A Comparison of Norms: Bicomplex Root and Ratio Tests and an Extension Theorem

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Abstract. A new root and ratio test for bicomplex power series produces the bicomplex extension theorem, where any complex power series with positive radius of convergence R extends, via a simple change of the complex domain variable z to the bicomplex variable ζ, to a bicomplex function analytic in a four real-dimensional ball with the same positive radius R. In this way, any complex-analytic function has a natural extension to a bicomplex-analytic function.

1. INTRODUCTION. For several centuries, mathematicians have sought a natural extension of the complex number system and corresponding function theory. Several different such extensions have been proposed, most notably the quaternions, which Sir William Rowan Hamilton first described in 1843, and the bicomplex numbers, which Corrado Segre described in 1892 [**12**]. Multiplicative noninvertibility of some of the bicomplex elements may have persuaded the mathematical community to focus on the quaternions as a more natural extension. But since the end of the 20th century, advances in the analytic qualities of bicomplex functions have shown that it is often advantageous to investigate bicomplex numbers, denoted $\mathbb{B}\mathrm{i}$ and described here, instead of quaternions. In particular, when using the standard difference quotient definition, the only functions on quaternions that are differentiable are linear functions (see [**13**]). In contrast, the collection of holomorphic bicomplex functions is rich. A host of compelling analytic properties extend outward to bicomplex functional analysis in often straightforward ways. These include generalizations of Euler's formula, the representation of complex-analytic functions as power series, and Cauchy's integral formula. See [**1, 3 – 6, 11**] for such results. The theory is also useful in applied settings (see [**2, 9, 10**]). In Theorem 6, this article gives the first appearance of bicomplex root and ratio tests. The main result is Theorem 7, which gives an extension theorem for power series on $\mathbb{B}\mathrm{i}$ and provides further support for the assertion that the theory of bicomplex functions is a natural and productive extension of complex function theory. Theorem 7 immediately provides the ability to define many bicomplex-analytic functions and formulate their properties in a way that hopes to inspire further study of bicomplex functions and related bicomplex topics.

Bicomplex numbers are of the form $\zeta = z_1 + jz_2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are complex numbers, $i^2 = -1$, and $j^2 = -1$. The set $\mathbb{B}\mathrm{i}$ is a four real-dimensional extension of the complex numbers $\mathbb{C}$ in the sense that $\mathbb{B}\mathrm{i}\big|_{z_2=0} = \mathbb{C}$. Here $\zeta = x_1 + iy_1 + jx_2 + ijy_2$, where $ij = ji$ so that $(ij)^2 = (-1)^2 = 1$. Given ζ and $\omega = w_1 + jw_2$ in $\mathbb{B}\mathrm{i}$,

$$\zeta + \omega = (z_1 + w_1) + j(z_2 + w_2) \text{ and}$$
$$\zeta \cdot \omega = (z_1 w_1 - z_2 w_2) + j(z_1 w_2 + w_1 z_2).$$

These equations agree with complex addition and multiplication when $\zeta, \omega \in \mathbb{C}$, and $\mathbb{B}\mathrm{i}$ is a commutative algebraic ring. The bicomplex Euclidean norm is as normally defined on four real dimensions, described on $\mathbb{B}\mathrm{i}$ via $|\zeta|^2 = z_1\overline{z}_1 + z_2\overline{z}_2$, where the conjugate of a complex number $z = x + iy$ is the usual $\overline{z} = x - iy$. Hence $|\zeta|^2 = |z_1|^2 + |z_2|^2 = |x_1|^2 + |y_1|^2 + |x_2|^2 + |y_2|^2$. All field properties are satisfied with one exception. Some bicomplex numbers (such as $\zeta = 1 + ji$) have no multiplicative inverse. However, when an inverse does exist, $(z_1 + jz_2)^{-1} = (z_1 - jz_2)/(z_1^2 + z_2^2)$.

This article also shows how the choice of norm on $\mathbb{B}\mathrm{i}$ is crucial when extending analytic theory in the most straightforward way. Two norms produce equivalent topologies but with different effectiveness, as they determine different formations of geometries such as different types of open balls. One is the Euclidean norm, and the other, which we call the hyperbolic norm, extends the standard characterization of a norm that requires output values to be nonnegative real numbers. Investigations have indicated that the hyperbolic norm, first investigated in [**8**] and outlined fully in [**1**] and [**5**], is the best choice for any study that uses $\mathbb{B}\mathrm{i}$ as a functional domain or (to study linear transformations) as underlying scalars. The results of this article persuasively support that fact.

2. THE EUCLIDEAN NORM. This section describes a natural investigation into bicomplex function theory via the geometry induced by the Euclidean norm. Bicomplex multiplication does not generally align itself well with a Euclidean geometry. For example, $|\zeta \cdot \omega| = |\zeta||\omega|$ is not always true, though it holds if ζ or ω is complex. In general,

$$\begin{aligned} |(z_1 + jz_2)(w_1 + jw_2)| &= |z_1(w_1 + jw_2) + jz_2(w_1 + jw_2)| \\ &\leq |z_1(w_1 + jw_2)| + |z_2(w_1 + jw_2)| \\ &= |z_1||w_1 + jw_2| + |z_2||w_1 + jw_2| \\ &= (|z_1| + |z_2|)|w_1 + jw_2| \\ &\leq \sqrt{2}(|z_1|^2 + |z_2|^2)^{1/2}|w_1 + jw_2| \text{ by Schwarz's inequality.} \end{aligned}$$

Hence, $|\zeta \cdot \omega| \leq \sqrt{2}|\zeta||\omega|$. In fact,

$$|(1 + ji)(1 + ji)| = |2 + 2ji| = \sqrt{8} = \sqrt{2}|1 + ji||1 + ji|,$$

and so the inequality is sharp. These calculations establish the following important algebraic result.

Lemma 1 (Product of Euclidean Norms [6, p. 7]). *For $\zeta, \omega \in \mathbb{B}\mathrm{i}$, $|\zeta\omega| \leq \sqrt{2}|\zeta||\omega|$. The equality $|\zeta\omega| = |\zeta||\omega|$ can hold and is guaranteed to hold whenever either ζ or ω is complex, but the inequality is the best possible.*

Bicomplex power series are functions of the form $f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - c)^n$ for $a_n, \zeta, c \in \mathbb{B}\mathrm{i}$, convergent in some bicomplex neighborhood about c. Absolute convergence implies convergence, as explained in the next theorem.

Theorem 2. *If $\sum \zeta_n$ is a bicomplex series and the real-valued series $\sum |\zeta_n|$ converges in the Euclidean norm (so that $\sum \zeta_n$ is said to converge absolutely), then $\sum \zeta_n$ also converges in the Euclidean norm.*

Proof. $\sum \zeta_n = \sum(z_n + jw_n) = \sum z_n + j\sum w_n$ converges in the Euclidean norm exactly when both $\sum z_n$ and $\sum w_n$ converge (as complex series). The theorem then follows once absolute convergence of the complex series $\sum z_n$ and $\sum w_n$ is verified. But $|\zeta_n| \geq |z_n|$ and $|\zeta_n| \geq |w_n|$ for each n. Apply the comparison test to get the result. ■

Theorem 3 (Euclidean Norm Weierstrass M-test). *Suppose $\{f_n\}$ is a sequence of bicomplex-valued functions on a ball $B(\zeta_0, R) \equiv \{\zeta : |\zeta - \zeta_0| < R\} \subset \mathbb{B}\mathrm{i}$ of positive radius R and center ζ_0, and suppose a sequence $\{M_n\}$ of positive numbers satisfies $|f_n(\zeta)| \leq M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$ for all $n \geq 1$ and $\zeta \in B(\zeta_0, R)$. Then $\sum_{n=1}^{\infty} f_n(\zeta)$ converges absolutely and uniformly on $B(\zeta_0, R)$.*

