Document of Hopping Robot Development

1 Introduction

This document includes most of the equations used in hopping robot development. One important message is that there maybe repeated uses of variables as we have included the equations of Penn Jerboa directly. **Please refer to the tables which are specifically decided to describe the parameters and variables in each model.**

Table 1: Nomenclature used in Simplified Model

g	gravitational constant
m	mass of point mass in spring mass system
l_0	natural length of spring
k	spring constant
l	length of spring in simplified model

2 Simplified Model

In this section, we are going to take a look at the simplified model of the hopping robot. One will observe that the simplified model will be based on the model of a one spring one mass system. Hence, we will derive the dynamic equations of the spring mass system in the first step. Note: in figure(1), m represents a point mass, hence l will be measuring from C.O.M. to the tip of the spring.

2.1 Lagrangian Dynamics

2.1.1 Flight Phase

The system is simply a free falling point mass

$$\begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= -g \end{aligned} \tag{1}$$

2.1.2 Stance Phase

Kinetic energy:

$$T = \frac{1}{2}(m\dot{l}^2 + ml^2\dot{\theta}) \tag{2}$$

Table 2: Nomenclature used in Penn Jerboa

g	gravitational constant
m_b	mass of robot body
ρ_l	natural length of spring
ρ_t	length of tail
l	length of spring
k	spring constant
τ_t	tail torque
τ_h	hip torque
x	horizontal position of C.O.M. of system
z	vertical position of C.O.M. of system
a_1	"pitch" coordinate
a_2	"shape" coordinate
θ_1	body-leg angle
θ_2	length of spring
θ_s	$= \theta_1$
ϕ_1	body-horizon angle
ϕ_2	body-tail angle
ξ	$:= \theta_1 - \phi_2$
$\ddot{\theta}$	$\begin{bmatrix}\ddot{\theta}_1 & \ddot{\theta}_2\end{bmatrix}^T$
$\ddot{a}$	$\begin{bmatrix}\ddot{a}_1 & \ddot{a}_2\end{bmatrix}^T$
T_s	duration of stance phase
k_t	tail gain of energy pumping
ω	$= \sqrt{\frac{k}{m_b}}$
g_1^v	$= \frac{k_t \dot{z}/\omega}{\sqrt{(\dot{z})^2+z^2}}$
J	$= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^T$, planar skew-symmetric matrix
$\bar{\beta}$	damping coefficient of spring in advanced model ($\bar{\beta} = \frac{b_{l_1}}{2m_1\omega}$)
χ	spring deflection: $(l - \rho_l)$
e_2	$= [0 \quad 1]$

Potential energy:

$$V = mgl\cos(\theta) + \frac{1}{2}k(l - l_0)^2 \tag{3}$$

The Lagrangian is defined as:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(m\dot{l}^2 + ml^2\dot{\theta}) - mgl\cos(\theta) - \frac{1}{2}k(l - l_0)^2 \end{aligned} \tag{4}$$

Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{l}} - \frac{\partial L}{\partial l} = 0 \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{6}$$

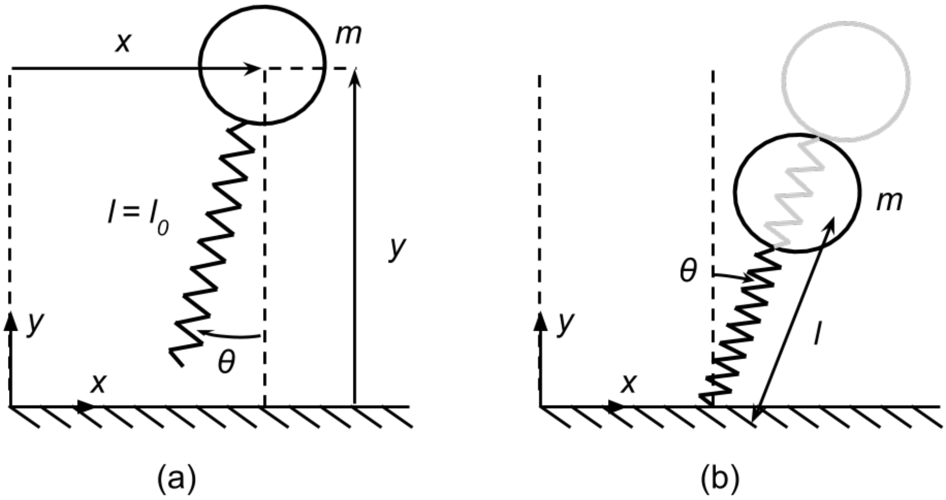

Figure 1: Simplified model.(a)Flight Phase, (b)Stance Phase

equation(5) becomes:

$$\ddot{l} = l\dot{\theta}^2 - g\cos(\theta) - \frac{k}{m}(l - l_0) \tag{7}$$

equation(6) becomes:

$$\ddot{\theta} = \frac{g}{l}\sin(\theta) - \frac{2\dot{l}\dot{\theta}}{l} \tag{8}$$

2.2 Comparison to the dynamics of Penn Jerboa

From equation(1),(7),(8):

$$\begin{gathered}
\begin{bmatrix} \ddot{\theta} \\ \ddot{l} \end{bmatrix} |_{stance} = \begin{bmatrix} \frac{g}{l}\sin(\theta) - \frac{2\dot{l}\dot{\theta}}{l} \\ l\dot{\theta}^2 - g\cos(\theta) - \frac{k}{m}(l - l_0) \end{bmatrix} \\
\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} |_{flight} = \begin{bmatrix} 0 \\ -g \end{bmatrix}
\end{gathered} \tag{9}$$

The dynamic equation(with control) of Penn Jerboa is as follow:

$$\begin{aligned}
\ddot{\theta}|_{stance} &= \begin{bmatrix} \frac{\tau_h}{m_b {\theta_2}^2} - \frac{2\dot{\theta}_2\dot{\theta}_s}{\theta_2} \\ \frac{k_s(\rho_l - \theta_2)}{m_b} + \theta_2\dot{\theta}_s^2 \end{bmatrix} + \frac{\tau_t}{\rho_t m_b} \begin{bmatrix} \sin(\xi/\theta_2) \\ -\cos(\xi) \end{bmatrix} \\
\ddot{a}|_{stance} &= \begin{bmatrix} -\tau_h \\ \tau_t \end{bmatrix} \\
\begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix} |_{flight} &= \begin{bmatrix} 0 \\ -g \end{bmatrix} + \frac{\tau_t}{\rho_t m_b} \begin{bmatrix} \sin(\phi_1 + \phi_2) \\ -\cos(\phi_1 + \phi_2) \end{bmatrix} \\
\ddot{a}|_{flight} &= \begin{bmatrix} 0 \\ \tau_t \end{bmatrix}
\end{aligned} \tag{10}$$

Under assumption 3 of Penn Jerboa, the gravity terms(red terms in equation(9)) are negligible compare to the compression force of the spring. In equation(10), the red terms are the external forces.By removing the red terms, we will observe that the dynamics of the Penn Jerboa is same as the spring mass system. In other words, the dynamics of the Penn Jerboa can be understood as simply adding torques on a spring mass system.

Table 3: Nomenclature used in Advanced Model

g	gravitational constant
m_1	mass of body
m_2	mass of tail
I_1	inertia of body
b_*	friction coefficient, where b_* can be b_1(air resistance on body: translation), b_θ(friction acting on hip joint), b_{ϕ_1}(air resistance on body: rotation), b_{ϕ_2}(friction acting on tail joint), b_{l_1}(damping in spring)
k	spring constant
l_{01}	natural length of spring
l_1	length of spring
l_2	length of tail
x_1	horizontal position of C.O.M. of body
x_2	horizontal position of tail mass
y_1	vertical position of C.O.M. of body
y_2	vertical position of tail mass
T_s	duration of stance phase
k_r	gain of Raibert Stepping Control
k_{rp}	proportional gain of Raibert Stepping Control
k_{rd}	derivative gain of Raibert Stepping Control
k_{tp}	proportional gain of tail orientation control in flight phase
k_{td}	derivative gain of tail orientation control in flight phase
k_{te}	gain of tail energy pumping in stance phase
k_{tm}	gain of tail momentum control
k_{hp}	proportional gain of hip orientaion control
k_{hd}	derivative gain of hip orientaion control
τ_t	tail torque acting on θ_3
τ_h	hip torque acting on θ_2
ϵ	saturation parameter for energy pumping control
ω	$= \sqrt{\frac{k}{m_1}}$

3 Advanced Model

We call the model in this section "advanced" because it almost include every physical parameters of the system. However there are assumptions that we are still using in the model, i.e. consider the spring to be massless and consider m_2 as a point mass

From figure(2), we can observe the relationship among the joint angles in two diferent phases.

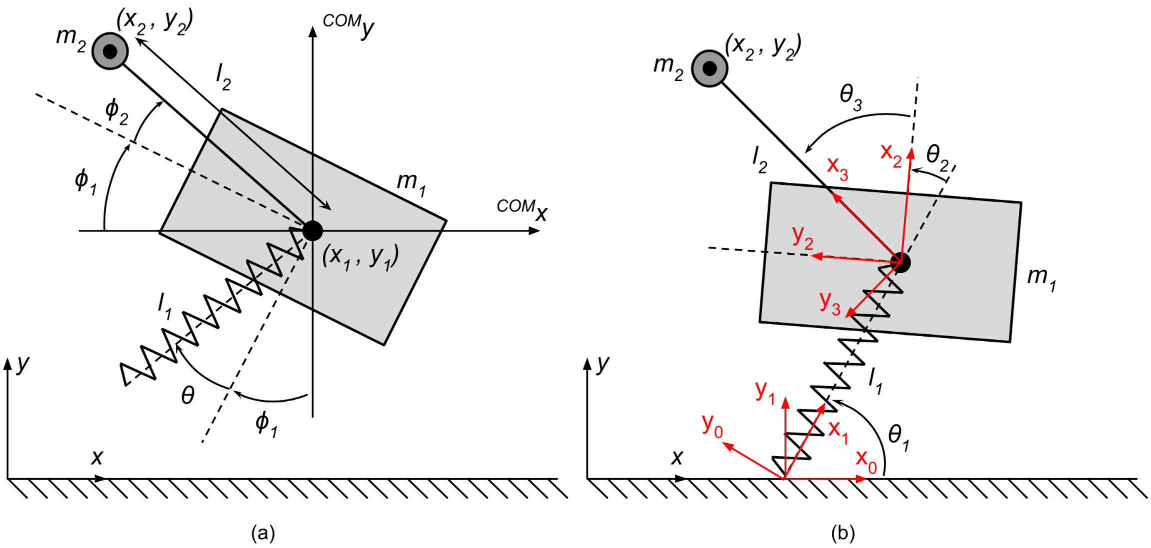

Figure 2: Advanced model.(a)Flight Phase, (b)Stance Phase

$$
\begin{aligned}
\theta_1 &= \frac{\pi}{2} - \theta - \phi_1 \\
\theta_2 &= \theta \\
\theta_3 &= \frac{\pi}{2} - \phi_2
\end{aligned} \tag{11}
$$

or,

$$
\begin{aligned}
\phi_1 &= \frac{\pi}{2} - \theta_1 - \theta_2 \\
\phi_2 &= \frac{\pi}{2} - \theta_3 \\
\theta &= \theta_2
\end{aligned} \tag{12}
$$

3.1 Lagrangian Dynamics

3.1.1 Flight Phase

Kinetic energy:

$$
\begin{aligned}
T =& \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} I_1 \dot{\phi}_1^2 \\
&+ \frac{1}{2} m_2 \left[\Big(\dot{x}_1 + l_2 (\dot{\phi}_1 + \dot{\phi}_2) \sin(\phi_1 + \phi_2) \Big)^2 + \Big(\dot{y}_1 + l_2 (\dot{\phi}_1 + \dot{\phi}_2) \cos(\phi_1 + \phi_2) \Big)^2 \right] \\
&+ \frac{1}{2} I_2 (\dot{\phi}_1 + \dot{\phi}_2)^2
\end{aligned} \tag{13}
$$

Potential energy:

$$
V = m_1 g y_1 + m_2 g \Big(y_1 + l_2 \sin(\phi_1 + \phi_2) \Big) \tag{14}
$$

The Lagrangian is defined as:

$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}I_1\dot{\phi}_1^2 \\
&+ \frac{1}{2}m_2\left[\Big(\dot{x}_1 + l_2(\dot{\phi}_1 + \dot{\phi}_2)\sin(\phi_1 + \phi_2)\Big)^2 + \Big(\dot{y}_1 + l_2(\dot{\phi}_1 + \dot{\phi}_2)\cos(\phi_1 + \phi_2)\Big)^2\right] \\
&+ \frac{1}{2}I_2(\dot{\phi}_1 + \dot{\phi}_2)^2 - m_1 g y_1 - m_2 g\Big(y_1 + l_2\sin(\phi_1 + \phi_2)\Big)
\end{aligned} \tag{15}$$

Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0 \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}_1} - \frac{\partial L}{\partial y_1} = 0 \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}_1} - \frac{\partial L}{\partial \phi_1} = 0 \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}_2} - \frac{\partial L}{\partial \phi_2} = 0 \tag{19}$$

equation(16) becomes:

$$(m_1 + m_2)\ddot{x}_1 + m_2 l_2\sin(\phi_1 + \phi_2)(\ddot{\phi}_1 + \ddot{\phi}_2) + m_2 l_2(\dot{\phi}_1 + \dot{\phi}_2)^2\cos(\phi_1 + \phi_2) = 0 \tag{20}$$

equation(17) becomes:

$$\begin{aligned}
&(m_1 + m_2)\ddot{y}_1 + m_2 l_2\cos(\phi_1 + \phi_2)(\ddot{\phi}_1 + \ddot{\phi}_2) \\
&- m_2 l_2(\dot{\phi}_1 + \dot{\phi}_2)^2\sin(\phi_1 + \phi_2) + (m_1 + m_2)g = 0
\end{aligned} \tag{21}$$

equation(18) becomes:

$$\begin{aligned}
&m_2 l_2\sin(\phi_1 + \phi_2)\ddot{x}_1 + m_2 l_2\cos(\phi_1 + \phi_2)\ddot{y}_1 \\
&+ (I_1 + I_2 + m_2 l_2^2)\ddot{\phi}_1 + (I_2 + m_2 l_2^2)\ddot{\phi}_2^2 + m_2 g l_2\cos(\phi_1 + \phi_2) = 0
\end{aligned} \tag{22}$$

equation(19) becomes:

$$\begin{aligned}
&m_2 l_2\sin(\phi_1 + \phi_2)\ddot{x}_1 + m_2 l_2\cos(\phi_1 + \phi_2)\ddot{y}_1 \\
&+ (I_2 + m_2 {l_2}^2)\ddot{\phi}_1 + (I_2 + m_2 {l_2}^2)\ddot{\phi}_2^2 + m_2 g l_2\cos(\phi_1 + \phi_2) = 0
\end{aligned} \tag{23}$$

Lastly, we have the equation of the hip joint

$$\ddot{\theta} = 0 \tag{24}$$

in matrix form:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = 0 \tag{25}$$

where,

$$M = \begin{bmatrix} m_1+m_2 & 0 & m_2 l_2\sin(\phi_1+\phi_2) & m_2 l_2\sin(\phi_1+\phi_2) & 0 \\ 0 & m_1+m_2 & m_2 l_2\cos(\phi_1+\phi_2) & m_2 l_2\cos(\phi_1+\phi_2) & 0 \\ m_2 l_2\sin(\phi_1+\phi_2) & m_2 l_2\cos(\phi_1+\phi_2) & I_1+I_2+m_2{l_2}^2 & I_2+m_2{l_2}^2 & 0 \\ m_2 l_2\sin(\phi_1+\phi_2) & m_2 l_2\cos(\phi_1+\phi_2) & I_2+m_2{l_2}^2 & I_2+m_2{l_2}^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

$$C = \begin{bmatrix} 0 & 0 & m_2 l_2\cos(\phi_1+\phi_2)(\dot{\phi_1}+\dot{\phi_2}) & m_2 l_2\cos(\phi_1+\phi_2)(\dot{\phi_1}+\dot{\phi_2}) & 0 \\ 0 & 0 & -m_2 l_2\sin(\phi_1+\phi_2)(\dot{\phi_1}+\dot{\phi_2}) & -m_2 l_2\sin(\phi_1+\phi_2)(\dot{\phi_1}+\dot{\phi_2}) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{27}$$

$$G = \begin{bmatrix} 0 \\ (m_1+m_2)g \\ m_2 g l_2\cos(\phi_1+\phi_2) \\ m_2 g l_2\cos(\phi_1+\phi_2) \\ 0 \end{bmatrix} \tag{28}$$

$$q = \begin{bmatrix} x_1 & y_1 & \phi_1 & \phi_2 & \theta \end{bmatrix}^T \tag{29}$$

With applied torque and friction:

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau - R\dot{q} \\ \ddot{q} = M(q)^{-1}(\tau - R\dot{q} - C(q,\dot{q})\dot{q} - G(q)) \end{aligned} \tag{30}$$

where,

$$\tau = \begin{bmatrix} 0 & 0 & 0 & \tau_t & \tau_h \end{bmatrix}^T \tag{31}$$

$$R = \begin{bmatrix} b_1 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & 0 \\ 0 & 0 & b_{\phi_1} & 0 & \\ 0 & 0 & 0 & b_{\phi_2} & 0 \\ 0 & 0 & 0 & 0 & b_\theta \end{bmatrix} \tag{32}$$

