Math2411 - Calculus II
Guided Lecture Notes
Improper Integrals

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Improper Integrals Introduction:

Our objective is to evaluate integrals over infinite intervals or to integrate functions unbounded functions, such as those with a vertical asymptote. For example,

$$\int_{x=1}^{\infty} \frac{1}{x^2}\, dx \quad \text{or} \quad \int_{x=0}^{x=1} \frac{1}{x}\, dx$$

See the graphs in the following figure to get a geometric perspective.

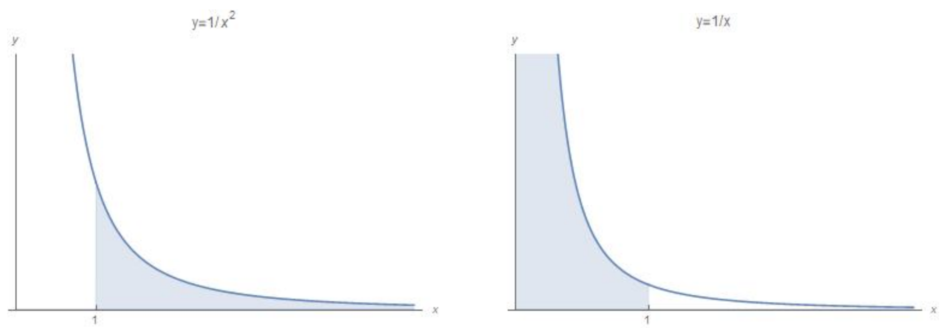

Figure 1: Integral With an Infinite Interval and Integrand With a Vertical Asymptote

Before we dig into the details we should probably consider the idea of non-terminating addition. In fact, normally, when we think of addition we think of a process which terminates, or comes to an end. For example

$$1 + 6 + 18 + 2 + 10 = 37.$$

Even our experience with integration can be considered a terminating addition process. The definite integral

$$\int_{x=a}^{x=b} f(x)\, dx$$

can be interpreted as adding up area under the graph of $y = f(x)$ starting at $x = a$ and finishing at $x = b$. See the following Figure 2.

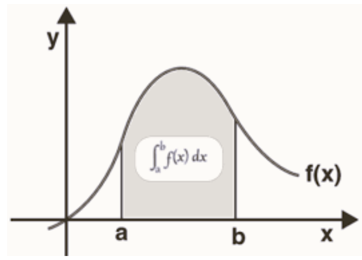

Figure 2: Adding Up Area Under the Graph $y = f(x)$

So what about non-terminating addition? It turns out we have already encountered non-terminating addition. For example, we should recall that we can write $1/3 = 0.33\overline{3}$. That can be written as follows:

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots .$$

I hope that nobody would protest about assigning the value 1/3 to the sum on the right. And if you're paying attention you might notice it is a non-terminating sum. That is where we are headed.

Here's another example based on one of Xeno's paradoxes. It states that motion is impossible. You've probably seen the argument.

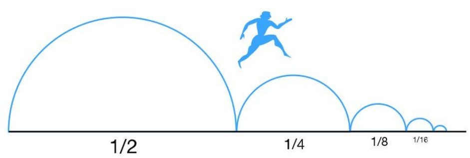

Figure 3: An Illustration of Xeno's Paradox

The logic goes as follows. In order to travel from one point to another point I must first travel 1/2 the distance, and then I must travel half of the remaining distance, or 1/4 the total distance. Continuing in this fashion I would next have to travel 1/8 the total distance, and then 1/16 of the total distance, etc. Since we can always cut the remaining distance in half, there are infinitely many steps to take and thus it is impossible to move from one point to another. We can describe the process in the paradox with an infinite non-terminating sum.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Now, of course, we know that motion is possible. So perhaps it makes sense to assign a value to the above sum. Any thoughts?

I think everyone would agree that the natural value to assign to the sum is 1. As we take more and more steps our total distance is approaching 1. So we write

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

A little thought should convince you that this sum is really no different than assigning the value 1/3 to the following sum.

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots$$

The Question: So what does this have to do with integration?

Integrals with Infinite Integration Limits:

It turns out that many applications in mathematics require integration over an infinite interval. For example, in probability theory we might consider the integral

$$\int_{x=0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Here we are adding up area under the graph $y = 1/(1+x^2)$ starting at $x = 0$. But we do not have a right endpoint to the interval and so the addition process does not terminate.