Proof. By the Cauchy criterion, for $\varepsilon > 0$ there exists $N > 0$ so that $\sum_{k=m+1}^{n} M_k < \varepsilon$ for $n > m > N$. For $\zeta \in B(\zeta_0, R)$ and for $n > m > N$, the sequence of partial sums $S_n(\zeta) = \sum_{k=1}^{n} f_k(\zeta)$ satisfies

$$|S_n(\zeta) - S_m(\zeta)| = \Big| \sum_{k=m+1}^{n} f_k(\zeta) \Big| \leq \sum_{k=m+1}^{n} |f_k(\zeta)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon.$$

Uniformly Cauchy implies uniform convergence, and so the series converges uniformly. The inequalities also show $\sum_{k=1}^{\infty} |f_k(\zeta)|$ converges (as its partial sums are Cauchy). ■

A ratio test that uses the Euclidean norm on bicomplex power series is slightly different from the one seen in real-valued calculus and in complex function theory. It can determine absolute convergence for the (real-valued) series $\sum |a_n(\zeta - c)^n|$, where $a_n, c, \zeta \in \mathbb{B}\mathrm{i}$. This test then serves to resolve questions about the radius of a ball about c over which the power series converges.

Theorem 4 (Euclidean Ratio Test). *For a given bicomplex power series $f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - c)^n$ with $a_n \neq 0$, define*[1]

$$R = \limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}.$$

When $R = \infty$, then $f(\zeta)$ is entire and the power series is absolutely convergent over all of $\mathbb{B}\mathrm{i}$. When $R > 0$ is finite, then the series is guaranteed to converge absolutely on $B(c, R/\sqrt{2}) = \{\zeta : |\zeta - c| < R/\sqrt{2}\}$.

Proof. A variable substitution $\omega = \zeta - c$ shows that, without loss of generality, $c = 0$. Suppose $0 < r < R$; find a positive integer N such that $r < |a_n|/|a_{n+1}|$ for $n \geq N$. Define $M = |a_N|r^N$. Then

$$|a_{N+1}|r^{N+1} = |a_{N+1}| \cdot r \cdot r^N < |a_N|r^N = M.$$

Similarly,

$$|a_{N+2}r^{N+2}| = |a_{N+2} \cdot r \cdot r^{N+1}| < |a_{N+1}|r^{N+1} < M.$$

Continuing in this manner, we conclude $|a_n|r^n \leq M$ for every $n \geq N$.

Also, because $|\zeta\omega| \leq \sqrt{2}|\zeta||\omega|$ for two noncomplex elements $\zeta, \omega \in \mathbb{B}\mathrm{i}$, the best we can say is that for each $n \geq N$,

$$|a_n\zeta^n| \leq \sqrt{2}|a_n| \cdot |\zeta^n| = \sqrt{2}\left(|a_n|\, r^n\right)|\zeta^n|/r^n \leq \sqrt{2}M|\zeta^n|/r^n$$

[1]Take note: the limit is of the ratio of the bicomplex moduli, not the modulus of the ratio. The result is problematic if the latter is used.

$$= \sqrt{2}M|\zeta^n/r^n| = \sqrt{2}M|(\zeta/r)^n|.$$

Hence $\sum_{n=0}^{\infty}|a_n\zeta^n|$ is dominated by $\sqrt{2}M\sum_{n=0}^{\infty}\left|\left(\frac{\zeta}{r}\right)^n\right| \leq M\sum_{n=0}^{\infty}\left(\frac{\sqrt{2}|\zeta|}{r}\right)^n$ and converges whenever the dominating series on the right converges. But that series is real geometric and converges whenever $\sqrt{2}|\zeta|/r < 1$. As r is any positive real with $r < R$, this proves the result. ■