3.1.2 Stance Phase

Kinetic energy:

$$\begin{aligned} T =& \frac{1}{2}m_1\dot{l}_1^2 + \frac{1}{2}m_1{l_1}^2\dot{\theta}_1^2 + \frac{1}{2}I_1(\dot{\theta}_1^2+\dot{\theta}_2^2) + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_2\dot{y}_2^2 \\ =& \frac{1}{2}m_1\dot{l}_1^2 + \frac{1}{2}m_1{l_1}^2\dot{\theta}_1^2 + \frac{1}{2}I_1(\dot{\theta}_1^2+\dot{\theta}_2^2) \\ &+ \frac{1}{2}m_2\Big(\dot{l}_1\cos(\theta_1) - l_1\dot{\theta}_1\sin(\theta_1) - l_2(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\sin(\theta_1+\theta_2+\theta_3)\Big)^2 \\ &+ \frac{1}{2}m_2\Big(\dot{l}_1\sin(\theta_1) - l_1\dot{\theta}_1\cos(\theta_1) - l_2(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\cos(\theta_1+\theta_2+\theta_3)\Big)^2 \end{aligned} \tag{33}$$

Potential energy:

$$V = m_1 g l_1 \sin(\theta_1) + m_2 g\Big(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2 + \theta_3\Big) + \frac{1}{2}(l_1 - l_{01})^2 \tag{34}$$

The Lagrangian is defined as:

$$\begin{aligned}
L =& T - V \\
=& \frac{1}{2}m_1 \dot{l}_1^2 + \frac{1}{2}m_1 {l_1}^2 \dot{\theta}_1^2 + \frac{1}{2}I_1(\dot{\theta}_1^2 + \dot{\theta}_2^2) \\
&+ \frac{1}{2}m_2\Big(\dot{l}_1 \cos(\theta_1) - l_1\dot{\theta}_1 \sin(\theta_1) - l_2(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)\sin(\theta_1 + \theta_2 + \theta_3)\Big)^2 \\
&+ \frac{1}{2}m_2\Big(\dot{l}_1 \sin(\theta_1) - l_1\dot{\theta}_1 \cos(\theta_1) - l_2(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)\cos(\theta_1 + \theta_2 + \theta_3)\Big)^2 \\
&- m_1 g l_1 \sin(\theta_1) - m_2 g\Big(l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2 + \theta_3\Big) - \frac{1}{2}(l_1 - l_{01})^2
\end{aligned} \tag{35}$$

Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0 \tag{36}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0 \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}_3} - \frac{\partial L}{\partial \theta_3} = 0 \tag{38}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{l}_1} - \frac{\partial L}{\partial l_1} = 0 \tag{39}$$

equation(36) becomes:

$$\begin{aligned}
0 =& \Big(m_1 {l_1}^2 + I_1 + m_2\Big({l_1}^2 + 2 l_1 l_2 \cos(\theta_2 + \theta_3) + {l_2}^2\Big)\Big)\ddot{\theta}_1 \\
&+ \Big(I_1 + m_2\Big(l_1 l_2 \cos(\theta_2 + \theta_3) + {l_2}^2\Big)\Big)\ddot{\theta}_2 \\
&+ m_2\Big(l_1 l_2 \cos(\theta_2 + \theta_3) + {l_2}^2\Big)\ddot{\theta}_3 \\
&- m_2 l_2 \sin(\theta_2 + \theta_3)\ddot{l}_1 \\
&+ 2 m_1 l_1 \dot{l}_1 \dot{\theta}_1 \\
&+ m_2\Big(\dot{l}_1 l_2 \cos(\theta_2 + \theta_3)(\dot{\theta}_1 + \dot{\theta}_3) - l_1 l_2 \sin(\theta_2 + \theta_3)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_3) - l_1 l_2 \sin(\theta_2 + \theta_3)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)\Big) \\
&+ m_1 g l_1 \cos\theta_1 + m_2 g l_1\Big(\cos(\theta_1) + \cos(\theta_1 + \theta_2 + \theta_3)\Big)
\end{aligned} \tag{40}$$

equation(37) becomes:

$$
\begin{aligned}
0 =&\Big(I_1 + m_2 l_2\Big(l_1\cos(\theta_2+\theta_3) + l_2\Big)\Big)\ddot{\theta}_1 + (I_1 + m_2 {l_2}^2)\ddot{\theta}_2 + m_2 {l_2}^2\ddot{\theta}_3 - m_2 l_2 \sin(\theta_2+\theta_3)\ddot{l}_1 \\
&+ m_2 l_2\Big(\dot{l}_1\cos(\theta_2+\theta_3)(\dot{\theta}_2+\dot{\theta}_3) + \dot{l}_1\dot{\theta}_1\cos(\theta_2+\theta_3) - l_1\dot{\theta}_1\sin(\theta_2+\theta_3)(\dot{\theta}_2+\dot{\theta}_3) \\
&+ \Big(\dot{l}_1\cos(\theta_2+\theta_3) + l_1\dot{\theta}_1\sin(\theta_2+\theta_3)\Big)(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\Big) \\
&+ m_2 g l_2 \cos(\theta_1+\theta_2+\theta_3)
\end{aligned}
\tag{41}
$$

equation(38) becomes:

$$
\begin{aligned}
0 =&\Big(m_2 l_2\Big(l_1\cos(\theta_2+\theta_3) + l_2\Big)\Big)\ddot{\theta}_1 + m_2 {l_2}^2\ddot{\theta}_2 + m_2 {l_2}^2\ddot{\theta}_3 - m_2 l_2 \sin(\theta_2+\theta_3)\ddot{l}_1 \\
&+ m_2 l_2\Big(\dot{l}_1\cos(\theta_2+\theta_3)(\dot{\theta}_2+\dot{\theta}_3) + \dot{l}_1\dot{\theta}_1\cos(\theta_2+\theta_3) - l_1\dot{\theta}_1\sin(\theta_2+\theta_3)(\dot{\theta}_2+\dot{\theta}_3) \\
&+ \Big(\dot{l}_1\cos(\theta_2+\theta_3) + l_1\dot{\theta}_1\sin(\theta_2+\theta_3)\Big)(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\Big) \\
&+ m_2 g l_2 \cos(\theta_1+\theta_2+\theta_3)
\end{aligned}
\tag{42}
$$

equation(39) becomes:

$$
\begin{aligned}
0 =&\Big(-m_2 l_2\sin(\theta_2+\theta_3)\Big)\ddot{\theta}_1 + \Big(-m_2 l_2\sin(\theta_2+\theta_3)\Big)\ddot{\theta}_2\Big(-m_2 l_2\sin(\theta_2+\theta_3)\Big)\ddot{\theta}_3 + (m_1+m_2)\ddot{l}_1 \\
&- m_2 l_2\cos(\theta_2+\theta_3)(\dot{\theta}_2+\dot{\theta}_3)(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3) - m_1 l_1 {\dot{\theta}_1}^2 \\
&- m_2\Big(l_1{\dot{\theta}_1}^2 + l_2\cos(\theta_2+\theta_3)\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\Big) \\
&m_1 g\sin(\theta_1) + m_2 g\sin(\theta_2) + k(l_1 - l_{01})
\end{aligned}
\tag{43}
$$

in matrix form:

$$
M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = 0 \tag{44}
$$

where,

$$
\begin{aligned}
M_{11} &= I_1 + m_1 {l_1}^2 + m_2\Big({l_1}^2 + 2 l_1 l_2 \cos(\theta_2 + \theta_3) + {l_2}^2\Big) \\
M_{12} &= I_1 + m_2\Big(l_1 l_2 \cos(\theta_2 + \theta_3) + {l_2}^2\Big) \\
M_{13} &= m_2\Big(l_1 l_2 \cos(\theta_2 + \theta_3) + {l_2}^2\Big) \\
M_{14} &= -m_2 l_2 \sin(\theta_2 + \theta_3) \\
M_{21} &= I_1 + m_2 l_2\Big(l_1 \cos(\theta_2 + \theta_3) + l_2\Big) \\
M_{22} &= I_1 + m_2 {l_2}^2 \\
M_{23} &= m_2 {l_2}^2 \\
M_{24} &= -m_2 l_2 \sin(\theta_2 + \theta_3) \\
M_{31} &= m_2 l_2\Big(l_1 \cos(\theta_2 + \theta_3) + l_2\Big) \\
M_{32} &= m_2 {l_2}^2 \\
M_{33} &= m_2 {l_2}^2 \\
M_{34} &= -m_2 l_2 \sin(\theta_2 + \theta_3) \\
M_{41} &= -m_2 l_2 \sin(\theta_2 + \theta_3) \\
M_{42} &= -m_2 l_2 \sin(\theta_2 + \theta_3) \\
M_{43} &= -m_2 l_2 \sin(\theta_2 + \theta_3) \\
M_{44} &= m_1 + m_2
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&C_{11} = 2m_1 l_1 \dot{l}_1 + m_2\Big(\dot{l}_1 l_2 \cos(\dot{\theta}_2 + \dot{\theta}_3) - l_1 l_2 \sin(\theta_2 + \theta_3) - l_1 l_2 \sin(\theta_2 + \theta_3)\Big) \\
&C_{12} = -m_2 l_1 l_2 \sin(\theta_2 + \theta_3) \\
&C_{13} = m_2\Big(\dot{l}_1 l_2 \cos(\theta_2 + \theta_3) - l_1 l_2 \sin(\theta_2 + \theta_3)\Big) \\
&C_{14} = 0 \\
&C_{21} = m_2 l_2\Big(2\dot{l}_2 \cos(\theta_2 + \theta_3) - l_1 \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) + l_1\dot{\theta}_1 \sin(\theta_2 + \theta_3)\Big) \\
&C_{22} = m_2 l_2\Big(2\dot{l}_1 \cos(\theta_2 + \theta_3) + l_1\dot{\theta}_1 \sin(\theta_2 + \theta_3)\Big) \\
&C_{23} = m_2 l_2\Big(2\dot{l}_1 \cos(\theta_2 + \theta_3) + l_1\dot{\theta}_1 \sin(\theta_2 + \theta_3)\Big) \\
&C_{24} = 0 \\
&C_{31} = m_2 l_2\Big(2\dot{l}_2 \cos(\theta_2 + \theta_3) - l_1 \sin(\theta_2 + \theta_3)(\dot{\theta}_2 + \dot{\theta}_3) + l_1\dot{\theta}_1 \sin(\theta_2 + \theta_3)\Big) \\
&C_{32} = m_2 l_2\Big(2\dot{l}_1 \cos(\theta_2 + \theta_3) + l_1\dot{\theta}_1 \sin(\theta_2 + \theta_3)\Big) \\
&C_{33} = m_2 l_2\Big(2\dot{l}_1 \cos(\theta_2 + \theta_3) + l_1\dot{\theta}_1 \sin(\theta_2 + \theta_3)\Big) \\
&C_{34} = 0 \\
&C_{41} = -(m_1 + m_2) l_1\dot{\theta}_1 - m_2 l_2 \cos(\theta_2 + \theta_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \\
&C_{42} = -m_2 l_2 \cos(\theta_2 + \theta_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \\
&C_{43} = -m_2 l_2 \cos(\theta_2 + \theta_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \\
&C_{44} = 0
\end{aligned} \tag{46}
$$

$$
G = \begin{bmatrix} m_1 g l_1 \cos(\theta_1) + m_2 g l_1\big(\cos(\theta_1) + \cos(\theta_1 + \theta_2 + \theta_3)\big) \\ m_2 g l_2 \cos(\theta_1 + \theta_2 + \theta_3) \\ m_2 g l_2 \cos(\theta_1 + \theta_2 + \theta_3) \\ m_1 g \sin(\theta_1) + m_2 g \sin(\theta_2) + k(l_1 - l_{01}) \end{bmatrix} \tag{47}
$$

$$
q = \begin{bmatrix}\theta_1 & \theta_2 & \theta_3 & l_1\end{bmatrix}^T \tag{48}
$$

With applied torque and friction:

$$
\begin{aligned}
M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) &= \tau - R\dot{q} \\
\ddot{q} &= M(q)^{-1}(\tau - R\dot{q} - C(q,\dot{q})\dot{q} - G(q))
\end{aligned} \tag{49}
$$

where,

$$
\tau = \begin{bmatrix}0 & \tau_h & \tau_t & 0\end{bmatrix}^T \tag{50}
$$

$$
R = \begin{bmatrix} b_\theta + b_{\phi_1} & 0 & 0 & 0 \\ 0 & b_\theta & 0 & 0 \\ 0 & 0 & b_{\phi_2} & 0 \\ 0 & 0 & 0 & b_{l_1} \end{bmatrix} \tag{51}
$$

3.2 Control Methods

3.2.1 Flight Phase

In flight phase, we have the Raibert stepping control from hip actuator and the tail orientation control from tail actuator.

$$\begin{aligned}\tau_h|_{flight} &= -k_{rp}\big((\theta+\phi_1)-\theta_{td}\big) - k_{rd}\big((\dot\theta+\dot\phi_1)-0\big)\\ \tau_t|_{flight} &= -k_{tp}(\phi_2-\phi_2^*) - k_{td}(\dot\phi_2 - 0)\end{aligned} \tag{52}$$

where,

$$\theta_{td} = \sin^{-1}\left(\Big(\frac{\dot x_1 T_s}{2} + k_r(\dot x_1 - \dot x_1^*)\Big)/l_{01}\right) \tag{53}$$

3.2.2 Stance Phase

It is a little bit different from the Penn Jerboa, we brake down the stance phase into two equal halves, i.e. compression phase and thrust phase. And we will discuss it in the coming sections. For the hip actuator, we have similar controller:

$$\tau_h|_{stance} = k_{hd}(\phi_1 - \phi_1^*) + k_{hd}(\dot\phi_1 - 0) \tag{54}$$

Compression Phase
Compression phase refers to the phase in which the spring is reducing in length, i.e. $\dot l_1 < 0$. In compression phase, we follow the idea of energy pumping using the tail to apply torque in phase-locked manner. However, we make some changes to the control law in Penn Jerboa.
Original form:

$$\tau_t|_{stance} = -\rho_t\theta_2 m_b \cdot g_1^v(\dot z) \tag{55}$$

Corrected form:

$$\tau_t|_{stance} = -\rho_t m_b \cdot g_1^v(\dot l) \tag{56}$$

in our notations:

$$\tau_t|_{stance,compression} = -m_1 l_2 \frac{k_{te}\dot l/\omega}{\sqrt{\left((l-l_{01})^2 + (\dot l/\omega)^2\right)} + \epsilon} \tag{57}$$

Note that we have removed the term θ_2, which represents the length of spring in Penn Jerboa model.

Please refer to **Section 4** for more information of the function $g_1^v(\dot z)$ in Penn Jerboa or the explicit form $\frac{k_{te}\dot l/\omega}{\sqrt{\left((l-l_{01})^2+\dot l^2\right)}+\epsilon}$ in our model.

Thrust Phase
Thrust phase refers to the phase in which the spring is increasing in length, i.e. $\dot l_1 > 0$. In thrust phase, we try to control the total angular momentum such that the system can become more stable when it switches from stance phase to flight phase.

$$\tau_t|_{stance,thrust} = k_{tm}(\dot\phi_1 + \dot\phi_2) \tag{58}$$

4 Energy Pumping in spring mass system

In this section, we will show the proof of finding the stable limit cycle of a forced oscillating spring mass system in a phase-locked manner.Please refer to table(2) for the symbols used.

Informal Proof
From (2) and (3) in Section 3 of The Penn Jerboa,

$$\dot{\boldsymbol{x}} = -\omega \boldsymbol{J}\boldsymbol{x} + \boldsymbol{e}_2^T(-2\bar{\beta}\omega x_2 + \tau/\omega), \tag{59}$$

$$\tau = k_t x_2/(||\boldsymbol{x}|| + \epsilon), \tag{60}$$

we have the equation of motion of the spring mass system:

$$\dot{x} = -\omega Jx + \bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x|| + \epsilon)}\bigg) x_2 e_2 \tag{61}$$

where $x = [x_1 \quad x_2]^T, x_2 = \dot{x_1}/\omega$. Hence,

$$\begin{aligned} x^T\dot{x} &= x_1\dot{x_1} + x_2\dot{x_2} \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}({x_1}^2 + {x_2}^2) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||x||^2 \end{aligned} \tag{62}$$

But at the same time, from equation (61), we have:

$$x^T\dot{x} = {x_2}^2(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x|| + \epsilon)}) \tag{63}$$

consider $||x||^* = \frac{k_t}{2\bar{\beta}\omega^2} - \epsilon$,
we have:

$$x^T\dot{x} \begin{cases} < 0 & \text{if } ||x|| > ||x||^* \\ = 0 & \text{if } ||x|| = ||x||^* \\ > 0 & \text{if } ||x|| < ||x||^* \end{cases} \tag{64}$$

In other words, from equation(62),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||x||^2 \begin{cases} < 0 & \text{if } ||x|| > ||x||^* \\ = 0 & \text{if } ||x|| = ||x||^* \\ > 0 & \text{if } ||x|| < ||x||^* \end{cases} \tag{65}$$

Intuitively, the norm decreases with time when it is larger than $||x||^*$; does not vary when it equals $||x||^*$; and increases with time when it is larger than $||x||^*$. Which simply shows that $||x||^* = \frac{k_t}{2\bar{\beta}\omega^2} - \epsilon$ is a stable limit cycle. Hence, by choosing $k_t = 2\bar{\beta}\omega^2\|x\|^*$, the desired controller is,

$$\begin{aligned} \tau &= k_t x_2/(||\boldsymbol{x}|| + \epsilon) \\ &= 2\bar{\beta}\omega^2\|x\|^* \frac{x_2}{\|x\| + \epsilon} \end{aligned} \tag{66}$$

Note that $\|\boldsymbol{x}\|$ can be replaced by x_2^{max}, i.e., the maximum value of x_2.

Formal Proof

Recall equation(61) $\dot{x} = -\omega Jx + (-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)})x_2 e_2$,. Consider:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)}\bigg) \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(\sqrt{{x_1}^2+{x_2}^2}+\epsilon)}\bigg) \\
=&\frac{k_t}{\omega}\cdot\frac{-1/2({x_1}^2+{x_2}^2)^{-1/2}(2x_1\dot{x_1}+2x_2\dot{x_2})}{(\sqrt{{x_1}^2+{x_2}^2}+\epsilon)^2} \\
=&-\frac{k_t}{\omega}\cdot\frac{({x_1}^2+{x_2}^2)^{-1/2}(x_1\dot{x_1}+x_2\dot{x_2})}{(\sqrt{{x_1}^2+{x_2}^2}+\epsilon)^2} \\
=&-\frac{k_t}{\omega}\cdot\frac{({x_1}^2+{x_2}^2)^{-1/2}x^T\dot{x}}{(\sqrt{{x_1}^2+{x_2}^2}+\epsilon)^2} \\
=&-\frac{k_t}{\omega}\cdot\frac{({x_1}^2+{x_2}^2)^{-1/2}\Big[{x_2}^2(-2\bar{\beta}\omega+\frac{k_t}{\omega(||x||+\epsilon)})\Big]}{(\sqrt{{x_1}^2+{x_2}^2}+\epsilon)^2} \quad \text{from(63)} \\
=&-\frac{k_t}{\omega}\cdot\frac{{x_2}^2({x_1}^2+{x_2}^2)^{-1/2}}{(\sqrt{{x_1}^2+{x_2}^2}+\epsilon)^2}\bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)}\bigg) \\
&\text{where } -\frac{k_t}{\omega}\cdot\frac{{x_2}^2({x_1}^2+{x_2}^2)^{-1/2}}{(\sqrt{{x_1}^2+{x_2}^2}+\epsilon)^2} \text{ is always negative} \\
&\text{except for } ||x||=0 \text{ or } ||x||=\frac{k_t}{2\bar{\beta}\omega^2}-\epsilon
\end{aligned} \tag{67}$$

Hence, we obtain the invariant set $-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)} = 0$, or after changing subject: $||x|| = \frac{k_t}{2\bar{\beta}\omega^2} - \epsilon$, which is of course the same result as the informal proof.

Now, define a Lyapunov function candidate:

$$\begin{aligned}
V &= \bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)}\bigg)^2 \\
&\begin{cases} = 0 & \text{if } ||x|| = ||x||^* \\ > 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{68}$$

$$
\begin{aligned}
\dot{V} &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)} \bigg)^2 \\
&= 2\bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)} \bigg)\frac{\mathrm{d}}{\mathrm{d}t}\bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)} \bigg) \\
&= 2\bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)} \bigg)\bigg(-\frac{k_t}{\omega}\cdot\frac{{x_2}^2({x_1}^2+{x_2}^2)^{-1/2}}{(\sqrt{{x_1}^2+{x_2}^2}+\epsilon)^2} \bigg)\bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)} \bigg) \quad \text{from (67)} \\
&= 2\bigg(-\frac{k_t}{\omega}\cdot\frac{{x_2}^2({x_1}^2+{x_2}^2)^{-1/2}}{(\sqrt{{x_1}^2+{x_2}^2}+\epsilon)^2} \bigg)\bigg(-2\bar{\beta}\omega + \frac{k_t}{\omega(||x||+\epsilon)} \bigg)^2 \\
&\begin{cases} = 0 & \text{if } ||x|| = \frac{k_t}{2\bar{\beta}\omega^2} - \epsilon \\ < 0 & \text{otherwise but not} ||x|| = 0 \end{cases}
\end{aligned} \tag{69}
$$

Hence $||x|| = \frac{k_t}{2\bar{\beta}\omega^2} - \epsilon$ is an attractive limit cycle.

5 Extended Stability Proof

5.1 Vertical Hopping with Full Stance Phase Under Feedback Control

In this section, we propose a proof to the stability of the vertical hopping system under feedback control through out the stance phase. The controller we use is basically the equation, i.e. (57) derived in early section.

$$
F|_{stance} = m\frac{k_{te}\dot{y}/\omega}{\sqrt{\left((y-y_0)^2+(\dot{y}/\omega)^2\right)}+\epsilon}. \tag{70}
$$

Define the following quantity,

$$
\delta y = \big|\|y\| - \|y\|^*\big|. \tag{71}
$$

From fig.3, and consider the distance from center to the state as the norm,

$$
\begin{aligned}
\delta y_{i+1} &= \delta y(t_3) \\
&= \delta y(t_2) \\
&= \gamma\delta y(t_1) \\
&= \gamma\delta y_i
\end{aligned} \tag{72}
$$

Where,
$\delta y(t_3) = \delta y(t_2)$ because of the energy conservation;
$\delta y(t_2) = \gamma\delta y(t_1)$ because the controller try to 'pull' or 'push' the state towards the limit cycle.
In summary,

$$
\begin{cases}
\|y_{i+1}\| - \|y\|^* = \gamma(\|y_i\| - \|y\|^*) & \text{for } \|y_i\| > \|y\|^* \\
\|y_{i+1}\| - \|y\|^* = 0 & \text{for } \|y_i\| = \|y\|^* \\
\|y_{i+1}\| - \|y\|^* = \gamma(\|y_i\| - \|y\|^*) & \text{for } \|y_i\| < \|y\|^*
\end{cases} \tag{73}
$$

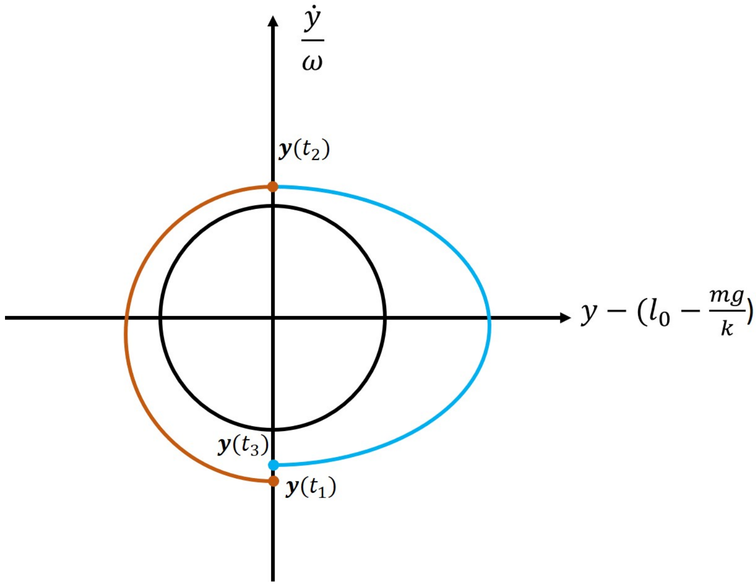

Figure 3: state-space plot

where,

$$\begin{cases} \gamma \in [0,1) & \text{for } \|y\| > \|y\|^* \\ \gamma = 1 & \text{for } \|y\| = \|y\|^* \\ \gamma \in (1,\infty) & \text{for } \|y\| < \|y\|^* \end{cases} \tag{74}$$

Fig.4 and fig.5 shows the conceptual plot of the return map based on the above equations. The plots are not exact but they show the possible shapes of the return maps. In the figures, yellow region indicates the possible location of the curve $\|y_{i+1}\| = \|y_i\|$ based on the arguments of the above equations.

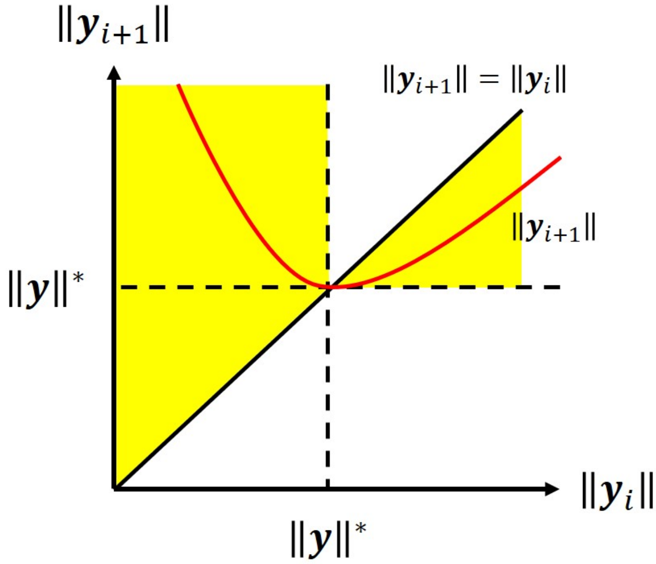

Figure 4: Conceptual diagram of the return map(case 1)

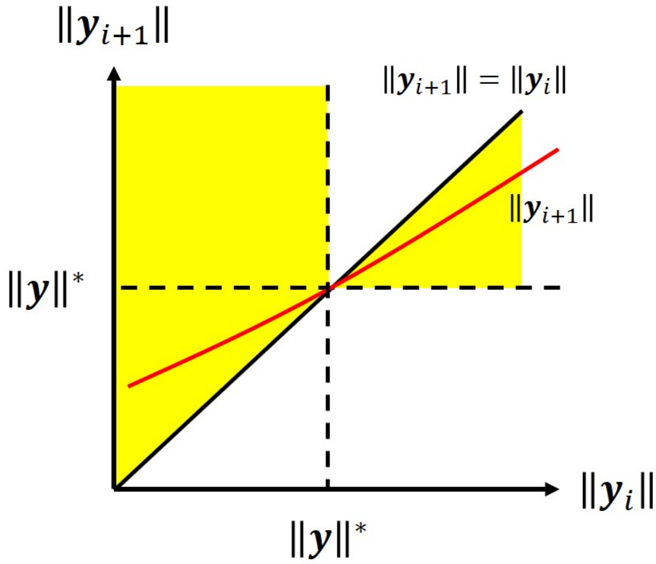

Figure 5: Conceptual diagram of the return map(case 2)

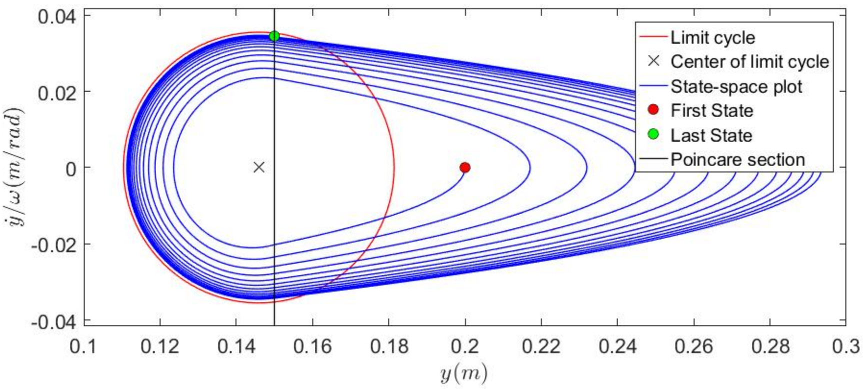

Figure 6: State-space plot showing outward convergence

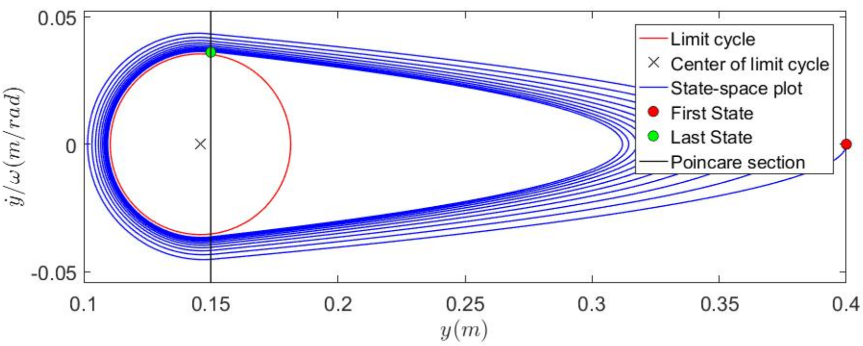

Figure 7: State-space plot showing inward convergence

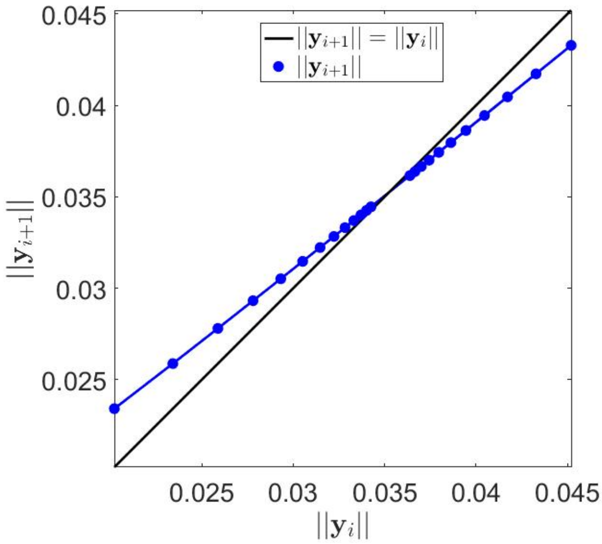

Figure 8: Return Map

5.2 Vertical Hopping with Half Stance Phase Under Feedback Control

In this section, we only control the compression phase while leaving thrust phase uncontrolled. With a large damping(fig.9), we cannot tell the stability or the convergence.[seems there is room for us to modify the feedback controller that can handle the damping coefficient, i.e. actively change the damping coefficient.]

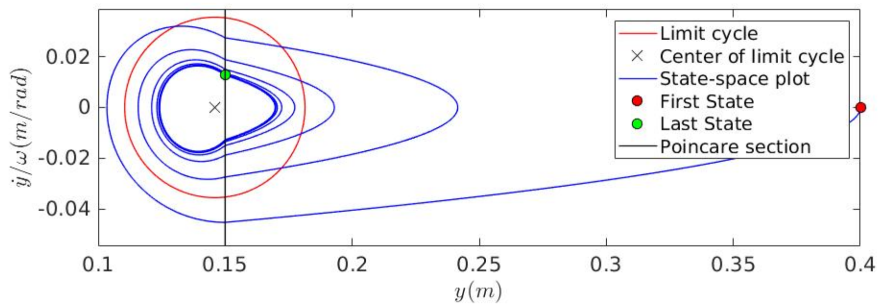

Figure 9: Return Map

6 Spring Compression Issue

In this section, we are going to find out the relationship between the maximum hopping height H, the mass m and the spring constant k for certain maximum spring compression Δw.

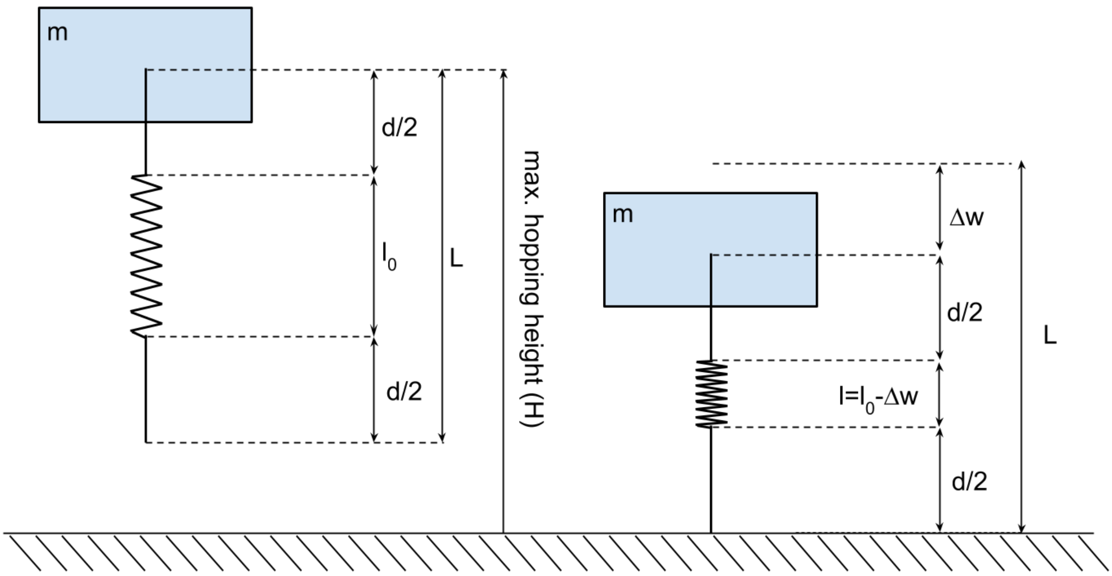

Figure 10: Model.(a)Flight Phase, (b)Stance Phase

We can simply consider the decrease in gravitational potential energy of the system will be used to increase the internal potential energy stored in the spring.

$$\begin{aligned} mgH &= mg(2 \times \frac{d}{2} + l_0 - \Delta w) + \frac{1}{2}k(\Delta w)^2 \\ H &= d + l_0 - \Delta w + \frac{k}{2mg}(\Delta w)^2 \end{aligned} \tag{75}$$

where,
d is non-spring length,
l_0 is natural length of spring,
L is the uncompressed leg length,
g is gravitational constant,
k is spring constant.

From figure(11) and figure(12), we can see that reducing mass can be a easier way to improve the hopping height compare to increasing spring constant. And with a larger maximum compression(fig.(12)), the maximum hopping height become significantly larger than that with a smaller maximum compression (figure(11)).
Note: Since our hopping robot has two legs, we can consider like each leg has spring constant 3000Nm and adding up two will give a total of 6000Nm.

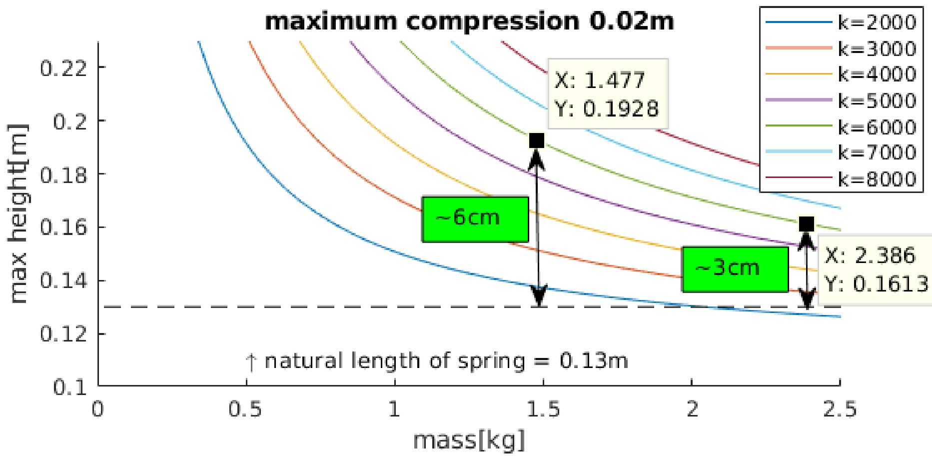

Figure 11: Maximum hopping height as a function of mass with different spring constants under the constraint: maximum compression = 0.02m.