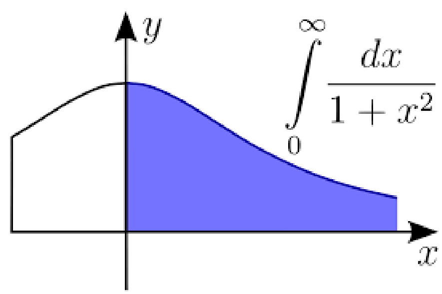

Figure 4: An Integral with Infinite Interval

Question: So how should we deal with such an integral? maybe we should approach this as we did with the non-terminating sums above?

Suppose we need to evaluate $\displaystyle\int_{x=a}^{\infty} f(x)\,dx$.

We will start with a proper Riemann integral

$$\int_{x=a}^{x=t} f(x)\,dx$$

and consider what happens to the value of the integral as the value t increases.

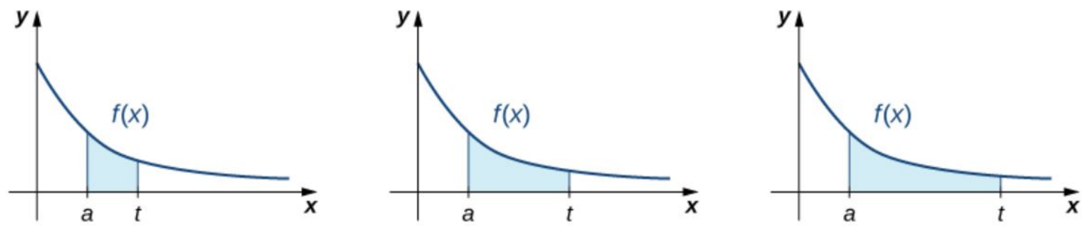

Figure 5: Dealing with an Integral with Infinite Interval of Integration

Formally we write

$$\int_{x=a}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{x=a}^{x=t} f(x)\,dx.$$

Continuing with the example from above we have

$$\int_{x=0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_{x=0}^{x=t} \frac{1}{1+x^2}\,dx.$$

Let's formalize this with a two step process.

- ***Step #1*:** Evaluate the proper integral $\displaystyle\int_{x=0}^{x=t} \frac{1}{1+x^2}\,dx$.

$$\begin{aligned}\int_{x=0}^{x=t} \frac{1}{1+x^2}\,dx &= \tan^{-1}(x)\Big|_{x=0}^{x=t} \\ &= \tan^{-1}(t) - \tan^{-1}(0) \\ &= \tan^{-1}(t)\end{aligned}$$

- ***Step #2*:** Evaluate the limit as $t \to \infty$.

$$\lim_{t\to\infty} \int_{x=a}^{x=t} f(x)\,dx = \lim_{t\to\infty} \tan^{-1}(t) = \frac{\pi}{2}$$

So as the value t increases the value of the integral is approaching $\pi/2$ and so we say the improper integral converges to $\pi/2$ and we write

$$\int_{x=0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Let's consider an example together.

Example 1. Evaluate $\displaystyle\int_{x=1}^{\infty} \frac{1}{x}\,dx$.

Workspace:

Solution:

We start by evaluating the proper integral $\int_{x=1}^{x=t} \frac{1}{x}\,dx$. This is easily seen to give

$$\int_{x=1}^{x=t} \frac{1}{x}\,dx = \ln(t)$$

Next we evaluate the limit.

$$\lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(t) = \infty$$

Since this limit is not finite, the value of the integral is not approaching any finite value as t increases and we say the improper integral diverges. That means we can not assign any finite value to this improper integral.

Let's have you try an example on your own.

Example 2. Evaluate $\int_{x=1}^{\infty} \frac{1}{x^2}\,dx$.

Workspace:

Solution:

We start by evaluating the proper integral $\int_{x=1}^{x=t} \frac{1}{x^2}\,dx$. This is easily seen to give

$$\int_{x=1}^{x=t} \frac{1}{x^2}\,dx = 1 - \frac{1}{t}$$

Next we evaluate the limit.

$$\lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$$

Since this limit is finite, the value of the integral is approaching a finite value as t increases and we say the improper integral converges to 1. That means we can write

$$\int_{x=1}^{\infty} \frac{1}{x^2}\,dx = 1.$$

Let's have you try another important example on your own.

Example 3. For which positive values p does the integral $\int_{x=1}^{\infty} \frac{1}{x^p}\,dx$ converge?.

