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On the Iteration of Quaternionic Moebius Transformations

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We describe the iterative behavior of Moebius transformations with values in the quaternions. A classification of the main conjugacy classes is given. The dynamics turns out to be similar to the complex case, with the exception of the case of a two dimensional fixed point manifold. For Moebius transformations mapping the unit ball onto itself an analogue to a theorem of Denjoy–Wolff holds.

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1. INTRODUCTION

Since complex analytic dynamical systems are quite well understood (see e.g. [1–3, 11]), it is natural to look at higher dimensions like quaternions. However, iteration theory in quaternions is still scarcely studied. Petek/Kozak [8, 10] consider the family of quadratic functions $q \to q^2 + c$, and Briggs/Bedding [4, 5] discuss the iteration of monogenic functions. For further information we also refer to [7].

We will focus on Moebius transformations, where a complete classification of the iterative behavior can be given. As Moebius transformations are the only conformal mappings in $\mathbf{R}^n$ with $n \geq 3$ (see [9]), this covers one of the central aspects of holomorphic functions.

In the sequel we denote Hamilton's quaternions by $\mathbf{H} := \{q = q_0 + q_1\imath + q_2\jmath + q_3\imath\jmath \mid q_0, q_1, q_2, q_3 \in \mathbf{R}\}$. They constitute an associative, non-commutative algebra over the reals. q_0 is the real part and $q_1\imath + q_2\jmath + q_3\imath\jmath$ the imaginary part of q, denoted by $\Re(q)$, $\Im(q)$ respectively. The conjugate quaternion $\overline{q}$ is defined by $\Re(q) - \Im(q)$ and $q\overline{q} = |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$ is the squared euclidean norm of q in $\mathbf{R}^4$. This implies the existence of an inverse $q^{-1} = \overline{q}|q|^{-2}$ if $q \neq 0$. The one point compactification $\hat{\mathbf{H}} := \mathbf{H} \cup \{\infty\}$ is obtained by stereographic projection in $\mathbf{R}^5$ as in the complex case. The fields $\mathbf{R}$ and $\mathbf{C}$ are canonically imbedded into $\mathbf{H}$.

As we are discussing successive applications of a function f we define the nth iterate recursively by $f^{(0)} := id$, $f^{(n+1)} := f \circ f^{(n)}$. If $f^{(n)}(q_0) = q_0$ for $n \in \mathbf{N}$ then q_0 is called a periodic point of period n and in particular a fixed point of f if $n = 1$. A periodic point is called attracting iff $|f^{(n)}(q) - q_0| < |q - q_0|$ in a neighborhood of q_0. The basin of attraction of q_0 is $\mathcal{A}(q_0) := \{q \in \hat{\mathbf{H}} \mid \lim_{n\to\infty} f^{(n)}(q) = q_0\}$.

2. MOEBIUS TRANSFORMATIONS

Our key interest functions are given by

DEFINITION 2.1 *Let* $a, b, c, d \in \mathbf{H}$ *with* $ad \neq 0$ *if* $c = 0$ *and* $b - ac^{-1}d \neq 0$ *if* $c \neq 0$. *The continuous extension to* $\hat{\mathbf{H}}$ *of the function*

$$f(q) := (aq+b)(cq+d)^{-1}, \qquad q \in \hat{\mathbf{H}}$$

is called a Moebius transformation in the quaternions.

Moebius transformations are closed under composition so we can focus on classes of conjugate functions, i.e., two functions f_1, f_2 are said to be conjugate under Moebius transformations if there exists a Moebius transformation T with $f_1 := T \circ f_2 \circ T^{-1}$. Fixed point structure and iterative behavior are invariant under conjugation.

If $c \neq 0$ a fixed point of the Moebius transformation always exists as solution of the equation

$$aq + b - qcq - qd = 0,$$

see [6].

Denote this fixed point by q_0 and define the Moebius transformation $T(q) = q^{-1} + q_0$ with the inverse $T^{-1}(q) = (q - q_0)^{-1}$ we get by straightforward computation

$$T^{-1} \circ f \circ T(q) = ((cq_0 + d)q + c)(a - q_0 c)^{-1}. \tag{2.1}$$

Hence, w.l.o.g. we can restrict ourselves to elements of conjugacy classes of the form

$$f(q) := (aq+b)d^{-1}, \qquad ad \neq 0. \tag{2.2}$$

Then we have

THEOREM 2.2 *Any Moebius transformation in the quaternions has exactly one, two or infinitely many fixed points in* $\hat{\mathbf{H}}$. *In the latter case, if the transformations are given as in* (2.2) *the fixed points form a two dimensional plane; in general a two dimensional sphere or plane.*

Proof From the representation (2.2) we obtain the fixed point equation as $aq - qd = -b$, which is a linear equation in $\mathbf{R}^4$. Let

$$a = (a_1, a_2, a_3, a_4) \qquad \text{and} \qquad d = (d_1, d_2, d_3, d_4)$$

with $a_i, b_i \in \mathbf{R}$, $i = 1 \cdots 4$, then the eigenvalues of the corresponding matrix are

$$(a_1 - d_1) \pm \left(\sqrt{a_2^2 + a_3^2 + a_4^2} \pm \sqrt{d_2^2 + d_3^2 + d_4^2}\right)\imath.$$

If $\Re(a) \neq \Re(d)$ or $|\Im(a)| \neq |\Im(d)|$ there exists a unique solution (i.e., a fixed point) in $\mathbf{H}$. Including ∞ these are two fixed points in $\hat{\mathbf{H}}$. If $\Re(a) = \Re(d)$ and $|\Im(a)| = |\Im(d)|$ there is a two dimensional fixed points plane or no fixed point in $\mathbf{H}$, depending on $-b$ being in the image of $q \to aq - qd$ or not. The general case of a fixed point sphere appears by conjugation.

The fixed point structure influences considerably the iterative behavior of a given Moebius transformation $q \to (aq+b)d^{-1}$. Using a theorem of Cayley we can achieve the coefficients a and d to be complex.

LEMMA 2.3 *Let $f(q) := (aq+b)d^{-1}$ be a linear Moebius transformation.*

(i) *If f has only one fixed point it is conjugate to*

$$\tilde{f}(q) = (\tilde{a}q + \tilde{b})\tilde{a}^{-1}$$

with $\tilde{a} \in \mathbf{C}$, $\tilde{b} = \tilde{b}_1 + \tilde{b}_2 \jmath$, $\tilde{b}_1, \tilde{b}_2 \in \mathbf{C}$ and $\tilde{b}_1 \neq 0$.

(ii) *If f has at least two fixed points it is conjugate to $\tilde{f}(q) = \tilde{a}q\tilde{d}^{-1}$ with $\tilde{a}, \tilde{d} \in \mathbf{C}$. If f has infinitely many fixed points then we can choose $\tilde{a} = \tilde{d}$.*

Proof According to a theorem of Cayley choose $r, s \in \mathbf{H}$ such that $r^{-1}ar \in \mathbf{C}$ and $s^{-1}ds \in \mathbf{C}$. With the Moebius transformation $T(q) := r^{-1}qs$ we obtain

$$T^{-1} \circ f \circ T(q) = ((r^{-1}ar)q + r^{-1}bs)(s^{-1}ds)^{-1} = (\tilde{a}q + \tilde{b})\tilde{d}^{-1}.$$

Analogously to the proof of Theorem 2.2 it follows that in case of exactly one or infinitely many fixed points $\tilde{a} = \tilde{d}$ or $\tilde{a} = \bar{\tilde{d}}$. In the latter case a subsequent conjugation $q \to \jmath q$ has to be applied to obtain again $\tilde{a} = \tilde{d}$. In the case of exactly one fixed point as in the proof of Theorem 2.2 we see that $\tilde{b}$ has the desired form.