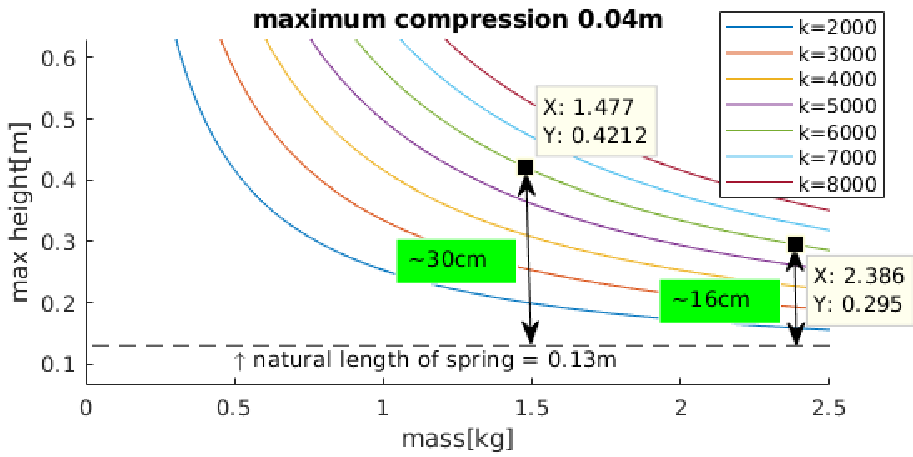

Figure 12: Maximum hopping height as a function of mass with different spring constants under the constraint: maximum compression = 0.04m.

7 Height Control Using Energy Pumping

In this section, we are going to find out the relationship between the gain k_t in energy pumping and the hopping height. To make things simple, we first use the veticle hopping spring mass system to do the analysis. Although it is different from the model used in Section(4), because we have the gravity effect in vertical orientation, the solution of the system is nearly the same. The only difference is the coordinate is shifted downward by mg/k(the natural compression).

some new variables used in this section:
y: height of the mass,
$\dot{y}$: vertical velocity of the mass,
k: spring constant,
m: mass,
ω: $\sqrt{\frac{k}{m}}$,
l_0: natural length of spring.
x: $= y - l_0 + mg/k$, displacement from the equilibrium position of mass under gravity

Ideally, according to the analysis in Section 4, we have the relationship between the gain k_t and the norm of $[(x) \quad \dot{x}/\omega]^T$ in the form $||x|| = \frac{k_t}{2\bar{\beta}\omega^2} - \epsilon$. For small ϵ, $||x|| = \frac{k_t}{2\bar{\beta}\omega^2}$ or equally, $\sqrt{x^2 + (\dot{x}/\omega)^2} = \frac{k_t}{2\bar{\beta}\omega^2}$. At the lowest position, the mass loses all its kinetic energy. So we have $\dot{x} = 0$ and hence $x = \frac{k_t}{2\bar{\beta}\omega^2}$. Which gives:

$$\begin{aligned} |y_{min} - l_0 + mg/k| &= \frac{k_t}{2\bar{\beta}\omega^2} \\ l_0 - y_{min} - mg/k &= \frac{mk_t}{b_{L_1}\omega} \\ l_0 - y_{min} &= \Big(\frac{m}{b_{L_1}}\sqrt{\frac{m}{k}}\Big)k_t + mg/k \end{aligned} \tag{76}$$

We can replace left hand side by Δw. Hence, $\Delta w = \Big(\frac{m}{b_{L_1}}\sqrt{\frac{m}{k}}\Big)k_t + mg/k$. We can then get the maximum hopping height by substituting it into (75).

$$H = d + l_0 - \left(\Big(\frac{m}{b_{L_1}}\sqrt{\frac{m}{k}}\Big)k_t + mg/k\right) + \frac{k}{2mg}\left(\Big(\frac{m}{b_{L_1}}\sqrt{\frac{m}{k}}\Big)k_t + mg/k\right)^2 \tag{77}$$

However, from the simulation result, the system in the stance phase tends to converge to a smaller limit cycle depending on how fast is the energy pump response before energy loss to damping.(figure(13)).

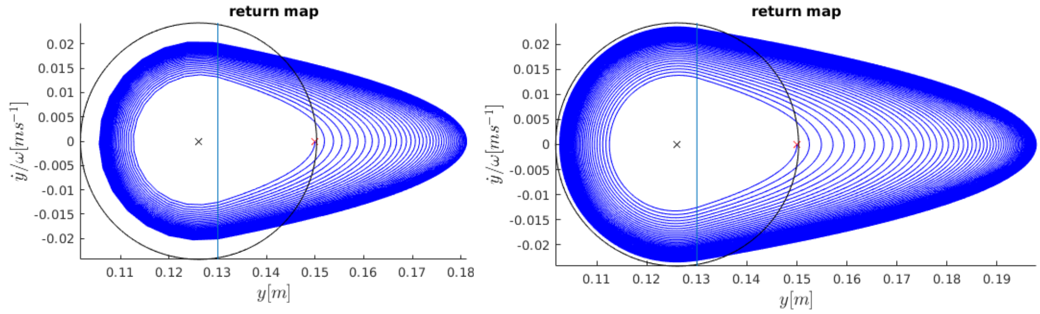

Figure 13: Return map. Sampling time is 0.005sec(left), 0.001sec(right). Black circle is the desired linit cycle $||x|| = \frac{k_t}{2\beta\omega^2}$. Vertical line indicates the natural length of leg. Red cross is the initial state. Blue cross is the centre of the limit cycle.

8 Front flip

In this section, we are going to review the technique that Marc Raibert used to do the flipping task. One thing different is that we are using tail to pump energy so that we are not able to inject energy to the robot in one single step before flipping like what Marc Raibert did. Instead, we can accumulate energy by using the tail and flip when we think the hopping height is high enough.

8.1 Angular speed for flipping

Time:
$T_{flight} : flight\ duration$
    $\dot{y}^* : vertical\ take\ off\ speed$

$$\begin{aligned} \frac{T_{flight}}{2} &= \frac{\dot{y}^*}{g} \\ T_{flight} &= \frac{2\dot{y}^*}{g} \end{aligned} \tag{78}$$

Angular speed:
$\dot{\phi}_1^* : angular\ speed\ at\ take\ off$
${\phi_1}^* : pitch\ down\ angle\ just\ before\ increasing\ \dot{\phi}_1^*$

$$\begin{aligned} \dot{\phi}_1^* T_{flight} &= 2\pi - \phi_1^* \\ \dot{\phi}_1^* &= \frac{(2\pi - \phi_1^*)}{T_{flight}} \end{aligned} \tag{79}$$

8.2 Control Method

Torque:
    I: inertia of body and tail

T_{thrust}: duration of thrust phase

$$\begin{aligned}\dot{\phi}_1^* &= \ddot{\phi}_1 T_{thrust} \\ &\approx \frac{\tau}{I} T_{thrust}\end{aligned} \tag{80}$$

Combining eqt(78)(9.2)(80),

$$\begin{aligned}\tau &\approx \frac{(2\pi - \phi_1{}^*)}{2\dot{y}^* T_{thrust}} Ig \\ &\approx \frac{(2\pi - \phi_1{}^*)}{2\sqrt{2g(y_{max} - l_0)}T_{thrust}} Ig\end{aligned} \tag{81}$$

pick a torque τ^* which is larger than certain value:

$$\tau = \tau^* > \frac{(2\pi - \phi_1{}^*)}{2\sqrt{2g(0.05)}T_{thrust}} Ig \tag{82}$$

where 0.05 maybe the minimum requirement of hopping height for front flip.

Then we can apply constant torque to accelerate ϕ_1 using the following scheme:

$$\tau \begin{cases} = \tau^* & \text{if } \dot{\phi_1} < \dot{\phi_1}^* \\ = 0 & \text{if } \dot{\phi_1} \geq \dot{\phi_1}^* \end{cases} \tag{83}$$

8.3 Simulation Result

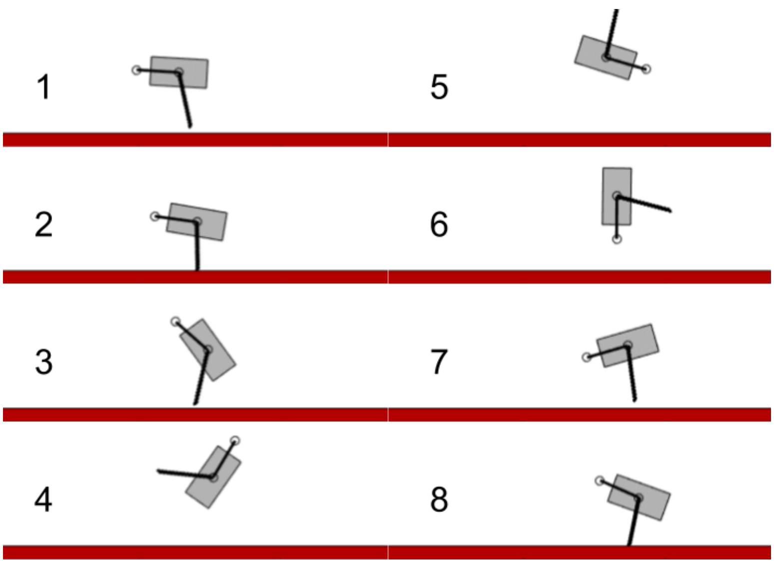

Figure 14: Front flip motion.

9 Front Flip with Tail Retraction

[prevent tail from touching ground + energy pumping in compression phase.]Now, we have a modification of the dynamic equation in flight phase such that the tail length is adjustable. Our idea is to reduce the tail length to half when the robot starts to flip such that the rotational inertia contributed by the tail along the pitch axis is greatly reduced. With the reduction in the rotational inertia under a fixed angular momentum, the system will rotate faster under the rule of conservation of angular momentum. We can than ensure the system rotate with the same angular speed in flight phase with a smaller angular speed before it take off.

9.1 Lagrangian Dynamics

9.1.1 Flight Phase

Kinetic energy:

$$\begin{aligned} T =& \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}I_1\dot{\phi}_1^2 + \cancel{\frac{1}{2}m_2\dot{l}_2^2} \\ &+ \frac{1}{2}m_2\left[\Big(\dot{x}_1 + l_2(\dot{\phi}_1 + \dot{\phi}_2)\sin(\phi_1+\phi_2)\Big)^2 + \Big(\dot{y}_1 + l_2(\dot{\phi}_1 + \dot{\phi}_2)\cos(\phi_1+\phi_2)\Big)^2\right] \\ &+ \frac{1}{2}I_2(\dot{\phi}_1 + \dot{\phi}_2)^2 \end{aligned} \tag{84}$$

Potential energy:

$$V = m_1 g y_1 + m_2 g\Big(y_1 + l_2\sin(\phi_1+\phi_2)\Big) \tag{85}$$

The Lagrangian is defined as:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}I_1\dot{\phi}_1^2 + \cancel{\frac{1}{2}m_2\dot{l}_2^2} \\ &+ \frac{1}{2}m_2\left[\Big(\dot{x}_1 + l_2(\dot{\phi}_1 + \dot{\phi}_2)\sin(\phi_1+\phi_2)\Big)^2 + \Big(\dot{y}_1 + l_2(\dot{\phi}_1 + \dot{\phi}_2)\cos(\phi_1+\phi_2)\Big)^2\right] \\ &+ \frac{1}{2}I_2(\dot{\phi}_1 + \dot{\phi}_2)^2 - m_1 g y_1 - m_2 g\Big(y_1 + l_2\sin(\phi_1+\phi_2)\Big) \end{aligned} \tag{86}$$

Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0 \tag{87}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}_1} - \frac{\partial L}{\partial y_1} = 0 \tag{88}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}_1} - \frac{\partial L}{\partial \phi_1} = 0 \tag{89}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}_2} - \frac{\partial L}{\partial \phi_2} = 0 \tag{90}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{l}_2} - \frac{\partial L}{\partial l_2} = 0 \tag{91}$$

equation(87) becomes:

$$\begin{aligned} &(m_1 + m_2)\ddot{x}_1 + m_2 l_2 \sin(\phi_1 + \phi_2)(\ddot{\phi}_1 + \ddot{\phi}_2) \\ &+ m_2 l_2(\dot{\phi}_1 + \dot{\phi}_2)^2 \cos(\phi_1 + \phi_2) + m_2 \dot{l}_2(\dot{\phi}_1 + \dot{\phi}_2) \sin(\phi_1 + \phi_2) = 0 \end{aligned} \tag{92}$$

equation(88) becomes:

$$\begin{aligned} &(m_1 + m_2)\ddot{y}_1 + m_2 l_2 \cos(\phi_1 + \phi_2)(\ddot{\phi}_1 + \ddot{\phi}_2) \\ &- m_2 l_2(\dot{\phi}_1 + \dot{\phi}_2)^2 \sin(\phi_1 + \phi_2) \\ &+ m_2 \dot{l}_2(\dot{\phi}_1 + \dot{\phi}_2) \cos(\phi_1 + \phi_2) + (m_1 + m_2)g = 0 \end{aligned} \tag{93}$$

equation(89) becomes:

$$\begin{aligned} &m_2 l_2 \sin(\phi_1 + \phi_2)\ddot{x}_1 + m_2 l_2 \cos(\phi_1 + \phi_2)\ddot{y}_1 \\ &+ (I_1 + I_2 + m_2 l_2^2)\ddot{\phi}_1 + (I_2 + m_2 l_2^2)\ddot{\phi}_2^2 \\ &+ m_2 \dot{l}_2 \sin(\phi_1 + \phi_2)\dot{x}_1 + m_2 \dot{l}_2 \cos(\phi_1 + \phi_2)\dot{y}_1 \\ &+ 2m_2 l_2 \dot{l}_2(\dot{\phi}_1 + \dot{\phi}_2) + mgl_2 \cos(\phi_1 + \phi_2) = 0 \end{aligned} \tag{94}$$

equation(90) becomes:

$$\begin{aligned} &m_2 l_2 \sin(\phi_1 + \phi_2)\ddot{x}_1 + m_2 l_2 \cos(\phi_1 + \phi_2)\ddot{y}_1 \\ &+ (I_2 + m_2 l_2^2)\ddot{\phi}_1 + (I_2 + m_2 l_2^2)\ddot{\phi}_2^2 \\ &+ m_2 \dot{l}_2 \sin(\phi_1 + \phi_2)\dot{x}_1 + m_2 \dot{l}_2 \cos(\phi_1 + \phi_2)\dot{y}_1 \\ &+ 2m_2 l_2 \dot{l}_2(\dot{\phi}_1 + \dot{\phi}_2) + mgl_2 \cos(\phi_1 + \phi_2) = 0 \end{aligned} \tag{95}$$

equation(91) becomes: !! need to re-derive, note that $l_2 = l_2(t)$.

$$\begin{aligned} &\cancel{m_2\ddot{l}_2 - m_2\ddot{x}_1 - m_2\ddot{y}_1} \\ &\cancel{-m_2 l_2\big(\sin(\phi_1 + \phi_2) + \cos(\phi_1 + \phi_2)\big)(\dot{\phi}_1 + \dot{\phi}_2)^2} \\ &\cancel{+m_2 g \sin(\phi_1 + \phi_2) = 0} \end{aligned} \tag{96}$$

Lastly, we have the equation of the hip joint

$$\ddot{\theta} = 0 \tag{97}$$

in matrix form:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = 0 \tag{98}$$

where,

$$M = \begin{bmatrix} m_1 + m_2 & 0 & m_2 l_2 \sin(\phi_1 + \phi_2) & m_2 l_2 \sin(\phi_1 + \phi_2) & 0 & 0 \\ 0 & m_1 + m_2 & m_2 l_2 \cos(\phi_1 + \phi_2) & m_2 l_2 \cos(\phi_1 + \phi_2) & 0 & 0 \\ m_2 l_2 \sin(\phi_1 + \phi_2) & m_2 l_2 \cos(\phi_1 + \phi_2) & I_1 + I_2 + m_2 {l_2}^2 & I_2 + m_2 {l_2}^2 & 0 & 0 \\ m_2 l_2 \sin(\phi_1 + \phi_2) & m_2 l_2 \cos(\phi_1 + \phi_2) & I_2 + m_2 {l_2}^2 & I_2 + m_2 {l_2}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{bmatrix} \tag{99}$$

$$
\begin{aligned}
C_{11} &= 0\\
C_{12} &= 0\\
C_{13} &= m_2 l_2 \cos(\phi_1 + \phi_2)(\dot{\phi_1} + \dot{\phi_2})\\
C_{14} &= m_2 l_2 \cos(\phi_1 + \phi_2)(\dot{\phi_1} + \dot{\phi_2})\\
C_{15} &= 0\\
C_{16} &= m_2(\dot{\phi}_1 + \dot{\phi}_2)\sin(\phi_1 + \phi_2)\\
C_{21} &= 0\\
C_{22} &= 0\\
C_{23} &= -m_2 l_2 \sin(\phi_1 + \phi_2)(\dot{\phi_1} + \dot{\phi_2})\\
C_{24} &= -m_2 l_2 \sin(\phi_1 + \phi_2)(\dot{\phi_1} + \dot{\phi_2})\\
C_{25} &= 0\\
C_{26} &= m_2(\dot{\phi}_1 + \dot{\phi}_2)\cos(\phi_1 + \phi_2)\\
C_{31} &= m_2 \dot{l}_2 \sin(\phi_1 + \phi_2)\\
C_{32} &= m_2 \dot{l}_2 \cos(\phi_1 + \phi_2)\\
C_{33} &= 2m_2 l_2 \dot{l}_2\\
C_{34} &= 2m_2 l_2 \dot{l}_2\\
C_{35} &= 0\\
C_{36} &= 0\\
C_{41} &= m_2 \dot{l}_2 \sin(\phi_1 + \phi_2)\\
C_{42} &= m_2 \dot{l}_2 \cos(\phi_1 + \phi_2)\\
C_{43} &= 2m_2 l_2 \dot{l}_2\\
C_{44} &= 2m_2 l_2 \dot{l}_2\\
C_{45} &= 0\\
C_{46} &= 0\\
C_{51} &= 0\\
C_{52} &= 0\\
C_{53} &= 0\\
C_{54} &= 0\\
C_{55} &= 0\\
C_{56} &= 0\\
C_{61} &= -m_2\\
C_{62} &= -m_2\\
C_{63} &= -m_2 l_2\big(\sin(\phi_1 + \phi_2) + \cos(\phi_1 + \phi_2)\big)(\dot{\phi_1} + \dot{\phi_2})\\
C_{64} &= -m_2 l_2\big(\sin(\phi_1 + \phi_2) + \cos(\phi_1 + \phi_2)\big)(\dot{\phi_1} + \dot{\phi_2})\\
C_{65} &= 0\\
C_{66} &= 0
\end{aligned}
\tag{100}
$$

$$G = \begin{bmatrix} 0 \\ (m_1+m_2)g \\ m_2gl_2\cos(\phi_1+\phi_2) \\ m_2gl_2\cos(\phi_1+\phi_2) \\ 0 \\ m_2g\sin(\phi_1+\phi_2) \end{bmatrix} \tag{101}$$

$$q = \begin{bmatrix} x_1 & y_1 & \phi_1 & \phi_2 & \theta & l_2 \end{bmatrix}^T \tag{102}$$

With applied torque and friction:

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) &= \tau - R\dot{q} \\ \ddot{q} &= M(q)^{-1}(\tau - R\dot{q} - C(q,\dot{q})\dot{q} - G(q)) \end{aligned} \tag{103}$$

where,

$$\tau = \begin{bmatrix} 0 & 0 & 0 & \tau_t & \tau_h \end{bmatrix}^T \tag{104}$$

$$R = \begin{bmatrix} b_1 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & 0 \\ 0 & 0 & b_{\phi_1} & 0 & \\ 0 & 0 & 0 & b_{\phi_2} & 0 \\ 0 & 0 & 0 & 0 & b_\theta \end{bmatrix} \tag{105}$$

9.1.2 Stance Phase

We have the same dynamic equation as before. We can change the tail length depending on the phase and use the same equation format to do the computation.

9.2 Control Method

From (9.2), we have $\dot{\phi}_1^* = \frac{(2\pi-\phi_1^*)}{T_{flight}}$
with ϕ_1^* being the desired angular speed for a robot with fixed tail length to rotate 360° in the mid air. Now we consider the conservation of angular momentum before and after take-off.

$$I_{stance}\dot{\phi_1}^* = I_{flight}\dot{\phi_1}^{**} \tag{106}$$

where, $\dot{\phi_1}^{**}$ being the desired angular speed just before take-off. When the tail is retracted, $I_{flight} < I_{stance}$,

$$\dot{\phi_1}^{**} = \frac{I_{stance}}{I_{flight}}\dot{\phi_1}^* < \dot{\phi_1}^*. \tag{107}$$

Rewrite (80),

$$\begin{aligned} \dot{\phi}_1^{**} &= \ddot{\phi}_1 T_{thrust} \\ &\approx \frac{\tau}{I_{stance}} T_{thrust} \end{aligned} \tag{108}$$

$$\begin{aligned} \tau &\approx \frac{I_{stance}\dot{\phi}_1^{**}}{T_{thrust}} \\ &= \frac{I_{stance}^2\dot{\phi}_1^*}{I_{flight}T_{thrust}} \\ &= \frac{I_{stance}^2\dot{\phi}_1^*(2\pi-\phi_1^*)}{I_{flight}T_{thrust}T_{flight}} \end{aligned} \tag{109}$$

9.3 Simulation Result

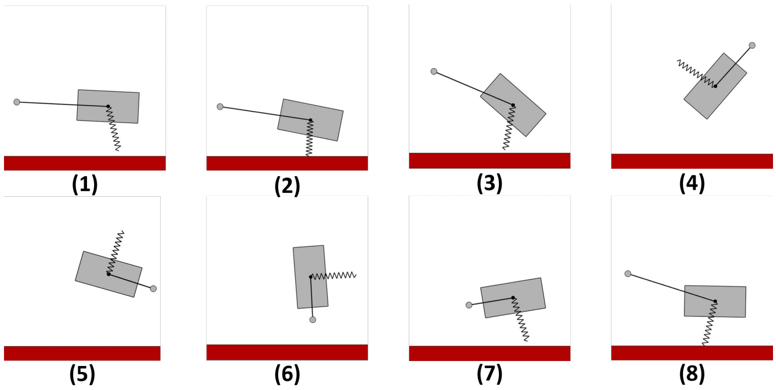

Figure 15: Front flip with tail retracted.

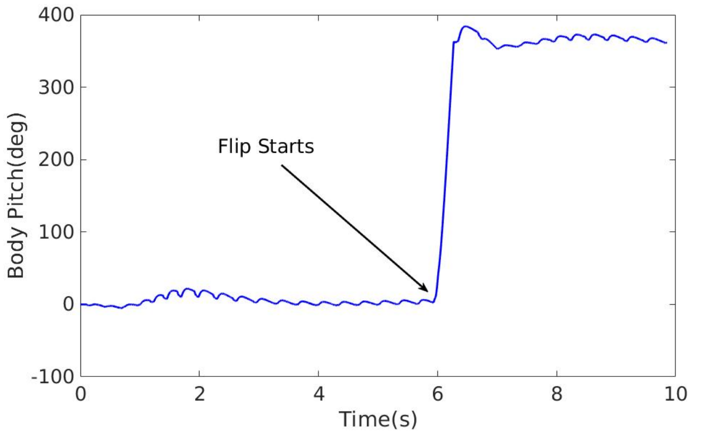

Figure 16: Body pitch angle increased from 0° to 360° in flight phase during flip.

10 Dynamic Equation of 3D Model in Flight Phase

In this section, we are going to introduce the 3D model of our hopping robot. At this stage, we only focus on the 3D dynamics in flight phase.

10.1 Model

Fig.19 shows the model of the 3D system. Note that there is no leg in the model because the massless leg will not contribute to the dynamics of the system. To make the derivation simple, we decomposed the flight phase dynamics into two parts: "internal dynamics" and "external dynamics". "Internal dynamics" refers to the dynamics of the system observed in frame 0. Frame 0 is a

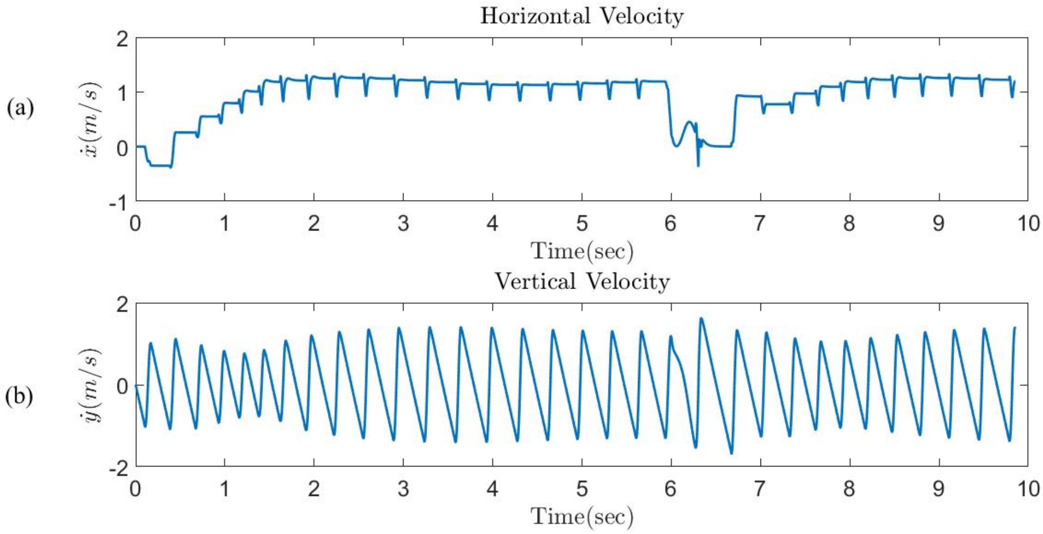

Figure 17: (a)Horizontal velocity.(b)Vertical velocity.

downward(-z direction) accelerating frame, or a transnational non-inertia frame. In this frame, there is no relative velocity among all particles. Besides, unlike rotating frame, it won't generate pseudo force(centrifugal force and Coriolis force). Hence one can simply consider the system being put in outer space where no gravity acts on the system. In such situation, only joint angles are variables while center of mass of the system is stationary. While "external dynamics" refers to the dynamics of frame 0, which only includes motions in x y z direction and under constant gravity in z-direction.

10.1.1 Important notes on the model

note that the box in the model has a different coordinate frame compare to frame 3. Below shows there relationships,

$$\begin{aligned} {}^{o}R_{b} &= {}^{o}R_{box}\ {}^{box}R_{b} \\ &= {}^{o}R_{box}\ {}^{b}R_{box}^{T} \end{aligned} \tag{110}$$

$where\ {}^{b}R_{box} = R_y(-\pi/2)R_z(-\pi)$

10.2 Equations

10.2.1 Recursive Newton-Euler method

For "internal dynamics", we apply recursive Newton-Euler method to get the analytic form of joint torques first

i	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	θ_1	0
2	$-\pi/2$	0	$\pi/2+\theta_2$	0
3	$-\pi/2$	0	θ_3	0
4	$\pi/2$	0	θ_4	0
5	$-\pi/2$	0	$-\pi/2$	0
6	$\pi/2$	0	θ_6	0
7	$-\pi/2$	0	0	d_7+l_2

Figure 18: DH table of 3D model.

Outward iteration: i from 0 to 6

$$\begin{aligned}
{}^{i+1}\omega_{i+1} &= {}^{i+1}R_i\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} \\
{}^{i+1}\dot{\omega}_{i+1} &= {}^{i+1}R_i\, {}^{i}\dot{\omega}_i + {}^{i+1}R_i\, {}^{i}\omega_i \times \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} \\
{}^{i+1}\dot{v}_{i+1} &= {}^{i+1}R_i({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1})) + {}^{i}\dot{v}_i \\
{}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1} \\
{}^{i+1}F_{i+1} &= m_{i+1}\, {}^{i+1}\dot{v}_{C_{i+1}} \\
{}^{i+1}N_{i+1} &= {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\omega_{i+1}
\end{aligned} \tag{111}$$

Inward iteration: i from 7 to 1

$$\begin{aligned}
{}^{i}f_i &= {}^{i}R_{i+1}\, {}^{i+1}f_{i+1} + {}^{i}F_i \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}R_{i+1}\, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}R_{i+1}\, {}^{i+1}f_{i+1} \\
\tau_i &= {}^{i}n_i^T\, {}^{i}\hat{Z}_i
\end{aligned} \tag{112}$$

[!] missing description of the variables in above equations.

[!] missing the assignment of the first element in the iteration

10.2.2 Dynamic Equation Formulation

After getting the joint torques(τ) from recursive Newton-Euler method, we can get the elements of the matrices: $M(q)_{6\times6}, C(q,\dot{q})_{6\times6}$. Since we are free from external contact, we don't have relative motion due to the effect of gravity. As a result, we can consider $G(q)_{6\times1}$ as a zero vector.

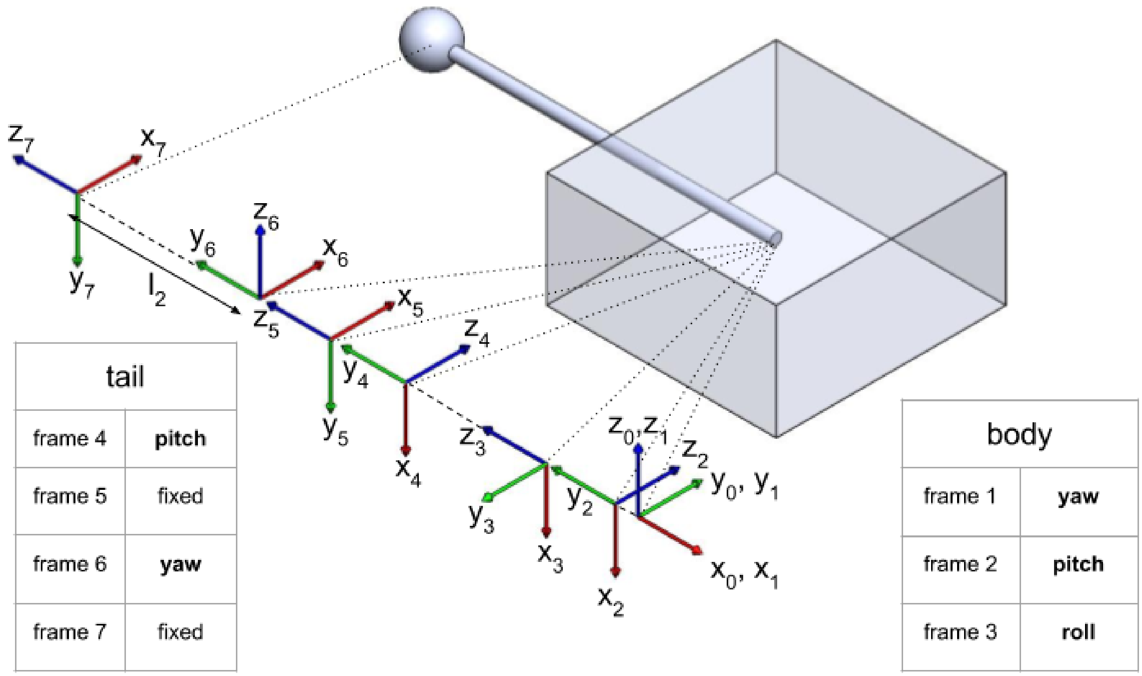

Figure 19: 3D model in flight phase. Frame 5 is a fix frame, hence there are only 6DOFs.

$$
\begin{aligned}
M_{i,j} &= \frac{\partial}{\partial \ddot{\theta}_j}\tau_i \\
C_{i,j} &= \frac{1}{2}\sum_{k=1}^{6}\Big(\frac{\partial M_{i,j}}{\partial \theta_k} + \frac{\partial M_{i,k}}{\partial \theta_j} - \frac{\partial M_{k,j}}{\partial \theta_i}\Big)\dot{\theta}_k \\
G_i &= 0
\end{aligned}
\tag{113}
$$

i	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	θ_1	0
2	$-\pi/2$	0	$\pi/2+\theta_2$	0
3	$-\pi/2$	0	θ_3	0
4	$\pi/2$	0	θ_4	0
5	$-\pi/2$	0	$-\pi/2$	0
6	$\pi/2$	0	θ_6	0
7	$-\pi/2$	0	0	d_7+l_2

Figure 20: DH table of the linkage.

