1. System of Linear Equations

As discussed in previous topics, a linear equation is typically of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_i \in \mathbb{R}, \forall i \in [1, \ldots n]$ and $b \in \mathbb{R}$.

And a system of linear equations hereafter will be represented as follows:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

where $a_{ij} \in \mathbb{R}, \forall i, j \in [1, \ldots n]$ and $b_i \in \mathbb{R}, \forall i \in [1,2, \ldots, n]$

In particular, we call the system to be homogenous if $b_i = 0$ for all i. Otherwise, the system is called heterogeneous for any non-zero b_i for all i.

Example 1:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 + 5x_2 + 6x_3 = 0 \\ 7x_1 + 8x_2 + 9x_3 = 0 \end{cases}$$

This is a homogenous system.

On the other hand,

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 + 5x_2 + 6x_3 = -1 \\ 7x_1 + 8x_2 + 9x_3 = 0 \end{cases}$$

Since there is one non-zero entry on the right hand side of the system, this is called heterogeneous system.

Theorem 1:
For a homogenous system, there exists at least one solution, known as zero or trivial solution. All variables simply equal zero.

It is particularly easy to verify Theorem 1 by assigning the value of zero to each of the variables of the system.

On the contrary, if the solution of the system is non-zero, we can name it non-trivial solution.

2. Solutions of System of Linear Equations

For the sake of easy presentation, we normally rewrite a given system of linear equations into a matrix equation form, for instance $Ax = b$.

We can solve the system, i.e. find the values for variable x's to satisfy the system, by some well-established methods! We will introduce those methods in the section.

Theoretically, we will generally have three types of solutions when solving a system.

Theorem 2:
A linear system has either
1. No solution;
2. Exactly one (unique) solution; or
3. Infinitely many solution.

Geometrically, we can visualize the ideas of Theorem 2 by the following cases and figures.

Case 1:

The two lines are parallel and not the same so there is no solution.

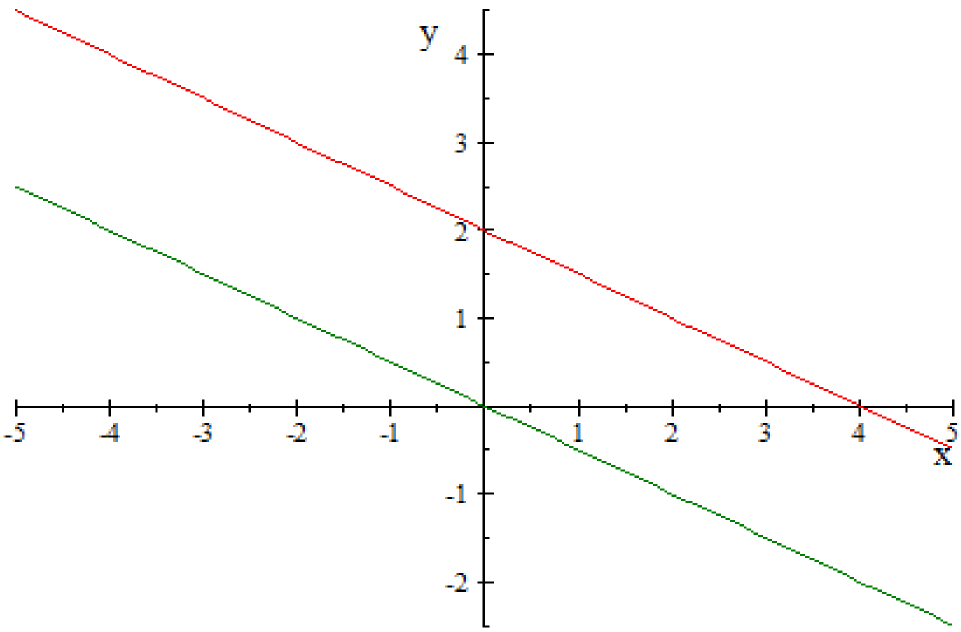

Figure 1: No Solution

Case 2:

The two lines intersect in a point so there is a unique solution.

