Appendix B

Metric and Banach spaces

In this appendix, we recall some basic definitions of functional analysis about metric and Banach spaces that are largely employed to prove existence results for partial differential equations. These definitions will in particular provide an answer to the question of "how close are two objects ?" The topological concepts and the concepts of the measure theory meet in this matter and there are numerous links between these two fields.

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B.1   Metric spaces

Definition B.1 *A metric space is a couple (X, d) where X is a set and d is a* metric *(or a* distance*) on X, that is a function $d : X \times X \to \mathbb{R}^+$ such that*

1. *$d(x, y) \geq 0$ (non-negativity)*

2. *$d(x, y) = 0$ if and only if $x = y$ (identity)*

3. *$d(x, y) = d(y, x)$ (symmetry)*

4. *$d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).*

Remark B.1 *Usually, only three conditions are used to define a distance function. Indeed, the first of these conditions is a property that follows from the other three, since:*

$$2d(x, y) = d(x, y) + d(y, x) \geq d(x, x) = 0 \,.$$

Definition B.2 *Let (X, d) be a metric space and r a stricly positive scalar value. At any point x in a metric space, we define the* open ball *(resp.* closed ball*) of radius r about x as the set $B(x, r) = \{y \in X : d(x, y) < r\}$ (resp. $B_c(x, r) = \{y \in X , d(x, y) \leq r\}$).*

These balls generate a topology on X, making it a *topological space* . A subset Y of X is called an *open* if it is a union of open balls, its complement is called a *closed*.

Given a non empty and bounded set A of a metric space (X, d). There exists a smallest scalar value δ such that

$$\forall (x, y) \in A \times A, \quad d(x, y) \leq \delta,$$

we denote $\delta(A)$ this value and we call it the diameter of A.

The notion of distance allows to define the concepts of the limit of a sequence and of a continuous function.

Definition B.3 *(a)* Convergence of sequences*: Let $(x_n)_{n \in \mathbb{N}}$ a sequence of elements of a metric set (X, d) and $l \in X$ a point. The sequence $(x_n)_{n \in \mathbb{N}}$ converges towards l if and only if:*

$$\forall \varepsilon > 0, \exists n_0, \ n \geq n_0 \Rightarrow d(x_n, l) < \varepsilon.$$

(b) continuity of functions*: Consider a function f between two metrics spaces (X, d_X) and (Y, d_Y) and a point $x_0 \in X$. Then, f is continuous at the point x_0 if:*

$$\forall \varepsilon > 0, \exists \delta > 0, / \, d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon.$$

Proposition B.1 (Composition of continuous functions) *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be three metric spaces and $f : X \to Y$, $g : Y \to Z$ two functions. Suppose f is continuous at a point x_0 and g at the point $f(x_0)$. Then the function $g \circ f$ is continuous at x_0.*

Proposition B.2 *Let (X, d_X) and (Y, d_Y) be two metric spaces, $f : X \to Y$ a function and $(x_n)_{n \in \mathbb{N}}$ a sequence of elements in X. Suppose $(x_n)_{n \in \mathbb{N}}$ converges to l and the function f is continuous at l. Then, the sequence of elements of Y defined by $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(l)$.*

We recall three topological concepts in topology.

Definition B.4 *Suppose (X, d) is a metric space and $A \subset X$ is an arbitrary subset of X. Then,*

(a) *The* interior *of A, denoted as $\overset{\circ}{A}$ is the set of points $x \in X$ such that there exists an open ball $B(x, r)$ $(r > 0)$ centered at x which is contained in A.*

(b) *The* closure *of A, denoted as $\bar{A}$, is the set of points $x \in X$ such that any open ball $B(x, r)$ $(r > 0)$ centered at x contains a point of A (the point may be x itself).*

(c) *A is* dense *in X if and only if $\bar{A} = X$.*

Remark B.2 *(i) the set A is open if and only if $A = \overset{\circ}{A}$. In particular, the interior of the empty set if the empty set.*

(ii) the set A is closed if and only if $\bar{A} = A$. In particular, the closure of the empty set is the empty set and the closure of A itself is A.

Proposition B.3 *Let A be a subset of a metric space (X, d). A point x belongs to $\bar{A}$ if and only if there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of elements of A such that $\lim_{n \to \infty} a_n = x$.*

Definition B.5 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X,d). We define the set of all closure values of $(x_n)_{n\in\mathbb{N}}$, denoted as $\mathrm{Clo}(x_n)$ as:*

$$\mathrm{Clo}(x_n) \stackrel{def}{=} \bigcap_{n\in\mathbb{N}} \bar{A}_n \qquad with \quad A_n \stackrel{def}{=} \{x_m, m \ge n\}.$$

Then , we have a concept very important in numerical analysis.

Theorem B.1 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X,d). A point $l \in X$ belongs to $\mathrm{Clo}(x_n)$ if and only if there exists a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ ($\forall n \in \mathbb{N}, \phi(n) > n$) such that:*

$$\lim_{n\to\infty} x_{\phi(n)} = l.$$

Definition B.6 *Let (X, d_X) and (Y, d_Y) be two metric spaces and let consider a function $f : X \to Y$. Then,*

(a) *the function is* uniformly continuous *if and only if:*

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall (x, x') \in X, \quad d_X(x, x') < \alpha \Rightarrow d_Y(f(x), f(x')) < \varepsilon,$$

(b) *the function f is locally Hölder continuous (satisfies a Hölder condition) of exponent $\alpha \in [0,1]$ if and only if there exists a nonnegative constant k such that:*

$$\forall (x, x') \in X \times X, d \quad {}_X(x, x') \le 1 \Rightarrow d_Y(f(x), f(x')) \le k\, d_X(x, x')^\alpha.$$

Remark B.3 *(see also Definition (C.1)). In the previous definition,*

(i) *If $\alpha = 1$, the function f is locally k-Lipschitz continuous.*

(ii) *If $\alpha = 1$ and if the condition $d_X(x, x') \le 1$ can be omited, the function f is globally k-Lipschitz continuous.*

Definition B.7 *Given a subset $A \subset X$ of a metric space (X,d) and $x \in X$. The* distance *between the point x and the set A is the infimum of the distances between the point and those in the set:*

$$d(x, A) \stackrel{def}{=} \inf_{a \in A} d(x, a).$$

Proposition B.4 *Given $A \subset (X, d)$. The function $d_A : X \to \mathbb{R}^+$, $x \mapsto d(x, A)$ is k-Lipschitz with the constant $k = 1$.*

B.2   Complete metric spaces.

Definition B.8 *Let (X, d) be a metric space. A Cauchy sequence in X is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X such that:*
$$\forall \varepsilon > 0, \exists\, n_0 \in \mathbb{N}, \forall n \ge n_0, \forall m \ge n_0, \quad d(x_n, x_m) \le \varepsilon.$$

In particular, any convergent sequence is a Cauchy sequence thanks to the triangle inequality.

Proposition B.5 *Any Cauchy sequence in a metric space is bounded.*

Proof.   If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, and for all $m \geq n_0$, we have $d(x_n, x_m) \leq 1 + \max_{(n,m)\in\{0,\dots n_0\}^2} d(x_n, x_m)$. The results follows. $\square$

Notice that given a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ and a continuous function f then $(f(x_n))_{n\in\mathbb{N}}$ is not necessarily a Cauchy sequence. However, we have

Theorem B.2 *Let (X, d_X) and (Y, d_Y) be two metric spaces and let consider a uniformely continuous function $f : (X, d_X) \rightarrow (Y, d_Y)$. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of X, then $f(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of F. The reciprocal one is not true.*

Proposition B.6 *We have two properties about convergent sequences and Cauchy sequences.*

(a) *In a metric space (X, d), any convergente sequence is a Cauchy sequence and conversely, any Cauchy sequence having a closure value l converges to l.*

(b) *In any metrix space (X, d), a Cauchy sequence which has convergent subsequence with limit l is itself convergent with the same limit l.*

These two properties yield one standard proof of the completness of the real numbers (with the Bolzano-Weierstrass theorem).

Definition B.9 *A metric space (X, d) in which every Cauchy sequence converges (has a limit in X) is called* complete. *A Polish space is a separable complete metric space.*

The advantage of working wih Cauchy sequences and with the completness concept is provided by the Banach spaces. The following theorem will be used to show the existence of a unique solution for ordinary differential equation (see Cauchy-Lipschitz theorem).

Theorem B.3 (Banach fixed-point) *Given $f : (X, d) \rightarrow (X, d)$ a contraction map, i.e. a function such that there exists a real number $k \in [0, 1[$ such that*

$$d(f(x), f(y)) \leq k\, d(x, y).$$

Then, there exists a unique fixed point z such that $f(z) = z$.

Proof.   **(Picard's iterative method)** Given $x_0 \in X$, we consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined as $x_{n+1} = f(x_n)$. We will show that this sequence converges toward the fixed point z, for all x_0. We write

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq k\, d(x_n, x_{n-1}).$$

and so on, leading to

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0).$$

Hence, for any couple of integer values (n, p) we have

$$
\begin{aligned}
d(x_{n+p}, x_n) &\leq \sum_{m=1}^{p} d(x_{n+m}, x_{n+m-1}) \\
&\leq d(x_1, x_0) \sum_{m=1}^{p} k^{n+m-1} \\
&\leq \frac{k^n}{1-k} d(x_1, x_0).
\end{aligned}
$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is thus a Cauchy sequence. Let z denotes its limit. As the function f is Lipschitz continuous, we obtain, by passing to the limit in the definition of the sequence $(x_n)_{n\in\mathbb{N}}$ that $z = f(z)$.

This proves the existence.

Let z_1 and z_2 be two solutions of the equation $z = f(z)$. According to the assumption on f, we have

$$d(z_1, z_2) \leq k\, d(z_1, z_2)\,.$$

Since $k < 1$, this ensures that $d(z_1, z_2) = 0$ thus $z_1 = z_2$. The results follows. □

Proposition B.7 *Let (X, d) be a metric space and A a subset of X. Then, the space (A, d) is complete if and only if A is a closed set of X.*

Theorem B.4 (Prolongation of uniformly continuous functions) *Let (X, d_X) and (Y, d_Y) be two metric spaces, $A \subset X$ a subset dense in X and $f : (A, d_X) \to (Y, d_Y)$ a uniformly continuous function. If Y is a complete space, then there exists a unique application uniformly continuous $\tilde{f} : (X, d_X) \to (Y, d_Y)$ such that $\tilde{f}_{|A} = f$.*

The following results are especially usueful to show properties that could not be established otherwise.

Lemma B.1 *Let (X, d) be a complete metric space and let $(u_n)_{n \in \mathbb{N}}$ a decreasing sequence of closed non emptysets of X which diameter tends to 0. Then, there exists an element $x \in X$ such that:*

$$\bigcap_{n \in \mathbb{N}} u_n = \{x\}\,.$$

Theorem B.5 (Baire) *Let (X, d) be a complete metric space. We have the following results.*

(a) *We consider a sequence $(u_n)_{n \in \mathbb{N}}$ of open subsets dense in X. Then, $\bigcap_{n \in \mathbb{N}} u_n$ is dense.*

(b) *If we consider a sequence $(v_n)_{n \in \mathbb{N}}$ of closed subsets of empty interior in X. Then, $\bigcup_{n \in \mathbb{N}} v_n$ is of empty interior.*

Corollary B.1 *Let $(v_n)_{n \in \mathbb{N}}$ be a sequence of closed sets of a complete metric space (X, d) whose union is X. Then, there exists an integer n_0 such that $\overset{\circ}{v_{n_0}} \neq 0$.*

Remark B.4 *The rational numbers $\mathbb{Q}$ are not a complete metric space. Indeed, $\mathbb{Q}$ corresponds to the countable union of closed sets of empty interior. There are sequences of rational numbers that converge in $\mathbb{R}$ to irrational numbers, or in other words, there are Cauchy sequence having no limit in $\mathbb{Q}$.*

Proposition B.8 *Given d and d' two distances uniformly equivalent on a metric space X.*

(a) *Every Cauchy sequence for d is a Cauchy sequence for d' (and conversely).*

(b) *If (X, d) is a complete metric space, so is (X, d').*

B.3 Boundedness and compactness

Definition B.10 *A metric space (X, d) is called* bounded *if there exists some number r such that*

$$d(x, y) \leq r\,, \qquad \forall x, y \in X\,.$$

The space (X, d) is called precompact *or* totally bounded *if for every $r > 0$ there exist finitely many open balls of radius r whose union covers X.*

Remark B.5 *Since the set of the centers of the balls of radius r is finite, it has a finite diameter. Then, using the triangle inequality, we conclude that every totally bounded space is bounded. The converse does not hold.*

Definition B.11 *A subset A of a metric space (X, d) is called* compact *if and only if every sequence $(x_n)_{n \in \mathbb{N}}$ of elements of A has a closure point (an adherent point) in A.*

Proposition B.9 *Given A a subset of a metric space (X, d). If A is compact, then A is closed and bounded.*

Remark B.6 *Two important remarks.*

(i) *In general, closed bounded subsets of a metric space are not compact ! Except in the peculiar case of the Euclidean space, $X = \mathbb{R}^d$, with the usual distance $d(x, y) = |x - y|$.*

$$closed + bounded \neq compact \,.$$

(ii) *Boundedness should not be confused with* finite. *The latter refers to the number of elements, not to how far the set extends.*

$$finiteness \Rightarrow boundedness \,, \qquad but \qquad boundedness \nRightarrow finiteness \,.$$

Proposition B.10 *Given A a compact subset of a metric space (X, d). If $B \subset A$ is a closed subset of X, then B is compact.*

Proposition B.11 (Product metric spaces) *If $(X_1, d_1), \ldots, (X_N, d_N)$ are metric spaces, we define*

$$X = X_1 \times \cdots \times X_N \qquad and \qquad d((x_1, \ldots, x_N), (y_1, \ldots, y_N)) = \max_{1 \le j \le N} d_j(x_j, y_j) \,.$$

Given $A_j \subset X_j$ a compact subset for every $1 \le j \le N$, then the subset $A = A_1 \times \cdots \times A_N$ is a compact subset of X. In particular, a finite product of compact spaces is a compact space.

We conclude this section, by enouncing several important theorems on compact metric spaces.

Theorem B.6 *A metric space (X, d) is compact if and only if for every collection $(U_\lambda)_{\lambda \in \mathbb{N}}$ of open subsets of X covering X, i.e., such that $X = \bigcup_{\lambda \in \mathbb{N}} U_\lambda$, there is a finite subset $(\lambda_j)_{1 \le j \le N}$ such that $X = \bigcup_{j=1}^{N} U_{\lambda_j}$ of elements of X admits a closure point.*

Lemma B.2 (Lebesgue) *Given a metric space (X, d) and a compact subset $A \subset X$. For every collection $(U_\lambda)_{\lambda \in \mathbb{N}}$ of open subsets of X covering A, there exists a real value $\alpha > 0$ such that:*

$$\forall x \in A, \quad \exists \lambda \in \mathbb{N} \quad / \quad B(x, \alpha) \subset U_\lambda \,.$$

Theorem B.7 (Heine) *Given (X, d_X) and (Y, d_Y) two metric spaces and $f : X \to Y$ a continuous function. Then, for every compact subset $A \subset X$, $f(A)$ is a compact subset of Y. Moreover, if X is compact then f is a uniformly continuous function.*

Corollary B.2 *Given $f : (X, d) \to (\mathbb{R}, |x - y|)$ a continuous function. If (X, d) is compact, then the function f has a minimum and a maximum:*

$$\exists x_m, x_M \in X \; such \; that \quad \forall x \in X, \quad f(x_m) \le f(x) \le f(x_M) \,.$$

Proposition B.12 *If (X, d) is a compact metric space, then (X, d) is complete.*

The following theorem of functional analysis is a criterion to decide whether a set of continuous functions from a compact metrix space into a metric space is compact.

Theorem B.8 (Ascoli) *Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric space. We consider a subset A of $C(X, Y)$ the set of continuous functions from X to Y endowed with the distance*

$$D(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Suppose we have:

(i) the subset A is equicontinuous, i.e.

$$\forall x \in X, \, \forall \varepsilon > 0, \, \exists \alpha > 0, \quad d(x, x') < \alpha \Rightarrow \forall f \in A, d_Y(f(x), f(x')) < \varepsilon,$$

(ii) for every $x \in X$, the set $\{f(x), \, f \in A\}$ is of compact closure.

Then, A is relatively compact, i.e. its closure is a compact set.

B.4 Normed and Banach spaces

B.4.1 Normed spaces

Definition B.12 *Let consider a vector space E on K ($K = \mathbb{R}$ or $K = \mathbb{C}$). A mapping $N : E \to \mathbb{R}^+$ is a seminorm on E if and only if the two following assertions are satisfied:*

(a) $N(x + y) \le N(x) + N(y)$,

(b) for every $\lambda \in K$, $N(\lambda x) = |\lambda| N(x)$.

A norm is a seminorm with the additional property: $N(x) = 0$ if and only if $x = 0$ (positive definiteness).

Let X denote a normed vector space, *i.e.* a vector space endowed with a norm $\| \cdot \|$. And we introduce the space of continuous functions:

Definition B.13 *Given $a, b \in \mathbb{R}$, with $a < b$ and $n \in \mathbb{N}$, we consider the space*

$$C^0([a, b], \mathbb{R}^n) \overset{def}{=} \{u : [a, b] \to \mathbb{R}^n, u \text{ is continuous at each } x \in [a, b]\}.$$

Since $[a, b]$ is a closed and bounded set, the function u must be bounded:

$$\|u\|_\infty = \sup_{x \in [a, b]} |u(x)| < +\infty.$$

Remark B.7 *The notion of Cauchy sequence can be extended to the topological vector spaces. To this end, we consider that for any open subset $\omega \ni 0$, there exists $N(\omega)$ such that for all $n, m \ge N(\omega)$, then $x_n - x_m \in \omega$.*

Theorem B.9 *Let (X, μ) be a measure space. For every real value $p \in [1, +\infty[$, we consider the space $L^p(X, d\mu)$. The map $f : L^p(X, d\mu) \to \mathbb{R}^+$ defined by*

$$f \mapsto \|f\|_{L^p} \overset{def}{=} \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} \qquad p < +\infty$$

$$f \mapsto \operatorname{ess\,sup}(|f|) \qquad p = +\infty$$

is a norm (see Appendix D for more information about measure theory).

Definition B.14 *Let E be a vector space and let N be a norm on E. The pair (E, N) is called a* normed space.

A normed space can be endowed with a distance function, to become a metric space.

Proposition B.13 *Let $(E, \| \cdot \|)$ be a normed space. The map $E \times E \to \mathbb{R}^+$, $(x, y) \mapsto \|x - y\|$ is a* distance *on E, called the distance associated to the norm $\| \cdot \|$.*

Remark B.8 *There are many fundamental differences between finite and infinite spaces. In finite dimensional space, all linear maps are continuous, i.e. there are first-degree homogeneous polynomials. Let consider the space E of $C^1([0, 1])$ continuous functions endowed with the norm $\sup_{x \in [0,1]} |f(x)|$ and let consider the map*

$$\delta_0' : E \to \mathbb{R} \qquad f \mapsto f'(0) \,.$$

This linear form is not continuous.
Another difference concerns the compactness of the unit ball.

Theorem B.10 *Let E be a normed vector space; the dimension of E is finite if and only if the closed unit ball of E is compact.*

Since we have defined a structure of metric space, we can introduce the notion of continuity. An we have the following series of results.

Theorem B.11 *Let E and F be two normed vector spaces and let $\ell : E \to F$ be a linear map. The map ℓ is Lipschitz continuous if and only if there exists an open set $U \subset E$ such that ℓ is bounded on U, i.e. such that*

$$\sup_{x \in U} \|\ell(x)\|_F \; < \; +\infty \,.$$

Theorem B.12 *Let $(E_j, \| \cdot \|_j)_{1 \le j \le N}$ be a family of normed vector spaces and let $(F, \| \cdot \|)$ be a normed vector space. We consider the N-linear map $f : E_1 \times \cdots \times E_N \to F$; f is a continuous map if and only if*

$$\sup_{\|x_j\|_{1 \le j \le n}} \|f(x_1, \dots, x_N)\| \; < \; +\infty \,.$$

Moreover, if f is continuous, then f is Lipschitz continuous on the bounded sets of $E_1 \times \cdots \times E_N$.

Proposition B.14 *Let E, F be two normed vector spaces. We denote by $\mathcal{L}(E, f)$ the set of all linear maps from E to F. Then, the map defined by*

$$\|\ell\|_{\mathcal{L}(E,F)} \; \overset{def}{=} \; \sup_{\|x\|_E \le 1} \|\ell(x)\|_F \,,$$

is a norm on $\mathcal{L}(E, F)$.

Proposition B.15 *Let E, F, G be three normed vector spaces and let (ℓ_1, ℓ_2) be an element of $\mathcal{L}(E, F) \times \mathcal{L}(F, G)$. The composition $\ell_2 \circ \ell_1$ belongs to $\mathcal{L}(E, G)$ and*

$$\|\ell_2 \circ \ell_1\|_{\mathcal{L}(E,G)} \; \le \; \|\ell_1\|_{\mathcal{L}(E,F)} \|\ell_2\|_{\mathcal{L}(F,G)} \,.$$

Remark B.9 *If $E = F = G$, we pose $\mathcal{L}(E) \overset{def}{=} \mathcal{L}(E, E)$. For all $(\ell_1, \ell_2) \in \mathcal{L}(E)^2$ we have then*

$$\|\ell_2 \circ \ell_1\|_{\mathcal{L}(E)} \; \le \; \|\ell_1\|_{\mathcal{L}(E)} \|\ell_2\|_{\mathcal{L}(E)} \,.$$

B.4.2 Banach spaces

Definition B.15 *Let $(E, \|\cdot\|)$ be a normed space. $(E, \|\cdot\|)$ is a Banach space if and only if the metric space (E, d) where d is the distance associated to the norm $\|\cdot\|$, i.e. $d(x, y) = \|x - y\|$, is a complete space.*

Remark B.10 *Any vector subspace of finite dimension of a normed vector space is a Banach space. For example, let E be a Banach space and X be a set; the set $\mathcal{B}(X, E)$ of the bounded functions from X to E endowed with the norm*

$$\|f\|_{\mathcal{B}(x,E)} \overset{def}{=} \sup_{x \in X} \|f(x)\|_E$$

is a Banach space.

The following result explains why the Lebesgue integration theory is extremely useful.

Theorem B.13 *Let (X, μ) be a measure space. For every scalar value $p \in [1, +\infty[$, the space $(L^p(X, d\mu), \|\cdot\|_{L^p})$ is a Banach space.*
Moreover, let E be a normed space and let F be a Banach space. Then, the space $\mathcal{L}(E, F)$ endowed with the norm defined in Proposition B.14 is a Banach space.

Proposition B.16 *Let E be a Banach space and let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements in E. Then, if the series $(\|x_n\|_E)_{n \in \mathbb{N}}$ is convergent, then the sequence $S_N = \sum_{0 \leq n \leq N} x_n$ will be convergent.*

We have another characterization of a Banach space.

Theorem B.14 *If E is a normed vector space such that for all sequence $(x_n)_{n \in \mathbb{N}}$ we have*

$$\sum_{n \in \mathbb{N}} \|x_n\|_E < +\infty \Rightarrow S_n \overset{def}{=} \sum_{n=0}^{N} x_n \ converges,$$

then the space E is a Banach space.

Proposition B.17 (Banach) *Let E, F be two Banach spaces and let $A \in \mathcal{L}(E, F)$. If A is a bijective map, then $A^{-1} \in \mathcal{L}(F, E)$.*

B.4.3 Inverse in $\mathcal{L}(E)$

Proposition B.18 *The set $U(E)$ of all invertible elements in $\mathcal{L}(E)$ is an open set and the map Inv defined by*

$$Inv : U(E) \to U(E) \qquad \ell \mapsto \ell^{-1}$$

is a continuous map.

Definition B.16 *Let E be a Banach space on $K = \mathbb{R}$ or $\mathbb{C}$ and let $u \in \mathcal{L}(E)$. The spectrum of u, denoted $\mathrm{Sp}(u)$ is the set of elements $\lambda \in K$ such that $u - \lambda Id$ is not invertible.*

Remark B.11 *In finite dimension, there is only one way for a linear map to be non invertible. Indeed, let ℓ be a linear map from E to E (a finite dimensional space), we know that*

$$\ell \ bijective \ map \ \Leftrightarrow \ell \ surjective \ map \ \Leftrightarrow \ell \ injective \ map.$$

Then, the reciprocal map ℓ^{-1} is a linear map.