$$
\begin{aligned}
M_{1,1} =& S_2\Big(I_{zz}S_2 + m_2S_4(d_7+l_2)^2(S_2S_4 - C_2C_3C_4) \\
&+ m_2C_4S_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)\Big) \\
&+ C_2\Bigg(S_3\Big(I_{yy}C_2S_3 - m_2C_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)\Big) \\
&+ C_3\Big(I_{xx}C_2C_3 - m_2C_4(d_7+l_2)^2(S_2S_4 - C_2C_3C_4) \\
&+ m_2S_4S_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)\Big)\Bigg) \\
M_{1,2} =& C_2\Bigg(C_3\Big(I_{xx}S_3 + m_2{C_4}^2S_3(d_7+l_2)^2 + m_2S_4S_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)\Big) \\
&- S_3\Big(I_{yy}C_3 + m_2C_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)\Big)\Bigg) + m_2C_4C_6S_2(d_7+l_2)^2(C_3S_6 - C_6S_3S_4) \\
M_{1,3} =& S_2(I_{zz} + m_2{S_4}^2(d_7+l_2)^2 + m_2{C_4}^2{S_6}^2(d_7+l_2)^2) - m_2C_2C_4C_6(d_7+l_2)^2(S_3S_6 + C_3C_6S_4) \\
M_{1,4} =& m_2C_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6) \\
M_{1,5} =& -M_2(d_7+l_2)^2(S_2S_4 - C_2C_3C_4) \\
M_{1,6} =& 0 \\
M_{2,1} =& S_3\Big(I_{xx}C_2C_3 - M_2C_4(d_7+l_2)^2(S_2S_4 - C_2C_3C_4) + M_2S_4S_6(d_7+l_2)^2(C_4S_2S_6 \\
&- C_2C_6S_3 + C_2C_3S_4S_6)\Big) - C_3\Big(I_{yy}C_2S_3 - M_2C_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)\Big) \\
M_{2,2} =& S_3\Big(I_{xx}S_3 + M_2C_4^2S_3(d_7+l_2)^2 + M_2S_4S_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)\Big) \\
&+ C_3\Big(I_{yy}C_3 + M_2C_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)\Big) \\
M_{2,3} =& M_2C_4C_6(d_7+l_2)^2(C_3S_6 - C_6S_3S_4) \\
M_{2,4} =& M_2C_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6) \\
M_{2,5} =& M_2C_4S_3(d_7+l_2)^2 \\
M_{2,6} =& 0 \\
M_{3,1} =& I_{zz}S_2 + M_2S_4(d_7+l_2)^2(S_2S_4 - C_2C_3C_4) \\
&+ M_2C_4S_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6) \\
M_{3,2} =& M_2C_4C_6(d_7+l_2)^2(C_3S_6 - C_6S_3S_4) \\
M_{3,3} =& I_{zz} + M_2{S_4}^2(d_7+l_2)^2 + M_2{C_4}^2{S_6}^2(d_7+l_2)^2 \\
M_{3,4} =& M_2C_4C_6S_6(d_7+l_2)^2 \\
M_{3,5} =& -M_2S_4(d_7+l_2)^2 \\
M_{3,6} =& 0
\end{aligned}
$$

$$
\begin{aligned}
M_{5,1} =& - M_2(d_7+l_2)^2(S_2S_4 - C_2C_3C_4)\\
M_{5,2} =& M_2C_4S_3(d_7+l_2)^2\\
M_{5,3} =& - M_2S_4(d_7+l_2)^2\\
M_{5,4} =& 0\\
M_{5,5} =& M_2(d_7+l_2)^2\\
M_{5,6} =& 0\\
M_{6,1} =& 0\\
M_{6,2} =& 0\\
M_{6,3} =& 0\\
M_{6,4} =& 0\\
M_{6,5} =& 0\\
M_{6,6} =& M_2
\end{aligned} \tag{115}
$$

$$
\begin{aligned}
C_{1,1} =& \dot{\theta}_3((C_2(C_3(I_{yy}C_2S_3 - M_2C_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))\\
&+ S_3(I_{yy}C_2C_3 + M_2C_2C_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)) - C_3(I_{xx}C_2S_3 + M_2C_2C_4^2S_3(d_7+l_2)^2\\
&+ M_2C_2S_4S_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)) - S_3(I_{xx}C_2C_3 - M_2C_4(d_7+l_2)^2(S_2S_4 - C_2C_3C_4)\\
&+ M_2S_4S_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))))/2\\
&- (M_2C_2C_4C_6S_2(d_7+l_2)^2(C_3S_6 - C_6S_3S_4))/2)\\
&+ \dot{d}_7((S_2(M_2S_4(S_2S_4 - C_2C_3C_4)(2d_7+2l_2)\\
&+ M_2C_4S_6(2d_7+2l_2)(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))/2\\
&- (C_2(C_3(M_2C_4(S_2S_4 - C_2C_3C_4)(2d_7+2l_2)\\
&- M_2S_4S_6(2d_7+2l_2)(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))\\
&+ M_2C_6S_3(2d_7+2l_2)(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))/2)\\
&+ \dot{\theta}_2((S_2(I_{zz}C_2 + M_2S_4(d_7+l_2)^2(C_2S_4 + C_3C_4S_2)\\
&+ M_2C_4S_6(d_7+l_2)^2(C_6S_2S_3 + C_2C_4S_6 - C_3S_2S_4S_6)))/2\\
&- (S_2(S_3(I_{yy}C_2S_3 - M_2C_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))\\
&+ C_3(I_{xx}C_2C_3 - M_2C_4(d_7+l_2)^2(S_2S_4 - C_2C_3C_4)\\
&+ M_2S_4S_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))))/2\\
&- (C_2(S_3(I_{yy}S_2S_3 + M_2C_6(d_7+l_2)^2(C_6S_2S_3 + C_2C_4S_6 - C_3S_2S_4S_6))\\
&+ C_3(I_{xx}C_3S_2 + M_2C_4(d_7+l_2)^2(C_2S_4 + C_3C_4S_2)\\
&- M_2S_4S_6(d_7+l_2)^2(C_6S_2S_3 + C_2C_4S_6 - C_3S_2S_4S_6))))/2\\
&+ (C_2(I_{zz}S_2 + M_2S_4(d_7+l_2)^2(S_2S_4 - C_2C_3C_4)\\
&+ M_2C_4S_6(d_7+l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))/2)\\
&+ M_2\dot{\theta}_6(d_7+l_2)^2({C_4}^2C_6{S_2}^2S_6 - {C_2}^2C_6{S_3}^2S_6 - C_2C_4{C_6}^2S_2S_3 + C_2C_4S_2S_3{S_6}^2\\
&- {C_2}^2C_3{C_6}^2S_3S_4 + {C_2}^2C_3S_3S_4{S_6}^2 + {C_2}^2{C_3}^2C_6{S_4}^2S_6 + 2C_2C_3C_4C_6S_2S_4S_6)\\
&+ M_2\dot{\theta}_4C_6(d_7+l_2)^2(S_2S_4 - C_2C_3C_4)(C_2S_3S_6 + C_4C_6S_2 + C_2C_3C_6S_4)
\end{aligned} \tag{116}
$$

$$
\begin{aligned}
C_{1,2} =& \dot{\theta}_3((C_2(I_{zz} + M_2S_4^2(d_7 + l_2)^2 + M_2C_4^2S_6^2(d_7 + l_2)^2))/2 \\
& - (C_2(S_3(I_{xx}S_3 + M_2C_4^2S_3(d_7 + l_2)^2 + M_2S_4S_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6)) \\
& + C_3(I_{yy}C_3 + M_2C_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6)) \\
& - S_3(I_{yy}S_3 + M_2C_6(d_7 + l_2)^2(C_6S_3 - C_3S_4S_6)) \\
& - C_3(I_{xx}C_3 + M_2C_3C_4^2(d_7 + l_2)^2 - M_2S_4S_6(d_7 + l_2)^2(C_6S_3 - C_3S_4S_6))))/2) \\
& + \dot{d}_7((C_2(C_3(M_2C_4^2S_3(2d_7 + 2l_2) + M_2S_4S_6(C_3C_6 + S_3S_4S_6)(2d_7 + 2l_2)) \\
& - M_2C_6S_3(C_3C_6 + S_3S_4S_6)(2d_7 + 2l_2)))/2 \\
& + (M_2C_4C_6S_2(C_3S_6 - C_6S_3S_4)(2d_7 + 2l_2))/2) \\
& + \dot{\theta}_1((S_2(I_{zz}C_2 + M_2S_4(d_7 + l_2)^2(C_2S_4 + C_3C_4S_2) \\
& + M_2C_4S_6(d_7 + l_2)^2(C_6S_2S_3 + C_2C_4S_6 - C_3S_2S_4S_6)))/2 \\
& - (S_2(S_3(I_{yy}C_2S_3 - M_2C_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)) \\
& + C_3(I_{xx}C_2C_3 - M_2C_4(d_7 + l_2)^2(S_2S_4 - C_2C_3C_4) \\
& + M_2S_4S_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))))/2 \\
& - (C_2(S_3(I_{yy}S_2S_3 + M_2C_6(d_7 + l_2)^2(C_6S_2S_3 + C_2C_4S_6 - C_3S_2S_4S_6)) \\
& + C_3(I_{xx}C_3S_2 + M_2C_4(d_7 + l_2)^2(C_2S_4 + C_3C_4S_2) \\
& - M_2S_4S_6(d_7 + l_2)^2(C_6S_2S_3 + C_2C_4S_6 - C_3S_2S_4S_6))))/2 \\
& + (C_2(I_{zz}S_2 + M_2S_4(d_7 + l_2)^2(S_2S_4 - C_2C_3C_4) \\
& + M_2C_4S_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))/2) \\
& - \dot{\theta}_2(S_2(C_3(I_{xx}S_3 + M_2C_4^2S_3(d_7 + l_2)^2 + M_2S_4S_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6)) \\
& - S_3(I_{yy}C_3 + M_2C_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6))) \\
& - M_2C_2C_4C_6(d_7 + l_2)^2(C_3S_6 - C_6S_3S_4)) \\
& - (M_2\dot{\theta}_6(d_7 + l_2)^2(2C_2C_3^2S_4 + 2C_2C_6^2S_4 + 2C_3C_4S_2 - 2C_3C_4C_6^2S_2 \\
& - 4C_2C_3^2C_6^2S_4 - 4C_2C_3C_6S_3S_6 - 2C_4C_6S_2S_3S_4S_6 + 2C_2C_3C_4^2C_6S_3S_6))/2 \\
& - M_2\dot{\theta}_4C_6(d_7 + l_2)^2(C_4^2C_6S_2S_3 - C_2C_3^2C_4S_6 - C_6S_2S_3 \\
& + C_3S_2S_4S_6 + C_2C_3C_4C_6S_3S_4)
\end{aligned}
\tag{117}
$$

$$
\begin{aligned}
C_{1,3} =&\dot{\theta}_1((C_2(C_3(I_{yy}C_2S_3 - M_2C_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))\\
&+ S_3(I_{yy}C_2C_3 + M_2C_2C_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6))\\
&- C_3(I_{xx}C_2S_3 + M_2C_2C_4^2S_3(d_7 + l_2)^2 + M_2C_2S_4S_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6))\\
&- S_3(I_{xx}C_2C_3 - M_2C_4(d_7 + l_2)^2(S_2S_4 - C_2C_3C_4)\\
&+ M_2S_4S_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))))/2\\
&- (M_2C_2C_4C_6S_2(d_7 + l_2)^2(C_3S_6 - C_6S_3S_4))/2)\\
&+ \dot{d}_7((S_2(M_2S_4^2(2d_7 + 2l_2) + M_2C_4^2S_6^2(2d_7 + 2l_2)))/2\\
&- (M_2C_2C_4C_6(S_3S_6 + C_3C_6S_4)(2d_7 + 2l_2))/2)\\
&+ \dot{\theta}_2((C_2(I_{zz} + M_2S_4^2(d_7 + l_2)^2 + M_2C_4^2S_6^2(d_7 + l_2)^2))/2\\
&- (C_2(S_3(I_{xx}S_3 + M_2C_4^2S_3(d_7 + l_2)^2 + M_2S_4S_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6))\\
&+ C_3(I_{yy}C_3 + M_2C_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6))\\
&- S_3(I_{yy}S_3 + M_2C_6(d_7 + l_2)^2(C_6S_3 - C_3S_4S_6)) - C_3(I_{xx}C_3 + M_2C_3C_4^2(d_7 + l_2)^2\\
&- M_2S_4S_6(d_7 + l_2)^2(C_6S_3 - C_3S_4S_6))))/2)\\
&+ M_2\dot{\theta}_4C_4C_6^2(d_7 + l_2)^2(S_2S_4 - C_2C_3C_4)\\
&+ M_2\dot{\theta}_6C_4C_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)\\
&- M_2\dot{\theta}_3C_2C_4C_6(d_7 + l_2)^2(C_3S_6 - C_6S_3S_4)
\end{aligned} \tag{118}
$$

$$
\begin{aligned}
C_{1,4} =&(M_2\dot{\theta}_6(d_7 + l_2)^2(2C_4C_6^2S_2 - 2C_4S_2 - 2C_2C_3S_4 + 2C_2C_3C_6^2S_4 + 2C_2C_6S_3S_6))/2\\
&+ (M_2\dot{d}_7C_6(2d_7 + 2l_2)(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))/2\\
&- M_2\dot{\theta}_2C_6(d_7 + l_2)^2(C_4^2C_6S_2S_3 - C_2C_3^2C_4S_6 - C_6S_2S_3 + C_3S_2S_4S_6 + C_2C_3C_4C_6S_3S_4)\\
&+ M_2\dot{\theta}_1C_6(d_7 + l_2)^2(S_2S_4 - C_2C_3C_4)(C_2S_3S_6 + C_4C_6S_2 + C_2C_3C_6S_4)\\
&+ M_2\dot{\theta}_3C_4C_6^2(d_7 + l_2)^2(S_2S_4 - C_2C_3C_4) - M_2\dot{\theta}_4C_6S_6(d_7 + l_2)^2(S_2S_4 - C_2C_3C_4)
\end{aligned} \tag{119}
$$

$$
\begin{aligned}
C_{1,5} =&(M_2\dot{\theta}_4(d_7 + l_2)^2(2C_4C_6^2S_2 - 2C_4S_2 - 2C_2C_3S_4 + 2C_2C_3C_6^2S_4 + 2C_2C_6S_3S_6))/2\\
&- (M_2\dot{d}_7(S_2S_4 - C_2C_3C_4)(2d_7 + 2l_2))/2\\
&- (M_2\dot{\theta}_2(d_7 + l_2)^2(2C_2C_3^2S_4 + 2C_2C_6^2S_4 + 2C_3C_4S_2 - 2C_3C_4C_6^2S_2 - 4C_2C_3^2C_6^2S_4\\
&- 4C_2C_3C_6S_3S_6 - 2C_4C_6S_2S_3S_4S_6 + 2C_2C_3C_4^2C_6S_3S_6))/2\\
&+ M_2\dot{\theta}_1(d_7 + l_2)^2(C_4^2C_6S_2^2S_6 - C_2^2C_6S_3^2S_6\\
&- C_2C_4C_6^2S_2S_3 + C_2C_4S_2S_3S_6^2 - C_2^2C_3C_6^2S_3S_4 + C_2^2C_3S_3S_4S_6^2 + C_2^2C_3^2C_6S_4^2S_6\\
&+ 2C_2C_3C_4C_6S_2S_4S_6) + M_2\dot{\theta}_3C_4C_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)
\end{aligned} \tag{120}
$$

$$
\begin{aligned}
C_{1,6} =&\dot{\theta}_3(M_2S_2(d_7+l_2)(C_4^2S_6^2+S_4^2)-(M_2C_2C_4C_6(S_3S_6+C_3C_6S_4)(2d_7+2l_2))/2)\\
&+\dot{\theta}_2((C_2(2M_2C_3(d_7+l_2)(S_3C_4^2+S_3S_4^2S_6^2+C_3C_6S_4S_6)\\
&-M_2C_6S_3(C_3C_6+S_3S_4S_6)(2d_7+2l_2)))/2\\
&+(M_2C_4C_6S_2(C_3S_6-C_6S_3S_4)(2d_7+2l_2))/2)\\
&-M_2\dot{\theta}_1(d_7+l_2)(C_2^2-C_2^2C_3^2-C_2^2C_6^2+C_4^2C_6^2+2C_2^2C_3^2C_6^2-C_2^2C_4^2C_6^2-C_2^2C_3^2C_4^2C_6^2\\
&+2C_2C_4C_6S_2S_3S_6+2C_2C_3C_4C_6^2S_2S_4+2C_2^2C_3C_6S_3S_4S_6-1)\\
&-(M_2\dot{\theta}_6(S_2S_4-C_2C_3C_4)(2d_7+2l_2))/2\\
&+(M_2\dot{\theta}_4C_6(2d_7+2l_2)(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6))/2
\end{aligned} \tag{121}
$$

$$
\begin{aligned}
C_{2,1} =&(M_2\dot{\theta}_6(d_7+l_2)^2(2C_2S_4-2C_2C_3^2S_4-2C_2C_6^2S_4+2C_3C_4C_6^2S_2+4C_2C_3^2C_6^2S_4\\
&+4C_2C_3C_6S_3S_6+2C_4C_6S_2S_3S_4S_6-2C_2C_3C_4^2C_6S_3S_6))/2\\
&-\dot{d}_7((S_3(M_2C_4(S_2S_4-C_2C_3C_4)(2d_7+2l_2)\\
&-M_2S_4S_6(2d_7+2l_2)(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6)))/2\\
&-(M_2C_3C_6(2d_7+2l_2)(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6))/2)\\
&-\dot{\theta}_1((S_2(I_{zz}C_2+M_2S_4(d_7+l_2)^2(C_2S_4+C_3C_4S_2)\\
&+M_2C_4S_6(d_7+l_2)^2(C_6S_2S_3+C_2C_4S_6-C_3S_2S_4S_6)))/2\\
&-(S_2(S_3(I_{yy}C_2S_3-M_2C_6(d_7+l_2)^2(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6))\\
&+C_3(I_{xx}C_2C_3-M_2C_4(d_7+l_2)^2(S_2S_4-C_2C_3C_4)\\
&+M_2S_4S_6(d_7+l_2)^2(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6))))/2\\
&-(C_2(S_3(I_{yy}S_2S_3+M_2C_6(d_7+l_2)^2(C_6S_2S_3+C_2C_4S_6-C_3S_2S_4S_6))\\
&+C_3(I_{xx}C_3S_2+M_2C_4(d_7+l_2)^2(C_2S_4+C_3C_4S_2)\\
&-M_2S_4S_6(d_7+l_2)^2(C_6S_2S_3+C_2C_4S_6-C_3S_2S_4S_6))))/2\\
&+(C_2(I_{zz}S_2+M_2S_4(d_7+l_2)^2(S_2S_4-C_2C_3C_4)\\
&+M_2C_4S_6(d_7+l_2)^2(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6)))/2)\\
&-\dot{\theta}_3((C_3(I_{yy}C_2C_3+M_2C_2C_6(d_7+l_2)^2(C_3C_6+S_3S_4S_6)))/2\\
&-(S_3(I_{yy}C_2S_3-M_2C_6(d_7+l_2)^2(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6)))/2+(I_{zz}C_2)/2\\
&+(S_3(I_{xx}C_2S_3+M_2C_2C_4^2S_3(d_7+l_2)^2+M_2C_2S_4S_6(d_7+l_2)^2(C_3C_6+S_3S_4S_6)))/2\\
&-(C_3(I_{xx}C_2C_3-M_2C_4(d_7+l_2)^2(S_2S_4-C_2C_3C_4)\\
&+M_2S_4S_6(d_7+l_2)^2(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6)))/2\\
&+(M_2S_4(d_7+l_2)^2(C_2S_4+C_3C_4S_2))/2\\
&+(M_2C_4S_6(d_7+l_2)^2(C_6S_2S_3+C_2C_4S_6-C_3S_2S_4S_6))/2)\\
&-M_2\dot{\theta}_4C_4C_6(d_7+l_2)^2(C_2S_6-C_2C_3^2S_6+C_4C_6S_2S_3+C_2C_3C_6S_3S_4)
\end{aligned} \tag{122}
$$

