Math2411 - Calculus II
Guided Lecture Notes
Divergence and Integral Tests

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Divergence and Integral Tests Introduction:

Our objective is to study some basic tests to determine whether an infinite series converges. So to get started, we recall geometric series

$$\sum_{n=0}^{\infty} cr^n = c = cr + cr^2 + cr^3 + \cdots$$

We were able to find a formula for the k^{th} partial sum s_k and use this to determine when the series converged and to what value does it converge.

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} c \cdot \left(\frac{1-r^{k-1}}{1-r}\right) = \frac{c}{1-r} \quad \text{if } -1 < r < r.$$

For all other values of r the geometric series diverges.

Unfortunately, a formula for the k^{th} partial sum s_k is usually too difficult to find. So we need other methods to check for convergence or divergence.

Divergence Test:

Can you think of a necessary condition on the terms of the series a_n for convergence of $\sum_{n=1}^{\infty} a_n$?

Suppose that a sequence $\sum_{n=1}^{\infty} a_n$ converges. Then $\lim_{k\to\infty} s_k = L$ for some finite number L. Then we have

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(s_k - s_{k-1}\right) = L - L = 0.$$

So if a series converges we conclude that the terms in the series much converge to zero. This is logically equivalent to the statement of the following theorem.

Theorem 5.8: Divergence Test

If $\lim\limits_{n\to\infty} a_n = c \neq 0$ or $\lim\limits_{n\to\infty} a_n$ does not exist, then the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

***Note*:** The converse of the theorem is not true. That is, if $\lim_{n\to\infty} a_n = 0$, we can conclude nothing about the convergence or divergence of the series. For example, $\lim_{n\to\infty} 1/n = 0$ while the series $\sum 1/n$ diverges. We will soon see that even though $\lim_{n\to\infty} 1/n^2 = 0$ the series $\sum 1/n^2$ converges.

Divergence Test Examples:

Example 1. Apply the divergence test to $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3n-1}$.

Workspace:

Solution:

We see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3} \neq 0.$$

So the series diverges by the Divergence Test.

Here is another example.

Example 2. Apply the divergence test to $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^4}$.

Workspace:

Solution:

We see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^4} = 0.$$

So the Divergence Test is inconclusive.

Here is another example.

Example 3. Apply the divergence test to $\sum_{n=0}^{\infty} e^{1/n^2}$.

Workspace:

Solution:

We see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{1/n^2} = e^0 = 1.$$

So the series diverges by the Divergence Test.

Integral Test:

The Integral Test is an important and powerful test and is based on the logic of direct comparison.

Let's return to a familiar example $\sum_{n=1}^{\infty} \frac{1}{n}$ and use the improper integral $\int_{x=1}^{\infty} \frac{1}{x}\, dx$ for comparison.

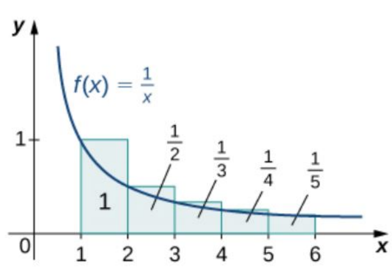

Figure 1: Comparison Method for Integral Test

We can see from the graphic that

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \int_{1}^{k+1} \frac{1}{x} dx.$$

Therefore, for each k, the kth partial sum S_k satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n} > \int_{1}^{k+1} \frac{1}{x} dx = \ln x \Big|_{1}^{k+1} = \ln(k+1) - \ln(1) = \ln(k+1).$$

Then since $\lim_{k\to\infty} \ln(1+k) = \infty$ we must have $\lim_{k\to\infty} S_k = \infty$ and the original infinite series diverges by the Integral Test.

Let's consider another example with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

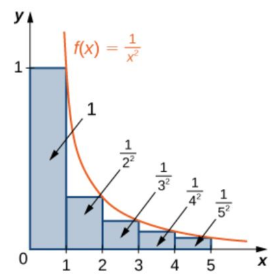

Figure 2: Comparison Method for Integral Test

We can see from the graphic that

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 1 + \int_{1}^{k} \frac{1}{x^2} dx.$$

Therefore, for each k, the kth partial sum S_k satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \int_{1}^{k} \frac{1}{x^2} dx = 1 - \frac{1}{x}\Big|_{1}^{k} = 1 - \frac{1}{k} + 1 = 2 - \frac{1}{k} < 2.$$

We conclude that the sequence of partial sums $\{S_k\}$ is bounded. We also see that $\{S_k\}$ is an increasing sequence:

$$S_k = S_{k-1} + \frac{1}{k^2} \text{ for } k \geq 2.$$

We will use an important fact about infinite sequences. If a sequence $\{S_k\}$ is increasing and bounded then it must converge. In this example is is clear that

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2}$$

is an increasing sequence. That is, $S_k \geq S_{k-1}$. We also see from above that the sequence of partial sums S_k is bounded above by 2. Therefore, the sequence of partial sums $\{S_k\}$ converges and we say the infinite series converges by the Integral Test.

This process can be summed up by the following theorem.

Theorem 5.9: Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

i. f is continuous,

ii. f is decreasing, and

iii. $f(n) = a_n$ for all integers $n \geq N$.

Then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_N^{\infty} f(x)dx$$

both converge or both diverge (see Figure 5.14).

Let's consider a few examples.

Integral Test Examples:

Example 4. Apply the Integral Test to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$.

