Uniform Convergence for Sequences of Iterates

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Iterates

Definition

Let $f : X \to X$; define the n^{th} iterate of f as

$$f^n := \overbrace{f \circ f \circ ... \circ f}^{n}$$

Example:

$$\text{If } f(z) = z^2\text{, then } f^3(z) = z^2 \circ z^2 \circ z^2 = (((z^2)^2)^2 = z^8$$

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Definition

We say $x \in X$ is a fixed point of $f\colon X \to X$ if $f(x) = x$.

Example:

If $f(z) = z^2$ then $f(0) = 0$. Therefore 0 is a fixed point.

Old Theorems

Denjoy-Wolff Theorem

If $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic but neither the identity nor a rotation, then there exists a unique point $a \in \overline{\mathbb{D}}$ so that φ^n converges to the constant function a uniformly on compact subsets of $\mathbb{D}$.

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We call a the Denjoy-Wolff point:

- If $a \in \mathbb{D}$, then $\varphi(a) = a$, and f has no other fixed points;
- If $a \in \partial\overline{\mathbb{D}}$, then $\lim_{r \to 1^-} \varphi(ra) = a$.

New Theorems

Theorem 1 (Cowen, Ko, Thompson, Tiang 2014)

Suppose $\varphi\colon \overline{\mathbb{D}} \to \mathbb{D}$ is analytic on $\mathbb{D}$ and continuous on the boundary, $\partial\mathbb{D}$. If the Denjoy-Wolff point $a \in \mathbb{D}$, then $\varphi^n \to a$ uniformly *on all of* $\mathbb{D}$ if and only if there is $N > 0$ such that $\varphi^N(\overline{\mathbb{D}}) \subseteq \mathbb{D}$.

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Theorem 2 (Cowen, Ko, Thompson, Tiang 2014)

Suppose $\varphi\colon \overline{\mathbb{D}} \to \mathbb{D}$ is analytic on $\mathbb{D}$ and continuous on the boundary, $\partial\mathbb{D}$, and has Denjoy-Wolff point a with $|a| = 1$ and $\varphi'(a) < 1$. If $\varphi^N(\mathbb{D}) \subseteq \mathbb{D} \cup \{a\}$ for some $N > 0$, then $\varphi^n \to a$ uniformly on all of $\mathbb{D}$.

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Definition

We say UCI (*uniformly convergent iterates*) holds for $f\colon X \to X$ if f^n *converges uniformly on all of X to a constant a*.

An Example

$\varphi(z) = \frac{1}{2}z + \frac{1}{2}$ has fixed point at 1 with $\varphi'(1) < 1$. Has UCI.

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Generalized Example

If φ is a linear fractional map with DW point $|a| = 1$ and $\varphi'(a) = 1$ but not an automorphism, then UCI.

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An Annoying Example

$\varphi(z) = \frac{(z+1)k+(z-1)}{(z+1)k-(z-1)}$ for $k > 1$ has $\varphi(1) = 1$ and $\varphi(-1) = -1$. NO UCI.

More New Theorems

Theorem 3 (Cowen, Ko, Thompson, Tiang 2014)

If

- $\varphi\colon \overline{\mathbb{D}} \to \mathbb{D}$ is analytic on $\mathbb{D}$,
- φ is continuous on $\partial\mathbb{D}$, and
- $\varphi^n(z) \to a$ uniformly on $\mathbb{D}$ and $|a| = 1$,

then a is the only fixed point of φ on the closed disk.

Theorem 4 (Glickfield, Kaschner 2018)

If

- $\varphi\colon \overline{\mathbb{D}} \to \mathbb{D}$ is analytic on $\mathbb{D}$ and
- $\varphi^n(z) \to a$ uniformly on $\mathbb{D}$ and $|a| = 1$,

then a is the only fixed point of φ on the closed unit disk.

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- bi-complex bi-disk and unit ball
- $(-1, 1)$
- $(-1, 1)^2$ (real bi-disk or unit square)
- $\{\vec{z} \in \mathbb{C}^2 \colon \|\vec{z}\| < 1\}$ (real unit ball)