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COMMON FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACES

Akbar Azam,[1] Brian Fisher,[2] and M. Khan[3]

[1] *Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan*
[2] *Department of Mathematics, University of Leicester, Leicester, England*
[3] *Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, Islamabad, Pakistan*

□ *We introduce complex valued metric spaces and obtain sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions.*

Keywords Common fixed point; Contractive type mapping; Complex valued metric space.

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1. INTRODUCTION AND PRELIMINARIES

Since the appearance of the Banach contraction mapping principle, a number of articles have been dedicated to the improvement and generalization of that result. Most of these deal with the generalizations of the contractive condition in metric spaces.

Ghaler [2] generalized the idea of metric space and introduced a 2-metric space which was followed by a number of papers dealing with this generalized space. Plenty of material is also available in other generalized metric spaces, such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, Quasi metric spaces, Quasi semi metric spaces, D-metric spaces, and cone metric spaces (see [1–11]). In this article, we introduce and study complex

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Address correspondence to Akbar Azam, Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, Islamabad 44000, Pakistan; E-mail: akbarazam@yahoo.com

valued metric spaces and established some fixed point results for mappings satisfying a rational inequality. The idea of complex valued metric spaces can be expolited to define complex valued normed spaces and complex valued Hilbert spaces; additionally, it offers numerous research activities in mathematical analysis.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \mathrm{Re}(z_1) \leqslant \mathrm{Re}(z_2), \quad \mathrm{Im}(z_1) \leqslant \mathrm{Im}(z_2).$$

It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:

$$\begin{aligned}
&\text{(i)}\ \mathrm{Re}(z_1) = \mathrm{Re}(z_2), \quad \mathrm{Im}(z_1) < \mathrm{Im}(z_2),\\
&\text{(ii)}\ \mathrm{Re}(z_1) < \mathrm{Re}(z_2), \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2),\\
&\text{(iii)}\ \mathrm{Re}(z_1) < \mathrm{Re}(z_2), \quad \mathrm{Im}(z_1) < \mathrm{Im}(z_2),\\
&\text{(iv)}\ \mathrm{Re}(z_1) = \mathrm{Re}(z_2), \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2).
\end{aligned}$$

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \implies |z_1| < |z_2|,$$
$$z_1 \preceq z_2, z_2 \prec z_3 \implies z_1 \prec z_3.$$

Definition 1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:

1. $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A \setminus X) \neq \phi$$

A is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{B(x, r) : x \in X, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on X.

Let x_n be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \longrightarrow x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space.

Lemma 2. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Proof. Suppose that $\{x_n\}$ converges to x. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 \prec c \in \mathbb{C}$ and there is a natural number N, such that

$$d(x_n, x) \prec c \quad \text{for all } n > N.$$

Therefore,

$$|d(x_n, x)| < |c| = \epsilon \quad \text{for all } n > N.$$

It follows that

$$|d(x_n, x)| \to 0 \text{ as } n \to \infty.$$

Conversely, suppose that $|d(x_n, x)| \to 0$ as $n \to \infty$. Then given $c \in \mathbb{C}$ with $0 \prec c$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \delta \implies z \prec c.$$

For this δ, there is a natural number N such that

$$|d(x_n, x)| < \delta \quad \text{for all } n > N.$$

This means that $d(x_n, x) \prec c$ for all $n > N$. Hence $\{x_n\}$ converges to x. $\square$

Lemma 3. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Proof. Suppose that $\{x_n\}$ is a Cauchy sequence. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$$

Then $0 \prec c \in \mathbb{C}$ and there is a natural number N, such that:

$$d(x_n, x_{n+m}) \prec c \quad \text{for all } n > N.$$

Therefore,

$$|d(x_n, x_{n+m})| \prec |c| = \epsilon \quad \text{for all } n > N.$$

It follows that

$$|d(x_n, x_{n+m})| \to 0 \text{ as } n \to \infty.$$

Conversely, suppose that $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$. For given $c \in \mathbb{C}$ with $0 \prec c$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \delta \implies z \prec c.$$

