1. Determinant of a Matrix

To begin with, we would need to define a new matrix expression to facilitate the developments of concepts and theories of this section!

Consider a square matrix A of size $n \times n$ such that $A = (a_{ij})_{n\times n}$. Given a couple of (i, j), we denote A_{ij} of A to be $(n-1) \times (n-1)$ matrix in which we seek to get rid of the i^{th} row and j^{th} column of A at the same time. See the subsequent example.

Example 1:

Given a 3×3 matrix A, where

$$A = \begin{pmatrix} 8 & 1 & 6 \\ -7 & 2.5 & 4 \\ 3/4 & -3 & 0 \end{pmatrix}$$

We have

$$A_{13} = \begin{pmatrix} -7 & 2.5 \\ 3/4 & -3 \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 8 & 6 \\ 3/4 & 0 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 8 & 6 \\ -7 & 4 \end{pmatrix}$$

You are recommended to work out the remaining i.e. $A_{11}, A_{12}, A_{21}, A_{23}, A_{31} \& A_{33}$. Would it work for non-square matrix?

With the definition of A_{ij} of A, it is time to formally define the ideas of determinant!

Definition 1:

Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix. If $n \geq 2$, we pick an arbitrary $i' \in [1, n]$ and define the determinant of A, $\det(A)$, as follows:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i'+j}\, a_{i'j} \det(A_{i'j})$$

In some books, they may denote the determinant of A as $|A|$. We will also use this expression throughout this course interchangeably.

Particularly, if $n = 2$, A simply equals the value of the only entry of A.

Example 2:

Consider a 1$\times$ 1 matrix A where $A = (1)$, the determinant of A is

$$\det(A) = |A| = |1| = 1$$

Consider a 2$\times$ 2 matrix, where $A = \begin{pmatrix} -7 & 2.5 \\ 3/4 & -3 \end{pmatrix}$, the determinant of A_{12} is

$$\det(A_{12}) = |A_{12}| = \left|\frac{3}{4}\right| = \frac{3}{4}$$

Now, you may question that, referring to Definition 1, if we choose different i out of n rows (or column), would the determinant of the concerned matrix be different in value?

The answer is ***NO!*** The required determinant will be independent of the choice of i. See next example!

Example 3:

Consider a 2× 2 matrix A, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. By Definition 1, we have

$$\det(A) = \sum_{j=1}^{2} (-1)^{i'+j}\, a_{i'j} \det(A_{i'j})$$

Suppose we take $i' = 1$, then

$$\begin{aligned} \det(A) &= \sum_{j=1}^{2} (-1)^{1+j}\, a_{1j} \det\left(A_{1j}\right) \\ &= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) \ = (-1)^2 (\,1)|4| + (-1)^3 (2)|3| \\ &= 4 - 6 = -2 \end{aligned}$$

What if $i' = 2$?

$$\begin{aligned} \det(A) &= \sum_{j=1}^{2} (-1)^{2+j}\, a_{2j} \det\left(A_{2j}\right) \\ &= (-1)^{2+1} a_{21} \det(A_{21}) + (-1)^{2+2} a_{22} \det(A_{22}) \ = (-1)^3 (\,3)|2| + (-1)^4 (4)|1| \\ &= -6 + 4 = -2 \end{aligned}$$

From the above results, we observe that the results are the same and independent of selection of i. To verify this observation, you could attempt the computations for the determinant of matrices of various sizes, say 3×3 , 4×4 or above.

By Definition 1, the formula for determinant computation is somewhat complicated especially for a matrix with large size! For a relatively smaller size, say 2×2 matrix, we could slightly simplify the expression.

Suppose A is a 2× 2 matrix, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a, b, c & d are real numbers. The determinant of A could be written in the following way,

$$\det(A) = |A| = ad - cb$$

1.1 Minor and Cofactor

Recall the discussions in the very beginning of this topic, we define a square matrix, say A and associated A_{ij} in which A_{ij} is a new matrix originated from A by removing the i^{th} row and j^{th} column of A. Now, we will subsequently define something new related to A_{ij}.

