EXAM 1 SOLUTION

. (10 pts. altogether) (a) (7 pts) What is the rank of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

Sol. to 1a): You apply the first phase of Gaussian elimination. The elementary row operations $r_3 - r_1 \to r_3$ and $r_4 - 2r_1 \to r_4$ will get everything below the $(1, 1)$ entry to be 0:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

The elementary row operation $r_4 - r_2 \to r_4$ will get everything below the $(2, 2)$ to be 0:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

The elementary row operation $r_3 - r_4 \to r_3$ will yield

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now it is in **row-echelon form**. We see that there are 3 pivots (or equivalently, three rows that are not all-zero). So the rank is 3.
Ans. to 1(a): 3.

(b) (3 points) Using part (a) find the nullity of A.

Sol. to 1(b)): The nullity is the number of columns (n) minus the rank. So it is $4-3 = 1$.

Ans. to 1(b): 1.

2. (10 pts.) Let

$$\mathcal{S} = \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad , \quad \begin{bmatrix} -5 \\ -10 \end{bmatrix} \}$$

determine whether the set $\mathcal{S}$ is linearly independent or linearly dependent. In case it is linearly dependent, write the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ explicitly as a non-trivial linear combination of the vectors in $\mathcal{S}$.

Sol. of 2: This is so simple that we can do it **by inspection**. The second vector is -5 times the first one, so:

$$\begin{bmatrix} -5 \\ -10 \end{bmatrix} = (-5) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad .$$

Moving everything to the left, we get

$$(5) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} -5 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

Ans. to 2: $\mathcal{S}$ is **linearly dependent** and the expression of $\mathbf{0}$ as a non-trivial linear combination of the vectors of $\mathcal{S}$ is:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (5) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} -5 \\ -10 \end{bmatrix} \quad .$$

Note: there are many (infintely many other) ways to do this, for example:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (10) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} -5 \\ -10 \end{bmatrix} \quad ,$$

and the **general way** is:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (5c) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (c) \begin{bmatrix} -5 \\ -10 \end{bmatrix} \quad , \quad c \neq 0 \quad .$$

If you can't do it by inspection, you form the matrix whose columns are the two vectors

$$\begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix}$$

and then you can use Gaussian elimination. The elementary row operation $r_2 - 2r_1 \rightarrow r_2$ will yield

$$\begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$$

From here you see that the second column is -5 the first column and by the **column-correspondence property** you get the same answer.

3. (10 pts altogether) Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Calculate the following matrix products, if they are defined, or explain why they don't make sense.

(a) (5 points) AB
Sol. to 3a):

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+0\cdot 0+1\cdot 1 & 1\cdot 0+0\cdot 1+1\cdot 0 \\ 0\cdot 1+1\cdot 0+0\cdot 1 & 0\cdot 0+1\cdot 1+0\cdot 0 \\ 1\cdot 1+0\cdot 0+0\cdot 1 & 1\cdot 0+0\cdot 1+1\cdot 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) (3 points) AB^T

Sol. to 3b): Undefined. You can't multiply a 3×3 matrix by a 2×3

matrix.

(c) (2 points) C^2

Sol 3c): $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+1\cdot -1 & 1\cdot 1+1\cdot 1 \\ -1\cdot 1+1\cdot -1 & -1\cdot 1+1\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

4. (10 pts.) For the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

compute the matrix A^8.

Sol. of 4):

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}^2 = \begin{bmatrix} -7 & 4 \\ -8 & -7 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} -7 & 4 \\ -8 & -7 \end{bmatrix}^2 = \begin{bmatrix} 17 & -56 \\ 112 & 17 \end{bmatrix} \quad .$$

5. (10 pts.) For the following matrix A finds its **reduced-row-echelon form**, R, and find an invertible matrix P such that $PA = R$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Sol. of 5:

We first bring the matrix to **reduced row echelon form**, taking careful note of the elementary row operations:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \overset{r_3 - r_1 \to r_3}{\to} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \overset{r_3 - r_2 \to r_3}{\to} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad .$$

Now it is in reduced-row-echelon form. This is the first part of the **answer**, R. TO get P we apply the above elementary row operations to the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{r_3 - r_1 \to r_3}{\to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \overset{r_3 - r_2 \to r_3}{\to} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad .$$

Ans. to 5:

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Note: Since A is not invertible (R is not I_3), there are more than one correct P. The R is always the same, regadless of the choice of the order of elementary row operations, but the P may be different. That's why it is good to check that $PA = R$, becuase there is more than one correct P that makes it come true.

6. (10 pts. altogether) In each case below, give an $m \times n$ matrix R in *reduced row echelon form* satisfying the given condition, or explain why it is impossible to do so.

(a)(4 pts) $m = 2$, $n = 3$ and the equation $R\mathbf{x} = \mathbf{c}$ has a solution for all $\mathbf{c}$.

