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The Denjoy–Wolff Theorem in the Open Unit Ball of a Strictly Convex Banach Space

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Let X be a complex strictly convex Banach space with an open unit ball B. For each compact, holomorphic and fixed-point-free mapping $f\colon B \to B$ there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ of iterates of f converges locally uniformly on B to the constant map taking the value ξ. © 1999 Academic Press

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1. INTRODUCTION

In 1926 Denjoy [8] and Wolff [33] proved that the iterates f^n of a fixed-point free holomorphic map $f\colon \Delta \to \Delta$ converge locally uniformly to a unique boundary point ξ. This convergence result was extended to the open unit ball of $\mathbb{C}^n$ [19, 21, 24] and to some other domains in $\mathbb{C}^n$ [1–3]. In the case of infinite dimensional Hilbert balls, Stachura [28] gave an example which shows that the Denjoy–Wolff theorem fails even for biholomorphic self-maps. But, if a fixed-point-free self-mapping f of the Hilbert ball is either firmly k_B-nonexpansive or an averaged mapping of the first or second kind, then for each point x the sequence of iterates $\{f^n(x)\}$ converges to a unique boundary point [15, 16, 25]. More information can be found in [31] and [32]. Recently, Chu and Mellon [7] proved that the Denjoy–Wolff theorem is still valid for a compact holomorphic fixed-point-free self-map of the open unit ball of a Hilbert space, and Kryczka and Kuczumow extended this result to k_B-nonexpansive self-mappings of the

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Hilbert ball [20]. In this paper we prove that such results are valid for holomorphic mappings on the open unit ball of a strictly convex Banach space and for k_B-nonexpansive mappings on the open unit ball of a uniformly convex Banach space. As a matter of fact, our results are new even in the finite-dimensional case.

2. PRELIMINARIES

All Banach spaces will be complex. B will always denote the open unit ball of a Banach space X and $B(z, r)$ will be the open ball of center z and radius $r > 0$ in X. If K is a subset of the Banach space X, then $\bar{K}$ denotes the norm closure of K in X. Let D be a bounded domain in a Banach space X. The Kobayashi distance in D is denoted by k_D. It is known that k_D is locally equivalent to the norm $\|\cdot\|$ [12, 16, 18]. We remark in passing that all distances assigned to a convex bounded domain D by Schwarz-Pick systems of pseudometrics [12, 16, 18] coincide [9, 23 and 30].

If D is also convex, then directly from the definition of k_D we obtain that for $z_1, z_2, w_1, w_2 \in D$, $0 \leqslant t \leqslant 1$, and $r > 0$ the inequalities $k_D(z_1, z_2) \leqslant r$ and $k_D(w_1, w_2) \leqslant r$ imply

$$k_D((1-t)\, z_1 + tw_1, (1-t)\, z_2 + tw_2) \leqslant r \tag{2.1}$$

[22].

If B is the open unit Hilbert ball, then k_B is given by the formula

$$k_B(w, z) = \arg\tanh\left(1 - \frac{(1-\|w\|^2)(1-\|z\|^2)}{|1-(w, z)|^2}\right)^{1/2} \tag{2.2}$$

where $w, z \in B$ [16, 17].

Remark 2.1. We will use the following notion in the Banach space X. If Y is the one dimensional subspace generated by $0 \neq w \in X$ and $z_1, z_2 \in Y$, let (z_1, z_2) denote the standard scalar product in Y generated by $w/\|w\|$. Now, if we consider k_B restricted to $(Y \times Y) \cap (B \times B)$, then k_B coincides with k_Δ [18], where Δ is the unit disc, and therefore we can use the formula (2.2).

We will also need the following property of k_B.

For $\tilde{z} \in B$ the index $\tilde{R}(\tilde{z})$ of $\tilde{z}$ is the supremum of the radii of all affine discs in B with center $\tilde{z}$. The following inequality is valid for all $w, z \in B$:

$$\arg\tanh\left(\frac{\|z-w\|}{\|z-w\| + 2\varphi(z)}\right) \leqslant k_B(z, w), \tag{2.3}$$

where

$$\varphi(z) = \widetilde{R}(\tilde{z}) \left[1 + \frac{\|z - \tilde{z}\|}{1 - \|\tilde{z}\|} \right] \tag{2.4}$$

[18].

Let D be a bounded domain in a Banach space X. A subset K of D is said to lie strictly inside D if $\operatorname{dist}(K, \partial D) > 0$. A mapping $f: D \to D$ is said to map D strictly inside D if $f(D)$ lies strictly inside D.

We say that $f: D \to D$ is k_D-nonexpansive if

$$k_D(f(w), f(z)) \leqslant k_D(w, z)$$

for all $w, z \in D$. Each holomorphic $f: D \to D$ is k_D-nonexpansive [12, 16, 18].

$Aut(D) \cap C^0(\bar{D})$ will denote the group of all those biholomorphisms φ of D onto itself such that φ and φ^{-1} have 1-1 continuous extensions to the boundary ∂D.

Now we recall a few facts about holomorphic mappings.

THEOREM 2.1 (Earle–Hamilton) [10]. *Let D be a bounded domain in a Banach space X. If a holomorphic $f: D \to D$ maps D strictly inside itself, then there exists $0 \leqslant s < 1$ such that*

$$k_D(f(w), f(z)) \leqslant s \cdot k_D(w, z)$$

for all w and z in D. Moreover, for any z in D the sequence of iterates $\{f^n(z)\}$ converges to the unique fixed point of f.

Hence, if D is a bounded convex domain, then by Theorem 2.1 the mapping $g_{t,z} = (1-t)\,z + t(\cdot): D \to D$ is a k_D-contraction for every $z \in D$ and $0 \leqslant t < 1$. Therefore for each k_D-nonexpansive mapping $f: D \to D$ the mapping $f_{t,z} = g_{t,z} \circ f = (1-t)\,z + tf: D \to D$ is a k_D-contraction and has exactly one fixed point which we denote by $h(t, z)$. Let us fix $z_0 \in D$ and $0 \leqslant t < 1$. The mapping $h(t, \cdot): D \to D$ is k_D-nonexpansive (holomorphic if f is holomorphic [6, 12]) as a limit of the sequence $\{f^n_{t,\cdot}(z_0)\}$.

THEOREM 2.2 [7]. *Let B be the open unit ball of a Banach space X and let $f: B \to B$ be a compact holomorphic map. Then every subsequence of the sequence $\{f^n\}$ of iterates of f has a subsequence converging locally uniformly to a holomorphic function $g: B \to \bar{B}$.*

THEOREM 2.3 (The strong maximum principle) [29]. *Let X be a strictly convex Banach space and let D be a domain in a Banach space Y. If B is the open unit ball in X and $f: D \to \bar{B}$ is holomorphic, then either $f(D) \subset B$ or f is constant.*

In [1] Abate introduced the following notion of horospheres in a bounded domain $D \subset X$ (see also [34]). For $z_0 \in D$, $\xi \in \partial D$, and $R > 0$ the small horosphere $E_{z_0}(\xi, R)$ and the big horosphere $F_{z_0}(\xi, R)$ of center ξ and radius R are defined by

$$E_{z_0}(\xi, R) = \{z \in D\colon \limsup_{w \to \xi} [k_D(z, w) - k_D(z_0, w)] < \tfrac{1}{2} \log R\}$$

and

$$F_{z_0}(\xi, R) = \{z \in D\colon \liminf_{w \to \xi} [k_D(z, w) - k_D(z_0, w)] < \tfrac{1}{2} \log R\}.$$

Horospheres are useful tools in investigations of holomorphic mappings [1, 4, 7, 13, 16, 21, 24, 26, 32, 34]. For our goal we introduce new horospheres in B by using sequences with limits on ∂B.

Let $z_0 \in B$, $\xi \in \partial B$, $R > 0$, $w_n \in B$, $n = 1, 2, \ldots$, and $\lim_n w_n = \xi$. Let us assume in addition that the limit

$$\lim_{n \to \infty} [k_B(z, w_n) - k_B(z_0, w_n)]$$

exists for each $z \in B$. The new horosphere $G(\xi, R, z_0, \{w_n\})$ in B is then defined as follows:

$$G(\xi, R, z_0, \{w_n\}) = \{z \in B\colon \lim_{n \to \infty} [k_B(z, w_n) - k_B(z_0, w_n)] < \tfrac{1}{2} \log R\}.$$

Now we recall a result on nonexpansive mappings in a metric space. Recall that a metric space (X, d) is called finitely compact if each bounded, closed and nonempty subset of X is compact. We say that $f\colon X \to X$ is nonexpansive if

$$d(f(x), f(y)) \leqslant d(x, y)$$

for all $x, y \in X$. If we consider the behavior of a sequence of iterates of a nonexpansive mapping on a finitely compact space X, then the basic result is due to Calka.

Theorem 2.4 [5]. *Let f be a nonexpansive mapping of a finitely compact metric space X into itself. If for some $x_0 \in X$ the sequence $\{f^n(x_0)\}$ contains a bounded subsequence, then for every $x \in X$ the sequence $\{f^n(x)\}$ is bounded.*

After applying the Calka theorem, the method of asymptotic centers [11, 14, 16], and either the Schauder theorem [27] or the Earle–Hamilton theorem, one can obtain the following result.

THEOREM 2.5 [20]. *Let X be a Banach space with the open unit ball B. If $f: B \to B$ is a compact k_B-nonexpansive mapping, then the following conditions are equivalent*:

(i) *f has a fixed point*;

(ii) *there exists $z \in B$ and a subsequence of its iterates $\{f^{n_i}(z)\}$ such that* $\sup_i \|f^{n_i}(z)\| < 1$;

(iii) *there exists $z \in B$ such that* $\sup_n \|f^n(z)\| < 1$;

(iv) *for each $z \in B$ we have* $\sup_n \|f^n(z)\| < 1$;

(v) *there exists a nonempty, closed, convex and f-invariant subset A of B such that* $\sup_{z \in A} \|z\| < 1$;

(vi) *there exists a nonempty f-invariant subset A of B such that* $\sup_{z \in A} \|z\| < 1$;

(vii) *there exists a sequence $\{z_n\}$ such that $z_n - f(z_n) \to 0$ and* $\sup_n \|z_n\| < 1$.

3. HOROSPHERES

We begin this section by proving an auxiliary lemma.

LEMMA 3.1. *Let X be a Banach space. Let $\{w_n\}$ be an arbitrary sequence in B with $\lim_n w_n = \xi \in \partial B$ and $w_n \neq 0$ for $n = 1, 2, \ldots$. Let $0 < \alpha < 1$, and let $Proj_n$ denote, for each n, the projection of X on the one dimensional space $lin(w_n)$ generated by w_n given by*

$$Proj_n(x) = f_n(x)\,\frac{w_n}{\|w_n\|}, \qquad x \in X,$$

where f_n is a linear functional with $\|f_n\| = 1$ and $f_n(w_n) = \|w_n\|$. If

$$z_n = Proj_n(\alpha\xi),$$

then

$$z_n \in B, \qquad \lim_{n\to\infty} z_n = \alpha\xi, \qquad \lim_{n\to\infty} k_B(z_n, \alpha\xi) = 0 \qquad \text{and}$$

$$\lim_{n\to\infty} (z_n, w_n) = \alpha. \tag{3.1}$$

Proof. It is obvious that $z_n \in B$. We also have

$$\|z_n - \alpha\xi\| = \left\| \alpha f_n(\xi) \frac{w_n}{\|w_n\|} - \alpha\xi \right\| \xrightarrow[n]{} 0.$$

Hence we get

$$\lim_{n \to \infty} k_B(z_n, \alpha\xi) = 0.$$

Next, taking $\alpha_n = \alpha f_n(\xi)$ we obtain

$$|\alpha_n - \alpha| = \left| \alpha f_n(\xi) - \alpha f_n\left(\frac{w_n}{\|w_n\|}\right)\right| \leqslant \left\| \xi - \frac{w_n}{\|w_n\|} \right\| \xrightarrow[n]{} 0.$$

Hence

$$(z_n, w_n) = \alpha_n \|w_n\| \xrightarrow[n]{} \alpha. \quad \blacksquare$$

This lemma enables us to establish the following result.