Workspace:

Solution:

We start by assuming $p \neq 1$ and evaluating the proper integral $\int_{x=1}^{x=t} \frac{1}{x^p}\,dx$. This is easily seen to give

$$\int_{x=1}^{x=t} \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p}\bigg|_{x=1}^{x=t} = \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$$

Next we evaluate the limit.

$$\lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{x}\,dx = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}.$$

If $p > 1$ then $1 - p < 0$ and the quantity $t^{1-p} = 1/t^{p-1} \to 0$ and the limit is finite. That is, when $p > 1$ we have

$$\lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{x}\,dx = \frac{1}{p-1}$$

and the improper integral converges to the value $1/(p-1)$.

However, if $p < 1$ then $1 - p > 0$ and the quantity $t^{1-p} \to \infty$ and the limit is infinite. So in this case the improper integral diverges.

We have already seen that the improper integral diverges when $p = 1$. So we can say the improper integral converges when $p > 1$ and diverges when $p \leq 1$.

Moreover, when $p > 1$ we can write

$$\int_{x=1}^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1}.$$

For example,

$$\int_{x=1}^{\infty} \frac{1}{x^{5/3}}\,dx = \frac{1}{5/3-1} = \frac{3}{2} \text{ since } p = 5/3 > 1.$$

Another example could be

$$\int_{x=1}^{\infty} \frac{1}{x^3/4}\,dx = \infty \text{ since } p = 3/4 \leq 1.$$

Let's move onto the next topic.

Integrals of Functions with Discontinuities:

In our previous calculus experience we have always integrated continuous functions. But what if our function f is not continuous on the interval $[a, b]$? Perhaps there is a vertical asymptote at $x = a$ or at $x = b$.

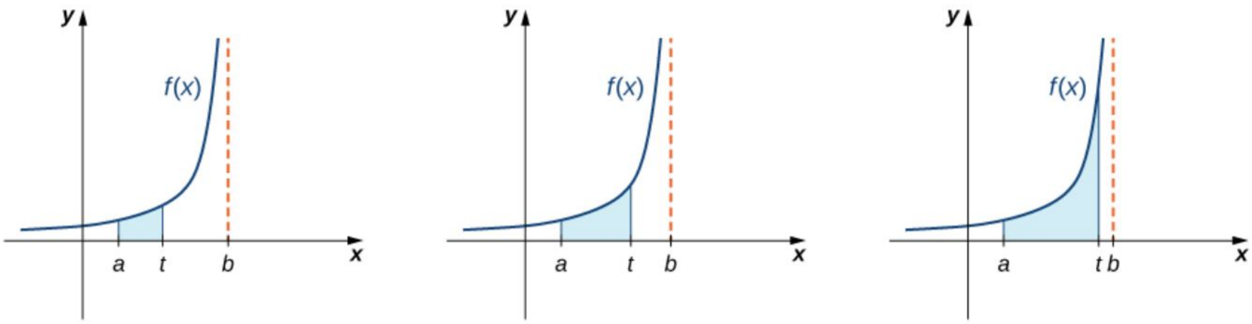

Figure 6: Dealing with an Integral of a Discontinuous Function

We capture the strategy in the following definition.

Definition

1. Let $f(x)$ be continuous over $[a, b)$. Then,

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx. \tag{3.19}$$

2. Let $f(x)$ be continuous over $(a, b]$. Then,

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx. \tag{3.20}$$

   In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. If $f(x)$ is continuous over $[a, b]$ except at a point c in (a, b), then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \tag{3.21}$$

   provided both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)dx$ diverges.

Let's work a concrete example.

Example 4. Evaluate $\displaystyle\int_{x=0}^{x=1} \frac{1}{\sqrt{x}}\, dx$.

Workspace:

Solution:

Notice there is a discontinuity (vertical asymptote) at $x = 0$. So we start by evaluating the proper integral $\int_{x=t}^{x=1} \frac{1}{\sqrt{x}}\, dx$ where $0 < t < 1$. This is easily seen to give

$$\int_{x=t}^{x=1} \frac{1}{\sqrt{x}}\, dx = 2\left(1 - \sqrt{t}\right)$$

Next we evaluate the limit.

$$\lim_{t \to 0^+} \int_{x=t}^{x=1} \frac{1}{\sqrt{x}}\, dx = \lim_{t \to 0^+} 2\left(1 - \sqrt{t}\right) = 2$$

Since this limit is finite, the value of the integral is approaching a finite value as t approaches zero and we say the improper integral converges to 2. That means we can write

$$\int_{x=0}^{x=1} \frac{1}{\sqrt{x}}\, dx = 2.$$

Let's have you try another example on your own.