Sol. to 6a): There many "correct solutions". One of them is:

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Explanation: The system $R\mathbf{x} = \mathbf{c}$, in *high-school language* is:

$$x_1 + 2x_3 = c_1 \quad ,$$

$$x_2 + 3x_3 = c_2 \quad .$$

Obviously you can solve it for **any** choice of real numbers c_1, c_2. x_3 is a free variable, and the general solution is $x_1 = c_1 - 2x_3, x_2 = c_2 - 3x_3, x_3 = x_3$, so in this system there are **infinitely many** solutions for all $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, but this is not what the question demnaded, only that there is **at least one** solution.

(b) (4 pts) $m = 2$, $n = 2$ and the equation $R\mathbf{x} = \mathbf{c}$ has a unique solution for all $\mathbf{c}$.

Sol. to 6b):

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Explanation: The system $R\mathbf{x} = \mathbf{c}$, in *high-school language* is:

$$x_1 = c_1 \quad ,$$

$$x_2 = c_2 \quad .$$

This system is so simple that it equals its **own solution**. Obviously there is a **unique** solution $x_1 = c_1, x_2 = c_2$ no matter what c_1, c_2 are. (There are no free variables, of course).

(c) (2 pts) $m = 3$, $n = 3$ and the equation $R\mathbf{x} = \mathbf{0}$ has no solution.

Sol. to 6c): impossible. $\mathbf{x} = \mathbf{0}$ **always** has a solution, namely $\mathbf{0}$!

7. (10 pts.) **Without first computing** A^{-1}, find $A^{-1}B$, if

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol, of 7: We perform Gaussian elimination on A, keeping track of the elementary row operations

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \overset{r_2 - 2r_1 \to r_2}{\to} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \overset{r_1 + 2r_2 \to r_1}{\to} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \overset{-r_2 \to r_2}{\to} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we **mimick** the same elementary row operations, in the **same order** starting with B:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \overset{r_2 - 2r_1 \to r_2}{\to} \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \end{bmatrix} \overset{r_1 + 2r_2 \to r_1}{\to} \begin{bmatrix} -1 & 3 & -4 \\ -1 & 2 & -3 \end{bmatrix} \overset{-r_2 \to r_2}{\to} \begin{bmatrix} -1 & 3 & -4 \\ 1 & -2 & 3 \end{bmatrix} \quad .$$

Ans. to 7:

$$A^{-1}B = \begin{bmatrix} -1 & 3 & -4 \\ 1 & -2 & 3 \end{bmatrix}$$

8. (10 pts. altogether , 2 each) **True** or **False**? Give a short explanation!

(a) For any $n \times n$ matrices A and B, if $AB = I_n$, then $BA = I_n$.

Sol. to 8(a): **True**. (theorem)

(b) If A and B are invertible 2×2 matrices, then so is $A + B$.

Sol. to 8(b): **False**. For example if $A = I_2$ and $B = -I_2$.

(c) The sum of *any* two $m \times n$ matrices is always defined.

Sol. to 8(c): **True**.

Note: Some people answered "False", since the two matrices "may not have the same m and n". In a different galaxy they may have been right, but the mathematical language in planet Earth (in the Solar System, Milky Way) implies when you say *two* $m \times n$ matrices, that we are talking about the **same** m and the **same** n.

(d) The product of *any* two 4×9 matrices is never well-defined.

Sol. to 8(d): **True**. For a matrix product to be well-defined the number of columns of the left-matrix must equal the number of rows of the right-matrix.

(e) The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only of $\mathbf{b}$ is a linear combination of the rows of A.

Sol. to 8(e): **False**. The correct statement is with "rows" replaced by columns.

9. (10 pts.) Let $\mathbf{u}$ be a solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}$ be a solution of $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{b}$ is a vector in R^m. Show that $\mathbf{u} + \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

Sol. of 9: We are told that

$$A\mathbf{u} = \mathbf{b} \quad , \quad A\mathbf{v} = \mathbf{0}$$

Now, by the distributive property and the data of the problem, we have

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}) = \mathbf{b} + \mathbf{0} \quad .$$

Since adding $\mathbf{0}$ does not change the vector, this equals

$$\mathbf{b} + \mathbf{0} = \mathbf{b} \quad .$$

We have just proved that

$$A(\mathbf{u} + \mathbf{v})) = \mathbf{b} \quad ,$$

but this means that $\mathbf{u} + \mathbf{v}$ is a solution of of $A\mathbf{x} = \mathbf{b}$.

10. (10 pts. altogether, 5 each)

(a) What does it mean to say that the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in R^n are *linearly independent*?

Sol. of 10(a) $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent if there is no way that the there are k real numbers $c_1, \ldots, c_k$ such that

$$c_1\mathbf{u_1} + \ldots + c_k\mathbf{u_k} = \mathbf{0}$$

unelss all of them are equal to 0, i.e. $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.

(b) What is meant by the *span* of a set of vectors $\mathcal{S} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$? Give the precise definition in one or more sentences.

Sol. of 10b) the span of $\mathcal{S} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is the **set** of all linear combinations. In other words, it is the set

$$\{c_1\mathbf{u_1} + \ldots + c_k\mathbf{u_k} : -\infty < c_1 < \infty\, , \, \ldots, -\infty < c_k < \infty\, \} \quad .$$