Example. The Euclidean ratio test proves $f(\zeta) = \sum_{n=0}^{\infty}\frac{\zeta^n}{n!}$ is entire, since

$$R = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty} n+1 = \infty.$$

Of course, as for the complex extension of the real function e^x, $f(\zeta) = \mathrm{e}^\zeta$. See [**5**, Section 6.2] for an Euler formula and more details on this function's beautiful properties on $\mathbb{B}$i. ■

A natural next question is, "Given a complex power series $\sum a_n(z-c)^n$ with positive radius of convergence, by changing z to a bicomplex variable ζ, over what largest ball $B(c,R)$ in $\mathbb{B}$i does the new power series converge?" How is this "radius of convergence" R formulated? Will this "ball of convergence" have the additional property that the power series will diverge on the complement of its closure? The next section uses a norm different from the Euclidean to resolve each of these issues.

3. THE HYPERBOLIC NORM. The bicomplex system is best thought of not in terms of its two complex components written as $\zeta = z_1 + z_2 j$, but instead in terms of two complex components in a so-called idempotent representation

$$\zeta = (z_1 - iz_2)\mathbf{e}_1 + (z_1 + iz_2)\mathbf{e}_2 \equiv \zeta_1\mathbf{e}_1 + \zeta_2\mathbf{e}_2\,.$$

Here $\mathbf{e}_1 = (1+ij)/2$ and $\mathbf{e}_2 = (1-ij)/2$, and $\zeta_1, \zeta_2 \in \mathbb{C}$ are the idempotent components. For $\zeta = \zeta_1\mathbf{e}_1 + \zeta_2\mathbf{e}_2$ and $\omega = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2$, the product

$$\zeta\omega = \zeta_1\omega_1\mathbf{e}_1 + \zeta_2\omega_2\mathbf{e}_2$$

computes componentwise! Hence $\zeta^n = \zeta_1^n\mathbf{e}_1 + \zeta_2^n\mathbf{e}_2$, and any nth root translates into the components (say for $n \in \mathbb{N}$) as

$$\zeta^{1/n} = \zeta_1^{1/n}\mathbf{e}_1 + \zeta_2^{1/n}\mathbf{e}_2 \text{ when } \zeta_1 \text{ and } \zeta_2 \text{ are nonnegative real.}$$

Simple algebra also proves two extremely important special items. First, ζ is complex if and only if $\zeta_1 = \zeta_2 = \zeta$. Second, the noninvertible elements ζ are exactly the values that have either $\zeta_1 = 0$ or $\zeta_2 = 0$.

A powerful so-called hyperbolic norm evolves from consideration of bicomplex numbers with real idempotent components. This norm equals the Euclidean norm when applied to complex numbers, but it outputs moduli for noncomplex numbers that are in general not real! Hence its properties form a generalization of the standard defining properties of a norm, obtained by replacing the nonnegative real numbers by a partially ordered set (a poset) of bicomplex numbers called the nonnegative hyperbolic numbers $\mathbb{H}^+$, which the following definition describes. To distinguish it from a norm that has the standard properties, mathematicians sometimes call the hyperbolic norm a poset-valued norm.

Definition (Hyperbolic Numbers). The hyperbolic numbers

$$\mathbb{H} = \{x + ijy :\ x, y \in \mathbb{R}\}$$

are a strict subset of $\mathbb{B}$i. The subset of nonnegative hyperbolic numbers is

$$\begin{aligned}\mathbb{H}^+ &= \{x + ijy : x \geq 0, |y| \leq x\}\\ &= \{(x+y)\mathbf{e}_1 + (x-y)\mathbf{e}_2 : x \geq 0, |y| \leq x\}\\ &= \{\eta_1\mathbf{e}_1 + \eta_2\mathbf{e}_2 : \eta_1 \geq 0, \eta_2 \geq 0\}.\end{aligned}$$

For any $\zeta = \zeta_1\mathbf{e}_1 + \zeta_2\mathbf{e}_2 \in \mathbb{B}$i, the hyperbolic norm is defined as

$$|\zeta|_{\mathbb{H}} = |\zeta_1|\mathbf{e}_1 + |\zeta_2|\mathbf{e}_2,$$

which is an element of $\mathbb{H}^+$ and satisfies $|\zeta \cdot \omega|_{\mathbb{H}} = |\zeta|_{\mathbb{H}} \cdot |\omega|_{\mathbb{H}}$ for any $\zeta, \omega \in \mathbb{B}$i.