Proposition B.19 *Let E be a Banach space on $\mathbb{K}$ and let u be an element of $\mathcal{L}(E)$. The spectrum of u is a closed set contained in the closed ball of radius $\|u\|_{\mathcal{L}(E)}$ and of center 0.*

The following result shows that the body is important

Proposition B.20 *Let E be a Banach space on $\mathbb{C}$ and let u be a map of $\mathcal{L}(E)$. The spectrum of u is not empty.*

B.4.4   Topological dual space

Definition B.17 *Let $(E, \|\cdot\|_E)$ be a normed vector space. The* topological dual space, *denoted E', is the Banach space of all continuous linear maps on E, i.e. $E' = \mathcal{L}(E, \mathbb{R})$. The space is endowed with the norm*

$$\|f\|_{E'} = \sup_{\substack{x \in E \\ \|x\|_E \leq 1}} f(x) = \sup_{\substack{x \in E \\ \|x\|_E = 1}} f(x) = \sup_{\substack{x \in E \\ x \neq 0}} \frac{f(x)}{\|x\|_E} \,.$$

In general, we adopt the notation $f(x) = \langle f, x \rangle_{E',E}$.

Theorem B.15 (Hahn-Banach) *Let $(E, \|\cdot\|_E)$ be a normed space and let consider $p : G \to \mathbb{R}$ a map such that*

$$\begin{aligned} p(\lambda x) &= \lambda p(x), & \forall x \in E \,, \forall \lambda > 0 \\ p(x+y) &\leq p(x) + p(y), & \forall x, y \in E \,. \end{aligned}$$

Let $G \subset E$ be a subspace and let $g : E \to \mathbb{R}$ be a linear form such that

$$g(x) \leq p(x), \qquad \forall x \in G \,.$$

Then there exists a linear extension f of g on the space E such that

$$f(x) \leq p(x) \quad \forall x \in E, \qquad f(x) = g(x), \forall x \in G \,.$$

The proof is given in [Brezis, 2005] and involves the Zorn's lemma (a formulation of the axiom of choice). The extension is in general not uniquely specified.

Corollary B.3 *Let $(E, \|\cdot\|_E)$ be a normed space. For all $x \in E$, we have*

$$\|x\|_E = \sup_{f \in E', \|f\|_{E'} \leq 1} \langle f, x \rangle_{E',E} = \max_{f \in E', \|f\|_{E'} \leq 1} \langle f, x \rangle_{E',E} \,.$$

Theorem B.16 *Let $p > 1$ be a scalar value and let B be the bilinear form defined by*

$$B : L^{p'} \times L^p \to K \qquad (g, f) \mapsto \int_\Omega f(x) g(x) d\mu(x) \,.$$

Then, the bilinear form B identifies the dual of $L^p(\Omega, d\mu)$ with $L^{p'}(\Omega, d\mu)$.

B.4.5   Separable normed spaces

The Hahn-Banach theorem has several important and useful consequences.

Corollary B.4 *Let $F \subset E$ be a vector subspace such that $\bar{F} \neq E$. Then, there exists a linear functional $f \in E'\backslash\{0\}$ such that $\langle f, x \rangle = 0$ for all $x \in \bar{F}$.*

Proposition B.21 *Let E be a normed vector space such that its dual E' is separable. Then, E is separable.*

An interesting application of Baire's theorem (see Section B.2) to Banach spaces concerns the *uniform boundedness principle*. The precise statement of this result is given by the following theorem.

Theorem B.17 (Banach-Steinhaus) *Let consider $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ two Banach spaces and $(l_n)_{n\in\mathbb{N}}$ a sequence of $\mathcal{L}(E,F)$. Suppose that the space E is complete and that for all $x \in E$, the limit of the sequence $(l_n(x))_{n\in\mathbb{N}}$ exists and is then denoted by $l(x)$. Then, the sequence $(l_n)_{n\in\mathbb{N}}$ is a bounded sequence of $\mathcal{L}(E,F)$ and thus l belongs to $\mathcal{L}(E,F)$. Furthermore, we have*

$$\|l\|_{\mathcal{L}(E,F)} \leq \liminf_{n\to\infty} \|l_n\|_{\mathcal{L}(E,F)}.$$

B.4.6   Compact operators

Definition B.18 *Let E, F be two Banach spaces. An element $u \in \mathcal{L}(E,F)$ is said to be* compact *if and only if for every bounded subset A, the closure of $u(A)$ is compact.*

Proposition B.22 *Let E, F be two Banach spaces and let consider $u \in \mathcal{L}(E,F)$. We have a characterization of a compact operator.*

