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Complex Laplacian and Derivatives of Bicomplex Functions

M. E. Luna-Elizarrarás · M. Shapiro · D. C. Struppa · A. Vajiac

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Abstract In this paper we study in detail the theory of bicomplex holomorphy, in the context of the several ways in which bicomplex numbers can be considered. In particular we will show how the notions of bicomplex derivability and bicomplex holomorphy can be interpreted in these different ways, and the consequences that can be derived.

Keywords Bicomplex derivability · Bicomplex differentiability · Bicomplex holomorphic functions · Complex and hyperbolic Laplacians

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M. E. Luna-Elizarrarás · M. Shapiro
Escuela Superior de Fisica y Matemáticas, Instituto Politécnico Nacional,
07338 México City, DF, Mexico
e-mail: eluna@esfm.ipn.mx

M. Shapiro
e-mail: shapiro@esfm.ipn.mx

D. C. Struppa (✉) · A. Vajiac
Schmid College of Science and Technology, Chapman University, One University Drive,
Orange, CA 92866, USA
e-mail: struppa@chapman.edu

A. Vajiac
e-mail: avajiac@chapman.edu

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1 Introduction

In the last several years, the theory of bicomplex numbers has enjoyed a renewed interest, mostly because of [2–4,9,11], and the theory of functions which are bicomplex holomorphic (or $\mathbb{BC}$-holomorphic as we will say in this paper) has been developed along many interesting directions, including extensions to several variables [16], extension to the case of multicomplex numbers [17], extension to pseudoanalytic functions theory [1,12]; elements of the geometric function theory for bicomplex functions can be found in [5]. The traditional text which lays out the foundations for the study of bicomplex numbers and of the functions defined on them is the monograph of Price [10], where some historical notes on these numbers are also given, while an elementary introduction of the first examples of $\mathbb{BC}$-holomorphic functions is given in [8]. In no way this brief review pretends to cover all the sources about bicomplex numbers and their functions.

A careful reading of the monograph of Price reveals many important intuitions, that have never been sufficiently explored, but if a critical commentary can be made, one should say that the many detailed computations which are carried out in [10], often appear to lack motivation, and there is no analysis of why such calculations end up being so successful. In a word, the theory of $\mathbb{BC}$-holomorphy is presented almost as isolated from the theory of differentiability for functions in four real variables, and from the theory of holomorphy for functions of two complex variables.

The major contribution we would like to offer with this paper is a detailed analysis of the theory of $\mathbb{BC}$-holomorphy, in the context of the several alternative ways in which bicomplex numbers can be considered. Let us be a bit more specific. To begin with, we consider the usual field of complex numbers $\mathbb{C}(\mathbf{i}) = \{x + \mathbf{i}y \ : \ x, y \in \mathbb{R}\}$. Note that we have made explicit the fact that we are using the imaginary unit $\mathbf{i}$, since a second imaginary unit $\mathbf{j}$ is now necessary to define bicomplex numbers. Indeed, we define the ring of bicomplex numbers, and we denote it by $\mathbb{BC}$, to be the ring of numbers of the form $Z = z_1 + \mathbf{j}z_2$, where both $z_1 = x_1 + \mathbf{i}y_1$ and $z_2 = x_2 + \mathbf{i}y_2$ are complex numbers in $\mathbb{C}(\mathbf{i})$. So, in a sense, a bicomplex number can be identified with a pair of complex numbers in $\mathbb{C}(\mathbf{i})$. However, a simple rearrangement of the terms shows that the same number Z can also be seen as a pair of complex numbers in $\mathbb{C}(\mathbf{j})$. Even more unexpected, if we denote by $\mathbf{k}$ the hyperbolic unit $\mathbf{ij}$ (by hyperbolic unit we simply mean that $\mathbf{k}^2 = 1$), and if we denote by $\mathbb{D}$ the ring of hyperbolic numbers defined by $\mathbb{D} := \{x + \mathbf{k}y \,\big|\, x, y \in \mathbb{R}\}$, then any bicomplex number can also be seen as a pair of hyperbolic numbers. The situation is actually even more complicated than it appears from this quick discussion, as it will be shown in detail in Sect. 2.

These three different structures (and as we mentioned, they are not the only ones we will be working with!) make it possible to interpret every definition in several different ways. It is our goal in this paper to exploit these multiple representations, and to show how the notions of $\mathbb{BC}$-derivability and of $\mathbb{BC}$-holomorphy can be reinterpreted in these settings, and what are the consequences that one may infer. As the reader will easily see, this approach will provide many interesting surprises and novelties, that were at a minimum not explicit (and in many cases not even contemplated) in [10].

The paper is fully self-contained and consists of six sections in addition to the introduction. Section 2 is dedicated to the definition of the ring of bicomplex numbers, and to the introduction of the various structures to which we have hinted above. We show, for example, that one can write bicomplex numbers in at least eight different ways, and that each different representation conveys a different flavor and has different consequences. In addition we recall that bicomplex numbers admit several different notions of conjugation, and correspondingly several different notions of modulus. We then come to the most important point in the theory of bicomplex numbers. Namely, while bicomplex numbers form a very nice commutative ring, they do not form a field, because not every non zero bicomplex number has a multiplicative inverse, and therefore we introduce the study of the zero-divisors in $\mathbb{BC}$. It turns out that these zero-divisors play a central role, which is at first just intuited by looking at the so-called idempotent representation of bicomplex numbers. This section concludes with a study of the various multiplicative structures that can be induced on $\mathbb{BC}$; specifically we will consider how $\mathbb{BC}$ can be considered as an $\mathbb{R}$-linear space, a $\mathbb{C}(\mathbf{i})$-linear space, a $\mathbb{C}(\mathbf{j})$-linear space or finally as a $\mathbb{D}$-module.

The core of the paper, however, begins with Sect. 3, where we study the notion of derivability and holomorphy for bicomplex valued functions of a bicomplex variable. For simplicity we limit ourselves, in this section, to the considerations that can be obtained by limiting ourselves to the various cartesian representations of bicomplex numbers (i.e., we do not make reference, in this section, to the idempotent representation of bicomplex numbers). A fundamental remark, in this section, is the observation that the notion of derivability in the complex and real cases, are formulated in ways that are completely parallel to the bicomplex case. However, some of the most natural consequences that we derive in the real or complex case cannot be established in the bicomplex case (at least not in a completely symmetric way). This is clarified in Corollary 3.5, and the subsequent Remark 3.6. It is finally in this Sect. 3 that we establish various equivalent conditions to the notion of derivability (conditions that lead to the introduction of suitable Cauchy–Riemann systems). Note that in the case of quaternionic and real Clifford analysis, the theory of derivability of the corresponding functions is rather different from our situation (see, for instance [7, 15]).

In Sect. 4 we examine in detail the interplay between real differentiability and the derivability of bicomplex functions. Specifically, in the first subsection of Sect. 4, we express the usual notion of differentiability for a bicomplex function (thought of as a function of four real variables) in terms of its complex or hyperbolic differentiability; as we have seen, a bicomplex function can be thought of as a function of two complex variables (in $\mathbb{C}(\mathbf{i})$ or in $\mathbb{C}(\mathbf{j})$) as well as a function of two hyperbolic variables, and therefore it is an interesting question to see how real differentiability relates to differentiability in terms of these other variables. As a consequence, we are able to express the real differentiability of a bicomplex function in four different languages: real, $\mathbb{C}(\mathbf{i})$, $\mathbb{C}(\mathbf{j})$, and hyperbolic.

But one may also want to look directly at bicomplex differentiability (i.e., differentiability when the function is regarded strictly as a function of one bicomplex variable). The result of this analysis gives the important equivalence, at least for $\mathcal{C}^1$ functions, between being $\mathbb{BC}$-derivable and being a solution of a system of three Cauchy–Riemann like systems of differential equations.

The far reaching consequences of this result are explored in more depth in Sect. 5. All of this is of significant independent interest, but it also acts as the prologue for

Sect. 6, where we study many variants of the Laplacian operator, in terms of the different variables which we can consider for a bicomplex function.

Finally, in Sect. 7, we go back to the issue of the meaning of bicomplex derivability and of bicomplex holomorphy. This time, however, we utilize the idempotent representation. As we had anticipated earlier in the paper, such a representation is quite powerful, and allows a deeper understanding of the notion of $\mathbb{BC}$-holomorphy.

This paper should be seen as a complement to the theoretical description given by Price in [10], and we hope it will stimulate further researches in an area that appears still considerably understudied. In particular, it is important to note how our results clarify the relationship between $\mathbb{BC}$-holomorphic functions and a suitable subset of holomorphic maps from $\mathbb{C}^2$ to $\mathbb{C}^2$, a topic which should be of considerable interest to specialists in several complex variables.

2 Bicomplex Numbers and Functions

In this section we introduce the bicomplex numbers and we recall those of their properties that are relevant to this paper. The set of *bicomplex* numbers is defined by

$$\mathbb{BC} := \{z_1 + \mathbf{j}\, z_2 \,\big|\, z_1, z_2 \in \mathbb{C}(\mathbf{i})\},$$

where $\mathbb{C}(\mathbf{i})$ is the set of complex numbers with the imaginary unit $\mathbf{i}$, and where $\mathbf{i}$ and $\mathbf{j}$ are distinct commuting imaginary units, i.e., $\mathbf{i} \neq \mathbf{j}$, $\mathbf{ij} = \mathbf{ji}$, and $\mathbf{i}^2 = \mathbf{j}^2 = -1$. Note that the set $\mathbb{H}$ of Hamilton quaternions can be introduced in the same way but requiring instead that $\mathbf{ij} = -\mathbf{ji}$; as it is well known, this leads to a very different theory.

The addition and multiplication of bicomplex numbers are defined by simply taking into account that $\mathbf{j}^2 = -1$. So, if $Z = z_1 + \mathbf{j}\, z_2$ and $W = w_1 + \mathbf{j}\, w_2$ are two bicomplex numbers, their sum is

$$Z + W = (z_1 + w_1) + \mathbf{j}(z_2 + w_2),$$

while their product is

$$Z \cdot W := (z_1 + \mathbf{j}z_2)(w_1 + \mathbf{j}w_2) = (z_1 w_1 - z_2 w_2) + \mathbf{j}(z_1 w_2 + z_2 w_1). \qquad (2.1)$$

Endowed with these two operations, the set $\mathbb{BC}$ turns into a commutative ring, but not into a field, since, as we will soon see, not all bicomplex numbers have multiplicative inverse.

Within the set $\mathbb{BC}$ there are several subsets which one could legitimately recognize as isomorphic to the field of complex numbers: one of them is the set of those bicomplex numbers with $z_2 = 0$, namely those bicomplex numbers Z of the form $Z = z_1 + \mathbf{j}0 = z_1$; this particular subset will be denoted, for obvious reasons, with the symbol $\mathbb{C}(\mathbf{i})$. Since $\mathbf{j}$ is another imaginary unit, we can define the set $\mathbb{C}(\mathbf{j})$ within $\mathbb{BC}$ as $\mathbb{C}(\mathbf{j}) := \{z_1 + \mathbf{j}z_2 \mid z_1, z_2 \in \mathbb{R}\}$. Of course, $\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$ are isomorphic fields but their immersion in $\mathbb{BC}$ creates an asymmetry, that will be of great importance in what follows.

The set of *hyperbolic* numbers is defined intrinsically (independent of $\mathbb{BC}$), as the set

$$\mathbb{D} := \{x + \mathbf{k}y \,\big|\, x, y \in \mathbb{R}\},$$

where $\mathbf{k}$ is a hyperbolic unit, i.e., $\mathbf{k}^2 = 1$, which commutes with both real numbers x and y. In some of the existing literature, hyperbolic numbers are also called *duplex*, *double* or *bireal* numbers. The addition and multiplication between hyperbolic numbers are defined in the obvious fashion, by replacing $\mathbf{k}^2$ by 1 whenever it occurs.

Within $\mathbb{BC}$ a hyperbolic unit $\mathbf{k}$ arises from the multiplication of the imaginary units $\mathbf{i}$ and $\mathbf{j}$, i.e., $\mathbf{k} = \mathbf{ij}$. Thus, we find that there is a subset in $\mathbb{BC}$ which is isomorphic as a ring to the set of hyperbolic numbers $\mathbb{D}$, and which inherits all the algebraic definitions, operations and properties from $\mathbb{BC}$.

While it is clear from the beginning that it is possible to write a bicomplex number in two different ways, depending on whether we factor the imaginary unit $\mathbf{i}$ or the imaginary unit $\mathbf{j}$, the introduction of the hyperbolic unit $\mathbf{k}$ shows that a bicomplex number defined as $Z = z_1 + \mathbf{j}z_2$ admits several other representations, which are helpful to highlight different aspects of the theory of bicomplex numbers. To clarify this comment, we consider the bicomplex number $Z = z_1 + \mathbf{j}z_2$, where the complex numbers z_1 and z_2 can be written, in terms of their real components, as $z_1 = x_1 + \mathbf{i}y_1$, $z_2 = x_2 + \mathbf{i}y_2$ with $x_1, \ y_1, \ x_2, \ y_2 \in \mathbb{R}$. Then any bicomplex number can be written in the following different ways:

$$\begin{aligned} Z &= (x_1 + \mathbf{i}y_1) + \mathbf{j}\,(x_2 + \mathbf{i}y_2) =: z_1 + \mathbf{j}\,z_2 && (2.2)\\ &= (x_1 + \mathbf{j}x_2) + \mathbf{i}\,(y_1 + \mathbf{j}y_2) =: \zeta_1 + \mathbf{i}\,\zeta_2 && (2.3)\\ &= (x_1 + \mathbf{k}y_2) + \mathbf{i}\,(y_1 - \mathbf{k}x_2) =: \mathfrak{z}_1 + \mathbf{i}\,\mathfrak{z}_2 && (2.4)\\ &= (x_1 + \mathbf{k}y_2) + \mathbf{j}\,(x_2 - \mathbf{k}y_1) =: \mathfrak{w}_1 + \mathbf{j}\,\mathfrak{w}_2 && (2.5)\\ &= (x_1 + \mathbf{i}y_1) + \mathbf{k}(y_2 - \mathbf{i}x_2) =: w_1 + \mathbf{k}\,w_2 && (2.6)\\ &= (x_1 + \mathbf{j}x_2) + \mathbf{k}(y_2 - \mathbf{j}y_1) =: \omega_1 + \mathbf{k}\,\omega_2 && (2.7)\\ &= x_1 + \mathbf{i}y_1 + \mathbf{j}x_2 + \mathbf{k}y_2\,, && (2.8) \end{aligned}$$

where clearly $z_1, z_2, w_1, w_2 \in \mathbb{C}(\mathbf{i})$, $\zeta_1, \zeta_2, \omega_1, \omega_2 \in \mathbb{C}(\mathbf{j})$, and $\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{w}_1, \mathfrak{w}_2 \in \mathbb{D}$.

Because there are two imaginary units in $\mathbb{BC}$, we can define three different conjugations on $\mathbb{BC}$:

(I) $\overline{Z} := \overline{z}_1 + \mathbf{j}\,\overline{z}_2$ (the bar-conjugation);

(II) $Z^\dagger := z_1 - \mathbf{j}\,z_2$ (the $\dagger$ -conjugation);

(III) $Z^* := \left(\overline{Z}\right)^\dagger = \overline{\left(Z^\dagger\right)} = \overline{z}_1 - \mathbf{j}\,\overline{z}_2$ (the $*$ -conjugation).

In a nutshell, for bicomplex numbers the *bar*-conjugation is with respect to the imaginary unit $\mathbf{i}$, the $\dagger$-conjugation is with respect to the imaginary unit $\mathbf{j}$, and the $*$-conjugation is with respect to both $\mathbf{i}$ and $\mathbf{j}$.

But within $\mathbb{BC}$ there is also a hyperbolic unit. Since the (hyperbolic) *conjugate* of a hyperbolic number $\mathfrak{z} = x + \mathbf{k}y$ is defined by $\mathfrak{z}^\diamond := x - \mathbf{k}y$, we obtain that for a bicomplex number of the form $\mathfrak{z} = x + \mathbf{ij}y$ we have $\overline{\mathfrak{z}} = \mathfrak{z}^\dagger = x - \mathbf{ij}y = \mathfrak{z}^\diamond$.

Note that working with just the complex numbers in $\mathbb{C}(\mathbf{j})$, we will use the $*$-conjugation as the intrinsic conjugation on $\mathbb{C}(\mathbf{j})$.

In the complex case the modulus of a complex number is intimately related with the complex conjugation. Similarly, accordingly to each of the three conjugations, three possible moduli arise:

$$|Z|_{\mathbf{i}}^2 := Z \cdot Z^{\dagger} = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i}); \tag{2.9}$$

$$|Z|_{\mathbf{j}}^2 := Z \cdot \overline{Z} = \zeta_1^2 + \zeta_2^2 \in \mathbb{C}(\mathbf{j}); \tag{2.10}$$

$$|Z|_{\mathbf{k}}^2 := Z \cdot Z^* = \mathfrak{z}_1^2 + \mathfrak{z}_2^2 = \mathfrak{w}_1^2 + \mathfrak{w}_2^2 \in \mathbb{D}, \tag{2.11}$$

and for a complex number z (whether it is in $\mathbb{C}(\mathbf{i})$ or $\mathbb{C}(\mathbf{j})$) we denote by $|z|$ its usual modulus. Of course these moduli are not $\mathbb{R}^+$-valued. The first two are complex-valued (in the sense of $\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$ respectively), while the last one is hyperbolic-valued. These moduli of bicomplex numbers have numerous important properties, which will be carefully studied elsewhere.

In the set of hyperbolic numbers $\mathbb{D}$, we have that $\mathfrak{z} \cdot \mathfrak{z}^{\diamond} = x^2 - y^2 \in \mathbb{R}$, so that it is possible to define the square of the *hyperbolic modulus* as $|\mathfrak{z}|_{\mathbb{D}}^2 := x^2 - y^2$. If we now regard the hyperbolic number $\mathfrak{z}$ as a bicomplex number, i.e., we write $\mathfrak{z} = z_1 + \mathbf{j} z_2 = (x_1 + 0\mathbf{i}) + \mathbf{j}(0 + \mathbf{i} y_2) \in \mathbb{BC}$, one obtains the following relations among its various bicomplex moduli:

$$|\mathfrak{z}|_{\mathbf{i}}^2 = |\mathfrak{z}|_{\mathbf{j}}^2 = |\mathfrak{z}|_{\mathbb{D}}^2 , \qquad |\mathfrak{z}|_{\mathbf{k}}^2 = \mathfrak{z}^2 . \tag{2.12}$$

Since none of these moduli is real valued, we can consider the Euclidean norm on $\mathbb{BC}$ in terms of the various ways in which we can interpret $\mathbb{BC}$ itself, namely as $\mathbb{C}^2(\mathbf{i})$, $\mathbb{C}^2(\mathbf{j})$, or finally as $\mathbb{R}^4$. Specifically, we can see $\mathbb{BC}$ as

$$\mathbb{C}^2(\mathbf{i}) := \mathbb{C}(\mathbf{i}) \times \mathbb{C}(\mathbf{i}) = \{(z_1, z_2) \,\big|\, z_1 + \mathbf{j} z_2 \in \mathbb{BC}\},$$

or as

$$\mathbb{C}^2(\mathbf{j}) := \mathbb{C}(\mathbf{j}) \times \mathbb{C}(\mathbf{j}) = \{(\zeta_1, \zeta_2) \,\big|\, \zeta_1 + \mathbf{i} \zeta_2 \in \mathbb{BC}\},$$

or as

$$\mathbb{R}^4 = \{(x_1, y_1, x_2, y_2) \,\big|\, (x_1 + \mathbf{i} y_1) + \mathbf{j}(x_2 + \mathbf{i} y_2) \in \mathbb{BC}\}.$$

Regardless of how we interpret $\mathbb{BC}$ we obtain that the Euclidean norm of a bicomplex number Z is defined by

$$|Z| := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{|\zeta_1|^2 + |\zeta_2|^2} = \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2} .$$

If we now denote by $\mathbb{D}^+$ the set

$$\mathbb{D}^+ = \{x + \mathbf{ij} y \,\big|\, x^2 - y^2 \geq 0, x \geq 0\},$$

then the Euclidean norm can also be related to the $\mathbb{D}^+$-valued modulus $|Z|_{\mathbf{k}}$ via the formula:

$$|Z| = \sqrt{\operatorname{Re}(|Z|_{\mathbf{k}}^2)}.$$

Using the triangle inequality, it is immediate to prove that

$$|Z \cdot W| \leq \sqrt{2}\,|Z| \cdot |W|, \qquad \forall Z, W \in \mathbb{BC}.$$

This apparently innocuous formula will play an essential role later on in the paper.

One important consequence of (2.9) is that $Z = z_1 + \mathbf{j}z_2$ is *invertible* if and only if $Z \cdot Z^\dagger = z_1^2 + z_2^2 \neq 0$, and the *inverse* of an invertible bicomplex number Z is given by the bicomplex number

$$Z^{-1} = \frac{Z^\dagger}{z_1^2 + z_2^2}\,.$$

If both z_1 and z_2 are non-zero but the sum $z_1^2 + z_2^2 = 0$, then Z has no inverse and in fact it is a *zero-divisor*, because $Z \cdot Z^\dagger = 0$. It is easy to show that all zero-divisors in $\mathbb{BC}$ are of the form: $Z = \lambda(1 \pm \mathbf{ij})$, where λ runs the whole set $\mathbb{C}(\mathbf{i}) \setminus \{0\}$. An equivalent description of zero-divisors can also be obtained using the other writings of Z. We denote the set of all zero-divisors in $\mathbb{BC}$ by $\mathfrak{S}$, and we set $\mathfrak{S}_0 := \mathfrak{S} \cup \{0\}$.

It turns out that there are two very special zero-divisors: the bicomplex numbers

$$\mathbf{e} := \frac{1 + \mathbf{ij}}{2} \qquad \text{and} \qquad \mathbf{e}^\dagger := \frac{1 - \mathbf{ij}}{2}\,,$$

which have the following properties: $\mathbf{e} \cdot \mathbf{e}^\dagger = 0$, $\mathbf{e}^2 = \mathbf{e}$, $(\mathbf{e}^\dagger)^2 = \mathbf{e}^\dagger$, $\mathbf{e} + \mathbf{e}^\dagger = 1$, $\mathbf{e} - \mathbf{e}^\dagger = \mathbf{ij}$. It can be easily shown that 0, 1, $\mathbf{e}$ and $\mathbf{e}^\dagger$, are actually the only idempotent bicomplex numbers.

These two bicomplex numbers allow us to describe a property that has no analogs for complex numbers, and that exemplifies one of the deepest peculiarities of the set of bicomplex numbers. Any bicomplex number $Z = z_1 + \mathbf{j}z_2 \in \mathbb{BC}$ can be written in the form:

$$Z = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger, \tag{2.13}$$

where $\beta_1 := z_1 - \mathbf{i}z_2$ and $\beta_2 := z_1 + \mathbf{i}z_2$ are uniquely defined complex numbers in $\mathbb{C}(\mathbf{i})$. Formula (2.13) is called the *idempotent representation* of the bicomplex number Z. The importance of this representation lies in the fact that addition and multiplication of bicomplex numbers can be realized term-by-term in the idempotent representation. Analogously, the Euclidean norm of Z in terms of the idempotent components can be computed term-by-term as

$$|Z| = \frac{1}{\sqrt{2}}\sqrt{|\beta_1|^2 + |\beta_2|^2}\,.$$

Consider now a bicomplex number Z of the form $Z = \zeta_1 + \mathbf{i}\,\zeta_2$, with $\zeta_1, \zeta_2 \in \mathbb{C}(\mathbf{j})$. Then a direct computation shows:

$$Z = \gamma_1 \mathbf{e} + \gamma_2 \mathbf{e}^\dagger := (\zeta_1 - \mathbf{j}\,\zeta_2)\mathbf{e} + (\zeta_1 + \mathbf{j}\,\zeta_2)\mathbf{e}^\dagger, \tag{2.14}$$

where $\gamma_1 := \zeta_1 - \mathbf{j}\,\zeta_2$ and $\gamma_2 := \zeta_1 + \mathbf{j}\zeta_2$ are complex numbers in $\mathbb{C}(\mathbf{j})$. So, we see that as a matter of fact every bicomplex number has two idempotent representations with complex coefficients, one with coefficients in $\mathbb{C}(\mathbf{i})$, and the other with coefficients in $\mathbb{C}(\mathbf{j})$:

$$Z = \beta_1\, \mathbf{e} + \beta_2\, \mathbf{e}^\dagger = \gamma_1\, \mathbf{e} + \gamma_2\, \mathbf{e}^\dagger. \tag{2.15}$$

Note that $Z\mathbf{e} = \beta_1 \mathbf{e} = \gamma_1 \mathbf{e}$ and $Z\mathbf{e}^\dagger = \beta_2 \mathbf{e}^\dagger = \gamma_2 \mathbf{e}^\dagger$, thus the uniqueness of the idempotent representation of bicomplex numbers does not consist in the fact that the coefficients β_1 and γ_1 (or β_2 and γ_2) are equal (indeed, this is most certainly not true!), but in the fact that the products $\beta_1 \mathbf{e}$ and $\gamma_1 \mathbf{e}$ (or $\beta_2 \mathbf{e}^\dagger$ and $\gamma_2 \mathbf{e}^\dagger$) are equal.

We conclude this section with a short mention of the various algebraic structure that multiplication induces on $\mathbb{BC}$, depending on how we consider it.

In particular, the fact that the rings $\mathbb{R}$, $\mathbb{C}(\mathbf{i})$, $\mathbb{C}(\mathbf{j})$, $\mathbb{D}$ are subrings of the ring $\mathbb{BC}$, implies that $\mathbb{BC}$ can be seen as a module over each one of these subrings, and of course, it is a module over itself. Since $\mathbb{R}$, $\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$ are fields, $\mathbb{BC}$ is a real linear space, a $\mathbb{C}(\mathbf{i})$ complex linear space and a $\mathbb{C}(\mathbf{j})$ complex linear space.

Formula (2.8) yields an isomorphism of real spaces between $\mathbb{BC}$ and $\mathbb{R}^4$, which maps the bicomplex numbers 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into the canonical basis of $\mathbb{R}^4$. Similarly, equation (2.2) yields an isomorphism between $\mathbb{BC}$ as a $\mathbb{C}(\mathbf{i})$ linear space and $\mathbb{C}^2(\mathbf{i})$, where the bicomplex numbers 1, $\mathbf{j}$ are mapped into the canonical basis of $\mathbb{C}^2(\mathbf{i})$. Seeing now $\mathbb{BC}$ as a $\mathbb{C}(\mathbf{j})$-linear space and using (2.3), we obtain an isomorphism which sends the bicomplex numbers 1 and $\mathbf{i}$ into the canonical basis in $\mathbb{C}^2(\mathbf{j})$.

Obviously these last two isomorphisms are different, which clarifies once again the fact that inside $\mathbb{BC}$, the "complex sets" $\mathbb{C}^2(\mathbf{i})$ and $\mathbb{C}^2(\mathbf{j})$ play different roles. A simple example that underscores this situation is the fact that the set $\{1, \mathbf{i}\}$ is linearly independent if we consider $\mathbb{BC}$ as a $\mathbb{C}(\mathbf{j})$-linear space, but the same set is linearly dependent in the $\mathbb{C}(\mathbf{i})$-linear space $\mathbb{BC}$.

Consider now the set $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$. With the component-wise addition inherited from $\mathbb{D}$, one sees that $\mathbb{D}^2$ is an additive abelian group. Defining also the component-wise multiplication by the scalars from $\mathbb{D}$, $\mathbb{D}^2$ becomes a hyperbolic module. Taking into account that $\mathbb{BC}$ is a module over $\mathbb{D}$ and that it is also a ring, we will say that $\mathbb{BC}$ is a commutative algebra over $\mathbb{D}$, or a $\mathbb{D}$-algebra. Finally note that the set $\mathbb{BC}$ is a bicomplex algebra. Finally note that the set $\mathbb{BC}$ is a bicomplex algebra.

3 Bicomplex Derivability and Bicomplex Holomorphy: The Case of Cartesian Representations

In this section we will study bicomplex valued functions of a bicomplex variable, and we will examine the notion of derivability and of holomorphy for such functions. In particular, we will confine ourselves to those properties that can be derived through the cartesian representations of bicomplex numbers that we have introduced in the previous section. We refer to Section 7 for the implications that can be derived by the idempotent representation of bicomplex functions. We will consider $\mathbb{BC}$ as a topological space endowed with the Euclidean topology of $\mathbb{R}^4$ and Ω an open set. Let F be a bicomplex function $F : \Omega \to \mathbb{BC}$ of one bicomplex variable

$$Z = x_1 + \mathbf{i}y_1 + \mathbf{j}x_2 + \mathbf{k}y_2.$$

Due to the various ways in which bicomplex numbers can be written, the function F inherits analogous representations, specifically:

$$\begin{aligned} F &= f_1 + \mathbf{j}f_2 = \rho_1 + \mathbf{i}\rho_2 = g_1 + \mathbf{k}g_2 = \gamma_1 + \mathbf{k}\gamma_2 = \mathfrak{f}_1 + \mathbf{i}\mathfrak{f}_2 = \mathfrak{g}_1 + \mathbf{j}\mathfrak{g}_2 \\ &= f_{11} + \mathbf{i}f_{12} + \mathbf{j}f_{21} + \mathbf{k}f_{22}, \end{aligned} \tag{3.1}$$

where f_1, f_2, g_1, g_2 are $\mathbb{C}(\mathbf{i})$-valued functions, $\rho_1, \rho_2, \gamma_1, \gamma_2$ are $\mathbb{C}(\mathbf{j})$-valued functions, $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{g}_1, \mathfrak{g}_2$ are hyperbolic-valued functions, and $f_{k\ell}$ are real-valued functions, all of them of a bicomplex variable Z. We will, at different times in this section, use all of these representations.

Take now a point $Z_0 \in \Omega$ and let $H = h_{11} + \mathbf{i}h_{12} + \mathbf{j}h_{21} + \mathbf{k}h_{22}$ be the increment. The partial derivatives of F with respect to the variables x_1, y_1, x_2, y_2 are defined as usual and we give their formulas in order to show some peculiarities of this case:

$$\begin{aligned} \frac{\partial F}{\partial x_1}(Z_0) &:= \lim_{h_{11}\to 0} \frac{F(Z_0 + h_{11}) - F(Z_0)}{h_{11}}, \\ \frac{\partial F}{\partial y_1}(Z_0) &:= \lim_{h_{12}\to 0} \frac{F(Z_0 + \mathbf{i}h_{12}) - F(Z_0)}{h_{12}}, \\ \frac{\partial F}{\partial x_2}(Z_0) &:= \lim_{h_{21}\to 0} \frac{F(Z_0 + \mathbf{j}h_{21}) - F(Z_0)}{h_{21}}, \\ \frac{\partial F}{\partial y_2}(Z_0) &:= \lim_{h_{22}\to 0} \frac{F(Z_0 + \mathbf{k}h_{22}) - F(Z_0)}{h_{22}}. \end{aligned} \tag{3.2}$$

Recalling formulas (2.2) and (2.3) we can write the increments as

$$\begin{aligned} H &= h_1 + \mathbf{j}h_2 := (h_{11} + \mathbf{i}h_{12}) + \mathbf{j}(h_{21} + \mathbf{i}h_{22}) \\ &= \kappa_1 + \mathbf{i}\kappa_2 := (h_{11} + \mathbf{j}h_{21}) + \mathbf{i}(h_{12} + \mathbf{j}h_{22}). \end{aligned}$$

This leads us to the

Definition 3.1 The complex partial derivatives of the bicomplex function F are defined as the following limits (if they exist):

$$F'_{z_1}(Z_0) := \lim_{h_1\to 0} \frac{F(Z_0 + h_1) - F(Z_0)}{h_1}, \tag{3.3}$$

$$F'_{z_2}(Z_0) := \lim_{h_2\to 0} \frac{F(Z_0 + \mathbf{j}h_2) - F(Z_0)}{h_2}, \tag{3.4}$$

$$F'_{\zeta_1}(Z_0) := \lim_{\kappa_1 \to 0} \frac{F(Z_0 + \kappa_1) - F(Z_0)}{\kappa_1}, \tag{3.5}$$

$$F'_{\zeta_2}(Z_0) := \lim_{\kappa_2 \to 0} \frac{F(Z_0 + \mathbf{i}\kappa_2) - F(Z_0)}{\kappa_2}. \tag{3.6}$$

Similarly we introduce the hyperbolic partial derivatives: if we recall formula (2.4) and write the increment as

$$H = \mathfrak{h}_1 + \mathbf{i}\mathfrak{h}_2 := (h_{11} + \mathbf{k}h_{22}) + \mathbf{i}(h_{12} + \mathbf{k}(-h_{21})),$$

then the hyperbolic partial derivatives are

$$\begin{aligned} F'_{\mathfrak{z}_1}(Z_0) &:= \lim_{\mathfrak{h}_1 \notin \mathfrak{S}_0(\mathbb{D}),\, \mathfrak{h}_1 \to 0} \frac{F(Z_0 + \mathfrak{h}_1) - F(Z_0)}{\mathfrak{h}_1}, \\ F'_{\mathfrak{z}_2}(Z_0) &:= \lim_{\mathfrak{h}_2 \notin \mathfrak{S}_0(\mathbb{D}),\, \mathfrak{h}_2 \to 0} \frac{F(Z_0 + \mathbf{i}\,\mathfrak{h}_2) - F(Z_0)}{\mathfrak{h}_2}, \end{aligned} \tag{3.7}$$

where with the symbol $\mathfrak{S}_0(\mathbb{D}) := \mathfrak{S}_0 \cap \mathbb{D}$ we indicate the set of the hyperbolic zero-divisors together with $0 \in \mathbb{D}$.

Example 3.2 From the point of view of the classic theory of functions of two complex variables one may think, because of the “symmetry” of the imaginary units **i** and **j**, and thus the analogy between the corresponding complex variables, that there is no reason to define all the complex partial derivatives (3.3)–(3.6). The following simple example illustrates that inside the bicomplex realm the differences are relevant. Consider to this purpose the function $F : \mathbb{BC} \to \mathbb{BC}$, $F(Z) = Z^{\dagger}$. It is immediate to see that

$$F'_{z_1}(Z_0) = 1, \qquad F'_{z_2}(Z_0) = -\mathbf{j}$$

and that neither $F'_{\zeta_1}(Z_0)$ nor $F'_{\zeta_2}(Z_0)$ exists for any $Z_0 \in \mathbb{BC}$.

Remark 3.3 Formulas (2.5)–(2.7) suggest the introduction of six more partial derivatives: four complex and two hyperbolic. But since there exists a direct relation between the corresponding variables, namely,

$$w_1 = z_1, \quad w_2 = -\mathbf{i}z_2, \quad \omega_1 = \zeta_1, \quad \omega_2 = -\mathbf{j}\zeta_2, \quad \mathfrak{w}_1 = \mathfrak{z}_1, \quad \mathfrak{w}_2 = -\mathbf{k}\mathfrak{z}_2,$$

it is easy to see that we do not get anything essentially new. The situation with the partial derivatives with respect to the variables arising from the idempotent representations of bicomplex numbers is on the other hand much more interesting, and will be discussed in detail later in this paper.

We continue now with the definition of the derivative of a bicomplex function $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ of one bicomplex variable Z as follows (see, for instance [10], and references therein):

Definition 3.4 The *derivative* $F'(Z_0)$ of the function F at a point $Z_0 \in \Omega$ is the limit, if it exists,

$$F'(Z_0) := \lim_{Z \to Z_0} \frac{F(Z) - F(Z_0)}{Z - Z_0} = \lim_{\mathfrak{S}_0 \not\ni H \to 0} \frac{F(Z_0 + H) - F(Z_0)}{H}, \tag{3.8}$$

for Z in the domain of F such that $H = Z - Z_0$ is an invertible bicomplex number. In this case, the function F is called *derivable* at Z_0.

Corollary 3.5 *The function F is derivable at Z_0 if and only if there exists a function α_{F,Z_0} such that*

$$\lim_{\mathfrak{S}_0 \not\ni H \to 0} \alpha_{F,Z_0}(H) = 0$$

and

$$F(Z_0 + H) - F(Z_0) = F'(Z_0) \cdot H + \alpha_{F,Z_0}(H)H \quad \text{for all } H \notin \mathfrak{S}_0. \tag{3.9}$$

Remark 3.6 It is necessary to make a comment here. Traditionally, see e.g. [18], p. 138 and p. 432, if h is either a real or a complex increment, the symbol $\mathfrak{o}(h)$ is used to indicate any expression of the form $\alpha(h)|h|$ for $\lim_{h \to 0} \alpha(h) = 0$. Since, both in the real and in the complex case, the expression $\frac{|h|}{h}$ remains bounded when $h \to 0$, it is clear that one could replace $\alpha(h)|h|$ by $\alpha(h)h$ in the expression of $\mathfrak{o}$. However, the situation is quite different in the bicomplex case. Here, the expression $\frac{|H|}{H}$ is not bounded when $H \to 0$, and therefore we need to carefully distinguish the two expressions. In accordance with the usual notation, we will always use $\mathfrak{o}(H)$ to denote a function of the form $\alpha(H)|H|$, and therefore the expression in the previous corollary is not, in general, $\mathfrak{o}(H)$. This distinction is at the basis of the notions of weak and strong Stoltz conditions for bicomplex functions, which are introduced by Price in [10]. We will discuss in detail how those conditions compare with our analysis elsewhere.

Remark 3.7 A bicomplex function F derivable at Z_0 enjoys the following property:

$$\lim_{H \notin \mathfrak{S}_0,\, H \to 0} (F(Z_0 + H) - F(Z_0)) = 0. \tag{3.10}$$

In other words, a function F, which is derivable at a point Z_0, enjoys some sort of weakened continuity at that point, in the sense that $F(Z)$ converges to $F(Z_0)$ as long as Z converges to Z_0 in such a way that $Z - Z_0$ is invertible. We will see later on that this restriction can be removed under reasonable assumptions.

The existence of the derivative $F'(Z_0)$ has important consequences, as we show below.

Theorem 3.8 *Consider a bicomplex function $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ derivable at $Z_0 \in \Omega$. Then the following statements are true:*

1. *The real partial derivatives $\frac{\partial F}{\partial x_\ell}(Z_0)$ and $\frac{\partial F}{\partial y_\ell}(Z_0)$ exist, for $\ell = 1, 2$.*

2. *The real partial derivatives satisfy the following identities:*

$$F'(Z_0) = \frac{\partial F}{\partial x_1}(Z_0) = -\mathbf{i}\frac{\partial F}{\partial y_1}(Z_0) = -\mathbf{j}\frac{\partial F}{\partial x_2}(Z_0) = \mathbf{k}\frac{\partial F}{\partial y_2}(Z_0). \tag{3.11}$$

Proof Because F is derivable at Z_0, the limit (3.8) exists no matter how H converges to 0, as long as H is invertible. Consider first H to be real, i.e., of the form $H = x_1 = x_1 + \mathbf{i}0 + \mathbf{j}0 + \mathbf{k}0 \to 0$, $x_1 \neq 0$, which is always invertible. Then, since there exists $F'(Z_0)$, the following limit also exists:

$$\lim_{x_1 \to 0} \frac{F(Z_0 + x_1) - F(Z_0)}{x_1}$$

and it coincides with $\frac{\partial F}{\partial x_1}(Z_0)$.

The rest of the proof follows by considering the three specific forms of the increment H along the units $\mathbf{i}, \mathbf{j}, \mathbf{k}$, i.e., $H = \mathbf{i}y_1$, with $y_1 \neq 0$; $H = \mathbf{j}x_2$, with $x_2 \neq 0$; $H = \mathbf{k}y_2$, with $y_2 \neq 0$, and noting that all are invertible bicomplex numbers. □

Let us write the bicomplex function as $F = f_{11} + \mathbf{i} f_{12} + \mathbf{j} f_{21} + \mathbf{k} f_{22}$ in terms of its real components, which are all real functions of a bicomplex variable. An immediate consequence of the theorem above is:

Corollary 3.9 *If F is derivable at Z_0, then the real Jacobi matrix of F at Z_0 is of the form*

$$\begin{pmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & c & b & a \end{pmatrix}, \tag{3.12}$$

where

$$\begin{aligned} a &:= \frac{\partial f_{11}}{\partial x_1} = \frac{\partial f_{12}}{\partial y_1} = \frac{\partial f_{21}}{\partial x_2} = \frac{\partial f_{22}}{\partial y_2}, \\ b &:= -\frac{\partial f_{11}}{\partial y_1} = \frac{\partial f_{12}}{\partial x_1} = -\frac{\partial f_{21}}{\partial y_2} = \frac{\partial f_{22}}{\partial x_2}, \\ c &:= -\frac{\partial f_{11}}{\partial x_2} = -\frac{\partial f_{12}}{\partial y_2} = \frac{\partial f_{21}}{\partial x_1} = \frac{\partial f_{22}}{\partial y_1}, \\ d &:= \frac{\partial f_{11}}{\partial y_2} = -\frac{\partial f_{12}}{\partial x_2} = -\frac{\partial f_{21}}{\partial y_1} = \frac{\partial f_{22}}{\partial x_1}, \end{aligned} \tag{3.13}$$

and all the partial derivatives are evaluated at the point Z_0.

Proof The proof relies on a direct computation using the equalities (3.11) written in terms of $f_{k\ell}$. □

Remark 3.10 The reader should notice that the special form of the real Jacobi matrix above encodes several Cauchy–Riemann type conditions on (certain pairs of) the real functions $f_{k\ell}$, a fact which we will exploit in detail below.

Remark 3.11 Every 4×4 matrix with real entries determines a linear transformation on $\mathbb{R}^4$. Of course, not all of them remain linear when $\mathbb{R}^4$ is seen as $\mathbb{BC}$, i.e., not all of them are $\mathbb{BC}$-linear. Those matrices which are $\mathbb{BC}$-linear are of the form (3.12). Taking into account that the entries of (3.12) are the values of the partial derivatives at Z_0, one may conclude that (3.12) determines a $\mathbb{BC}$-linear operator acting on the tangential $\mathbb{BC}$-linear module at the point Z_0. Moreover, in the matrix (3.12) there are hidden the two linearities with respect to $\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$, in the following sense: fixing the identifications

$$(x_1 + \mathbf{i}y_1, x_2 + \mathbf{i}y_2) = (z_1, z_2) \longleftrightarrow (x_1, y_1, x_2, y_2)$$

and

$$(x_1 + \mathbf{j}x_2, y_1 + \mathbf{j}y_2) = (\zeta_1, \zeta_2) \longleftrightarrow (x_1, y_1, x_2, y_2)$$

we have two different identifications $\mathbb{C}^2(\mathbf{i}) \leftrightarrow \mathbb{R}^4$ and $\mathbb{C}^2(\mathbf{j}) \leftrightarrow \mathbb{R}^4$. It follows that a 4×4 matrix with real entries determines not only a real transformation but a $\mathbb{C}(\mathbf{i})$-linear one if and only if it is of the form

$$\begin{pmatrix} l & -m & u & -v \\ m & l & v & u \\ t & -s & g & -h \\ s & t & h & g \end{pmatrix}.$$

Similarly, a 4×4 matrix with real entries represents a $\mathbb{C}(\mathbf{j})$-linear transformation if and only if it is if the form

$$\begin{pmatrix} A & B & -E & -F \\ C & D & -G & -H \\ E & F & A & B \\ G & H & C & D \end{pmatrix}.$$

Thus the structure of the matrix (3.12) includes both complex structures.

As a next step, we investigate the consequence of the existence of $F'(Z_0)$ in terms of the complex variables $z_1, z_2 \in \mathbb{C}(\mathbf{i})$, where we write the bicomplex variable as $Z = z_1 + \mathbf{j}z_2$. We prove the following

Theorem 3.12 *Consider a bicomplex function $F = f_1 + \mathbf{j} f_2$ derivable at Z_0. Then we have:*

1. *The $\mathbb{C}(\mathbf{i})$-complex partial derivatives $F'_{z_\ell}(Z_0)$ exist, for $\ell = 1, 2$.*
2. *The complex partial derivatives above verify the identity:*

$$F'(Z_0) = F'_{z_1}(Z_0) = -\mathbf{j}F'_{z_2}(Z_0), \tag{3.14}$$

which is equivalent to the $\mathbb{C}(\mathbf{i})$-complex Cauchy–Riemann system *for* F *(at* Z_0*), also called the* generalized Cauchy–Riemann system *in* [13]:

$$f'_{1,z_1}(Z_0) = f'_{2,z_2}(Z_0), \qquad f'_{1,z_2}(Z_0) = -f'_{2,z_1}(Z_0). \tag{3.15}$$

Proof As in the proof of Theorem 3.8, the limit of the difference quotient must be the same regardless of the path on which H approaches zero, as long as H is invertible. Let us choose $H = h_1 = h_1 + \mathbf{j}\,0 \to 0$, and note that if $Z_0 = z_{01} + \mathbf{j}z_{02}$ then $Z_0 + H = (z_{01} + h_1) + \mathbf{j}z_{02}$. Then, since $F'(Z_0)$ exists, the following limit also exists:

$$\lim_{h_1 \to 0} \frac{F(Z_0 + h_1) - F(Z_0)}{h_1} = F'_{z_1}(Z_0).$$

Similarly, taking $H = \mathbf{j}\,h_2 \to 0$ we get:

$$F'(Z_0) = \lim_{h_2 \to 0} \frac{F(Z_0 + \mathbf{j}h_2) - F(Z_0)}{\mathbf{j}h_2} = -\mathbf{j}F'_{z_2}(Z_0).$$

In conclusion, the complex partial derivatives of F with respect to z_1 and z_2 exist (at Z_0), and they verify the equality

$$F'(Z_0) = F'_{z_1}(Z_0) = -\mathbf{j}F'_{z_2}(Z_0). \tag{3.16}$$

If we write $F = f_1 + \mathbf{j}f_2$, the complex partial derivatives of F at Z_0 are therefore given by:

$$F'_{z_1}(Z_0) = f'_{1,z_1}(Z_0) + \mathbf{j}f'_{2,z_1}(Z_0), \qquad F'_{z_2}(Z_0) = f'_{1,z_2}(Z_0) + \mathbf{j}f'_{2,z_2}(Z_0).$$

Now it is immediate to see that equality (3.16) is equivalent to the $\mathbb{C}(\mathbf{i})$-*complex* Cauchy–Riemann conditions for F at Z_0. □

Notice that the symbol f'_{1,z_1} (and the other similarly defined symbols) denotes in our situation the "authentic" complex partial derivative, not the formal operation

$$\frac{\partial f}{\partial z_1} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - \mathbf{i}\frac{\partial f}{\partial y_1}\right)$$

defined on $\mathcal{C}^1$-functions. At the same time, these results indicate that bicomplex functions which are derivable in a domain are related to holomorphic mappings of two complex variables. We will come back in more detail to this issue at the end of this section.

Corollary 3.13 *Let* $F = f_1 + \mathbf{j}f_2$ *be a bicomplex function derivable at* Z_0*, then the real components of the functions* $f_1 = f_{11} + \mathbf{i}f_{12}$ *and* $f_2 = f_{21} + \mathbf{i}f_{22}$ *verify the* *usual real Cauchy–Riemann system (at Z_0) associated with each complex variable $z_1 = x_1 + \mathbf{i}y_1$ and $z_2 = x_2 + \mathbf{i}y_2$; in complex notation this is equivalent to*

$$\frac{\partial F}{\partial \bar{z}_1}(Z_0) = \frac{\partial F}{\partial \bar{z}_2}(Z_0) = 0, \tag{3.17}$$

i.e.,

$$\frac{\partial f_1}{\partial \bar{z}_1}(Z_0) = \frac{\partial f_1}{\partial \bar{z}_2}(Z_0) = \frac{\partial f_2}{\partial \bar{z}_1}(Z_0) = \frac{\partial f_2}{\partial \bar{z}_2}(Z_0) = 0, \tag{3.18}$$

where the symbols $\frac{\partial}{\partial \bar{z}_1}$ and $\frac{\partial}{\partial \bar{z}_2}$ are the commonly used formal operations on $\mathbb{C}(\mathbf{i})$-valued functions of z_1 and z_2.

Proof Let us employ the equalities (3.11) involving the real partial derivatives of $F = f_1 + \mathbf{j} f_2$ at Z_0. The second equality in (3.11) is

$$\frac{\partial F}{\partial x_1}(Z_0) = -\mathbf{i}\frac{\partial F}{\partial y_1}(Z_0), \tag{3.19}$$

which is equivalent to (for simplicity we eliminate the explicit reference to Z_0)

$$\frac{\partial f_{11}}{\partial x_1} + \mathbf{i}\frac{\partial f_{12}}{\partial x_1} + \mathbf{j}\frac{\partial f_{21}}{\partial x_1} + \mathbf{ij}\frac{\partial f_{22}}{\partial x_1} = -\mathbf{i}\left(\frac{\partial f_{11}}{\partial y_1} + \mathbf{i}\frac{\partial f_{12}}{\partial y_1} + \mathbf{j}\frac{\partial f_{21}}{\partial y_1} + \mathbf{ij}\frac{\partial f_{22}}{\partial y_1}\right).$$

Because the functions $f_{k\ell}$ are real, this is equivalent to the system

$$\begin{aligned} \frac{\partial f_{11}}{\partial x_1} &= \frac{\partial f_{12}}{\partial y_1}, \qquad & \frac{\partial f_{11}}{\partial y_1} &= -\frac{\partial f_{12}}{\partial x_1}, \\ \frac{\partial f_{21}}{\partial x_1} &= \frac{\partial f_{22}}{\partial y_1}, \qquad & \frac{\partial f_{21}}{\partial y_1} &= -\frac{\partial f_{22}}{\partial x_1}. \end{aligned} \tag{3.20}$$

These are the real Cauchy–Riemann conditions for the complex functions $f_1 = f_{11} + \mathbf{i} f_{12}$ and $f_2 = f_{21} + \mathbf{i} f_{22}$, with respect to the complex variable $z_1 = x_1 + \mathbf{i}y_1$. In complex notation, if we write the equality (3.19) in terms of the complex differential operator $\frac{\partial}{\partial \bar{z}_1} = \frac{\partial}{\partial x_1} + \mathbf{i}\frac{\partial}{\partial y_1}$, it becomes equivalent to the first part of (3.17).

We repeat this reasoning for the equality

$$-\mathbf{j}\frac{\partial F}{\partial x_2}(Z_0) = \mathbf{ij}\frac{\partial F}{\partial y_2}(Z_0), \tag{3.21}$$

which, after division by $-\mathbf{j}$, is equivalent to the second equality in (3.17). This leads to the conclusion that f_1 and f_2 verify the real Cauchy–Riemann system with respect to the variable z_2 at Z_0. □

Let us now express the bicomplex variable as $Z = \zeta_1 + \mathbf{i}\,\zeta_2$, and the bicomplex function as $F = \rho_1 + \mathbf{i}\rho_2$, where $\rho_1 = f_{11} + \mathbf{j} f_{21}$ and $\rho_2 = f_{12} + \mathbf{j} f_{22}$ are $\mathbb{C}(\mathbf{j})$-valued functions. We prove the following

Theorem 3.14 *Consider a bicomplex function F derivable at Z_0. Then*

1. *The $\mathbb{C}(\mathbf{j})$-complex partial derivatives $F'_{\zeta_\ell}(Z_0)$ exist, for $\ell = 1, 2$.*
2. *The complex partial derivatives above verify the equality:*

$$F'(Z_0) = F'_{\zeta_1}(Z_0) = -\mathbf{i}F'_{\zeta_2}(Z_0), \tag{3.22}$$

which is equivalent to the $\mathbb{C}(\mathbf{j})$-complex Cauchy–Riemann system *(at Z_0):*

$$\rho'_{1,\zeta_1}(Z_0) = \rho'_{2,\zeta_2}(Z_0), \qquad \rho'_{1,\zeta_2}(Z_0) = -\rho'_{2,\zeta_1}(Z_0). \tag{3.23}$$

Proof In the definition of $F'(Z_0)$ we consider the limit of the difference quotient as $H \to 0$ first through invertible values of the form $H = \kappa_1 + \mathbf{i}0$, and then through values of the form $H = 0 + \mathbf{i}\kappa_2$, where $\kappa_1, \kappa_2 \in \mathbb{C}(\mathbf{j})$. A computation as in Theorem 3.12 leads to the existence of the complex partial derivatives of F at Z_0 with respect to the variables $\zeta_1, \zeta_2 \in \mathbb{C}(\mathbf{j})$, and one can show that they verify the equation:

$$F'(Z_0) = F'_{\zeta_1}(Z_0) = -\mathbf{i}F'_{\zeta_2}(Z_0), \tag{3.24}$$

where

$$F'_{\zeta_1}(Z_0) = \rho'_{1,\zeta_1}(Z_0) + \mathbf{i}\rho'_{2,\zeta_1}(Z_0), \qquad F'_{\zeta_2}(Z_0) = \rho'_{1,\zeta_2}(Z_0) + \mathbf{i}\rho'_{2,\zeta_2}(Z_0).$$

Now it is immediate to see that the equality (3.24) is equivalent to the system (3.23). □

Corollary 3.15 *If $F = \rho_1 + \mathbf{i}\rho_2$ is bicomplex derivable at Z_0, then the real components of the functions $\rho_1 = f_{11} + \mathbf{j}f_{21}$ and $\rho_2 = f_{12} + \mathbf{j}f_{22}$ verify the usual real Cauchy–Riemann system (at Z_0) associated to each complex variable $\zeta_1 = x_1 + \mathbf{j}x_2$ and $\zeta_2 = y_1 + \mathbf{j}y_2$; in complex notation this is equivalent to*

$$\frac{\partial F}{\partial \zeta_1^*}(Z_0) = \frac{\partial F}{\partial \zeta_2^*}(Z_0) = 0, \tag{3.25}$$

i.e.,

$$\frac{\partial \rho_1}{\partial \zeta_1^*}(Z_0) = \frac{\partial \rho_1}{\partial \zeta_2^*}(Z_0) = \frac{\partial \rho_2}{\partial \zeta_1^*}(Z_0) = \frac{\partial \rho_2}{\partial \zeta_2^*}(Z_0) = 0. \tag{3.26}$$

Here

$$\frac{\partial}{\partial \zeta_1^*} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial \zeta_2^*} := \frac{1}{2}\left(\frac{\partial}{\partial y_1} + \mathbf{j}\frac{\partial}{\partial y_2}\right).$$

Proof The following equalities from (3.11) among the real partial derivatives of F

$$\frac{\partial F}{\partial x_1}(Z_0) = -\mathbf{j}\frac{\partial F}{\partial x_2}(Z_0), \qquad -\mathbf{i}\frac{\partial F}{\partial y_1}(Z_0) = \mathbf{ij}\frac{\partial F}{\partial y_2}(Z_0) \tag{3.27}$$

are equivalent to (3.25). □

Remark 3.16 We showed in Example 3.2 that it is possible for a bicomplex function to have partial derivatives with respect to z_1, z_2, and not to have partial derivatives with respect to ζ_1, ζ_2. Now we see that if the function is bicomplex derivable then such a situation is not possible: a bicomplex derivable function has complex partial derivatives with respect to the $\mathbb{C}(\mathbf{i})$ complex variables z_1, z_2 as well as with respect to the $\mathbb{C}(\mathbf{j})$ variables ζ_1, ζ_2. Moreover, formulas (3.14) and (3.22) show that such derivatives are related by

$$F'_{z_1}(Z_0) = F'_{\zeta_1}(Z_0) = -jF'_{z_2}(Z_0) = -iF'_{\zeta_2}(Z_0).$$

We express now the bicomplex variable $Z = \mathfrak{z}_1 + \mathbf{i}\mathfrak{z}_2$, where $\mathfrak{z}_1 = x_1 + \mathbf{k}y_2$ and $\mathfrak{z}_2 = y_1 + \mathbf{k}(-x_2)$ are hyperbolic numbers, and the function $F = \mathfrak{f}_1 + \mathbf{i}\mathfrak{f}_2$, where $\mathfrak{f}_1 = f_{11} + \mathbf{k}f_{22}$ and $\mathfrak{f}_2 = f_{12} + \mathbf{k}(-f_{21})$. We prove the following

Theorem 3.17 *Consider a bicomplex function $F = \mathfrak{f}_1 + \mathbf{i}\mathfrak{f}_2$ derivable at a point Z_0. Then*

1. *The hyperbolic partial derivatives $F'_{\mathfrak{z}_\ell}$ exist, for $\ell = 1, 2$.*
2. *The partial derivatives above verify the equality*

$$F'(Z_0) = F'_{\mathfrak{z}_1}(Z_0) = -\mathbf{i}F'_{\mathfrak{z}_2}(Z_0),$$

which is equivalent to the following Cauchy–Riemann type system for the hyperbolic components of a bicomplex derivable function:

$$\mathfrak{f}'_{1,\mathfrak{z}_1}(Z_0) = \mathfrak{f}'_{2,\mathfrak{z}_2}(Z_0), \qquad \mathfrak{f}'_{1,\mathfrak{z}_2}(Z_0) = -\mathfrak{f}'_{2,\mathfrak{z}_1}(Z_0). \tag{3.28}$$

Proof The existence of the hyperbolic partial derivatives of $F = F(\mathfrak{z}_1, \mathfrak{z}_2)$ is obtained in a similar fashion as above, letting $H \to 0$ through invertible hyperbolic values $H = \mathfrak{h}_1 + \mathbf{i}0$ and then on paths of the form $H = \mathbf{i}\mathfrak{h}_2$, where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are invertible hyperbolic numbers. We obtain also:

$$F'(Z_0) = F'_{\mathfrak{z}_1}(Z_0) = -\mathbf{i}F'_{\mathfrak{z}_2}(Z_0),$$

which is equivalent to the system (3.28). □

Corollary 3.18 *If $F = \mathfrak{f}_1 + \mathbf{i}\mathfrak{f}_2$ is derivable at Z_0 then the real components of the hyperbolic functions $\mathfrak{f}_1 = f_{11} + \mathbf{k}f_{22}$ and $\mathfrak{f}_2 = f_{12} + \mathbf{k}(-f_{21})$ verify the* Cauchy–Riemann type systems *with respect to both variables $\mathfrak{z}_1, \mathfrak{z}_1 \in \mathbb{D}$; in hyperbolic terms this is equivalent to:*

$$\frac{\partial F}{\partial \mathfrak{z}_1^\diamond}(Z_0) = \frac{\partial F}{\partial \mathfrak{z}_2^\diamond}(Z_0) = 0, \tag{3.29}$$

i.e.,

$$\frac{\partial \mathfrak{f}_1}{\partial \mathfrak{z}_1^\diamond}(Z_0) = \frac{\partial \mathfrak{f}_1}{\partial \mathfrak{z}_2^\diamond}(Z_0) = \frac{\partial \mathfrak{f}_2}{\partial \mathfrak{z}_1^\diamond}(Z_0) = \frac{\partial \mathfrak{f}_2}{\partial \mathfrak{z}_2^\diamond}(Z_0) = 0, \tag{3.30}$$

where

$$\frac{\partial}{\partial \mathfrak{z}_1^{\diamond}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \mathbf{k}\frac{\partial}{\partial y_2}\right), \qquad \frac{\partial}{\partial \mathfrak{z}_2^{\diamond}} = \frac{1}{2}\left(\frac{\partial}{\partial y_1} + \mathbf{k}\frac{\partial}{\partial x_2}\right). \tag{3.31}$$

Proof We use once again the following equalities from (3.11):

$$\frac{\partial F}{\partial x_1}(Z_0) = \mathbf{k}\frac{\partial F}{\partial y_2}(Z_0), \qquad \frac{\partial F}{\partial y_1}(Z_0) = -\mathbf{k}\frac{\partial F}{\partial x_2}(Z_0), \tag{3.32}$$

where $\mathbf{k} = \mathbf{ij}$. Recalling formulas (3.31) we obtain equality (3.29). □

Remark 3.19 Appealing again to formulas (2.5)–(2.7) which deal with the complex variables $w_1, w_2, \omega_1, \omega_2$ and the hyperbolic variables $\mathfrak{w}_1, \mathfrak{w}_2$, one may wonder: what about the Cauchy–Riemann conditions with respect to the corresponding partial derivatives? Remark 3.3 explains how they can be obtained directly from the previous statements. We omit the details.

Definition 3.20 Let F be a bicomplex function defined on a non-empty open set $\Omega \subset \mathbb{BC}$. If F has bicomplex derivative at each point of Ω, we will say that F is a *bicomplex holomorphic*, or $\mathbb{BC}$-holomorphic, function.

Thus for a $\mathbb{BC}$-holomorphic function F all the conclusions made in this section hold in the whole domain. Theorem 3.12 says that F is holomorphic with respect to z_1 for any z_2 fixed and F is holomorphic with respect to z_2 for any z_1 fixed. Thus, see for instance [6, pages 4-5], F is holomorphic in the classical sense of two complex variables. This implies immediately many quite useful properties of F, in particular, it is of class $\mathcal{C}^{\infty}(\Omega)$, and the reader may compare this with Remark 3.7 where we were able, working with just one point, not with a domain, to state a weakened continuity at the point.

4 Interplay Between Real Differentiability and Derivability of Bicomplex Functions

4.1 Real Differentiability in Complex and Hyperbolic Terms

We begin now assuming that Ω is an open set in $\mathbb{BC}$ and $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ is a bicomplex function of class $\mathcal{C}^1(\Omega)$. We are going to work with $\mathbb{BC}$-holomorphic functions and we want to determine the place that $\mathbb{BC}$-holomorphic functions occupy among the bicomplex $\mathcal{C}^1$ functions. The condition $F \in \mathcal{C}^1(\Omega, \mathbb{BC})$ ensures that F is real differentiable for any $Z \in \Omega$, i.e., that

$$\begin{aligned} F(Z+H) - F(Z) = {} & \frac{\partial F}{\partial x_1}(Z)h_{11} + \frac{\partial F}{\partial y_1}(Z)h_{12} \\ & + \frac{\partial F}{\partial x_2}(Z)h_{21} + \frac{\partial F}{\partial y_2}(Z)h_{22} + \mathfrak{o}(H), \end{aligned} \tag{4.1}$$

where $Z = x_1 + \mathbf{i}y_1 + \mathbf{j}x_2 + \mathbf{k}y_2$, $H = h_{11} + \mathbf{i}h_{12} + \mathbf{j}h_{21} + \mathbf{k}h_{22}$. As a matter of fact, this formula does not depend on how the function F and the variable Z are written, and therefore it will be quite helpful to analyze the structure of such functions.

First of all, let us write (4.1) in terms of the $\mathbb{C}(\mathbf{i})$-complex variables $h_1 := h_{11} + \mathbf{i}h_{12}$ and $h_2 := h_{21} + \mathbf{i}h_{22}$, so that:

$$h_{11} = \frac{h_1 + \overline{h}_1}{2}, \qquad h_{12} = \frac{h_1 - \overline{h}_1}{2\mathbf{i}},$$
$$h_{21} = \frac{h_2 + \overline{h}_2}{2}, \qquad h_{22} = \frac{h_2 - \overline{h}_2}{2\mathbf{i}}.$$

Grouping adequately the terms in (4.1), we obtain:

$$\begin{aligned} F(Z+H) - F(Z) = {} & \frac{1}{2}\left(\frac{\partial F}{\partial x_1}(Z) - \mathbf{i}\frac{\partial F}{\partial y_1}(Z)\right) \cdot h_1 \\ & + \frac{1}{2}\left(\frac{\partial F}{\partial x_1}(Z) + \mathbf{i}\frac{\partial F}{\partial y_1}(Z)\right) \cdot \overline{h}_1 + \frac{1}{2}\left(\frac{\partial F}{\partial x_2}(Z) - \mathbf{i}\frac{\partial F}{\partial y_2}(Z)\right) \cdot h_2 \\ & + \frac{1}{2}\left(\frac{\partial F}{\partial x_2}(Z) + \mathbf{i}\frac{\partial F}{\partial y_2}(Z)\right) \cdot \overline{h}_2 + \mathfrak{o}(H). \end{aligned}$$

If we employ the $\mathbb{C}(\mathbf{i})$-complex variables $z_1 = x_1 + \mathbf{i}y_1$ and $z_2 = x_2 + \mathbf{i}y_2$, and the usual complex differential operators:

$$\frac{\partial}{\partial z_1} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \mathbf{i}\frac{\partial}{\partial y_1}\right), \qquad \frac{\partial}{\partial \overline{z}_1} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \mathbf{i}\frac{\partial}{\partial y_1}\right),$$
$$\frac{\partial}{\partial z_2} := \frac{1}{2}\left(\frac{\partial}{\partial x_2} - \mathbf{i}\frac{\partial}{\partial y_2}\right), \qquad \frac{\partial}{\partial \overline{z}_2} := \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \mathbf{i}\frac{\partial}{\partial y_2}\right),$$

we obtain the formula:

$$\begin{aligned} F(Z+H) - F(Z) = {} & \frac{\partial F}{\partial z_1}(Z) \cdot h_1 + \frac{\partial F}{\partial \overline{z}_1}(Z) \cdot \overline{h}_1 \\ & + \frac{\partial F}{\partial z_2}(Z) \cdot h_2 + \frac{\partial F}{\partial \overline{z}_2}(Z) \cdot \overline{h}_2 + \mathfrak{o}(H). \end{aligned} \tag{4.2}$$

We emphasize that (4.2) does not express a new notion; indeed, this is simply the condition of real differentiability for a $\mathcal{C}^1$ bicomplex function, although it is now expressed in $\mathbb{C}(\mathbf{i})$ complex terms.

