + \cdots\right) \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

The previous theorem tells us that the new series will have the same radius of convergence, and so the series will converge on the interval $(-1, 1)$. However, integrating or differentiating a power series can change convergence or divergence at the endpoints of the interval of convergence. So we check the endpoints.

- $x = -1$: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+1}}{2n+1} = -\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ is a convergent alternating series.
- $x = 1$: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ is a convergent alternating series.

So the interval of convergence is $[-1, 1]$ and we have for $-1 \le x \le 1$ that

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Let's try another example.

Example 4. We can show that the function $f(x) = \sin(x)$ can be represented as the following power series with interval of convergence $(-\infty, \infty)$.

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

Can we use this series to find a power series representation of the function $\cos(x)$? Yes.

Workspace:

Solution:

We know that $\cos(x) = \frac{d}{dx}\Big[\sin(x)\Big]$. It follows that the power series for $\cos(x)$ can be found by differentiating the power series for $\sin(x)$.

$$\begin{aligned}\frac{d}{dx}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots\right] &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\end{aligned}$$

The previous theorem tells us that the new series will have the same radius of convergence, and so the series will converge on the interval $(-\infty, \infty)$. We have for all x-values that

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$$

Hopefully we can see that power series are fairly easy to manipulate power series to discover power series for certain desired functions. Let's consider one more example along with an application.

Example 5. Use the power series for $\sin(x)$ centered at $a = 0$ to evaluate the integral $\displaystyle\int_{x=0}^{x=1} \frac{\sin(x)}{x}\,dx$.

Workspace:

Workspace Continued:

Solution:

We have the power series representation

$$\frac{\sin(x)}{x} = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!}$$

We can also check that this series converges for all x-values. We can use this to solve the difficult problem of evaluating the following integral.

$$\int_{x=0}^{x=1}\frac{\sin(x)}{x}\,dx$$

The difficulty is that there is no elementary antiderivative for the function $\sin(x)/x$ Fortunately, we can integrate power series (all power series correspond to continuous functions) and so we have

$$\begin{aligned}
\int_{x=0}^{x=1}\frac{\sin(x)}{x}\,dx &= \int_{x=0}^{x=1}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!}\right)dx \\
&= \int_{x=0}^{x=1} 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\frac{x^{10}}{11!}+\cdots\,dx \\
&= x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}-\frac{x^7}{7\cdot 7!}+\frac{x^9}{9\cdot 9!}-\cdots\Bigg|_{x=0}^{x=1} \\
&= \left(1-\frac{1}{3\cdot 3!}+\frac{1}{5\cdot 5!}-\frac{1}{7\cdot 7!}+\frac{1}{9\cdot 9!}-\cdots\right)-\Big(0-0+0-0+0-0+\cdots\Big) \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)(2n+1)!}
\end{aligned}$$

This is an alternating series and so we can easily approximate the value of the integral and have an easy analysis of the error in our approximation. For example, consider the 3^{th} partial sum.

$$S_6 = \sum_{n=0}^{3}(-1)^n\frac{1}{(2n+1)(2n+1)!} = 1-\frac{1}{3\cdot 3!}+\frac{1}{5\cdot 5!}-\frac{1}{7\cdot 7!} = 0.9460827664$$

We know that the error can be analyzed as follows.

$$|R_3| = \left|S-S_3\right| < \frac{1}{9\cdot 9!} = 0.000000306.$$

$$\vdots$$

....Solution Continued

$$\vdots$$

So we have the following range of values:

$$0.9460824602 < \int_{x=0}^{x=1} e^{-x^2}\,dx < 0.9460830726$$

and our estimate S_3 is accurate to at least five decimal places. The computer generated result is

$$\int_{x=0}^{x=1} \frac{\sin(x)}{x}\,dx = 0.9460830704.$$

We can make a few more comments about this power series expansion for $f(x) = \sin(x)/x$.

Discussion:

Notice that we would describe the domain of $f(x) = \sin(x)/x$ as $(-\infty, 0) \cup (0, \infty)$ since we can not divide by zero. But the power series representation of $f(x)$ can be evaluated at $x = 0$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}\Bigg|_{x=0} = 1 - 0 + 0 - 0 + 0 - 0 + \cdots = 1.$$

So the power series representation for $f(x) = \sin(x)/x$ has domain $\mathbb{R} = (-\infty, \infty)$ and so we have "extended" the domain of the original function $f(x)$. This is interesting! Moreover, recall from Calculus I that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

and since power series represent continuous functions we would need to have the function value at $x = 0$ equal to 1. The value of the power series at $x = 0$ matches this requirement exactly.

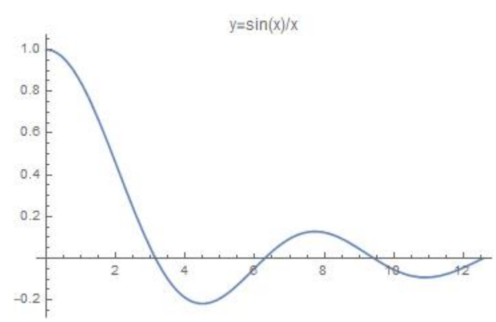

Figure 1: Graph of $y = \sin(x)/x$.

We can see that our knowledge of Taylor series makes us pretty powerful when it comes to solving certain difficult problems.

$$\vdots$$

Discussion:

$$\vdots$$

Note that the same process would not work for the function $g(x) = \cos(x)/x$. If we use the power series expansion of $g(x)$ we would have

$$\frac{\cos(x)}{x} = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = \frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots .$$

This is not a power series (because of the first term) and so results about power series do not apply. Moreover, the series can not be evaluated at $x = 0$ and the domain of $g(x)$ can not be extended using this power series. This is not surprising since $f(x) = \cos(x)/x$ has a vertical asymptote at $x = 0$.

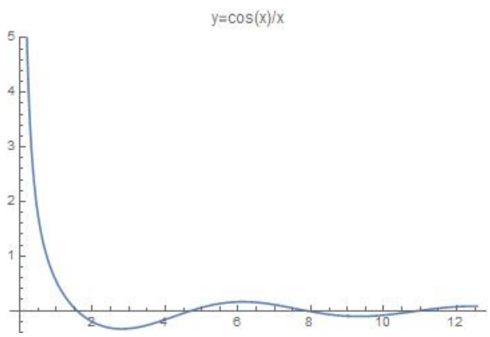

Figure 2: Graph of $y = \cos(x)/x$.

We can freely multiply a power series by a quantity bx^m where $m \geq 0$. But when multiplying by other quantities we must be much more careful about the resulting series.

Please let me know if you have any questions, comments, or corrections!