  (i) *u is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ of elements of E, the sequence $(u(x_n))_{n\in\mathbb{N}}$ has a closure value.*

  (ii) *u is compact if and only if the closure of $(B_E(0,1))$ is compact.*

Theorem B.18 *Let consider E, F, G three Banach spaces.*

   - *The set of compact elements $u \in \mathcal{L}(E,F)$ is a closed vector subspace of $\mathcal{L}(E,F)$.*

   - *Suppose $u \in \mathcal{L}(E,F)$ and $v \in \mathcal{L}(F,G)$, then the operator $u \circ v$ is compact if either u or v is compact.*

B.4.7   Stone-Weierstrass theorem

The Stone-Weierstrass theorem has practical relevance, for instance in polynomial approximation. We give a real version of this theorem.

Theorem B.19 (Stone-Weierstrass) *Let consider a subset $\mathcal{B}$ of $C^0(K, \mathbb{R})$, K being a compact space, such that*

  (i) *$\mathcal{B}$ is an algebra (stability by addition, multiplication, multiplication by a scalar value),*

  (ii) *$\mathcal{B}$ contains constants,*

 (iii) *points are separated in $\mathcal{B}$ ($\forall x, y \in K$, $x \neq y$, then $f(x) \neq f(y)$ for a certain $f \in \mathcal{B}$).*

Then, $\mathcal{B}$ is dense in $C^0(K, \mathbb{R})$.

Another well-known statement of the approximation theorem, due to the German mathematician K. Weierstrass (1815, 1897), reads

Theorem B.20 (Weierstrass) *The vector space of polynomial is dense in $C^0([0,1])$ for the topology related to the uniform convergence.*

We recall that

Definition B.19 *The sequence $(f_n)_{n\in\mathbb{N}}$ is of real-valued functions $f_n : X \to \mathbb{R}$ is uniformely convergent with a limit $f \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\forall x \in X\,, \; \forall n \geq N\,, \quad |f_n(x) - f(x)| < \varepsilon\,.$$

We have the following result about the uniform convergence.

Theorem B.21 (Dini) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $C^0(X, \mathbb{R})$ such that, for all $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ is a monotonically increasing bounded sequence. If $g(x) = \lim_{n\to\infty} f_n(x)$ is a continuous function on X, then the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ toward g is uniform, i.e.*

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - g(x)| = 0\,.$$

Lemma B.3 *If X is a compact space, then the Banach space $C^0(X, \mathbb{K})$ is separable.*

Remark B.12 *Banach's theorem B.3 provides an elegant way of solving evolution equations, the Cauchy-Lipschitz theorem C.1 (cf. Appendix C) is an example of such result.*

B.5 Schwartz space, Fréchet space

In functional analysis, Schwartz and Fréchet spaces are special topological vector spaces that are generalizations of Banach spaces.

Definition B.20 *The Schwartz space of the regular rapidly decreasing functions is defined as:*

$$S(\mathbb{R}^d, \mathbb{R}) = \{f \in C^0(\mathbb{R}^d, \mathbb{R}) \; / \; (1 + |x|^k)\frac{\partial^m f}{\partial x^m} \in C^0\,, \forall k \in \mathbb{N}\,, \forall m = (m_1, \ldots, m_d) \in \mathbb{N}^d\}\,.$$

It is a vector space and we introduce the *supremum seminorm*:

$$|f_{k,m}| = \sup_{x\in\mathbb{R}^d}\left((1 + |x|^k)\left|\frac{\partial^m f}{\partial x^m}\right|\right)\,.$$

Definition B.21 *A topological vector space E is a* Fréchet space *if*

(i) *it is complete as a uniform space,*

(ii) *its topology may be induced by a coutable family of seminorms $|\cdot|_i$, $i \in \mathbb{N}^*$ such that*

$$|x_i| = 0\,, \; \forall i \in \mathbb{N}^* \; \Leftrightarrow \; x = 0\,.$$

Fréchet spaces are endowed with a distance, for example:

$$d(f, g) = \sum_i 2^{-i}\frac{|f - g|_i}{1 + |f - g|_i}\,.$$

Trivially, every Banach space is a Fréchet space as the norm induces a translation invariant metric and the space is complete with respect to this metric.