10/17/13

NUMERICAL DIFFENTIATION & INTEGRATION

LET $f'(x) = \frac{df(x)}{dx}$

MATHEMATICALLY, TO FIRST ORDER

ASK STUDENTS $\longrightarrow$ $$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

$$f'(x) \cong \frac{f(x+h) - f(x)}{h} \quad \text{FOR SMALL } h$$

OR $$f'(x) \cong \frac{f(x+h) - f(x)}{h} + O(h^2)$$

WE CAN CREATE A MORE ACCURATE APPROXIMATION (STILL FIRST ORDER) USING A TAYLOR SERIES:

$$f(x_0+h) \cong \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x_0)$$

OR $$f(x_0+h) \cong \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x_0) + O(h^{n+1})$$

ASK STUDENTS $\longrightarrow$ SO $$f(x+h) \cong f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + O(h^3)$$

NOW, USING $-h$ SO THAT WE CAN DIFFERENCE ABOUT x_0:

ASK STUDENTS $\longrightarrow$ $$f(x_0-h) \cong f(x_0) - h f'(x_0) + \frac{h^2}{2} f''(x_0) - O(h^3)$$

So,

$$f(x_0+h) - f(x_0-h) =$$

$$2h f'(x_0) + 2\,O(h^3) = 2h f'(x_0) + O(h^3)$$

$$\Rightarrow f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{O(h^3)}{h}$$

$\boxed{\text{ASK STUDENTS}}$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - O(h^2)$$

SAME AS "+"

$\Rightarrow$ NOW, OUR ERROR TERM IS $O(h^2)$, WHEREAS IT WAS $O(h)$ USING ONLY $f(x_0+h)$.

REMINDER, IF THE STUDENTS WANT TO SEE IT:

$$f(x_0+h) = f(x_0) + h f'(x_0) + O(h^2)$$

$$\frac{f(x_0+h) - f(x_0)}{h} - \frac{O(h^2)}{h} = f'(x_0)$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} + O(h)$$

SECOND-ORDER DERIVATIVES

SECOND ORDER DERIVATIVES CAN BE APPROXIMATED SIMILARLY USING A TAYLOR EXPANSION:

$$f''(x_0) \cong \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2} + O(h^2)$$

WE CAN SEE THAT THE SMALLER h IS, THE MORE PRECISE THE DERIVATIVE WILL BE. $\Longrightarrow$

DISCRETE DERIVATIVES ON A MESH WITH PYTHON → GO TO NOTEBOOK.

EMPHASIZE

- ACCURACY VS. MESH SIZE
- NOISY NATURE OF DISCRETE DERIVATIVES.