Math3810 - Probability
Section 001 - Fall 2025
Notes: Derived Distribution for $Y = aX + b$

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The Problem:

Let X be a continuous random variable with probability density function (PDF) $f_X(x)$. We define a new random variable:

$$Y = aX + b,$$

where $a \neq 0$ and b are constants. We are interested in the distribution and density function $f_Y(y)$ for the random variable Y. The PDF of Y can be found using the change-of-variable formula:

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}.$$

Or the derivation can be completed using the distribution function $F_Y(y)$.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) \\ &= P\Big(X \leq \frac{y-b}{a}\Big), \quad \text{if } a > 0, \\ &= F_X\Big(\frac{y-b}{a}\Big), \end{aligned}$$

or

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) \\ &= P\Big(X \geq \frac{y-b}{a}\Big), \quad \text{if } a < 0, \\ &= 1 - F_X\Big(\frac{y-b}{a}\Big). \end{aligned}$$

And then we have using the chain rule that

$$f_Y(y) = \frac{d}{dy}F_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}.$$

Then by properties of expected value and variance we have

$$E[Y] = aE[X] + b, \quad Var(Y) = a^2Var(X).$$

Special Example: Normal Distribution

Suppose that $X \sim N(\mu, \sigma^2)$. Then

$$Y = aX + b \sim N(a\mu + b, a^2\sigma^2).$$

To see this, using the formula above we have for all $y \in \mathbb{R}$ that

$$\begin{aligned} f_Y(y) &= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \\ &= \frac{1}{|a|\sigma\sqrt{2\pi}} Exp\left[-\frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}\right] \\ &= \frac{1}{|a|\sigma\sqrt{2\pi}} Exp\left(-\frac{\Big(y - (a\mu + b)\Big)^2}{2a^2\sigma^2}\right) \end{aligned}$$

which is the density for a normal distribution $N\big(a\mu + b, a^2\sigma^2\big)$. We see that the normal distribution is closed under this affine transformation $Y = aX + b$.

Example 1: Simple Transformation Let $X \sim N(10, 4)$ and define $Y = 3X - 5$.

- Mean: $\mathbb{E}[Y] = 3 \cdot 10 - 5 = 25$
- Variance: $\text{Var}(Y) = 3^2 \cdot 4 = 36$
- Standard deviation: $\sigma_Y = \sqrt{36} = 6$
- Conclusion: $Y \sim N(25, 36)$

Example 2: Negative Scaling Let $X \sim N(5, 9)$ and $Y = -2X + 7$.

- Mean: $\mathbb{E}[Y] = -2 \cdot 5 + 7 = -3$
- Variance: $\text{Var}(Y) = (-2)^2 \cdot 9 = 36$
- Standard deviation: $\sigma_Y = 6$
- Conclusion: $Y \sim N(-3, 36)$

Example 3: Standardizing a Normal Random Variable Let $X \sim N(\mu, \sigma^2)$ and define

$$Z = \frac{X - \mu}{\sigma}.$$

- Mean: $\mathbb{E}[Z] = \frac{\mathbb{E}[X] - \mu}{\sigma} = 0$
- Variance: $\text{Var}(Z) = \frac{\text{Var}(X)}{\sigma^2} = 1$
- Conclusion: $Z \sim N(0, 1)$, the standard normal variable.

Let's use this for

$$X \sim N(\mu = 12, \sigma^2 = 16)$$

so that $\sigma = 4$. We want to compute

$$P(X < 15).$$

Then let

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 12}{4} \sim N(0, 1)$$

and we have

$$P(X < 15) = P\left(\frac{X - 12}{4} < \frac{15 - 12}{4}\right) = P(Z < 0.75)$$

$$P(Z < 0.75) = \Phi(0.75) \approx 0.7734$$

Answer:

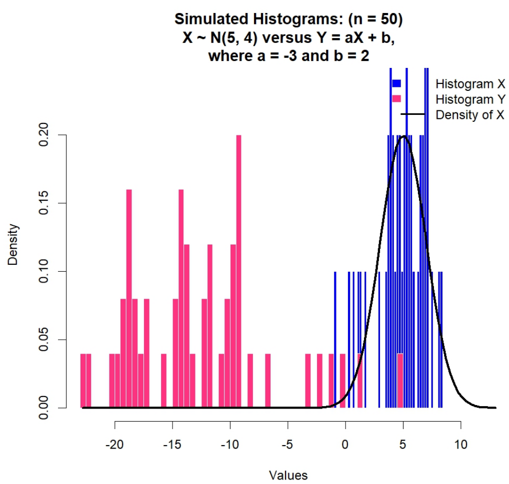
$$\boxed{P(X < 15) \approx 0.773}$$

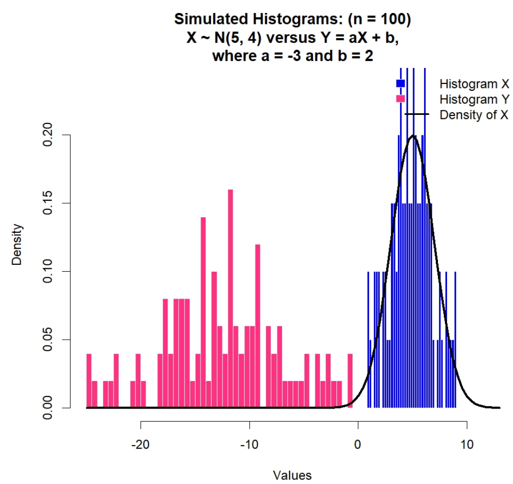
Summary:

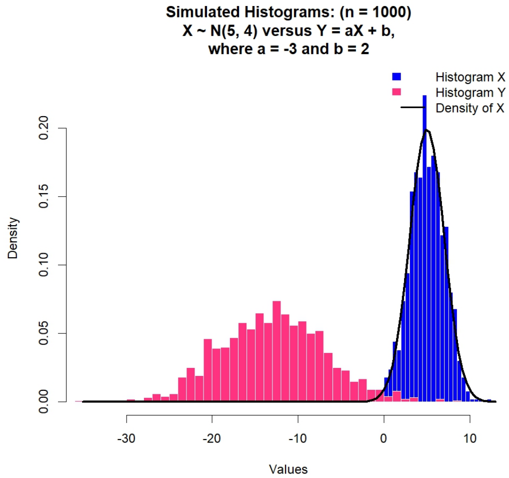
- An affine transformation $Y = aX + b$ scales and shifts the distribution of X.
- Mean and variance transform as $\mathbb{E}[Y] = a\mathbb{E}[X] + b$, $\text{Var}(Y) = a^2\text{Var}(X)$.
- The normal distribution is closed under affine transformations: $Y \sim N(a\mu + b, a^2\sigma^2)$.
- Special cases include scaling by negatives and standardization.

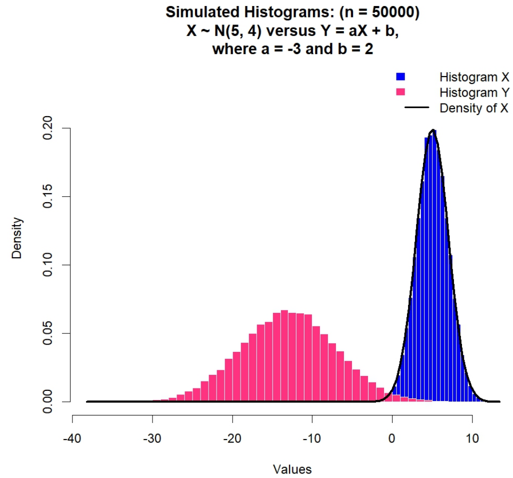
Simulations:

We will simulate the density and distribution after the transformation $Y = aX + b$ using R.

