Math3810 - Probability
Section 001 - Fall 2025
Practice Problems: WLLN and CLT (Solutions)

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics - Dr. Robert Rostermundt

Part A: Weak Law of Large Numbers

1. Basic WLLN Verification. Let $X_1, X_2, \ldots$ be i.i.d. with

$$\mathbb{P}(X_i = 2) = \tfrac{1}{2}, \qquad \mathbb{P}(X_i = -1) = \tfrac{1}{2}.$$

Define $S_n = X_1 + \cdots + X_n$ and $\bar{X}_n = S_n/n$.

  (a) Compute $E[X_1]$ and $\text{Var}(X_1)$.

  (b) Use Chebyshev's inequality to show that $\bar{X}_n \to E[X_1]$ in probability.

2. WLLN for Random Variables with Increasing Variance. Let X_n be independent with

$$E[X_n] = 1, \qquad \text{Var}(X_n) = \frac{1}{n}.$$

Does the LLN hold for $\bar{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k$?

3. Sample Proportion Convergence. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. Bernoulli(p). Use the WLLN to show that the sample proportion

$$\hat{p}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

converges in probability to p. Then compute n such that

$$\mathbb{P}\big(|\hat{p}_n - p| > 0.05\big) < 0.01.$$

4. WLLN When Moments Do Not Exist. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. with density

$$f(x) = \frac{1}{x^2}, \qquad x \geq 1.$$

Does the Weak Law hold for $\bar{X}_n$? Explain.

5. WLLN for Non-Identically Distributed Variables. Suppose X_k are independent with

$$E[X_k] = 0, \qquad \text{Var}(X_k) = \frac{1}{k}.$$

Consider $Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k$. Does $Y_n \to 0$ in probability?

Part B: Central Limit Theorem

6. CLT for Bernoulli Variables. Let $X_i \sim \text{Bernoulli}(p)$. Derive the CLT approximation for

$$\mathbb{P}\left(\frac{\hat{p}_n - p}{\sqrt{p(1-p)/n}} \leq z \right).$$

Approximate

$$\mathbb{P}(|\hat{p}_n - p| < 0.02)$$

when $p = 0.3$ and $n = 400$.

7. CLT for Poisson Variables. Let $X_i \sim \text{Poisson}(\lambda)$. Approximate

$$\mathbb{P}(S_{200} \leq 230),$$

where $S_{200} = \sum_{i=1}^{200} X_i$, using the CLT with continuity correction.

8. CLT Approximation Error. If $X_i \sim N(\mu, \sigma^2)$, redo the CLT derivation and identify the limiting distribution of S_n. Explain why the CLT is unnecessary here.

9. CLT for Uniform Distribution. Let $X_i \sim \text{Uniform}(0,1)$. Approximate

$$\mathbb{P}(S_{50} > 30),$$

where $S_{50} = X_1 + \cdots + X_{50}$.

10. Lindeberg Condition Check. Let X_k be independent with

$$\mathbb{P}(X_k = k) = \frac{1}{2k}, \qquad \mathbb{P}(X_k = -k) = \frac{1}{2k}, \qquad \mathbb{P}(X_k = 0) = 1 - \frac{1}{k}.$$

(a) Compute $E[X_k]$ and $\text{Var}(X_k)$.

(b) Let $S_n = \sum_{k=1}^{n} X_k$. Does the CLT apply? Check whether the Lindeberg condition holds.

11. CLT + Delta Method. Let $X_i \sim \text{Exponential}(1)$. Approximate the distribution of

$$\sqrt{n}\big(\log(\bar{X}_n) - \log(1) \big).$$

12. CLT for Non-Rectangular Domains. Let (X_i, Y_i) be i.i.d. points uniformly distributed on the unit disk. Consider

$$R_n = \frac{1}{n} \sum_{i=1}^{n} \sqrt{X_i^2 + Y_i^2}.$$

(a) Compute $\mu = E[\sqrt{X_1^2 + Y_1^2}]$.

(b) Using the CLT, approximate the distribution of $\sqrt{n}(R_n - \mu)$.

