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Article *in* Nonlinear Dynamics · April 2013

DOI: 10.1007/s11071-012-0686-6

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Nonlinear Dyn (2013) 72:17–26
DOI 10.1007/s11071-012-0686-6

ORIGINAL PAPER

The generalized M–J sets for bicomplex numbers

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Received: 27 June 2012 / Accepted: 12 November 2012 / Published online: 6 December 2012

Abstract We explained the theory about bicomplex numbers, discussed the precondition of that addition and multiplication are closed in bicomplex number mapping of constructing generalized Mandelbrot–Julia sets (abbreviated to M–J sets), and listed out the definition and constructing arithmetic of the generalized Mandelbrot–Julia sets in bicomplex numbers system. And we studied the connectedness of the generalized M–J sets, the feature of the generalized Tetrabrot, and the relationship between the generalized M sets and its corresponding generalized J sets for bicomplex numbers in theory. Using the generalized M–J sets for bicomplex numbers constructed on computer, the author not only studied the relationship between the generalized Tetrabrot sets and its corresponding generalized J sets, but also studied their fractal feature, finding that: (1) the bigger the value of the escape time is, the more similar the 3-D generalized J sets and its corresponding 2-D J sets are; (2) the generalized Tetrabrot set contains a great deal information of constructing its corresponding 3-D generalized J sets; (3) both the generalized Tetrabrot sets and its corresponding cross section make a feature of axis symmetry; and (4) the bigger the value of the escape time is, the more similar the cross section and the generalized Tetrabrot sets are.

Keywords Bicomplex number system · Generalized M–J sets · Generalized Tetrabrot sets · Connectedness · Symmetry

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1 Introduction

Twenty years have passed, and people have lucubrated the M–J sets from complex mapping $z \leftarrow z^{\alpha} + c(\alpha = 2)$ [1, 2]. Basing on the characteristics of the appearance structure of the M–J sets for $\alpha \in \mathbb{R}$, Lakhtakia [3] and Gujar [4, 5] et al. brought forward a few hypotheses and Glynn [6] found the symmetry evolution of the generalized M set for the phase angle $\theta \in [-\pi, \pi)$. Dhurandhar [7] et al. discussed the fractal feature of the generalized J sets [8] for $\alpha < 0$. The authors [9] discussed the overlapping embedment topology distribution structure and discontinuity evolution laws of the generalized M–J sets. Sasmor [10] analyzed the discontinuity evolution of generalized M–J sets for a rational index number when the phase angle $\theta \in [-\pi, \pi)$. Romera [11, 12] et al. have explored overlapping embedment relation of "petal" at the Misiurewicz point in the generalized M set [13]. Geum [14] and the authors [15] have studied structure and distributing of periodicity "petal" and topological law of periodicity orbits of the generalized M sets. But all researches above are studied in a complex plane. If we review the fractal feature of the M–J sets in hyperdimension complex number space [16–19], there would be more significance in theory and in an application perspective. So, Norton [20, 21], Heidrich [22], and Lakner [23] et al. brought forward the arithmetic on how to construct M–J sets in a ternary number and quaternions. Kantor [24] and Chatelin et al. [25] established the hypercomplex number space [26]. Gematam [27] and Bedding et al. [28] researched the dynamics of the quadratic polynomial in the quaternion and pointed out that this mapping was a lack of actual physical background. Wang and Ge [29] introduced the perturbations into the quasi-sine Fibonacci M set. We note that another definition of the M set for the quaternion was introduced by Holbrook [30]. This definition give the M set in 3-D, which is not a slice of the quadratic M set. Based on the definition of the quaternion mentioned above, Rochon [31] used a commutative generalization of the complex numbers called bicomplex numbers [32–34] to give a new version of the M set called the Tetrabrot set [35] in 3-D and generalized M set in 4-D. Based on the above research, this paper not only studied the relationship between the generalized Tetrabrot set and its corresponding generalized J set, but also studied their fractal feature.

2 Bicomplex number systems

Each bicomplex generalize the M–J set to be constructed and requires a complex number system. A bicomplex number is defined as follows:

$$\mathbb{C}_2 = \left\{a + b\mathrm{i}_1 + c\mathrm{i}_2 + d\mathrm{j} : \mathrm{i}_1^2 = \mathrm{i}_2^2 = -1, \mathrm{j}^2 = 1\right\}, \quad (1)$$

where $a, b, c, d \in \mathbb{R}$; i_1, i_2 and j are imaginary units, and $\mathrm{i}_2\mathrm{j} = \mathrm{j}\mathrm{i}_2 = -\mathrm{i}_1$, $\mathrm{i}_1\mathrm{j} = \mathrm{j}\mathrm{i}_1 = -\mathrm{i}_2$, $\mathrm{i}_2\mathrm{i}_1 = \mathrm{i}_1\mathrm{i}_2 = \mathrm{j}$. Any bicomplex number system used to construct a generalized M–J set must be closed under addition and multiplication. This is due to the iterative function $z \leftarrow z^\alpha + c (\alpha \in \mathbb{R})$, which contains only addition and multiplication. A bicomplex number is of the form

$$(a + b\mathrm{i}_1) + (c + d\mathrm{i}_1)\mathrm{i}_2 = z_1 + z_2\mathrm{i}_2, \quad (2)$$

where $z_1, z_2 \in \mathbb{C}_1 = \{x + y\mathrm{i}_1 : \mathrm{i}_1^2 = -1\}$. Thus, $\mathbb{C}_2$ can be viewed as the complexification of $\mathbb{C}_1$; the multiplication of two bicomplex can be defined as

$$(z_1 + z_2\mathrm{i}_2)(z_3 + z_4\mathrm{i}_2) = (z_1z_3 - z_2z_4) + (z_1z_4 + z_2z_3)\mathrm{i}_2. \quad (3)$$

Based on the definition for the bicomplex number, we can obtain some operation rules of the bicomplex number as follows:

$$z_1 + z_2\mathrm{i}_2 = z_3 + z_4\mathrm{i}_2 \quad \Leftrightarrow \quad z_1 = z_3 \quad \text{and} \quad z_2 = z_4, \quad (4)$$

$$(z_1 + z_2\mathrm{i}_2) + (z_3 + z_4\mathrm{i}_2) = (z_1 + z_3) + (z_2 + z_4)\mathrm{i}_2. \quad (5)$$

According to Eq. (3) and Eq. (5), we know that multiplication and addition of a bicomplex number mapping $z \leftarrow z^\alpha + c$ are closed in $\mathbb{C}_2$ when $\alpha = 2, 3, 4, \ldots$.

According to the expression of the bicomplex number, the general definition of multiplication and addition can also be defined. Given the two bicomplex numbers

$$s = s_1 \cdot 1 + s_2 \cdot \mathrm{i}_1 + s_3 \cdot \mathrm{i}_2 + s_4 \cdot \mathrm{j}$$

and

$$t = t_1 \cdot 1 + t_2 \cdot \mathrm{i}_1 + t_3 \cdot \mathrm{i}_2 + t_4 \cdot \mathrm{j},$$

where $s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4 \in R$. Addition is defined as

$$s + t = (s_1 + t_1) \cdot 1 + (s_2 + t_2) \cdot \mathrm{i}_1 + (s_3 + t_3) \cdot \mathrm{i}_2 + (s_4 + t_4) \cdot \mathrm{j}, \quad (6)$$

multiplication is defined as

$$\begin{aligned} s \cdot t = {} & (s_1t_1 - s_2t_2 - s_3t_3 + s_4t_4) \\ & + (s_1t_2 + s_2t_1 - s_3t_4 - s_4t_3)\mathrm{i}_1 \\ & + (s_1t_3 - s_2t_4 + s_3t_1 - s_4t_2)\mathrm{i}_2 \\ & + (s_1t_4 + s_2t_3 + s_3t_2 + s_4t_1)\mathrm{j}. \end{aligned} \quad (7)$$

Lemma 1 [31] *Let $w = z_1 + z_2\mathrm{i}_2 \in \mathbb{C}_2$. Then w is noninvertible iff*

$$z_1^2 + z_2^2 = (z_1 - z_2\mathrm{i}_1)(z_1 + z_2\mathrm{i}_1) = 0.$$

Every bicomplex number $z_1 + z_2\mathrm{i}_2$ has the following unique idempotent representation:

$$z_1 + z_2\mathrm{i}_2 = (z_1 - z_2\mathrm{i}_1)e_1 + (z_1 + z_2\mathrm{i}_1)e_2, \quad (8)$$

where $e_1 = (1 + \mathrm{j})/2$, $e_2 = (1 - \mathrm{j})/2$. Equation (8) is very useful because: addition, multiplication, and division can be done term by term according to Eq. (8). Also, a bicomplex number will be noninvertible iff $z_1 - z_2 \mathrm{i}_1 = 0$ or $z_1 + z_2 \mathrm{i}_1 = 0$. According to Eq. (8), the definition of the Descartes product for a bicomplex number can be given.

Definition 1 we say that $X \subseteq \mathbb{C}_2$ is a $\mathbb{C}_2$-Cartesian set determined by X_1 and X_2 if

$$\begin{aligned} X &= X_1 \times_e X_2 \\ &= \{z_1 + z_2 \mathrm{i}_2 \in \mathbb{C}_2 : z_1 + z_2 \mathrm{i}_2 \\ &= w_1 e_1 + w_2 e_2,\ (w_1, w_2) \in X_1 \times X_2\}. \end{aligned}$$

If X_1 and X_2 are domains of $\mathbb{C}_1$, then $X_1 \times_e X_2$ is also a domain of $\mathbb{C}_2$. Then a manner to construct a natural "disc" in $\mathbb{C}_2$ is to take the $\mathbb{C}_2$-Cartesian product of two discs in $\mathbb{C}_1$. Hence, we define the natural "disc" of $\mathbb{C}_2$ as follows:

$$\begin{aligned} D(0, r) &= B^1(0, r) \times_e B^1(0, r) \\ &= \{z_1 + z_2 \mathrm{i}_2 : z_1 + z_2 \mathrm{i}_2 \\ &= w_1 e_1 + w_2 e_2,\ |w_1| < r,\ |w_2| < r\}, \end{aligned}$$

where $B^n(0, r)$ is the open ball of $\mathbb{C}_1^n \simeq \mathbb{C}^n$ with radius r.

3 The generalized M–J sets for bicomplex numbers space

3.1 The definition and constructing arithmetic of the generalized M–J sets

The iteration should begin with the critical point of f if the generalized M sets are constructed from the bicomplex number map $f : z \leftarrow z^\alpha + c(\alpha = 2, 3, 4, \ldots; z, c \in \mathbb{C}_2)$. When $\alpha > 1$, the critical point of f is 0, so we let $z_0 = 0$, then $z_1 = c, z_2 = c^\alpha + c, \ldots$. So the initial iterating point is chosen as $z_0 = c$. But what should be noticeable is that if c is chosen as the initial point when $\alpha \in [0, 1]$, the images obtained are not the real M sets. The reason is that f does not have any critical points when $\alpha = 1$, so there is no point in discussing the trajectory of the critical point; When $0 \le \alpha < 1$, the critical point is ∞, the parameter c is not on the trajectory of ∞, therefore, the resulting image of the iteration from c as the initial point are not the generalized M sets actually.

Definition 2 Suppose $f : z \leftarrow z^\alpha + c(\alpha = 2, 3, 4, \ldots; z, c \in \mathbb{C}_2)$ is a bicomplex number map in bicomplex number space $\mathbb{C}_2$. M_2 is the set of bicomplex number c in $\mathbb{C}_2$ whose trajectory is limited, i.e.,

$$\begin{aligned} \mathrm{M}_2 &= \{c \in \mathbb{C}_2 : \{f^k(c)\}_{k=1}^{\infty} \text{ has boundary}\} \\ &= \{c \in \mathbb{C}_2 : c, c^\alpha + c, (c^\alpha + c)^\alpha \\ &\quad + c, \ldots \nrightarrow \infty, k \to \infty\}, \end{aligned}$$

then M_2 is called the generalized M set corresponding to a bicomplex number map f.

For $f : z \leftarrow z^\alpha + c(\alpha = 2, 3, 4, \ldots; z, c \in \mathbb{C}_2)$, if ω satisfies $f(\omega) = \omega$, then ω is called the fixed point of f. If there is the minimum positive integer p, which is greater than 1 and p and satisfies $f^p(\omega) = \omega$, we say that ω is the p-periodicity point of f. If the complex differential quotient $(f^p)'(\omega) = \lambda$ and $|\lambda| > 1$, then we call the point ω is repelling. From the famous Montel theorem [34], we know that J_2, which is the generalized J set of f is the closure consisting of exclusive periodic points. If $c = 0$, then $f(z) = z^\alpha$ and $f^k(z) = z^{\alpha^k}$, and the points which satisfy $f^p(\omega) = \omega$ are $\{\exp(\frac{2\pi q \mathrm{i}}{\alpha^p - 1}) : 0 \le q < |\alpha^p - 1| - 1\}$. If let $|\alpha| > 1$, then these points satisfy $|(f^p)'(z)| = |\alpha^p| > 1$. So, these points are repelling and J_2 is the unit pellet $|z| = 1$ in the bicomplex number space $\mathbb{C}_2$. It is obvious that when $k \to \infty$, if $|z| < 1$, then $f^k(z) \to 0$ or ∞; if $|z| > 1$, then $f^k(z) \to \infty$ or 0; but if $|z| = 1$, then $f^k(z)$ is always on J_2. When iterated, J_2 is the boundary of the point sets which converge to 0 and ∞ separately. It is certain that J_2 is not fractal in such special case. If c is a small bicomplex number, then $f(z) = z^\alpha + c$. It is easy to make out that if z is also small, then $f^k(z) \to \omega$ or ∞, where ω is a fixed point of f near by the origin. But if z is a great number, then $f^k(z) \to \infty$ or ω. Now it appears that J_2 is the fractal curve surface, though J_2 is the boundary of two different kinds of sets.

Definition 3 Suppose $f : z \leftarrow z^\alpha + c(\alpha = 2, 3, 4, \ldots; z, c \in \mathbb{C}_2)$ is a bicomplex number map in a bicomplex number space $\mathbb{C}_2$. F_2 is the set of bicomplex number z in $\mathbb{C}_2$ whose trajectory is limited, i.e.,

$$\mathrm{F}_2 = \{z \in \mathbb{C}_2 : \{|f^k(z)|\}_{k=1}^{\infty} \text{ has boundary}\},$$

then the set is called filled generalized J set corresponding to the bicomplex mapping f. The boundary of F_2 is called the generalized J set of the bicomplex mapping f, which denoted by J_2, i.e.,

$$\mathrm{J}_2 = \partial \mathrm{F}_2.$$

Definitions 2 and 3 are theory foundations to construct M_2 and F_2 using the escaping-time algorithm [2]. The method of constructing the bicomplex generalized M–J sets is given as shown below:

(1) For the dynamics system $\{\mathbb{C}_2, f\}$, set up the view window W ($W \subset \mathbb{C}_{2(c)}$ or $\mathbb{C}_{2(z)}$, $\mathbb{C}_{2(c)}$ and $\mathbb{C}_{2(z)}$ denote parameter bicomplex number space and dynamics bicomplex number space, respectively), the escaping-radius R and escaping-time limit N.

(2) Define the escaping-time function

$$T(x) = \begin{cases} k, & \left|f^k(x)\right| \geq R \quad \text{and} \quad \left|f^i(x)\right| < R \\ & (i = 1, 2, \ldots, k-1; k \leq N) \\ 0, & \left|f^i(x)\right| < R \quad (i = 1, 2, \ldots, N). \end{cases}$$

(3) $\exists c_0 \in W$ and $W \subset \mathbb{C}_{2(c)}$. Let $z_0 = c$ and calculate $T(z_k)$; $\exists z_0 \in W$ and $W \subset \mathbb{C}_{2(z)}$, calculate $T(z_k)$ for given parameter c.

(4) If $T(z_k) = 0$, then $z_k \in M_f$ or F_2. If $T(z_k) \neq 0$, then $z_k \in \overline{\mathrm{M}}_2$ or $\overline{\mathrm{F}}_2$.

(5) Repeating processes (3) and (4) until all of the points in the view window W are exhausted, we can get the generalized M–J sets in the bicomplex number space.

3.2 The connectedness of the generalized M sets

Theorem 1 [2, 36] *The generalized* M *set* M_1 *of the complex mapping* $f : z \leftarrow z^\alpha + c (\alpha = 2, 3, 4, \ldots; z, c \in \mathbb{C}_1)$ *is connected.*

The next lemma is a characterization of the bicomplex number generalized M set M_2 using only M_1, which is a generalized M set in a complex plane $\mathbb{C}_1$. This lemma will be useful to prove that M_2 is also a connected set.

Lemma 2 $\mathrm{M}_2 = \mathrm{M}_1 \times_e \mathrm{M}_1$.

Proof First, we prove that $\mathrm{M}_2 \subseteq \mathrm{M}_1 \times_e \mathrm{M}_1$.

Let $c \in \mathbb{C}_2$. For $f : z \leftarrow z^\alpha + c (\alpha = 2, 3, 4, \ldots; z, c \in \mathbb{C}_2)$ and $f_c^{\circ n}(z) := (f_c^{\circ(n-1)} \circ f_c)(z)$, according to Definition 2, we can know that $f_c^{\circ n}(0)$ has boundary $\forall n \in \mathbb{N}$. Besides, when $\alpha = 2, 3, 4, \ldots$, we have

$$\begin{aligned} f_c(z) &= z^\alpha + c \\ &= \left[(z_1 - z_2 \mathrm{i}_1)^\alpha + (c_1 - c_2 \mathrm{i}_1)\right] e_1 \\ &\quad + \left[(z_1 + z_2 \mathrm{i}_1)^\alpha + (c_1 + c_2 \mathrm{i}_1)\right] e_2, \end{aligned}$$

where $z = (z_1 - z_2 \mathrm{i}_1) e_1 + (z_1 + z_2 \mathrm{i}_1) e_2$, $c = (c_1 - c_2 \mathrm{i}_1) e_1 + (c_1 + c_2 \mathrm{i}_1) e_2$. Then

$$f_c^{\circ n}(z) = f_{c_1 - c_2 \mathrm{i}_1}^{\circ n}(z_1 - z_2 \mathrm{i}_1) e_1 + P_{c_1 + c_2 \mathrm{i}_1}^{\circ n}(z_1 + z_2 \mathrm{i}_1) e_2.$$

By hypothesis, $f_c^{\circ n}(0) = f_{c_1 - c_2 \mathrm{i}_1}^{\circ n}(0) e_1 + f_{c_1 + c_2 \mathrm{i}_1}^{\circ n}(0) e_2$ is bounded $\forall n \in \mathbb{N}$. Hence, $f_{c_1 - c_2 \mathrm{i}_1}^{\circ n}(0)$ and $f_{c_1 + c_2 \mathrm{i}_1}^{\circ n}(0)$ are also bounded. Then $c_1 - c_2 \mathrm{i}_1$, $c_1 + c_2 \mathrm{i}_1 \in \mathrm{M}_1$ and $c = (c_1 - c_2 \mathrm{i}_1) e_1 + (c_1 + c_2 \mathrm{i}_1) e_2 \in \mathrm{M}_1 \times_e \mathrm{M}_1$.

Conversely, if we take $c \in \mathrm{M}_1 \times_e \mathrm{M}_1$, we have $c = (c_1 - c_2 \mathrm{i}_1) e_1 + (c_1 + c_2 \mathrm{i}_1) e_2$ with $c_1 - c_2 \mathrm{i}_1, c_1 + c_2 \mathrm{i}_1 \in \mathrm{M}_1$. Hence, $f_{c_1 - c_2 \mathrm{i}_1}^{\circ n}(0) f_{c_1 + c_2 \mathrm{i}_1}^{\circ n}(0)$ are also bounded $\forall n \in \mathbb{N}$. Then $f_c^{\circ n}(0)$ is bounded $\forall n \in \mathbb{N}$, that is $c \in \mathrm{M}_2$. □

Theorem 2 *The generalized* M *set* M_2 *is connected.*

Proof Define a mapping e as follows:

$$\begin{aligned} \mathbb{C}_1^2 = \mathbb{C}_1 \times \mathbb{C}_1 &\xrightarrow{e} \mathbb{C}_1 \times_e \mathbb{C}_1 \\ &= \mathbb{C}_2, (z_1, z_2) \mapsto z_1 e_1 + z_2 e_2. \end{aligned}$$

The mapping e is clearly a homeomorphism. Then if X_1 and X_2 are connected subsets of $\mathbb{C}_1$, we have that $e(X_1 \times X_2) = X_1 \times_e X_2$ is also connected. Now, by Lemma 2 and Theorem 1, if we let $X_1 = X_2 = \mathrm{M}_1$, M_2 is connected. □

3.3 The generalized Tetrabrot set

In the previous section, we studied the generalized M set M_2 in 4-D. Now we will establish the generalized Tetrabrot set in 3-D using the definition of M_2. We construct the generalized M set in the subspace $a + b\mathrm{i}_1 + c\mathrm{i}_2 (a, b, c \in \mathbb{R})$, and preserve the generalized Tetrabrot set on two perpendicular complex planes.

Definition 4 The generalized Tetrabrot set in a bicomplex number space can be defined as follows:

$$\begin{aligned} \mathrm{T} = \big\{ c = c_1 + c_2 \mathrm{i}_2 \in \mathbb{C}_2 : \mathrm{Im}(c_2) = 0 \\ \text{and } \forall n \in \mathbb{N},\ \exists f_c^{\circ n}(0) \text{ is bounded} \big\}. \end{aligned}$$

The next theorem is useful to construct the generalized Tetrabrot set by computer.

Theorem 3 *$\mathrm{M}_2 \subset \overline{D}(0,2) \subset \overline{B^2(0,2)}$, where $\overline{D}(0,2) = \overline{B^1(0,2)} \times_e \overline{B^1(0,2)}$, moreover, the radius 2 is the best possible in each case.*

Proof By Lemma 2, $\mathrm{M}_2 = \mathrm{M}_1 \times_e \mathrm{M}_1$. Moreover, $\overline{D}(0,2) = \overline{B^1(0,2)} \times_e \overline{B^1(0,2)}$ and $\mathrm{M}_1 \subset \overline{B^1(0,2)}$ with a point of M_1, which touches the boundary of this disc [36]. Then $\mathrm{M}_2 \subset \overline{D}(0,2)$ and the radius 2 is the best possible. Finally, it is proven by Price [37] that $\overline{D}(0,2) \subset \overline{B^2(0,2)}$ with points of $\overline{D}(0,2)$ which touch the boundary of $\overline{B^2(0,2)}$.

Based on Theorem 3, it is possible to compute the infinite divergence layers of the generalized Tetrabrot set from the number of iterations needed to have $|f_c^{\circ n}(0)| > 2$. We have to remark here that each divergence layer whose value of $T(x)$ is bigger will hide the others whose value of $T(x)$ is smaller. $\square$

3.4 The generalized M set and its corresponding generalized J set F_2

Now we study the relationship between the generalized M set M_2 and its corresponding generalized filled J set F_2 in a bicomplex number space.

Theorem 4 [2, 36] *Let F_1 is the generalized filled J set corresponding to the complex mapping $f: z \leftarrow z^\alpha + c (\alpha = 2, 3, 4, \ldots; z, c \in \mathbb{C}_1)$, then $c \in \mathrm{M}_1 \Leftrightarrow \mathrm{F}_1$ is connected.*

The next lemma gives a characterization of F_2 in terms of F_1. This lemma can be used to prove a theorem corresponding to Theorem 4 for the bicomplex numbers.

Lemma 3 $\mathrm{F}_{2,c} = \mathrm{F}_{2,(c_1 - c_2 \mathrm{i}_1)e_1 + (c_1 + c_2 \mathrm{i}_1)e_2} = \mathrm{F}_{c_1 - c_2 \mathrm{i}_1} \times_e \mathrm{F}_{c_1 + c_2 \mathrm{i}_1}$.

The proof is along the same lines as the proof of the Lemma 2.

Theorem 5 *$c \in \mathrm{M}_2 \Leftrightarrow \mathrm{F}_{2,c}$ is connected.*

Proof By Lemma 3, we know that $\mathrm{F}_{2,c} = \mathrm{F}_{c_1 - c_2 \mathrm{i}_1} \times_e \mathrm{F}_{c_1 + c_2 \mathrm{i}_1}$. Also, by the homeomorphism in the proof of Theorem 2, $\mathrm{F}_{c_1 - c_2 \mathrm{i}_1} \times \mathrm{F}_{c_1 + c_2 \mathrm{i}_1}$ is connected if and only if $\mathrm{F}_{c_1 - c_2 \mathrm{i}_1} \times_e \mathrm{F}_{c_1 + c_2 \mathrm{i}_1}$ is connected. Then $\mathrm{F}_{c_1 - c_2 \mathrm{i}_1} \times_e \mathrm{F}_{c_1 + c_2 \mathrm{i}_1}$ is connected if and only if $\mathrm{F}_{c_1 - c_2 \mathrm{i}_1}$ and $\mathrm{F}_{c_1 + c_2 \mathrm{i}_1}$ are connected. Hence, by Theorem 4, $\mathrm{F}_{2,c}$ is connected if and only if $c_1 - c_2 \mathrm{i}_1, c_1 + c_2 \mathrm{i}_1 \in \mathrm{M}_1$. Therefore, by Lemma 2, $\mathrm{F}_{2,c}$ is connected if and only if $c = (c_1 - c_2 \mathrm{i}_1)e_1 + (c_1 + c_2 \mathrm{i}_1)e_2 \in \mathrm{M}_2$. $\square$

Theorem 6 *The generalized Tetrabrot set can also be characterized as follows:*

$$\mathrm{T} = \bigcup_{y \in [-m,m]} \left\{(\mathrm{M}_1 - y\mathrm{i}_1) \cap (\mathrm{M}_1 + y\mathrm{i}_1) + y\mathrm{i}_2\right\},$$

where $m := \sup\{q \in \mathbb{R} : \exists p \in \mathbb{R}, p + q\mathrm{i}_1 \in \mathrm{M}_1\}$.

Proof By Definition 4, $\mathrm{T} = \{c = c_1 + c_2 \mathrm{i}_2 \in \mathbb{C}_2 : \mathrm{Im}(c_2) = 0$, and $\forall n \in \mathbb{N}, \exists f_c^{\circ n}(0)$ is bounded$\}$. If $c_1 = c_{11} + c_{12}\mathrm{i}_1$ and $c_2 = c_{21} + c_{22}\mathrm{i}_1$, where $c_{11}, c_{12}, c_{21}, c_{22} \in \mathbb{R}$, then $c = (c_1 - c_2 \mathrm{i}_1)e_1 + (c_1 + c_2 \mathrm{i}_1)e_2$. Now if $\mathrm{Im}(c_2) = 0$, we have $c_2 = c_{21} + 0\mathrm{i}_1$ and, therefore, $c = (c_1 - c_{21}\mathrm{i}_1)e_1 + (c_1 + c_{21}\mathrm{i}_1)e_2$ whenever $\mathrm{Im}(c_2) = 0$. Hence, $\mathrm{T} = \{c = c_1 + c_2 \mathrm{i}_2 \in \mathbb{C}_2 : \mathrm{Im}(c_2) = 0, \exists f_c^{\circ n}(0)$ is bounded $\forall n \in \mathbb{N}\} = \{(c_1 - c_{21}\mathrm{i}_1)e_1 + (c_1 + c_{21}\mathrm{i}_1)e_2 : f_{c_1 - c_{21}\mathrm{i}_1}^{\circ n}(0)$ and $f_{c_1 + c_{21}\mathrm{i}_1}^{\circ n}(0)$ are bounded $\forall n \in \mathbb{N}\}$. To continue the proof, we need to remark the following fact: $\forall z \in \mathbb{C}_1$,

$$\left\{c \in \mathbb{C}_1 : f_{c+z}^{\circ n}(0) \text{ is bounded } \forall n \in \mathbb{N}\right\} = \mathrm{M}_1 - z.$$

By definition, $f_{c_1 - c_{21}\mathrm{i}_1}^{\circ n}(0)$ and $f_{c_1 + c_{21}\mathrm{i}_1}^{\circ n}(0)$ are bounded $\forall n \in \mathbb{N}$ if and only if $c_1 - c_{21}\mathrm{i}_1, c_1 + c_{21}\mathrm{i}_1 \in \mathrm{M}_1$. That is to say $f_{c_1 - c_{21}\mathrm{i}_1}^{\circ n}(0)$ and $f_{c_1 + c_{21}\mathrm{i}_1}^{\circ n}(0)$ are bounded $\forall n \in \mathbb{N}$ if and only if $c_1 \in (\mathrm{M}_1 - c_{21}\mathrm{i}_1) \cap (\mathrm{M}_1 + c_{21}\mathrm{i}_1)$. Hence, if we express $(c_1 - c_{21}\mathrm{i}_1)e_1 + (c_1 + c_{21}\mathrm{i}_1)e_2 = c_1 + c_{21}\mathrm{i}_2 = c_{11} + c_{12}\mathrm{i}_1 + c_{21}\mathrm{i}_2$, the generalized Tetrabrot set can be characterized as follows:

$$\begin{aligned}\mathrm{T} &= \big\{c_{11} + c_{12}\mathrm{i}_1 + c_{21}\mathrm{i}_2 : \\ &\qquad c_{11} + c_{12}\mathrm{i}_1 \in (\mathrm{M}_1 - c_{21}\mathrm{i}_1) \cap (\mathrm{M}_1 + c_{21}\mathrm{i}_1)\big\} \\ &= \bigcup_{y \in \mathbb{R}} \big\{\big[(\mathrm{M}_1 - y\mathrm{i}_1) \cap (\mathrm{M}_1 + y\mathrm{i}_1)\big] + y\mathrm{i}_2\big\}.\end{aligned}$$

$\mathrm{M}_1 \subset \{z \in \mathbb{C}_1 : |\mathrm{Im}(z)| \le m\}$, so $(\mathrm{M}_1 - y\mathrm{i}_1) \cap (\mathrm{M}_1 + y\mathrm{i}_1) = \emptyset$ whenever $y \in [-m, m]^c$. Hence, it is possible to express the generalize Tetrabrot set more precisely as

$$\mathrm{T} = \bigcup_{y\in[-m,m]} \{[(\mathrm{M}_1 - y\mathrm{i}_1) \cap (\mathrm{M}_1 + y\mathrm{i}_1)] + y\mathrm{i}_2\}.$$

Moreover, $(\mathrm{M}_1 - y\mathrm{i}_1) \cap (\mathrm{M}_1 + y\mathrm{i}_1) \neq \emptyset$ $\forall y \in [-m, m]$. To see this, we just have to prove that $E_y := \{c = c_{11} + 0\mathrm{i}_1 + y\mathrm{i}_2 : f_c^{on}(0)$ is bounded $\forall n \in \mathbb{N}\}$ and is nonempty $\forall y \in [-m, m]$. Because $E_y \subset \{c = c_{11} + c_{12}\mathrm{i}_1 + c_{21}\mathrm{i}_2 : f_c^{on}(0)$ is bounded, $\forall n \in \mathbb{N}\} = \{c = c_{11} + c_{12}\mathrm{i}_1 + c_{21}\mathrm{i}_2 : c_{11} + c_{12}\mathrm{i}_1 \in (\mathrm{M}_1 - c_{21}\mathrm{i}_1) \cap (\mathrm{M}_1 + c_{21}\mathrm{i}_1)\}$. In fact, the set E_y is the escaping time algorithm for the generalized M set iterates, with the imaginary part in "i_2" fixed at y. By the compactness and the symmetry of the generalized M set M_1, there must exist x_m such that $x_m - m\mathrm{i}_2, x_m + m\mathrm{i}_2 \in E_m$. Therefore, because M_1 is connected, we must have $\forall y \in [-m, m], \exists E_y \neq \emptyset$.

Definitions 2 and 3 give the definition of the generalized M set M_2 and its corresponding generalized filled J set F_2 in bicomplex number space. Now we will put forward the generalized filled J set $\mathrm{L}_{2,c}$ corresponding to the generalized Tetrabrot set. □

Definition 5 $\forall c \in \mathbb{C}_2$, the associated generalized filled J set for the generalized Tetrabrot set is defined as follows:

$$\mathrm{L}_{2,c} = \{z = z_1 + z_2\mathrm{i}_2 \in \mathbb{C}_2 : \mathrm{Im}(z_2) = 0,\ f_c^{on}(0) \text{ is bounded } \forall n \in \mathbb{N}\}.$$

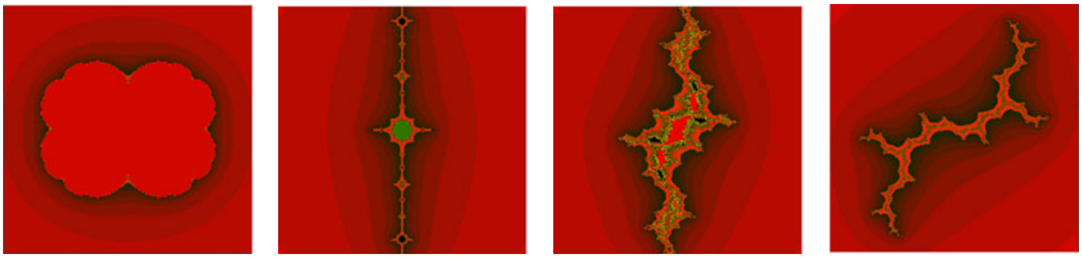
4 Experiment and results

In 1993, Carleson [38] et al. remarked this interesting fact: The M set contains a great deal of information of constructing the J set, that is to say, the M set is the diagram catalogue set corresponding to the J set. Now we will study the relationship between the generalized Tetrabrot set and its corresponding generalized J set by computer experiments. To obtain a good visual effect, our figures are a rotation by 90° of the original figures anticlockwise.

Figure 1 shows the corresponding J sets whose initial points are chosen differently from the M set for $\alpha = 2$. Figure 1 and the structure of M set indicate that there is great difference in shape and connectedness for the J sets corresponding to different points in M set, and the local boundary of the M set is similar to the J set corresponding to the point on M set. It seems that the boundary of the M set is woven together by its tiny corresponding J sets, namely there is obvious "family comparability" between the fractal corresponding to the local boundary points of the parameter space and geometry of the local boundary.

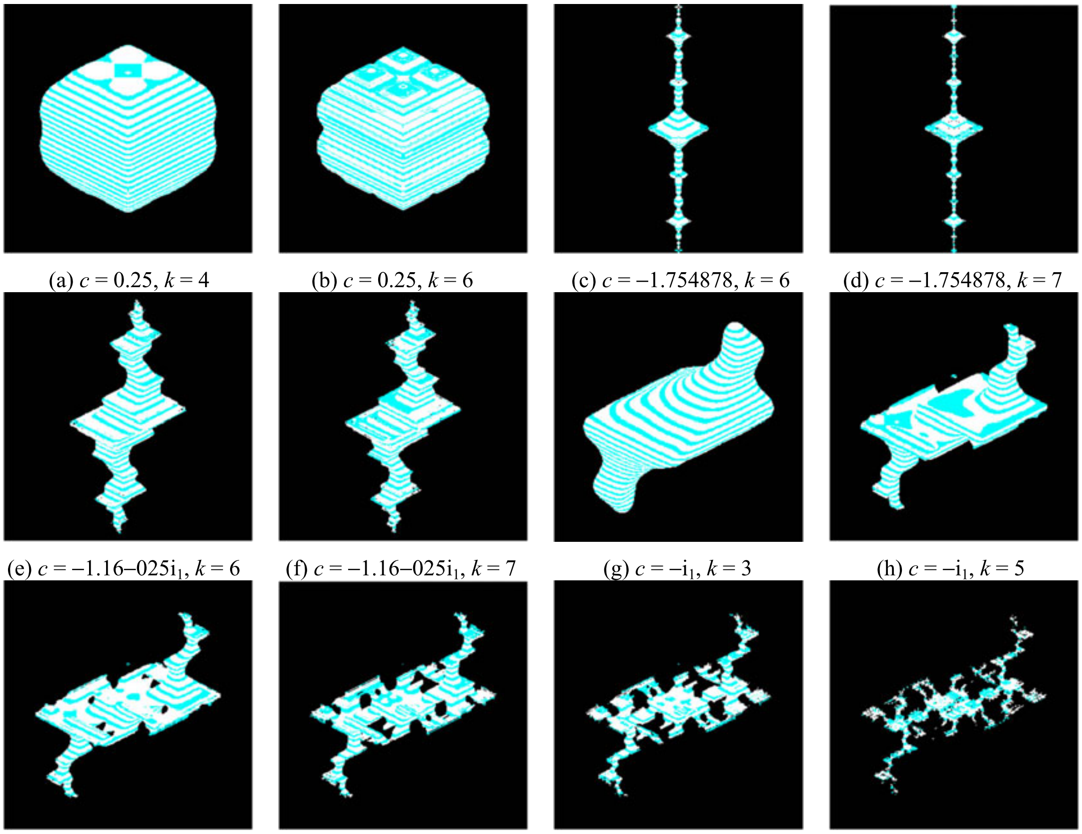
Figure 2 shows the 3-D generalized J sets whose initial points are chosen differently from the generalized Tetrabrot set. Figures 2(a) and 2(b) have the same parameter of c, so do Figs. 2(c) and 2(d), Figs. 2(e) and 2(f), Figs. 2(g)–2(l), but the escaping time is different (that is to say, the value k of the escaping time function is different), and they respectively associate with the J set, which are given in Figs. 1(a), 1(b), 1(c), and 1(d), so it generalizes the J set in a 3-D bicomplex number space. Figure 3 shows the generalized J sets whose initial points is a fixed point on the generalized M set for $\alpha = 3$. Figures 3(b)–3(d) have a different escaping time, that is, the generalized J sets in a complex plane (Fig. 3(a)) are generalized in a 3-D bicomplex number space.