Workspace:

Solution:

We will compare

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} \text{ and } \int_1^{\infty} \frac{1}{\sqrt{2x-1}} dx.$$

Since

$$\begin{aligned} \int_1^{\infty} \frac{1}{\sqrt{2x-1}} dx &= \lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{2x-1}} dx = \lim_{b \to \infty} \sqrt{2x-1}\Big|_1^b \\ &= \lim_{b \to \infty} \left[\sqrt{2b-1} - 1\right] = \infty, \end{aligned}$$

the integral $\int_1^{\infty} 1/\sqrt{2x-1}\, dx$ diverges, and therefore

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$$

diverges.

We now look to an important type of series called a p-series.

Definition

For any real number p, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a ***p*-series**.

Fortunately we can come to a general solution about p-series.

A p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

Consider a few examples.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

   Since $p = 5/4 > 1$ the infinite series converges as a p-series.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Since $p = 1/2 \leq 1$ the infinite series diverges as a p-series.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

   Since $p = 3 > 1$ the infinite series converges as a p-series.

***Note*:** Observe that we do not necessarily know the the value of a convergent p-series.

Let's argue our result about p-series.

Discussion:

If $p > 0$, then $f(x) = 1/x^p$ is a positive, continuous, decreasing function. Therefore, for $p > 0$, we use the integral test, comparing

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ and } \int_1^{\infty} \frac{1}{x^p} dx.$$

We have already considered the case when $p = 1$. Here we consider the case when $p > 0$, $p \neq 1$. For this case,

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^p} dx = \lim_{b \to \infty} \frac{1}{1-p} x^{1-p} \Big|_1^b = \lim_{b \to \infty} \frac{1}{1-p} \left[b^{1-p} - 1 \right].$$

Because

$$b^{1-p} \to 0 \text{ if } p > 1 \text{ and } b^{1-p} \to \infty \text{ if } p < 1,$$

we conclude that

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}.$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $0 < p < 1$.

Estimation of Series Value:

Theorem 5.10: Remainder Estimate from the Integral Test

Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

i. f is continuous,

ii. f is decreasing, and

iii. $f(n) = a_n$ for all integers $n \geq 1$.

Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

$$S_N + \int_{N+1}^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)dx.$$

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_{N}^{\infty} f(x)dx.$$

This is known as the **remainder estimate**.

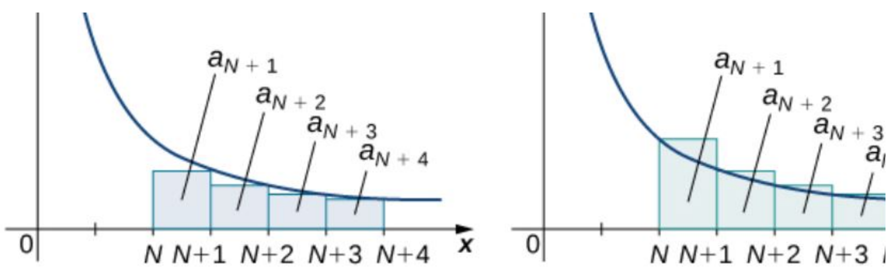

Figure 3: Visualization of Partial Sum Estimate

We can see that

$$\begin{aligned}
R_N &= a_{N+1} + a_{N+2} + a_{N+3} + \cdots \leq \int_{x=N}^{\infty} f(x)\, dx \\
R_N &= a_{N+1} + a_{N+2} + a_{N+3} + \cdots \geq \int_{x=N+1}^{\infty} f(x)\, dx
\end{aligned}$$

So the integral is either an overestimate of R_N or an underestimate of R_n.

Let's work an example.

Example 5. Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^3}$. It turns out that, even though we know the series converges as a p-series, to this day mathematicians have been unable to determine an explicit form the value of the series. So we must estimate the value. Let's use the partial sum S_{10}.

Workspace:

Solution:

We can compute the following.

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} \approx 1.19753.$$

By the remainder estimate, we know

$$R_N < \int_N^{\infty} \frac{1}{x^3} dx.$$

We have

$$\int_{10}^{\infty} \frac{1}{x^3} dx = \lim_{b \to \infty} \int_{10}^{b} \frac{1}{x^3} dx = \lim_{b \to \infty} \left[-\frac{1}{2x^2} \right]_N^b = \lim_{b \to \infty} \left[-\frac{1}{2b^2} + \frac{1}{2N^2} \right] = \frac{1}{2N^2}.$$

$$\vdots$$

....Solution Continued

$$\vdots$$

Then we have the error for our calculation as

$$R_{10} < \frac{1}{2(10)^2} = 0.005$$

So we conclude that

$$1.19253 < \sum_{n=1}^{\infty} \frac{1}{n^3} < 1.20253$$

If we want a more accurate estimation we simply need to add more terms. For example, if we want our error $R_N < 0.00001$ we would could solve

$$R_N < \frac{1}{2N^2} < \frac{1}{100000} \quad \Longrightarrow \quad N > \sqrt{50000} \approx 223.6$$

Then the partial sum $S_{224} = \sum_{n=0}^{224} \frac{1}{n^3}$ would be within 0.00001 of the true value of the series. After computing S_{224} we conclude

$$1.20204 < \sum_{n=1}^{\infty} \frac{1}{n^3} < 1.20206$$

In fact, since the partial sums are increasing we can do better and conclude

$$S_{224} < \sum_{n=1}^{\infty} \frac{1}{n^3} < S_{224} + 0.00001 \quad \Longrightarrow \quad 1.202046983 < \sum_{n=1}^{\infty} \frac{1}{n^3} < 1.202056983$$

Since the series converges we can continue to add more and more terms and become closer and closer to the true value of the series.

This concludes our notes on the Divergence Test and the Integral Test.

Please let me know if you have any questions, comments, or corrections!