Solutions

1. Basic WLLN Verification. *(a)* The expectation is computed directly from the definition:

$$E[X_1] = \sum_x x\, \mathbb{P}(X_1 = x) = 2 \cdot \tfrac{1}{2} + (-1) \cdot \tfrac{1}{2} = \tfrac{1}{2}.$$

Next compute the second moment:

$$E[X_1^2] = 2^2 \cdot \tfrac{1}{2} + (-1)^2 \cdot \tfrac{1}{2} = \tfrac{1}{2}(4) + \tfrac{1}{2}(1) = \tfrac{5}{2}.$$

Hence

$$\mathrm{Var}(X_1) = E[X_1^2] - (E[X_1])^2 = \tfrac{5}{2} - \left(\tfrac{1}{2}\right)^2 = \frac{9}{4}.$$

(b) Since the X_i are i.i.d.,

$$E[\bar{X}_n] = E[X_1] = \tfrac{1}{2}, \qquad \mathrm{Var}(\bar{X}_n) = \frac{\mathrm{Var}(X_1)}{n} = \frac{9}{4n}.$$

By Chebyshev's inequality,

$$\mathbb{P}\left(|\bar{X}_n - \tfrac{1}{2}| > \varepsilon\right) \le \frac{\mathrm{Var}(\bar{X}_n)}{\varepsilon^2} = \frac{9}{4n\varepsilon^2}.$$

As $n \to \infty$, this bound converges to 0, so

$$\bar{X}_n \xrightarrow{P} \tfrac{1}{2}.$$

—

2. WLLN for Random Variables with Increasing Variance. First compute the mean:

$$E[\bar{X}_n] = \frac{1}{n} \sum_{k=1}^{n} E[X_k] = 1.$$

Using independence,

$$\mathrm{Var}(\bar{X}_n) = \frac{1}{n^2} \sum_{k=1}^{n} \mathrm{Var}(X_k) = \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k}.$$

Since $\sum_{k=1}^{n} \frac{1}{k} \sim \log n$, we obtain

$$\mathrm{Var}(\bar{X}_n) \sim \frac{\log n}{n^2} \to 0.$$

Chebyshev's inequality then implies

$$\bar{X}_n \xrightarrow{P} 1.$$

Thus the Weak Law holds even though the individual variances do not vanish.

—

3. Sample Proportion Convergence. For Bernoulli(p) variables,

$$E[X_i] = p, \qquad \mathrm{Var}(X_i) = p(1-p).$$

Hence

$$E[\hat{p}_n] = p, \qquad \mathrm{Var}(\hat{p}_n) = \frac{p(1-p)}{n}.$$

By the WLLN,

$$\hat{p}_n \xrightarrow{P} p.$$

For the probability bound, Chebyshev's inequality gives

$$\mathbb{P}(|\hat{p}_n - p| > 0.05) \leq \frac{p(1-p)}{n(0.05)^2}.$$

To make this less than 0.01, it suffices that

$$\frac{p(1-p)}{n(0.05)^2} < 0.01 \implies n > \frac{p(1-p)}{0.000025}.$$

This bound is conservative but does not rely on any normal approximation.

—

4. WLLN When Moments Do Not Exist. We compute

$$E[X_1] = \int_1^\infty x \frac{1}{x^2}\, dx = \int_1^\infty \frac{1}{x}\, dx = \infty.$$

Since the expectation does not exist, the classical WLLN does not apply. In fact, this heavy-tailed distribution allows rare but extremely large observations that dominate the sample average, preventing stabilization of $\bar{X}_n$.

—

5. WLLN for Non-Identically Distributed Variables. We have

$$E[Y_n] = 0, \qquad \mathrm{Var}(Y_n) = \frac{1}{n^2}\sum_{k=1}^n \frac{1}{k} \sim \frac{\log n}{n^2} \to 0.$$

Applying Chebyshev's inequality,

$$\mathbb{P}(|Y_n| > \varepsilon) \leq \frac{\mathrm{Var}(Y_n)}{\varepsilon^2} \to 0.$$

Thus

$$Y_n \xrightarrow{P} 0.$$

—

6. CLT for Bernoulli Variables. By the CLT,

$$\frac{\hat{p}_n - p}{\sqrt{p(1-p)/n}} \Rightarrow N(0,1).$$

For $p = 0.3$ and $n = 400$,

$$\sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.21}{400}} \approx 0.0229.$$

