Math2411 - Calculus II
Guided Lecture Notes
Sequences and Series

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Sequences and Series Introduction:

Our objective is to study the basics of ***infinite sequences*** and ***infinite series***. So to get started, what is an infinite sequence?

Definition

An **infinite sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$a_1, a_2, \ldots, a_n, \ldots .$$

The subscript n is called the **index variable** of the sequence. Each number a_n is a **term** of the sequence. Sometimes sequences are defined by **explicit formulas**, in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Sequence Examples:

Example 1. Consider the sequence $\{a_n\}$ where $a_n = 2^n$ for natural numbers n.

$$\{2^n\}_{n\in\mathbb{N}} = \{2, 4, 8, 16, 32, \ldots\}$$

Notice that we will use curly set brackets for sequence notation. And as above we can describe a sequence either by some explicit rule, or by listing enough of the sequence elements to understand the pattern (if there is indeed a pattern). Here is a graph of the sequence.

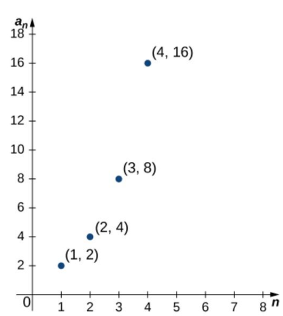

Figure 1: Graph of the sequence where $a_n = 2^n$.

Here is another example.

Example 2. The Fibonnaci sequence is one of the most famous sequences in all of mathematics. This is defined recursively as follows.

$$a_n = a_{n-1} + a_{n-2}, \quad \text{where } a_0 = a_1 = 1$$

Notice that the sequence is not defined as an explicit formula involving n. That is, we are not given a rule $a_n = f(n)$. Rather, the n^{th} term in the sequence is determined by the previous two terms. Here is the sequence.

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$$

Here is another simpler example.

Example 3. Define a sequence as follows so that $a_n = \dfrac{n}{n^2+1}$.

$$\{a_n\}_{n\in\mathbb{N}} = \left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \dots\right\}$$

Here is another example.

Example 4. Here is is one of the most important sequences in all of mathematics. Let k be any real number and define

$$a_n = \left(1 + \frac{k}{n}\right)^n$$

So we have

$$\{a_n\}_{n=1}^{\infty} = \left\{1 + k, \left(1 + \frac{k}{2}\right)^2, \left(1 + \frac{k}{3}\right)^3, \left(1 + \frac{k}{4}\right)^4, \dots\right\}$$

Limiting Values of Sequences:

The main question we will ask about sequences is whether of not the sequence converges to some value or diverges.

Definition

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence**. In this case, we write

$$\lim_{n\to\infty} a_n = L.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence**.

There is a nice graphic that reflects this idea.

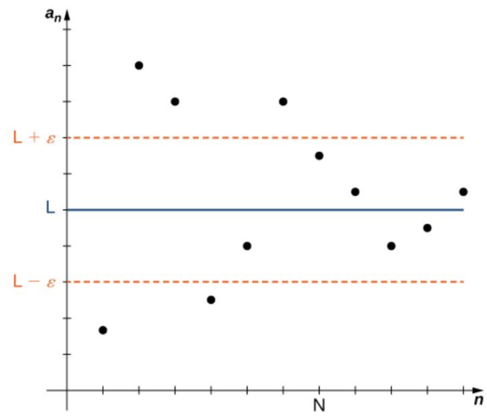

Figure 2: Graphic Representation of a Sequence $\{a_n\}$ Converging to L.

The above definition is a somewhat informal definition that we will use in this class. However, we should realize that this definition can be formalized to be perfectly precise as follows.

Definition

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0$, there exists an integer N such that $|a_n - L| < \varepsilon$ if $n \geq N$. The number L is the limit of the sequence and we write

$$\lim_{n\to\infty} a_n = L \, or \, a_n \to L.$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We will not be working with this formal definition in this class. So how will we determine if a sequence converges or diverges? We will be exclusively evaluating explicit sequences and the following theorem will be our main tool for determining convergence or divergence.

Theorem 5.1: Limit of a Sequence Defined by a Function

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L,$$

then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = L.$$

This theorem tells us that we can use all of our previous knowledge of function limits to evaluate sequence limits.

Example 5. Determine if the sequence $\{a_n\}$, where $a_n = n/(n^2+1)$, converges or diverges.

Workspace:

Solution:

We see that $a_n = f(n)$ for the function $f(x) = x/(x^2+1)$. Then our knowledge of growth rates tells us that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+1} = 0.$$

Therefore, the sequence $\{a_n\}$ converges to zero.