$$
\begin{aligned}
C_{2,2} = &\dot{d}_7((S_3(M_2C_4^2S_3(2d_7+2l_2)+M_2S_4S_6(C_3C_6+S_3S_4S_6)(2d_7+2l_2)))/2\\
&+(M_2C_3C_6(C_3C_6+S_3S_4S_6)(2d_7+2l_2))/2)\\
&+\dot{\theta}_3((S_3(I_{xx}C_3+M_2C_3C_4^2(d_7+l_2)^2-M_2S_4S_6(d_7+l_2)^2(C_6S_3-C_3S_4S_6)))/2\\
&+(C_3(I_{xx}S_3+M_2C_4^2S_3(d_7+l_2)^2+M_2S_4S_6(d_7+l_2)^2(C_3C_6+S_3S_4S_6)))/2\\
&-(S_3(I_{yy}C_3+M_2C_6(d_7+l_2)^2(C_3C_6+S_3S_4S_6)))/2\\
&-(C_3(I_{yy}S_3+M_2C_6(d_7+l_2)^2(C_6S_3-C_3S_4S_6)))/2)\\
&-M_2\dot{\theta}_6(d_7+l_2)^2(C_3^2C_6S_6-C_3C_6^2S_3S_4+C_3S_3S_4S_6^2-C_6S_3^2S_4^2S_6)\\
&+M_2\dot{\theta}_4C_4C_6S_3(d_7+l_2)^2(C_3S_6-C_6S_3S_4)
\end{aligned}
\tag{123}
$$

$$
\begin{aligned}
C_{2,3} = &\dot{\theta}_2((S_3(I_{xx}C_3+M_2C_3C_4^2(d_7+l_2)^2-M_2S_4S_6(d_7+l_2)^2(C_6S_3-C_3S_4S_6)))/2\\
&+(C_3(I_{xx}S_3+M_2C_4^2S_3(d_7+l_2)^2+M_2S_4S_6(d_7+l_2)^2(C_3C_6+S_3S_4S_6)))/2\\
&-(S_3(I_{yy}C_3+M_2C_6(d_7+l_2)^2(C_3C_6+S_3S_4S_6)))/2\\
&-(C_3(I_{yy}S_3+M_2C_6(d_7+l_2)^2(C_6S_3-C_3S_4S_6)))/2)\\
&-\dot{\theta}_1((C_2(I_{zz}+M_2S_4^2(d_7+l_2)^2+M_2C_4^2S_6^2(d_7+l_2)^2))/2\\
&-(S_3(I_{yy}C_2S_3-M_2C_6(d_7+l_2)^2(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6)))/2\\
&+(C_3(I_{yy}C_2C_3+M_2C_2C_6(d_7+l_2)^2(C_3C_6+S_3S_4S_6)))/2\\
&+(S_3(I_{xx}C_2S_3+M_2C_2C_4^2S_3(d_7+l_2)^2+M_2C_2S_4S_6(d_7+l_2)^2(C_3C_6+S_3S_4S_6)))/2\\
&-(C_3(I_{xx}C_2C_3-M_2C_4(d_7+l_2)^2(S_2S_4-C_2C_3C_4)\\
&+M_2S_4S_6(d_7+l_2)^2(C_4S_2S_6-C_2C_6S_3+C_2C_3S_4S_6)))/2\\
&+(M_2C_4C_6S_2(d_7+l_2)^2(S_3S_6+C_3C_6S_4))/2)+(M_2\dot{d}_7C_4C_6(C_3S_6-C_6S_3S_4)(2d_7+2l_2))/2\\
&+(M_2\dot{\theta}_4C_6^2S_3(2S_4^2-2)(d_7+l_2)^2)/2-M_2\dot{\theta}_3C_4C_6(d_7+l_2)^2(S_3S_6+C_3C_6S_4)\\
&+M_2\dot{\theta}_6C_4C_6(d_7+l_2)^2(C_3C_6+S_3S_4S_6)
\end{aligned}
\tag{124}
$$

$$
\begin{aligned}
C_{2,4} = &(M_2\dot{d}_7C_6(C_3C_6+S_3S_4S_6)(2d_7+2l_2))/2\\
&-M_2C_4C_6(d_7+l_2)^2(\dot{\theta}_1C_2S_6+\dot{\theta}_2C_6S_4-\dot{\theta}_4S_3S_6+\dot{\theta}_3C_4C_6S_3-\dot{\theta}_2C_3S_3S_6-\dot{\theta}_1C_2C_3^2S_6\\
&-\dot{\theta}_2C_3^2C_6S_4+\dot{\theta}_1C_4C_6S_2S_3+\dot{\theta}_1C_2C_3C_6S_3S_4)\\
&-(M_2\dot{\theta}_6(d_7+l_2)^2(2S_3S_4-2C_6^2S_3S_4+2C_3C_6S_6))/2;
\end{aligned}
\tag{125}
$$

$$
\begin{aligned}
C_{2,5} = &(M_2(d_7+l_2)^2(2\dot{\theta}_2S_6C_6+2\dot{\theta}_1C_2S_4-2\dot{\theta}_4S_3S_4-2\dot{\theta}_4C_3C_6S_6-2\dot{\theta}_2C_3S_3S_4+2\dot{\theta}_3C_3C_4C_6^2\\
&-2\dot{\theta}_1C_2C_3^2S_4-2\dot{\theta}_1C_2C_6^2S_4-4\dot{\theta}_2C_3^2C_6S_6-2\dot{\theta}_2C_4^2C_6S_6+2\dot{\theta}_4C_6^2S_3S_4+4\dot{\theta}_1C_2C_3^2C_6^2S_4\\
&+2\dot{\theta}_2C_3^2C_4^2C_6S_6+2\dot{\theta}_1C_3C_4C_6^2S_2+4\dot{\theta}_2C_3C_6^2S_3S_4+4\dot{\theta}_1C_2C_3C_6S_3S_6+2\dot{\theta}_3C_4C_6S_3S_4S_6\\
&+2\dot{\theta}_1C_4C_6S_2S_3S_4S_6-2\dot{\theta}_1C_2C_3C_4^2C_6S_3S_6))/2+(M_2\dot{d}_7C_4S_3(2d_7+2l_2))/2
\end{aligned}
\tag{126}
$$

$$
\begin{aligned}
C_{2,6} =& M_2\dot{\theta}_2(d_7 + l_2)(C_3^2C_6^2 + C_4^2S_3^2 + S_3^2S_4^2S_6^2 + 2C_3C_6S_3S_4S_6) \\
&- \dot{\theta}_1((S_3(M_2C_4(S_2S_4 - C_2C_3C_4)(2d_7 + 2l_2) \\
&- M_2S_4S_6(2d_7 + 2l_2)(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))/2 \\
&- (M_2C_3C_6(2d_7 + 2l_2)(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))/2) \\
&+ (M_2\dot{\theta}_4C_6(C_3C_6 + S_3S_4S_6)(2d_7 + 2l_2))/2 + (M_2\dot{\theta}_6C_4S_3(2d_7 + 2l_2))/2 \\
&+ (M_2\dot{\theta}_3C_4C_6(C_3S_6 - C_6S_3S_4)(2d_7 + 2l_2))/2
\end{aligned} \tag{127}
$$

$$
\begin{aligned}
C_{3,1} =& \dot{d}_7((M_2S_4(S_2S_4 - C_2C_3C_4)(2d_7 + 2l_2))/2 \\
&+ (M_2C_4S_6(2d_7 + 2l_2)(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))/2) \\
&- \dot{\theta}_1((C_2(C_3(I_{yy}C_2S_3 - M_2C_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)) \\
&+ S_3(I_{yy}C_2C_3 + M_2C_2C_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6)) \\
&- C_3(I_{xx}C_2S_3 + M_2C_2C_4^2S_3(d_7 + l_2)^2 + M_2C_2S_4S_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6)) \\
&- S_3(I_{xx}C_2C_3 - M_2C_4(d_7 + l_2)^2(S_2S_4 - C_2C_3C_4) \\
&+ M_2S_4S_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))))/2 \\
&- (M_2C_2C_4C_6S_2(d_7 + l_2)^2(C_3S_6 - C_6S_3S_4))/2) \\
&+ \dot{\theta}_2((C_3(I_{yy}C_2C_3 + M_2C_2C_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6)))/2 \\
&- (S_3(I_{yy}C_2S_3 - M_2C_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))/2 \\
&+ (I_{zz}C_2)/2 \\
&+ (S_3(I_{xx}C_2S_3 + M_2C_2C_4^2S_3(d_7 + l_2)^2 + M_2C_2S_4S_6(d_7 + l_2)^2(C_3C_6 + S_3S_4S_6)))/2 \\
&- (C_3(I_{xx}C_2C_3 - M_2C_4(d_7 + l_2)^2(S_2S_4 - C_2C_3C_4) \\
&+ M_2S_4S_6(d_7 + l_2)^2(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6)))/2 \\
&+ (M_2S_4(d_7 + l_2)^2(C_2S_4 + C_3C_4S_2))/2 \\
&+ (M_2C_4S_6(d_7 + l_2)^2(C_6S_2S_3 + C_2C_4S_6 - C_3S_2S_4S_6))/2) \\
&+ M_2\dot{\theta}_4C_6(d_7 + l_2)^2(C_2C_3C_6 - C_2C_3C_4^2C_6 + C_4C_6S_2S_4 + C_2S_3S_4S_6) \\
&+ M_2\dot{\theta}_6C_4(d_7 + l_2)^2(C_2S_3 - C_2C_6^2S_3 + C_4C_6S_2S_6 + C_2C_3C_6S_4S_6)
\end{aligned} \tag{128}
$$

$$
\begin{aligned}
C_{3,2} =& \dot{\theta}_1((C_2(S_3(I_{xx}S_3 + M_2C_4^2S_3(d_7+l_2)^2 + M_2S_4S_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)) \\
&+ C_3(I_{yy}C_3 + M_2C_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)) \\
&- S_3(I_{yy}S_3 + M_2C_6(d_7+l_2)^2(C_6S_3 - C_3S_4S_6)) \\
&- C_3(I_{xx}C_3 + M_2C_3C_4^2(d_7+l_2)^2 - M_2S_4S_6(d_7+l_2)^2(C_6S_3 - C_3S_4S_6))))/2 \\
&+ (I_{zz}C_2)/2 + (M_2S_4(d_7+l_2)^2(C_2S_4 + C_3C_4S_2))/2 \\
&+ (M_2C_4S_6(d_7+l_2)^2(C_6S_2S_3 + C_2C_4S_6 - C_3S_2S_4S_6))/2 \\
&+ (M_2C_4C_6S_2(d_7+l_2)^2(S_3S_6 + C_3C_6S_4))/2) - \dot{\theta}_2((S_3(I_{xx}C_3 + M_2C_3C_4^2(d_7+l_2)^2 \\
&- M_2S_4S_6(d_7+l_2)^2(C_6S_3 - C_3S_4S_6)))/2 + (C_3(I_{xx}S_3 + M_2C_4^2S_3(d_7+l_2)^2 \\
&+ M_2S_4S_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)))/2 \\
&- (S_3(I_{yy}C_3 + M_2C_6(d_7+l_2)^2(C_3C_6 + S_3S_4S_6)))/2 \\
&- (C_3(I_{yy}S_3 + M_2C_6(d_7+l_2)^2(C_6S_3 - C_3S_4S_6)))/2) \\
&- M_2\dot{\theta}_4C_6(d_7+l_2)^2(C_3S_4S_6 - C_6S_3 + C_4^2C_6S_3) \\
&+ M_2\dot{\theta}_6C_4(d_7+l_2)^2(C_3C_6^2 - C_3 + C_6S_3S_4S_6) \\
&+ (M_2\dot{d}_7C_4C_6(C_3S_6 - C_6S_3S_4)(2d_7+2l_2))/2
\end{aligned} \tag{129}
$$

$$
\begin{aligned}
C_{3,3} =& \dot{d}_7((M_2S_4^2(2d_7+2l_2))/2 + (M_2C_4^2S_6^2(2d_7+2l_2))/2) \\
&+ M_2C_4C_6(d_7+l_2)^2(\dot{\theta}_4C_6S_4 + \dot{\theta}_6C_4S_6)
\end{aligned} \tag{130}
$$

$$
\begin{aligned}
C_{3,4} =& M_2C_6(d_7+l_2)^2(\dot{\theta}_2C_6S_3 - \dot{\theta}_4S_4S_6 + \dot{\theta}_1C_2C_3C_6 + \dot{\theta}_3C_4C_6S_4 - \dot{\theta}_2C_3S_4S_6 - \dot{\theta}_2C_4^2C_6S_3 \\
&+ \dot{\theta}_1C_4C_6S_2S_4 + \dot{\theta}_1C_2S_3S_4S_6 - \dot{\theta}_1C_2C_3C_4^2C_6) + M_2\dot{\theta}_6C_4(d_7+l_2)^2(C_6^2 - 1) \\
&+ (M_2\dot{d}_7C_4C_6S_6(2d_7+2l_2))/2
\end{aligned} \tag{131}
$$

$$
\begin{aligned}
C_{3,5} =& (M_2C_4(d_7+l_2)^2(2\dot{\theta}_4C_6^2 - 2\dot{\theta}_4 - 2\dot{\theta}_2C_3 + 2\dot{\theta}_1C_2S_3 + 2\dot{\theta}_2C_3C_6^2 + 2\dot{\theta}_3C_4C_6S_6 - 2\dot{\theta}_1C_2C_6^2S_3 \\
&+ 2\dot{\theta}_1C_4C_6S_2S_6 + 2\dot{\theta}_2C_6S_3S_4S_6 + 2\dot{\theta}_1C_2C_3C_6S_4S_6))/2 - (M_2\dot{d}_7S_4(2d_7+2l_2))/2
\end{aligned} \tag{132}
$$

$$
\begin{aligned}
C_{3,6} =& \dot{\theta}_1((M_2S_4(S_2S_4 - C_2C_3C_4)(2d_7+2l_2))/2 \\
&+ (M_2C_4S_6(2d_7+2l_2)(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))/2) \\
&+ M_2\dot{\theta}_3(d_7+l_2)(C_4^2S_6^2 + S_4^2) - (M_2\dot{\theta}_6S_4(2d_7+2l_2))/2 \\
&+ (M_2\dot{\theta}_2C_4C_6(C_3S_6 - C_6S_3S_4)(2d_7+2l_2))/2 + (M_2\dot{\theta}_4C_4C_6S_6(2d_7+2l_2))/2
\end{aligned} \tag{133}
$$

$$
\begin{aligned}
C_{4,1} =& (M_2\dot{d}_7C_6(2d_7+2l_2)(C_4S_2S_6 - C_2C_6S_3 + C_2C_3S_4S_6))/2 \\
&+ M_2\dot{\theta}_6C_6(d_7+l_2)^2(C_2S_3S_6 + C_4C_6S_2 + C_2C_3C_6S_4) \\
&- M_2\dot{\theta}_3C_6(d_7+l_2)^2(C_2C_3C_6 - C_2C_3C_4^2C_6 + C_4C_6S_2S_4 + C_2S_3S_4S_6) \\
&- M_2\dot{\theta}_1C_6(d_7+l_2)^2(S_2S_4 - C_2C_3C_4)(C_2S_3S_6 + C_4C_6S_2 + C_2C_3C_6S_4) \\
&+ M_2\dot{\theta}_2C_4C_6(d_7+l_2)^2(C_2S_6 - C_2C_3^2S_6 + C_4C_6S_2S_3 + C_2C_3C_6S_3S_4)
\end{aligned} \tag{134}
$$

$$
\begin{aligned}
C_{4,2} =& M_2C_4C_6(d_7+l_2)^2(\dot\theta_1C_2S_6+\dot\theta_2C_6S_4-\dot\theta_2C_3S_3S_6-\dot\theta_1C_2C_3^2S_6-\dot\theta_2C_3^2C_6S_4\\
&+\dot\theta_1C_4C_6S_2S_3+\dot\theta_1C_2C_3C_6S_3S_4)-M_2\dot\theta_6C_6(d_7+l_2)^2(C_3S_6-C_6S_3S_4)\\
&+(M_2\dot d_7C_6(C_3C_6+S_3S_4S_6)(2d_7+2l_2))/2\\
&+M_2\dot\theta_3C_6(d_7+l_2)^2(C_3S_4S_6-C_6S_3+C_4^2C_6S_3)
\end{aligned}
\tag{135}
$$

$$
\begin{aligned}
C_{4,3} =& (M_2\dot d_7C_4C_6S_6(2d_7+2l_2))/2\\
&-M_2C_6(d_7+l_2)^2(\dot\theta_2C_6S_3-\dot\theta_6C_4C_6+\dot\theta_1C_2C_3C_6+\dot\theta_3C_4C_6S_4-\dot\theta_2C_3S_4S_6-\dot\theta_2C_4^2C_6S_3\\
&+\dot\theta_1C_4C_6S_2S_4+\dot\theta_1C_2S_3S_4S_6-\dot\theta_1C_2C_3C_4^2C_6)
\end{aligned}
\tag{136}
$$

$$
C_{4,4} = M_2\dot d_7C_6^2(d_7+l_2)-M_2\dot\theta_6S_6C_6(d_7+l_2)^2 \tag{137}
$$

$$
\begin{aligned}
C_{4,5} =& M_2C_6(d_7+l_2)^2(\dot\theta_3C_4C_6-\dot\theta_4S_6-\dot\theta_2C_3S_6+\dot\theta_1C_4C_6S_2+\dot\theta_1C_2S_3S_6+\dot\theta_2C_6S_3S_4\\
&+\dot\theta_1C_2C_3C_6S_4)
\end{aligned}
\tag{138}
$$

$$
\begin{aligned}
C_{4,6} =& M_2C_6(d_7+l_2)(\dot\theta_4C_6+\dot\theta_2C_3C_6+\dot\theta_3C_4S_6-\dot\theta_1C_2C_6S_3+\dot\theta_1C_4S_2S_6+\dot\theta_2S_3S_4S_6\\
&+\dot\theta_1C_2C_3S_4S_6)
\end{aligned}
\tag{139}
$$

