Math3810 - Probability
Section 001 - Fall 2025
Notes: Continuity of Probability Measure

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Introduction: Suppose that we flip a fair coin (or even a biased coin) infinitely many times, what is the probability that we never see heads in infinitely many tosses? Our intuition suggests that the probability should be zero. But in order to formally answer this question we will use continuity of probability measures.

Theorem 1 (Continuity)**.** *Let* $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ *be a probability space and* $A_1, A_2, A_3, A_4, \ldots$ *be a countable sequence of events.*

1. *If the sequence is increasing so that* $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ *then*

$$\mathbb{P}\left(\bigcup_n A_n\right) = \lim_{n\to\infty} \mathbb{P}\big(A_n\big).$$

2. *If the sequence is decreasing so that* $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ *then*

$$\mathbb{P}\left(\bigcap_n A_n\right) = \lim_{n\to\infty} \mathbb{P}\big(A_n\big).$$

The proof of this result is fairly straightforward using the countable additivity of a probability measure.

The Solution: Now back to our original problem. Let A be the event that no heads appear in the infinite sequence of coin tosses. Then let A_n be the event that the first n tosses do not contain a head. Then it is easy to see that the sequence $A_1, A_2, A_3, \ldots$ is decreasing since $A_k \supseteq A_{k+1}$ for all k. If there are no heads in the first $k+1$ tosses then certainly there were no heads in the first k tosses. Additionally, we have

$$A = \bigcap_n A_n.$$

So by continuity we see that

$$\mathbb{P}\big(A\big) = \mathbb{P}\left(\bigcap_n A_n\right) = \lim_{n\to\infty} \mathbb{P}\big(A_n\big) = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0.$$

Notice that this probability will be the same even with a biased coin where probability p of tails on a single toss satisfies $0 < p < 1$. The same probability of zero will apply to any specific sequence of head and tails in an infinite sequence of coin tosses.