Alternatively we could apply L'Hopital's rule.

$$\lim_{x\to\infty} \frac{x}{x^2+1} \quad = \quad \frac{\infty}{\infty} \quad \text{form}$$

$$\downarrow \quad \text{by L'Hopital's Rule}$$

$$\lim_{x\to\infty} \frac{x}{x^2+1} \quad = \quad \lim_{x\to\infty} \frac{1}{2x} = \frac{1}{\infty} \quad \text{form}$$

So $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0.$

Let's consider another example.

Example 6. Determine if the sequence $\{a_n\}$, where $a_n = 2n^3/(5n^3 - n^2 + 1)$, converges or diverges.

Workspace:

Solution:

We see that $a_n = f(n)$ for the function $f(x) = 2x^3/(5x^3 - x^2 + 1)$. Then our knowledge of growth rates tells us that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+1} = \frac{2}{5}.$$

Therefore, the sequence $\{a_n\}$ converges to $2/5$.

Alternatively we could apply algebraic techniques.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^3}{5x^3 - x^2 + 1} &= \lim_{x\to\infty} \frac{x^3}{x^3} \cdot \left(\frac{2}{5 - \frac{1}{x} + \frac{1}{x^3}} \right) \\
&= \lim_{x\to\infty} \frac{2}{5 - \frac{1}{x} + \frac{1}{x^3}} \\
&= \frac{2}{5 - 0 + 0} \\
&= \frac{2}{5}
\end{aligned}$$

So $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^3}{5n^3 - n^2 + 1} = \frac{2}{5}$.

Let's try another example.

Example 7. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = 7n/\ln(n+1)$, converges or diverges.

Workspace:

Solution:

We see that $a_n = f(n)$ for the function $f(x) = 7x/\ln(x+1)$. Then our knowledge of growth rates tells us that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{7x}{\ln(x+1)} = \infty.$$

So the sequence $\{a_n\}$ diverges to ∞.

Alternatively we could apply L'Hopital's rule.

$$\begin{aligned}\lim_{x\to\infty} \frac{7x}{\ln(x+1)} &= \frac{\infty}{\infty} \quad \text{form} \\ &\downarrow \ \text{by L'Hopital's Rule} \\ \lim_{x\to\infty} 7(x+1) &= \infty\end{aligned}$$

So $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{7n}{\ln(n+1)} = \infty$ and the sequence diverges.

Let's try another extremely important example.

Example 8. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \left(1 + \dfrac{k}{n}\right)^n$, converges or diverges.

Workspace:

Solution:

We see that $a_n = f(n)$ for the function $f(x) = \left(1 + \frac{k}{x}\right)^x$. Then our knowledge of limits from Calculus I tells us that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(1 + \frac{k}{x}\right)^x = e^k.$$

So the sequence $\{a_n\}$ converges to the exponential e^k.

Alternatively we could apply L'Hopital's rule to verify the above limit, but from this point on we will define

$$\lim \left(1 + \frac{k}{n}\right)^n := e^k.$$

And now one more example that will be important for us in this course.

Example 9. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = cr^n$ for some real numbers r and c, converges or diverges. This sequence is called a ***geometric sequence***.

Workspace:

Solution:

The sequence is listed as

$$\{a_n\}_{n=0}^{\infty} = \{c, cr, cr^2, cr^3, cr^4, \dots\}.$$

The defining property of this sequence is the common ration between consecutive terms.

$$\dots, cr^{k-1}, \underbrace{cr^k, cr^{k+1}}_{a_{k+1}=ra_k}, \dots$$

The limiting value of this sequence depends on the value of r.

- If $-1 < r < 1$, then $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} cr^k = 0$.
- If $r = 1$, then $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} c1^k = c$.
- If $r > 1$, then $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} cr^k = \infty$.
- If $r \leq -1$, then $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} cr^k$ does not exist.

A Theoretical Result (Optional):

The following content is important for theoretical arguments.

Definition

A sequence $\{a_n\}$ is increasing for all $n \geq n_0$ if

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is decreasing for all $n \geq n_0$ if

$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is a **monotone sequence** for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

We then have the following result about convergence of monotone sequences.

Theorem 5.6: Monotone Convergence Theorem

If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geq n_0$, then $\{a_n\}$ converges.

A proof of the theorem would require theoretical tools beyond the scope of this class. But the following diagram helps to understand why the theorem should be true.

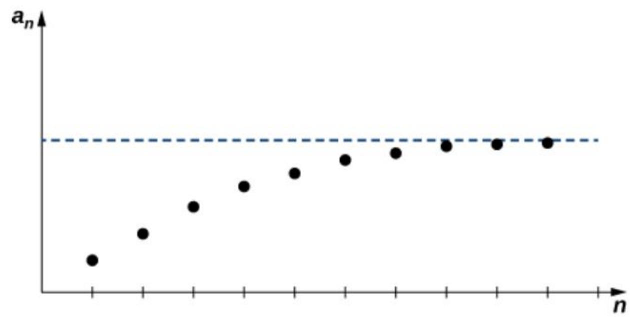

Figure 3: Graphical Representation of Monotone Convergence Theorem

The theorem means if we can show a sequence is bounded and either increasing or decreasing, then we can conclude the sequence converges. But the theorem does not tell us about the actual limiting value of the sequence. It just states that a limit exists.