If at least one additional fixed point $q_0 \in \mathbf{H}$ exists, a subsequent conjugation with $q \to q - q_0$ eliminates $\tilde{b}$.

The preceding lemma is the key tool to understand the iterative behavior of the different conjugacy classes.

THEOREM 2.4 *Let $f(q) = (aq+b)d^{-1}$ be a linear Moebius transformation with coefficients according to Lemma 2.3.*

(i) *If f has exactly one fixed point, then $\mathcal{A}(\infty) = \hat{\mathbf{H}}$ (parabolic case).*

(ii) *If $f(q) = aqd^{-1}$ with exactly two fixed points and*
(a) *if $|ad^{-1}| > 1$ then ∞ is an attracting fixed point with $\mathcal{A}(\infty) = \hat{\mathbf{H}} \setminus \{0\}$ (hyperbolic/loxodromic case 1)*
(b) *if $|ad^{-1}| < 1$ then 0 is an attracting fixed point with $\mathcal{A}(0) = \mathbf{H}$ (hyperbolic/loxodromic case 2)*
(c) *if $|ad^{-1}| = 1$ then the sequence of iterates converges only for fixed points; there exists a periodic point with period $n \in \mathbf{N}$ iff $(a/|a|)^{2n} = 1$ and $a^n = d^n$, in which case $f^{(n)} = id$ (elliptic case 1).*

(iii) *If f has infinitely many fixed points, the sequence of iterates converges only for fixed points; there exists a periodic point with period $n \in \mathbf{N}$ iff $(a/|a|)^{2n} = 1$ in which case $f^{(n)} = id$ (elliptic case 2).*

Proof In case (i), the nth iterate of f has the form

$$f^{(n)}(q) = a^n q a^{-n} + \sum_{i=0}^{n-1} a^i b a^{-i-1}.$$

Let $q = q_1 + q_2 j$, $q_1, q_2 \in \mathbf{C}$ and $b = b_1 + b_2 j$, $b_1, b_2 \in \mathbf{C}$ and $b_1 \neq 0$. Then

$$f^{(n)}(q) = q_1 + nb_1 a^{-1} + \left(a^n \overline{a^{-n}} q_2 + \sum_{i=0}^{n-1} a^i \overline{a^{-i-1}} b_2 \right) j,$$

thus $\lim_{n \to \infty} f^{(n)}(q) = \infty$.

Assertions (ii), (a), (b) follow immediately from the representation. If $|ad^{-1}| = 1$, then with arbitrary $q = q_1 + q_2 j$, $q_1, q_2 \in \mathbf{C}$ we have

$$f^{(n)}(q) = a^n q d^{-n} = a^n (q_1 + q_2 j) d^{-n} = (ad^{-1})^n q_1 + (\overline{ad^{-1}})^n q_2 j.$$

Hence, periodicity occurs only under the given conditions.

In case (iii) we have the representation $f(q) = aqa^{-1}$, then with arbitrary $q = q_1 + q_2 j$, $q_1, q_2 \in \mathbf{C}$ we obtain

$$f^{(n)}(q) = q_1 + \left(\frac{a^2}{|a|^2} \right)^n q_2 j,$$

this again yields the periodicity condition.

3. MOEBIUS TRANSFORMATIONS OF THE UNIT BALL

The iteration theory of holomorphic functions mapping the unit circle into itself is governed by a theorem of Denjoy–Wolff, stating that either all iterates converge to a point in the closure of the unit circle, or the function is conjugate to a rotation (see [11]).

In case of Moebius transformations

$$f(q) = a(q - q_0)(1 + \overline{q}_0)^{-1} d^{-1}, \qquad a, d, q, q_0 \in \mathbf{H}, \quad |ad^{-1}| = 1, \quad |q_0| < 1 \tag{3.1}$$

mapping the unit ball in $\mathbf{R}^4$ onto itself, an analogous result holds.

