2.3 Geometric function theory in the disk

It is elementary to prove that the ***automorphisms*** of the unit disk, that is, the one-to-one analytic maps of the disk onto itself, are just the functions

$$\varphi(z) = \lambda \frac{a-z}{1-\overline{a}z}$$

where $|\lambda| = 1$ and $|a| < 1$. In terms of the mapping, $\lambda = -\varphi'(0)/|\varphi'(0)|$ and $a = \varphi^{-1}(0)$. When $\lambda = 1$, this automorphism is an involution, that is, $\varphi^{-1} = \varphi$, that exchanges 0 and a. It is easy to check that for any automorphism ψ of the disk

$$1 - |\psi(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2} \tag{2.3.1}$$

where $a = \psi^{-1}(0)$ and $|z| \leq 1$. The group of automorphisms of the disk will be denoted Aut(D).

Every disk automorphism is an automorphism of the Riemann sphere and has two fixed points on the sphere, counting multiplicity. The automorphisms are classified according to the location of their fixed points: elliptic if one fixed point is in the disk and the other is in the complement of the closed disk, for example $\varphi(z) = iz$ which has fixed points 0 and ∞, hyperbolic if both fixed points are on the unit circle, for example $\varphi(z) = (z + .5)/(1 + .5z)$ which has fixed points ± 1, and parabolic if there is one fixed point on the unit circle (of multiplicity 2), for example, $\varphi(z) = [(1+i)z - i]/[iz + 1 - i]$ which has fixed point 1. (The other combinations of locations cannot occur as fixed points of a disk automorphism.) Dynamically, the elliptic automorphisms are "rotations" around the fixed point, hyperbolic automorphisms are flows from one fixed point to the other, and the

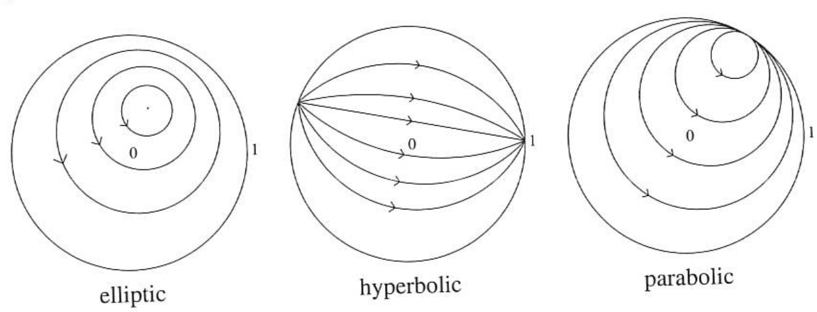

FIGURE 2.3 **Flow lines of automorphisms.**

parabolic automorphisms are flows around the disk from the fixed point and back to it from the other side. A hyperbolic automorphism is conformally equivalent to translation (parallel to the edges) in a strip or dilation in the right halfplane. A parabolic automorphism is conformally equivalent to translation (parallel to the edge) in a halfplane.

From the point of view of analytic functions, the usual Euclidean metric on the disk is inappropriate. The automorphisms of the disk are isometries with respect to the Poincaré metric in which the length of a curve γ is

$$\int_\gamma \frac{|dz|}{1-|z|^2}$$

or with respect to the (equivalent) pseudohyperbolic metric in which the distance between two points ζ_1, ζ_2 is

$$\left|\frac{\zeta_1-\zeta_2}{1-\overline{\zeta_1}\zeta_2}\right|$$

so these metrics are more useful than the Euclidean metric.

One view of automorphisms is that they represent a change of variables. In the study of a problem, we frequently wish to change variables to highlight a particular aspect of the problem. That the automorphisms preserve pseudohyperbolic distances gives a restriction on how much can be accomplished in changing variables. Exercise 2.3.1 illustrates how much flexibility we have; in particular, it illustrates the transitivity of $\mathrm{Aut}(D)$.

It is frequently helpful to give theorems in a form which is invariant under the application of automorphisms. The invariant form of the Schwarz Lemma is the following. Many of the results of this section are consequences of this basic inequality.

THEOREM 2.39 ***(Schwarz–Pick Theorem)***
If φ is an analytic map of the disk into itself, then

$$\left|\frac{\varphi(w)-\varphi(z)}{1-\overline{\varphi(w)}\varphi(z)}\right| \le \left|\frac{w-z}{1-\overline{w}z}\right|$$

and if equality holds for any $z \neq w$, then φ is an automorphism of the disk.

PROOF Let $u=\varphi(w)$. Since φ maps the disk into itself so does

$$\left(\frac{u-\varphi(z)}{1-\overline{u}\varphi(z)}\right)$$

Since $(w-z)/(1-\overline{w}z)$ is zero at $z=w$ and has modulus 1 on the unit circle, if we define ψ by

$$\left(\frac{u-\varphi(z)}{1-\overline{u}\varphi(z)}\right)=\left(\frac{w-z}{1-\overline{w}z}\right)\psi(z)$$

then ψ is analytic in the disk and by the maximum modulus theorem, satisfies $|\psi(z)|\le 1$ for all z in D, which is the inequality we were to prove. If equality holds for some z in D with $z \neq w$, then $|\psi(z)| = 1$, so ψ is constant and

$$\left(\frac{u-\varphi(z)}{1-\overline{u}\varphi(z)}\right) = \lambda\left(\frac{w-z}{1-\overline{w}z}\right)$$

from which the equality condition follows. ∎

Geometrically, the Schwarz–Pick Theorem says that the image of the pseudohyperbolic disk $D(w,r)$ under φ is contained in $D(\varphi(w),r)$. A helpful interpretation of the Schwarz–Pick Theorem is that in the Poincaré or pseudohyperbolic metrics, a map of the disk into itself is a contraction (although not necessarily a strict contraction). As a corollary, we get an upper bound on the modulus of $\varphi(z)$.

COROLLARY 2.40

If φ is an analytic map of the disk into itself, then

$$|\varphi(z)| \leq \frac{|z| + |\varphi(0)|}{1 + |z||\varphi(0)|}$$

PROOF By Equation (2.3.1) we have

$$1 - \left|\frac{\varphi(0)-\varphi(z)}{1-\overline{\varphi(0)}\varphi(z)}\right|^2 = \frac{(1-|\varphi(0)|^2)(1-|\varphi(z)|^2)}{|1-\overline{\varphi(0)}\varphi(z)|^2}$$

so that

$$\begin{aligned}\left|\frac{\varphi(0)-\varphi(z)}{1-\overline{\varphi(0)}\varphi(z)}\right|^2 &\geq 1 - \frac{(1-|\varphi(0)|^2)(1-|\varphi(z)|^2)}{(1-|\varphi(0)||\varphi(z)|)^2} \\ &= \frac{(|\varphi(z)|-|\varphi(0)|)^2}{(1-|\varphi(0)||\varphi(z)|)^2}\end{aligned}$$

The Schwarz–Pick Theorem (Theorem 2.39) gives

$$\left|\frac{\varphi(0)-\varphi(z)}{1-\overline{\varphi(0)}\varphi(z)}\right| \leq \left|\frac{0-z}{1-\overline{0}z}\right| = |z|$$

Thus

$$|z| \geq \frac{||\varphi(z)|-|\varphi(0)||}{1-|\varphi(0)||\varphi(z)|} \geq \frac{|\varphi(z)|-|\varphi(0)|}{1-|\varphi(0)||\varphi(z)|}$$

from which the result follows. ∎

In particular, notice that the estimate of Corollary 2.40 shows that for any analytic map φ of the disk into itself we have

$$\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1-|\varphi(0)|}{1+|\varphi(0)|}$$

an observation which is relevant to Julia's lemma below.

For $k > 0$ and ζ in the unit circle, let

$$E(k, \zeta) = \{z \in D : |\zeta - z|^2 \leq k(1 - |z|^2)\}$$

A computation shows that $E(k, \zeta)$ is a closed disk internally tangent to the circle at ζ with center $\frac{1}{1+k}\zeta$ and radius $\frac{k}{1+k}$. The boundary circle of this disk is called an ***oricycle***. The parabolic automorphisms that fix ζ map these oricycles onto themselves and are distinguished from all other non-identity maps of the disk by this property in the sense that is made precise in Exercises 2.3.7 and 2.3.8. Moreover, these oricyles play an important role for arbitrary self-maps of the disk, as described by the following geometric lemma.

LEMMA 2.41 (Julia's Lemma)
Suppose ζ is in the unit circle and

$$d(\zeta) = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|}$$

is finite where the lower limit is taken as z approaches ζ unrestrictedly in D. Suppose $\{a_n\}$ is a sequence along which this lower limit is achieved and for which $\varphi(a_n)$ converges to η. Then $|\eta| = 1$ and for every z in D

$$\frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} \leq d(\zeta) \frac{|\zeta - z|^2}{1 - |z|^2}$$

Moreover, if equality holds for some z in D, then φ is an automorphism of the disk.

The geometric interpretation of this result is that φ maps each disk $E(k, \zeta)$ into the corresponding disk $E(kd(\zeta), \eta)$.

PROOF We are assuming $a_n \to \zeta$ and $\varphi(a_n) \to \eta$ with

$$d(\zeta) = \lim_{n \to \infty} \frac{1 - |\varphi(a_n)|}{1 - |a_n|} < \infty$$

Clearly $|\eta| = 1$. The Schwarz–Pick Theorem (2.39) gives, for all z in D,

$$1 - \left| \frac{\varphi(z) - \varphi(a_n)}{1 - \varphi(z)\overline{\varphi(a_n)}} \right|^2 \geq 1 - \left| \frac{z - a_n}{1 - \overline{a_n} z} \right|^2$$

which by the fundamental identity for automorphisms (Equation (2.3.1)) is equivalent to

$$\frac{(1 - |\varphi(z)|^2)(1 - |\varphi(a_n)|^2)}{|1 - \varphi(z)\overline{\varphi(a_n)}|^2} \geq \frac{(1 - |a_n|^2)(1 - |z|^2)}{|1 - \overline{a_n} z|^2}$$

or

$$\frac{|1 - \varphi(z)\overline{\varphi(a_n)}|^2}{1 - |\varphi(z)|^2} \leq \frac{(1 - |\varphi(a_n)|^2)|1 - \overline{a_n} z|^2}{(1 - |a_n|^2)(1 - |z|^2)}$$

Letting n go to ∞ we obtain

$$\frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} = \frac{|1 - \overline{\eta}\varphi(z)|^2}{1 - |\varphi(z)|^2} \leq d(\zeta)\frac{|1 - \overline{\zeta}z|^2}{1 - |z|^2} = d(\zeta)\frac{|\zeta - z|^2}{1 - |z|^2}$$

as desired.

The proof of the equality condition is left as an exercise (Exercise 2.3.9). ∎

The quantity $d(\zeta)$ plays an important role in the study of the geometry of self-maps of the disk. While $d(\zeta)$ may be $+\infty$, as a consequence of Julia's Lemma (or by the estimate following the proof of Corollary 2.40), it must always be strictly greater than 0.

The geometric interpretation of Julia's Lemma is particularly satisfying in the case $\zeta = \eta$. Clearly, in this case, the point ζ deserves to be called a fixed point, but since we have not assumed (and do not want to) that φ is continuous on the boundary, we need to extend our notion of fixed point to include the case of a fixed point on the unit circle.

DEFINITION 2.42 *If φ is an analytic mapping of the disk into itself and b is a point of the closed disk, we will call b a* ***fixed point of*** *φ if*

$$\lim_{r \to 1} \varphi(rb) = b$$

The Schwarz–Pick Theorem implies that an analytic function on the disk has at most one fixed point inside the disk, but analytic functions can have many fixed points on the circle. The fixed point set must have measure zero and for a univalent function, the fixed point set has capacity zero, but, at least for compact sets, these are the only requirements (see [CoP82] and Exercise 2.3.13). The Schwarz–Pick Theorem tells us about the behavior of an analytic function φ near an interior fixed point: φ maps pseudohyperbolic disks centered at the fixed point into other (smaller) pseudohyperbolic disks centered at the fixed point. Julia's Lemma gives a similar statement for a fixed point ζ on the boundary when $d(\zeta)$ is finite: φ maps internally tangent disks at ζ into (other) internally tangent disks at ζ.

For ζ on the unit circle and $\alpha > 1$ we define a ***nontangential approach region at*** ζ by

$$\Gamma(\zeta, \alpha) = \{z \in D : |z - \zeta| < \alpha(1 - |z|)\}$$

Of course, the term "nontangential" refers to the fact that the boundary curves of $\Gamma(\zeta, \alpha)$ have a corner at ζ, with angle less than π (see Exercise 2.3.11). A function f is said to have a ***nontangential limit at*** ζ if $\lim_{z \to \zeta} f(z)$ exists in each nontangential region $\Gamma(\zeta, \alpha)$.

DEFINITION 2.43 *We say φ has a* ***finite angular derivative*** *at ζ on the unit circle if there is η on the circle so that $(\varphi(z) - \eta)/(z - \zeta)$ has a finite nontangential limit as $z \to \zeta$. When it exists (as a finite complex number), this limit is denoted $\varphi'(\zeta)$.*

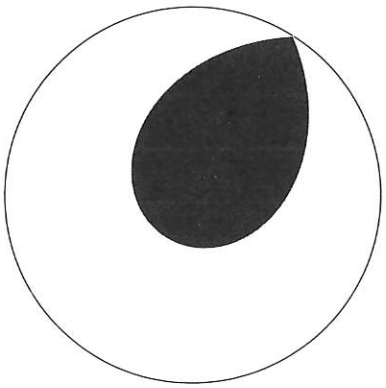

FIGURE 2.4
A typical nontangential approach region.

Our next result, the Julia–Carathéodory Theorem, is a circle of ideas which makes precise the relationship between the angular derivative $\varphi'(\zeta)$, the limit of $\varphi'(z)$ at ζ, and the quantity $d(\zeta)$ from Julia's Lemma.

THEOREM 2.44 (Julia–Carathéodory Theorem)
For $\varphi : D \to D$ analytic and ζ in ∂D, the following are equivalent:

(1) $d(\zeta) = \liminf_{z\to\zeta}(1-|\varphi(z)|)/(1-|z|) < \infty$, *where the limit is taken as z approaches ζ unrestrictedly in D.*

(2) *φ has finite angular derivative $\varphi'(\zeta)$ at ζ.*

(3) *Both φ and φ' have (finite) nontangential limits at ζ, with $|\eta| = 1$ for $\eta = \lim_{r\to 1}\varphi(r\zeta)$.*

Moreover, when these conditions hold, we have $\lim_{r\to 1}\varphi'(r\zeta) = \varphi'(\zeta) = d(\zeta)\overline{\zeta}\eta$ *and $d(\zeta)$ is the nontangential limit* $\lim_{z\to\zeta}(1-|\varphi(z)|)/(1-|z|)$.

The proof uses the following simple lemma about nontangential approach regions in D.

LEMMA 2.45
Given $1 < \alpha < \beta$, let $\delta = (\beta-\alpha)/(\alpha+\alpha\beta)$. If z is in $\Gamma(\zeta,\alpha)$ and $|\lambda| \le \delta|\zeta - z|$, then $z+\lambda$ is in $\Gamma(\zeta,\beta)$.

PROOF We have

$$\begin{aligned}|z+\lambda-\zeta| &\le |z-\zeta| + |\lambda| \\ &< \alpha(1-|z|) + \delta|\zeta - z| \\ &\le \alpha(1-|z|) + \delta\alpha(1-|z|)\end{aligned}$$

$$= (\alpha + \delta\alpha)(1 - |z|)$$

But since $|\lambda| \le \delta|\zeta - z|$ and $|\zeta - z| < \alpha(1 - |z|)$ we get

$$1 - |z + \lambda| \ge 1 - |z| - |\lambda| \ge (1 - |z|)(1 - \delta\alpha)$$

Thus,

$$|z + \lambda - \zeta| \le (\alpha + \delta\alpha)(1 - |z|) \le \frac{\alpha + \delta\alpha}{1 - \delta\alpha}(1 - |z + \lambda|)$$

Since $\beta = (\alpha + \delta\alpha)/(1 - \delta\alpha)$, the conclusion follows. ∎

PROOF (of Theorem 2.44) We will show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

For (1) $\Rightarrow$ (2) recall that by Lemma 2.41 there exists η on the unit circle so that for all z in D

$$\frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} \le d(\zeta)\frac{|\zeta - z|^2}{1 - |z|^2}$$

We first consider the radial limit of $(\varphi(z) - \eta)/(z - \zeta)$ at ζ. Now

$$\frac{1 - |\varphi(r\zeta)|}{1 - r}\frac{1 + r}{1 + |\varphi(r\zeta)|} \le \frac{|\eta - \varphi(r\zeta)|^2}{1 - |\varphi(r\zeta)|^2}\frac{1 - r^2}{(1 - r)^2}$$

$$\le d(\zeta)\frac{|\zeta - r\zeta|^2}{1 - r^2}\frac{1 - r^2}{(1 - r)^2} = d(\zeta)$$

Since $d(\zeta)$ is the lower limit of $(1 - |\varphi(z)|)/(1 - |z|)$ at $z = \zeta$ we must have

$$\lim_{r \to 1} \frac{1 - |\varphi(r\zeta)|}{1 - r} = d(\zeta) \tag{2.3.2}$$

and $\lim_{r \to 1} |\varphi(r\zeta)| = 1$. Furthermore, since

$$\frac{(1 - |\varphi(r\zeta)|)^2}{(1 - r)^2} \le \frac{|\eta - \varphi(r\zeta)|^2}{(1 - r)^2} \le d(\zeta)\frac{1 - |\varphi(r\zeta)|^2}{1 - r^2}$$

we have

$$\lim_{r \to 1} \frac{|\eta - \varphi(r\zeta)|}{1 - r} = d(\zeta) \tag{2.3.3}$$

Comparing Equations (2.3.2) and (2.3.3) yields

$$\lim_{r \to 1} \frac{1 - |\varphi(r\zeta)|}{|\eta - \varphi(r\zeta)|} = 1$$

and a computation with this shows that $\arg(1 - \overline{\eta}\varphi(r\zeta))$ tends to 0 as r goes to 1. Using Equation (2.3.3) we see that

$$\lim_{r \to 1} \frac{\eta - \varphi(r\zeta)}{\zeta - r\zeta} = d(\zeta)\overline{\zeta}\eta$$

To finish, we must extend this from radial convergence to nontangential convergence. To this end, fix an arbitrary nontangential approach region $\Gamma(\zeta, \alpha)$. For z in $\Gamma(\zeta, \alpha)$, we have $|\zeta - z| < \alpha(1 - |z|) \leq \alpha(1 - |z|^2)$ so by Julia's Lemma

$$\frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} \leq d(\zeta)\frac{|\zeta - z|^2}{1 - |z|^2} \leq \alpha|\zeta - z|d(\zeta)$$

This implies

$$\frac{|\eta - \varphi(z)|}{|\zeta - z|} \leq \alpha d(\zeta)(1 + |\varphi(z)|)\frac{1 - |\varphi(z)|}{|\eta - \varphi(z)|} \leq 2\alpha d(\zeta)$$

Thus $(\eta - \varphi(z))/(\zeta - z)$ is bounded in $\Gamma(\zeta, \alpha)$, and since it has radial limit $d(\zeta)\eta\overline{\zeta}$ at ζ, Lindelöf's Theorem ([Dur70, p. 6]) shows that it tends to the same limit in $\Gamma(\zeta, \beta)$ for any $\beta < \alpha$. Since α, and hence β, is arbitrary, we are done.

Next we show (2) $\Rightarrow$ (3). Suppose φ has finite angular derivative at ζ and $\eta = \lim_{r \to 1} \varphi(r\zeta)$. Fix a nontangential approach region $\Gamma(\zeta, \alpha)$ and fix w in this region. If r is small enough that $\{w + re^{i\theta} : 0 \leq \theta \leq 2\pi\}$ lies in D then by the Cauchy Integral Formula for $(\varphi - \eta)'(w)$ we have

$$\begin{aligned}\varphi'(w) = (\varphi - \eta)'(w) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\varphi(w + re^{i\theta}) - \eta}{re^{i\theta}}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\varphi(w + re^{i\theta}) - \eta}{w + re^{i\theta} - \zeta}\right)\left(\frac{w + re^{i\theta} - \zeta}{re^{i\theta}}\right) d\theta\end{aligned}$$

If we choose $r = \delta|w - \zeta|$ where $\delta = (1 + 2\alpha)^{-1}$ then Lemma 2.45 guarantees that the circle $w + re^{i\theta}$ is contained in $\Gamma(\zeta, \beta)$ where $\beta = 2\alpha$. It follows from Condition (2) that

$$\frac{\varphi(w + re^{i\theta}) - \eta}{w + re^{i\theta} - \zeta}$$

is bounded, independent of θ and w in $\Gamma(\zeta, \alpha)$. Since

$$|(w + re^{i\theta} - \zeta)/(re^{i\theta})| = |1 + (w - \zeta)/re^{i\theta}| \leq 1 + 1/\delta$$

we have φ' bounded in $\Gamma(\zeta, \alpha)$. Moreover, setting $w = t\zeta$ for $0 < t < 1$ and letting t tend to 1, we can use the bounded convergence theorem together with the fact that $(\varphi(z) - \eta)/(z - \zeta)$ approaches $\varphi'(\zeta)$ nontangentially to conclude that $\lim_{t \to 1} \varphi'(t\zeta) = \varphi'(\zeta)$. The boundedness of φ' in $\Gamma(\zeta, \alpha)$ and this radial convergence imply nontangential convergence in any smaller approach region. Since α is arbitrary we are done.

Finally, we show (3) $\Rightarrow$ (1). Let $M < \infty$ be such that $|\varphi'(r\zeta)| \leq M$ for $r > 0$. Then

$$|\eta - \varphi(r\zeta)| = \left|\int_r^1 \varphi'(t\zeta)\, dt\right| \leq M(1 - r)$$

Hence

$$\frac{1-|\varphi(r\zeta)|}{1-|r\zeta|} \le \frac{|\eta-\varphi(r\zeta)|}{1-r} \le M$$

and $d(\zeta)$, being a lower limit, is finite.

In the proof of (1) $\Rightarrow$ (2), we saw that $(\eta-\varphi(z))/(\zeta-z)$ converges to $d(\zeta)\overline{\zeta}\eta$ as z tends to ζ nontangentially. This is the same as saying that $(1-\overline{\eta}\varphi(z))/(1-\overline{\zeta}z)$ converges to $d(\zeta)$ as z tends to ζ nontangentially. In particular, since $d(\zeta)$ is positive, $|1-\overline{\eta}\varphi(z)|/|1-\overline{\zeta}z|$ also converges to $d(\zeta)$ and

$$\lim_{z\to\zeta} \frac{1-\overline{\eta}\varphi(z)}{|1-\overline{\eta}\varphi(z)|} \Big/ \frac{1-\overline{\zeta}z}{|1-\overline{\zeta}z|} = 1$$

As a consequence, we see that when z approaches ζ nontangentially, $\varphi(z)$ approaches η nontangentially also. Nontangential convergence of z to ζ implies

$$\left|\mathrm{Im}\,\frac{1-\overline{\zeta}z}{|1-\overline{\zeta}z|}\right| \le C\mathrm{Re}\,\frac{1-\overline{\zeta}z}{|1-\overline{\zeta}z|}$$

for some constant C, so

$$\lim_{z\to\zeta} \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{|1-\overline{\eta}\varphi(z)|} \Big/ \frac{\mathrm{Re}\,(1-\overline{\zeta}z)}{|1-\overline{\zeta}z|} = 1$$

or

$$\lim_{z\to\zeta} \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{\mathrm{Re}\,(1-\overline{\zeta}z)} = \lim_{z\to\zeta} \frac{|1-\overline{\eta}\varphi(z)|}{|1-\overline{\zeta}z|} = d(\zeta)$$

Finally, the nontangential convergence implies

$$\lim_{z\to\zeta} \frac{\mathrm{Re}\,(1-\overline{\zeta}z)}{1-|z|} = 1 \quad\text{and}\quad \lim_{z\to\zeta} \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{1-|\varphi(z)|} = 1$$

so

$$\lim_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} = \lim_{z\to\zeta} \frac{\mathrm{Re}\,(1-\overline{\eta}\varphi(z))}{\mathrm{Re}\,(1-\overline{\zeta}z)} = d(\zeta)$$

as z approaches ζ nontangentially, as we wished to prove. ∎

The final conclusion shows that if φ has finite angular derivative at ζ then $|\varphi'(\zeta)| = d(\zeta)$. Moreover, since $d(\zeta) = \infty$ if and only if φ does not have finite angular derivative at ζ we are justified in writing $|\varphi'(\zeta)| = d(\zeta)$ even when $d(\zeta) = \infty$ and we will do so whenever convenient. Because of this identification the number $d(\zeta)$, whether finite (and positive) or infinite, is itself sometimes referred to as the angular derivative at ζ. It is useful to notice that the infimum of the angular derivative is always attained:

PROPOSITION 2.46
If φ is an analytic map of the disk into itself, then

$$\inf_{\zeta\in\partial D}|\varphi'(\zeta)| = \liminf_{|z|\to 1}\frac{1-|\varphi(z)|}{1-|z|} = |\varphi'(\zeta_0)|$$

for some ζ_0 in the unit circle.

PROOF Set $\beta = \inf_{\zeta\in\partial D}|\varphi'(\zeta)|$. If we assume $|\varphi'(\zeta)| > \beta$ for all ζ in the circle, we may find for each ζ an open disk $\Delta(\zeta)$ with center ζ and radius $\epsilon(\zeta) > 0$ such that

$$\frac{1-|\varphi(z)|}{1-|z|} > \beta + \epsilon(\zeta)$$

for all z in $\Delta(\zeta)\cap D$. Cover ∂D by a finite subcollection $\Delta(\zeta_1), \Delta(\zeta_2), \ldots, \Delta(\zeta_m)$ and find $r_0 > 0$ so that the annulus $\{r_0 < |z| < 1\}$ is contained in $\cup_{j=1}^{m}\Delta(\zeta_j)$. Set $\epsilon = \min\{\epsilon(\zeta_j) : j = 1, 2, \ldots, m\}$. Then on this annulus

$$\frac{1-|\varphi(z)|}{1-|z|} > \beta + \epsilon$$

which contradicts the definition of β. ∎

From the proof of the Julia–Carathéodory Theorem, we see that $d(\zeta) < \infty$ implies that $\varphi'(\zeta) = d(\zeta)\overline{\zeta}\eta$ where η is the radial limit of φ at ζ. This says $\arg\varphi'(\zeta) = \arg\eta - \arg\zeta$ which leads, by the standard argument, to the conclusion that φ is conformal at ζ whenever $d(\zeta) < \infty$. In particular, note that at a fixed point of φ on the unit circle, then $\varphi'(\zeta) = d(\zeta) > 0$. Another geometric consequence of the Julia–Carathéodory Theorem is contained in the following corollary.

COROLLARY 2.47
If $|\zeta| = 1$ and ζ is a fixed point of φ with $\varphi'(\zeta) < \infty$, then for any nontangential approach region $\Gamma(\zeta,\alpha)$ there is $r < 1$ such that φ is univalent on $\Gamma(\zeta,\alpha)\cap\{z : r < |z| < 1\}$.

PROOF We know that $\varphi'(z)$ has nontangential limit equal to $\varphi'(\zeta)$ at ζ. Fix a nontangential approach region $\Gamma(\zeta,\alpha)$ and choose r sufficiently close to 1 so that $|\varphi'(z) - \varphi'(\zeta)| < \frac{1}{2}|\varphi'(\zeta)|$ for z in $\Gamma(\zeta,\alpha)\cap\{r < |z| < 1\}$. If both z_1 and z_2 lie in $\Gamma(\zeta,\alpha)\cap\{r < |z| < 1\}$ then so does the line segment L joining them and

$$|\varphi(z_2) - \varphi(z_1) - \varphi'(\zeta)(z_2 - z_1)| = \left|\int_L (\varphi'(z) - \varphi'(\zeta))\,dz\right| \le \frac{1}{2}|\varphi'(\zeta)||z_2 - z_1|$$

so that $\varphi(z_2) \ne \varphi(z_1)$ unless $z_1 = z_2$. ∎

Our next result identifies a distinguished fixed point on the unit circle in the case φ has no fixed points in D and shows, for all fixed points ζ of φ on the unit circle except this one, that $d(\zeta) > 1$. In other words, except for this distinguished fixed point, $\varphi' > 1$ at fixed points on the circle; this inequality is improved in Exercise 2.3.15.

THEOREM 2.48 (Wolff's Lemma)
If φ is an analytic map of the disk into itself that has no fixed points in D, then there is a unique fixed point a of φ on the unit circle with $d(a) \leq 1$. If φ, not the identity, has a fixed point in D, then $d(\zeta) > 1$ for all fixed points ζ of φ on the unit circle.

PROOF First assume φ has no fixed point in D. Choose any sequence r_n increasing to 1 and consider the maps $f_n \equiv r_n\varphi : D \to r_nD$. Since f_n is continuous as a map of $r_n\overline{D}$ into itself, the Brouwer Fixed Point Theorem guarantees that f_n has a fixed point a_n in D. By passing to a subsequence, we may assume the sequence $\{a_n\}$ converges to a point a in $\overline{D}$. If $|a| < 1$, then the continuity of φ near a shows $\varphi(a_n)$ converges to $\varphi(a)$ and, since r_n increases to 1, this means $r_n\varphi(a_n)$ tends to $\varphi(a)$ also. But $r_n\varphi(a_n) = a_n$ so we must have $\varphi(a) = a$, contradicting the assumption that φ has no fixed points in the open disk. Thus $|a| = 1$.

We claim that a is a fixed point of φ and that $d(a) \leq 1$. This second fact is immediate:

$$\liminf_{z\to a} \frac{1-|\varphi(z)|}{1-|z|} \leq \liminf_{n\to\infty} \frac{1-|\varphi(a_n)|}{1-|a_n|} = \liminf_{n\to\infty} \frac{1-\frac{|a_n|}{r_n}}{1-|a_n|} \leq 1$$

and therefore $d(a) \leq 1$. Since $f_n(a_n) = a_n$, the Schwarz–Pick Theorem (Theorem 2.39) shows that for arbitrary w in D,

$$1 - \left|\frac{a_n - f_n(w)}{1-\overline{a_n}f_n(w)}\right|^2 \geq 1 - \left|\frac{a_n - w}{1-\overline{a_n}w}\right|^2$$

Equation (2.3.1) and some algebra show this is equivalent to

$$\frac{|1-\overline{a_n}f_n(w)|^2}{1-|f_n(w)|^2} \leq \frac{|1-\overline{a_n}w|^2}{1-|w|^2} \tag{2.3.4}$$

On recalling that $f_n = r_n\varphi$ and letting n tend to infinity in Inequality (2.3.4), we obtain

$$\frac{|a-\varphi(w)|^2}{1-|\varphi(w)|^2} \leq \frac{|a-w|^2}{1-|w|^2}$$

for all w in D. Thus $\varphi(a) = a$ and the claim is verified.

Uniqueness follows from Julia's Lemma (2.41): if ζ_1 and ζ_2 are fixed points of φ on the unit circle with $d(\zeta_1) \leq 1$ and $d(\zeta_2) \leq 1$, choose k_1 and k_2 so that the oricycles $\partial E(k_1, \zeta_1)$ and $\partial E(k_2, \zeta_2)$ are tangent to each other at w in D (see Figure 2.5). But then $\varphi(w)$ is in $\overline{E(k_1, \zeta_1)} \cap \overline{E(k_2, \zeta_2)} = \{w\}$, contradicting the hypothesis that φ is fixed point free in D.

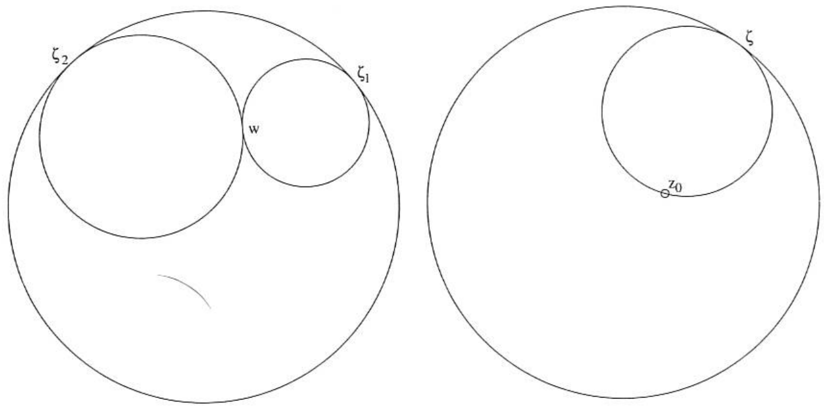

FIGURE 2.5
Oricycles at contractive fixed points.

Similarly, suppose φ has a fixed point z_0 in D and ζ is a fixed point of φ on the unit circle with $d(\zeta) \leq 1$. If ρ is chosen so that the oricycle $\partial E(\rho, \zeta)$ passes through z_0 (see Figure 2.5), then in the inequality in Julia's Lemma (2.41), we have $\eta = \zeta$, $d(\zeta) = 1$ and equality holds for $z = z_0$. This means that φ is an automorphism of the disk that fixes z_0 and ζ which is impossible since φ is not the identity. ∎

We will use this theorem to investigate the iterates φ_n of a fixed point free map φ, where $\varphi_1 = \varphi$ and $\varphi_{n+1} = \varphi \circ \varphi_n$, $n = 1, 2, \ldots$. Since the analytic maps of D into D form a normal family, every sequence of iterates of φ contains a subsequence which converges uniformly on compact subsets of D. By the maximum modulus theorem any such limit is either an analytic map of D into D or a constant function of modulus 1. Our next lemma shows that only one constant function is possible among the subsequential limits of $\{\varphi_n\}$.

LEMMA 2.49

Suppose φ is an analytic map of the disk into itself that has no fixed points in D and let a be the unique fixed point of the circle with $d(a) \leq 1$. The only constant function that can appear as a limit of iterates of φ is $f(z) \equiv a$.

PROOF Suppose φ_{n_j} converges uniformly on compact subsets of D to the constant w in the closed disk and assume $w \neq a$. We may find a neighborhood V of w in $\overline{D}$ and ρ sufficiently small so that $E(\rho, a)$ and V are disjoint. By Julia's Lemma, $\varphi_n(E(\rho, a)) \subset E(\rho, a)$ for all n since $d(a) \leq 1$. Hence for any z in $E(\rho, a)$, $\varphi_n(z)$ is not in V for all n, which contradicts the assumption that φ_{n_j} converges to w. ∎

We also note for future reference that if φ (not the identity map) has a fixed point a in D, then $\varphi_n(a) = a$ for every n and the only constant function among the subsequential limits of φ_n is $f(z) \equiv a$.

LEMMA 2.50
Let φ be an analytic map of the disk into itself and suppose $\{\varphi_n\}$ has a subsequence which converges to a non-constant function. Then φ is an automorphism of the disk.