THEOREM 3.2. *For every $\xi \in \partial B$ and $0 < \alpha < 1$ we have*

$$\lim_{w \to \xi} [k_B(\alpha\xi, w) - k_B(0, w)] = -k_B(0, \alpha\xi). \tag{3.2}$$

Proof. Let us observe (see (2.2)) that

$$k_B(0, w) = \arg\tanh \|w\| = \frac{1}{2} \log \frac{1 + \|w\|}{1 - \|w\|} = \frac{1}{2} \log \frac{(1 + \|w\|)^2}{1 - \|w\|^2} \tag{3.3}$$

for $w \in B$, and that

$$k_B(0, \alpha\xi) = \arg\tanh \|\alpha\xi\| = \frac{1}{2} \log \frac{1 - \alpha^2}{(1 - \alpha)^2}. \tag{3.4}$$

Now let us take an arbitrary sequence $\{w_n\}$ in B with $w_n \neq 0$ for all n and $\lim_n w_n = \xi$. For each n, let $Proj_n$ denote the projection of X on the one dimensional space $lin(w_n)$ given in Lemma 3.1. If $z_n = Proj_n(\alpha\xi)$, then by

(3.1) we get $z_n \in B$, $\lim_{n\to\infty} z_n = \alpha\xi$, and $\lim_{n\to\infty} k_B(z_n, \alpha\xi) = 0$. Therefore we need to prove that

$$\lim_{n\to\infty} [k_B(z_n, w_n) - k_B(0, w_n)] = -k_B(0, \alpha\xi), \tag{3.5}$$

in place of (3.2). Since $w_n, z_n \in lin(w_n)$ for every n we have (see(2.2))

$$\begin{aligned} k_B(z_n, w_n) &= \arg\tanh\left(1 - \frac{(1-\|z_n\|^2)(1-\|w_n\|^2)}{|1-(z_n, w_n)|^2}\right)^{1/2} \\ &= \frac{1}{2}\log \frac{\{|1-(z_n, w_n)| + [|1-(z_n, w_n)|^2 - (1-\|z_n\|^2)(1-\|w_n\|^2)]^{1/2}\}^2}{(1-\|z_n\|^2)(1-\|w_n\|^2)}. \end{aligned} \tag{3.6}$$

By (3.3) and (3.6) we get

$$\begin{aligned} &k_B(z_n, w_n) - k_B(0, w_n) \\ &\quad = \frac{1}{2}\log \frac{\{|1-(z_n, w_n)| + [|1-(z_n, w_n)|^2 - (1-\|z_n\|^2)(1-\|w_n\|^2)]^{1/2}\}^2}{(1-\|z_n\|^2)(1+\|w_n\|)^2} \end{aligned}$$

and by (3.1) and (3.4) this implies that

$$\lim_{n\to\infty} [k_B(z_n, w_n) - k_B(0, w_n)] = \frac{1}{2}\log\frac{(1-\alpha)^2}{1-\alpha^2} = -k_B(0, \alpha\xi)$$

Thus the formula (3.5) is valid. ∎

Remark 3.1. If X is a Hilbert space, then the limit

$$\lim_{w\to\xi} [k_B(z, w) - k_B(0, w)] = \frac{1}{2}\log\frac{|1-(z, \xi)|^2}{(1-\|z\|^2)}$$

exists for every $z \in B$ and $\xi \in \partial B$ and the ellipsoid [13, 16, 26]

$$E(\xi, R) = \left\{ z \in B\colon \frac{|1-(z, \xi)|^2}{1-\|z\|^2} < R \right\},$$

where $(\cdot, \cdot)$ is the scalar product in X, is equal to the horospheres

$$E_0(\xi, R) = G(\xi, R, 0, \{w_n\}) = F_0(\xi, R),$$

where $\lim_n w_n = \xi$ [34].

Remark 3.2. If X is a separable Banach space, $\xi \in \partial B$, $w_n \in B$ for $n = 1, 2, \ldots$, and $\lim_n w_n = \xi$, then by a standard diagonalization procedure there exists a subsequence $\{w_{n_k}\}$ such that all the limits

$$\lim_{k \to \infty} [k_B(z, w_{n_k}) - k_B(z_0, w_{n_k})], \qquad z, z_0 \in B,$$

exist, and therefore all the horospheres $G(\xi, R, z_0, \{w_{n_k}\})$ are well defined.

THEOREM 3.3. *Let X be a Banach space with the open unit ball B. Let ξ and $\{w_n\}$ be fixed and let $G(\xi, R, z_0, \{w_n\})$ exist for each $z_0 \in B$ and $R > 0$. Then the horospheres $G(\xi, R, z_0, \{w_n\})$ have the following properties*:

(i) $\emptyset \neq E_{z_0}(\xi, R) \subset G(\xi, R, z_0, \{w_n\}) \subset F_{z_0}(\xi, R)$ *for every* $z_0 \in B$, $\xi \in \partial B$ *and* $R > 0$;

(ii) $G(\xi, R, z_0, \{w_n\})$ *is convex*;

(iii) *for every* $0 < R_1 < R_2$ *we have* $[\overline{G(\xi, R_1, z_0, \{w_n\})} \cap B] \subset G(\xi, R_2, z_0, \{w_n\})$;

(iv) *for every* $R > 1$ *we have* $B(z_0, \frac{1}{2} \log R) \subset G(\xi, R, z_0, \{w_n\})$;

(v) *for every* $R < 1$ *we have* $G(\xi, R, z_0, \{w_n\}) \cap B(z_0, -\frac{1}{2} \log R) = \emptyset$;

(vi) $\bigcup_{R>0} G(\xi, R, z_0, \{w_n\}) = B$ *and* $\bigcap_{R>0} G(\xi, R, z_0, \{w_n\}) = \emptyset$;

(vii) $\xi \in \bigcap_{R>0} \overline{G(\xi, R, z_0, \{w_n\})} \cap \bar{B} \subset \partial B$;

(viii) *if X is strictly convex, then* $\bigcap_{R>0} \overline{G(\xi, R, z_0, \{w_n\})} \cap \bar{B} = \{\xi\}$;

(ix) *if* $\varphi, \varphi^{-1} \in Aut(B) \cap C^0(\bar{B})$, *then* $\varphi(G(\xi, R, z_0, \{w_n\})) = G(\varphi(\xi), R, \varphi(z_0), \{\varphi(w_n)\})$;

(x) *if* $z_1 \in B$ *and* $\lim_{n \to \infty} [k_B(z_1, w_n) - k_B(z_0, w_n)] = \frac{1}{2} \log L$, *then* $G(\xi, R, z_1, \{w_n\}) \subset G(\xi, LR, z_0, \{w_n\})$.

Proof. (i) The inclusions $E_{z_0}(\xi, R) \subset G(\xi, R, z_0, \{w_n\}) \subset F_{z_0}(\xi, R)$ for every $z_0 \in B$, $\xi \in \partial B$ and $R > 0$ are obvious. Next, by Theorem 3.2 the horosphere $G(\xi, R, 0, \{w_n\})$ is nonempty for every $R > 0$. Now it is sufficient to observe that by (3.2) we get

$$\begin{aligned}
&\lim_{n \to \infty} [k_B(\alpha\xi, w_n) - k_B(z_0, w_n)] \\
&\qquad = \lim_{n \to \infty} [k_B(\alpha\xi, w_n) - k_B(0, w_n)] + \lim_{n \to \infty} [k_B(0, w_n) - k_B(z_0, w_n)] \\
&\qquad = -k_B(0, \alpha\xi) + k_B(0, z_0)
\end{aligned}$$

for each $0 < \alpha < 1$.

(ii) It is sufficient to apply (2.1).

(iii)–(vi) are obvious.

(vii) Apply Theorem 3.2.

(vii) See(ii), (iii) and (vii).

(ix) Obvious.

(x) By the equality

$$\lim_{n\to\infty} [k_B(z, w_n) - k_B(z_0, w_n)]$$

$$= \lim_{n\to\infty} [k_B(z, w_n) - k_B(z_1, w_n)] + \lim_{n\to\infty} [k_B(z_1, w_n) - k_B(z_0, w_n)]$$

$$= \lim_{n\to\infty} [k_B(z, w_n) - k_B(z_1, w_n)] + \tfrac{1}{2}\log L,$$

we get

$$G(\xi, R, z_1, \{w_n\}) \subset G(\xi, LR, z_0, \{w_n\}),$$

as claimed. ∎

4. THE DENJOY–WOLFF THEOREM

THEOREM 4.1. *Let X be a strictly convex Banach space with the open unit ball B. Let $f: B \to B$ be a compact holomorphic map with no fixed point in B. Then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ of iterates of f converges locally uniformly on B to the constant map taking the value ξ.*

Proof. Since f is fixed-point free, the limit ξ of every convergent sequence $\{w_n\} = \{h(t_n, z_n)\}$ $(t_n \to 1,\ 0 < t_n < 1,\ z_n \in B$ for $n = 1, 2, ...)$ belongs to ∂B. Let X_0 denote a separable closed subspace of X which contains $f(B)$ and set $B_0 = B \cap X_0$. Then $f_{|B_0}$ is holomorphic [4]. By Remark 3.2 we can assume without loss of generality that

$$\lim_{n\to\infty} [k_{B_0}(z, w_n) - k_{B_0}(z_0, w_n)]$$

exists for all $z, z_0 \in B_0$. Now, for $z_0 \in B_0$ let $G(\xi, R, z_0, \{w_n\})$ denote the horosphere in X_0. Let us observe that for $z \in B_0$ we have

$$\begin{aligned}
&\lim_{n\to\infty} [k_{B_0}(f(z), w_n) - k_{B_0}(0, w_n)] \\
&\quad = \lim_{n\to\infty} [k_{B_0}(f(z), f_{t_n, z_n}(z)) + k_{B_0}(f_{t_n, z_n}(z), f_{t_n, z_n}(w_n)) - k_{B_0}(0, w_n)] \\
&\quad \leqslant \lim_{n\to\infty} [k_{B_0}(z, w_n) - k_{B_0}(0, w_n)].
\end{aligned}$$

The above inequality implies that

$$f(G(\xi, R, 0, \{w_n\})) \subset G(\xi, R, 0, \{w_n\})$$

for arbitrary $R > 0$. Let us take $z \in B_0$. By Theorem 2.5 we have

$$\lim_{n\to\infty} \|f^n(z)\| = 1.$$

If $A \subset \partial B$ denotes the set of all accumulation points of the sequence $\{f^n(z)\}$, then by the compactness of f the set A is nonempty and by Theorems 2.2 and 2.3, A is independent of the choice of z and therefore by Theorem 3.3(viii)

$$\emptyset \neq A \subset \bigcap_{R>0} \overline{G(\xi, R, 0, \{w_n\})} \cap \partial B = \{\xi\}.$$

Hence $\lim_n f^n(z) = \xi$ and by Theorems 2.2 and 2.3 the sequence $\{f^n\}$ is locally uniformly convergent on B to the constant map ξ. ▮

Directly from the proof of Theorem 4.1 we also deduce the following result.

COROLLARY 4.2. *Let X be a strictly convex Banach space with the open unit ball B. Let $f\colon B \to B$ be a compact holomorphic map with no fixed point in B. Then there exists $\xi \in \partial B$ such that $\{h(t, \cdot)\}$ tends uniformly on B to the constant map ξ and the sequence $\{f^n\}$ is locally uniformly convergent to the same constant map ξ.*

Before we consider k_B-nonexpansive mappings in uniformly convex Banach spaces we prove the following lemma.

LEMMA 4.3. *Let X be a uniformly convex Banach space with the open unit ball B. If $w_n, z_n \in B$ for $n = 1, 2, \ldots$, $\lim_n z_n = \xi \in \partial B$ and $\sup_n k_B(z_n, w_n) < \infty$, then $\|z_n - w_n\| \to 0$.*

Proof. Let us observe that by the uniform convexity of X the index $\tilde{R}(\tilde{z})$ of $\tilde{z}$ tends to 0 when $\|\tilde{z}\| \to 1$. Taking $0 < \alpha < 1$ and $\tilde{z} = \alpha\xi$ we get (see (2.4))

$$\varphi(z_n) = \tilde{R}(\tilde{z})\left[1 + \frac{\|z_n - \tilde{z}\|}{1 - \|\tilde{z}\|}\right]$$

$$= \tilde{R}(\tilde{z})\left[1 + \frac{\|z_n - \alpha\xi\|}{1 - \alpha}\right] \xrightarrow[n]{} 2\tilde{R}(\tilde{z}),$$

and hence $\varphi(z_n)$ is arbitrarily small for α sufficiently close to 1 and all n sufficiently large. Therefore the inequalities (see (2.3))

$$\operatorname{arg\,tanh}\left(\frac{\|z_n - w_n\|}{\|z_n - w_n\| + 2\varphi(z_n)}\right) \leqslant k_B(z_n, w_n) \leqslant \sup_n k_B(z_n, w_n) < \infty$$

yield

$$\lim_{n \to \infty} \|z_n - w_n\| = 0. \quad \blacksquare$$

Next we need the following observation.

Remark 4.1. Let X be a Banach space with the open unit ball B and the Kobayashi distance k_B. Let X_0 be a closed separable subspace of X with the open unit ball $B_0 = B \cap X_0$, $\xi \in \partial B_0$, $w_n \in B_0$ for $n = 1, 2, \ldots$, and $\lim_n w_n = \xi$. Then by Remark 3.2 there exists a subsequence $\{w_{n_k}\}$ such that all the limits

$$\lim_{k \to \infty} [k_B(z, w_{n_k}) - k_B(z_0, w_{n_k})], \qquad z, z_0 \in B_0,$$

exist. Therefore in place of the horospheres G with respect to the Kobayashi distance k_{B_0} we can analogously define the horospheres

$$G_{B_0, k_B}(\xi, R, z_0, \{w_{n_k}\}) = \{z \in B_0 \colon \lim_{k \to \infty} [k_B(z, w_{n_k}) - k_B(z_0, w_{n_k})] < \tfrac{1}{2} \log R\}$$

in B_0. The horospheres G_{B_0, k_B} have the properties (ii), (iii), (vi), (vii), (viii) (with B replaced by B_0) given in Theorem 3.3.

Now we are ready to prove the following theorem.

THEOREM 4.4. *Let X be a uniformly convex Banach space with the open unit ball B. Let $f\colon B \to B$ be a compact k_B-nonexpansive map with no fixed point in B. Then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ of iterates of f converges locally uniformly on B to the constant map taking the value ξ.*

Proof. Applying Lemma 4.3 instead of Theorems 2.2 and 2.3, and replacing in B_0 the horospheres with respect to k_{B_0} with the horospheres G_{B_0, k_B} (see Remark 4.1), we can repeat the proof of Theorem 4.1 to obtain the convergence of $\{f^n(z)\}$ to ξ for each $z \in B_0$. The k_B-nonexpansiveness, the compactness of f and Lemma 4.3 imply the locally uniform convergence of $\{f^n\}$ on B. ∎

COROLLARY 4.5. *Let X be a uniformly convex Banach space with the open unit ball B. Let $f: B \to B$ be a compact k_B-nonexpansive map with no fixed point in B. Then there exists $\xi \in \partial B$ such that $\{h(t, \cdot)\}$ tends uniformly on B to the constant map ξ and the sequence $\{f^n\}$ is locally uniformly convergent to the same constant map ξ.*

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