Math2411 - Calculus II
Guided Lecture Notes
Divergence and Integral Tests

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## Divergence and Integral Tests Introduction:

Our objective is to study some basic tests to determine whether an infinite series converges. So to get started, we recall geometric series

$$\sum_{n=0}^{\infty} cr^n = c = cr + cr^2 + cr^3 + \cdots$$

We were able to find a formula for the $k^{th}$ partial sum $s_k$ and use this to determine when the series converged and to what value does it converge.

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} c \cdot \left(\frac{1-r^{k-1}}{1-r}\right) = \frac{c}{1-r} \quad \text{if } -1 < r < r.$$

For all other values of $r$ the geometric series diverges.

Unfortunately, a formula for the $k^{th}$ partial sum $s_k$ is usually too difficult to find. So we need other methods to check for convergence or divergence.

## Divergence Test:

Can you think of a necessary condition on the terms of the series $a_n$ for convergence of $\sum_{n=1}^{\infty} a_n$?

Suppose that a sequence $\sum_{n=1}^{\infty} a_n$ converges. Then $\lim_{k\to\infty} s_k = L$ for some finite number $L$. Then we have

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \Big(s_k - s_{k-1}\Big) = L - L = 0.$$

So if a series converges we conclude that the terms in the series much converge to zero. This is logically equivalent to the statement of the following theorem.

**Theorem 5.8: Divergence Test**

If $\lim\limits_{n\to\infty} a_n = c \neq 0$ or $\lim\limits_{n\to\infty} a_n$ does not exist, then the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

***Note*:** The converse of the theorem is not true. That is, if $\lim_{n\to\infty} a_n = 0$, we can conclude nothing about the convergence or divergence of the series. For example, $\lim_{n\to\infty} 1/n = 0$ while the series $\sum 1/n$ diverges. We will soon see that even though $\lim_{n\to\infty} 1/n^2 = 0$ the series $\sum 1/n^2$ converges.

## Divergence Test Examples:

**Example 1.** Apply the divergence test to $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3n-1}$.

***Workspace:***

Here is another example.

**Example 2.** Apply the divergence test to $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^4}$.

***Workspace:***

Here is another example.

**Example 3.** Apply the divergence test to $\displaystyle\sum_{n=0}^{\infty} e^{1/n^2}$.

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## Integral Test:

The Integral Test is an important and powerful test and is based on the logic of direct comparison.

Let's return to a familiar example $\sum_{n=1}^{\infty} \frac{1}{n}$ and use the improper integral $\int_{x=1}^{\infty} \frac{1}{x}\,dx$ for comparison.

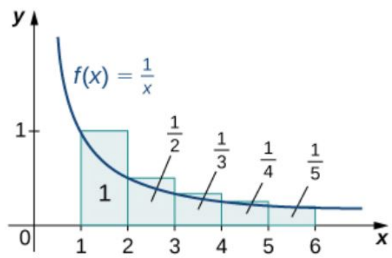


Figure 1: Comparison Method for Integral Test

We can see from the graphic that

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \int_{1}^{k+1} \frac{1}{x} dx.$$

Therefore, for each $k$, the $k$th partial sum $S_k$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n} > \int_{1}^{k+1} \frac{1}{x} dx = \ln x \Big|_{1}^{k+1} = \ln(k+1) - \ln(1) = \ln(k+1).$$

Then since $\lim_{k\to\infty} \ln(1+k) = \infty$ we must have $\lim_{k\to\infty} S_k = \infty$ and the original infinite series diverges by the Integral Test.

