2. The Classification Theorem

Having ruled out the possibility of wandering domains, we now know that every component of the Fatou set is periodic or preperiodic. Our next goal is to classify periodic components. We have already seen some examples: domains containing an attracting fixed point, and Siegel disks. There are various other possibilities.

A periodic component U of period n of the Fatou set $\mathcal{F}$ is called *parabolic* if there is on its boundary a neutral fixed point ζ for R^n with multiplier 1, such that all points in U converge to ζ under iteration by R^n. The domains $U, R(U), ..., R^{n-1}(U)$ form a *parabolic cycle*. Their union is the immediate basin of attraction associated with an attracting petal at ζ.

A periodic component U of period n of the Fatou set is called a *Herman ring* (or an *Arnold ring*) if it is doubly connected and R^n is conjugate to either a rotation on an annulus or to a rotation followed by an inversion. We shall see in Chapter VI that Herman rings can occur and that moreover there can be only finitely many of them. The definition of Siegel disk is similarly extended to include periodic components U that are simply connected, on which some R^n is conjugate to a rotation. Siegel disks and Herman rings are referred to as *rotation domains.*

THEOREM 2.1. *Suppose U is a periodic component of the Fatou set $\mathcal{F}$. Then exactly one of the following holds:*

1. *U contains an attracting periodic point.*
2. *U is parabolic.*
3. *U is a Siegel disk.*
4. *U is a Herman ring.*

Proof. We may assume U is fixed by R. Since $\mathcal{J}$ has more than two points, U is hyperbolic. The proof will be organized as a series of lemmas in the context of analytic maps of hyperbolic domains. Let $\rho = \rho_U$ denote the hyperbolic metric on U. The first alternative of Theorem 2.1 is covered by the following lemma.

LEMMA 2.2. *Suppose U is hyperbolic, $f : U \to U$ is analytic, and f is not an isometry with respect to the hyperbolic metric. Then either $f^n(z) \to \partial U$ for all $z \in U$, or else there is an attracting fixed point for f in U to which all orbits converge.*

Proof. Since f is not an isometry, $\rho(f(z), f(w)) < \rho(z, w)$ for all $z, w \in U$. In particular for any compact set $K \subset U$, there is a constant $k = k(K) < 1$ such that

$$\rho(f(z), f(w)) \leq k\rho(z, w), \qquad z, w \in K.$$

Suppose there is $z_0 \in U$ whose iterates $z_n = f^n(z_0)$ visit some compact subset L of U infinitely often. Take K to be a compact neighborhood of $L \cup f(L)$. Then $\rho(z_{m+2}, z_{m+1}) \leq k\rho(z_{m+1}, z_m)$ whenever $z_m \in L$, and this occurs infinitely often, so $\rho(z_{n+1}, z_n) \to 0$. Thus by continuity any cluster point $\zeta \in L$ of the sequence $\{z_n\}$ is fixed by f, and in fact is an attracting fixed point since $\rho(f(z), \zeta) \leq k\rho(z, \zeta)$ in some neighborhood of ζ. Since the iterates of f form a normal family, they converge on U to ζ. □

The third and fourth alternatives of Theorem 2.1 are covered by the following.

LEMMA 2.3. *Suppose U is hyperbolic, $f : U \to U$ is analytic, and f is an isometry with respect to the hyperbolic metric. Then exactly one of the following holds:*

1. *$f^n(z) \to \partial U$ for all $z \in U$.*
2. *$f^m(z) = z$ for all $z \in U$ and some fixed $m \geq 1$.*
3. *U is conformally a disk, and f is conjugate to an irrational rotation.*
4. *U is conformally an annulus, and f is conjugate to an irrational rotation or to a reflection followed by an irrational rotation.*
5. *U is conformally a punctured disk, and f is conjugate to an irrational rotation.*

Proof. By an isometry, we mean at the local level, so that the lift of f to the universal covering space Δ is a hyperbolic isometry.

Suppose first that U is simply connected. Let φ map U conformally onto the open unit disk Δ. Then $S = \varphi \circ f \circ \varphi^{-1}$ is a conformal self-map of Δ, a Möbius transformation. If S has fixed points on the unit circle, then $|S^n| \to 1$ on Δ, and (1) holds. If S has a fixed point in the disk, we may assume it is at the origin. Then S is a rotation, and either (2) or (3) holds.

Now assume U is not simply connected. Let $\psi : \Delta \to U$ be the universal covering map, and let $\mathcal{G}$ be the associated group of covering transformations, the group of conformal self-maps g of Δ satisfying $\psi \circ g = \psi$. The lift of f to the unit disk via ψ is a Möbius transformation F, which satisfies $\psi \circ F = f \circ \psi$. Let Γ be the group obtained by adjoining F to $\mathcal{G}$.

Assume first Γ discrete (orbits accumulate only on $\partial\Delta$). Since no iterate f^k of f is the identity on U, no iterate F^k of F belongs to $\mathcal{G}$. Since Γ is discrete, this implies $gF^k(0) \to \partial\Delta$ uniformly in $g \in \mathcal{G}$. Hence $f^k(z_0) \to \partial U$, and (1) holds.

The final possibility is that Γ is not discrete. Let $\overline{\Gamma}$ be the closure of Γ in the (Lie) group of conformal self-maps of Δ, and let Γ_0 be the connected component of $\overline{\Gamma}$ containing the identity. If $g \in \mathcal{G}$ then also $FgF^{-1} \in \mathcal{G}$, since

$$\psi \circ (F \circ g \circ F^{-1}) = f \circ \psi \circ g \circ F^{-1} = f \circ \psi \circ F^{-1} = \psi \circ F \circ F^{-1} = \psi.$$

It follows that $\overline{\Gamma}$, and hence Γ_0, also conjugates $\mathcal{G}$ to itself. Since $\mathcal{G}$ is discrete and Γ_0 is connected, $hgh^{-1} = g$ for all $h \in \Gamma_0$ and $g \in \mathcal{G}$, and every $g \in \mathcal{G}$ commutes with every $h \in \Gamma_0$.

LEMMA. *If A and B are two conformal self-maps of the open unit disk Δ which commute, and A is not the identity, then B belongs to the one-parameter subgroup generated by A.*

Proof. There are three cases:

CASE 1. Suppose A has a fixed point in Δ. We may assume the fixed point is 0, so that $A(z) = e^{i\theta}z$. Then $e^{i\theta}B(0) = (AB)(0) = (BA)(0) = B(0)$. Since A is not the identity, $B(0) = 0$, and B has the form $e^{i\phi}z$.

CASE 2. Suppose A has two fixed points on $\{|z| = 1\}$ which are different. We can map the problem to the right half-plane with the fixed points going to 0 and ∞, and $A(z) = \lambda z$ for some $\lambda > 0$, $\lambda \neq 1$. As above, B either fixes each of $0, \infty$ or interchanges them. In the second case, $B(z) = \mu/z$ for some $\mu > 0$ and does not commute with A. Therefore B fixes these points and $B(z) = \mu z$.

CASE 3. Suppose A has one fixed point on $\{|z| = 1\}$. Again map the problem to the right half-plane with ∞ fixed. Then one sees easily that $A(z) = z + \lambda i$ for some real $\lambda \neq 0$ and $B(z) = z + i\mu$ for some $\mu \in \mathbf{R}$. □

Möbius transformations corresponding to these three cases are called *elliptic*, *hyperbolic*, and *parabolic*, respectively. It is easy to show that every such transformation preserving the disk (except the identity) is one of these three types.

Proof of Lemma 2.3 (continued). Choose $h \in \Gamma_0$ which is not the identity. Since $\mathcal{G}$ commutes with h it belongs to the one-parameter group generated by h. Since $\mathcal{G}$ is discrete and infinite we conclude that $\mathcal{G}$ has the form $\{g^n\}_{-\infty}^{\infty}$. This means that the fundamental group of U is isomorphic to the integers, and U is doubly connected. Since U is hyperbolic, U cannot be a punctured plane, and U is either an annulus or a punctured disk. One of the alternatives (2), (4) or (5) must hold. □

There is one more technical obstacle to surmount before completing the proof of the classification theorem. It is handled by the following variant of the "snail lemma". The reason for the nomenclature will become apparent from the proof, which stems from Section 54 of **[Fa2]**.

LEMMA 2.4. *Suppose U is hyperbolic, $f : U \to U$ is analytic on U and across ∂U, and $f^n(z_0) \to \partial U$ for some $z_0 \in U$. Then there is a fixed point $\zeta \in \partial U$ for f such that $f^n(z) \to \zeta$ for all $z \in U$. Either ζ is an attracting fixed point, or ζ is a parabolic fixed point with multiplier $f'(\zeta) = 1$.*

Proof. By Theorem I.4.3, the spherical distances between the iterates z_n and z_{n+1} of z_0 tend to 0. Thus the limit set of $\{z_n\}$ is a connected subset of ∂U, and furthermore, by continuity of f, any limit point is a fixed point for f. Since the fixed points of f are isolated, we conclude that $z_n \to \zeta$ for some fixed point $\zeta \in \partial U$ of R. The orbit of every other $z \in U$ also converges to ζ, since it remains a bounded hyperbolic distance from the orbit of z_0. The fixed point ζ is not repelling, since $z_n \to \zeta$.

Suppose $f'(\zeta) = e^{2\pi i\theta}$ where θ is rational. Then U is contained in the basin of attraction associated with one of the petals at ζ. Since the local rotation f at ζ induces a cyclic permutation of the petals at ζ, and since f leaves U invariant, in fact f induces the identity permutation, and $f'(\zeta) = 1$.

Suppose $f'(\zeta) = e^{2\pi i\theta}$, where θ is irrational. Assume $\zeta = 0$. Let $z_0 \in U$, and let V be a relatively compact subdomain of U such that V is simply connected and z_0 and $z_1 = f(z_0)$ belong to V. Since f is univalent near 0 and $f^n \to 0$ uniformly on V, we can assume each f^n is univalent on V. Then

$$\varphi_n(z) = \frac{f^n(z)}{f^n(z_0)}, \qquad z \in V,$$

is also univalent on V, $\varphi_n(z_0) = 1$, and $0 \notin \varphi_n(V)$. Let ψ be the Riemann map from Δ to V, $\psi(0) = z_0$. Then $h_n(\zeta) = \varphi_n(\psi(\zeta)) - 1$ is univalent on Δ, $h_n(0) = 0$, $h_n'(0) = \varphi_n'(z_0)\psi'(0)$, and h_n omits -1. Thus the function $h_n/h_n'(0)$ belongs to $\mathcal{S}$ and omits $-1/h_n'(0)$. The Koebe one-quarter theorem implies $|h_n'(0)| \le 4$. Since $\mathcal{S}$ is a normal family, the sequence $\{h_n/h_n'(0)\}$ is normal on Δ, as is $\{h_n\}$. Consequently $\{\varphi_n\}$ is normal on V.

We claim that all limit functions of $\{\varphi_n\}$ are nonconstant. For if $|\varphi_n - 1| < \delta$ then $f^n(V)$ would be included in a narrow angle with vertex 0, and if this angle were smaller than $\theta/3$ then since $f(z) \cong e^{i\theta}z$ near 0, $f^{n+1}(V)$ would be disjoint from $f^n(V)$, contrary to hypothesis. Thus $\varphi_n'(z_0)$ is bounded away from 0. From Koebe's theorem again we deduce that there is $\rho > 0$ such that $\varphi_n(V)$ contains a disk centered at 1 of radius ρ. Thus $f^n(V)$ contains a disk centered at z_n of radius $\rho|z_n|$.

Choose N so that the disks of radius $\rho/2$ centered at the points $e^{2\pi i m\theta}$, $0 \le m \le N$, cover an annulus containing the unit circle. Since $z_{m+1} = e^{2\pi i\theta}z_m + o(|z_m|)$, the disks centered at z_m of radius $\rho|z_m|$, $n \le m \le n+N$, cover an open annulus containing z_n and z_{n+1} for n large. Hence $\cup f^n(V)$ contains a punctured neighborhood of 0, and 0 is an isolated point of ∂U. But then Theorem II.6.2 implies that f is conjugate to a rotation about 0, contradicting $f^n(z) \to 0$. We conclude that $f'(\zeta)$ cannot be irrational. □

Proof of Theorem 2.1 (continued). The proof is now complete, in view of the following observations. Since R^n has degree > 1, no power of R can coincide with the identity, and case (2) of Lemma 2.3 is ruled out. Since $\mathcal{J}$ has no isolated points, case (5) of Lemma 2.3 is also impossible for R. Finally, since there are no attracting periodic points in $\mathcal{J}$, Lemma 2.4 produces a parabolic fixed point in ∂U whenever $f^n(z) \to \partial U$. □

EXAMPLE. Suppose R is a finite Blaschke product of degree $d \ge 2$. We have seen in Section III.1 that the Julia set $\mathcal{J}$ of R is either the entire unit circle or a totally disconnected subset of the circle. There are now four possibilities. If R has an attracting fixed point on $\partial\Delta$, or if R has a parabolic fixed point on $\partial\Delta$ with only one attracting petal, then $\mathcal{J}$ is totally disconnected. If R has an attracting fixed point $\zeta \in \Delta$, then $1/\bar{\zeta} \in \overline{\mathbb{C}}\backslash\overline{\Delta}$ is also an attracting fixed point, and $\mathcal{J}$ separates the basins of attraction, so $\mathcal{J} = \partial\Delta$. The only remaining possibility is that R has a parabolic fixed point on $\partial\Delta$ with two attracting petals, in which case again $\mathcal{J} = \partial\Delta$. One can check that each of these four cases already occurs for Blaschke products of degree two. In all cases, the periodic points of R that are not fixed are repelling.

3. The Wolff–Denjoy Theorem

Before leaving this area let us prove the following beautiful related theorem of J. Wolff (1926) and A. Denjoy (1926). This brief proof was discovered by A.F. Beardon (1990).

THEOREM 3.1. *Let $f : \Delta \to \Delta$ be analytic, and assume f is not an elliptic Möbius transformation nor the identity. Then there is $\alpha \in \overline{\Delta}$ such that $f^n(z) \to \alpha$ for all $z \in \Delta$.*

Proof. The theorem is easy if f is a Möbius transformation, and so we assume that f is not an isometry with respect to the hyperbolic metric, and further that $f(0) \ne 0$. If the orbit of 0 visits any compact set infinitely often, Lemma 2.2 provides a fixed point in Δ. Thus we may assume that the orbit of 0 accumulates on $\partial\Delta$. The problem is to show that the orbit can accumulate at only one point α of $\partial\Delta$. This is again easy if f extends continuously to $\partial\Delta$. The main point of the theorem is that no continuity is assumed.

Define $f_\varepsilon(z) = (1-\varepsilon)f(z)$, which maps to a compact subset of Δ. Let z_ε be the fixed point of f_ε, and let D_ε be the hyperbolic disk centered at z_ε with radius $\rho(0, z_\varepsilon)$. Since f_ε is contracting, $f_\varepsilon(D_\varepsilon) \subset$

D_ε. Now D_ε is a euclidean disk with 0 on its boundary. Any limit D of the D_ε's is a euclidean disk with 0 on its boundary, and $f(D) \subset D$. Thus the point of tangency of D and $\partial\Delta$ is the only possible limit point on $\partial\Delta$ of the orbit of 0. □

Another approach to the Wolff–Denjoy theorem provides an illuminating application of the Herglotz formula. We replace Δ by the upper half-plane $\mathbf{H}$. We are now considering $f = u + iv$ as a mapping of $\mathbf{H}$ to itself, with $v = \operatorname{Im} f > 0$, and we must show that either every orbit tends to ∞, or else every orbit is bounded. The Herglotz representation of the positive harmonic function v in the upper half-plane is

$$v(x,y) = cy + y\int_{-\infty}^{\infty} \frac{d\sigma(s)}{(x-s)^2+y^2}, \qquad y > 0, \tag{3.1}$$

where $c \geq 0$ and σ is a positive measure satisfying

$$\int_{-\infty}^{+\infty} \frac{d\sigma(s)}{1+s^2} < \infty.$$

We may assume $\sigma \neq 0$.

Suppose $c \geq 1$. From (3.1) we have $v(x,y) > y$, so that the half-planes $\{y > y_0\}$ are invariant under f. These correspond to the invariant tangent disk D to $\partial\Delta$ in the preceding proof. Let $z_n = x_n + iy_n$ be an orbit. Then either y_n increases to $+\infty$, or else y_n increases to some finite limit value y_∞, in which case (3.1) shows that $|x_n| \to \infty$. In any event $|z_n| \to \infty$.

Now suppose $0 \leq c < 1$. In this case orbits are bounded. To see this, we use the Herglotz formula for f,

$$f(z) = cz + \int_{-\infty}^{+\infty} \frac{d\sigma(s)}{s-z} + b, \qquad z \in \mathbf{H},$$

where b is real. The problem is to obtain an estimate of the form $|f(z) - cz| = o(z)$ for large $|z|$. This can be done by expressing f as

$$\begin{aligned} f(z) &= cz + \int_{|s|\geq A} \left[\frac{1}{s-z} - \frac{1}{s}\right] d\sigma(s) + \int_{-A}^{A} \frac{d\sigma(s)}{s-z} + b' \\ &= cz + z\int_{|s|\geq A} \frac{d\sigma(s)}{s(s-z)} + \mathcal{O}(1) \end{aligned}$$

as $|z| \to \infty$ and by estimating carefully the integral that appears here.

V

Critical Points and Expanding Maps

Critical points and their forward orbits play a key role in complex dynamical systems. The forward orbit of the critical points is dense in the boundary of any Siegel disk and Herman ring. If the critical points and their iterates stay away from the Julia set, the mapping is expanding on the Julia set, and the Julia set becomes more tractable.

1. Siegel Disks

Let CP denote the (finite) set of critical points of R. The *postcritical set* of R is defined to be the forward orbit $\cup_{n\geq 0} R^n(CP)$ of the critical points. We denote by CL the closure of the postcritical set of R. This set is important because on its complement all branches of R^{-n}, $n \geq 1$, are locally defined and analytic.

As we have already seen for polynomials (Section III.4), many basic properties of the dynamics and structure of $\mathcal{J}$ are determined by the critical points and the postcritical set. By Theorems III.2.2 and III.2.3, each attracting and parabolic cycle of components of the Fatou set $\mathcal{F}$ contains a critical point. Siegel disks and Herman rings contain no critical point, however in a certain sense they can be associated to critical points. We begin with the following theorem.