Math2411 - Calculus II
Guided Lecture Notes
The Ratio and Root Tests

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The Ratio and Root Tests Introduction:

Many important infinite series can not be evaluated using the techniques we have studied so far. For example, the following series does not satisfy the criteria of any of our current convergence tests.

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

Our objective is to develop tests to handle this type of series. The first test is called the ***Ratio Test***.

The Ratio Test

Theorem 5.16: Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

Let's work a few examples.

Ratio Test Examples:

Example 1. Determine whether $\sum_{n=o}^{\infty} \frac{2^n}{n!}$ converges or diverges.

Workspace:

Here is another example.

Example 2. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converges or diverges.

Workspace:

Here is another example.

Example 3. Determine whether $\sum_{n=1}^{\infty}(-1)^n \frac{(n!)^2}{(2n)!}$ converges or diverges.

Workspace:

Here is another example.

Example 4. Determine whether the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges or diverges.

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We have another test which can be useful when the Ratio Test is difficult to use. It is called the Root Test.

The Root Test:

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}.$$

The Ratio Test will require significant simplification to evaluate the limit. Fortunately, there is another test that is well-suited for this type of series. It is called the ***Root Test***.

Theorem 5.17: Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|}.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

Root Test Examples:

Let's work through an example.

Example 5. Determine if the infinite series $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$ converges or diverges.

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Example 6. Determine if the infinite series $\sum_{n=1}^{\infty} \frac{(-12)^n}{n}$ converges or diverges.

Workspace:

Example 7. Determine if the infinite series $\sum_{n=1}^{\infty} \frac{(-5)^{1+2n}}{2^{5n-3}}$ converges or diverges.

Workspace:

Let's consider an example from the section on Alternating Series Test

Example 8. Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{n+1}\right)^{n^2}$ converges or diverges.

Workspace:

Choosing a Convergence Test

Problem-Solving Strategy: Choosing a Convergence Test for a Series

Consider a series $\sum_{n=1}^{\infty} a_n$. In the steps below, we outline a strategy for determining whether the series converges.

1. Is $\sum_{n=1}^{\infty} a_n$ a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a $p-$series or geometric series? If so, check the power p or the ratio r to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If we are interested in absolute convergence, proceed to step 3, considering the series of absolute values $\sum_{n=1}^{\infty} |a_n|$.
3. Is the series similar to a $p-$series or geometric series? If so, try the comparison test or limit comparison test.
4. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = b_n^n$, try the root test first. Otherwise, try the ratio test first.
5. Use the divergence test. If this test does not provide any information, try the integral test.