Math2411 - Calculus II
Guided Lecture Notes
Taylor Series

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Taylor Series Introduction:

Our objective is to extend a Taylor polynomial for a function f centered at a point a to an infinite series called a **Taylor series**. First recall our definition of a Taylor polynomial.

Definition

If f has n derivatives at $x = a$, then the nth Taylor polynomial for f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth Taylor polynomial for f at 0 is known as the nth Maclaurin polynomial for f.

Then a Taylor series is a power series with the same coefficients as a Taylor polynomial.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Since a Taylor series is just a power series all of our knowledge of power series applies to Taylor series. But with a Taylor series for a function $f(x)$ we have to make sure that the Taylor series actually converges to the correct function value $f(x)$. Fortunately we can determine this using the remainder formula. Recall the following.

Theorem 6.7: Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n + 1$ times on an interval I containing the real number a. Let p_n be the nth Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x)$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

If there exists a real number M such that $\left| f^{(n+1)}(x) \right| \le M$ for all $x \in I$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x in I.

Then we have the following theorem.

Theorem 6.8: Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n \to \infty} R_n(x) = 0$$

for all x in I.

Let's work an example.

Example 1. Show that the Taylor series for $f(x) = \sin(x)$ centered at the point $a = 0$ converges to $\sin(x)$ for all x-values.

Note: Recall the Taylor series for $f(x) = \sin(x)$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

Workspace:

Let's now spend a little time with the binomial series.

Example 2. Apply the Taylor series for $f(x) = (1+x)^r$ centered at the point $a = 0$ to estimate the value of $\sqrt{1.5}$.

Workspace:

Example 3. Let's look at another application of the binomial series. In particular, we want to solve the equation for the period of an oscillating pendulum. If T is the period, L is the length of the pendulum, g is acceleration due to gravity, θ_{max} is the maximum angle of swing, and $k = \sin(\theta_{max}/2)$ we have

$$T = 4\sqrt{\frac{L}{g}} \int_{\theta=0}^{\theta=\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2(\theta)}} \, d\theta.$$
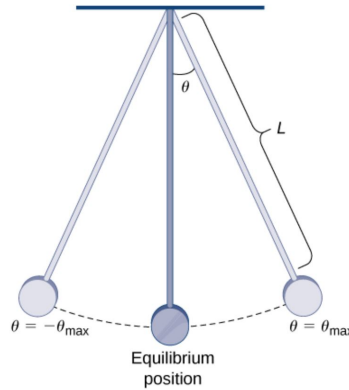

Figure 1: Oscillating Undamped Pendulum

Workspace:

Example 4. We can show that the function $f(x) = e^x$ can be represented as the following power series with interval of convergence $(-\infty, \infty)$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Can we use this series to find a Taylor series representation of the function e^{-x^2}? Can we then use this Taylor series to evaluate the integral $\displaystyle\int_{x=0}^{x=1} e^{-x^2}\, dx$? Yes to both.

Workspace: