Math3810 - Probability
Section 001 - Fall 2025
Notes: Joint Distribution Practice Solutions

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics - Dr. Robert Rostermundt

A. Discrete Joint Distributions

Problem 1. Two fair dice are rolled. Let X be the minimum and Y the maximum.

(a) Find the joint pmf $p_{X,Y}(x, y)$.

(b) Compute the marginal pmfs $p_X(x)$ and $p_Y(y)$.

(c) Are X and Y independent?

Solution:

Given the four-point distribution, the marginals follow by summing the joint probabilities. Independence fails because $p_{X,Y}(0,0) \neq p_X(0)p_Y(0)$. (Full explicit computations omitted for brevity unless you'd like them fully spelled out.)

Problem 2. Let N be Bernoulli(1/2). Conditional on $N = n$:

- if $n = 0$: X and Y are independent Bernoulli(1/4),
- if $n = 1$: X and Y are independent Bernoulli(3/4).

(a) Find the joint pmf $p_{X,Y}(x, y)$.

(b) Find the marginals.

(c) Are X and Y independent?

Solution:

The constant is

$$c^{-1} = \int_0^1 \int_0^1 (x+y)\,dy\,dx = 1.$$

Thus $c = 1$. The marginal:

$$f_X(x) = \int_0^1 (x+y)\,dy = x + \frac{1}{2}.$$

Probability:

$$\mathbb{P}(Y > X/2) = \int_0^1 \int_{x/2}^1 (x+y)\,dy\,dx.$$

This evaluates to

$$= \int_0^1 \left[\frac{1}{2}(1-x/2)^2 + x(1-x/2)\right] dx = \frac{17}{24}.$$

The conditional:

$$f_{Y|X}(y \mid x) = \frac{x+y}{x+\frac{1}{2}}.$$

Problem 3. Suppose (X, Y) has pmf $p_{X,Y}(x, y) = c(x+y)$ over $x \in \{1, 2\}, \quad y \in \{1, 2, 3\}$.

(a) Find c.

(b) Find the marginal pmfs.

(c) Compute $P(X < Y)$.

Solution:

$$E[X] = \sum_{x=0}^{3} \left(x \sum_{y=0}^{3} p(x,y) \right).$$

Only the given entries contribute:

$$E[X] = 0 \cdot 0.1 + 0 \cdot 0.1 + 1 \cdot 0.1 + 2 \cdot 0.2 = 0.5.$$

Similarly

$$E[Y] = 0 \cdot 0.1 + 0 \cdot 0.1 + 2 \cdot 0.1 + 3 \cdot 0.2 = 0.8.$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y].$$

Compute:

$$E[XY] = 0 \cdot 0.1 + 0 \cdot 0.1 + 2 \cdot 0.1 + 6 \cdot 0.2 = 1.4.$$

Thus

$$\mathrm{Cov}(X,Y) = 1.4 - (0.5)(0.8) = 1.0.$$

Problem 4. A biased coin has probability p of heads. Let X be the number of heads in the first two flips and Y the number in the first three.

(a) Find the joint pmf $p_{X,Y}(x,y)$.

(b) Find the marginals.

(c) Find $p_{Y|X}(y \mid x)$.

Solution:

$$\mathbb{P}(X > 1) = \int_1^2 \int_0^{2-x} k \, dy \, dx = k \int_1^2 (2-x) \, dx = k \left[2x - \frac{x^2}{2} \right]_1^2 = 0.5k.$$

$$E[Y] = \int_0^2 \int_0^{2-x} yk \, dy \, dx = k \int_0^2 \frac{1}{2}(2-x)^2 dx = k \cdot \frac{4}{3}.$$

Problem 5. Let (X,Y) have joint pmf 1/8 on the eight points

$$(0,0), (0,1), (1,0), (1,1), (0,2), (2,0), (2,1), (1,2).$$

(a) Create the joint distribution table.

(b) Compute marginals.

(c) Are X and Y independent?

Solution:

$$f_X(x) = \int_0^2 (x^2 + y)\,dy = x^2 \cdot 2 + \frac{1}{2}(2^2) = 2x^2 + 2.$$

$$f_Y(y) = \int_0^1 (x^2 + y)\,dx = \frac{1}{3} + y.$$

$$\mathbb{P}(X > Y) = \int_0^1 \int_y^1 (x^2 + y)\,dx\,dy.$$

Evaluate:

$$= \int_0^1 \left[\frac{1}{3}(1 - y^3) + y(1 - y)\right] dy = \frac{29}{45}.$$

B. Continuous Joint Distributions

Problem 6. Suppose $f(x, y) = k(x + y), \quad 0 < x < 1,\ 0 < y < 1.$

(a) Find k.

(b) Find the marginals.

(c) Compute $P(X < Y)$.

Solution:

The region is the triangle $0 < y < x < 1$.

$$k^{-1} = \int_0^1 \int_0^x xye^{-x^2} dy\,dx = \int_0^1 xe^{-x^2} \frac{x^2}{2} dx = \frac{1}{2} \int_0^1 x^3 e^{-x^2} dx.$$

Use $u = x^2$:

$$= \frac{1}{4} \int_0^1 ue^{-u} du = \frac{1}{4}(1 - 2/e).$$

Then invert to get k.

Problem 7. Let (X, Y) uniform on $0 < y < x < 2$.

(a) Find $f_{X,Y}(x, y)$.

(b) Find the marginals.

(c) Compute $P(Y < 1)$.

(d) Find $f_{Y|X}(y \mid x)$.

Solution:

Support: $x \in \{1, 2, 3\}$, $y \in \{1, 2, 3\}$ with $x + y \leq 4$:
Pairs are $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)$.

$$c^{-1} = \sum (xy) = 1 + 2 + 3 + 2 + 4 + 3 = 15.$$

$$p_Y(y) = \sum_x cxy.$$

Example: $p_Y(2) = c(1 \cdot 2 + 2 \cdot 2) = 6c$.

$$\mathbb{P}(X < Y) = c[(1, 2), (1, 3), (2, 3 \text{ invalid})].$$

Independence fails because support is triangular.

Problem 8. Let $X \sim \text{Exp}(1)$, $Y \sim \text{Exp}(2)$ be independent random variables.

(a) Find the joint pdf.

(b) Compute $P(X < Y)$.

(c) Find the pdf of $Z = X + Y$.

(d) Find $f_{X,Z}(x, z)$.

Solution:

$$f_{X,Y}(x, y) = c(x + y)e^{-(x+y)}.$$
$$c^{-1} = \int_0^\infty \int_0^\infty (x + y)e^{-(x+y)} dydx = 2.$$

Thus $c = 1/2$.

$$\text{Cov}(X, Y) = 0$$

because X and Y are independent (the joint factorizes).

$$\mathbb{P}(X > 1, Y < 2) = \left(e^{-1}\right)\left(1 - e^{-2}\right).$$

Problem 9. Suppose $f_{X,Y}(x, y) = 6(1 - y), \quad 0 < x < y < 1$.

(a) Verify the pdf is valuid.

(b) Find both marginals.

(a) Compute $P(Y - X > 1/4)$.

Solution:

Normalizing triangle:

$$k^{-1} = \int_0^2 \int_0^{1-x/2} dy\,dx = \int_0^2 (1 - \tfrac{x}{2})dx = 1.$$

Compute marginals by integrating 1 over the slice. Expectation:

$$E[X] = \int_0^2 x(1 - x/2)dx = \frac{2}{3}.$$

Problem 10. (X, Y) jointly normal with

$$E[X] = 0,\ E[Y] = 2,\ \mathrm{Var}(X) = 1,\ \mathrm{Var}(Y) = 4,\ \mathrm{Cov}(X, Y) = 1.$$

(a) Write the joint pdf.

(b) Find the marginals.

(c) $X \mid Y = y$.

(d) Compute $P(X > 0, Y > 2)$.

Solution:

Region $x > 0$, $0 < y < 1/x$:

$$c^{-1} = \int_0^\infty \int_0^{1/x} e^{-(x+y)} dy\,dx = \int_0^\infty e^{-x}(1 - e^{-1/x})dx.$$

This integral is convergent. Then marginals follow by integrating.

C. Mixed Discrete–Continuous

Problem 11. $N \sim \mathrm{Poisson}(\lambda)$ and $X|N = n \sim \mathrm{Gamma}(n + 1, 1)$.

(a) Find the joint distribution.

(b) Find the marginal of X.

(c) Compute $P(N = n \mid X = x)$.

Solution:

Geometric region area $= 1/2$.

$$c^{-1} = \int_0^1 \int_0^{1-x} 6xy\, dy\, dx = 1.$$

$$P(X + Y < 1/2) = \int_0^{1/2} \int_0^{1/2-x} 6xy\, dy\, dx.$$

Problem 12. With prob. 1/2: $(X, Y) = (0, U)$, $U \sim \text{Unif}(0, 2)$. With prob. 1/2: $(X, Y) = (V, 0)$, $V \sim \text{Unif}(0, 2)$.

(a) Find the joint distribution.

(b) Find both marginals.

(c) Compute $P(X = 0 \text{ or } Y = 0)$.

Solution:

Points are $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2)$.

$$c^{-1} = 1 + 2 + 3 + 2 + 4 = 12.$$

Then compute marginals and the event $|X - Y| \leq 1$ by summation.

D. Additional Problems (Double Integrals / Non-Rectangular Regions)

Problem 13. Let $f_{X,Y}(x, y) = k(x^2 + y^2)$ on the quarter disk

$$x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 4.$$

(a) Find k.

(b) Find $f_X(x)$.

(c) Compute $P(X + Y < 1)$.

da) Find $f_{Y|X}(y \mid x)$.

Solution:

Region is quarter circle radius 2.

$$k^{-1} = \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) dy\, dx.$$

Convert to polar:

$$k^{-1} = \int_0^{\pi/2} \int_0^2 r^2 r\, dr\, d\theta = \frac{\pi}{8}(2^4) = 2\pi.$$

Thus

$$k = \frac{1}{2\pi}.$$

Marginal:

$$f_X(x) = \int_0^{\sqrt{4-x^2}} \tfrac{1}{2\pi}(x^2 + y^2)\, dy.$$

$$\mathbb{P}(X + Y < 1) = \int_0^1 \int_0^{1-x} \tfrac{1}{2\pi}(x^2 + y^2)\, dy\, dx.$$

Conditional:

$$f_{Y|X}(y \mid x) = \frac{x^2 + y^2}{\displaystyle\int_0^{\sqrt{4-x^2}} (x^2 + y^2) dy}.$$

Problem 14. Let $f(x, y) = c(2x + 3y)$ on the triangle

$$0 < x < 2, \quad 0 < y < 1 - x/2.$$

(a) Find c.

(b) Find $f_Y(y)$.

(c) Compute $P(Y < X/2)$.

(d) Find $f_{X|Y}(x \mid y)$.

Solution:

Triangle $0 < x < 2$, $0 < y < 1 - x/2$.

$$c^{-1} = \int_0^2 \int_0^{1-x/2} (2x + 3y)\, dy\, dx.$$

Compute inside:

$$\int_0^{1-x/2} (2x + 3y) dy = 2x(1 - x/2) + \tfrac{3}{2}(1 - x/2)^2.$$

Integrate over x on $(0, 2)$ to obtain c.
Marginal:

$$f_Y(y) = \int_{x=0}^{2-2y} c(2x + 3y)\, dx.$$

$$\mathbb{P}(Y < X/2) = \int_0^2 \int_0^{\min(1-x/2, x/2)} c(2x + 3y)\, dy\, dx.$$

Conditional:

$$f_{X|Y}(x \mid y) = \frac{c(2x + 3y)}{\int_0^{2-2y} c(2u + 3y)\, du}.$$

Problem 15. Let $p(x, y) = c(x + y)$ for integer lattice points

$$x \geq 1,\ y \geq 1,\ x + y \leq 6.$$

(a) Find c.

(b) Find $p_Y(y)$.

(c) Compute $P(X > Y)$.

(d) Are X, Y independent?

Solution:

Support: integers $x, y \geq 1$, $x + y \leq 6$.
Compute

$$c^{-1} = \sum_{x=1}^{5} \sum_{y=1}^{6-x} (x+y) = \sum_{s=2}^{6} \sum_{x=1}^{s-1} s = \sum_{s=2}^{6} s(s-1) = 70.$$

Thus

$$c = \frac{1}{70}.$$

$$p_Y(y) = \sum_{x=1}^{5-y} \frac{x+y}{70}.$$

$$\mathbb{P}(X > Y) = \sum_{x>y} \frac{x+y}{70}.$$

Not independent: support is triangular.

Problem 16. Suppose $f_{X,Y}(x, y) = \lambda^2 e^{-\lambda(x+2y)}$ on the wedge

$$0 < x, \; 0 < y < x.$$

(a) Normalize the pdf.

(b) Compute $f_X(x)$.

(c) Compute $P(Y < X/3)$.

(d) Compute $E[Y]$.

Solution:

Region $0 < y < x < \infty$.
Check normalization:

$$\int_0^\infty \int_0^x \lambda^2 e^{-\lambda(x+2y)} dy\, dx = \int_0^\infty \lambda^2 e^{-\lambda x} \left[\frac{1}{2}e^{-\lambda x} - \frac{1}{2}e^{-2\lambda x}\right] dx.$$

Simplify:

$$= \frac{\lambda^2}{2} \int_0^\infty \left(e^{-2\lambda x} - e^{-3\lambda x}\right) dx = \frac{\lambda^2}{2}\left(\frac{1}{2\lambda} - \frac{1}{3\lambda}\right) = \frac{1}{12}.$$

Thus multiply by 12 to normalize; true pdf is

$$12\lambda^2 e^{-\lambda(x+2y)}.$$

Marginal:

$$f_X(x) = \int_0^x 12\lambda^2 e^{-\lambda(x+2y)} dy = 12\lambda^2 e^{-\lambda x} \frac{1 - e^{-2\lambda x}}{2\lambda}.$$

$$\mathbb{P}(Y < X/3) = \int_0^\infty \int_0^{x/3} 12\lambda^2 e^{-\lambda(x+2y)} dy\, dx.$$

$$E[Y] = \int_0^\infty \int_0^x y \cdot 12\lambda^2 e^{-\lambda(x+2y)} dy\, dx.$$

Please let me know if you have any questions, comments, or corrections!