$$
\begin{aligned}
C_{5,1} =& -(M_2\dot d_7(S_2S_4-C_2C_3C_4)(2d_7+2l_2))/2\\
&-M_2\dot\theta_1(d_7+l_2)^2(C_4^2C_6S_2^2S_6-C_2^2C_6S_3^2S_6-C_2C_4C_6^2S_2S_3+C_2C_4S_2S_3S_6^2\\
&-C_2^2C_3C_6^2S_3S_4+C_2^2C_3S_3S_4S_6^2+C_2^2C_3^2C_6S_4^2S_6+2C_2C_3C_4C_6S_2S_4S_6)\\
&-(M_2\dot\theta_2(d_7+l_2)^2(2C_2S_4-2C_2C_3^2S_4-2C_2C_6^2S_4+2C_3C_4C_6^2S_2+4C_2C_3^2C_6^2S_4\\
&+4C_2C_3C_6S_3S_6+2C_4C_6S_2S_3S_4S_6-2C_2C_3C_4^2C_6S_3S_6))/2\\
&-M_2\dot\theta_4C_6(d_7+l_2)^2(C_2S_3S_6+C_4C_6S_2+C_2C_3C_6S_4)\\
&-M_2\dot\theta_3C_4(d_7+l_2)^2(C_2S_3-C_2C_6^2S_3+C_4C_6S_2S_6+C_2C_3C_6S_4S_6)
\end{aligned}
\tag{140}
$$

$$
\begin{aligned}
C_{5,2} =& (M_2\dot d_7C_4S_3(2d_7+2l_2))/2\\
&-(M_2(d_7+l_2)^2(2\dot\theta_2S_6C_6-2\dot\theta_3C_3C_4+2\dot\theta_1C_2S_4-2\dot\theta_4C_3C_6S_6-2\dot\theta_2C_3S_3S_4\\
&+2\dot\theta_3C_3C_4C_6^2-2\dot\theta_1C_2C_3^2S_4-2\dot\theta_1C_2C_6^2S_4-4\dot\theta_2C_3^2C_6S_6-2\dot\theta_2C_4^2C_6S_6+2\dot\theta_4C_6^2S_3S_4\\
&+4\dot\theta_1C_2C_3^2C_6^2S_4+2\dot\theta_2C_3^2C_4^2C_6S_6+2\dot\theta_1C_3C_4C_6^2S_2+4\dot\theta_2C_3C_6^2S_3S_4+4\dot\theta_1C_2C_3C_6S_3S_6\\
&+2\dot\theta_3C_4C_6S_3S_4S_6+2\dot\theta_1C_4C_6S_2S_3S_4S_6-2\dot\theta_1C_2C_3C_4^2C_6S_3S_6))/2
\end{aligned}
\tag{141}
$$

$$
\begin{aligned}
C_{5,3} =& -M_2C_4(d_7+l_2)^2(\dot\theta_4C_6^2-\dot\theta_2C_3+\dot\theta_1C_2S_3+\dot\theta_2C_3C_6^2+\dot\theta_3C_4C_6S_6-\dot\theta_1C_2C_6^2S_3\\
&+\dot\theta_1C_4C_6S_2S_6+\dot\theta_2C_6S_3S_4S_6+\dot\theta_1C_2C_3C_6S_4S_6)-(M_2\dot d_7S_4(2d_7+2l_2))/2
\end{aligned}
\tag{142}
$$

$$
\begin{aligned}
C_{5,4} =& - M_2 C_6 (d_7 + l_2)^2 (\dot{\theta}_3 C_4 C_6 - \dot{\theta}_4 S_6 - \dot{\theta}_2 C_3 S_6 + \dot{\theta}_1 C_4 C_6 S_2 + \dot{\theta}_1 C_2 S_3 S_6 + \dot{\theta}_2 C_6 S_3 S_4 \\
&+ \dot{\theta}_1 C_2 C_3 C_6 S_4)
\end{aligned} \tag{143}
$$

$$
C_{5,5} = M_2 \dot{d}_7 (d_7 + l_2) \tag{144}
$$

$$
C_{5,6} = M_2 (d_7 + l_2)(\dot{\theta}_6 - \dot{\theta}_3 S_4 + \dot{\theta}_2 C_4 S_3 - \dot{\theta}_1 S_2 S_4 + \dot{\theta}_1 C_2 C_3 C_4) \tag{145}
$$

$$
\begin{aligned}
C_{6,1} =& \dot{\theta}_2 ((S_3 (M_2 C_4 (S_2 S_4 - C_2 C_3 C_4)(2d_7 + 2l_2) \\
&- M_2 S_4 S_6 (2d_7 + 2l_2)(C_4 S_2 S_6 - C_2 C_6 S_3 + C_2 C_3 S_4 S_6)))/2 \\
&- (M_2 C_3 C_6 (2d_7 + 2l_2)(C_4 S_2 S_6 - C_2 C_6 S_3 + C_2 C_3 S_4 S_6))/2) \\
&- \dot{\theta}_3 ((M_2 S_4 (S_2 S_4 - C_2 C_3 C_4)(2d_7 + 2l_2))/2 \\
&+ (M_2 C_4 S_6 (2d_7 + 2l_2)(C_4 S_2 S_6 - C_2 C_6 S_3 + C_2 C_3 S_4 S_6))/2) \\
&+ M_2 \dot{\theta}_1 (d_7 + l_2)(C_2^2 - C_2^2 C_3^2 - C_2^2 C_6^2 + C_4^2 C_6^2 + 2C_2^2 C_3^2 C_6^2 - C_2^2 C_4^2 C_6^2 - C_2^2 C_3^2 C_4^2 C_6^2 \\
&+ 2C_2 C_4 C_6 S_2 S_3 S_6 + 2C_2 C_3 C_4 C_6^2 S_2 S_4 + 2C_2^2 C_3 C_6 S_3 S_4 S_6 - 1) \\
&+ (M_2 \dot{\theta}_6 (S_2 S_4 - C_2 C_3 C_4)(2d_7 + 2l_2))/2 \\
&- (M_2 \dot{\theta}_4 C_6 (2d_7 + 2l_2)(C_4 S_2 S_6 - C_2 C_6 S_3 + C_2 C_3 S_4 S_6))/2
\end{aligned} \tag{146}
$$

$$
\begin{aligned}
C_{6,2} =& - \dot{\theta}_1 ((C_2 (2M_2 C_3 (d_7 + l_2)(S_3 C_4^2 + S_3 S_4^2 S_6^2 + C_3 C_6 S_4 S_6) \\
&- M_2 C_6 S_3 (C_3 C_6 + S_3 S_4 S_6)(2d_7 + 2l_2)))/2 + (M_2 C_4 C_6 S_2 (C_3 S_6 - C_6 S_3 S_4)(2d_7 + 2l_2))/2) \\
&- M_2 \dot{\theta}_2 (d_7 + l_2)(C_3^2 C_6^2 + C_4^2 S_3^2 + S_3^2 S_4^2 S_6^2 + 2C_3 C_6 S_3 S_4 S_6) \\
&- (M_2 \dot{\theta}_4 C_6 (C_3 C_6 + S_3 S_4 S_6)(2d_7 + 2l_2))/2 \\
&- (M_2 \dot{\theta}_6 C_4 S_3 (2d_7 + 2l_2))/2 - (M_2 \dot{\theta}_3 C_4 C_6 (C_3 S_6 - C_6 S_3 S_4)(2d_7 + 2l_2))/2
\end{aligned} \tag{147}
$$

$$
\begin{aligned}
C_{6,3} =& M_2 \dot{\theta}_6 S_4 (d_7 + l_2) - \dot{\theta}_1 (M_2 S_2 (d_7 + l_2)(C_4^2 S_6^2 + S_4^2) \\
&- (M_2 C_2 C_4 C_6 (S_3 S_6 + C_3 C_6 S_4)(2d_7 + 2l_2))/2) - M_2 \dot{\theta}_3 (d_7 + l_2)(C_4^2 S_6^2 + S_4^2) \\
&- (M_2 \dot{\theta}_2 C_4 C_6 (C_3 S_6 - C_6 S_3 S_4)(2d_7 + 2l_2))/2 - (M_2 \dot{\theta}_4 C_4 C_6 S_6 (2d_7 + 2l_2))/2
\end{aligned} \tag{148}
$$

$$
\begin{aligned}
C_{6,4} =& - M_2 C_6 (d_7 + l_2)(\dot{\theta}_4 C_6 + \dot{\theta}_2 C_3 C_6 + \dot{\theta}_3 C_4 S_6 - \dot{\theta}_1 C_2 C_6 S_3 + \dot{\theta}_1 C_4 S_2 S_6 + \dot{\theta}_2 S_3 S_4 S_6 \\
&+ \dot{\theta}_1 C_2 C_3 S_4 S_6)
\end{aligned} \tag{149}
$$

$$
C_{6,5} = - M_2 (d_7 + l_2)(\dot{\theta}_6 - \dot{\theta}_3 S_4 + \dot{\theta}_2 C_4 S_3 - \dot{\theta}_1 S_2 S_4 + \dot{\theta}_1 C_2 C_3 C_4) \tag{150}
$$

$$
C_{6,6} = 0 \tag{151}
$$

$$
G = \mathbf{0}_{6\times 1} \tag{152}
$$

11 Control of 3D Model in Flight Phase

In this section, we are going to touch on the control of the 3D model. Beforehand, let me clarify two things. Firstly, in flight phase, there are only three active joints: tail pitch, tail yaw, and tail extension. However, we have six variables that are of concern to us: body yaw, body pitch, body row, tail yaw, tail pitch, and tail extension. Secondly, depending on the task, we actually need to choose different sets of actuators to control different sets of joints, e.g. using tail pitch and tail extension can control body pitch plus tail pitch and tail extension together.

11.1 modeling

$$\boldsymbol{H}^0 = \boldsymbol{J}^0\boldsymbol{\omega}^0 + m_r\boldsymbol{\xi}^0 \times \dot{\boldsymbol{\xi}}^0 \tag{153}$$

$$\boldsymbol{H}^b = \boldsymbol{J}^b\boldsymbol{\omega} + m_r\boldsymbol{\xi}^b \times (\boldsymbol{R}_b^0)^T\dot{\boldsymbol{R}}_b^0\boldsymbol{\xi}^b + m_r\boldsymbol{\xi}^b \times \dot{\boldsymbol{\xi}}^b \tag{154}$$

$$m_r = \frac{m_1 m_2}{m_1 + m_2} \tag{155}$$

$$\begin{aligned}\boldsymbol{\xi}^b &= \boldsymbol{\rho}_t^b - \boldsymbol{\rho}_b^b \\ &= \boldsymbol{\rho}_t^b \\ &= \boldsymbol{r}^b\end{aligned} \tag{156}$$

Refer to Fig. 19. To be more general, we consider there is an offset($d_5 \neq 0$) at frame 4.

$$\boldsymbol{r}^b = (l_2 + d_7)\begin{bmatrix} -S_4C_6 \\ S_6 \\ C_4C_6 + \frac{d_5}{l_2+d_7} \end{bmatrix} \tag{157}$$

$$\dot{\boldsymbol{r}}^b = (l_2 + d_7)\begin{bmatrix} S_4S_6\dot{\theta}_6 - C_4C_6\dot{\theta}_4 \\ C_6\dot{\theta}_6 \\ -S_4C_6\dot{\theta}_4 - C_4S_6\dot{\theta}_6 \end{bmatrix} + \dot{d}_7\begin{bmatrix} -S_4C_6 \\ S_6 \\ C_4C_6 \end{bmatrix} \tag{158}$$

for $\dot{d}_7 = 0$,

$$\dot{\boldsymbol{r}}^b = (l_2 + d_7)\begin{bmatrix} S_4S_6\dot{\theta}_6 - C_4C_6\dot{\theta}_4 \\ C_6\dot{\theta}_6 \\ -S_4C_6\dot{\theta}_4 - C_4S_6\dot{\theta}_6 \end{bmatrix} \tag{159}$$

$$\boldsymbol{r}^b \times \dot{\boldsymbol{r}}^b = (l_2+d_7)^2\begin{bmatrix} -S_4S_6C_6\dot{\theta}_4 - C_4{S_6}^2\dot{\theta}_6 - C_4{C_6}^2\dot{\theta}_6 - \frac{d_5}{l_2+d_7}C_6\dot{\theta}_6 \\ -{S_4}^2{C_6}^2\dot{\theta}_4 - S_4C_4S_6C_6\dot{\theta}_6 + S_4C_4S_6C_6\dot{\theta}_6 - {C_4}^2{C_6}^2\dot{\theta}_4 + \frac{d_5}{l_2+d_7}(S_4S_6\dot{\theta}_6 - C_4C_6\dot{\theta}_4) \\ -S_4{C_6}^2\dot{\theta}_6 - S_4{S_6}^2\dot{\theta}_6 + C_4S_6C_6\dot{\theta}_4 \end{bmatrix} \tag{160}$$

$$\boldsymbol{r}^b \times \dot{\boldsymbol{r}}^b = (l_2 + d_7)^2\begin{bmatrix} -S_4S_6C_6 & -C_4 - \frac{d_5}{l_2+d_7}C_6 \\ -{C_6}^2 - \frac{d_5}{l_2+d_7}C_4C_6 & \frac{d_5}{l_2+d_7}S_4S_6 \\ C_4S_6C_6 & -S_4 \end{bmatrix}\begin{bmatrix} \dot{\theta}_4 \\ \dot{\theta}_6 \end{bmatrix} \tag{161}$$

$$\boldsymbol{F} = m_r(l_2 + d_7)^2\begin{bmatrix} -S_4S_6C_6 & -C_4 - \frac{d_5}{l_2+d_7}C_6 \\ -{C_6}^2 - \frac{d_5}{l_2+d_7}C_4C_6 & \frac{d_5}{l_2+d_7}S_4S_6 \\ C_4S_6C_6 & -S_4 \end{bmatrix} \tag{162}$$

Kinematics equation under the constraint of conservation of angular momentum(Tomizuka 2013):

$$\begin{aligned} \mathbf{0} &= \boldsymbol{I_c}\boldsymbol{\omega^b} \ + \boldsymbol{F}\dot{\boldsymbol{\phi}} \\ &= \boldsymbol{I_c}\boldsymbol{\omega^b} \ + \boldsymbol{F}\dot{\boldsymbol{\theta}}_{tail} \end{aligned} \tag{163}$$

$$\boldsymbol{\omega^b} = -\boldsymbol{I}_c{}^{-1}\boldsymbol{F}\dot{\boldsymbol{\theta}}_{tail} \tag{164}$$

$$\boldsymbol{Q}\dot{\boldsymbol{\theta}}_{body} = -\boldsymbol{I}_c{}^{-1}\boldsymbol{F}\dot{\boldsymbol{\theta}}_{tail} \tag{165}$$

$$\dot{\boldsymbol{\theta}}_{body} = -\boldsymbol{Q}^{-1}\boldsymbol{I}_c{}^{-1}\boldsymbol{F}\dot{\boldsymbol{\theta}}_{tail} \tag{166}$$

$$\dot{\boldsymbol{\theta}}_{body} = \boldsymbol{J}_{bt}\dot{\boldsymbol{\theta}}_{tail} \tag{167}$$

$$\dot{\boldsymbol{\theta}}_{body} = \boldsymbol{J}_{bt}\dot{\boldsymbol{\theta}}_{tail} \tag{168}$$

where,

$$\boldsymbol{I}_c = \boldsymbol{I}^b - m_r(\boldsymbol{r}^b \times \boldsymbol{r}^b \times) \tag{169}$$

$$\boldsymbol{J}_{bt} = -\boldsymbol{Q}^{-1}\boldsymbol{I}_c{}^{-1}\boldsymbol{F} \tag{170}$$

$$\boldsymbol{Q} = \begin{bmatrix} {}^0\boldsymbol{T}_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, {}^0\boldsymbol{T}_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, {}^0\boldsymbol{T}_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix} \tag{171}$$

Q maps body angular velocity in body frame to the body yaw-pitch-roll velocity in frame 0.
From the SFB tail analysis,

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \boldsymbol{J}_{mt} \begin{bmatrix} \dot{\theta}_{pitch} \\ \dot{\theta}_{yaw} \end{bmatrix} \tag{172}$$

$$\dot{\boldsymbol{\theta}}_{motor} = \boldsymbol{J}_{mt}\dot{\boldsymbol{\theta}}_{tail} \tag{173}$$

$$\boldsymbol{J}_{mt} = \begin{bmatrix} 1 & \frac{w \sec^2 \theta_{yaw}}{4r\sqrt{1-\left(\frac{w}{2r}\tan\theta_{yaw}\right)^2}} \\ -1 & \frac{w \sec^2 \theta_{yaw}}{4r\sqrt{1-\left(\frac{w}{2r}\tan\theta_{yaw}\right)^2}} \end{bmatrix} \tag{174}$$

$$\dot{\boldsymbol{\theta}}_{tail} = \boldsymbol{J}_{mt}^{\#}\dot{\boldsymbol{\theta}}_{motor} \tag{175}$$

Combining (168) and (175),

$$\begin{aligned} \dot{\boldsymbol{\theta}}_{body} &= \boldsymbol{J}_{bt}\boldsymbol{J}_{mt}^{\#}\dot{\boldsymbol{\theta}}_{motor} \\ &= \boldsymbol{J}_{bm}\dot{\boldsymbol{\theta}}_{motor} \end{aligned} \tag{176}$$

12 Observations that help tuning gains

Observation:

System accumulating body orientation angle (ϕ_1) after every step.

Reason:

It may be caused by the slow damp out of angular speed of the system in the thrust phase. Try tune up the gain k_{tm}.

13 Observations that help flipping

Observation:

System can rotate forward but cannot land properly. Reason:
In main.m there is a parameter: genIn.landingGauge which determines when the leg starts to point to the ground for landing. One can adjust this parameter so that the leg can move earlier.
One can also increase the gain of the P control for the leg movement such that the leg can move to desired position at a faster rate.