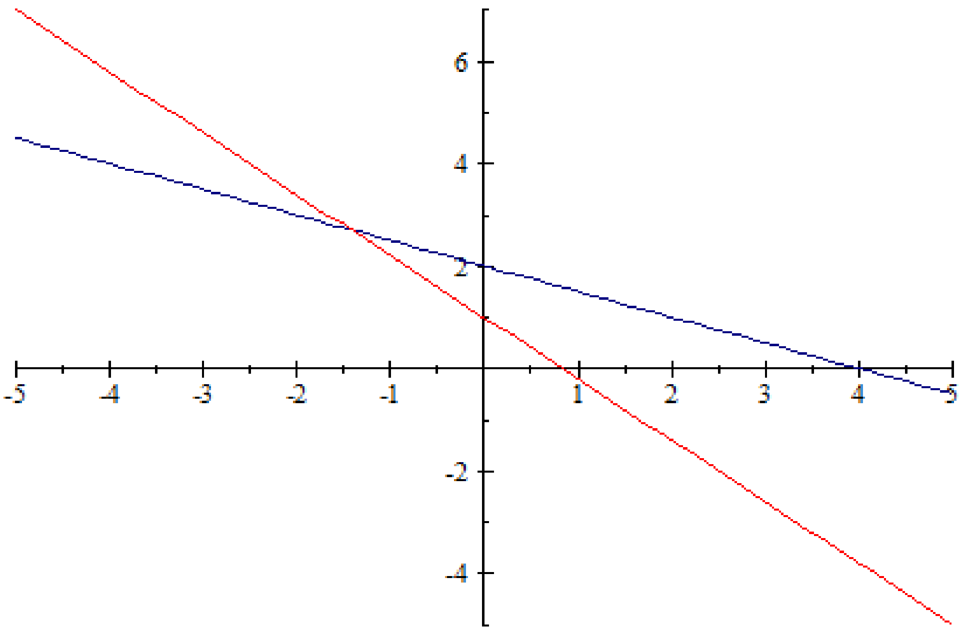

Figure 2 Unique (Exactly one) Solution

Case 3:

When there are more than one line overlapping and it results in infinitely many overlapping points along the line as the solutions.

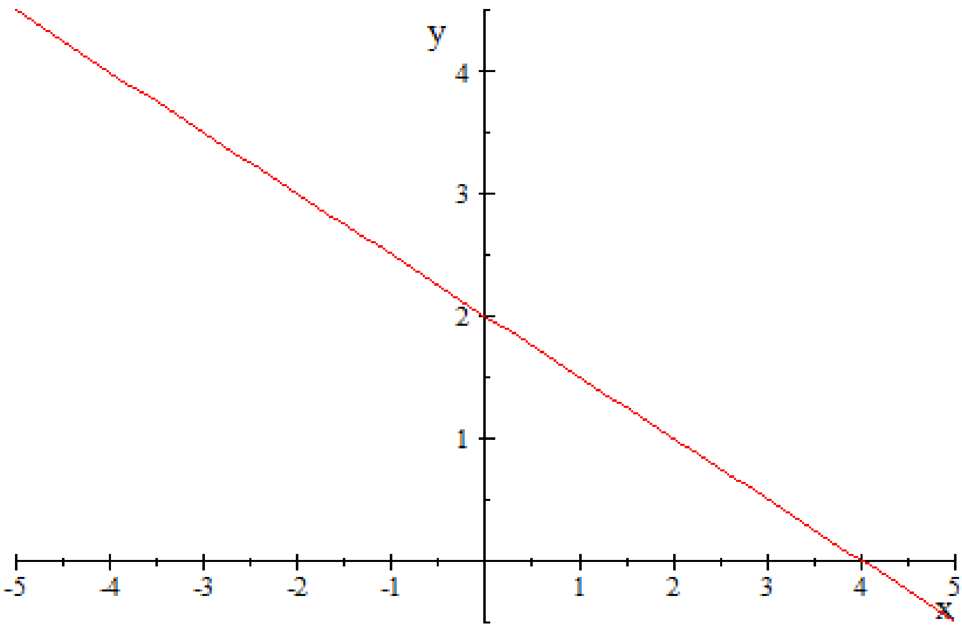

Figure 3 Infinitely Many Solutions

If a linear system contains at least one solution, we say that the system is consistent; otherwise it is inconsistent. The following theorem characterizes the consistency of the system in relation to **rank**.

