Question sheet 2

(5) let $S_n = \underbrace{X_1 + X_2 \cdots + X_{12}}$

~~$f(x_i$~~ the ~~round~~ sum of rounding errors.

$E(X_i) = 0$

$Var(X_i) = \frac{1}{12}$

From the Central limit theorem

we have that for large n, S_n is

approximately $N(n\mu, n\sigma^2)$

For $n = 12$

$N(12 \cdot 0, 1) = N(0,1) = \cancel{S_n} \; X_1$

We need to find ~~$P(x$~~ $P(X_1 \geq 1)$

$P(X_1 = k) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{2}}$

So $P(X_i \geq 1) = \frac{1}{2\sqrt{\pi}} \int_1^\infty e^{-\frac{x^2}{2}} dx$

(5) (a) $E(X_n) = x_0 + E(Y_1) + E(Y_2) \dots + E(Y_n)$

$$E(Y_i) = p - (1-p) = 2p - 1$$

$$\therefore E(X_n) = \underline{\underline{x_0 + n(2p-1)}}$$

$$\text{Variance} = E(X_n^2) - \left[E(X_n)\right]^2$$

$$\cancel{E(X_n^2) = x_0 + E(Y_i^2)}$$

(b) $P(X_n = n+k)$ when $X_0 = k$

is the same as telling

$P(X_n = n)$ when $X_0 = 0$

Which means that all of $q_i = 1$

$$\therefore \quad P(X_n = n) = p^n \quad \square$$

Question sheet 3

① (a) this follows from the Chapman-Kolmogorov theorem which states

$$P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n)}$$ ⇒ matrix multiplication.

where $P_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$

$$\therefore \quad P^{(m+n)} = P^{(m)} P^{(n)} \quad \square$$

(b) $\lambda_i^{(n)} = P(X_n = i) = \sum_j P(X_n = i \mid X_0 = j) \cdot P(X_0 = j)$

$$= \cancel{\lambda^{(0)}} \cdot \sum_j \lambda_j^{(0)} P_{ji}^{(n)} = \left(\lambda^{(0)} P^{(n)}\right)_i$$

$\therefore \quad \lambda^{(n)} = \lambda^{(0)} P^{(n)}$ for every i

(2) State j is accessible from i: $P_{ij} \neq 0$, >0

States i and j communicate: ~~$P_{ij}, P_{ji} \geq 0$~~
if we can reach i from j, then we can reach j from i.

To show that the second relation is an equivalence relation we need to show that is reflexive, transitive and symmetric.

~~reflexivity~~:

i and j communicate: $P(X_n = j \mid X_{\cancel{n-1}\,k} = i) > 0$

and

$$P(X_n = i \mid X_{\cancel{n-1}\,k} = j) > 0$$

$k < n$

The reflexivity and ~~transitivity~~ symmetry follows from the definition.

~~Now we~~ w

~~Wow, the~~

Now I have to show transitivity.

i.e if $P(X_m = i \mid X_n = j)$ and

$P(X_n = j \mid X_0 = k)$ then

$P(X_m = i \mid X_0 = k)$

this follows from the definition of markov chains.

Here closed class means ~~a disjoint set~~ the set of disjoint sets of states.

(3)

$$P_{ij}^{(n)} = \sum_{r=0}^{n} f_{ij}^{(r)} P_{ij}^{(n-r)}$$

ie; we will go to j from i for the first time

at the r^{th} transition

if we multiply now by z^n

$$z^{(n)} P_{ij}^{(n)} = \sum_{r=0}^{n} f_{ij}^{(r)} P_{ij}^{(n-r)} \cdot z^{(n)}$$

Summing over n

$$\sum_{n=0}^{\infty} P_{ij}^{(n)} z^n = P_{ij}(z) = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} f_{ij}^{(r)} P_{ij}^{(n-r)}\right) z^{(n)}$$

(4) X_n is a finite state markov chain

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$$

We need to show that

1) $\pi_j \geq 0$ for all $j \in S$ and $\sum_{j\in S} \pi_j = 1$

2) ~~$\pi = \pi P$~~ $\pi_j = \sum_{i\in S} \pi_i P_{ij}$

since $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} \geq 0$

We need to show that $\sum_j \pi_j = 1$