1. (10 pts. altogether) (a) (7 pts) What is the rank of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

(b) (3 points) Using part (a) find the nullity of A.

2. (10 pts.) Let

$$\mathcal{S} = \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad , \quad \begin{bmatrix} -5 \\ -10 \end{bmatrix} \},$$

determine whether the set $\mathcal{S}$ is linearly independent or linearly dependent. In case it is linearly dependent, write the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ explicitly as a non-trivial linear combination of the vectors in $\mathcal{S}$.

3. (10 pts altogether) Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Calculate the following matrix products, if they are defined, or expained why they don't make sense.

(a) (5 points) AB

(b) (3 points) AB^T

(c) (2 points) C^2

4. (10 pts.) For the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

compute the matrix A^8.

5. (10 pts.) For the following matrix A finds its **reduced-row-echelon form**, R, and find an invertible matrix P such that $PA = R$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

6. (10 pts. altogether) In each case below, give an $m \times n$ matrix R in *reduced row echelon form* satisfying the given condition, or explain why it is impossible to do so.

(a)(4 pts) $m = 2$, $n = 3$ and the equation $R\mathbf{x} = \mathbf{c}$ has a solution for all $\mathbf{c}$.

(b) (4 pts) $m = 2$, $n = 2$ and the equation $R\mathbf{x} = \mathbf{c}$ has a unique solution for all $\mathbf{c}$.

(c) (2 pts) $m = 3$, $n = 3$ and the equation $R\mathbf{x} = \mathbf{0}$ has no solution.

7. (10 pts.) **Without first computing** A^{-1}, find $A^{-1}B$, if

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

8. (10 pts. altogether , 2 each) **True** or **False**? Give a short explanation!
(a) For any $n \times n$ matrices A and B, if $AB = I_n$, then $BA = I_n$.

(b) If A and B are invertible 2×2 matrices, then so is $A + B$.

(c) The sum of *any* two $m \times n$ matrices is always defined.

(d) The product of *any* two 4×9 matrices is never well-defined.

(e) The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only of $\mathbf{b}$ is a linear combination of the rows of A.

9. (10 pts.) Let $\mathbf{u}$ be a solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}$ be a solution of $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{b}$ is a vector in R^m. Show that $\mathbf{u} + \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

10. (10 pts. altogether, 5 each) What does it mean to say that the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in R^n are *linearly independent*? Give the precise definition in one or more sentences.

(b) What is meant by the *span* of a set of vectors $\mathcal{S} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$? Give the precise definition in one or more sentences.