

Math2411 - Calculus II
 Guided Lecture Notes
 Trigonometric Substitution

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Trigonometric Substitution Introduction:

Our objective is to integrate function involving square roots of differences and sums of squares.

$$\sqrt{a^2 - x^2} \quad \sqrt{x^2 - a^2} \quad \sqrt{x^2 + a^2}$$

We will need a few basic trig identities.

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1 & \sin(2x) = \sin^2(x) &= \frac{1 - \cos(2x)}{2} \\ \cos^2(x) &= \frac{1 + \cos(2x)}{2} & \tan^2(x) &= 1 - \sec^2(x)\end{aligned}$$

Integrals involving $\sqrt{a^2 - x^2}$:

Let's consider an example together. The general strategy is to make a substitution $x = a \sin(\theta)$.

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a \sin \theta)^2} && \text{Let } x = a \sin \theta \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \text{ Simplify.} \\ &= \sqrt{a^2 - a^2 \sin^2 \theta} && \text{Factor out } a^2. \\ &= \sqrt{a^2(1 - \sin^2 \theta)} && \text{Substitute } 1 - \sin^2 x = \cos^2 x. \\ &= \sqrt{a^2 \cos^2 \theta} && \text{Take the square root.} \\ &= |a \cos \theta| \\ &= a \cos \theta.\end{aligned}$$

Then our square root quantity is converted into a simple trig function. Here is a right triangle for reference.

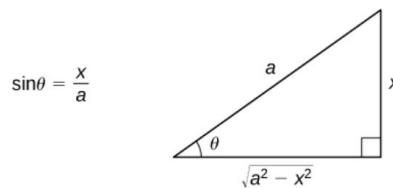


Figure 1: Reference triangle for $\sqrt{a^2 - x^2}$.

Example 1. Evaluate $\int \sqrt{9 - x^2} dx$.

Workspace:

Let's now have you work an example.

Example 2. Evaluate the integral $\int x^3 \sqrt{1 - x^2} dx$.

Workspace:

We can also solve his using u -sub. See if you can solve this using u -sub.

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Integrals involving $\sqrt{x^2 + a^2}$:

Let's consider integrals with a term $\sqrt{x^2 + a^2}$. We let $x = a \tan(\theta)$ and build a reference triangle.

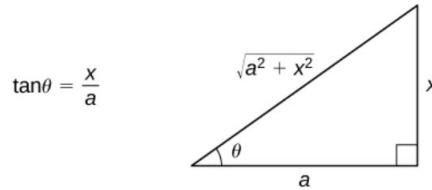


Figure 2: Reference triangle for $\sqrt{x^2 + a^2}$

We can use the following problem solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a \tan\theta$ and $dx = a \sec^2\theta d\theta$. This substitution yields

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a \tan\theta)^2} = \sqrt{a^2(1 + \tan^2\theta)} = \sqrt{a^2 \sec^2\theta} = |a \sec\theta| = a \sec\theta.$$

(Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a \sec\theta| = a \sec\theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from [Figure 3.7](#) to rewrite the result in terms of x . You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

Example 3. Calculate the length of the curve $y = x^2$ on the interval $[0, 1/2]$. Our arclength formula gives us

$$s = \int_{x=a}^{x=b} \sqrt{1 + [f'(x)]^2} dx = \int_{x=0}^{x=1/2} \sqrt{1 + 4x^2} dx.$$

This looks like a tricky integral since there is no obvious u -substitution. We try a trig-sub. We let $x = \frac{1}{2} \tan(\theta)$ so that $dx = \frac{1}{2} \sec^2(\theta) d\theta$. Now continue on your own.

Note: Notice that the quantity $\sqrt{1 + 4x^2}$ suggests a reference triangle with leg lengths of 1 and $2x$ giving a hypotenuse with length of $\sqrt{1 + 4x^2}$. Build your own reference triangle.

Workspace:

Workspace Cont.:

Example 4. Evaluate the integral $\int \frac{1}{\sqrt{1+x^2}} dx$.

Start by letting $x = \tan(\theta)$ and forming the reference triangle.