THEOREM 3.1 *Let f be a Moebius transformation, as given in* (3.1), *mapping the unit ball onto itself. Then one of the following alternatives holds*:

(i) *Apart from the fixed points no convergence of the iterates occurs. In this case there is a fixed point in the unit ball and f is conjugate to a rotation.*
(ii) *All iterates converge towards a unique point at the boundary of the unit ball. In this case there is no fixed point of f in the unit ball.*

Proof First we prove the convergence of the iterates towards a unique point of the boundary of the unit ball if all fixed points of the mapping are assumed to lie at the boundary.

If f has exactly one fixed point at the boundary (parabolic case) then we infer from Theorem 2.4(i) that convergence of the sequence of iterates takes place from each initial value inside the unit ball.

If f has exactly two fixed points then we assume without loss of generality that 1 is one of these fixed points. Otherwise we apply a conjugation with $T(q) = \tilde{q}^{-1} q$, where $\tilde{q}$ is one of the fixed points, hence $|\tilde{q}| = 1$. This ensures the conjugate mapping again to be of type (3.1).

Since 1 is fixed point of (3.1) we infer the relation

$$a = d(1-\overline{q_0})(1-q_0)^{-1}. \tag{3.2}$$

Together with (2.1) and using the second fixing point as in the proof of Lemma 2.3(ii) we obtain the representation

$$\tilde{f}(q) = \tilde{a}q\tilde{d}^{-1}, \qquad \tilde{a} = d(1-\overline{q_0}), \qquad \tilde{d} = a + d\overline{q_0}. \tag{3.3}$$

Assume now that $|\tilde{a}\tilde{d}^{-1}| = 1$.

From (3.2), (3.3) we infer

$$\begin{aligned} 1 = |\tilde{a}\tilde{d}^{-1}| &= |(d(1-\overline{q_0})(1-q_0)^{-1} + d\overline{q_0})(1-\overline{q_0})^{-1}d^{-1}| \\ &= |d(1-\overline{q_0}) + d\overline{q_0}(1-\overline{q_0}))[\overline{(1-q_0)}(1-q_0)]^{-1}| = \frac{1-|q_0|^2}{|1-q_0|^2}, \end{aligned} \tag{3.4}$$

hence together with (3.2) and (3.1) we obtain $\tilde{a} = \tilde{d}$. But then by Lemma 2.3(ii) the mapping under consideration has infinitely many fixed points.

So we must have $|\tilde{a}\tilde{d}^{-1}| \neq 1$ and by Theorem 2.4(ii), (a) or (b) it follows that one fixed point is attracting all points from inside the unit ball.

If f has infinitely many fixed points at the boundary of the unit ball we again may assume that 1 is one of these fixed points and as before we deduce that here f is conjugate to a mapping $q \to \tilde{a}q\tilde{a}^{-1}$. Obviously all reals are fixed points here. From the transformation in (2.1) with the fixed point $q_0 = 1$ we infer that $r^{-1} + 1$, $r \in \mathbf{R}$ describes an infinite subset of fixed points of the considered Moebius transformation of the unit ball. This contradicts our assumption that all fixed points are at the boundary of the unit ball, hence the situation of infinitely many fixed points at the boundary cannot appear.

Now we assume that a fixed point exists outside the boundary of the unit ball.

If a Moebius transformation has a fixed point in the unit ball then it also has another fixed point outside the unit ball and vice versa. If one fixed point is attracting, by Theorem 2.4 all initial values except the other fixed point must be attracted. This is impossible since the open unit ball and the complement of the closed unit ball, respectively, are invariant sets under the iteration of Moebius transformations of type (3.1). Hence, convergence cannot occur. Conjugating with an appropriate Moebius transformation of the unit ball results in a Moebius transformation of the unit ball having zero as a fixed point. Due to a theorem of Cayley these transformations are rotations around zero.

Since every Moebius transformation has at least one fixed point, the proof is complete.

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