Comparing Fig. 1(d) with Figs. 2(g)–2(l), Fig. 3(a) with Figs. 3(b)–3(d), we know that the bigger the value k of the escape time is, the more similar the 3-D generalized J sets and its corresponding 2-D J sets are (for example, Fig. 2(l) is similar to Fig. 1(d), Fig. 3(d) is similar to Fig. 3(a)), conversely, the difference is



(a) $c = 0.25$ (b) $c = -1.754878$ (c) $c = -1.16–025\mathrm{i}_1$ (d) $c = -\mathrm{i}_1$

Fig. 1 The corresponding J sets whose initial points are chosed differently from the M set

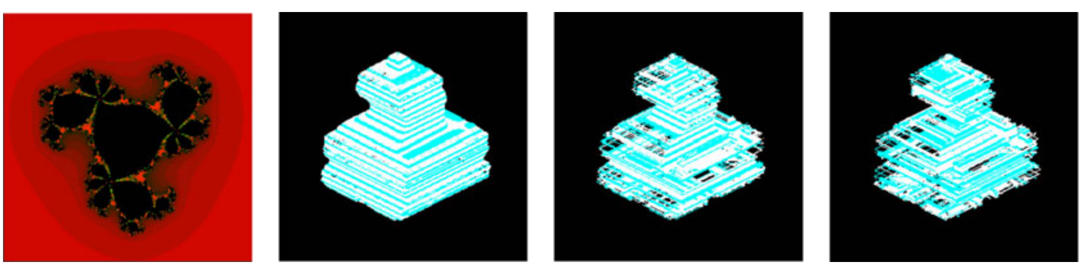


(a) $c = 0.25$, $k = 4$ (b) $c = 0.25$, $k = 6$ (c) $c = -1.754878$, $k = 6$ (d) $c = -1.754878$, $k = 7$

(e) $c = -1.16\text{–}025\mathrm{i}_1$, $k = 6$ (f) $c = -1.16\text{–}025\mathrm{i}_1$, $k = 7$ (g) $c = -\mathrm{i}_1$, $k = 3$ (h) $c = -\mathrm{i}_1$, $k = 5$

(i) $c = -\mathrm{i}_1$, $k = 6$ (j) $c = -\mathrm{i}_1$, $k = 7$ (k) $c = -\mathrm{i}_1$, $k = 8$ (l) $c = -\mathrm{i}_1$, $k = 10$

Fig. 2 The 3-D generalized J sets whose initial points are chosed differently from the generalized Tetrabrot set

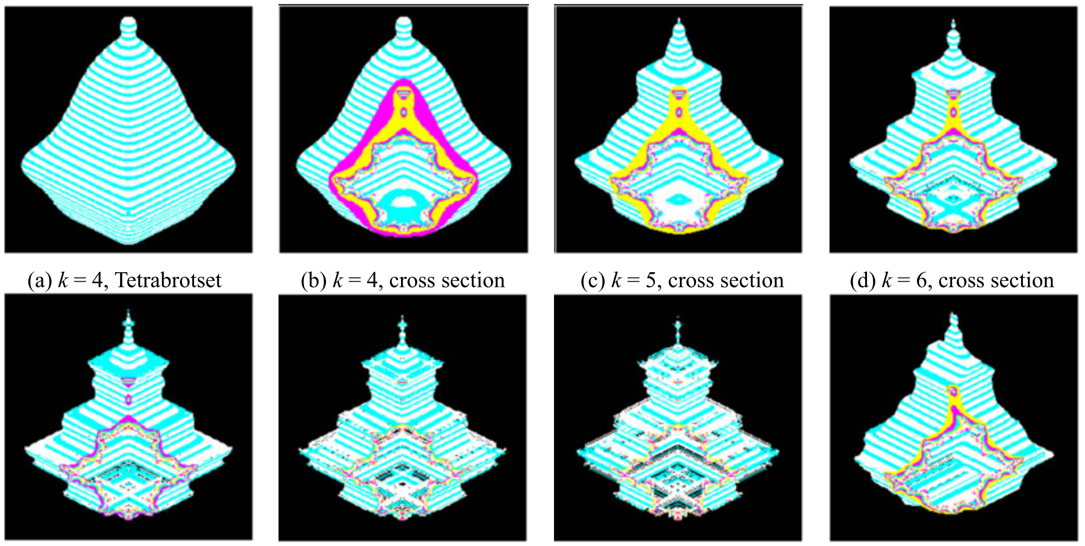


(a) 2-D generalized J set (b) 3-D generalized J set, $k = 4$ (c) 3-D generalized J set, $k = 6$ (d) 3-D generalized J set, $k = 7$

Fig. 3 The generalized J sets whose initial points takes $c = 0.5363637\text{–}0.1545455\mathrm{i}_1$ from the generalized M set for $\alpha = 3$

greater. When $\alpha \geq 3$, a lot of computer experiments show the same results as mentioned above. So, we can conclude that the points in the generalized Tetrabrot set and their corresponding 3-D generalized J set have the one-to-one corresponding relationship, namely the generalized Tetrabrot set contains a great deal information of constructing the corresponding 3-D generalized J set.

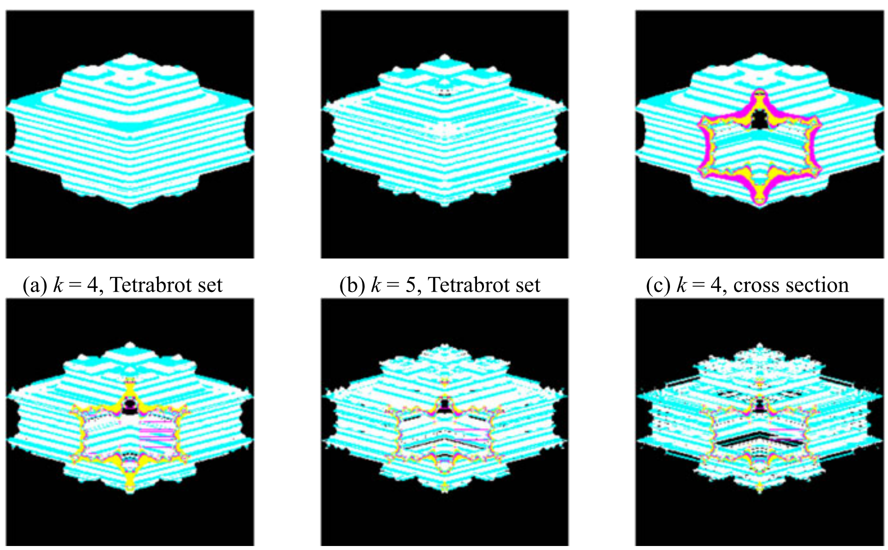
The generalized Tetrabrot set and its corresponding cross section are studied in the following sections. Figures 4, 5, 6 show that the 3-D generalized Tetrabrot set is rotatably symmetric about a given axis, which parallels to the x axis, and there is not a relationship between the 3-D generalized Tetrabrot set and its corresponding 2-D generalized M set in structure. The gen-