For this δ, there is a natural number N such that:

$$|d(x_n, x_{n+m})| < \delta \quad \text{for all } n > N.$$

That is $d(x_n, x_{n+m}) \prec c$ for all $n > N$ and so $\{x_n\}$ is a Cauchy sequence. □

2. AN EXTENSION OF THE BANACH FIXED POINT THEOREM

Theorem 4. *Let (X, d) be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfy:*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx) d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then S, T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define

$$
\begin{aligned}
x_{2k+1} &= Sx_{2k} \\
x_{2k+2} &= Tx_{2k+1}, \quad k = 0, 1, 2, \ldots
\end{aligned}
$$

Then,

$$
\begin{aligned}
d(x_{2k+1}, x_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\
&\precsim \lambda d(x_{2k}, x_{2k+1}) + \frac{\mu d(x_{2k+1}, Tx_{2k+1}) d(x_{2k}, Sx_{2k})}{1 + d(x_{2k}, x_{2k+1})} \\
&\precsim \lambda d(x_{2k}, x_{2k+1}) + \frac{\mu d(x_{2k+1}, x_{2k+2}) d(x_{2k}, x_{2k+1})}{1 + d(x_{2k}, x_{2k+1})} \\
&\precsim \lambda d(x_{2k}, x_{2k+1}) + \mu d(x_{2k+1}, x_{2k+2}), \\
&\qquad \text{since } d(x_{2k}, x_{2k+1}) \precsim 1 + d(x_{2k}, x_{2k+1}) \\
&\precsim \frac{\lambda}{1-\mu} d(x_{2k}, x_{2k+1}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
d(x_{2k+2}, x_{2k+3}) &= d(Sx_{2k+2}, Tx_{2k+1}) \\
&\precsim \lambda(x_{2k+2}, x_{2k+1}) + \frac{\mu d(x_{2k+1}, Tx_{2k+1}) d(x_{2k+2}, Sx_{2k+2})}{1 + d(x_{2k+2}, x_{2k+1})} \\
&\precsim \lambda(x_{2k+2}, x_{2k+1}) + \frac{\mu d(x_{2k+1}, x_{2k+2}) d(x_{2k+2}, x_{2k+3})}{1 + d(x_{2k+1}, x_{2k+2})} \\
&\precsim \lambda(x_{2k+2}, x_{2k+1}) + \mu d(x_{2k+2}, x_{2k+3}) \\
&\precsim \frac{\lambda}{1-\mu} (x_{2k+2}, x_{2k+1}).
\end{aligned}
$$

Now with $h = \lambda/(1-\mu)$, we have

$$
\begin{aligned}
d(x_{n+1}, x_{n+2}) &\precsim h d(x_n, x_{n+1}) \\
&\precsim \cdots \precsim h^{n+1} d(x_0, x_1).
\end{aligned}
$$

So for any $m > n$,

$$
\begin{aligned}
d(x_n, x_m) &\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\precsim [h^n + h^{n+1} + \cdots + h^{m-1}] d(x_0, x_1) \\
&\precsim \left[\frac{h^n}{1-h}\right] d(x_0, x_1)
\end{aligned}
$$

and so

$$|d(x_m,\ x_n)| \precsim \frac{h^n}{1-h}|d(x_0, x_1)| \to 0, \quad \text{as } m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$. It follows that $u = Su$, otherwise $d(u, Su) = z > 0$ and we would then have

$$\begin{aligned} z &\precsim d(u, x_{2k+2}) + d(x_{2k+2}, Su) \\ &\precsim d(u, x_{2k+2}) + d(Tx_{2k+1}, Su) \\ &\precsim d(u, x_{2k+2}) + \lambda d(x_{2k+1}, u) + \frac{\mu d(x_{2k+1}, Tx_{2k+1}) d(u, Su)}{1 + d(u, x_{2k+1})} \\ &\precsim d(u, x_{2k+2}) + \lambda d(x_{2k+1}, u) + \frac{\mu d(x_{2k+1}, x_{2k+2}) z}{1 + d(u, x_{2k+1})}. \end{aligned}$$

This implies that

$$|z| \leqslant |d(u, x_{2k+2})| + \lambda |d(x_{2k+1}, u)| + \frac{\mu |d(x_{2k+1}, x_{2k+2})||z|}{|1 + d(u, x_{2k+1})|}.$$

That is $|z| = 0$, a contradiction and, hence, $u = Su$. It follows similarly that $u = Tu$.

We now show that S and T have unique common fixed point. For this, assume that u^* in X is a second common fixed point of S and T . Then

$$\begin{aligned} d(u, u^*) &= d(Su, Tu^*) \\ &\precsim \lambda d(u, u^*) + \frac{\mu d(u, Su) d(u^*, Tu^*)}{1 + d(u, u^*)} \\ &\precsim \lambda d(u, u^*). \end{aligned}$$

This implies that $u^* = u$, completing the proof of the theorem. □

Corollary 5. *Let (X, d) be a complete complex valued metric space and let the mapping $T : X \to X$ satisfy:*

$$d(Tx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Tx) d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then T has a unique fixed point.

Corollary 6. *Let (X, d) be a complete complex valued metric space and $T : X \to X$ satisfy:*

$$d(T^n x, T^n y) \precsim \lambda d(x, y) + \frac{\mu d(x, T^n x) d(y, T^n y)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then T has a unique fixed point.

Proof. By Corollary 5 we obtain $v \in X$ such that

$$T^n v = v$$

The result then follows from the fact that

$$\begin{aligned} d(Tv, v) &= d(TT^n v, T^n v) = d(T^n Tv, T^n v) \\ &\precsim \lambda d(Tv, v) + \frac{\mu d(Tv, T^n Tv) d(v, T^n v)}{1 + d(Tv, v)} \\ &\precsim \lambda d(Tv, v) + \frac{\mu d(Tv, TT^n v) d(v, v)}{1 + d(Tv, v)} = \lambda d(Tv, v). \end{aligned}$$ □

Example 7. Let

$$X_1 = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Re}(z) \leqslant 1,\ \operatorname{Im}(z) = 0\},$$
$$X_2 = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Im}(z) \leqslant 1,\ \operatorname{Re}(z) = 0\}$$

and let $X = X_1 \cup X_2$. Then with $z = x + iy$, define

$$Tz = \begin{cases} ix & \text{if } z \in X_1 \\ \frac{1}{2} y & \text{if } z \in X_2. \end{cases}$$

If d_u is usual metric on X then T is not contractive as

$$d_u(Tz_{1,} Tz_2) = |x_1 - x_2| = d_u(z_{1,} z_2) \quad \text{if } z_1, z_2 \in X_1.$$

Therefore, the Banach contraction theorem is not valid to find the unique fixed point 0 of T. To apply the corollary, consider a complex valued metric $d : X \times X \longrightarrow \mathbb{C}$ as follows:

$$d(z_1, z_2) = \begin{cases} \frac{2}{3}|x_1 - x_2| + \frac{i}{2}|x_1 - x_2|, & \text{if } z_1, z_2 \in X_1, \\ \frac{1}{2}|y_1 - y_2| + \frac{i}{3}|y_1 - y_2|, & \text{if } z_1, z_2 \in X_2, \\ \left(\frac{2}{3}x_1 + \frac{1}{2}y_2\right) + i\left(\frac{1}{2}x_1 + \frac{1}{3}y_2\right), & \text{if } z_1 \in X_1, \ \ z_2 \in X_2, \\ \left(\frac{1}{2}y_1 + \frac{2}{3}y_2\right) + i\left(\frac{1}{3}y_1 + \frac{1}{2}x_2\right), & \text{if } z_1 \in X_2, \ \ z_2 \in X_1, \end{cases}$$

where $z_1 = x_1 + iy_1$, $z_1 = x_2 + iy_2 \in X$. Then (X, d) is a complete complex valued metric space and

$$d(Tz_1, Tz_2) = \frac{3}{4} d(z_1, z_2) \quad \text{for all } z_1, z_2 \in X.$$

Example 8. Let $X = C([1, 3], \mathbb{R})$, $a > 0$ and for every $x, y \in X$ let

$$M_{xy} = \max_{t \in [1,3]} |x(t) - y(t)|,$$
$$d(x, y) = M_{xy}\sqrt{1 + a^2}\, e^{i \tan^{-1} a}.$$

