Math2411 - Calculus II
Guided Lecture Notes
Area Between Curves

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Area Between Curves Introduction

Suppose that we want to determine the area between two curves $y = f(x)$ and $y = g(x)$ on an interval $[a, b]$.

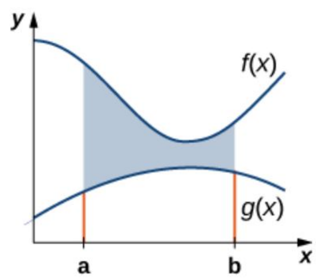

Figure 1: Region between graphs $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$

It is very natural to approach the problem by subtracting the area below the lower graph (in this example $y = g(x)$) from the area under the upper graph (in this example $y = f(x)$) to have

$$\text{Area} = \int_{x=a}^{x=b} f(x)\,dx - \int_{x=a}^{x=b} g(x)\,dx \quad \xRightarrow[\text{of Integrals}]{\text{By Properties}} \quad \text{Area} = \int_{x=a}^{x=b} f(x) - g(x)\,dx.$$

Another more general approach is to create a familiar type estimate of the area using rectangles.

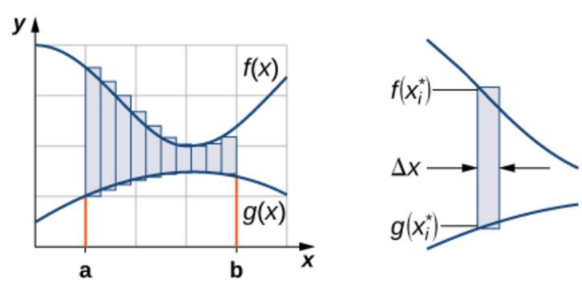

Figure 2: Estimated area between graphs $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$.

If we improve the estimate by letting the number of rectangles increase we have the same formula that we just saw above.

Theorem 2.1: Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b [f(x) - g(x)]dx. \tag{2.1}$$

If we label the upper graph as $y = upper(x)$ and the lower graph as $y = lower(x)$, we can rewrite the formula 2.1 and also interpret the integral geometrically (as we've seen before) as a sum of the areas of "very, very, thin" rectangles.

$$\int_{x=a}^{x=b} [upper(x) - lower(x)]\, dx \xRightarrow{\text{Can Be Interpreted As}} \overset{\text{Sum}}{\overset{\downarrow}{\int_{x=a}^{x=b}}} \underbrace{\overset{\text{Rectangle Height}}{\overset{\downarrow}{[upper(x) - lower(x)]}} \qquad \overset{\text{Rectangle Width}}{\overset{\downarrow}{dx}}}_{\text{Rectangle Area}}$$

Area Between Curves Examples:

Let's work some examples.

Example 1. Find the area between the curves $y = 9 - x^2/4$ and $y = 6 - x$.

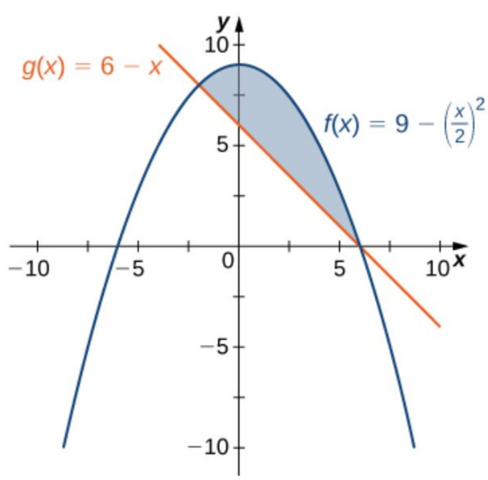

Figure 3: Region between graphs $y = 9 - x^2/4$ and $y = 6 - x$.

Start by finding the interval. At which x-values do the curves intersect?

We solve the following equation.

$$\begin{aligned}
9 - x^2/4 &= 6 - x \\
x^2 - 4x - 12 &= 0 \\
(x-6)(x+2) &= 0
\end{aligned}$$

So the interval will be $[-2, 6]$. On this interval the upper graph is $y = 9 - x^2/4$ and the lower graph is $y = 6 - x$ and so the integral is

$$\text{Area} = \int_{x=a}^{x=b} \left[upper(x) - lower(x) \right] dx = \int_{x=-2}^{x=6} \left[\left(9 - \frac{x^2}{4} \right) - \left(6 - x \right) \right] dx$$

We evaluate the integral.

$$\begin{aligned}
\int_{x=-2}^{x=6} \left[\left(9 - \frac{x^2}{4} \right) - \left(6 - x \right) \right] dx &= \int_{x=-2}^{x=6} \left[-\frac{x^2}{4} + x + 3 \right] dx \\
&= -\frac{x^3}{12} + \frac{x^2}{2} + 3x \Bigg|_{x=-2}^{x=6} \\
&= \left(-\frac{6^3}{12} + \frac{6^2}{2} + 3 \cdot 6 \right) - \left(-\frac{(-2)^3}{12} + \frac{(-2)^2}{2} + 3 \cdot (-2) \right) \\
&= \frac{64}{3}
\end{aligned}$$

So the area between the two graphs is Area=64=3.

Let's have you try an example on your own.

Example 2. Find the area between the curves $y = x + 4$ and $y = 3 - x/2$ on the interval $[1, 4]$.

Workspace:

Example 3. Find the area between the curves $y = \sin(x)$ and $y = \cos(x)$ on the interval $[0, \pi]$.

Workspace:

Let's have you try another example on your own.

Example 4. Find the area between the curves $y = \sqrt{x}$ and $y = 3/2 - x/2$ and $y = 0$.

Workspace: