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Research Article

Fixed Point Theorems in Quaternion-Valued Metric Spaces

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The aim of this paper is twofold. First, we introduce the concept of quaternion metric spaces which generalizes both real and complex metric spaces. Further, we establish some fixed point theorems in quaternion setting. Secondly, we prove a fixed point theorem in normal cone metric spaces for four self-maps satisfying a general contraction condition.

1. Introduction and Preliminaries

A metric space can be thought as very basic space having a geometry, with only a few axioms. In this paper we introduce the concept of quaternion metric spaces. The paper treats material concerning quaternion metric spaces that is important for the study of fixed point theory in Clifford analysis. We introduce the basic ideas of quaternion metric spaces and Cauchy sequences and discuss the completion of a quaternion metric space.

In what follows we will work on $\mathbb{H}$, the skew field of quaternions. This means we can write each element $x \in \mathbb{H}$ in the form $q = x_0 + x_1 i + x_2 j + x_3 k$, $x_n \in \mathbb{R}$, where $1, i, j, k$ are the basis elements of $\mathbb{H}$ and $n = 1, 2, 3$. For these elements we have the multiplication rules $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $kj = -jk = -i$, and $ki = -ik = j$. The conjugate element $\overline{x}$ is given by $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$. The quaternion modulus has the form of $|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$.

Quaternions can be defined in several different equivalent ways. Notice the noncommutative multiplication, their novel feature; otherwise, quaternion arithmetic has special properties. There is also more abstract possibilty of treating quaternions as simply quadruples of real numbers $[x_0, x_1, x_2, x_3]$, with operation of addition and multiplication suitably defined. The components naturally group into the imaginary part (x_1, x_2, x_3), for which we take this part as a vector and the purely real part, x_0, which called a scalar. Sometimes, we write a quaternion as $[v, x_0]$ with $v = (x_1, x_2, x_3)$.

Here, we give the following forms:

$$\begin{aligned} q &:= [v, x_0], \quad v \in \mathbb{R}^3;\ x_0 \in \mathbb{R} \\ &= [(x_1, x_2, x_3), x_0]; \quad x_0, x_1, x_2, x_3 \in \mathbb{R} \\ &= x_0 + x_1 i + x_2 j + x_3 k. \end{aligned} \tag{1}$$

Thus a quaternion q may be viewed as a four-dimensional vector (x_0, x_1, x_2, x_3).

For more information about quaternion analysis, we refer to [1–4] and others.

Define a partial order $\precsim$ on $\mathbb{H}$ as follows.

$q_1 \precsim q_2$ if and only if $\mathrm{Re}(q_1) \le \mathrm{Re}(q_2)$, $\mathrm{Im}_s(q_1) \le \mathrm{Im}_s(q_2)$, $q_1, q_2 \in \mathbb{H}$; $s = i, j, k$, where $\mathrm{Im}_i = x_1$; $\mathrm{Im}_j = x_2$; $\mathrm{Im}_k = x_3$. It follows that $q_1 \precsim q_2$, if one of the following conditions is satisfied.

(i) $\mathrm{Re}(q_1) = \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_1}(q_1) = \mathrm{Im}_{s_1}(q_2)$, where $s_1 = j, k$; $\mathrm{Im}_i(q_1) < \mathrm{Im}_i(q_2)$.

(ii) $\mathrm{Re}(q_1) = \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_2}(q_1) = \mathrm{Im}_{s_2}(q_2)$, where $s_2 = i, k$; $\mathrm{Im}_j(q_1) < \mathrm{Im}_j(q_2)$.

(iii) $\mathrm{Re}(q_1) = \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_3}(q_1) = \mathrm{Im}_{s_3}(q_2)$, where $s_3 = i, j$; $\mathrm{Im}_k(q_1) < \mathrm{Im}_k(q_2)$.

(iv) $\mathrm{Re}(q_1) = \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_1}(q_1) < \mathrm{Im}_{s_1}(q_2)$; $\mathrm{Im}_i(q_1) = \mathrm{Im}_i(q_2)$.

(v) $\mathrm{Re}(q_1) = \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_2}(q_1) < \mathrm{Im}_{s_2}(q_2)$; $\mathrm{Im}_j(q_1) = \mathrm{Im}_j(q_2)$.

(vi) $\mathrm{Re}(q_1) = \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_3}(q_1) < \mathrm{Im}_{s_3}(q_2)$; $\mathrm{Im}_k(q_1) = \mathrm{Im}_k(q_2)$.

(vii) $\mathrm{Re}(q_1) = \mathrm{Re}(q_2)$; $\mathrm{Im}_s(q_1) < \mathrm{Im}_s(q_2)$.

(viii) $\mathrm{Re}(q_1) < \mathrm{Re}(q_2)$; $\mathrm{Im}_s(q_1) = \mathrm{Im}_s(q_2)$.

(ix) $\mathrm{Re}(q_1) < \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_1}(q_1) = \mathrm{Im}_{s_1}(q_2)$; $\mathrm{Im}_i(q_1) < \mathrm{Im}_i(q_2)$.

(x) $\mathrm{Re}(q_1) < \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_2}(q_1) = \mathrm{Im}_{s_2}(q_2)$; $\mathrm{Im}_j(q_1) < \mathrm{Im}_j(q_2)$.

(xi) $\mathrm{Re}(q_1) < \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_3}(q_1) = \mathrm{Im}_{s_3}(q_2)$; $\mathrm{Im}_k(q_1) < \mathrm{Im}_k(q_2)$.

(xii) $\mathrm{Re}(q_1) < \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_1}(q_1) < \mathrm{Im}_{s_1}(q_2)$; $\mathrm{Im}_i(q_1) = \mathrm{Im}_i(q_2)$.

(xiii) $\mathrm{Re}(q_1) < \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_2}(q_1) < \mathrm{Im}_{s_2}(q_2)$; $\mathrm{Im}_j(q_1) = \mathrm{Im}_j(q_2)$.

(xiv) $\mathrm{Re}(q_1) < \mathrm{Re}(q_2)$; $\mathrm{Im}_{s_3}(q_1) < \mathrm{Im}_{s_3}(q_2)$; $\mathrm{Im}_k(q_1) = \mathrm{Im}_k(q_2)$.

(xv) $\mathrm{Re}(q_1) < \mathrm{Re}(q_2)$; $\mathrm{Im}_s(q_1) < \mathrm{Im}_s(q_2)$.

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Remark 1. In particular, we will write $q_1 \precnsim q_2$ if $q_1 \neq q_2$ and one from (i) to (xvi) is satisfied. Also, we will write $q_1 \prec q_2$ if only (xv) is satisfied. It should be remarked that

$$q_1 \precsim q_2 \Longrightarrow |q_1| \leq |q_2|. \tag{2}$$

Remark 2. The conditions from (i) to (xv) look strange but these conditions are natural generalizations to the corresponding conditions in the complex setting (see [5]). So, the number of these conditions is related to the number of units in the working space. For our quaternion setting we have four units (one real and three imaginary); then we have 2^4 conditions. But in the complex setting there were 2^2 conditions.

Azam et al. in [5] introduced the definition of the complex metric space as follows.

Definition 3 ([5]). Let X be a nonempty set and suppose that the mapping $d_{\mathbb{C}} : X \times X \to \mathbb{C}$ satisfies the following.

(d_1) $0 < d_{\mathbb{C}}(x, y)$, for all $x, y \in X$ and $d_{\mathbb{C}}(x, y) = 0$ if and only if $x = y$.

(d_2) $d_{\mathbb{C}}(x, y) = d_{\mathbb{C}}(y, x)$ for all $x, y \in X$.

(d_3) $d_{\mathbb{C}}(x, y) \precsim d_{\mathbb{C}}(x, z) + d_{\mathbb{C}}(z, y)$ for all $x, y, z \in X$.

Then $(X, d_{\mathbb{C}})$ is called a complex metric space.

Now, we extend the above definition to Clifford analysis.

Definition 4. Let X be a nonempty set. Suppose that the mapping $d_{\mathbb{H}} : X \times X \to \mathbb{H}$ satisfies

(1) $0 \precsim d_{\mathbb{H}}(x, y)$ for all $x, y \in X$ and $d_{\mathbb{H}}(x, y) = 0$ if and only if $x = y$,

(2) $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(y, x)$ for all $x, y \in X$,

(3) $d_{\mathbb{H}}(x, y) \precsim d_{\mathbb{H}}(x, z) + d_{\mathbb{H}}(z, y)$ for all $x, y, z \in X$.

Then $d_{\mathbb{H}}$ is called a quaternion valued metric on X, and $(X, d_{\mathbb{H}})$ is called a quaternion valued metric space.

Example 5. Let $d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ be a quaternion valued function defined as $d_{\mathbb{H}}(p, q) = |a_0 - b_0| + i|a_1 - b_1| + j|a_2 - b_2| + k|a_3 - b_3|$, where $p, q \in \mathbb{H}$ with

$$p = a_0 + a_1 i + a_2 j + a_3 k, \qquad q = b_0 + b_1 i + b_2 j + b_3 k; \\ a_s, b_s \in \mathbb{R}; \quad s = 1, 2, 3. \tag{3}$$

Then $(X, d_{\mathbb{H}})$ is a quaternion metric space.

Now, we give the following definitions.

Definition 6. Point $x \in X$ is said to be an interior point of set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{H}$ such that

$$B(x, r) = \{y \in X : d_{\mathbb{H}}(x, y) \prec r\} \subseteq A. \tag{4}$$

Definition 7. Point $x \in X$ is said to be a limit point of $A \subseteq X$ whenever for every $0 \prec r \in \mathbb{H}$

$$B(x, r) \cap (A - \{x\}) \neq \phi. \tag{5}$$

Definition 8. Set A is called an open set whenever each element of A is an interior point of A. Subset $B \subseteq X$ is called a closed set whenever each limit point of B belongs to B. The family

$$F = \{B(x, r) : x \in X,\ 0 \prec r\} \tag{6}$$

is a subbase for Hausdroff topology τ on X.

Definition 9. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $q \in \mathbb{H}$ with $0 \prec q$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_{\mathbb{H}}(x_n, x) \prec q$, then $\{x_n\}$ is said to be convergent if $\{x_n\}$ converges to the limit point x; that is, $x_n \to x$ as $n \to \infty$ or $\lim_n x_n = x$. If for every $q \in \mathbb{H}$ with $0 \prec q$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_{\mathbb{H}}(x_n, x_{n+m}) \prec q$, then $\{x_n\}$ is called Cauchy sequence in $(X, d_{\mathbb{H}})$. If every Cauchy sequence is convergent in $(X, d_{\mathbb{H}})$, then $(X, d_{\mathbb{H}})$ is called a complete quaternion valued metric space.

2. Convergence in Quaternion Metric Spaces

In this section we give some auxiliary lemmas using the concept of quaternion metric spaces; these lemmas will be used to prove some fixed point theorems of contractive mappings.

Lemma 10. *Let $(X, d_{\mathbb{H}})$ be a quaternion valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d_{\mathbb{H}}(x_n, x)| \to 0$ as $n \to \infty$.*

Proof. Suppose that $\{x_n\}$ converges to x. For a given real number $\varepsilon > 0$, let

$$q = \frac{\varepsilon}{2} + i\frac{\varepsilon}{2} + j\frac{\varepsilon}{2} + k\frac{\varepsilon}{2}. \tag{7}$$

Then $0 \prec q \in \mathbb{H}$ and there is a natural number N such that $d_{\mathbb{H}}(x_n, x) \prec q$ for all $n > N$.

Therefore, $|d_{\mathbb{H}}(x_n, x)| < |q| = \varepsilon$ for all $n > N$. Hence $|d_{\mathbb{H}}(x_n, x)| \to 0$ as $n \to \infty$.

Conversely, suppose that $|d_{\mathbb{H}}(x_n, x)| \to 0$ as $n \to \infty$. Then, given $q \in \mathbb{H}$ with $0 \prec q$, there exists a real number $\delta > 0$, such that, for $h \in \mathbb{H}$,

$$|h| < \delta \Longrightarrow h < q. \tag{8}$$

For this δ, there is a natural number N such that $|d_{\mathbb{H}}(x_n, x)| < \delta$ for all $n > N$. Implying that $d_{\mathbb{H}}(x_n, x) \prec q$ for all $n > N$, hence $\{x_n\}$ converges to x. □

Lemma 11. *Let $(X, d_{\mathbb{H}})$ be a quaternion valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d_{\mathbb{H}}(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Proof. Suppose that $\{x_n\}$ is a Cauchy sequence. For a given real number $\varepsilon > 0$, let

$$q = \frac{\varepsilon}{2} + i\frac{\varepsilon}{2} + j\frac{\varepsilon}{2} + k\frac{\varepsilon}{2}. \tag{9}$$

Then, $0 \prec q \in \mathbb{H}$ and there is a natural number $\mathbb{N}$ such that $d_{\mathbb{H}}(x_n, x_{n+m}) \prec q$ for all $n > N$. Therefore $|d_{\mathbb{H}}(x_n, x_{n+m})| < |q| = \varepsilon$ for all $n > N$. Hence $|d_{\mathbb{H}}(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

Conversely, suppose that $|d_{\mathbb{H}}(x_n, x_{n+m})| \to 0$ as $n \to \infty$. Then, given $q \in \mathbb{H}$ with $0 \prec q$, there exists a real number $\delta > 0$, such that for $h \in \mathbb{H}$, we have that

$$|h| < \delta \Longrightarrow h \prec q. \tag{10}$$

For this δ, there is a natural number N such that $|d_{\mathbb{H}}(x_n, x_{n+m})| < \delta$ for all $n > N$, which implies that $d_{\mathbb{H}}(x_n, x_{n+m}) \prec q$ for all $n > N$. Hence $\{x_n\}$ is a Cauchy sequence. This completes the proof of Lemma 11. □

Definition 12. Let $(X, d_{\mathbb{H}})$ be a complete quaternion valued metric space. For all $x, y \in X$, $d_{\mathbb{H}}(x, y)$ represents $\|x - y\|$. A quaternion valued metric space $(X, d_{\mathbb{H}})$ is said to be metrically convex if X has the property that, for each $x, y \in X$ with $x \neq y$, there exists $z \in X$, $x \neq y \neq z$ such that

$$\left|d_{\mathbb{H}}(x, z) + d_{\mathbb{H}}(z, y)\right| = \left|d_{\mathbb{H}}(x, y)\right|. \tag{11}$$

The following lemma finds immediate applications which is straightforward from [6].

Lemma 13. *Let $(X, d_{\mathbb{H}})$ be a metrically convex quaternion valued metric space and K a nonempty closed subset of X. If $x \in K$ and $y \in K$, then there exists a point $z \notin \partial K$ (where ∂K stands for the boundary of K) such that*

$$d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(x, z) + d_{\mathbb{H}}(z, y). \tag{12}$$

Definition 14. Let K be a nonempty subset of a quaternion valued metric space $(X, d_{\mathbb{H}})$ and let $F, T : K \to X$ satisfy the condition

$$\begin{aligned} &\varphi\left[\left|d_{\mathbb{H}}(Fx, Fy)\right|\right] \\ &\quad \leq b\left\{\varphi\left[\left|d_{\mathbb{H}}(Tx, Fx)\right|\right] + \varphi\left[\left|d_{\mathbb{H}}(Ty, Fy)\right|\right]\right\} \\ &\qquad + c\min\left\{\varphi\left[\left|d_{\mathbb{H}}(Tx, Fy)\right|\right], \varphi\left[\left|d_{\mathbb{H}}(Ty, Fx)\right|\right]\right\}. \end{aligned} \tag{13}$$

For all $x, y \in K$, with $x \neq y$, $b, c \geq 0$, $2b + c < 1$ and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing continuous function for which the following property holds:

$$\varphi(t) = 0 \Longleftrightarrow t = 0. \tag{14}$$

We call function F satisfying condition (13) generalized T-contractive.

Motivated by [7, 8], we construct the following definition.

Definition 15. Let K be a nonempty subset of a quaternion valued metric space $(X, d_{\mathbb{H}})$ and $F, T : K \to X$. The pair $\{F, T\}$ is said to be weakly commuting if, for each $x, y \in K$ such that $x = Fy$ and $Ty \in K$, we have

$$d_{\mathbb{H}}(Tx, FTy) \precsim d_{\mathbb{H}}(Ty, Fy). \tag{15}$$

It follows that

$$\left|d_{\mathbb{H}}(Tx, FTy)\right| \leq \left|d_{\mathbb{H}}(Ty, Fy)\right|. \tag{16}$$

Remark 16. It should be remarked that Definition 15 extends and generalizes the definition of weakly commuting mappings which are introduced in [7].

3. Common Fixed Point Theorems in Quaternion Analysis

In this section, we prove common fixed point theorems for two pairs of weakly commuting mappings on complete quaternion metric spaces. The obtained results will be proved using generalized contractive conditions.

Now, we give the following theorem.

Theorem 17. *Let $(X, d_{\mathbb{H}})$ be a complete quaternion valued metric space, K a nonempty closed subset of X, and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ an increasing continuous function satisfying* (13) *and* (14). *Let $F, T : K \to X$ be such that F is generalized T-contractive satisfying the conditions:*

(i) $\partial K \subseteq TK$, $\quad FK \subseteq TK$,

(ii) $Tx \in \partial K \Rightarrow Fx \in K$,

(iii) *F and T are weakly commuting mappings,*

(iv) *T is continuous at K,*

then there exists a unique common fixed point z in K, such that $z = Tz = Fz$.

Proof. We construct the sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Let $x \in \partial K$. Then there exists a point $x_0 \in K$ such that $x = Tx_0$ as $\partial K \subseteq TK$. From $Tx_0 \in \partial K$ and the implication $Tx \in \partial K \Rightarrow Fx \in K$, we conclude that $Fx_0 \in K \cap FK \subseteq TK$. Now, let $x_1 \in K$ be such that

$$y_1 = Tx_1 = Fx_0 \in K. \tag{17}$$

Let $y_2 = Fx_1$ and assume that $y_2 \in K$; then

$$y_2 \in K \cap FK \subseteq TK, \tag{18}$$

which implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Suppose $y_2 \notin K$; then there exists a point $p \in \partial K$ (using Lemma 13) such that

$$d_{\mathbb{H}}(Tx_1, p) + d_{\mathbb{H}}(p, y_2) = d_{\mathbb{H}}(Tx_1, y_2). \tag{19}$$

Since $p \in \partial K \subseteq TK$, there exists a point $x_2 \in K$ such that $p = Tx_2$ and so

$$d_{\mathbb{H}}(Tx_1, Tx_2) + d_{\mathbb{H}}(Tx_2, y_2) = d_{\mathbb{H}}(Tx_1, y_2). \tag{20}$$

Let $y_3 = Fx_2$. Thus, repeating the forgoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{n+1} = Fx_n$,

(ii) $y_n \in K \Rightarrow y_n = Tx_n$, or

(iii) $y_n \notin K \Rightarrow Tx_n \in \partial K$,

$$d_{\mathbb{H}}(Tx_{n-1}, Tx_n) + d_{\mathbb{H}}(Tx_n, y_n) = d_{\mathbb{H}}(Tx_{n-1}, y_n). \tag{21}$$

We denote

$$\begin{aligned} P &= \{Tx_i \in \{Tx_n\} : Tx_i = y_i\}, \\ Q &= \{Tx_i \in \{Tx_n\} : Tx_i \neq y_i\}. \end{aligned} \tag{22}$$

Obviously, the two consecutive terms of $\{Tx_n\}$ cannot lie in Q.

Let us denote $t_n = d_{\mathbb{H}}(Tx_n, Tx_{n+1})$. We have the following three cases.

Case 1. If $Tx_n, Tx_{n+1} \in P$, then

$$\begin{aligned} \varphi(|t_n|) &= \varphi[|d_{\mathbb{H}}(Tx_n, Tx_{n+1})|] \\ &= \varphi[|d_{\mathbb{H}}(Fx_{n-1}, Fx_n)|] \\ &\leq b\{\varphi[|d_H(Tx_{n-1}, Fx_{n-1})|] + \varphi[|d_{\mathbb{H}}(Tx_n, Fx_n)|]\} \\ &\quad + c\min\{\varphi[|d_{\mathbb{H}}(Tx_{n-1}, Fx_n)|], \\ &\qquad \varphi[|d_{\mathbb{H}}(Tx_n, Fx_{n-1})|]\} \\ &= b[\varphi(|t_{n-1}|) + \varphi(|t_n|)]. \end{aligned} \tag{23}$$

Thus,

$$\varphi(|t_n|) \leq \left(\frac{b}{1-b}\right)\varphi(|t_{n-1}|). \tag{24}$$

Case 2. If $Tx_n \in P$, $Tx_{n+1} \in Q$, note that

$$d_{\mathbb{H}}(Tx_n, Tx_{n+1}) + d_{\mathbb{H}}(Tx_{n+1}, y_{n+1}) = d_{\mathbb{H}}(Tx_n, y_{n+1}) \tag{25}$$

or

$$d_{\mathbb{H}}(Tx_n, Tx_{n+1}) \precsim d_{\mathbb{H}}(Tx_n, y_{n+1}) = d_{\mathbb{H}}(y_n, y_{n+1}), \tag{26}$$

which implies that

$$|d_{\mathbb{H}}(Tx_n, Tx_{n+1})| \leq |d_{\mathbb{H}}(y_n, y_{n+1})|. \tag{27}$$

Hence

$$\begin{aligned} \varphi(|t_n|) &\leq \varphi[|d_{\mathbb{H}}(y_n, y_{n+1})|] = \varphi[|d_{\mathbb{H}}(Fx_{n-1}, Fx_n)|] \\ &\leq b\{\varphi[|d_{\mathbb{H}}(Tx_{n-1}, Fx_{n-1})|] + \varphi[|d_{\mathbb{H}}(Tx_n, Fx_n)|]\} \\ &\quad + c\min\{\varphi[|d_{\mathbb{H}}(Tx_{n-1}, Fx_n)|], \\ &\qquad \varphi[|d_{\mathbb{H}}(Tx_n, Fx_{n-1})|]\} \\ &= b[\varphi(|t_{n-1}|) + \varphi(|d_{\mathbb{H}}(y_n, y_{n+1})|)]. \end{aligned} \tag{28}$$

Therefore,

$$\varphi(|d_{\mathbb{H}}(y_n, y_{n+1})|) \leq \left(\frac{b}{1-b}\right)\varphi(|t_{n-1}|). \tag{29}$$

Hence,

$$\varphi(|t_n|) \leq \varphi(|d_{\mathbb{H}}(y_n, y_{n+1})|) \leq \left(\frac{b}{1-b}\right)\varphi(|t_{n-1}|). \tag{30}$$

Case 3. If $x_n \in Q$, $Tx_{n+1} \in P$, so $Tx_{n-1} \in P$. Since Tx_n is a convex linear combination of Tx_{n-1} and y_n, it follows that

$$d_{\mathbb{H}}(Tx_n, Tx_{n+1}) \precsim \max\{d_{\mathbb{H}}(Tx_{n-1}, Tx_{n+1}), d_{\mathbb{H}}(y_n, Tx_{n+1})\}. \tag{31}$$

If $d_{\mathbb{H}}(Tx_{n-1}, Tx_{n+1}) \precsim d_{\mathbb{H}}(y_n, Tx_{n+1})$, then $d_{\mathbb{H}}(Tx_n, Tx_{n+1}) \precsim d_{\mathbb{H}}(y_n, Tx_{n+1})$ implying that

$$|d_{\mathbb{H}}(Tx_n, Tx_{n+1})| \leq |d_{\mathbb{H}}(y_n, Tx_{n+1})|. \tag{32}$$

Hence,

$$\begin{aligned} \varphi(|t_n|) &= \varphi(|d_{\mathbb{H}}(Tx_n, Tx_{n+1})|) \\ &\leq \varphi(|d_{\mathbb{H}}(y_n, Tx_{n+1})|) \\ &= \varphi(|d_{\mathbb{H}}(y_n, y_{n+1})|) \\ &= \varphi(|d_{\mathbb{H}}(Fx_{n-1}, Fx_n)|) \\ &\leq b\{\varphi[|d_{\mathbb{H}}(Tx_{n-1}, Fx_{n-1})|] \\ &\qquad + \varphi[|d_{\mathbb{H}}(Tx_n, Fx_n)|]\} \\ &\quad + c\min\{\varphi[|d_{\mathbb{H}}(Tx_{n-1}, Fx_n)|], \\ &\qquad \varphi[|d_{\mathbb{H}}(Tx_n, Fx_{n-1})|]\} \\ &= b[\varphi(|d_{\mathbb{H}}(Tx_{n-1}, y_n)|) + \varphi(|t_n|)] \\ &\quad + c\min\{\varphi[|d_{\mathbb{H}}(Tx_{n-1}, Tx_{n+1})|], \\ &\qquad \varphi[|d_{\mathbb{H}}(Tx_n, y_n)|]\}. \end{aligned} \tag{33}$$

It follows that

$$(1-b)\,\varphi\left(\left|t_n\right|\right) \le b\varphi\left(\left|d_{\mathbb{H}}\left(Tx_{n-1}, y_n\right)\right|\right) + c\varphi\left(\left|d_{\mathbb{H}}\left(Tx_n, y_n\right)\right|\right). \tag{34}$$

Since

$$d_{\mathbb{H}}\left(Tx_{n-1}, y_n\right) \succsim d_{\mathbb{H}}\left(Tx_n, y_n\right) \quad \text{as } Tx_n \in Q, \tag{35}$$

then

$$\varphi\left(\left|d_{\mathbb{H}}\left(Tx_{n-1}, y_n\right)\right|\right) \ge \varphi\left(d_{\mathbb{H}}\left(Tx_n, y_n\right)\right). \tag{36}$$

Therefore,

$$\begin{aligned}(1-b)\,\varphi\left(\left|t_n\right|\right) &\le (b+c)\,\varphi\left(\left|d_{\mathbb{H}}\left(Tx_{n-1}, y_n\right)\right|\right)\\ \Longrightarrow \varphi\left(\left|t_n\right|\right) &\le \left(\frac{b+c}{1-b}\right)\varphi\left(\left|d_{\mathbb{H}}\left(Tx_{n-1}, y_n\right)\right|\right).\end{aligned} \tag{37}$$

Now, proceeding as in Case 2 (because $Tx_{n-1} \in P$, $Tx_n \in Q$), we obtain that

$$\varphi\left(\left|t_n\right|\right) \le \left(\frac{b+c}{1-b}\right)\left(\frac{b}{1-b}\right)\varphi\left(\left|t_{n-2}\right|\right). \tag{38}$$

Also from (31), if $d_{\mathbb{H}}(y_n, Tx_{n+1}) \precsim d_{\mathbb{H}}(Tx_{n-1}, Tx_{n+1})$, then

$$d_{\mathbb{H}}\left(Tx_n, Tx_{n+1}\right) \precsim d_{\mathbb{H}}\left(Tx_{n-1}, Tx_{n+1}\right) \tag{39}$$

which implies that $|d_{\mathbb{H}}(Tx_n, Tx_{n+1})| \le |d_{\mathbb{H}}(Tx_{n-1}, Tx_{n+1})|$; hence

$$\begin{aligned}\varphi\left(\left|t_n\right|\right) &= \varphi\left(\left|d_{\mathbb{H}}\left(Tx_n, Tx_{n+1}\right)\right|\right)\\ &\le \varphi\left(\left|d_{\mathbb{H}}\left(Tx_{n-1}, Tx_{n+1}\right)\right|\right)\\ &= \varphi\left(\left|d_{\mathbb{H}}\left(Fx_{n-2}, Fx_n\right)\right|\right)\\ &\le b\left\{\varphi\left[\left|d_{\mathbb{H}}\left(Tx_{n-2}, Fx_{n-2}\right)\right|\right]\right.\\ &\qquad \left.+\varphi\left[\left|d_{\mathbb{H}}\left(Tx_n, Fx_n\right)\right|\right]\right\}\\ &\quad + c\min\left\{\varphi\left[\left|d_{\mathbb{H}}\left(Tx_{n-2}, Fx_n\right)\right|\right],\right.\\ &\qquad\qquad \left.\varphi\left[\left|d_{\mathbb{H}}\left(Tx_n, Fx_{n-2}\right)\right|\right]\right\}\\ &= b\left[\varphi\left(\left|t_{n-2}\right|\right) + \varphi\left(\left|t_n\right|\right)\right] + c\varphi\left(\left|t_{n-1}\right|\right).\end{aligned} \tag{40}$$

Therefore, noting that, by Case 2, $\varphi(|t_{n-1}|) < \varphi(|t_{n-2}|)$, we conclude that

$$\varphi\left(\left|t_n\right|\right) \le \left(\frac{b+c}{1-b}\right)\varphi\left(\left|t_{n-2}\right|\right). \tag{41}$$

Thus, in all cases we get either

$\varphi(|t_n|) \le ((b+c)/(1-b))\varphi(|t_{n-1}|)$
or $\varphi(|t_n|) \le ((b+c)/(1-b))\varphi(|t_{n-2}|)$;
for $n = 1$, we have $\varphi(|t_1|) \le ((b+c)/(1-b))\varphi(|t_0|)$;
for $n = 2$, we have $\varphi(|t_2|) \le ((b+c)/(1-b))\varphi(|t_1|) \le ((b+c)/(1-b))^2\varphi(|t_0|)$;
by induction, we get $\varphi(|t_n|) \le ((b+c)/(1-b))^n\varphi(|t_0|)$. Letting $n \to \infty$, we have $\varphi(|t_n|) \to 0$ and by (14), we have

$$\left|t_n\right| = \left|d_{\mathbb{H}}\left(Tx_n, Tx_{n+1}\right)\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{42}$$

so that $\{Tx_n\}$ is a Cauchy sequence and hence it converges to point z in K. Now there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that it is contained in P. Without loss of generality, we may denote $\{Tx_{n_k}\} = \{Tx_n\}$. Since T is continuous, $\{TTx_n\}$ converges to Tz.

We now show that T and F have common fixed point ($Tz = Fz$). Using the weak commutativity of T and F, we obtain that

$$Tx_n = Fx_{n-1}, \qquad Tx_{n-1} \in K; \tag{43}$$

then

$$d_{\mathbb{H}}\left(TTx_n, FTx_{n-1}\right) \precsim d_{\mathbb{H}}\left(Fx_{n-1}, Tx_{n-1}\right) = d_{\mathbb{H}}\left(Tx_n, Tx_{n-1}\right). \tag{44}$$

This implies that

$$\left|d_{\mathbb{H}}\left(TTx_n, FTx_{n-1}\right)\right| \le \left|d_H\left(Tx_n, Tx_{n-1}\right)\right|. \tag{45}$$

On letting $n \to \infty$, we obtain

$$d_{\mathbb{H}}\left(Tz, FTx_{n-1}\right) \longrightarrow 0, \tag{46}$$

which means that

$$\left\{FTx_{n-1}\right\} \longrightarrow Tz \quad \text{as } n \longrightarrow \infty. \tag{47}$$

Now, consider

$$\begin{aligned}&\varphi\left(\left|d_{\mathbb{H}}\left(FTx_{n-1}, Fz\right)\right|\right)\\ &\quad\le b\left\{\varphi\left[\left|d_{\mathbb{H}}\left(TTx_{n-1}, FTx_{n-1}\right)\right|\right] + \varphi\left[\left|d_{\mathbb{H}}\left(Tz, Fz\right)\right|\right]\right\}\\ &\qquad + c\min\left\{\varphi\left[\left|d_{\mathbb{H}}\left(TTx_{n-1}, Fz\right)\right|\right], \varphi\left[\left|d_{\mathbb{H}}\left(Tz, FTx_{n-1}\right)\right|\right]\right\}.\end{aligned} \tag{48}$$

Letting $n \to \infty$ yields

$$\varphi\left(\left|d_{\mathbb{H}}\left(Tz, Fz\right)\right|\right) \le b\varphi\left[\left|d_{\mathbb{H}}\left(Tz, Fz\right)\right|\right], \tag{49}$$

a contradiction, thus giving $\varphi(|d_{\mathbb{H}}(Tz, Fz)|) = 0$ which implies $|d_{\mathbb{H}}(Tz, Fz)| = 0$, so that $d_{\mathbb{H}}(Tz, Fz) = 0$ and hence $Tz = Fz$.

To show that $Tz = z$, consider

$$\begin{aligned}&\varphi\left(\left|d_{\mathbb{H}}\left(Tx_n, Tz\right)\right|\right)\\ &\quad= \varphi\left(\left|d_{\mathbb{H}}\left(Fx_{n-1}, Fz\right)\right|\right)\\ &\quad\le b\left\{\varphi\left[\left|d_{\mathbb{H}}\left(Tx_{n-1}, Fx_{n-1}\right)\right|\right] + \varphi\left[\left|d_{\mathbb{H}}\left(Tz, Fz\right)\right|\right]\right\}\\ &\qquad + c\min\left\{\varphi\left[\left|d_{\mathbb{H}}\left(Tx_{n-1}, Fz\right)\right|\right], \varphi\left[\left|d_{\mathbb{H}}\left(Tz, Fx_{n-1}\right)\right|\right]\right\}.\end{aligned} \tag{50}$$

Letting $n \to \infty$, we obtain

$$\varphi\left(\left|d_{\mathbb{H}}\left(z, Tz\right)\right|\right) \le c\varphi\left[\left|d_{\mathbb{H}}\left(z, Tz\right)\right|\right], \tag{51}$$

a contradiction, thereby giving $\varphi(|d_{\mathbb{H}}(z,Tz)|) = 0$ which implies $|d_{\mathbb{H}}(z,Tz)| = 0$, so that $d_{\mathbb{H}}(z,Tz) = 0$ and hence $z = Tz$. Thus, we have shown that $z = Tz = Fz$, so z is a common fixed point of F and T. To show that z is unique, let w be another fixed point of F and T; then

$$\begin{aligned}
&\varphi\left(\left|d_{\mathbb{H}}(w,z)\right|\right)\\
&\quad= \varphi\left(\left|d_{\mathbb{H}}(Fw,Fz)\right|\right)\\
&\quad\leq b\left\{\varphi\left[\left|d_{\mathbb{H}}(Tw,Fw)\right|\right] + \varphi\left[\left|d_{\mathbb{H}}(Tz,Fz)\right|\right]\right\}\\
&\qquad+ c\min\left\{\varphi\left[\left|d_{\mathbb{H}}(Tw,Fz)\right|\right], \varphi\left[\left|d_{\mathbb{H}}(Tz,Fw)\right|\right]\right\}\\
&\quad= c\varphi\left(\left|d_{\mathbb{H}}(w,z)\right|\right),
\end{aligned} \tag{52}$$

a contradiction, therefore giving $\varphi(|d_{\mathbb{H}}(w,z)|) = 0$ which implies that $|d_{\mathbb{H}}(w,z)| = 0$, so that $d_{\mathbb{H}}(w,z) = 0$; thus $w = z$. This completes the proof. □

Remark 18. If $\operatorname{Im}_s d_{\mathbb{H}}(x,y) = 0$, $s = i,j,k$ then $d_{\mathbb{H}}(x,y) = d(x,y)$ and therefore, we obtain the same results in [9]. So, our theorem is more general than Theorem 3.1 in [9] for a pair of weakly commuting mappings.

Using the concept of commuting mappings (see, e.g., [10]), we can give the following result.

Theorem 19. *Let $(X,d_{\mathbb{H}})$ be a complete quaternion valued metric space, K a nonempty closed subset of X, and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ an increasing continuous function satisfying* (13) *and* (14). *Let $F,T: K\to X$ be such that F is generalized T-contractive satisfying the conditions:*

(i) *$\partial K\subseteq TK$, $FK\subseteq TK$,*

(ii) *$Tx\in\partial K\Rightarrow Fx\in K$,*

(iii) *F and T are commuting mappings,*

(iv) *T is continuous at K;*

then there exists a unique common fixed point z in K such that $z = Tz = Fz$.

Proof. The proof is very similar to the proof of Theorem 17 with some simple modifications; so it will be omitted. □

Corollary 20. *Let $(X,d_{\mathbb{C}})$ be a complete complex valued metric space, K^* a nonempty closed subset of X, and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ an increasing continuous function satisfying* (14) *and*

$$\begin{aligned}
&\varphi\left[\left|d_{\mathbb{C}}(Fx,Fy)\right|\right]\\
&\quad\leq b\left\{\varphi\left[\left|d_{\mathbb{C}}(Tx,Fx)\right|\right] + \varphi\left[\left|d_{\mathbb{C}}(Ty,Fy)\right|\right]\right\}\\
&\qquad+ c\min\left\{\varphi\left[\left|d_{\mathbb{C}}(Tx,Fy)\right|\right], \varphi\left[\left|d_{\mathbb{C}}(Ty,Fx)\right|\right]\right\},
\end{aligned} \tag{53}$$

for all $x,y\in K^*$, with $x\neq y$; $b,c\geq 0$, $2b+c<1$.

Let $F,T: K^*\to X$ be such that F is generalized T-contractive satisfying the conditions:

(i) *$\partial K^*\subseteq TK^*$, $FK^*\subseteq TK^*$,*

(ii) *$Tx\in\partial K^*\Rightarrow Fx\in K^*$,*

(iii) *F and T are weakly mappings,*

(iv) *T is continuous at K^*;*

then there exists a unique common fixed point z in K^* such that $z = Tz = Fz$.

Proof. Since each element $x\in\mathbb{H}$ can be written in the form $q = x_0 + x_1 i + x_2 j + x_3 k$, $x_n\in\mathbb{R}$, where $1, i, j, k$ are the basis elements of $\mathbb{H}$ and $n = 1,2,3$. Putting $x_2 = x_3 = 0$, we obtain an element in $\mathbb{C}$. So, the proof can be obtained from Theorem 17 directly. □

4. Fixed Points in Normal Cone Metric Spaces

In this section, we prove a fixed point theorem in normal cone metric spaces, including results which generalize a result due to Huang and Zhang in [11] as well as a result due to Abbas and Rhoades [12]. The obtained result gives a fixed point theorem for four mappings without appealing to commutativity conditions, defined on a cone metric space.