Example 5. Evaluate $\int_{x=0}^{x=1} \frac{1}{x}\, dx$.

Workspace:

Solution:

Notice there is a discontinuity (vertical asymptote) at $x = 0$. So we start by evaluating the proper integral $\int_{x=t}^{x=1} \frac{1}{x}\,dx$ where $0 < t < 1$. This is easily seen to give

$$\int_{x=t}^{x=1} \frac{1}{x}\,dx = -\ln(t)$$

Next we evaluate the limit.

$$\lim_{t\to 0^+} \int_{x=t}^{x=1} \frac{1}{x}\,dx = \lim_{t\to 0^+} -\ln(t) = \infty$$

Since this limit is infinite, the value of the integral is not approaching a finite value as t approaches zero and we say the improper integral diverges (to ∞). That means we can write

$$\int_{x=0}^{x=1} \frac{1}{x}\,dx = \infty.$$

Let's have you try an important example.

Important Examples:

Example 6. Evaluate $\int_{x=-1}^{x=1} \frac{1}{x^3}\,dx$.

Workspace:

Solution:

Notice there is a discontinuity (vertical asymptote) at $x = 0$.

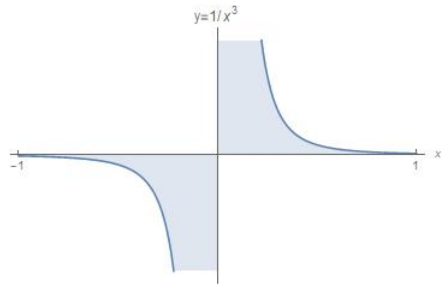

Figure 7: Improper Integral $\int_{x=-1}^{x=1} \frac{1}{x^3}\,dx$

It is tempting to use symmetry. In fact, if we use symmetry arguments from Calculus I what would believe is the value of the improper integral? Does the integral converge?

Since there is a discontinuity at $x = 0$ we will write the improper integral as

$$\int_{x=-1}^{x=1} \frac{1}{x^3}\,dx = \int_{x=-1}^{x=0} \frac{1}{x^3}\,dx + \int_{x=0}^{x=1} \frac{1}{x^3}\,dx$$

Consider the integral from $x = 0$ to $x = 1$. We start by evaluating the proper integral $\int_{x=t}^{x=1} \frac{1}{x^3}\,dx$ where $0 < t < 1$. This is easily seen to give

$$\int_{x=t}^{x=1} \frac{1}{x^3}\,dx = -\frac{1}{2x^2}\bigg|_{x=t}^{x=1} = \frac{1}{2t^2} - \frac{1}{2}.$$

Next we evaluate the limit.

$$\lim_{t\to 0^+} \int_{x=t}^{x=1} \frac{1}{x^3}\,dx = \lim_{t\to 0^+} \frac{1}{2t^2} - \frac{1}{2} = \infty$$

Since this limit is infinite, the value of the integral is not approaching a finite value as t approaches zero and we say the improper integral diverges (to ∞). That means we can write

$$\int_{x=0}^{x=1} \frac{1}{x^3}\,dx = \infty.$$

By symmetry we can be assured that the integral from $x = -1$ to $x = 0$ also diverges.

$\vdots$

....Solution Continued

$$\vdots$$

In this case,

$$\int_{x=-1}^{x=0} \frac{1}{x^3}\,dx = -\infty.$$

But $\infty - \infty$ is an indeterminate form and so we can not say the improper integral converges. In fact, since at least one of the integrals diverges we must say the original improper integral diverges. So we have to be careful about applying symmetry arguments to improper integrals. See point 3 in the definition given on page 8.

Let's have you try another important example.

Example 7. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$.

Workspace:

Solution:

We look at a graphic view of the problem.

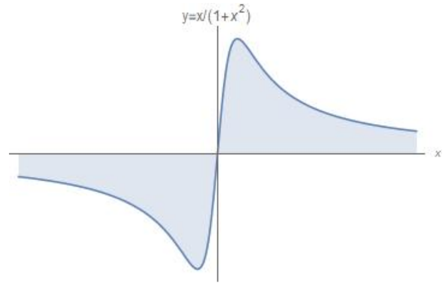

Figure 8: Improper Integral $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$

It is tempting to use symmetry. In fact, if we use symmetry arguments from Calculus I what would believe is the value of the improper integral? Does the integral converge?