Definition 2:

If A is a square matrix, the **minor** of the entry of the the i^{th} row and j^{th} column, denoted as M_{ij} is the determinant of A_{ij}.

Example 4:

Consider a 3×3 matrix, where $A = \begin{pmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{pmatrix}$. We wish to find the minor of 2nd row and 3rd column. By definition 2, we have

$$A_{23} = \begin{pmatrix} 1 & 4 \\ -1 & 9 \end{pmatrix}$$

$$\therefore M_{23} = |A_{23}| = \begin{vmatrix} 1 & 4 \\ -1 & 9 \end{vmatrix} = 13$$

Definition 3:

If A is a square matrix and M_{ij} is a minor of associated with the entry of i^{th} row and j^{th} column, the **cofactor** associated with the entry of i^{th} row and j^{th} column, C_{ij}, is given by multiplication $(-1)^{i+j}$ with M_{ij}. Mathematically, it is defined as

$$\begin{aligned} C_{ij} &= (-1)^{i+j} \cdot M_{ij} \\ &= (-1)^{i+j} \cdot |A_{ij}| \end{aligned}$$

1.2 Properties of Determinants

In this section, we collectively assume A is a $n \times n$ square matrix such that $A = (a_{ij})_{n\times n}$. We will present some important properties of determinants.

1. A has rank n if and only if $\det(A) \neq 0$

2. In last sections, we put much emphasis on expansion of a determinant by row. As a matter of fact, we can also expand and compute the determinant of the same matrix by column expansion. Analogous to Definition 1, we have the first property shown as follows.

   For matrix A where $n \geq 2$, we arbitrarily pick a $j' \in [1, n]$. The determinant of A will be given by

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j'} a_{ij'} \det(A_{ij'})$$

   Also, the choice of j' will not alter the value of determinant.

Example 5:
We re-take the 2× 2 matrix of Example 3, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. In this time, we seek to expand the matrix in term of column to calculate the determinant.

Suppose we choose $j' = 1$, then

$$\det(A) = \sum_{i=1}^{2} (-1)^{i+1} a_{i1} \det(A_{i1})$$
$$= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{2+1} a_{21} \det(A_{21}) = (-1)^2 (1)|4| + (-1)^3 (3)|2| = 4 - 6 = -2$$

We can clearly see that the result is identically the same as the one in Example 3.

You are highly suggested to check with the result when $j' = 2$. You will find that it equals -2 as well!

3. The determinant of a matrix A is equal to the determinant of transpose of matrix A, i.e.

$$\det(A) = \det(A^T)$$

   We just state without proving this property. Following with the last property, the central idea follows with the fact that expansion of determinant of a matrix by column is equivalent to that by row.

4. The determinant of matrix A will equal zero, i.e. $\det(A) = 0$ if any of the following hold(s):

   a. A has a zero row ~~or~~ or column;

   b. A has two equal row or column; and

   c. A has two (or more) proportional rows or columns. r(A) >= 2

5. When it comes to the determinant of matrix after carrying out elementary row operations (discussed in previous topic), we can summarize the corresponding properties as follows:

   a. If we interchange two row or column of A, the determinant of A is multiplied by -1.

   b. If we multiple a row of A by a non-zero real number, say k, the determinant of A is multiplied by k.

   c. If we add k times one row (or column) of A to another row (or column), the determinant of the resulting matrix does not change.

6. If matrix A is of row-echelon form (see topic 1 for more details), then determinant of A equals

$$\det(A) = \prod_{i=1}^{n} a_{ii} = a_{11} \cdot a_{22} \cdot a_{33} \dots a_{nn}$$

7. In addition to matrix A, suppose there exists another square matrix B of the same size as A, the determinant of product A & B is given by

$$\det(AB) = \det(A) \cdot \det(B)$$

2. Inverse of a Matrix

You should be well familiar with the fact that for each non-zero real number, say a, there exists a number b, named reciprocal of a such that

$$a \cdot b = 1 \text{ and } b \cdot a = 1$$

For the non-zero real number a, it has one and only one *reciprocal* and, symbolically, we take

$$a^{-1} \text{ or } {}^{1}\!/_{a}$$

to represent.