Let E be a real Banach space. A subset P of E is called a cone if and only if

(a) P is closed and nonempty and $P\neq 0$;

(b) $a,b\in\mathbb{R}$, $a,b\geq 0$, $a,b\in P$ implies that $ax+by\in P$;

(c) $P\bigcap(-P) = \{0\}$.

Given cone $P\subset E$, we define a partial ordering $\leq$ with respect to P by $x\leq y$ if and only if $y-x\in P$. Cone P is called normal if there is a number $K_1>0$ such that, for all $x,y\in E$,

$$0\leq x\leq y \text{ implies } \|x\|\leq K_1\|y\|. \tag{54}$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x\ll y$ stands for $y-x\in\operatorname{int}P$ (interior of P). We will write $x<y$ to indicate that $x\ll y$ but $x\neq y$.

Definition 21 (see [11]). Let X be a nonempty set. Suppose that the mapping $d: X\times X\to E$ satisfies

(d_1) $0\leq d(x,y)$ for all $x,y\in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d_2) $d(x,y) = d(y,x)$ for all $x,y\in X$;

(d_3) $d(x,y)\leq d(x,z)+d(z,y)$ for all $x,y,z\in X$.

Then, d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 22 (see [11]). Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X, and $x\in X$. For every $c\in E$ with $0\ll c$, we say that $\{x_n\}$ is

(1) a Cauchy sequence if there is an N such that, for all $n,m>N$, $d(x_n,x_m)\ll c$;

(2) a convergent sequence if there is an N such that, for all $n>N$, $d(x_n,x)\ll c$; for some $x\in X$, $d(x,y)\leq d(x,z)+d(z,y)$ for all $x,y,z\in X$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Now, we give the following result.

Theorem 23. *Let (X,d) be a complete cone metric space and P a normal cone with normal constant K_1. Suppose that the mappings f, g, f_1, g_1 are four self-maps of X satisfying*

$$\begin{aligned}
&\lambda d(fx,gy)+(1-\lambda)d(f_1x,g_1y)\\
&\quad\leq \alpha d(x,y)+\beta[\lambda(d(x,fx)+d(y,gy))\\
&\qquad\qquad+(1-\lambda)(d(x,f_1x)+d(y,g_1y))]\\
&\qquad+\gamma[\lambda(d(x,gy)+d(y,fx))\\
&\qquad\qquad+(1-\lambda)(d(x,g_1x)+d(y,f_1x))]
\end{aligned}\tag{55}$$

for all $x,y\in X$, where $0\leq\lambda\leq 1$ and $\alpha,\beta,\gamma\geq 0$ with $\alpha+2\beta+2\gamma<1$. Then, f, g, f_1, and g_1 have a unique common fixed point in X.

Proof. If $\lambda=0$ or $\lambda=1$, the proof is already known from [12]. So, we consider the case when $0<\lambda<1$. Suppose x_0 is an arbitrary point of X, and define $\{x_n\}$ by $x_{2n+1}=fx_{2n}=f_1x_{2n}$ and $x_{2n+2}=gx_{2n+1}=g_1x_{2n+1}$; $n=0,1,2,\ldots$. Then, we have

$$\begin{aligned}
&d(x_{2n+1},x_{2n+2})\\
&\quad=d(fx_{2n},gx_{2n+1})\\
&\quad=\lambda d(fx_{2n},gx_{2n+1})+(1-\lambda)d(fx_{2n},gx_{2n+1})\\
&\quad=\lambda d(fx_{2n},gx_{2n+1})+(1-\lambda)d(f_1x_{2n},g_1x_{2n+1})\\
&\quad\leq \alpha d(x_{2n},x_{2n+1})\\
&\qquad+\beta[\lambda(d(x_{2n},fx_{2n})+d(x_{2n+1},gx_{2n+1}))\\
&\qquad\qquad+(1-\lambda)(d(x_{2n},f_1x_{2n})+d(x_{2n+1},g_1x_{2n+1}))]\\
&\qquad+\gamma[\lambda(d(x_{2n},gx_{2n+1})+d(x_{2n+1},fx_{2n}))\\
&\qquad\qquad+(1-\lambda)(d(x_{2n},g_1x_{2n})+d(x_{2n+1},f_1x_{2n}))]\\
&\quad=(\alpha+\beta+\gamma)d(x_{2n},x_{2n+1})+(\beta+\gamma)d(x_{2n+1},x_{2n+2});
\end{aligned}\tag{56}$$

this yields that

$$d(x_{2n+1},x_{2n+2})\leq\delta d(x_{2n},x_{2n+1}),\quad \text{where } \delta=\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}<1.\tag{57}$$

Therefore, for all n, we deduce that

$$d(x_{n+1},x_{n+2})\leq\delta d(x_n,x_{n+1})\leq\cdots\leq\delta^{n+1}d(x_0,x_1).\tag{58}$$

Now, for $m>n$, we obtain that

$$d(x_m,x_n)\leq\frac{\delta^n}{1-\delta}d(x_0,x_1).\tag{59}$$

From (54), we have

$$\|d(x_m,x_n)\|\leq\frac{K_1\delta^n}{1-\delta}\|d(x_0,x_1)\|,\tag{60}$$

which implies that $d(x_m,x_n)\to 0$ as $n,m\to\infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete there exists a point $p\in X$ such that $x_n\to p$ as $n\to\infty$. Now using (55), we obtain that

$$\begin{aligned}
&\lambda d(p,gp)+(1-\lambda)d(p,g_1p)\\
&\quad\leq\lambda[d(p,x_{2n+1})+d(x_{2n+1},gp)]\\
&\qquad+(1-\lambda)[d(p,x_{2n+1})+d(x_{2n+1},g_1p)]\\
&\quad\leq d(p,x_{2n+1})+\alpha d(x_{2n},p)\\
&\qquad+\beta[d(x_{2n},x_{2n+1})+\lambda\, d(p,gp)+(1-\lambda)d(p,g_1p)]\\
&\qquad+\gamma[d(x_{2n},p)+\lambda\, d(p,gp)\\
&\qquad\qquad+(1-\lambda)d(p,g_1p)+d(p,x_{2n+1})].
\end{aligned}\tag{61}$$