PROOF We assume φ_{n_j} converges to g uniformly on compact subsets of the disk, where g is non-constant. Of course, $g(D) \subset D$. Set $m_j = n_{j+1} - n_j$ and choose a subsequence $\varphi_{m_{j_k}}$ which converges to, say, h. On the one hand $\varphi_{m_{j_k}} \circ \varphi_{n_{j_k}}$ converges to $h \circ g$, but also $\varphi_{m_j} \circ \varphi_{n_j} = \varphi_{n_{j+1}}$ which converges to g. Thus h is the identity on the range of g, a set with more than one point so, by the Schwarz Lemma, h is the identity on all of D. Passing to a further subsequence if necessary, we may assume

$$\varphi_{m_{j_k}-1} \to f$$

and hence

$$\varphi_{m_{j_k}} = \varphi_{m_{j_k}-1} \circ \varphi \to f \circ \varphi$$

so that $f \circ \varphi$ is the identity. Since $f(D) \subset D$ so that also $\varphi \circ \varphi_{m_{j_k}-1} \to \varphi \circ f$, we have $\varphi \circ f$ is the identity. This implies φ is a one-to-one map of the disk onto itself. ∎

We use these lemmas to prove the Denjoy–Wolff theorem.

THEOREM 2.51 (Denjoy–Wolff Theorem)
If φ, not the identity and not an elliptic automorphism of D, is an analytic map of the disk into itself, then there is a point a in $\overline{D}$ so that the iterates φ_n of φ converge to a uniformly on compact subsets of D.

PROOF We consider first the case that φ is not an automorphism of D. By Lemmas 2.49 and 2.50 the only function that can appear as a limit of any sequence of iterates of φ is a constant function, where the constant is either the interior fixed point of φ, if there is one, or the unique unimodular a with $d(a) \leq 1$. Since $\{\varphi_n\}$ is a normal family, the entire sequence of iterates must converge to this constant, uniformly on compact subsets of the disk.

If φ is a hyperbolic or parabolic automorphism of D we still know that there is a unique fixed point a of the unit circle with $\varphi'(a) \leq 1$. It is not difficult to show directly (Exercise 2.3.2 and 2.3.3) that in this case, φ_n converges uniformly on compact subsets to a. ∎

DEFINITION 2.52 *The limit point a of Theorem 2.51 will be referred to as the* ***Denjoy–Wolff point of*** φ.

The preceding few results show that if φ is a map with no fixed point in D, then there is unique fixed point a on the unit circle with $d(a) \leq 1$ and that the iterates of φ converge to this fixed point. On the other hand, if φ, not the identity or an elliptic automorphism, has a fixed point a in D, then the Schwarz Lemma implies a is unique and $|\varphi'(a)| < 1$ and Theorem 2.48 shows that φ has no fixed points on the boundary with $d(\zeta) \leq 1$. In this case, the iterates of φ also converge to the fixed point a. Thus, we see that the Denjoy–Wolff point of φ can be described as the unique fixed point of φ in $\overline{D}$ with $|\varphi'(a)| \leq 1$. In the next section, we will study the iteration of functions in the disk. We can use the geometric ideas of this section to better understand the nature of the iteration that occurs in the Denjoy–Wolff theorem. We will pay particular attention to the behavior of the mapping in the neighborhood of the Denjoy–Wolff point.

Exercises

2.3.1 (a) Show that if $z_1 \neq z_2$ and $w_1 \neq w_2$ are points of D for which

$$\left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right| = \left|\frac{w_1 - w_2}{1 - \overline{w_1}w_2}\right|$$

then there is a unique automorphism φ of the disk so that $\varphi(z_1) = w_1$ and $\varphi(z_2) = w_2$.

(b) Show that if ζ_1, ζ_2, and ζ_3 are distinct points arranged counterclockwise on the unit circle and η_1, η_2, and η_3 are also distinct points arranged counterclockwise on the unit circle, then there is a unique automorphism φ of the disk so that $\varphi(\zeta_j) = \eta_j$ for $j = 1, 2, 3$.

2.3.2 Suppose ψ is an automorphism of the upper halfplane Π^+ which fixes 0 and ∞ only. Show that $\psi(w) = tw$ for some $t > 0, t \neq 1$. (Hint: ψ is a linear fractional transformation, so $\psi(w) = (aw+b)/(cw+d)$ where we may assume a, b, c,and d are real since ψ maps the extended real axis to itself.) Show that, consequently, any hyperbolic automorphism of the disk is conformally equivalent to a dilation of Π^+ and that its iterates converge to its Denjoy–Wolff point as asserted in Theorem 2.51.

2.3.3 Suppose ψ is an automorphism of Π^+ which fixes ∞ only. Show that $\psi(w) = w+b$ for some real $b \neq 0$. Conclude that a parabolic automorphism of the disk is conformally equivalent to a translation of Π^+ and that its iterates converge to its Denjoy–Wolff point as asserted in Theorem 2.51.

2.3.4 Suppose φ is a map of the disk into itself that is analytic in a neighborhood of the closed disk and has Denjoy–Wolff point a. Suppose ψ is an automorphism of the disk with $\psi(b) = a$. Show that $\widetilde{\varphi}$ defined by $\widetilde{\varphi}(z) = \psi^{-1}(\varphi(\psi(z)))$ has Denjoy–Wolff point b and $\widetilde{\varphi}'(b) = \varphi'(a)$. Find a formula relating $\widetilde{\varphi}''(b)$ to $\varphi''(a)$.

2.3.5 (a) Show that an elliptic disk automorphism φ is conformally equivalent to λz where $\lambda = \varphi'(a)$ and a is the fixed point of φ in D.

(b) Show that if $a = (a_1, 0')$ then $\mathcal{E}(a, r)$ is an ellipsoid and identify its center $c = (c_1, 0')$. Also show that $\mathcal{E}(a, r) \cap [e_1]$ is a disk of radius $\sim r(1 - |a|^2)$ while $\mathcal{E}(a, r) \cap \{z_1 = c_1\}$ is a ball of radius $\sim r\sqrt{1 - |a|^2}$ for r small. By virtue of (a) these observations give the shape of an arbitrary $\mathcal{E}(a, r)$.