Infinite Series:

Next we will be studying ***infinite series***. So what is an infinite series? It is simply a non-terminating discrete sum. We can think of it as an infinite sum of the terms in an infinite sequence.

Infinite Series Examples:

Example 10. Determine if the following infinite series converges.

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots$$

Workspace:

Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the kth **partial sum** of the infinite series. The partial sums form a sequence $\{S_k\}$. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we have the **divergence of a series**.

Workspace Cont.:

Solution:

Recognizing this series as corresponding to the decimal expansion of $1/3$ we can see that the series will converge to $1/3$. But let's be more formal and use the given definition.

$$s_k = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k}.$$

Multiplying both sides by $1/10$ we have

$$s_k/10 = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots + \frac{3}{10^{k+1}}.$$

It follows that

$$s_k - \frac{s_k}{10} = \frac{9s_k}{10} = \frac{3}{10} - \frac{3}{10^{k+1}} \quad \Longrightarrow \quad s_k = \frac{10}{9}\left(\frac{3}{10} - \frac{3}{10^{k+1}}\right) = \frac{1}{3} - \frac{1}{3 \cdot 10^k}.$$

To keep things as concrete as possible let's see what this means. If we have added $k = 5$ terms then the 5^{th} partial sum is

$$s_5 = \frac{1}{3} - \frac{1}{3 \cdot 10^5} = 0.33333.$$

If we have added $k = 10$ terms then the 10^{th} partial sum is

$$s_{10} = \frac{1}{3} - \frac{1}{3 \cdot 10^{10}} = 0.3333333333.$$

If we have added $k = 100$ terms then the 100^{th} partial sum is

$$s_{100} = \frac{1}{3} - \frac{1}{3 \cdot 10^{100}} = 0.\underbrace{33333\ldots3333}_{\text{One Hundred 3's}}.$$

It should be clear that as we add more and more terms we are getting arbitrarily close to the value $1/3$.

More formally,

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1}{3} - \frac{1}{3 \cdot 10^k} = \frac{1}{3} - 0 = \frac{1}{3}.$$

So the infinite series converges to $s = 1/3$.

This example is a specific case of what's known as a geometric series. Let's consider the general case for a geometric series (one of the most important sequences).

Example 11. Determine if the following infinite series converges.

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + cr^4 + \cdots$$

Workspace:

Solution:

Let's be very careful and use the formal definition.

$$s_k = c + cr + cr^2 + cr^3 + cr^4 + \cdots + cr^k.$$

Multiplying both sides by 1/10 we have

$$rs_k = cr + cr^2 + cr^3 + cr^4 + \cdots + cr^{k+1}.$$

It follows that

$$s_k - \frac{s_k}{10} = (1-r)s_k = c - cr^{k+1} \quad \Longrightarrow \quad s_k = c\left(\frac{1-r^{k+1}}{1-r}\right).$$

Now formally,

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} c\left(\frac{1-r^{k+1}}{1-r}\right) = c\left(\frac{1}{1-r}\right) \quad \text{if} \quad -1 < r < 1.$$

Otherwise the limit does not exist. So the geometric series converges to $s = \dfrac{c}{1-r}$ only when $|r| < 1$. In this case, we write

$$\sum_{n=0}^{\infty} cr^k = \frac{c}{1-r}.$$

Otherwise, the series diverges. This is a fact that should be memorized from this point onwards.

Let's try another important example.

Example 12. Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

Note*:** This is an important infinite series known as the ***Harmonic Series.

Workspace:

Solution:

Let's group terms of the sum as follows.

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geq 1/2}+\underbrace{\frac{1}{5}+\cdots+\frac{1}{8}}_{\geq 1/2}+\underbrace{\frac{1}{9}+\cdots+\frac{1}{16}}_{\geq 1/2}+\underbrace{\frac{1}{17}+\cdots+\frac{1}{32}}_{\geq 1/2}+\cdots$$

It is easy to see that the sequence of partial sums $\{s_k\}$ is increasing and not bounded above. Therefore we have

$$\lim_{k\to\infty} s_k = \infty$$

and the infinite series diverges to ∞.

Discussion:

We will discover that for most infinite series it may be practically impossible to discover a formula for the k^{th} partial sum. So we will need to spend a good deal of time developing tests for convergence and divergence of infinite series that don't require knowledge of any formula for the partial sums. This content will occupy a good deal of time in the next part of the course. See you there.

Please let me know if you have any questions, comments, or corrections!