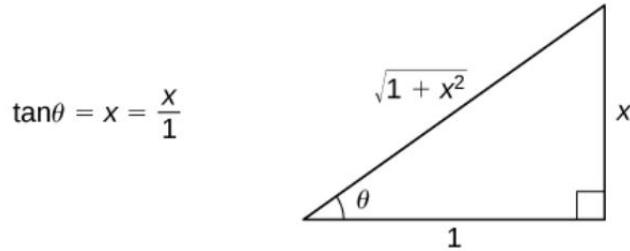


Figure 3: Reference triangle for $\sqrt{1+x^2}$

Now solve the integral by yourself.

Workspace:

Integrals involving $\sqrt{x^2 - a^2}$:

Example 5. Evaluate the integral $\int_{x=3}^{x=5} \sqrt{x^2 - 9} dx$.

The geometry suggests we let $x = 3 \sec(\theta)$ and so then have $dx = 3 \sec(\theta) \tan(\theta) d\theta$.

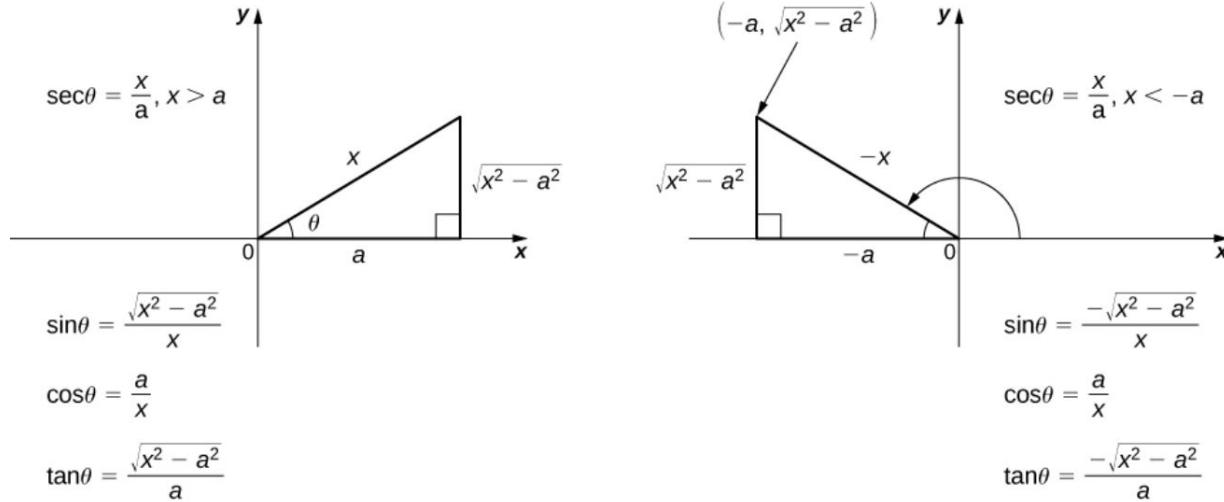


Figure 4: Reference triangle for $\sqrt{x^2 - a^2}$

Notice there are different reference triangles depending on whether the x -values are positive (so we have $x > a$) or negative (so we have $x < -a$). The main consequence is when x is positive we have $\sqrt{x^2 - a^2} = a \tan(\theta)$. When x is negative we have $\sqrt{x^2 - a^2} = -a \tan(\theta)$.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a \sec \theta$ and $dx = a \sec \theta \tan \theta d\theta$. This substitution yields

$$\sqrt{x^2 - a^2} = \sqrt{(a \sec \theta)^2 - a^2} = \sqrt{a^2(\sec^2 \theta + 1)} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta|.$$
 For $x \geq a$, $|a \tan \theta| = a \tan \theta$ and for $x \leq -a$, $|a \tan \theta| = -a \tan \theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from [Figure 3.9](#) to rewrite the result in terms of x . You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

Let's try an example on your own.

Example 6. Evaluate the integral $\int \frac{1}{\sqrt{x^2 - 4}} dx$, assuming that $x < -2$.

Note: The domain of the integrand $f(x) = 1/\sqrt{x^2 - 4}$ is $(-\infty, -2) \cup (2, \infty)$.

Workspace: