1.6 Idempotent representations of bicomplex numbers

It turns out that there are two very special zero-divisors.

Proposition 1.6.1. *The bicomplex numbers*

$$\mathbf{e} := \frac{1+\mathbf{ij}}{2} \quad \textit{and} \quad \mathbf{e}^\dagger := \frac{1-\mathbf{ij}}{2}$$

have the properties:

$$\mathbf{e}\cdot\mathbf{e}^\dagger = 0$$

(thus, each of them is a zero-divisor);

$$\mathbf{e}^2 = \mathbf{e}, \qquad (\mathbf{e}^\dagger)^2 = \mathbf{e}^\dagger$$

(thus, they are idempotents);

$$\mathbf{e}+\mathbf{e}^\dagger = 1, \qquad \mathbf{e}-\mathbf{e}^\dagger = \mathbf{ij}\,.$$

The properties of the idempotents $\mathbf{e}$ and $\mathbf{e}^\dagger$ cause many strange phenomena. One of them is the following

Corollary 1.6.2. *There holds:*

$$\begin{aligned} \mathbf{i}\,\mathbf{e} &= -\mathbf{j}\,\mathbf{e}, & \mathbf{i}\,\mathbf{e}^\dagger &= \mathbf{j}\,\mathbf{e}^\dagger\,, \\ \mathbf{k}\,\mathbf{e} &= \mathbf{e}, & \mathbf{k}\mathbf{e}^\dagger &= -\mathbf{e}^\dagger\,. \end{aligned} \tag{1.21}$$

The next property has no analogs for complex numbers, and it exemplifies one of the deepest peculiarities of the set of bicomplex numbers. For any bicomplex number $Z = z_1 + \mathbf{j}z_2 \in \mathbb{BC}$ we have:

$$\begin{aligned} Z = z_1 + \mathbf{j}z_2 &= \frac{z_1 - \mathbf{i}z_2 + z_1 + \mathbf{i}z_2}{2} + \mathbf{j}\frac{z_2 + \mathbf{i}z_1 + z_2 - \mathbf{i}z_1}{2} \\ &= \frac{z_1 - \mathbf{i}z_2}{2} + \frac{z_1 + \mathbf{i}z_2}{2} + \mathbf{ij}\frac{z_1 - \mathbf{i}z_2}{2} - \mathbf{ij}\frac{z_1 + \mathbf{i}z_2}{2} \\ &= (z_1 - \mathbf{i}z_2)\frac{1+\mathbf{ij}}{2} + (z_1 + \mathbf{i}z_2)\frac{1-\mathbf{ij}}{2}, \end{aligned}$$

that is,

$$Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger, \tag{1.22}$$

where $\beta_1 := z_1 - \mathbf{i}z_2$ and $\beta_2 := z_1 + \mathbf{i}z_2$ are complex numbers in $\mathbb{C}(\mathbf{i})$. Formula (1.22) is called the $\mathbb{C}(\mathbf{i})$-*idempotent representation* of the bicomplex number Z.

It is obvious that since β_1 and β_2 are both in $\mathbb{C}(\mathbf{i})$, then $\beta_1\mathbf{e}+\beta_2\mathbf{e}^\dagger = 0$ if and only if $\beta_1 = 0 = \beta_2$. This implies that the above idempotent representation of the bicomplex number Z is unique: indeed, assume that $Z \neq 0$ has two idempotent representations, say,

$$Z = \beta_1\,\mathbf{e} + \beta_2\,\mathbf{e}^\dagger = \beta_1'\,\mathbf{e} + \beta_2'\,\mathbf{e}^\dagger,$$

then $0 = (\beta_1 - \beta_1')\,\mathbf{e} + (\beta_2 - \beta_2')\,\mathbf{e}^\dagger$ and thus $\beta_1 = \beta_1'$, $\beta_2 = \beta_2'$.

The following proposition shows the advantage of using the idempotent representation of bicomplex numbers in all algebraic operations.

Proposition 1.6.3. *The addition and multiplication of bicomplex numbers can be realized "term-by-term" in the idempotent representation* (1.22). *Specifically, if* $Z = \beta_1\,\mathbf{e} + \beta_2\,\mathbf{e}^\dagger$ *and* $W = \nu_1\,\mathbf{e} + \nu_2\,\mathbf{e}^\dagger$ *are two bicomplex numbers, then*

$$\begin{aligned} Z + W &= (\beta_1 + \nu_1)\,\mathbf{e} + (\beta_2 + \nu_2)\,\mathbf{e}^\dagger, \\ Z \cdot W &= (\beta_1\nu_1)\,\mathbf{e} + (\beta_2\nu_2)\,\mathbf{e}^\dagger, \\ Z^n &= \beta_1^n\,\mathbf{e} + \beta_2^n\,\mathbf{e}^\dagger. \end{aligned}$$

The proof of the formulas in the proposition above relies simply on the rather specific properties of the numbers $\mathbf{e}$ and $\mathbf{e}^\dagger$. For example, let us prove the second property:

$$\begin{aligned} Z \cdot W &= \left(\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger\right) \cdot \left(\nu_1\mathbf{e} + \nu_2\mathbf{e}^\dagger\right) \\ &= \beta_1\mathbf{e} \cdot \nu_1\mathbf{e} + \beta_1\mathbf{e} \cdot \nu_2\mathbf{e}^\dagger + \beta_2\mathbf{e}^\dagger \cdot \nu_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \cdot \nu_2\mathbf{e}^\dagger \\ &= \beta_1\nu_1 \cdot \mathbf{e} + \beta_1\nu_2 \cdot 0 + \beta_2\nu_1 \cdot 0 + \beta_2\nu_2 \cdot \mathbf{e}^\dagger \\ &= \beta_1\nu_1 \cdot \mathbf{e} + \beta_2\nu_2 \cdot \mathbf{e}^\dagger. \end{aligned}$$

We used the fact that $\mathbf{e}$ and $\mathbf{e}^\dagger$ are idempotents, i.e., each of them squares to itself, and that their product is zero.

We showed after formula (1.22) that the coefficients β_1 and β_2 of the idempotent representation are uniquely defined complex numbers. But this refers to the complex numbers in $\mathbb{C}(\mathbf{i})$, and the paradoxical nature of the idempotents $\mathbf{e}$ and $\mathbf{e}^\dagger$ manifests itself as follows.

Take a bicomplex number Z written in the form $Z = \zeta_1 + \mathbf{i}\,\zeta_2$, with $\zeta_1, \zeta_2 \in \mathbb{C}(\mathbf{j})$. Then a direct computation shows:

$$Z = \alpha_1\mathbf{e} + \alpha_2\mathbf{e}^\dagger := (\zeta_1 - \mathbf{j}\,\zeta_2)\mathbf{e} + (\zeta_1 + \mathbf{j}\,\zeta_2)\mathbf{e}^\dagger\,, \tag{1.23}$$

where $\alpha_1 := \zeta_1 - \mathbf{j}\,\zeta_2$ and $\alpha_2 := \zeta_1 + \mathbf{j}\,\zeta_2$ are complex numbers in $\mathbb{C}(\mathbf{j})$. So, we see that as a matter of fact every bicomplex number has two idempotent representations with COMPLEX coefficients, one with coefficients in $\mathbb{C}(\mathbf{i})$, and the other with coefficients in $\mathbb{C}(\mathbf{j})$:

$$Z = \beta_1\,\mathbf{e} + \beta_2\,\mathbf{e}^\dagger = \alpha_1\,\mathbf{e} + \alpha_2\,\mathbf{e}^\dagger. \tag{1.24}$$

Let us find out which is the relation between them. One has that

$$\mathbf{e}Z = \beta_1\,\mathbf{e} = \alpha_1\,\mathbf{e}$$

and

$$\mathbf{e}^\dagger Z = \beta_2\,\mathbf{e}^\dagger = \alpha_2\,\mathbf{e}^\dagger,$$

thus the authentic uniqueness consists of the fact that not the coefficients β_1 and α_1 (or β_2 and α_2) are equal, but the products $\beta_1\mathbf{e}$ and $\alpha_1\mathbf{e}$ (or $\beta_2\mathbf{e}^\dagger$ and $\alpha_2\mathbf{e}^\dagger$) are equal respectively. What is more, $\beta_1\mathbf{e} = \alpha_1\mathbf{e}$ is equivalent to $(\beta_1 - \alpha_1)\mathbf{e} = 0$, but since $\mathbf{e}$ is a zero-divisor, then $\beta_1 - \alpha_1$ is also a zero-divisor, that is, $\beta_1 - \alpha_1 = A \cdot \mathbf{e}^\dagger$, where A can be chosen either in $\mathbb{C}(\mathbf{i})$ or in $\mathbb{C}(\mathbf{j})$. The latter is justified with the following reasoning. Take β_1, β_2 to be $\beta_1 = c_1 + \mathbf{i}d_1$, $\beta_2 = c_2 + \mathbf{i}d_2$, then

$$\begin{aligned} Z &= \beta_1\,\mathbf{e} + \beta_2\,\mathbf{e}^\dagger = (c_1 + \mathbf{i}\,d_1)\,\mathbf{e} + (c_2 + \mathbf{i}\,d_2)\,\mathbf{e}^\dagger \\ &= c_1\,\mathbf{e} - \mathbf{j}\,d_1\,\mathbf{e} + c_2\,\mathbf{e}^\dagger + \mathbf{j}\,d_2\,\mathbf{e}^\dagger \\ &= \mathbf{e}\cdot(c_1 - \mathbf{j}\,d_1) + \mathbf{e}^\dagger\cdot(c_2 + \mathbf{j}\,d_2) \\ &= \mathbf{e}\cdot\alpha_1 + \mathbf{e}^\dagger\cdot\alpha_2\,, \end{aligned}$$

where $\alpha_1 = c_1 - \mathbf{j}\,d_1$, $\alpha_2 = c_2 + \mathbf{j}\,d_2$; thus

$$\begin{aligned} \beta_1 - \alpha_1 &= c_1 + \mathbf{i}\,d_1 - c_1 + \mathbf{j}\,d_1 = d_1\,(\mathbf{i} + \mathbf{j}) \\ &= \mathbf{i}\,d_1\,(1 - \mathbf{i}\mathbf{j}) \\ &= 2\,d_1\,\mathbf{i}\,\mathbf{e}^\dagger\ = 2\,d_1\,\mathbf{j}\,\mathbf{e}^\dagger. \end{aligned}$$

Example 1.6.4. Consider the bicomplex number:

$$Z = (1 + \mathbf{i}) + \mathbf{j}\,(3 - 2\mathbf{i}) =: z_1 + \mathbf{j}\,z_2\,.$$

Then $\beta_1 = z_1 - \mathbf{i}\,z_2 = -1 - 2\mathbf{i}$ and $\beta_2 = z_1 + \mathbf{i}\,z_2 = 3 + 4\mathbf{i}$, so in the first idempotent representation we have:

$$Z = (-1 - 2\mathbf{i})\mathbf{e} + (3 + 4\mathbf{i})\mathbf{e}^\dagger\,.$$

Now we write the same bicomplex number as

$$Z = (1 + 3\mathbf{j}) + \mathbf{i}\,(1 - 2\mathbf{j}) =: \zeta_1 + \mathbf{i}\,\zeta_2\,.$$

Then $\alpha_1 = \zeta_1 - \mathbf{j}\zeta_2 = -1 + 2\mathbf{j}$ and $\alpha_2 = \zeta_1 + \mathbf{j}\zeta_2 = 3 + 4\mathbf{j}$. The second idempotent representation of Z is then

$$Z = (-1 + 2\mathbf{j})\mathbf{e} + (3 + 4\mathbf{j})\mathbf{e}^\dagger\,.$$

Thus in this situation $\beta_1 = -1 - 2\mathbf{i} = c_1 + \mathbf{i}d_1$, $\beta_2 = 3 + 4\mathbf{i} = c_2 - \mathbf{i}d_2$, $\alpha_1 = -1 + 2\mathbf{j} = c_1 - \mathbf{j}d_1$, $\alpha_2 = 3 + 4\mathbf{j} = c_2 + \mathbf{j}d_2$ and as we know it should be that

$$\beta_1 - \alpha_1 = d_1(\mathbf{i} + \mathbf{j}).$$

Since

$$\beta_1 - \alpha_1 = -2\,(\mathbf{i} + \mathbf{j}) = -4\,\mathbf{i}\,\mathbf{e}^\dagger = -4\,\mathbf{j}\,\mathbf{e}^\dagger,$$

one obtains $d_1 = -2$, which coincides with the value of d_1 in this example. □

Let us see now how the conjugations and moduli manifest themselves in idempotent representations. Take $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger = \alpha_1\mathbf{e} + \alpha_2\mathbf{e}^\dagger$, with β_1 and β_2 in $\mathbb{C}(\mathbf{i})$, α_1 and α_2 in $\mathbb{C}(\mathbf{j})$. Then it is immediate to see that

$$\begin{aligned}\overline{Z} &= \overline{\beta}_2\mathbf{e} + \overline{\beta}_1\mathbf{e}^\dagger = \alpha_2\mathbf{e} + \alpha_1\mathbf{e}^\dagger;\\ Z^\dagger &= \beta_2\mathbf{e} + \beta_1\mathbf{e}^\dagger = \alpha_2^\dagger\mathbf{e} + \alpha_1^\dagger\mathbf{e}^\dagger;\\ Z^* &= \overline{\beta}_1\mathbf{e} + \overline{\beta}_2\mathbf{e}^\dagger = \alpha_1^\dagger\mathbf{e} + \alpha_2^\dagger\mathbf{e}^\dagger.\end{aligned}$$

Hence, the squares of all the three moduli become:

$$\begin{aligned}|Z|_{\mathbf{j}}^2 &= Z \cdot \overline{Z}\\ &= \left(\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger\right) \cdot \left(\overline{\beta}_2\mathbf{e} + \overline{\beta}_1\mathbf{e}^\dagger\right)\\ &= \beta_1\overline{\beta}_2\mathbf{e} + \overline{\beta_1\overline{\beta}_2}\mathbf{e}^\dagger\\ &= \left(\alpha_1\mathbf{e} + \alpha_2\mathbf{e}^\dagger\right) \cdot \left(\alpha_2\mathbf{e} + \alpha_1\mathbf{e}^\dagger\right)\\ &= \alpha_1\alpha_2\mathbf{e} + \alpha_1\alpha_2\mathbf{e}^\dagger = \alpha_1\alpha_2 \in \mathbb{C}(\mathbf{j});\end{aligned}$$

$$\begin{aligned}|Z|_{\mathbf{i}}^2 &= Z \cdot Z^\dagger\\ &= \left(\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger\right) \cdot \left(\beta_2\mathbf{e} + \beta_1\mathbf{e}^\dagger\right)\\ &= \beta_1\beta_2\mathbf{e} + \beta_1\beta_2\mathbf{e}^\dagger \;=\; \beta_1\beta_2\\ &= \left(\alpha_1\mathbf{e} + \alpha_2\mathbf{e}^\dagger\right) \cdot \left(\alpha_2^\dagger\mathbf{e} + \alpha_1^\dagger\mathbf{e}^\dagger\right)\\ &= \alpha_1\alpha_2^\dagger\mathbf{e} + \left(\alpha_1\alpha_2^\dagger\right)^\dagger\mathbf{e}^\dagger \in \mathbb{C}(\mathbf{i});\end{aligned}$$

$$\begin{aligned}|Z|_{\mathbf{k}}^2 &= Z \cdot Z^*\\ &= \left(\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger\right) \cdot \left(\overline{\beta}_1\mathbf{e} + \overline{\beta}_2\mathbf{e}^\dagger\right)\\ &= \beta_1\overline{\beta}_1\mathbf{e} + \beta_2\overline{\beta}_2\mathbf{e}^\dagger \;=\; |\beta_1|^2\mathbf{e} + |\beta_2|^2\mathbf{e}^\dagger\\ &= \left(\alpha_1\mathbf{e} + \alpha_2\mathbf{e}^\dagger\right) \cdot \left(\alpha_1^\dagger\mathbf{e} + \alpha_2^\dagger\mathbf{e}^\dagger\right)\\ &= \alpha_1\alpha_1^\dagger\mathbf{e} + \alpha_2\alpha_2^\dagger\mathbf{e}^\dagger \;=\; |\alpha_1|^2\mathbf{e} + |\alpha_2|^2\mathbf{e}^\dagger \in \mathbb{D}^+.\end{aligned}$$

Observe that in the formulas for $|Z|_{\mathbf{k}}^2$ the idempotent coefficients are non-negative real numbers and we will see soon that this is a characteristic property of non-negative hyperbolic numbers. Observe also that given $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger = \alpha_1\mathbf{e} + \alpha_2\mathbf{e}^\dagger$ with β_1, β_2 in $\mathbb{C}(\mathbf{i})$ and α_1, α_2 in $\mathbb{C}(\mathbf{j})$, then

$$|Z| = \frac{1}{\sqrt{2}}\sqrt{|\beta_1|^2 + |\beta_2|^2} = \frac{1}{\sqrt{2}}\sqrt{|\alpha_1|^2 + |\alpha_2|^2}.$$

We can characterize now the invertibility of bicomplex numbers in terms of the idempotent representations.

Theorem 1.6.5. *Given a bicomplex number $Z \neq 0$, $Z = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger = \alpha_1 \mathbf{e} + \alpha_2 \mathbf{e}^\dagger$, with β_1 and β_2 in $\mathbb{C}(\mathbf{i})$, α_1 and α_2 in $\mathbb{C}(\mathbf{j})$, the following are equivalent:*

1. *Z is invertible;*
2. *$\beta_1 \neq 0$ and $\beta_2 \neq 0$;*
3. *$\alpha_1 \neq 0$ and $\alpha_2 \neq 0$.*

Whenever this holds the inverse of Z is given by

$$Z^{-1} = \beta_1^{-1}\mathbf{e} + \beta_2^{-1}\mathbf{e}^\dagger = \alpha_1^{-1}\mathbf{e} + \alpha_2^{-1}\mathbf{e}^\dagger.$$

Proof. It follows using items (6) and (7) from Theorem 1.5.1 together with the idempotent expressions for $|Z|_{\mathbf{i}}^2$ and $|Z|_{\mathbf{j}}^2$. □

Again, we have a "dual" description of zero-divisors in terms of the idempotent decompositions.

Corollary 1.6.6. *Given a bicomplex number $Z \neq 0$, $Z = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger = \alpha_1 \mathbf{e} + \alpha_2 \mathbf{e}^\dagger$, with β_1 and β_2 in $\mathbb{C}(\mathbf{i})$, α_1 and α_2 in $\mathbb{C}(\mathbf{j})$, the following are equivalent:*

1. *Z is a zero-divisor;*
2. *$\beta_1 = 0$ and $\beta_2 \neq 0$ or $\beta_1 \neq 0$ and $\beta_2 = 0$;*
3. *$\alpha_1 = 0$ and $\alpha_2 \neq 0$ or $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$.*

This means that any zero-divisor can be written in one of the following forms:

$$\begin{aligned}
Z &= \beta_1 \mathbf{e} && \text{with} \quad \beta_1 \in \mathbb{C}(\mathbf{i}) \setminus \{0\}; \\
Z &= \beta_2 \mathbf{e}^\dagger && \text{with} \quad \beta_2 \in \mathbb{C}(\mathbf{i}) \setminus \{0\}; \\
Z &= \alpha_1 \mathbf{e} && \text{with} \quad \alpha_1 \in \mathbb{C}(\mathbf{j}) \setminus \{0\}; \\
Z &= \alpha_2 \mathbf{e}^\dagger && \text{with} \quad \alpha_2 \in \mathbb{C}(\mathbf{j}) \setminus \{0\}.
\end{aligned}$$

One can ask if there are more idempotents in $\mathbb{BC}$, not only $\mathbf{e}$ and $\mathbf{e}^\dagger$ (of course the trivial idempotents 0 and 1 do not count). Assume that a bicomplex number $Z = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger$, with β_1 and β_2 being complex numbers either in $\mathbb{C}(\mathbf{i})$ or $\mathbb{C}(\mathbf{j})$, is an idempotent: $Z^2 = Z$. Then

$$\beta_1^2 \mathbf{e} + \beta_2^2 \mathbf{e}^\dagger = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger$$

and

$$\beta_1^2 = \beta_1 \qquad \text{and} \qquad \beta_2^2 = \beta_2,$$

which gives:

$$\beta_1 \in \{0, 1\}, \qquad \beta_2 \in \{0, 1\}.$$

Hence, combining all possible choices we have at most four candidates for idempotents in $\mathbb{BC}$:

$$Z_1 = 0 \cdot \mathbf{e} + 0 \cdot \mathbf{e}^\dagger = 0,$$

$$Z_2 = 1 \cdot \mathbf{e} + 1 \cdot \mathbf{e}^\dagger = 1,$$
$$Z_3 = 1 \cdot \mathbf{e} + 0 \cdot \mathbf{e}^\dagger = \mathbf{e},$$
$$Z_4 = 0 \cdot \mathbf{e} + 1 \cdot \mathbf{e}^\dagger = \mathbf{e}^\dagger.$$

Thus, one concludes that $\mathbf{e}$ and $\mathbf{e}^\dagger$ are the only non-trivial idempotents in $\mathbb{BC}$.

Remark 1.6.7. *The formulas*

$$Z = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger \qquad \text{and} \qquad Z^\dagger = \beta_2 \mathbf{e} + \beta_1 \mathbf{e}^\dagger,$$

with β_1 and β_2 in $\mathbb{C}(\mathbf{i})$, allow us to express the idempotent components of a bicomplex number in terms of the bicomplex number itself. Indeed:

$$\beta_1 = \beta_1 \mathbf{e} + \beta_1 \mathbf{e}^\dagger = Z\mathbf{e} + Z^\dagger \mathbf{e}^\dagger;$$
$$\beta_2 = \beta_2 \mathbf{e}^\dagger + \beta_2 \mathbf{e} = Z\mathbf{e}^\dagger + Z^\dagger \mathbf{e}.$$

Writing now the number Z with coefficients in $\mathbb{C}(\mathbf{j})$, $Z = \gamma_1 \mathbf{e} + \gamma_2 \mathbf{e}^\dagger$, we get a similar pair of formulas:

$$\gamma_1 = \gamma_1 \mathbf{e} + \gamma_1 \mathbf{e}^\dagger = Z\mathbf{e} + \overline{Z}\mathbf{e}^\dagger;$$
$$\gamma_2 = \gamma_2 \mathbf{e}^\dagger + \gamma_2 \mathbf{e} = \overline{Z}\mathbf{e} + Z\mathbf{e}^\dagger.$$

1.7 Hyperbolic numbers inside bicomplex numbers

Although the hyperbolic numbers had been found long ago and although we wrote about them at the beginning of the chapter, we believe that it would be instructive for the reader to have an intrinsic description of the properties of hyperbolic numbers, and only then to show how they can be obtained by appealing to bicomplex numbers.

For a hyperbolic number $\mathfrak{z} = x + \mathbf{k}y$, its (hyperbolic) *conjugate* $\mathfrak{z}^\diamond$ is defined by

$$\mathfrak{z}^\diamond := x - \mathbf{k}y\,.$$

The reader immediately notices that

$$\mathfrak{z} \cdot \mathfrak{z}^\diamond = x^2 - y^2 \in \mathbb{R}\,, \tag{1.25}$$

which yields the notion of the square of the *(intrinsic) modulus* of $\mathfrak{z}$:

$$|\mathfrak{z}|^2_{hyp} := x^2 - y^2\,,$$

which is a real number (it could be negative!).

If both x and y are non-zero real numbers, but $x^2 - y^2 = 0$, then the corresponding hyperbolic number $\mathfrak{z} = x + \mathbf{k}y$ is a zero-divisor, since its conjugate is non-zero, but the product is zero: $\mathfrak{z} \cdot \mathfrak{z}^{\diamond} = 0$. All zero-divisors in $\mathbb{D}$ are characterized by $x^2 = y^2$, i.e., $x = \pm y$, thus they are of the form

$$\mathfrak{z} = \lambda(1 \pm \mathbf{k})$$

for any $\lambda \in \mathbb{R} \setminus \{0\}$.

The idempotent representation of the hyperbolic number $\mathfrak{z} = x + \mathbf{k}y \in \mathbb{D}$ is

$$\mathfrak{z} = (x + y)\mathbf{e} + (x - y)\mathbf{e}^{\diamond}, \tag{1.26}$$

where $\mathbf{e} = \dfrac{1}{2}(1 + \mathbf{k})$, $\mathbf{e}^{\diamond} = \dfrac{1}{2}(1 - \mathbf{k})$. We consciously use the same letter $\mathbf{e}$ that was used for the idempotent representation in $\mathbb{BC}$ since, as we will soon show, the two representations coincide in $\mathbb{BC}$. Direct analogs of Proposition 1.6.1 and Proposition 1.6.3 can be reformulated in this case.

Whenever there is no danger of confusion, we will denote the coefficients of the idempotent representation of a hyperbolic number $\mathfrak{z}$ by $s := x + y$ and $t := x - y$, so that we have:

$$\mathfrak{z} = s\mathbf{e} + t\mathbf{e}^{\diamond}. \tag{1.27}$$

Observe that

$$|\mathfrak{z}|^2_{hyp} = x^2 - y^2 = (x + y)(x - y) = st.$$

Let us show now how these properties are related with their bicomplex antecedents. We are interested in bicomplex numbers $Z = z_1 + \mathbf{j}\, z_2$ with $\operatorname{Im}(z_1) = 0 = \operatorname{Re}(z_2)$, that is, our hyperbolic numbers are of the form $\mathfrak{z} = x_1 + \mathbf{ij}\, y_2$ and the hyperbolic unit is $\mathbf{k} = \mathbf{ij}$. Then the $\diamond$-conjugation operation is consistent with the bicomplex conjugations $\dagger$ and *bar* in the following way:

$$\mathfrak{z}^{\diamond} = ((x_1 + \mathbf{i}0) + \mathbf{j}(0 + \mathbf{i}y_2))^{\dagger} = \overline{((x_1 + \mathbf{i}0) + \mathbf{j}(0 + \mathbf{i}y_2))} = x_1 - \mathbf{k}y_2.$$

For this reason, from this point on we will not write the hyperbolic conjugate of $\mathbf{e}$ as $\mathbf{e}^{\diamond}$ anymore, but we will use the bicomplex notation $\mathbf{e}^{\dagger}$.

For a general bicomplex number, the three moduli have been defined in Section 1.4. Let us see what happens if they are evaluated on a generic hyperbolic number $\mathfrak{z} = x_1 + \mathbf{k}y_2$. Considering it as $\mathfrak{z} = z_1 + \mathbf{j}z_2 := (x_1 + \mathbf{i}0) + \mathbf{j}(0 + \mathbf{i}y_2) \in \mathbb{BC}$, we have:

$$|\mathfrak{z}|^2_{\mathbf{i}} = z_1^2 + z_2^2 = x_1^2 - y_2^2 = |\mathfrak{z}|^2_{hyp}.$$

Recalling that the definition of $|\cdot|_{\mathbf{i}}$ involves the $\dagger$-conjugation, the definition of $|\cdot|_{\mathbf{j}}$ involves the *bar*-conjugation and that on hyperbolic numbers both conjugations coincide, we see that on hyperbolic numbers both moduli reduce to the intrinsic modulus of hyperbolic numbers:

$$|\mathfrak{z}|^2_{\mathbf{i}} = |\mathfrak{z}|^2_{\mathbf{j}} = |\mathfrak{z}|^2_{hyp}. \tag{1.28}$$

This is not the case of the third modulus: the hyperbolic-valued modulus of $Z = \mathfrak{z}$ is different than the intrinsic modulus of $\mathfrak{z}$. Indeed, we have:

$$|\mathfrak{z}|_{\mathbf{k}}^2 = Z \cdot Z^* = Z \cdot Z = Z^2 = \mathfrak{z}^2 . \tag{1.29}$$

In (1.28) we have a relation between the squares of the three moduli $|\mathfrak{z}|_{\mathbf{i}}$, $|\mathfrak{z}|_{\mathbf{j}}$ and $|\mathfrak{z}|_{hyp}$ for hyperbolic numbers. The question now is how to define the modulus $|\mathfrak{z}|_{hyp}$ itself, which obviously should be defined as the square root of $x_1^2 - y_2^2$. Note that some authors consider the non-negative values of $x_1^2 - y_2^2$ only.

It is instructive to analyze the situation more rigorously and to understand if we have other options for choosing an appropriate value of the intrinsic modulus. Although we work here with hyperbolic numbers, at the same time one can think about bicomplex numbers also as of possible values of the square roots of a hyperbolic number. So let us consider the solutions in $\mathbb{BC}$ of the equation $Z^2 = R$ for a given real number R. Write $Z = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger$, then the equation $Z^2 = R$ is equivalent to

$$\beta_1^2 \mathbf{e} + \beta_2^2 \mathbf{e}^\dagger = R\mathbf{e} + R\mathbf{e}^\dagger$$

which is equivalent to

$$\beta_1^2 = R; \quad \beta_2^2 = R.$$

If $R = 0$, then the only solution is $Z = 0$. If R is positive, then

$$\beta_1 = \pm\sqrt{R}; \quad \beta_2 = \pm\sqrt{R},$$

and we get four solutions:

$$\sqrt{R}; \quad -\sqrt{R}; \quad \mathbf{k}\sqrt{R}; \quad -\mathbf{k}\sqrt{R}.$$

These are all the solutions in $\mathbb{BC}$, and they are real or hyperbolic numbers.

If R is negative, then one gets:

$$\beta_1 = \pm\mathbf{i}\sqrt{-R}, \quad \beta_2 = \pm\mathbf{i}\sqrt{-R}$$

giving the following solutions:

$$\begin{aligned} &\mathbf{i}\sqrt{-R}; \quad -\mathbf{i}\sqrt{-R}; \\ &\mathbf{ik}\sqrt{-R} = -\mathbf{j}\sqrt{-R}; \\ &-\mathbf{ik}\sqrt{-R} = \mathbf{j}\sqrt{-R}. \end{aligned}$$

Thus, for $R < 0$ the equation $Z^2 = R$ has four solutions none of which is a hyperbolic number; two of them are complex numbers in $\mathbb{C}(\mathbf{i})$ and the remaining two are complex numbers in $\mathbb{C}(\mathbf{j})$.

Returning to the intrinsic modulus $|\mathfrak{z}|_{hyp}$ of a hyperbolic number $\mathfrak{z}$ we see that in case $x_1^2 - y_2^2 > 0$ this modulus can be taken as a positive real number

$\sqrt{x_1^2 - y_2^2}$ or even as a hyperbolic number $\pm \mathbf{k}\sqrt{x_1^2 - y_2^2}$. But if $x_1^2 - y_2^2 < 0$, then there are no solutions in $\mathbb{D}$, the candidates should be taken as complex (in $\mathbb{C}(\mathbf{i})$ or in $\mathbb{C}(\mathbf{j})$) numbers.

It is instructive to note that in case $x_1^2 - y_2^2 > 0$ the positive real number $\sqrt{x_1^2 - y_2^2}$ coincides with the equal values of $|\mathfrak{z}|_{\mathbf{i}}$ and $|\mathfrak{z}|_{\mathbf{j}}$ as defined in Section 1.4.

When $x_1^2 - y_2^2 < 0$, then $|\mathfrak{z}|_{hyp}$ can be chosen either as $|\mathfrak{z}|_{\mathbf{i}} \in \mathbb{C}(\mathbf{i})$ or as $|\mathfrak{z}|_{\mathbf{j}} \in \mathbb{C}(\mathbf{j})$ (recall that we have agreed to take, in both cases, the value of the square root which is in the upper half plane); as formula (1.29) shows, it cannot be chosen as $|\mathfrak{z}|_{\mathbf{k}}$.

1.7.1 The idempotent representation of hyperbolic numbers

Recall that the "hyperbolic" idempotents $\mathbf{e}$ and $\mathbf{e}^\diamond$ in (1.26) and the "bicomplex" idempotents $\mathbf{e}$ and $\mathbf{e}^\dagger$ are the same bicomplex numbers (which are hyperbolic numbers!). Here $(x + y)$ and $(x - y)$ correspond to the idempotent "coordinates" β_1 and β_2 of a bicomplex number. Indeed, considering $\mathfrak{z} = z_1 + \mathbf{j}z_2 := (x_1 + \mathbf{i}0) + \mathbf{j}(0 + \mathbf{i}y_2) \in \mathbb{BC}$, its idempotent representation is

$$\begin{aligned}\mathfrak{z} &= \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger = (z_1 - \mathbf{i}z_2)\mathbf{e} + (z_1 + \mathbf{i}z_2)\mathbf{e}^\dagger \\ &= (x_1 - \mathbf{i}(\mathbf{i}y_2))\mathbf{e} + (x_1 + \mathbf{i}(\mathbf{i}y_2))\mathbf{e}^\dagger = (x_1 + y_2)\mathbf{e} + (x_1 - y_2)\mathbf{e}^\dagger .\end{aligned}$$

Recall also that we have defined the set $\mathbb{D}^+$ of non-negative hyperbolic numbers as

$$\mathbb{D}^+ = \left\{x + \mathbf{k}y \mid x^2 - y^2 \geq 0,\ \ x \geq 0\right\} .$$

The first of the defining inequalities gives the two systems:

$$\begin{cases} x - y & \geq & 0, \\ x + y & \geq & 0, \end{cases} \qquad \text{or} \qquad \begin{cases} x - y & \leq & 0, \\ x + y & \leq & 0, \end{cases}$$

but the condition $x \geq 0$ eliminates the second system; hence, the set $\mathbb{D}^+$ can be described as

$$\mathbb{D}^+ = \left\{x + \mathbf{k}y \,\middle|\, x \geq 0;\ \ |y| \leq x\right\} ,$$

or as

$$\mathbb{D}^+ = \{\nu\mathbf{e} + \mu\mathbf{e}^\dagger \mid \nu,\, \mu \geq 0\}.$$

Thus positive hyperbolic numbers are those hyperbolic numbers whose both idempotent components are non-negative, that somehow explains the origin of the name.

In Fig. 1.7.1 the points (x, y) correspond to the hyperbolic numbers $\mathfrak{z} = x + \mathbf{k}y$. One sees that, geometrically, the hyperbolic positive numbers are situated in the quarter plane denoted by $\mathbb{D}^+$. The quarter plane symmetric to it with respect to the origin corresponds to the negative hyperbolic numbers. The other points correspond to those hyperbolic numbers which cannot be called either positive or negative.

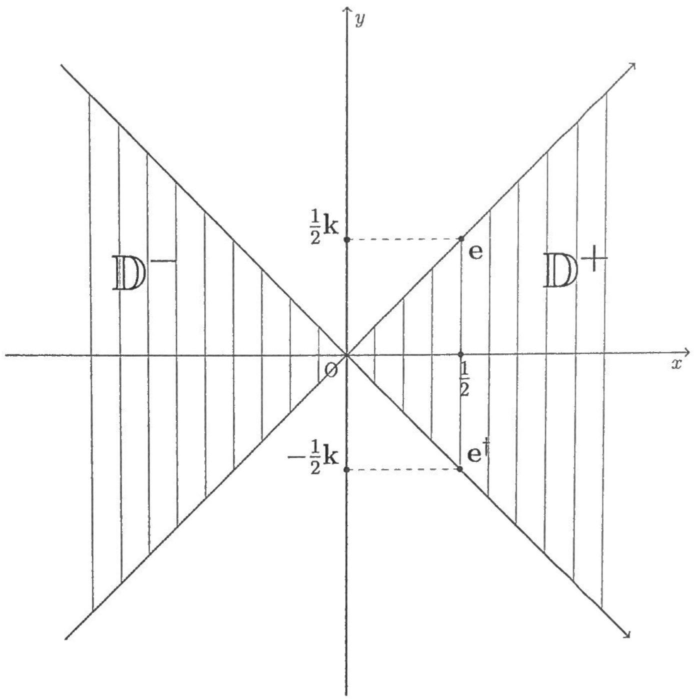

Figure 1.7.1: THE POSITIVE AND NEGATIVE HYPERBOLIC NUMBERS.

Analogously, the non-positive hyperbolic numbers form the set

$$\mathbb{D}^- = \left\{x + \mathbf{k}y \;\middle|\; x \leq 0;\ \ |y| \leq |x|\right\},$$

or equivalently

$$\mathbb{D}^- = \{\nu\mathbf{e} + \mu\mathbf{e}^\dagger \mid \nu,\, \mu \leq 0\}.$$

We will say sometimes that the hyperbolic number $\mathfrak{z} = \nu\mathbf{e} + \mu\mathbf{e}^\dagger$ is semi-positive if one of the coefficients μ and ν is positive and the other is zero.

We mentioned already that $\mathbb{D}^+$ plays an analogous role as non-negative real numbers, and now we illustrate this by computing the square roots of a hyperbolic number in $\mathbb{D}^+$. Take $\mathfrak{z} \in \mathbb{D}^+$, then $\mathfrak{z} = \mu\mathbf{e} + \nu\mathbf{e}^\dagger$ with $\mu,\, \nu \in \mathbb{R}^+ \cup \{0\}$, and it is easy to see that all the four hyperbolic numbers

$$\pm\sqrt{\mu}\,\mathbf{e} \pm \sqrt{\nu}\,\mathbf{e}^\dagger$$

square to $\mathfrak{z}$, but only one of them is a non-negative hyperbolic number: $\sqrt{\mu}\,\mathbf{e} + \sqrt{\nu}\,\mathbf{e}^\dagger$.

We are now in a position to define the meaning of the symbol $|Z|_{\mathbf{k}}$ for any bicomplex number $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger$. Indeed, we have obtained that $|Z|^2_{\mathbf{k}} = |\beta_1|^2\mathbf{e} + |\beta_2|^2\mathbf{e}^\dagger$ which is a non-negative hyperbolic number, hence the modulus $|Z|_{\mathbf{k}}$ can be taken as

$$|Z|_{\mathbf{k}} := |\beta_1|\mathbf{e} + |\beta_2|\mathbf{e}^\dagger \in \mathbb{D}^+ .$$

We will come back to this in the next chapter considering the notion of $\mathbb{BC}$ as a bicomplex normed module where the norm will be $\mathbb{D}^+$-valued. Meanwhile we can complement the above reasoning solving the equation

$$|\mathfrak{z}|_{\mathbf{k}} = \mathfrak{w}$$

where $\mathfrak{z}$ is an unknown hyperbolic number and $\mathfrak{w}$ is in $\mathbb{D}^+$. Writing $\mathfrak{z}$ and $\mathfrak{w}$ in the idempotent form $\mathfrak{z} = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger$ and $\mathfrak{w} = \gamma_1\mathbf{e} + \gamma_2\mathbf{e}^\dagger$ we infer easily a series of conclusions:

- If $\mathfrak{w} = 0$, then $\mathfrak{z} = 0$ is a unique solution.
- If $\mathfrak{w}$ is a semi-positive hyperbolic number, that is, $\mathfrak{w}$ is a positive zero-divisor: $\gamma_1 = 0$ and $\gamma_2 > 0$ or $\gamma_1 > 0$ and $\gamma_2 = 0$, then the solutions are also zero-divisors although not necessarily semi-positive:

$$\mathfrak{z} = \pm\gamma_2\mathbf{e}^\dagger \qquad \text{or} \qquad \mathfrak{z} = \pm\gamma_1\mathbf{e}\,,$$

  respectively.
- If $\mathfrak{w}$ is positive but not semi-positive: $\gamma_1 > 0$ and $\gamma_2 > 0$, then all four solutions are

$$\mathfrak{z} = \pm\gamma_1\mathbf{e} \pm \gamma_2\mathbf{e}^\dagger .$$

1.8 The Euclidean norm and the product of bicomplex numbers

We know already that for any two bicomplex numbers Z, W one has:

$$|Z \cdot W| \le \sqrt{2}\,|Z| \cdot |W|. \tag{1.30}$$

Note that this inequality is sharp since taking $Z = \mathbf{e}$, $W = \mathbf{e}$, one has:

$$|\mathbf{e} \cdot \mathbf{e}| = |\mathbf{e}| = \frac{1}{\sqrt{2}}$$

and

$$\sqrt{2}\,|\mathbf{e}| \cdot |\mathbf{e}| = \frac{1}{\sqrt{2}}.$$

But for particular bicomplex numbers we can say more.

Proposition 1.8.1. *If $U = u_1 + \mathbf{j}\,u_2 \in \mathbb{BC}$ is an arbitrary bicomplex number, but Z is a complex number in $\mathbb{C}(\mathbf{i})$ or $\mathbb{C}(\mathbf{j})$, or Z is a hyperbolic number, then*

a) *if $Z \in \mathbb{C}(\mathbf{i})$ or $\mathbb{C}(\mathbf{j})$, then $|Z \cdot U| = |Z| \cdot |U|$;*

In the same way we can begin with the $\mathbb{C}(\mathbf{j})$-valued quadratic form $\zeta_1^2 + \zeta_2^2$ and get the same $\mathbb{BC}$ which now will be seen as a $\mathbb{C}(\mathbf{j})$-algebra.

Finally, if we begin with the $\mathbb{D}$-valued quadratic form

$$\mathfrak{z}_1^2 + \mathfrak{z}_2^2$$

acting on the $\mathbb{D}$-algebra $\mathbb{D}^2$, then any of the two imaginary units, $\mathbf{i}$ or $\mathbf{j}$, will arise giving the factorizations into two factors each of which is $\mathbb{D}$-two-dimensional.

2.6 A partial order on the set of hyperbolic numbers

2.6.1 Definition of the partial order

We have noticed already a deep similarity between the role of non-negative hyperbolic numbers inside $\mathbb{D}$ and the role of non-negative real numbers inside $\mathbb{R}$. It turns out that this similarity can be extended and the (partial) notions "greater than" and "less than" can be introduced on hyperbolic numbers. Take two hyperbolic numbers $\mathfrak{z}_1$ and $\mathfrak{z}_2$; if their difference $\mathfrak{z}_2 - \mathfrak{z}_1 \in \mathbb{D}^+$, that is, the difference is a non-negative hyperbolic number, then we write $\mathfrak{z}_2 \succeq \mathfrak{z}_1$ or $\mathfrak{z}_1 \preceq \mathfrak{z}_2$ and we say that $\mathfrak{z}_2$ is $\mathbb{D}$-greater than or equal to $\mathfrak{z}_1$, or that $\mathfrak{z}_1$ is $\mathbb{D}$-less than or equal to $\mathfrak{z}_2$.

Writing these hyperbolic numbers in their idempotent form $\mathfrak{z}_1 = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger$ and $\mathfrak{z}_2 = \gamma_1\mathbf{e} + \gamma_2\mathbf{e}^\dagger$, with real numbers β_1, β_2, γ_1 and γ_2, we have that

$$\mathfrak{z}_1 \preceq \mathfrak{z}_2 \quad \text{if and only if} \quad \gamma_1 \geq \beta_1 \quad \text{and} \quad \gamma_2 \geq \beta_2.$$

On Figure 2.6.1, $\mathfrak{z}_0 = x_0 + \mathbf{k}y_0$ is an arbitrary hyperbolic number, and one can see that the entire plane is divided into four quarters: the quarter plane of hyperbolic numbers which are $\mathbb{D}$-greater than or equal to $\mathfrak{z}_0$ ($\mathfrak{z} \succeq \mathfrak{z}_0$); the quarter plane of hyperbolic numbers which are $\mathbb{D}$-less than or equal to $\mathfrak{z}_0$ ($\mathfrak{z} \preceq \mathfrak{z}_0$); and the two quarter planes where the hyperbolic numbers are not $\mathbb{D}$-comparable with $\mathfrak{z}_0$ (neither $\mathfrak{z} \succeq \mathfrak{z}_0$ nor $\mathfrak{z} \preceq \mathfrak{z}_0$ holds).

Thus we have introduced a binary relation on $\mathbb{D}$ which is, obviously, reflexive: for any $\mathfrak{z} \in \mathbb{D}$, $\mathfrak{z} \preceq \mathfrak{z}$; transitive: if $\mathfrak{z}_1 \preceq \mathfrak{z}_2$ and $\mathfrak{z}_2 \preceq \mathfrak{z}_3$, then $\mathfrak{z}_1 \preceq \mathfrak{z}_3$; and antisymmetric: if $\mathfrak{z}_1 \preceq \mathfrak{z}_2$ and $\mathfrak{z}_2 \preceq \mathfrak{z}_1$, then $\mathfrak{z}_1 = \mathfrak{z}_2$. Since this relation is applicable not for any pair of elements in $\mathbb{D}$, then the relation $\preceq$ defines a partial order on $\mathbb{D}$. As can be expected, this partial order extends the total order $\leq$ on $\mathbb{R}$: if one takes two real numbers x_1 and x_2, $x_1 \leq x_2$, and considers them as hyperbolic numbers with zero imaginary parts, then $x_1 \preceq x_2$.

In case $\mathfrak{z}_2 - \mathfrak{z}_1 \in \mathbb{D}^+ \setminus \{0\}$ we write $\mathfrak{z}_2 \succ \mathfrak{z}_1$ and we say that $\mathfrak{z}_2$ is $\mathbb{D}$-greater than $\mathfrak{z}_1$, or we write $\mathfrak{z}_1 \prec \mathfrak{z}_2$ and say that $\mathfrak{z}_1$ is $\mathbb{D}$-less than $\mathfrak{z}_2$. This implies that $\mathfrak{z} \in \mathbb{D}^+$ is equivalent to $\mathfrak{z} \succeq 0$ and that $\mathfrak{z} \in \mathbb{D}^+ \setminus \{0\}$ is equivalent to $\mathfrak{z} \succ 0$; $\mathfrak{z} \in \mathbb{D}^-$ is equivalent to $\mathfrak{z} \preceq 0$ and $\mathfrak{z} \in \mathbb{D}^- \setminus \{0\}$ is equivalent to $\mathfrak{z} \prec 0$.

As a matter of fact, we can trace an analogy with the future and past cones in a two-dimensional space with the Minkowski metric. Specifically, let us identify a

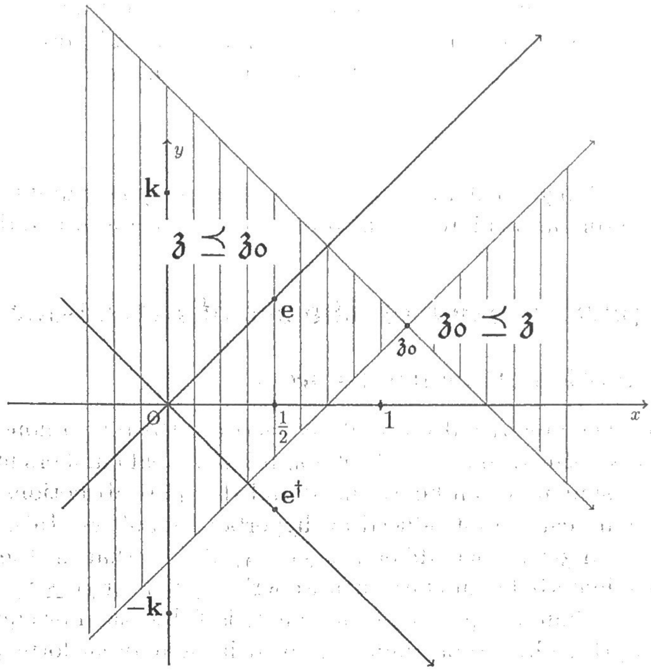

Figure 2.6.1: A PARTIAL ORDER ON $\mathbb{D}$

one-dimensional time with the x-axis and a one-dimensional space with the y-axis, both axes being embedded into $\mathbb{D}$ and assume the speed of light to be equal to one; then the set of zero-divisors with positive real part, $x > 0$ is nothing more than the future of a ray of light which was sent from the origin towards either direction. Analogously, the set of zero-divisors with negative real part, $x < 0$, represent the past of the same ray of light.

Thus, positive hyperbolic numbers represent the future cone, i. e., they correspond to the events which are in the future of the origin; the negative hyperbolic numbers, that is, those which after multiplying by (-1) become positive, represent the past cone, i.e., they correspond to the events in the past of the origin.

Accepting this interpretation, we see how the set of hyperbolic numbers which are greater than a given hyperbolic number Z represents the events in the future of the event which is represented by Z.

2.6.2 Properties of the partial order

Let us describe here some consequences of the the definition of the partial order $\preceq$. We combine them into three groups. Let $\mathfrak{z}$, $\mathfrak{w}$ and $\mathfrak{y}$ be hyperbolic numbers.

(I) CONSEQUENCES OF THE DEFINITION ITSELF

- If $\mathfrak{z}$ and $\mathfrak{y}$ are comparable with respect to $\preceq$, then precisely one of the following relations holds:
$$\mathfrak{z} \prec \mathfrak{y} \quad \text{or} \quad \mathfrak{z} \succ \mathfrak{y} \quad \text{or} \quad \mathfrak{z} = \mathfrak{y}.$$
- The inequalities $\mathfrak{z} \prec \mathfrak{y}$ and $\mathfrak{y} \preceq \mathfrak{w}$ imply that $\mathfrak{z} \prec \mathfrak{w}$.
- The inequalities $\mathfrak{z} \preceq \mathfrak{y}$ and $\mathfrak{y} \prec \mathfrak{w}$ imply that $\mathfrak{z} \prec \mathfrak{w}$.

(II) CONNECTIONS BETWEEN THE ADDITION AND THE ORDER ON $\mathbb{D}$

- $\mathfrak{z} \prec \mathfrak{w}$ implies that $\mathfrak{z} + \mathfrak{y} \prec \mathfrak{w} + \mathfrak{y}$.
- $\mathfrak{z} \preceq \mathfrak{w}$ implies that $\mathfrak{z} + \mathfrak{y} \preceq \mathfrak{w} + \mathfrak{y}$.
- $0 \prec \mathfrak{z}$ implies that $-\mathfrak{z} \prec 0$.
- $\mathfrak{z}_1 \preceq \mathfrak{z}_2$ and $\mathfrak{y} \preceq \mathfrak{w}$ imply that $\mathfrak{z}_1 + \mathfrak{y} \preceq \mathfrak{z}_2 + \mathfrak{w}$.
- $\mathfrak{z}_1 \preceq \mathfrak{z}_2$ and $\mathfrak{y} \prec \mathfrak{w}$ imply that $\mathfrak{z}_1 + \mathfrak{y} \prec \mathfrak{z}_2 + \mathfrak{w}$.

(III) CONNECTIONS BETWEEN MULTIPLICATION AND PARTIAL ORDER ON $\mathbb{D}$

- If $\mathfrak{z}$ and $\mathfrak{y}$ are non-negative hyperbolic numbers, then so is their product:
$$\mathfrak{z} \cdot \mathfrak{y} \in \mathbb{D}^+.$$
- If $\mathfrak{z}$ and $\mathfrak{y}$ are strictly positive hyperbolic numbers, then so is their product:
$$\mathfrak{z} \cdot \mathfrak{y} \in \mathbb{D}^+ \setminus \{0\}.$$
- If $\mathfrak{z}$ and $\mathfrak{y}$ are strictly negative hyperbolic numbers, then their product is strictly positive:
$$\mathfrak{z} \cdot \mathfrak{y} \succ 0.$$
- If one of $\mathfrak{z}$ and $\mathfrak{y}$ is strictly positive and another is strictly negative, then their product is strictly negative:
$$\mathfrak{z} \cdot \mathfrak{y} \prec 0.$$
- If $\mathfrak{z} \prec \mathfrak{y}$ and $\mathfrak{w} \succ 0$, then $\mathfrak{z} \cdot \mathfrak{w} \prec \mathfrak{y} \cdot \mathfrak{w}$.
- If $\mathfrak{z} \prec \mathfrak{y}$ and $\mathfrak{w} \prec 0$, then $\mathfrak{z} \cdot \mathfrak{w} \succ \mathfrak{y} \cdot \mathfrak{w}$.
- If $\mathfrak{z}$ is a (strictly) positive hyperbolic number, then it is invertible and its inverse is also positive: if $\mathfrak{z} \succ 0$ and $\mathfrak{z} \prec \mathfrak{y}$, then $\mathfrak{y}^{-1} \succ 0$ and $\mathfrak{y}^{-1} \prec \mathfrak{z}^{-1}$.

Example 2.6.2.1. Let us illustrate the above properties solving for $\mathfrak{z} = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger$ in $\mathbb{D}$ the inequality
$$|\mathfrak{z}|_{\mathbf{k}} \preceq \mathfrak{w}, \tag{2.20}$$

where $\mathfrak{w} = \gamma_1 \mathbf{e} + \gamma_2 \mathbf{e}^\dagger$ is in $\mathbb{D}^+$ and $|\cdot|_\mathbf{k}$ is the $\mathbb{D}$-valued modulus. If $\mathfrak{w} = 0$ the unique solution is $\mathfrak{z} = 0$. Hence, consider $\mathfrak{w} \in \mathbb{D}^+ \setminus \{0\}$. The inequality (2.20) is equivalent to

$$|\beta_1|\mathbf{e} + |\beta_2|\mathbf{e}^\dagger \preceq \gamma_1 \mathbf{e} + \gamma_2 \mathbf{e}^\dagger$$

which in turn is equivalent to the system

$$\begin{cases} |\beta_1| \leq \gamma_1, \\ |\beta_2| \leq \gamma_2. \end{cases}$$

Thus the solutions of (2.20) are hyperbolic numbers $\mathfrak{z} = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger$ with $-\gamma_1 \leq \beta_1 \leq \gamma_1$ and $-\gamma_2 \leq \beta_2 \leq \gamma_2$. This means that the inequality (2.20) is equivalent to the double hyperbolic inequality $-\mathfrak{w} \preceq \mathfrak{z} \preceq \mathfrak{w}$. □

We introduce now the notion of a hyperbolic interval (or hyperbolic segment). Given two hyperbolic numbers $\mathfrak{a}$ and $\mathfrak{b}$, $\mathfrak{a} \preceq \mathfrak{b}$, we set

$$[\mathfrak{a}, \mathfrak{b}]_\mathbb{D} := \{\mathfrak{z} \in \mathbb{D} \,|\, \mathfrak{a} \preceq \mathfrak{z} \preceq \mathfrak{b}\} .$$

Consider now two particular cases:

- Let $\mathfrak{a} = \mathbf{k}$ and $\mathfrak{b} = 1$. Since obviously $\mathbf{k} = \mathbf{e} - \mathbf{e}^\dagger \preceq 1 = \mathbf{e} + \mathbf{e}^\dagger$, then the interval $[\mathbf{k}, 1]_\mathbb{D}$ is well defined. The inequality

$$\mathbf{k} \preceq \mathfrak{z} = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger \preceq 1$$

  gives:

$$1 \leq \beta_1 \leq 1, \qquad -1 \leq \beta_2 \leq 1.$$

  It turns out that in this case the hyperbolic interval is a one-dimensional set. See the Figure 2.6.2.

- Take now $\mathfrak{a} = \mathbf{k}$ and $\mathfrak{b} = 2$ (obviously $\mathbf{k} \prec 2$). In this case the hyperbolic interval is given by

$$[\mathbf{k}, 2]_\mathbb{D} = \{\mathfrak{z} = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger \,|\, 1 \leq \beta_1 \leq 2 \text{ and } -1 \leq \beta_2 \leq 2\},$$

  and it is now a two-dimensional set. See Figure 2.6.3.

2.6.3 $\mathbb{D}$-bounded subsets in $\mathbb{D}$.

Given a subset $\mathcal{A}$ in $\mathbb{D}$, we define as usual the notion of $\mathbb{D}$-upper and $\mathbb{D}$-lower bounds, as well as the notions of a set being $\mathbb{D}$-bounded from above, from below, and finally of a $\mathbb{D}$-bounded set. There are some fine points here. If $\mathcal{A}$ has a $\mathbb{D}$-upper or a $\mathbb{D}$-lower bound α, then this means that for any $a \in \mathcal{A}$ there holds that a is comparable with α and $a \preceq \alpha$ or $\alpha \preceq a$. But this does not mean that the elements of $\mathcal{A}$ are necessarily comparable between them; the same happens taking two $\mathbb{D}$-upper of $\mathbb{D}$-lower bounds α and β they are not always comparable.

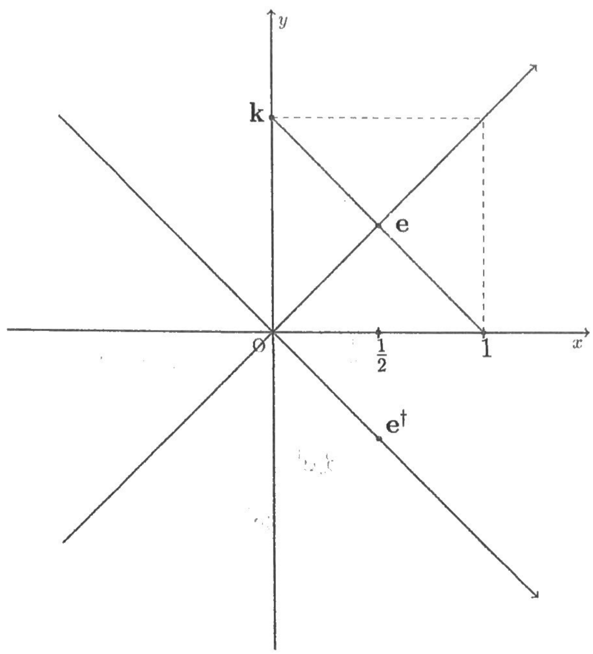

Figure 2.6.2: THE HYPERBOLIC SEGMENT $[\mathbf{k}, 1]_{\mathbb{D}}$.

If $\mathcal{A} \subset \mathbb{D}$ is a set $\mathbb{D}$-bounded from above, we define the notion of its $\mathbb{D}$-supremum, denoted by $\sup_{\mathbb{D}} \mathcal{A}$, to be the least upper bound for $\mathcal{A}$, and its $\mathbb{D}$-infimum $\inf_{\mathbb{D}} \mathcal{A}$ to be the greatest lower bound for $\mathcal{A}$. The "least" upper bound here means that $\sup_{\mathbb{D}} \mathcal{A} \preceq \alpha$ for any $\mathbb{D}$-upper bound α even if not all of the $\mathbb{D}$-upper bounds are comparable. Similarly the meaning of the "greatest" lower bound is understood. Of course, every non-empty set of hyperbolic numbers which is $\mathbb{D}$-bounded from above has its $\mathbb{D}$-supremum, and if it is $\mathbb{D}$-bounded from below, then it has its $\mathbb{D}$-infimum. This can be seen immediately if one notes that there are more convenient expressions for these notions. Given a set $\mathcal{A} \subset \mathbb{D}$, consider the sets $\mathcal{A}_1 := \{ a_1 \mid a_1\mathbf{e} + a_2\mathbf{e}^\dagger \in \mathcal{A}\}$ and $\mathcal{A}_2 := \{ a_2 \mid a_1\mathbf{e} + a_2\mathbf{e}^\dagger \in \mathcal{A}\}$. If $\mathcal{A}$ is $\mathbb{D}$-bounded from above, then the $\sup_{\mathbb{D}} \mathcal{A}$ can be computed by the formula

$$\sup_{\mathbb{D}}\mathcal{A} = \sup \mathcal{A}_1 \cdot \mathbf{e} + \sup \mathcal{A}_2 \cdot \mathbf{e}^\dagger .$$

If $\mathcal{A}$ is $\mathbb{D}$-bounded from below, then the $\inf_{\mathbb{D}} \mathcal{A}$ can be computed by the formula

$$\inf_{\mathbb{D}}\mathcal{A} = \inf \mathcal{A}_1 \cdot \mathbf{e} + \inf \mathcal{A}_2 \cdot \mathbf{e}^\dagger .$$

The above formulas explain a very peculiar character of the partial order on $\mathbb{D}$. Note only that although two $\mathbb{D}$-upper (or $\mathbb{D}$-lower) bounds can be incomparable nevertheless they are always comparable with $\sup_{\mathbb{D}} \mathcal{A}$ (or with $\inf_{\mathbb{D}} \mathcal{A}$).

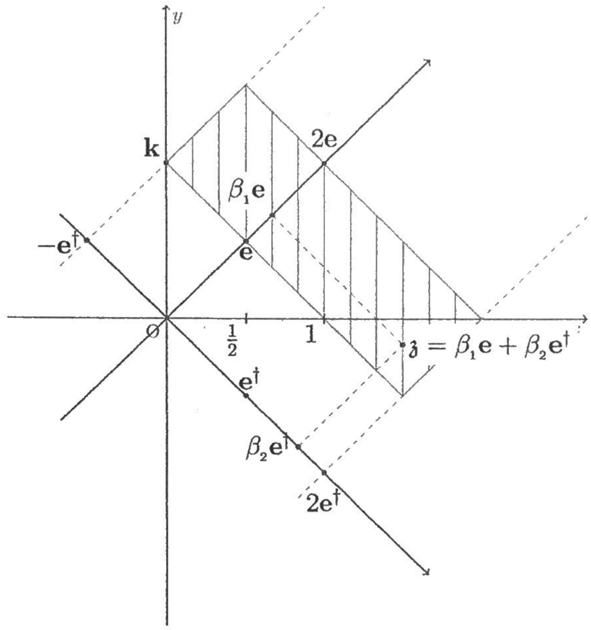

Figure 2.6.3: THE HYPERBOLIC INTERVAL $[\mathbf{k}, 2]_{\mathbb{D}}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathbb{D}$, denote by $-\mathcal{A}$ the set of all the elements in $\mathcal{A}$ multiplied by -1, and denote by $\mathcal{A}+\mathcal{B}$ the set of all sums $\mathfrak{z}+\mathfrak{y}$ with $\mathfrak{z}\in\mathcal{A}$ and $\mathfrak{y}\in\mathcal{B}$; define in the same fashion the set $\mathcal{A}\cdot\mathcal{B}$. The reader is invited to prove the following properties.

- A set $\mathcal{A}$ is $\mathbb{D}$-bounded from above (or from below) if and only if the set $-\mathcal{A}$ is $\mathbb{D}$-bounded from below (or from above); for such sets it holds that
$$\inf_{\mathbb{D}}(-\mathcal{A}) = -\sup_{\mathbb{D}}\mathcal{A}; \quad \sup_{\mathbb{D}}(-\mathcal{A}) = -\inf_{\mathbb{D}}\mathcal{A}.$$
- If $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{D}$-bounded from below, then so is $\mathcal{A}+\mathcal{B}$ and for such sets one has:
$$\inf_{\mathbb{D}}(\mathcal{A}) + \inf_{\mathbb{D}}(\mathcal{B}) = \inf_{\mathbb{D}}(\mathcal{A}+\mathcal{B}).$$
If $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{D}$-bounded from above, then so is $\mathcal{A}+\mathcal{B}$ and for such sets one has:
$$\sup_{\mathbb{D}}(\mathcal{A}) + \sup_{\mathbb{D}}(\mathcal{B}) = \sup_{\mathbb{D}}(\mathcal{A}+\mathcal{B}).$$
- Assume that $\mathcal{A}$ and $\mathcal{B}$ are subsets of the set $\mathbb{D}^+$ of non-negative hyperbolic numbers. If $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{D}$-bounded from below, then so is $\mathcal{A}\cdot\mathcal{B}$ and for such

sets one has:

$$\inf_{\mathbb{D}}(\mathcal{A}\cdot\mathcal{B}) = \inf_{\mathbb{D}}(\mathcal{A})\cdot\inf_{\mathbb{D}}(\mathcal{B})\,.$$

If $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{D}$-bounded from above, then so is $\mathcal{A}\cdot\mathcal{B}$ and for such sets one has:

$$\sup_{\mathbb{D}}(\mathcal{A}\cdot\mathcal{B}) = \sup_{\mathbb{D}}(\mathcal{A})\cdot\sup_{\mathbb{D}}(\mathcal{B})\,.$$

2.7 The hyperbolic norm on $\mathbb{BC}$

We know already, for any bicomplex $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger$, the formula

$$|Z|_{\mathbf{k}} := |\beta_1|\mathbf{e} + |\beta_2|\mathbf{e}^\dagger\,.$$

That is, we have the map

$$|\cdot|_{\mathbf{k}} : \mathbb{BC} \longrightarrow \mathbb{D}^+$$

with the properties:

(I) $|Z|_{\mathbf{k}} = 0$ if and only if $Z = 0$;

(II) $|Z\cdot W|_{\mathbf{k}} = |Z|_{\mathbf{k}}\cdot|W|_{\mathbf{k}}$ for any $Z,\ W \in \mathbb{BC}$;

(III) $|Z+W|_{\mathbf{k}} \preceq |Z|_{\mathbf{k}} + |W|_{\mathbf{k}}$.