Hence

$$\mathbb{P}(|\hat{p}_n - p| < 0.02) \approx \mathbb{P}\left(|Z| < \frac{0.02}{0.0229}\right) = \mathbb{P}(|Z| < 1.746) \approx 0.919.$$

—

7. CLT for Poisson Variables. If $X_i \sim \text{Poisson}(1)$, then

$$S_{200} \sim \text{Poisson}(200), \qquad E[S_{200}] = 200, \quad \text{Var}(S_{200}) = 200.$$

Using the normal approximation with continuity correction,

$$\mathbb{P}(S_{200} \leq 230) \approx \mathbb{P}\left(Z \leq \frac{230.5 - 200}{\sqrt{200}}\right) = \mathbb{P}(Z \leq 2.162) \approx 0.985.$$

—

8. CLT Approximation Error. If $X_i \sim N(\mu, \sigma^2)$, then the sum

$$S_n = \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$$

exactly, for every n. Thus the limiting distribution obtained from the CLT coincides with the exact finite-sample distribution. In this case the CLT provides no approximation—it is already exact.

—

9. CLT for Uniform Distribution. For $X_i \sim U(0,1)$,

$$E[X_i] = \tfrac{1}{2}, \qquad \text{Var}(X_i) = \tfrac{1}{12}.$$

Therefore

$$E[S_{50}] = 25, \qquad \text{Var}(S_{50}) = \frac{50}{12}.$$

By the CLT,

$$\mathbb{P}(S_{50} > 30) \approx \mathbb{P}\left(Z > \frac{30 - 25}{\sqrt{50/12}}\right) = \mathbb{P}(Z > 2.45) \approx 0.0071.$$

—

10. Lindeberg Condition Check. *(a)* By symmetry,

$$E[X_k] = 0.$$

The variance is

$$\mathrm{Var}(X_k) = E[X_k^2] = k^2 \left(\frac{1}{2k} + \frac{1}{2k} \right) = k.$$

(b) Let $S_n = \sum_{k=1}^{n} X_k$. Then

$$\mathrm{Var}(S_n) = \sum_{k=1}^{n} \mathrm{Var}(X_k) = \sum_{k=1}^{n} k \sim \frac{n^2}{2}.$$

Thus the natural scaling for a CLT would be S_n/n.

However, with probability $1/k$, the variable X_k takes values of size k. These jumps are *not negligible* relative to the standard deviation of S_n, which is of order n. Consequently, for any $\varepsilon > 0$,

$$\mathcal{E}\left[X_k^2 \mathbf{1}_{\{|X_k|>\varepsilon n\}}\right] \approx k^2 \cdot \frac{1}{k} = k,$$

so the Lindeberg condition fails. Hence the CLT does *not* apply.

—

11. CLT + Delta Method. For $X_i \sim \mathrm{Exponential}(1)$,

$$E[X_i] = 1, \qquad \mathrm{Var}(X_i) = 1.$$

By the CLT,

$$\sqrt{n}(\bar{X}_n - 1) \Rightarrow N(0,1).$$

Let $g(x) = \log x$, which is differentiable at $x = 1$ with $g'(1) = 1$. By the delta method,

$$\sqrt{n}(\log \bar{X}_n - \log 1) \Rightarrow N\left(0, (g'(1))^2\right) = N(0,1).$$

Thus the asymptotic distribution is standard normal.

—

12. CLT for Non-Rectangular Domains. *(a)* Let $R = \sqrt{X^2 + Y^2}$. For a point uniformly distributed in the unit disk, the radial density is

$$f_R(r) = 2r, \qquad 0 \le r \le 1.$$

Hence

$$E[R] = \int_0^1 r(2r)\,dr = \frac{2}{3}.$$

(b) We compute

$$E[R^2] = \int_0^1 r^2(2r)\,dr = \frac{1}{2},$$

so

$$\mathrm{Var}(R) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}.$$

Since the R_i are i.i.d., the CLT yields

$$\sqrt{n}(R_n - \tfrac{2}{3}) \Rightarrow N\left(0, \frac{1}{18}\right).$$