Suppose there exists a linear system such that $Ax = b$ where A is a coefficient matrix of size $n \times n$ and $\tilde{A}$ is an augmented matrix of A correspondingly. $r(A)$ denote the rank of matrix A

Theorem 3:

Linear system has

1. No solution iff $r(A) < r(\tilde{A})$;
2. Exactly one solution iff $r(A) = r\left(\tilde{A}\right) = n$;
3. Infinitely many solution iff $r(A) = r\left(\tilde{A}\right) < n$

Example 2:

Consider the following systems:

I. $\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + x_2 = -2 \end{cases}$

II. $\begin{cases} -6x_1 - 4x_2 = -6 \\ 3x_1 + 2x_2 = 3 \end{cases}$

III. $\begin{cases} -6x_1 - 4x_2 = 0 \\ 3x_1 + 2x_2 = 3 \end{cases}$

And the augmented matrices are shown below.

I. $\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & -2 \end{array}\right)$

II. $\left(\begin{array}{cc|c} -6 & -4 & -6 \\ 3 & 2 & 3 \end{array}\right)$

III. $\left(\begin{array}{cc|c} -6 & -4 & 0 \\ 3 & 2 & 3 \end{array}\right)$

Applying elementary row operations, we yield (reduced) echelon forms for each of these augmented matrices. They are

I. $\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$

II. $\left(\begin{array}{cc|c} 1 & {}^{2}/_{3} & 1 \\ 0 & 0 & 0 \end{array}\right)$

III. $\left(\begin{array}{cc|c} 1 & {}^{2}/_{3} & 0 \\ 0 & 0 & 1 \end{array}\right)$

Based on the results, we can summarize the relationship between rank and the consistency of respective systems.

System	**Rank$A$** $r(A)$	**Rank$\widetilde{A}$** $r(\widetilde{A})$	n	**No. of solution (Consistency conditions)**
I.	2	2	2	1
II.	1	1	2	Infinitely many solution
III.	1	2	2	No Solution

3. Gaus(sian) Elimination

Suppose that we are given a linear system $Ax = b$. We again adopt A and $\tilde{A}$ to be coefficient matrix and augmented matrix of A respectively.

When we carry out elementary row operations, we attempt to transform $\tilde{A}$ to another new matrix, namely $\tilde{A}'$. The corresponding linear system of $\tilde{A}'$ will carry the same solution when compared with that of $\tilde{A}$. It turns out that elementary row operations will not alter the nature of the solution of a system. We say that $\tilde{A}$ and $\tilde{A}'$ are row equivalent.

Example 3:

Consider the following system

$$\begin{cases} 2x_1 - x_2 - 2x_3 = 2 \\ x_1 + 2x_2 + 3x_3 = 4 \end{cases}$$

Clearly, the augmented matrix of this system is

$$\tilde{A} = \left(\begin{array}{ccc|c} 2 & -1 & -2 & 2 \\ 1 & 2 & 3 & 4 \end{array}\right)$$

And, further, we find the subsequent matrices are all row equivalent to $\tilde{A}$,

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & -1 & -2 & 2 \end{array}\right), \left(\begin{array}{ccc|c} 2 & -1 & -2 & 2 \\ 2 & 4 & 6 & 8 \end{array}\right) \text{ and } \left(\begin{array}{ccc|c} 2 & -1 & -2 & 2 \\ 4 & 3 & 4 & 10 \end{array}\right).$$

You could work out the elementary row operations to see how to arrive at each row-equivalent matrix to $\tilde{A}$ above respectively.
Actually, we can justify that the systems correspond to each of the above row-equivalent matrix will have the same solution as the given system.

From Example 3, we observe that the solution of a system will preserve when we attempt to convert a system $Ax = b$ to associated system $A'x = b'$ whose augmented matrix is in row echelon form. The reason is that we can "spot" the nature of the solution of a system with ease when the augmented matrix is resulted in echelon form!

Example 4:

Consider the following linear system

$$\begin{cases} 2x_1 + x_2 + 6x_3 = 3 \\ x_1 \quad + 5x_3 = 1 \\ 8x_2 + 3x_3 = 2 \\ 3x_3 = 4 \end{cases}$$

The augmented matrix of such system is

$$\left(\begin{array}{ccc|c} 2 & 1 & 6 & 3 \\ 1 & 0 & 5 & 1 \\ 0 & 8 & 3 & 2 \\ 0 & 0 & 3 & 4 \end{array}\right)$$

which can be converted to another matrix that is indeed row-equivalent in echelon form, see

$$\left(\begin{array}{ccc|c} 2 & 1 & 6 & 3 \\ 0 & -1/2 & 2 & -1/2 \\ 0 & 0 & 12 & 1 \\ 0 & 0 & 0 & -107/12 \end{array}\right)$$

And the corresponding linear system is

$$\begin{cases} 2x_1 + x_2 + 6x_3 = 3 \\ -\frac{1}{2}x_2 + 2x_3 = -\frac{1}{2} \\ 12x_3 = 1 \\ 0 = -\frac{107}{12} \end{cases}$$

Apparently, by inspection, the above system has no solution. (Why??)

Example 5:
Consider the following linear system

$$\begin{cases} 2x_1 + x_2 + 6x_3 = 3 \\ x_1 \quad + 5x_3 = 1 \\ \quad 3x_2 \quad = 4 \end{cases}$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & 6 & 3 \\ 1 & 0 & 5 & 1 \\ 0 & 3 & 0 & 4 \end{array}\right)$$

Converting the augmented matrix to echelon form, we obtain

$$\left(\begin{array}{ccc|c} 2 & 1 & 6 & 3 \\ 0 & -1/2 & 2 & -1/2 \\ 0 & 0 & 12 & 1 \end{array}\right)$$

And the associated linear system is

$$\begin{cases} 2x_1 + x_2 + 6x_3 = 3 \\ (-1/2)x_2 + 2x_3 = -1/2 \\ 12x_3 = 1 \end{cases}$$

Seems that it is not trival enough to see the solution for the system! We seek to work it out by <u>backward substitution</u> and start with the equation $12x_3 = 1$. We firstly obtain $x_3 = \frac{1}{12}$. Next, we substitute the result to the equation $(-1/2)x_2 + 2x_3 = -1/2$, we get $x_2 = \frac{4}{3}$. Lastly, we plug in the values of x_2 and x_3 into the first equation $2x_1 + x_2 + 6x_3 = 3$ and obtain $x_1 = \frac{7}{12}$ at the end.

To conclude, the procedure solving the linear system in Example 5 is called Gauss(ian) Elimination!

Example 6:

Solve $\begin{cases} x_1 + \ \ x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0 \end{cases}$ by Gauss(ian) Elimination. We have augmented matrix as follows

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

After a series of elementary row operations (the intermediate steps are left as exercise), we arrive at the echelon form like this

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -{}^{7}/_{2} & -{}^{17}/_{2} \\ 0 & 0 & 1 & 3 \end{array}\right)$$

This matrix corresponds to the linear system

$$\begin{cases} x_1 + \ \ x_2 + 2x_3 = 9 \\ x_2 - ({}^{7}/_{2})x_3 = -{}^{17}/_{2} \\ x_3 = 3 \end{cases}$$

By backward substitution, we finally acquire $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$.

4. Gauss(ian)-Jordan Elimination

In this section, we will try to extend the application of Gauss(ian) Elimination in the hope of computing the inverse of a matrix.

Suppose we are given a 3×3 matrix A where $A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{pmatrix}$. By inspection, we may not be able to spot and identify the inverse of A, symbolically A^{-1}. Due to this reason, we are motivated to derive an effective means to compute A^{-1}, if any.