(c) Interpret Exercise 2.5.2 in terms of these ellipsoids.

Notes

Early contributors to the study of the Schwarz Lemma in several variables include K. Reinhardt, C. Carathéodory and H. Cartan; see [Di89] for a discussion of some of the history of and references for generalizations of the Schwarz Lemma.

The automorphisms in B_2 were first described by H. Poincaré in [Poi07]. The discussion of $\mathrm{Aut}(B_N)$ given here was greatly influenced by the exposition in [Ru80].

Theorem 2.76 is due to M. Hervé [He63]; see also [Ru78].

2.6 Julia–Carathéodory theory in the ball

In this section we will discuss the analogues of the ideas of Section 2.3 for the ball. As much as possible our treatment here will parallel the treatment in the disk, although a few new technical details will be needed. For ζ in ∂B_N we will continue to use the notation $d(\zeta)$ for $\liminf_{z \to \zeta}(1 - |\varphi(z)|)/(1 - |z|)$ where φ is an analytic map of the ball B_N into itself. Julia's Lemma (Lemma 2.41) generalizes to the following:

***LEMMA 2.77** (Julia's Lemma in B_N)*
Suppose ζ is in ∂B_N with $d(\zeta) < \infty$. Suppose $a_n \to \zeta$ satisfies

$$\lim_{n\to\infty} \frac{1 - |\varphi(a_n)|}{1 - |a_n|} = d(\zeta)$$

and $\lim_{n\to\infty} \varphi(a_n) = \eta$ where η is in ∂B_N. Then for every z in B_N

$$\frac{|1 - \langle \varphi(z), \eta\rangle|^2}{1 - |\varphi(z)|^2} \le d(\zeta)\frac{|1 - \langle z, \zeta\rangle|^2}{1 - |z|^2}$$

PROOF By the Schwarz–Pick Theorem in B_N (Exercise 2.5.2) we have

$$\frac{|1 - \langle \varphi(z), \varphi(a_n)\rangle|^2}{(1 - |\varphi(z)|^2)(1 - |\varphi(a_n)|^2)} \le \frac{|1 - \langle z, a_n\rangle|^2}{(1 - |z|^2)(1 - |a_n|^2)}$$

or

$$\frac{|1 - \langle \varphi(z), \varphi(a_n)\rangle|^2}{1 - |\varphi(z)|^2} \le \frac{1 - |\varphi(a_n)|^2}{1 - |a_n|^2}\frac{|1 - \langle z, a_n\rangle|^2}{1 - |z|^2}$$

Letting $n \to \infty$ gives the conclusion. ∎

As in the one variable case, this lemma has an appealing geometric interpretation. Set $E(k, \zeta) = \{z \in B_N : |1 - \langle z, \zeta \rangle|^2 \leq k(1 - |z|^2)\}$. In the special case $\zeta = e_1$ a computation shows that this is equivalent to

$$\left| z_1 - \frac{1}{1+k} \right|^2 + \frac{k}{1+k}|z'|^2 \leq \left(\frac{k}{1+k} \right)^2$$

which is an ellipsoid tangent at e_1 with center $\frac{1}{1+k}e_1$. Its intersection with $[e_1]$ is a disk of radius $\frac{k}{1+k}$,while its intersection with $z_1 = \frac{1}{1+k}$ is a ball of radius $\sqrt{\frac{k}{1+k}}$. Because unitary maps preserve inner products, $U(E(k, \zeta)) = E(k, U\zeta)$ for any unitary U. Thus in general $E(k, \zeta)$ is an ellipsoid internally tangent to the unit sphere at ζ with center $\frac{1}{1+k}\zeta$. Julia's Lemma says φ maps each ellipsoid $E(k, \zeta)$ into the corresponding ellipsoid $E(d(\zeta)k, \eta)$.

One important technical complication for the Julia–Carathéodory theory in B_N arises from the appropriate analogue for nontangential approach regions when $N > 1$. For many of the basic function theory results in B_N, approach within a Koranyi approach region replaces nontangential approach. These regions are defined for $\alpha > 1$ by

$$\Gamma(\zeta, \alpha) = \{z \in B_N : |1 - \langle z, \zeta \rangle| < \frac{\alpha}{2}(1 - |z|^2)\}$$

Setting $N = 1$ we see that, since

$$\frac{\alpha}{2}(1 - |z|^2) \sim \alpha(1 - |z|)$$

for $|z|$ near 1, in essence, this gives our previous definition for a nontangential approach region in the disk, and we will use the same notation. However, when $N > 1$ these regions have a perhaps unexpected geometric property. Since $U\Gamma(\zeta, \alpha) = \Gamma(U\zeta, \alpha)$, we can concentrate on understanding $\Gamma(e_1, \alpha)$. An easy computation shows that while the intersection of $\Gamma(e_1, \alpha)$ with the complex line through e_1 is a standard nontangential approach region in the disk $B_N \cap [e_1]$, in other directions $\Gamma(e_1, \alpha)$ permits tangential approach to ∂B_N. In particular, the intersection of $\Gamma(e_1, \alpha)$ with $\{z : \text{Im } z_1 = 0\}$ is the ball

$$(x_1 - 1/\alpha)^2 + |z'|^2 < (1 - 1/\alpha)^2$$

containing e_1 in its boundary. We say f has ***admissible limit at*** ζ if it has a limit $f^*(\zeta)$ along every curve lying in some Koranyi region $\Gamma(\zeta, \alpha)$.

In the definition of a Koranyi approach region one is comparing the distance (in the Euclidean metric) from z to $\zeta + T^C(\zeta)$, where $T^C(\zeta)$ is the maximal complex subspace of the tangent space to ∂B_N at ζ, with the distance from z to ∂B_N. In one variable this point of view degenerates, since the maximal complex subspace of