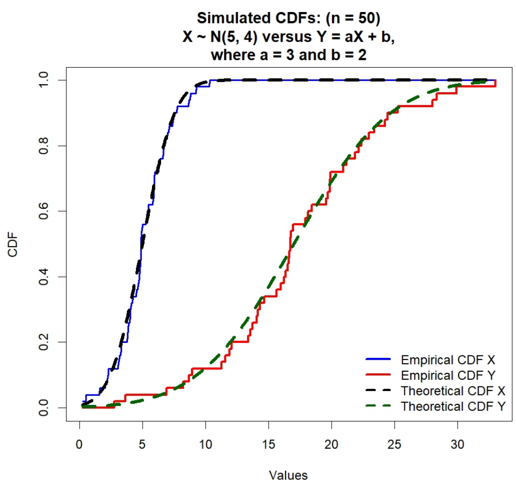


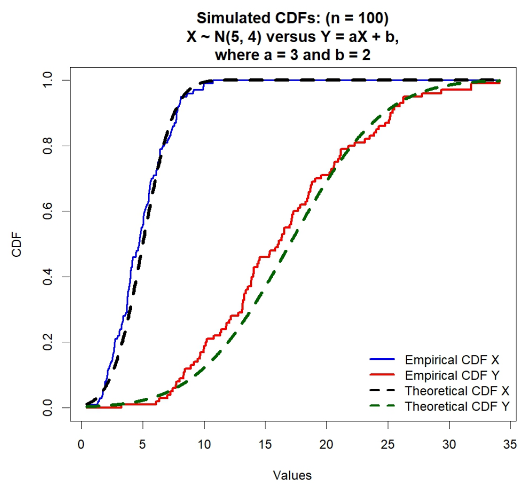


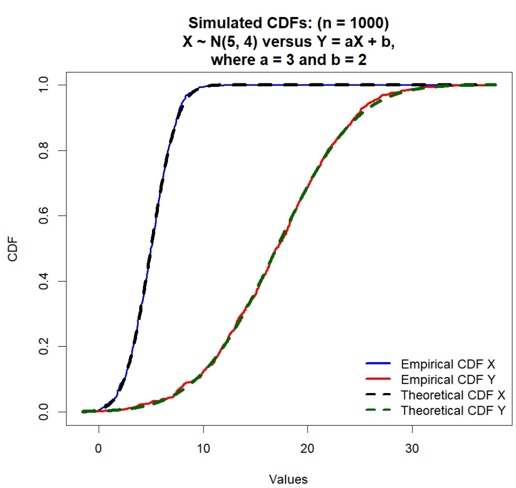


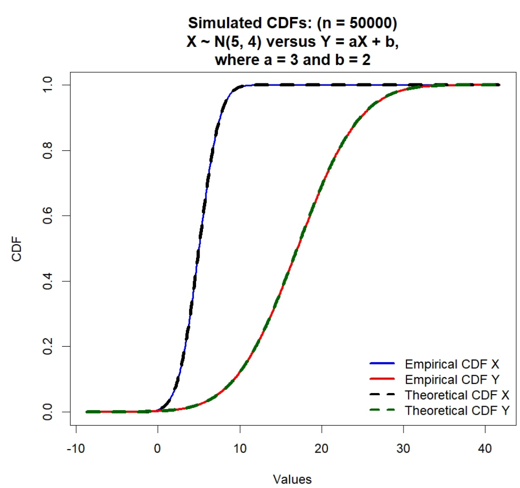


Here are simulations for the distribution function.









The R Code:

Here is the R code used for the above simulations.

```
################################################
# This simulates a linear transformation Y=aX+b
# for a normal variable X

set.seed(123)

mu <- 5
sigma <- 2
a <- -3
b <- 2
n <- 50000

X <- rnorm(n, mu, sigma)
Y <- a * X + b

### X density range and maximum height
x_density_range <- c(mu - 4*sigma, mu + 4*sigma)
max_density_X <- dnorm(mu, mean = mu, sd = sigma)

### Combine ranges so x-axis covers BOTH X density and Y histogram
x_min <- min(x_density_range[1], min(Y), min(X))
x_max <- max(x_density_range[2], max(Y), max(X))

### --- FIRST HISTOGRAM: X ---
hist(X, breaks = 50, prob = TRUE,
     main = paste0(
       "Simulated Histograms: (n = ", n, ")\n",
       "X ~ N(", mu, ", ", sigma^2, ") versus Y = aX + b,\n",
       "where a = ", a, " and b = ", b
     ),
     xlab = "Values",
     col = "blue",
# rgb(0.2, 0.4, 1, 0.4),   # transparent blue
     border = "white",
     xlim = c(x_min, x_max),
     ylim = c(0, max_density_X * 1.2))

### --- SECOND HISTOGRAM: Y (OVERLAY) ---
hist(Y, breaks = 50, prob = TRUE,
     col = rgb(1, 0.2, 0.5, 1),   # transparent pink
     border = "white",
     add = TRUE)                   # <-- the missing piece!

### --- OVERLAY DENSITY OF X ---
curve(dnorm(x, mean = mu, sd = sigma),
      from = x_min, to = x_max,
```

```
      add = TRUE, lwd = 3, col = "black")

legend("topright",
       legend = c("Histogram X", "Histogram Y", "Density of X"),
       fill = c("blue", rgb(1,0.2,0.5,1), NA),  # fill for histograms
       border = c("white", "white", NA),
       lty = c(0, 0, 1),       # solid line for the density
       col = c(NA, NA, "black"), # black color for density line
       lwd = c(NA, NA, 2),      # optional: make the density line thicker
       bty = "n")


########################################################################
# This simulates the distrbution functions of X and Y=aX+b
#######################################################################


mu <- 5
sigma <- 2
a <- 3
b <- 2
n <- 50000

X <- rnorm(n, mu, sigma)
Y <- a * X + b


### --- EMPIRICAL CDFs ---
ecdf_X <- ecdf(X)
ecdf_Y <- ecdf(Y)


### --- RANGE FOR PLOT ---
x_min <- min(X, Y)
x_max <- max(X, Y)
x_vals <- seq(x_min, x_max, length.out = 1000)


### --- PLOT CDFs ---
plot(x_vals, ecdf_X(x_vals), type = "l", lwd = 2, col = "blue",
     ylim = c(0,1), xlab = "Values", ylab = "CDF",
     main = paste0("Simulated CDFs: (n = ", n, ")\n",
                   "X ~ N(", mu, ", ", sigma^2, ") versus Y = aX + b,\n",
                   "where a = ", a, " and b = ", b))
lines(x_vals, ecdf_Y(x_vals), lwd = 3, col = "red")


### --- THEORETICAL CDFs ---
lines(x_vals, pnorm(x_vals, mean = mu, sd = sigma), lwd = 4,
col = "black", lty = 2)         # X
lines(x_vals, pnorm(x_vals, mean = a*mu + b, sd = a*sigma), lwd = 4,
col = "darkgreen", lty = 2)  # Y
```

```
legend("bottomright",
       legend = c("Empirical CDF X", "Empirical CDF Y", "Theoretical CDF X",
       "Theoretical CDF Y"),
       col = c("blue", "red", "black", "darkgreen"), lwd = 3,
       lty = c(1,1,2,2), bty = "n")
```

Please let me know if you have any questions, comments, or corrections!