Assume the desired A^{-1} is of the following form.

$$A^{-1} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

As a matter of fact, we have identity $AA^{-1} = I$ which is

$$\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The key is to find out all of the $x_{ij}\ \forall i,j \in [1,2,3]$. In other words, this is essentially to solve the following 3 linear systems at a time.

$$\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

By using Gauss(ian) Elimination, we ought to be able to solve the above systems one by one. Now, we take the first system as an illustrative example

$$\left(\begin{array}{ccc|c} 0 & 2 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Originally, we use Gauss(ian) Elimination to solve the system by backward substitution. However, we will present a new approach for tackling this system. We keep on carrying out elementary row operations to make the left hand side of the augmented matrix to become reduced echelon form (refer to Topic 1) such that

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The left hand side of the above matrix becomes reduced echelon form and the advantage is that we can spot the solution directly and very easily. This technique is called Gauss(ian)-Jordan Elimination.

Now, we can directly tell the solution to be $x_{11} = -4$, $x_{21} = \frac{1}{2}$ and $x_{31} = 1$. The remaining two systems can be solved likewise.

Once we solve all of the concerned linear systems by using Gauss(ian) and

Gauss(ian)-Jordan Elimination, we will obtain the inverse of A,

$$A^{-1} = \begin{pmatrix} -4 & 1 & -4 \\ -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

It is interesting to note that the left hand sides of all augmented matrices are all the same. Collectively speaking, we can consider the following presentation to compute the A^{-1}.

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \dots \dots \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 1 & -4 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

And we can generally start with $(A|I)$ and apply elementary row operations to arrive at the form $(I|A^{-1})$, i.e.

$$(A|I) \Rightarrow \dots \dots \Rightarrow (I|A^{-1})$$

You may ask why we introduce the Gauss(ian) Jordan Elimination for inverse of a matrix. You will find the answer very soon.

Consider a linear system of the subsequent form, $Ax = b$, where, as usual, A is coefficient matrix, x is decision variable and b is a constant matrix or column vector. In particular, consider that inverse of A, A^{-1}, exists. In other words, $AA^{-1} = I$ and $A^{-1}A = I$. By the given system, we have

$$\begin{gathered} Ax = b \\ A^{-1}Ax = A^{-1}b \\ Ix = A^{-1}b \\ \therefore x = A^{-1}b \end{gathered}$$

To conclude this section, if we can compute A^{-1} by some effective way (say Gauss(ian)-Jordan Elimination), we will be able compute the required solution for a system provided that A^{-1} does exist!

5. Cramer's Rule

This section is quite straightforward. We start with stating a theorem as follows.

Theorem 4:

Suppose there exists a linear system of the form $Ax = b$ where A is an $n \times n$ matrix, x is a vector of decision variable and b is a constant column matrix (vector). When $\det(A)$ is non-zero, the system has a unique solution

$$x_1 = \frac{\det(A_1)}{\det(A)}$$
$$x_2 = \frac{\det(A_2)}{\det(A)}$$
$$\vdots$$
$$\vdots$$
$$x_n = \frac{\det(A_n)}{\det(A)}$$

where $A_i \; \forall i \, \epsilon [1, n]$ is the matrix obtained from A by replacing the i^{th} column of A with the column vector b while all of the other columns of A remain unchanged.

Example 7:

Consider the linear system $\begin{cases} -5x_1 + 4x_2 = 8 \\ 3x_1 - 2x_2 = 6 \end{cases}$. We attempt to solve the system by Cramer's Rule.

From the system, we are clear to note that $A = \begin{pmatrix} -5 & 4 \\ 3 & -2 \end{pmatrix}$ and $\det(A) = -2$.

On the other hand, $A_1 = \begin{pmatrix} 8 & 4 \\ 6 & -2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} -5 & 8 \\ 3 & 6 \end{pmatrix}$. Their corresponding determinants are

$$\det(A_1) = 8(-2) - 6(4) = -40$$

and $\det(A_2) = -5(6) - 3(8) = -54$

By Theorem 4, we have

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{-2} = 20$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-54}{-2} = 27$$

Theorem 5:
Suppose that A is an $n \times n$ matrix and a linear system has n equations as well as n variables. Then, the linear system has a unique solution iff $\det(A) \neq 0$.

We can notably find that Theorems 4 & 5 are closely related!