1 Preliminaries

In this research, we'll investigate questions that may go something like this:

- When does UCI (Uniform Convergence of Iterates) hold?
- When is the fixed point unique?
- What happens when we look at different domains including but not limited to
  - $\mathbb{D}^2$ (bi-disk)
  - $\{\vec{z} \in \mathbb{C}^2 : \|\vec{z} < 1\|\}$ (unit ball)
  - bicomplex bi-disk and unit ball
  - $(-1, 1)$
  - $(-1, 1)^2$ (real bi-disk or unit square)
  - real unit ball

What interesting questions! But it might be nice to have some tools and motivations to find their answers. Let's look at some.

1.1 Relevant Theorems and Previous Work

1.1.1 Old Stuff

First, a wall of definitions:

Definition 1.1. *Let $f : X \to X$; define the n^{th} iterate of f as*

$$f^n := \overbrace{f \circ f \circ ... \circ f}^{n}$$

Definition 1.2. *We say $x \in X$ is a fixed point of $f\colon X \to X$ if $f(x) = x$.*

Definition 1.3. *We say UCI (*uniformly convergent iterates*) holds for $f\colon X \to X$ if f^n converges uniformly on all of X to a constant a.*

Amazing! Now, a (very important) theorem:

Theorem 1.4. ***The Denjoy-Wolff Theorem:*** *If $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic but neither the identity nor a rotation, then there exists a unique point $a \in \overline{\mathbb{D}}$ so that φ^n converges to the constant function a uniformly on compact subsets of $\mathbb{D}$.*

We call a the Denjoy-Wolff point:

- If $a \in \mathbb{D}$, then $\varphi(a) = a$, and f has no other fixed points;
- If $a \in \partial\overline{\mathbb{D}}$, then $\lim\limits_{r \to 1^-} \varphi(ra) = a$.

1.1.2 New Stuff

For some crazy reason, the current body of work in this area is quite limited. So limited that [1] and some research done by Kaschner and Glickfield at the MRC are basically all we have to work off of (or so we think). Let's throw that information here.

Theorem 1.5. *Theorem 1 (Cowen, Ko, Thompson, Tiang 2014) Suppose $\varphi\colon \overline{\mathbb{D}} \to \mathbb{D}$ is analytic on $\mathbb{D}$ and continuous on the boundary, $\partial\mathbb{D}$. If the Denjoy-Wolff point $a \in \mathbb{D}$, then $\varphi^n \to a$ uniformly* on all of $\mathbb{D}$ *if and only if there is $N > 0$ such that $\varphi^N(\overline{\mathbb{D}}) \subseteq \mathbb{D}$.*

Theorem 1.6. *Theorem 2 (Cowen, Ko, Thompson, Tiang 2014) Suppose $\varphi\colon \overline{\mathbb{D}} \to \mathbb{D}$ is analytic on $\mathbb{D}$ and continuous on the boundary, $\partial\mathbb{D}$, and has Denjoy-Wolff point a with $|a| = 1$ and $\varphi'(a) < 1$. If $\varphi^N(\mathbb{D}) \subseteq \mathbb{D} \cup \{a\}$ for some $N > 0$, then $\varphi^n \to a$ uniformly on all of $\mathbb{D}$.*

1.2 Examples??

2 Bicomplex Numbers

Insert some fun stuff on bicomplex numbers here

3 Attempt at proving the metric space part of Dr. T's paper

Before I attempt to prove the thing, here is/are some notation/definitions:

- $\boldsymbol{e}_1 = \dfrac{1+ij}{2}$, $\boldsymbol{e}_2 = \dfrac{1-ij}{2}$
- $\mathbb{H} = \{x + ijy \ : \ x, y \in \mathbb{R}\}$
  - $\mathbb{H} \subset \mathbb{B}$
- $\mathbb{H}^+ = \{x + ijy \ : \ x \geq 0, |y| \leq x\} = \{(x+y)\boldsymbol{e}_1 + (x-y)\boldsymbol{e}_2 \ : \ x \geq 0, |y| \leq x\}$
- *I am declaring this to be my bicomplex unit-disky thing...may not be the best notation or the best definition* $\mathbb{D}^+ = \{x + ijy \ : \ x \geq 0, |y| \leq x, x^2 + y^2 \leq 1\}$
- The partial order: Let $\zeta = (x_1 + y_1)\boldsymbol{e}_1 + (x_1 - y_1)\boldsymbol{e}_2$ and $\omega = (x_2 + y_2)\boldsymbol{e}_1 + (x_2 - y_2)\boldsymbol{e}_2$. Then, $\zeta \prec \omega$ when $x_1 + y_1 < x_2 + y_2$ and $x_1 - y_1 < x_2 - y_2$.

Theorem 3.1. *Suppose $\varphi : \mathbb{D}^+ \to \mathbb{D}^+$ is analytic and continuous on $\partial\mathbb{D}^+$. Suppose there is a Denjoy-Wolff Theorem for $\mathbb{B}$ and that there exists a Denjoy-Wolff point a of φ in $\mathbb{D}^+$. If $\varphi_n \to a$ uniformly, then there exists an $N > 0$ such that $\varphi_N(\overline{\mathbb{D}}^+) \subseteq \mathbb{D}^+$.*

Proof. *This first bit is the bit I am most sketched out about* Let M be the minimum distance between a and $\partial\mathbb{D}^+$. Since $\varphi_n \to a$ uniformly on $\mathbb{D}^+$, we know that for all $\zeta \in \mathbb{D}^+$, for some $\varepsilon \in \mathbb{H}^+$ with $\varepsilon \succ 0$, there exists some $N > 0$ such that for all $\zeta \in \mathbb{D}^+$, $|\varphi_n(\zeta) - a| \prec \varepsilon$ for all $n \geq N$.

Let $M \in \mathbb{H}^+$ and $\varepsilon = \dfrac{M}{2}$. Also let $b_1, b_2 \in \partial\mathbb{D}^+$ where $|b_1| = |b_2| = 1$ and suppose $\varphi_N(b_1) = b_2$. This implies that for all $\epsilon \succ 0$, there exists $\delta \succ 0$ such that $|b_1 - \zeta| \prec \delta$ implies that $|b_2 - \varphi_n(\zeta)| \prec \varepsilon$. Let $\epsilon = \varepsilon$. This gives the following:

$$\begin{aligned} M \preceq |b_2 - a| &= |b_2 - \varphi_N(\zeta) + \varphi_N(\zeta) - a| \\ &\preceq |b_2 - \varphi_N(\zeta)| + |\varphi_N(\zeta) - a| \\ &\prec \varepsilon + \varepsilon = M. \end{aligned}$$

This is a lie, so $\varphi_N(\overline{\mathbb{D}}^+) \subseteq \mathbb{D}^+$.

■

References

[1] Carl C. Cowen, Eungil Ko, Derek Thompson, and Feng Tian. Spectra of some weighted composition operators onh2. *Acta Scientiarum Mathematicarum*, 82(12):221–234, 2016.