We will write the improper integral as

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{x=0} \frac{x}{1+x^2}\,dx + \int_{x=0}^{\infty} \frac{x}{1+x^2}\,dx$$

Consider the integral from $x = 0$ to $+\infty$. We start by evaluating the proper integral $\int_{x=0}^{x=t} \frac{x}{1+x^2}\,dx$ where $0 < t$. This is easily seen to give

$$\int_{x=0}^{x=t} \frac{x}{1+x^2}\,dx = -\frac{1}{2}\ln(1+x^2)\Big|_{x=0}^{x=t} = \frac{1}{2}\ln(1+t^2).$$

Next we evaluate the limit.

$$\lim_{t\to\infty} \int_{x=0}^{x=t} \frac{x}{1+x^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\ln(1+t^2) = \infty$$

Since this limit is infinite, the value of the integral is not approaching a finite value as t increases and we say the improper integral diverges (to ∞). That means we can write

$$\int_{x=0}^{\infty} \frac{x}{1+x^2}\,dx = \infty.$$

By symmetry we can be assured that the integral on the interval $(-\infty, 0)$ also diverges.

$\vdots$

....Solution Continued

$$\vdots$$

In this case,

$$\int_{-\infty}^{x=0} \frac{x}{1+x^2}\,dx = -\infty.$$

But $\infty - \infty$ is an indeterminate form and so we can not say the improper integral converges. In fact, since at least one of the integrals diverges we must say the original improper integral diverges. So we have to be careful about applying symmetry arguments to improper integrals.

Direct Comparison:

Consider the following example.

Example 8. Determine whether the following improper integral converges or diverges.

$$\int_{x=2}^{\infty} \frac{x^3}{x^4 - x - 1}\,dx$$

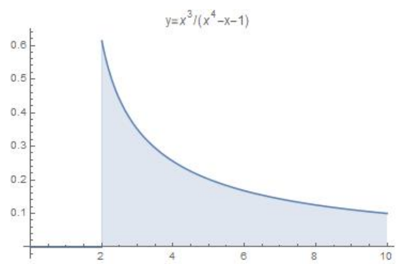

Figure 9: Area under the curve $y = \frac{x^3}{x^4-x-1}$

Workspace:

Solution:

We would like to avoid using partial fractions on the integrand because it is difficult to factor degree=4 polynomials. In fact, all of the zeros to this polynomial are quite ugly. One of the zeros is

$$\begin{aligned} x &= -\frac{1}{2}\sqrt{-4\left(\frac{2}{3(9+\sqrt{849})}\right)^{1/3} + \frac{\left(\frac{1}{2}(9+\sqrt{849})\right)^{1/3}}{3^{2/3}}} + \cdots \\ &\cdots + \frac{1}{2}\sqrt{4\left(\frac{2}{3(9+\sqrt{849})}\right)^{1/3} - \frac{\left(\frac{1}{2}(9+\sqrt{849})\right)^{1/3}}{3^{2/3}} - \frac{2}{\sqrt{-4\left(\frac{2}{3(9+\sqrt{849})}\right)^{1/3} + \frac{\left(\frac{1}{2}(9+\sqrt{849})\right)^{1/3}}{3^{2/3}}}}} \end{aligned}$$

So let's find a useful improper integral for a comparison. We can use the following fact that for all $x \geq 2$ we have

$$\frac{x^3}{x^4 - x - 1} > \frac{x^3}{x^4} = \frac{1}{x}.$$

Looking at the following graphic, it follows that

$$\int_{x=2}^{x=t} \frac{x^3}{x^4 - x - 1}\, dx > \int_{x=2}^{x=t} \frac{1}{x}\, dx$$

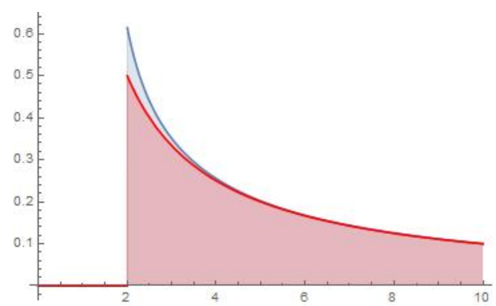

Figure 10: Area under the curve $y = \frac{x^3}{x^4-x-1}$ versus $y = \frac{1}{x}$.

Because $\displaystyle\int_{x=2}^{\infty} \frac{1}{x}\, dx$ diverges to ∞, we can conclude that $\displaystyle\int_{x=2}^{\infty} \frac{x^3}{x^4 - x - 1}\, dx$ also diverges to ∞ by direct comparison.

Please let me know if you have any questions, comments, or corrections!