A partial ordering on $\mathbb{H}^+$ now follows. For two elements $\zeta = \eta_1\mathbf{e}_1 + \eta_2\mathbf{e}_2$ and $\omega = \psi_1\mathbf{e}_1 + \psi_2\mathbf{e}_2$ in $\mathbb{H}^+$, define the hyperbolic inequality $\zeta <_{\mathbb{H}} \omega$ when both $\eta_1 < \psi_1$ and $\eta_2 < \psi_2$. A less-than-or-equal-to partial ordering is similarly defined. The hyperbolic norm of any bicomplex number is a hyperbolic number. That norm can therefore be used via this hyperbolic inequality to define open balls of bicomplex numbers that have hyperbolic radii. The balls are of the form

$$B_{\mathbb{H}}(c, R) = \{\zeta : |\zeta - c|_{\mathbb{H}} <_{\mathbb{H}} R\},$$

where the center $c \in \mathbb{B}$i and the hyperbolic radius $R \in \mathbb{H}^+$. The collection of balls $B_{\mathbb{H}}(c, R)$ can also be thought of as the neighborhood basis open sets in a topology induced by the hyperbolic norm.

Any bicomplex series $\sum \zeta_n = (\sum \zeta_{1,n})\mathbf{e}_1 + (\sum \zeta_{2,n})\mathbf{e}_2$, where $\zeta_n = \zeta_{1,n}\mathbf{e}_1 + \zeta_{2,n}\mathbf{e}_2$, converges if and only if both $\sum \zeta_{1,n}$ and $\sum \zeta_{2,n}$ converge as complex series. For example,

$$\sum_{n=0}^{\infty} \zeta^n = \Big(\sum_{n=0}^{\infty} \zeta_1^n\Big)\mathbf{e}_1 + \Big(\sum_{n=0}^{\infty} \zeta_2^n\Big)\mathbf{e}_2$$

converges over $\{\zeta = \zeta_1\mathbf{e}_1 + \zeta_2\mathbf{e}_2 : |\zeta_1| < 1 \text{ and } |\zeta_2| < 1\} = \{\zeta : |\zeta|_{\mathbb{H}} < 1\mathbf{e}_1 + 1\mathbf{e}_2 = 1\} = B_{\mathbb{H}}(0, 1)$. Significantly, the Euclidean norm ratio test guarantees convergence only on $B(0, 1/\sqrt{2})$, and so an improvement should be possible. That improvement comes when using the hyperbolic norm instead of the Euclidean norm.

Definition (Hyperbolic Convergence). A functional sequence $f_n : \Omega \to \mathbb{B}$i on a bicomplex domain[2] Ω *converges in the hyperbolic norm* to $f : \Omega \to \mathbb{B}$i when, for any $\zeta \in \Omega$ and for $\varepsilon \in \mathbb{H}^+$ with $\varepsilon >_{\mathbb{H}} 0$, there exists $N > 0$ such that $|f_n(\zeta) - f(\zeta)|_{\mathbb{H}} <_{\mathbb{H}} \varepsilon$ for $n \geq N$. The value for N can depend on ε and ζ, and if it does not depend on ζ, the convergence is said to be *uniform*.

Because the hyperbolic norm is calculated as a sum of the complex norms on each idempotent component, hyperbolic convergence of any functional sequence $f_n = f_{1,n}\mathbf{e}_1 + f_{2,n}\mathbf{e}_2$ is equivalent to the convergence of both sequences $f_{1,n}$ and $f_{2,n}$ in $\mathbb{C}$. A bicomplex series converges in the hyperbolic norm when its sequence of partial sums does, and so basic theorems regarding complex convergence such as the Cauchy criteria and the comparison test generalize to hold for bicomplex series, too. See [**5**, Chapter 10] for a full foundational exposition. For example, we have:

Theorem 5 (Hyperbolic Weierstrass M-test [5, p. 204]). *Suppose $|\zeta_n|_{\mathbb{H}} \leq_{\mathbb{H}} M_n$ for $M_n \in \mathbb{H}^+$. Then hyperbolic convergence of $\sum M_n$ implies hyperbolic uniform and absolute convergence of $\sum \zeta_n$.*

[2] Here, $\Omega = \Omega_1\mathbf{e}_1 \oplus \Omega_2\mathbf{e}_2$, where $\Omega_i\mathbf{e}_i = \{z\mathbf{e}_i : z \in \Omega_i\}$, $i = 1, 2$, for complex domains Ω_i.

Proof. Simply apply the corresponding comparison tests to both complex series in the idempotent representation for $\sum |\zeta_n|_{\mathbb{H}}$ and $\sum M_n$. ■

The superb monograph [**5**] has a result associated with a small part of the next theorem but without an ability to calculate an associated radius of convergence. In particular, varieties of the enumerated sets of Theorem 6 appear in its Proposition 10.4.2. However, Theorem 6 is the first appearance of a powerful bicomplex root test, defining the radius of convergence R for a bicomplex power series via a limit supremum calculation. A new ratio test also follows.

Theorem 6 (Hyperbolic Root and Ratio Tests). *For a given bicomplex function power series $f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - c)^n = \sum_{n=0}^{\infty}(a_{1,n}\mathbf{e}_1 + a_{2,n}\mathbf{e}_2)(\zeta - c)^n$, define the nonnegative hyperbolic radius R as the radius of convergence, as well as the limit supremum in its calculation, by*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|_{\mathbb{H}}^{1/n} \equiv \limsup_{n\to\infty} |a_{1,n}|^{1/n}\mathbf{e}_1 + \limsup_{n\to\infty} |a_{2,n}|^{1/n}\mathbf{e}_2.$$

In a similar and consistent manner, when a_n is invertible, set[3]

$$R = \limsup_{n\to\infty} \frac{|a_n|_{\mathbb{H}}}{|a_{n+1}|_{\mathbb{H}}} \equiv \limsup_{n\to\infty} \left|\frac{a_{1,n}}{a_{1,n+1}}\right| \mathbf{e}_1 + \limsup_{n\to\infty} \left|\frac{a_{2,n}}{a_{2,n+1}}\right| \mathbf{e}_2.$$

When the radius of convergence R is infinity in both idempotent components, then $f(\zeta)$ is entire, and the power series is absolutely hyperbolic convergent over all of $\mathbb{B}\mathrm{i}$. Otherwise, the series hyperbolically converges absolutely in the ball $B_{\mathbb{H}}(c, R) = \{\zeta : |\zeta|_{\mathbb{H}} <_{\mathbb{H}} R\}$ and diverges on the complement of its closure. Furthermore, $B_{\mathbb{H}}(c, R)$ is one of five basic types, written in terms of the idempotent components of $R = R_1\mathbf{e}_1 + R_2\mathbf{e}_2 \in \mathbb{H}^+$ and $c = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 \in \mathbb{B}\mathrm{i}$:

1. *$0 < R_1, R_2 < \infty$, and $B_{\mathbb{H}}(c, R) = \{\zeta_1\mathbf{e}_1 + \zeta_2\mathbf{e}_2 : |\zeta_1 - c_1| < R_1$ and $|\zeta_2 - c_2| < R_2\}$;*
2. *R_1 or R_2 equals infinity and the other value is finite, and (for example, in the case $R_1 = \infty$) $B_{\mathbb{H}}(c, R) = \{\zeta_1\mathbf{e}_1 + \zeta_2\mathbf{e}_2 : \zeta_1 \in \mathbb{C}$ and $|\zeta_2 - c_2| < R_2\}$;*
3. *R_1 or R_2 equals 0 and the other value is nonzero finite, and (for example, in the case $R_1 = 0$) $B_{\mathbb{H}}(c, R) = \{\zeta_1\mathbf{e}_1 + \zeta_2\mathbf{e}_2 : \zeta_1 = c_1$ and $|\zeta_2 - c_2| < R_2\}$;*
4. *R_1 or R_2 equals 0 and the other value is infinite, and (for example, in the case $R_1 = 0$) $B_{\mathbb{H}}(c, R) = \{\zeta_1\mathbf{e}_1 + \zeta_2\mathbf{e}_2 : \zeta_1 = c_1$ and $\zeta_2 \in \mathbb{C}\}$;*
5. *$R = 0$, and $B_{\mathbb{H}}(c, 0) = \{c\}$.*

Proof. Simply apply the complex root and ratio tests to both idempotent components in the two calculations. The result immediately follows. ■

The hyperbolic root test implies the following important theorem.

Theorem 7 (Hyperbolic Extension Theorem). *Any complex-valued function expressed as a power series $f(z) = \sum_{n=0}^{\infty} a_n(z - c)^n$, where $z, a_n, c \in \mathbb{C}$, over a positive-radius (or infinite) disk of convergence $D(c, R) = \{z \in \mathbb{C} : |z - c| < R\}$ can be extended into the four-dimensional bicomplex space as the same power series.*

[3]Of course, as for the complex ratio test, this ratio test's value of R will always agree with that of the root test, whenever the ratio test's limit supremum can be calculated.

Simply replace the complex variable z with the bicomplex variable ζ in the power series expression to define the bicomplex function $f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - c)^n$. The extended bicomplex power series converges in the ball $B_{\mathbb{H}}(c, R) = \{\zeta : |\zeta - c|_{\mathbb{H}} < R\}$ with the same radius, and it diverges on the complement of its closure.

Proof. In this situation, R is real, and then its idempotent expansion is $R = R\mathbf{e}_1 + R\mathbf{e}_2$. Similarly, $a_n \in \mathbb{C}$ has idempotent expansion $a_n = a_n\mathbf{e}_1 + a_n\mathbf{e}_2$, and then the hyperbolic root test's result follows from the same complex root test's limit supremum value calculated on both idempotent components. That result thereby shows that (the largest) ball of convergence is $B_{\mathbb{H}}(c, R)$ and the power series diverges on the complement of its closure. ■

Why is the hyperbolic extension theorem interesting? Two reasons are immediately apparent. First, it shows that every complex-analytic function extends to a well-defined bicomplex-analytic function, and therefore a host of current results in complex-analytic function theory will extend to the bicomplex case, giving a foundation for the rich study of a broader class of bicomplex-analytic functions. Second, any equation that relates two complex-analytic functions extends immediately to the corresponding bicomplex-analytic functions' equation. Simply replace $z \in \mathbb{C}$ with $\zeta \in \mathbb{B}\mathrm{i}$ in the functions' power series representations. The bicomplex equation results from the hyperbolic extension theorem. The theorem eliminates the need for the sorts of tedious bicomplex algebra calculations that were previously used to establish such identities. Here are eight simple but nontrivial examples[4] for $\zeta \in \mathbb{B}\mathrm{i}$:

- $\cos^2\zeta + \sin^2\zeta = 1$
- $\mathrm{e}^{i\zeta} = \cos\zeta + i\sin\zeta$
- $\cos\zeta = (\mathrm{e}^{i\zeta} + \mathrm{e}^{-i\zeta})/2$
- $\sin\zeta = -i(\mathrm{e}^{i\zeta} - \mathrm{e}^{-i\zeta})/2$
- $f'(\zeta) = \cos\zeta$ when $f(\zeta) = \sin\zeta$
- $\tan\zeta = \sin\zeta \,/\cos\zeta$
- $\cosh\zeta = (\mathrm{e}^{\zeta} + \mathrm{e}^{-\zeta})/2$
- $\mathrm{e}^{\mathrm{Log}\zeta} = \zeta$, when ζ is invertible

4. CONCLUSIONS. The analysis shows the superiority of analyzing convergence, as well as providing algebraic properties, of bicomplex functions using the hyperbolic norm as opposed to the Euclidean norm. The Euclidean norm will give meaningful results, as it did here for its Euclidean ratio test, but those results typically fall short of the parallel results for the hyperbolic norm. In the case of the extension theorem, the extension to a ball in $\mathbb{B}\mathrm{i}$ of a convergent complex power series is realized fully in terms of the hyperbolic norm, and that norm provides a maximal radius of convergence for the power series.

The superiority of the hyperbolic norm over the Euclidean norm is a recurring theme when working with bicomplex functions. Results from complex function theory, applied to each idempotent component, allow for powerful generalizations to bicomplex function theory when using the hyperbolic norm. Proofs of the generalizations often follow from the complex-analytic result applied to each idempotent component. Simply put, one should expect complete mathematical fields that use complex scalars, such

[4]The last example involves the bicomplex single-valued analytic logarithm function, which is defined as the power series extension of the complex-valued logarithm function in terms of its principal branch.

as functional analysis, linear algebra, and operator theory, to extend to corresponding studies with bicomplex scalars when using the hyperbolic norm, whereas the Euclidean norm will not provide generalizations that give optimal conclusions. This sentiment is clearly supported by the hyperbolic extension theorem and the simplicity of the proofs of Theorems 6 and 7, which follow directly from the corresponding analysis on each hyperbolic component. Those results show how every complex-analytic function extends to a bicomplex-analytic function, and thus analytic function theory extends to $\mathbb{B}\mathrm{i}$ in the same way that the theory of real power series extends to $\mathbb{C}$. It is also supported by the known results (see [**5**]) that:

- bicomplex-analytic functions decompose into hyperbolic components that are each one-variable complex-analytic;
- a bicomplex Cauchy integral representation exists as a direct generalization of the complex Cauchy integral representation;
- a corresponding Lebesgue path integral can be defined in terms of the corresponding Lebesgue integration on the two hyperbolic components;
- any finite-dimensional or infinite-dimensional bicomplex matrix decomposes into the sum of its two corresponding hyperbolic component complex matrices in such a way that bicomplex matrix multiplication can be calculated simply as matrix multiplication within each separate hyperbolic component; and then
- a corresponding hyperbolic operator poset norm can be defined in terms of the standard operator norm on each hyperbolic component.

Of course, not every convergent bicomplex power series has complex coefficients, and so not every bicomplex power series is an extension of a complex power series. The collection of bicomplex power series is much more populated, just as the collection of complex power series is more populated than that of real power series. We should thereby expect the study of such bicomplex-analytic functions to be more robust than the study of complex-analytic functions. Several important discussions of bicomplex series with complex coefficients have already been studied and published; see, for example [**7**]. Many points regarding convergence in these discussions, as well as the formulaic relationships between bicomplex-analytic functions, follow easily from the hyperbolic extension theorem.

A primary hope is that this article introduces bicomplex analysis as a fascinating and fairly new mathematical subfield, one that is not yet well known and which basically originated in 1991 with Price's text [**6**]. Especially as more and more applications of the bicomplex system are published as in [**2, 9, 10**], the itemized list above shows the appropriateness of choosing bicomplex function theory as a standard in mathematical function theory, linear algebra, and operator theory research, in the same way that complex function theory currently serves as that standard. In this way, a corresponding second hope is that this article spurs further research into bicomplex function theory and bicomplex operator theory, which should prove valuable to the mathematical community.

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