Math3810 - Probability
Section 001 - Fall 2025
Practice Problems: WLLN and CLT

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Part A: Weak Law of Large Numbers

1. Basic WLLN Verification. Let $X_1, X_2, \dots$ be i.i.d. with

$$\mathbb{P}(X_i = 2) = \tfrac{1}{2}, \qquad \mathbb{P}(X_i = -1) = \tfrac{1}{2}.$$

Define $S_n = X_1 + \cdots + X_n$ and $\bar{X}_n = S_n/n$.

(a) Compute $E[X_1]$ and $\mathrm{Var}(X_1)$.

(b) Use Chebyshev's inequality to show that $\bar{X}_n \to E[X_1]$ in probability.

2. WLLN for Random Variables with Increasing Variance. Let X_n be independent with

$$E[X_n] = 1, \qquad \mathrm{Var}(X_n) = \frac{1}{n}.$$

Does the LLN hold for $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$?

3. Sample Proportion Convergence. Let $X_1, \dots, X_n, \dots$ be i.i.d. Bernoulli(p). Use the WLLN to show that the sample proportion

$$\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

converges in probability to p. Then compute n such that

$$\mathbb{P}\big(|\hat{p}_n - p| > 0.05\big) < 0.01.$$

4. WLLN When Moments Do Not Exist. Let $X_1, \dots, X_n, \dots$ be i.i.d. with density

$$f(x) = \frac{1}{x^2}, \qquad x \geq 1.$$

Does the Weak Law hold for $\bar{X}_n$? Explain.

5. WLLN for Non-Identically Distributed Variables. Suppose X_k are independent with

$$E[X_k] = 0, \qquad \mathrm{Var}(X_k) = \frac{1}{k}.$$

Consider $Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k$. Does $Y_n \to 0$ in probability?

Part B: Central Limit Theorem

6. CLT for Bernoulli Variables. Let $X_i \sim \mathrm{Bernoulli}(p)$. Derive the CLT approximation for

$$\mathbb{P}\left(\frac{\hat{p}_n - p}{\sqrt{p(1-p)/n}} \le z\right).$$

Approximate

$$\mathbb{P}(|\hat{p}_n - p| < 0.02)$$

when $p = 0.3$ and $n = 400$.

7. CLT for Poisson Variables. Let $X_i \sim \mathrm{Poisson}(\lambda)$. Approximate

$$\mathbb{P}(S_{200} \le 230),$$

where $S_{200} = \sum_{i=1}^{200} X_i$, using the CLT with continuity correction.

8. CLT Approximation Error. If $X_i \sim N(\mu, \sigma^2)$, redo the CLT derivation and identify the limiting distribution of S_n. Explain why the CLT is unnecessary here.

9. CLT for Uniform Distribution. Let $X_i \sim \mathrm{Uniform}(0,1)$. Approximate

$$\mathbb{P}(S_{50} > 30),$$

where $S_{50} = X_1 + \cdots + X_{50}$.

10. Lindeberg Condition Check. Let X_k be independent with

$$\mathbb{P}(X_k = k) = \frac{1}{2k}, \qquad \mathbb{P}(X_k = -k) = \frac{1}{2k}, \qquad \mathbb{P}(X_k = 0) = 1 - \frac{1}{k}.$$

(a) Compute $E[X_k]$ and $\mathrm{Var}(X_k)$.

(b) Let $S_n = \sum_{k=1}^{n} X_k$. Does the CLT apply? Check whether the Lindeberg condition holds.

11. CLT + Delta Method. Let $X_i \sim \mathrm{Exponential}(1)$. Approximate the distribution of

$$\sqrt{n}\big(\log(\bar{X}_n) - \log(1)\big).$$

12. CLT for Non-Rectangular Domains. Let (X_i, Y_i) be i.i.d. points uniformly distributed on the unit disk. Consider

$$R_n = \frac{1}{n}\sum_{i=1}^{n}\sqrt{X_i^2 + Y_i^2}.$$

(a) Compute $\mu = E[\sqrt{X_1^2 + Y_1^2}]$.

(b) Using the CLT, approximate the distribution of $\sqrt{n}(R_n - \mu)$.