The role of the number 1 in general and ordinary arithmetic is analogous in matrix world by the identity matrix I (see topic 1 for more details). In view of this, we have the following definition.

Definition 4:

Let A and B be $n \times n$ matrices. If $AB = I$ and $BA = 1$, then B is an inverse of A and denoted as A^{-1}.

Certainly, we can say that A is an inverse of B and denoted as B^{-1} in a similar manner. Notice that we have to confine the definition to square matrix only!

Example 6:

We consider a 2×2 case, $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ is an inverse of $\begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}$. Since we can check that

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \text{ and}$$

$$\begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

By the same token, consider a 3×3 case.

$\begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix}$ is an inverse of $\begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix}$. Since we can check that

$$\begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix}\begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \text{ and}$$

$$\begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

At this point, you may question whether we can always find an inverse for any square matrix! The answer is ***NO!***

Theorem 1:

A square matrix has at most one inverse.

Proof:

Suppose that A is a square matrix and further suppose that B and C are inverses of A so we have

$$\begin{cases} AB = I \\ BA = I \end{cases} \text{ and } \begin{cases} AC = I \\ CA = I \end{cases}$$

We start with considering $AB = I$ and multiplying matrix C on the both sides. We have

$$CAB = CI = C$$

On the other hand, we re-arrange the term a bit,

$$CAB = (CA)B = IB = B$$

Let's recall that matrix multiplication is associative, the result follows that $B = C$.

This result also suggests why we need to state both $AB = I$ and $BA = 1$ in Definition 4!

To facilitate the development of Inverse hereafter, we have the following definitions,

Definition 5:
A square matrix A is said to be
1. *Singular* if it does not have an inverse.

2. *Non-singular* if it has an inverse.

Following Definitions 4 & 5, we also call a square matrix that has an inverse is *invertible*. Hence, for any invertible matrix A,

$$AA^{-1} = I \text{ and } A^{-1}A = 1$$

We notice that if A is invertible (or non-singular), A^{-1} is also invertible with A being the inverse correspondingly. In other words,

$$(A^{-1})^{-1} = A$$

i.e. A and A^{-1} are inverses of each other.

2.1 Properties and Theorems of Inverses

1. Let A and B be $n \times n$ square matrices. Then $(AB)^{-1} =$ A^-1B^-1

This result can actually be extended to a more general result for any number of multiplication of matrices.

Theorem 2:
Let $A_1, A_2, A_3, \dots, A_k$ be invertible matrices of the same size. The multiplication of the k matrices is invertible and given by

$$(A_1 A_2 \dots A_{k-1} A_k)^{-1} = {A_k}^{-1} {A_{k-1}}^{-1} \dots {A_2}^{-1} {A_1}^{-1}$$

2. Let A be an $n \times n$ square matrix. Then $(A^T)^{-1} = (A^{-1})^T$.

3. Let A be an $n \times n$ invertible matrix. Then

$$\det(A^{-1}) = \frac{1}{\det(A)} = [\det(A)]^{-1}$$

4. Let A, B and C be $n \times n$ square matrices. If A is invertible and $AB = AC$, then B is equal to C.

So far, we have discussed quite a number of properties and theorems for matrix inverse. However, we are yet to know how to compute an inverse of a matrix. We attempt to achieve this goal. We define

Definition 6:
For an $n \times n$ square matrix A, the inverse of A, namely A^{-1}, is given by

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & & \ddots & \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T$$

where
1) $C_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$. By section 1.1, C_{ij} is called cofactor.
2) A_{ij} is an $(n-1) \times (n-1)$ square matrix in which we eliminate the i^{th} row and j^{th} column of A at the same time.

Example 7:

Re-visit example 6 and consider the inverse of $\begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix}$. By using Definition 6, we have $\det(A) = 1$ (Check it yourself!)

And, we have the following

$\det(A_{11}) = \begin{vmatrix} -5 & 1 \\ 11 & -2 \end{vmatrix} = -1$ Thus, $C_{11} = (-1)^{1+1}(-1) = -1$

which is essentially the same as the final result of example 6!