Now, using (54) and (61), we obtain that

$$\begin{aligned}
&|\lambda|\,\|d(p,gp)\|\\
&\quad\leq\|\lambda d(p,gp)+(1-\lambda)d(p,g_1p)\|\\
&\quad\leq\frac{K_1}{1-\beta-\gamma}\{\|d(p,x_{n+1})\|+\alpha\|d(x_{n+1},p)\|\\
&\qquad\qquad+\beta\|d(x_n,x_{n+1})\|\\
&\qquad\qquad+\gamma\|d(x_n,p)\|+\gamma\|d(p,x_{n+1})\|\}.
\end{aligned}\tag{62}$$

Therefore

$$\begin{aligned}
&\|d(p,gp)\|\\
&\quad\leq(K_1\{\|d(p,x_{n+1})\|+\alpha\|d(x_{n+1},p)\|+\beta\|d(x_n,x_{n+1})\|\\
&\qquad\qquad+\gamma\|d(x_n,p)\|+\gamma\|d(p,x_{n+1})\|\})\\
&\qquad\times(|\lambda|(1-\beta-\gamma))^{-1}.
\end{aligned}\tag{63}$$

Since the right hand side of the above inequality approaches zero as $n\to 0$, hence $\|d(p,gp)\|=0$, and then $p=gp$. Also, we have

$$\begin{aligned}
&|1-\lambda|\,\|d(p,g_1p)\|\\
&\quad\leq\|\lambda d(p,gp)+(1-\lambda)d(p,g_1p)\|\\
&\quad\leq\frac{K_1}{1-\beta-\gamma}\{\|d(p,x_{n+1})\|+\alpha\|d(x_{n+1},p)\|\\
&\qquad\qquad+\beta\|d(x_n,x_{n+1})\|+\gamma\|d(x_n,p)\|\\
&\qquad\qquad+\gamma\|d(p,x_{n+1})\|\},
\end{aligned}\tag{64}$$

which implies that

$$\begin{aligned}
&\|d(p, g_1 p)\| \\
&\quad \leq (K_1 \{\|d(p, x_{n+1})\| + \alpha \|d(x_{n+1}, p)\| + \beta \|d(x_n, x_{n+1})\| \\
&\qquad + \gamma \|d(x_n, p)\| + \gamma \|d(p, x_{n+1})\|\}) \\
&\qquad \times (|1-\lambda|(1-\beta-\gamma))^{-1}.
\end{aligned} \tag{65}$$

Letting $n \to 0$, we deduce that $\|d(p, g_1 p)\| = 0$, and then $p = g_1 p$. Similarly, by replacing g by f and g_1 by f_1 in (61), we deduce that

$$\begin{aligned}
&\|d(p, fp)\| \\
&\quad \leq (K_1 \{\|d(p, x_{n+1})\| + \alpha \|d(x_{n+1}, p)\| + \beta \|d(x_n, x_{n+1})\| \\
&\qquad + \gamma \|d(x_n, p)\| + \gamma \|d(p, x_{n+1})\|\}) \\
&\qquad \times (|\lambda|(1-\beta-\gamma))^{-1},
\end{aligned} \tag{66}$$

where $\|d(p, fp)\| = 0$, as $n \to 0$. Hence $\|d(p, fp)\| = 0$, and then $p = fp$. Also,

$$\begin{aligned}
&\|d(p, f_1 p)\| \\
&\quad \leq (K_1 \{\|d(p, x_{n+1})\| + \alpha \|d(x_{n+1}, p)\| + \beta \|d(x_n, x_{n+1})\| \\
&\qquad + \gamma \|d(x_n, p)\| + \gamma \|d(p, x_{n+1})\|\}) \\
&\qquad \times (|1-\lambda|(1-\beta-\gamma))^{-1}.
\end{aligned} \tag{67}$$

Letting $n \to 0$, we deduce that $\|d(p, f_1 p)\| = 0$, and then $p = f_1 p$.

To prove uniqueness, suppose that q is another fixed point of f, g, g_1, and f_1, and then

$$\begin{aligned}
d(p,q) &= \lambda d(p,q) + (1-\lambda) d(p,q) \\
&= \lambda d(fp, gq) + (1-\lambda) d(f_1 p, g_1 q) \\
&\leq \alpha d(p,q) \\
&\quad + \beta [\lambda d(p, fp) + (1-\lambda) d(p, f_1 p) \\
&\qquad + \lambda d(q, gq) + (1-\lambda) d(q, g_1 q)] \\
&\quad + \gamma [\lambda d(p, gq) + (1-\lambda) d(p, g_1 q) \\
&\qquad + \lambda d(q, fp) + (1-\lambda) d(q, f_1 p)] \\
&= (\alpha + 2\gamma) d(p,q),
\end{aligned} \tag{68}$$

which gives $d(p,q) = 0$, and hence $p = q$. This completes the proof. □

Now, we give the following result.

Corollary 24. *Let (X, d) be a complete cone metric space and P a normal cone with normal constant K_1. Suppose that the mappings f, g, f_1, g_1 are four self-maps of X satisfying*

$$\begin{aligned}
&\lambda d(f^m x, g^n y) + (1-\lambda) d(f_1^m x, g_1^n y) \\
&\quad \leq \alpha d(x, y) + \beta [\lambda (d(x, f^m x) + d(y, g^n y)) + (1-\lambda) \\
&\qquad \times (d(x, f_1^m x) + d(y, g_1^n y))] \\
&\quad + \gamma [\lambda (d(x, g^n y) + d(y, f^m x)) \\
&\qquad + (1-\lambda)(d(x, g_1^n x) + d(y, f_1^m x))]
\end{aligned} \tag{69}$$

for all $x, y \in X$, where $0 \leq \lambda \leq 1$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$. Then, f, g, f_1, and g_1 have a unique common fixed point in X; $m, n > 1$.

Proof. Inequality (69) is obtained from (55) by setting $f \equiv f^m$, $f_1 \equiv f_1^m$, and $g \equiv g^m$, $g_1 \equiv g_1^m$. Then the result follows from Theorem 23. □

Remark 25. If we put $\lambda = 0$ or $\lambda = 1$ in Theorem 23, we obtain Theorem 2.1 in [12].

Remark 26. It should be remarked that the quaternion metric space is different from cone metric space which is introduced in [11] (see also [12–19]). The elements in $\mathbb{R}$ or $\mathbb{C}$ form an algebra while the same is not true in $\mathbb{H}$.

Open Problem. It is still an open problem to extend our results in this paper for some kinds of mappings like biased, weakly biased, weakly compatible mappings, compatible mappings of type (T), and m-weak** commuting mappings. See [20–30] for more studies on these mentioned mappings.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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