(a) $k = 4$, Tetrabrotset (b) $k = 4$, cross section (c) $k = 5$, cross section (d) $k = 6$, cross section

(e) $k = 7$, cross section (f) $k = 8$, cross section (g) $k = 9$, cross section (h) $k = 6$, $c_3 = 0.1$, cross section

Fig. 4 The Tetrabrot set and its cross section with different escaping time for $\alpha = 2$



(a) $k = 4$, Tetrabrot set (b) $k = 5$, Tetrabrot set (c) $k = 4$, cross section

(d) $k = 5$, cross section (e) $k = 6$, cross section (f) $k = 7$, cross section

Fig. 5 The generalized Tetrabrot set and its cross section with different escaping time for $\alpha = 3$

eralized Tetrabrot set is constructed in a 3-D space, so any higher divergence layer hides the lower, thus we cannot observe the structure of the lower divergence layers. In order to observe the configuration of the generalized Tetrabrot set, the cross section with different escaping time is constructed. Figures 4(a)–4(g) show the Tetrabrot set and its cross section with different escaping time for $\alpha = 2$; we can see that the bigger the value k of the escape time is, the more similar the cross section and the Tetrabrot set are, and these cross sections are symmetric over a given axis which parallels with the x axis. When $c_3 \neq 0$, Fig. 4(d) dissatisfies the definition of the Tetrabrot, namely Fig. 4(h) is not the cross section of the Tetrabrot, so it loses the symmetry. Observing the generalized Tetrabrot set and its cross section (Fig. 5) with different escaping time for $\alpha = 3$, we can see that the generalized Tetrabrot set is rotatably symmetric about two given axes, which parallel

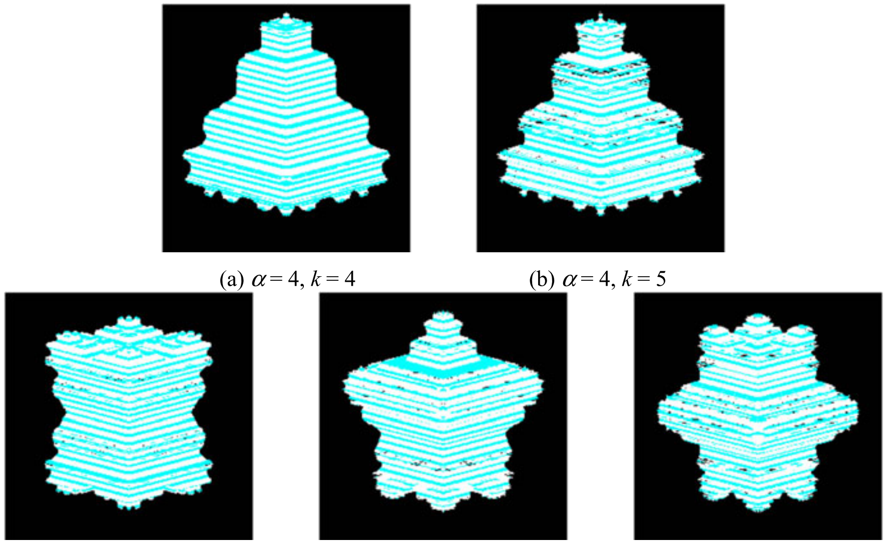


Fig. 6 The generalized Tetrabrot set with different escaping time for $\alpha > 3$

(a) $\alpha = 4, k = 4$ (b) $\alpha = 4, k = 5$ (c) $\alpha = 5, k = 4$ (d) $\alpha = 6, k = 4$ (e) $\alpha = 7, k = 4$

to the x axis and y axis, respectively, except has the same character mentioned above. Observing the generalized Tetrabrot set as shown in Fig. 6 with different escaping time for $\alpha > 3$, it is obvious that when α is an odd number, the generalized Tetrabrot set is rotatablely symmetric about two given axis which parallel to x axis and y axis, respectively; contrarily, when α is an even number, the generalized Tetrabrot set is rotatably symmetric about a given axis, which parallels to the x axis.

5 Conclusion

In this paper, we explained the theory about bicomplex numbers, discussed the precondition of addition and multiplication are closed in bicomplex number mapping of constructing generalized M–J sets, and listed out the definition and constructing arithmetic of the generalized M–J sets in a bicomplex number system. And we studied the connectedness of the generalized M–J sets, the feature of the generalized Tetrabrot, and the relationship between the generalized M sets and its corresponding generalized J sets for bicomplex numbers in theory. Using the generalized M–J sets for bicomplex numbers constructed on computer, the author not only studied the relationship between the generalized Tetrabrot sets and its corresponding generalized J sets, but also studied their fractal feature, finding that: (1) the bigger the value of the escape time is, the more similar the 3-D generalized J sets and its corresponding 2-D J sets are; (2) the generalized Tetrabrot set contains a great deal information of constructing its corresponding 3-D generalized J sets; (3) both the generalized Tetrabrot sets and its corresponding cross section make a feature of axis symmetry; and (4) the bigger the value of the escape time is, the more similar the cross section and the generalized Tetrabrot sets are.

Acknowledgements This research is supported by the National Natural Science Foundation of China (Nos. 61173183, 60973152, and 60573172), the Superior University Doctor Subject Special Scientific Research Foundation of China (No. 20070141014), Program for Liaoning Excellent Talents in University (No. LR2012003), the National Natural Science Foundation of Liaoning Province (No. 20082165), and the Fundamental Research Funds for the Central Universities (No. DUT12JB06).

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