Let's consider another example with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

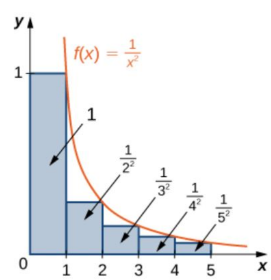


Figure 2: Comparison Method for Integral Test

We can see from the graphic that

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 1 + \int_1^k \frac{1}{x^2} dx.$$

Therefore, for each $k$, the $k$th partial sum $S_k$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \int_1^k \frac{1}{x^2} dx = 1 - \frac{1}{x}\Big|_1^k = 1 - \frac{1}{k} + 1 = 2 - \frac{1}{k} < 2.$$

We conclude that the sequence of partial sums $\{S_k\}$ is bounded. We also see that $\{S_k\}$ is an increasing sequence:

$$S_k = S_{k-1} + \frac{1}{k^2} \text{ for } k \geq 2.$$

We will use an important fact about infinite sequences. If a sequence $\{S_k\}$ is increasing and bounded then it must converge. In this example is is clear that

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2}$$

is an increasing sequence. That is, $S_k \geq S_{k-1}$. We also see from above that the sequence of partial sums $S_k$ is bounded above by 2. Therefore, the sequence of partial sums $\{S_k\}$ converges and we say the infinite series converges by the Integral Test.

This process can be summed up by the following theorem.

**Theorem 5.9: Integral Test**

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms $a_n$. Suppose there exists a function $f$ and a positive integer $N$ such that the following three conditions are satisfied:

i. $f$ is continuous,

ii. $f$ is decreasing, and

iii. $f(n) = a_n$ for all integers $n \geq N$.

Then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_N^{\infty} f(x) dx$$

both converge or both diverge (see Figure 5.14).

Let's consider a few examples.

## Integral Test Examples:

**Example 4.** Apply the Integral Test to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$.

***Workspace:***

We now look to an important type of series called a $p$-series.

**Definition**

For any real number $p$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a ***p*-series**.

Fortunately we can come to a general solution about $p$-series.

A $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

Consider a few examples.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

   Since $p = 5/4 > 1$ the infinite series converges as a $p$-series.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

   Since $p = 1/2 \leq 1$ the infinite series diverges as a $p$-series.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

   Since $p = 3 > 1$ the infinite series converges as a $p$-series.

## Estimation of Series Value:

**Theorem 5.10: Remainder Estimate from the Integral Test**

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function $f$ satisfying the following three conditions:

i. $f$ is continuous,

ii. $f$ is decreasing, and

iii. $f(n) = a_n$ for all integers $n \geq 1$.

Let $S_N$ be the $N$th partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers $N$,

$$S_N + \int_{N+1}^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)dx.$$

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_{N}^{\infty} f(x)dx.$$

This is known as the **remainder estimate**.

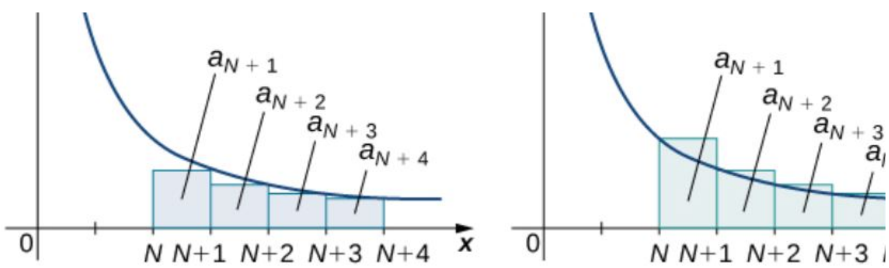


Figure 3: Visualization of Partial Sum Estimate

We can see that

$$\begin{aligned} R_N &= a_{N+1} + a_{N+2} + a_{N+3} + \cdots \leq \int_{x=N}^{\infty} f(x)\,dx \\ R_N &= a_{N+1} + a_{N+2} + a_{N+3} + \cdots \geq \int_{x=N+1}^{\infty} f(x)\,dx \end{aligned}$$

So the integral is either an overestimate of $R_N$ or an underestimate of $R_n$.

Let's work an example.

**Example 5.** Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^3}$. It turns out that, even though we know the series converges as a $p$-series, to this day mathematicians have been unable to determine an explicit form the value of the series. So we must estimate the value. Let's use the partial sum $S_{10}$.

***Workspace:***