5.2 The Euclidean topology on $\mathbb{BC}$

Since we have already endowed $\mathbb{BC}$ with the Euclidean norm, which is associated with the identifications $\mathbb{BC} = \mathbb{R}^4 = \mathbb{C}^2(\mathbf{i}) = \mathbb{C}^2(\mathbf{j})$, we will consider the topological space $(\mathbb{BC}, \tau_{euc})$ where τ_{euc} is the Euclidean topology on $\mathbb{R}^4$: its basis consists of all open balls in $\mathbb{R}^4$. Since for any bicomplex numbers Z and W there holds:

$$|Z + W| \leq |Z| + |W|;$$
$$|Z \cdot W| \leq \sqrt{2}\,|Z| \cdot |W|,$$

then the operations of addition and of multiplication are continuous with respect to τ_{euc}, and we can speak about the respective linearity of the topological space $(\mathbb{BC}, \tau_{euc})$; more exactly, it is a real, a $\mathbb{C}(\mathbf{i})$-complex and a $\mathbb{C}(\mathbf{j})$-complex linear topological space, but also a hyperbolic and a bicomplex linear topological module.

Besides the Euclidean open balls we can consider also open bicomplex balls with non-zero-divisor radius:

$$\left\{ Z \mid |Z|_{\mathbf{k}} \prec \gamma \ \text{ with } \ \gamma \in \mathbb{D}^+_{inv} \right\}.$$

Geometrically, such a ball can be seen as a bidisk in $\mathbb{C}^2$, thus all such balls form another basis in the topology τ_{euc}.

In Section 5.1 we introduced two formally different types of convergent sequences. Now we realize they are the same convergence with respect to the Euclidean topology but one of the definitions deals with the Euclidean basis of it and the other deals with the bicomplex balls with non-zero-divisor radius. Thus, in accordance with the problem we are faced, we will use one or another basis of the topology, depending on which one of them is more appropriate.

5.3 Bicomplex functions

Given a set $\Omega \subset \mathbb{BC}$, any mapping $F : \Omega \to \mathbb{BC}$ will be called a bicomplex function of the bicomplex variable $Z \in \Omega$. Since both Z and $F(Z)$ are bicomplex numbers, each of them admits any of the representations (see (1.3)–(1.9) and (1.24)), then F can be interpreted in different ways: as a mapping from $\mathbb{C}^2(\mathbf{i})$ to $\mathbb{C}^2(\mathbf{i})$, from $\mathbb{C}^2(\mathbf{j})$ to $\mathbb{C}^2(\mathbf{j})$, from $\mathbb{C}^2(\mathbf{i})$ generated by the idempotent representation to itself, etc. All these mappings are different but they all coincide when the bicomplex structure is considered.

We illustrate the above with an example. Consider the bicomplex function $F(Z) = Z^2$ and several mappings generated by it. To do this we write:

$$\begin{aligned} F(Z) = Z^2 &= (z_1 + \mathbf{j}z_2)^2 = (z_1^2 - z_2^2) + \mathbf{j}(2z_1z_2) \\ &= (\gamma_1 + \mathbf{i}\gamma_2)^2 = (\gamma_1^2 - \gamma_2^2) + \mathbf{i}(2\gamma_1\gamma_2) \\ &= (\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger)^2 = \beta_1^2\mathbf{e} + \beta_2^2\mathbf{e}^\dagger \\ &= (\mathfrak{z}_1 + \mathbf{i}\mathfrak{z}_2)^2 = (\mathfrak{z}_1^2 - \mathfrak{z}_2^2) + \mathbf{i}(2\mathfrak{z}_1\mathfrak{z}_2). \end{aligned}$$

Copyright © 2015. Birkhäuser Boston. All rights reserved.

Luna-Elizarrarás, M. Elena, et al. <i>Bicomplex Holomorphic Functions : The Algebra, Geometry and Analysis of Bicomplex Numbers</i>, Birkhäuser Boston, 2015. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/butler/detail.action?docID=4189334.
Created from butler on 2019-09-02 07:42:10.

Euclidean norm

This generates the following maps:

$$\begin{aligned}
\mathbb{C}^2(\mathbf{i}) \ni (z_1, z_2) &\mapsto (z_1^2 - z_2^2, 2z_1z_2) \in \mathbb{C}^2(\mathbf{i});\\
\mathbb{C}^2(\mathbf{j}) \ni (\gamma_1, \gamma_2) &\mapsto (\gamma_1^2 - \gamma_2^2, 2\gamma_1\gamma_2) \in \mathbb{C}^2(\mathbf{j});\\
\mathbb{C}^2(\mathbf{i}) \ni (\beta_1, \beta_2) &\mapsto (\beta_1^2, \beta_2^2) \in \mathbb{C}^2(\mathbf{i});\\
\mathbb{D}^2 \ni (\mathfrak{z}_1, \mathfrak{z}_2) &\mapsto (\mathfrak{z}_1^2 - \mathfrak{z}_2^2, 2\mathfrak{z}_1\mathfrak{z}_2) \in \mathbb{D}^2 .
\end{aligned}$$

Let Z_0 be a point in the closure of Ω. The function F has the limit A at Z_0 if for any $\epsilon > 0$ there exists $\delta > 0$ such that the condition $|Z - Z_0| < \delta$ implies that $|F(Z) - A| < \epsilon$. As usual in metric spaces, it is equivalent to say that for any sequence $\{Z_n\}_{n\in\mathbb{N}} \subset \Omega$ such that $\lim\limits_{n\to\infty} Z_n = Z_0$, the sequence $\{F(Z_n)\}_{n\in\mathbb{N}}$ converges to A. We obtain immediately:

(I) if the limit $\lim\limits_{Z\to Z_0} F(Z)$ exists, it is unique;

(II) if the limit $\lim\limits_{Z\to Z_0} F(Z)$ exists, then the function F is bounded in a Euclidean ball with center in Z_0 and it is $\mathbb{D}$-bounded in a bicomplex ball with a non-zero-divisor radius;

(III) if $\lim\limits_{Z\to Z_0} F(Z) = A \notin \mathfrak{S}_0$, then there exists a ball B with center in Z_0 such that for all $Z \in B$, $F(Z) \notin \mathfrak{S}_0$;

(IV) if $\lim\limits_{Z\to Z_0} F(Z) = A$, $\lim\limits_{Z\to Z_0} G(Z) = B$, then the sum, the product and the quotient (if $B \notin \mathfrak{S}_0$) have limits at Z_0 and the usual formulas hold.

A bicomplex function is *continuous* at a point $Z_0 \in \Omega \subset \mathbb{BC}$, if $\lim\limits_{Z\to Z_0} F(Z)$ exists and

$$\lim_{Z\to Z_0} F(Z) = F(Z_0) .$$

Then we say that a bicomplex function $F : \Omega \to \mathbb{BC}$, where $\Omega \subset \mathbb{BC}$, is continuous on Ω if and only if F is continuous at every $Z_0 \in \Omega$.

As in the complex case, it is easy to prove that if two functions are continuous at a point, then their sum and product are also continuous at that point. Moreover, if the second function takes at Z_0 an invertible value, then the quotient is continuous at this point also. Furthermore, the composition of continuous functions is continuous.

Bicomplex sequences and notions of convergence, limits and continuity, have also been studied in works such as [41, 45, 84]. Applications to dynamical systems, e.g. Mandelbrot and Julia sets in the bicomplex setup, have been developed in [29, 59, 60, 62, 91, 92, 93, 102].

Copyright © 2015. Birkhäuser Boston. All rights reserved.

Luna-Elizarrarás, M. Elena, et al. <i>Bicomplex Holomorphic Functions : The Algebra, Geometry and Analysis of Bicomplex Numbers</i>, Birkhäuser Boston, 2015. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/butler/detail.action?docID=4189334.
Created from butler on 2019-09-02 07:42:10.