Define $T : X \to X$ by

$$T(x(t)) = 4 + \int_1^t (x(u) + u^2) e^{u-1} du, \quad t \in [1, 3].$$

For every $x, y \in X$

$$\begin{aligned} d(Tx, Ty) &= M_{TxTy}\sqrt{1 + a^2}\, e^{i \tan^{-1} a} = \max_{t \in [1,3]} |Tx(t) - Ty(t)| \sqrt{1 + a^2}\, e^{i \tan^{-1} a} \\ &\precsim \int_1^3 \max_{t \in [1,3]} |(x(u) - y(u))| e^2 \sqrt{1 + a^2}\, e^{i \tan^{-1} a} du \\ &\precsim 2e^2 d(x, y). \end{aligned}$$

Similarly,

$$d(T^n x, T^n y) \precsim e^{2n} \frac{2^n}{n!} d(x, y).$$

Note that

$$e^{2n}\frac{2^n}{n!} = \begin{cases} 109 & \text{if } n = 2 \\ 1987 & \text{if } n = 4 \\ 1.31 & \text{if } n = 37 \\ 0.53 & \text{if } n = 38. \end{cases}$$

Thus for $\lambda = 0.53$, $\mu = 0$, $n = 38$, all conditions of Corollary 6 are satisfied and so T has a unique fixed point, which is the unique solution of the integral equation:

$$x(t) = 4 + \int_1^t (x(u) + u^2)e^{u-1}du, \quad t \in [1, 3]$$

or the differential equation:

$$x'(t) = (x + t^2)e^{t-1}, \quad t \in [1, 3], \quad x(1) = 4.$$

Theorem 9. *Let $X = C([a, b], \mathbb{R}^n)$, $a > 0$ and $d : X \times X \to \mathbb{C}$ is defined as follows:*

$$d(x, y) = \max_{t\in[a,b]} \|x(t) - y(t)\|_\infty \sqrt{1 + a^2}e^{i \tan^{-1} a}.$$

Consider the Urysohn integral equations

$$x(t) = \int_a^b K_1(t, s, x(s))ds + g(t), \tag{1}$$

$$x(t) = \int_a^b K_2(t, s, x(s))ds + h(t), \tag{2}$$

where $t \in [a, b] \subset \mathbb{R}$, $x, g, h \in X$.

Suppose that $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_x, G_x \in X$ for each $x \in X$, where

$$F_x(t) = \int_a^b K_1(t, s, x(s))ds, \quad G_x(t) = \int_a^b K_2(t, s, x(s))ds \quad \text{for all } t \in [a, b].$$

If there exist nonnegative reals λ, μ with $\lambda + \mu < 1$ such that for every $x, y \in X$

$$\|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1 + a^2}e^{i \tan^{-1} a} \precsim \lambda A(x, y)(t) + \mu B(x, y)(t)$$

for all $x, y \in X$, where

$$A(x,y)(t) = \|x(t) - y(t)\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a},$$
$$B(x,y)(t) = \frac{\|F_x(t) + g(t) - x(t)\|_\infty \|G_y(t) + h(t) - y(t)\|_\infty}{1 + d(x,y)} \sqrt{1+a^2}e^{i\tan^{-1}a},$$

then the system of integral equations (1) and (2) have a unique common solution.

Proof. Define $S, T : X \to X$ by

$$Sx = F_x + g, \quad Tx = G_x + h.$$

Then

$$d(Sx, Ty) = \max_{t\in[a,b]} \|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a},$$
$$d(x, Tx) = \max_{t\in[a,b]} \|F_x(t) + g(t) - x(t)\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}$$

and

$$d(y, Ty) = \max_{t\in[a,b]} \|G_y(t) + h(t) - y(t)\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}.$$

It is easily seen that

$$d(Sx, Ty) \precsim \lambda d(x,y) + \frac{\mu d(x,Tx) d(y,Sy)}{1 + d(x,y)},$$

for every $x, y \in X$. By Theorem 4, the Urysohn integral Eqs. (1) and (2) have a unique common solution. □

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