The first two properties are clear. Let us prove (III).

$$\begin{aligned}
|Z+W|_{\mathbf{k}} &= |(\beta_1+\nu_1)\cdot\mathbf{e} + (\beta_2+\nu_2)\cdot\mathbf{e}^\dagger|_{\mathbf{k}} \\
&= |\beta_1+\nu_1|\cdot\mathbf{e} + |\beta_2+\nu_2|\cdot\mathbf{e}^\dagger \\
&\preceq (|\beta_1|+|\nu_1|)\cdot\mathbf{e} + (|\beta_2|+|\nu_2|)\cdot\mathbf{e}^\dagger \\
&= |Z|_{\mathbf{k}} + |W|_{\mathbf{k}}.
\end{aligned}$$

The three properties (I)–(III) manifest again the analogy between real positive numbers and hyperbolic positive numbers; now we see that the hyperbolic modulus of a bicomplex number has exactly the same properties as the real modulus of a complex number whenever the partial order $\preceq$ is used instead of $\leq$.

Because of properties (I)–(III) we will say that $|\cdot|_{\mathbf{k}}$ is the hyperbolic-valued ($\mathbb{D}$-valued) norm on the $\mathbb{BC}$-module $\mathbb{BC}$.

It is instructive to compare (II) with (1.13) where the norm of the product and the product of the norms are related with an inequality. We believe that one could say that the hyperbolic norm of bicomplex numbers is better suited to the algebraic structure of the latter although, of course, one has to allow hyperbolic values for the norm.

Remark 2.7.1. (1) *Since for any $Z \in \mathbb{BC}$ it holds that*

$$|Z|_{\mathbf{k}} \preceq \sqrt{2}\cdot|Z|\,, \tag{2.21}$$

with $|Z|$ the Euclidean norm of Z, then one has:

$$|Z \cdot W|_{\mathbf{k}} \preceq \sqrt{2} \cdot |Z| \cdot |W|_{\mathbf{k}}.$$

In contrast with property (II) *above, this inequality involves both the Euclidean and the hyperbolic norms.*

(2) *Take $\mathfrak{z}_1$ and $\mathfrak{z}_2$ in $\mathbb{D}^+$, then clearly*

$$\mathfrak{z}_1 \preceq \mathfrak{z}_2 \quad \text{implies that} \quad |\mathfrak{z}_1| \leq |\mathfrak{z}_2|. \tag{2.22}$$

(3) *Note that the definition of hyperbolic norm for a bicomplex number Z does not depend on the choice of its idempotent representation. We have used, for $Z \in \mathbb{BC}$, the idempotent representation $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger$, with β_1 and β_2 in $\mathbb{C}(\mathbf{i})$. If we had started with the idempotent representation $Z = \gamma_1\mathbf{e} + \gamma_2\mathbf{e}^\dagger$, with γ_1 and γ_2 in $\mathbb{C}(\mathbf{j})$, then we would have arrived at the same definition of the hyperbolic norm since $|\beta_1| = |\gamma_1|$ and $|\beta_2| = |\gamma_2|$.*

(4) *The comparison of the Euclidean norm $|Z|$ and the $\mathbb{D}$-valued norm $|Z|_{\mathbf{k}}$ of a bicomplex number Z gives:*

$$||Z|_{\mathbf{k}}| = \frac{1}{\sqrt{2}}\sqrt{|\beta_1|^2 + |\beta_2|^2} = |Z| \tag{2.23}$$

where the left-hand side is the Euclidean norm of a hyperbolic number.

2.7.1 Multiplicative groups of hyperbolic and bicomplex numbers

The set $\mathbb{R}^+$ of strictly positive real numbers is a multiplicative group. For the set $\mathbb{D}^+$ of non-negative hyperbolic numbers the situation is more delicate although it preserves some analogies.

Introduce the set $\mathbb{D}^+_{inv} := \mathbb{D}^+ \setminus \mathfrak{S}_0$ of all strictly positive hyperbolic numbers. This set has the following properties:

- if λ_1 and λ_2 are in $\mathbb{D}^+_{inv}$, then $\lambda_1 \cdot \lambda_2 \in \mathbb{D}^+_{inv}$;
- $1 \in \mathbb{D}^+_{inv}$;
- if $\lambda \in \mathbb{D}^+_{inv}$, then λ is invertible in $\mathbb{D}$ and $\lambda^{-1} \in \mathbb{D}^+_{inv}$.

Hence, $\mathbb{D}^+_{inv}$ is a multiplicative group with respect to the hyperbolic multiplication and thus it is an exact analogue of $\mathbb{R}^+$; what is more, $\mathbb{R}^+$ is a subgroup of $\mathbb{D}^+_{inv}$.

The zero-divisors in $\mathbb{D}^+$ are either of the form $\mathfrak{z} = \lambda\mathbf{e}$ or of the form $\mathfrak{z} = \mu\mathbf{e}^\dagger$ where λ and μ are positive real numbers; so we will use the notations $\mathbb{D}^+_{\mathbf{e}} := \{\lambda\mathbf{e} \mid \lambda > 0\}$ and $\mathbb{D}^+_{\mathbf{e}^\dagger} := \{\mu\mathbf{e}^\dagger \mid \mu > 0\}$. Both sets of semi-positive hyperbolic numbers are closed under hyperbolic multiplication but neither of them contains the number one.

origin and radius $\frac{a_0}{\sqrt{2}}$, contained in the complex line $\mathbb{BC}_{\mathbf{e}}$. Similarly, if $\gamma_0 = b_0\mathbf{e}^\dagger$, the sphere $\mathbb{S}_{b_0\mathbf{e}^\dagger}$ is the circumference with center at the origin and radius $\frac{b_0}{\sqrt{2}}$, contained in the complex line $\mathbb{BC}_{\mathbf{e}^\dagger}$. Finally, if γ_0 is not a zero-divisor, recall that the sphere $\mathbb{S}_{a_0\mathbf{e}+b_0\mathbf{e}^\dagger}$ is the surface of a torus:

$$\mathbb{S}_{a_0\mathbf{e}+b_0\mathbf{e}^\dagger} = \left\{ Z = \beta_1 \cdot \mathbf{e} + \beta_2 \cdot \mathbf{e}^\dagger \mid |\beta_1| = a_0, \ , \ |\beta_2| = b_0 \right\} .$$

We are now ready to describe what a bicomplex ball is.

If γ_0 is not a zero-divisor, i.e., $a_0 \neq 0$ and $b_0 \neq 0$, then

$$\mathbb{B}_{\gamma_0} = \left\{ Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \mid |\beta_1| < a_0, \, |\beta_2| < b_0 \right\} ,$$

i.e., if we are looking at $\mathbb{BC}$ as $\mathbb{C}^2(\mathbf{i})$ with the idempotent coordinates, then the bicomplex ball $\mathbb{B}_{\gamma_0}$ is the bicomplex form of writing for the usual bidisk centered at the origin and with bi-radius (a_0, b_0).

If γ_0 is a zero-divisor, then we cannot define the ball in the same way because none of the inequalities $|\beta_1| < 0$ or $|\beta_2| < 0$ has solutions. So we define in this case the ball $\mathbb{B}_{\gamma_0}$ to be one of the two disks: one is located in $\mathbb{BC}_{\mathbf{e}}$ with center at the origin and radius $\frac{a_0}{\sqrt{2}}$, and the other is located in $\mathbb{BC}_{\mathbf{e}^\dagger}$ with center at the origin and radius $\frac{b_0}{\sqrt{2}}$.

It is worth noting that for a bicomplex ball $\mathbb{B}_{\gamma_0}$, with γ_0 not a zero-divisor, the respective bicomplex sphere $\mathbb{S}_{\gamma_0}$ is not its topological boundary but it is its distinguished, or Shilov, boundary.

4.5 Multiplicative groups of bicomplex spheres

It is known that in the study of Euclidean spaces $\mathbb{R}^n$, the cases of $n = 2$ and $n = 4$ are peculiar for many reasons but in particular because the corresponding unitary spheres $\mathbb{S}^1$ in $\mathbb{R}^2$ and $\mathbb{S}^3$ in $\mathbb{R}^4$ are multiplicative groups; this is thanks to the complex numbers multiplication in $\mathbb{R}^2$ and the quaternionic multiplication in $\mathbb{R}^4$.

Let us consider some analogues of the above facts related to the bicomplex multiplication and bicomplex spheres with a hyperbolic radius:

$$\mathbb{S}_\lambda := \left\{ Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \mid |Z|_{\mathbf{k}} = \lambda \right\}$$

where $\lambda \in \mathbb{D}^+$. The multiplicative property of the hyperbolic modulus

$$|Z \cdot W|_{\mathbf{k}} = |Z|_{\mathbf{k}} \cdot |W|_{\mathbf{k}} \tag{4.37}$$

will be crucial for the reasoning below.

Obviously, $\mathbb{S}_0 = \{0\}$. Consider the bicomplex unitary sphere $\mathbb{S}_1$; clearly