Note that in Sect. 3 we introduced the symbols $F'_{z_1}(Z)$ and $F'_{z_2}(Z)$ instead of the symbols $\frac{\partial F}{\partial z_1}(Z)$ and $\frac{\partial F}{\partial z_2}(Z)$ because the former are complex partial derivatives, defined, as usual, as limits of suitable difference quotients, meanwhile the latter indicates well known operators acting on $\mathcal{C}^1$-functions. The relationship between these two notions is clarified by the following definition and theorem.

Definition 4.1 A bicomplex function F is called $\mathbb{C}(\mathbf{i})$-complex differentiable if

$$F(Z+H) - F(Z) = F'_{z_1}(Z) \cdot h_1 + F'_{z_2}(Z) \cdot h_2 + \mathfrak{o}(H).$$

Theorem 4.2 *A $\mathcal{C}^1$ bicomplex function F is $\mathbb{C}(\mathbf{i})$ complex differentiable if and only if both its components f_1, f_2 are holomorphic functions in the sense of two complex variables.*

Proof The partial derivative $F'_{z_1}(Z)$ exists in Ω if and only if the operator $\frac{\partial}{\partial \bar{z}_1}$ (which one can think of as a dual to the operator $\frac{\partial}{\partial z_1}$) annihilates the function F; that is, if and only if F is holomorphic as a function of z_1; this can be proved by taking $h_2 = 0$ and $h_1 \neq 0$ in (4.2). What remains is

$$F(Z+H) - F(Z) = F(Z+h_1) - F(Z) = \frac{\partial F}{\partial z_1}(Z) \cdot h_1 + \frac{\partial F}{\partial \bar{z}_1}(Z) \cdot \overline{h}_1 + \mathfrak{o}(H),$$

and since $\frac{\overline{h}_1}{h_1}$ has no limit when $h_1 \to 0$, then we conclude that $F'_{z_1}(Z)$ exists if and only if $\frac{\partial F}{\partial \bar{z}_1}(Z) = 0$. In this case, $F'_{z_1}(Z) = \frac{\partial F}{\partial z_1}(Z)$.

Similarly, the partial derivative $F'_{z_2}(Z)$ exists in Ω if and only if the operator $\frac{\partial}{\partial \bar{z}_2}$ annihilates the function F; this is because we can take now $h_1 = 0, h_2 \neq 0$ in (4.2). Therefore $F'_{z_2}(Z) = \frac{\partial F}{\partial z_2}(Z)$.

Finally, we can assume both conditions

$$\frac{\partial F}{\partial \bar{z}_1}(Z) = \frac{\partial F}{\partial \bar{z}_2}(Z) = 0$$

to be fulfilled in Ω, with (4.2) becoming

$$F(Z+H) - F(Z) = F'_{z_1}(Z) \cdot h_1 + F'_{z_2}(Z) \cdot h_2 + \mathfrak{o}(H). \tag{4.3}$$

This concludes the proof. □

Note that for an arbitrary $\mathbb{C}(\mathbf{i})$ complex differentiable bicomplex function, in general, there is no relation between its complex partial derivatives.

Very similar calculations can be made if we write the bicomplex number as $Z = \zeta_1 + \mathbf{i}\zeta_2$, where now $\kappa_1 := h_{11} + \mathbf{j}h_{21}$ and $\kappa_2 := h_{12} + \mathbf{j}h_{22}$. If one follows the steps indicated above, one eventually obtains the analog of (4.2):

$$\begin{aligned} F(Z+H) - F(Z) = & \frac{\partial F}{\partial \zeta_1}(Z)\kappa_1 + \frac{\partial F}{\partial \overline{\zeta}_1}(Z)\overline{\kappa}_1 \\ & + \frac{\partial F}{\partial \zeta_2}(Z)\kappa_2 + \frac{\partial F}{\partial \overline{\zeta}_2}(Z)\overline{\kappa}_2 + \mathfrak{o}(H), \end{aligned} \tag{4.4}$$

where the expressions $\frac{\partial}{\partial \zeta_1}$ (and similar) are the usual complex differential operators in $\mathbb{C}(\mathbf{j})$. Again, this simply represents the real differentiability of a $\mathcal{C}^1$ bicomplex function in $\mathbb{C}(\mathbf{j})$ complex terms.

The same analysis as above applies to the equation (4.4). In particular, the definition of the $\mathbb{C}(\mathbf{j})$ complex differentiability of a bicomplex function is

$$F(Z+H)-F(Z)=F'_{\zeta_1}(Z)\cdot\kappa_1+F'_{\zeta_2}(Z)\cdot\kappa_2+\mathfrak{o}(H).$$

Notice that Remark 3.3 explains why the other complex variables $w_1, w_2, \omega_1, \omega_2$ do not present any interest in the analysis of the increment of the function.

Our last step consists in writing $Z=\mathfrak{z}_1+\mathbf{i}\mathfrak{z}_2$, where $\mathfrak{z}_1 := x_1+\mathbf{k}\,y_2$ and $\mathfrak{z}_2 := y_1+\mathbf{k}(-x_2)$ are hyperbolic variables. The hyperbolic increments are $\mathfrak{h}_1 := h_{11}+\mathbf{k}h_{22}$ and $\mathfrak{h}_2 := h_{12}+\mathbf{k}(-h_{21})$. Then, using the formulas

$$\begin{aligned} h_{11} &= \frac{\mathfrak{h}_1+\mathfrak{h}_1^\diamond}{2}, \qquad & h_{22} &= \frac{\mathfrak{h}_1-\mathfrak{h}_1^\diamond}{2\mathbf{k}}\,, \\ h_{12} &= \frac{\mathfrak{h}_2+\mathfrak{h}_2^\diamond}{2}, \qquad & h_{21} &= -\frac{\mathfrak{h}_2-\mathfrak{h}_2^\diamond}{2\mathbf{k}}\,, \end{aligned}$$

we regroup the right-hand side of (4.1) as follows:

$$\begin{aligned} F(Z+H)-F(Z) &= \frac{\partial F}{\partial x_1}(Z)\cdot\frac{\mathfrak{h}_1+\mathfrak{h}_1^\diamond}{2}+\frac{\partial F}{\partial y_1}(Z)\cdot\frac{\mathfrak{h}_2+\mathfrak{h}_2^\diamond}{2} \\ &\quad -\frac{\partial F}{\partial x_2}(Z)\cdot\frac{\mathfrak{h}_2-\mathfrak{h}_2^\diamond}{2\mathbf{k}}+\frac{\partial F}{\partial y_2}(Z)\cdot\frac{\mathfrak{h}_1-\mathfrak{h}_1^\diamond}{2\mathbf{k}}+\mathfrak{o}(H) \\ &= \frac{1}{2}\left(\frac{\partial F}{\partial x_1}(Z)+\mathbf{k}\frac{\partial F}{\partial y_2}(Z)\right)\mathfrak{h}_1+\frac{1}{2}\left(\frac{\partial F}{\partial x_1}(Z)-\mathbf{k}\frac{\partial F}{\partial y_2}(Z)\right)\mathfrak{h}_1^\diamond \\ &\quad +\frac{1}{2}\left(\frac{\partial F}{\partial y_1}(Z)-\mathbf{k}\frac{\partial F}{\partial x_2}(Z)\right)\mathfrak{h}_2+\frac{1}{2}\left(\frac{\partial F}{\partial y_1}(Z)+\mathbf{k}\frac{\partial F}{\partial x_2}(Z)\right)\mathfrak{h}_2^\diamond \\ &\quad +\mathfrak{o}(H). \end{aligned}$$

The “hyperbolic” differential operators appearing above are consistent with the ones in the context of hyperbolic analysis. For the hyperbolic variable $\mathfrak{z}=x+\mathbf{k}\,y$, the formal hyperbolic partial derivatives are given by the formulas:

$$\frac{\partial}{\partial\mathfrak{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathbf{k}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial\mathfrak{z}^\diamond}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathbf{k}\frac{\partial}{\partial y}\right),$$

where $\mathfrak{z}^\diamond = x-\mathbf{k}\,y$ is the hyperbolic conjugate of $\mathfrak{z}$. Therefore, in terms of the hyperbolic variables $\mathfrak{z}_1 = x_1+\mathbf{k}\,y_2$ and $\mathfrak{z}_2 = y_1+\mathbf{k}(-x_2)$ and the corresponding hyperbolic differential operators, we obtain:

$$F(Z+H)-F(Z)=\frac{\partial F}{\partial\mathfrak{z}_1}(Z)\mathfrak{h}_1+\frac{\partial F}{\partial\mathfrak{z}_1^\diamond}(Z)\mathfrak{h}_1^\diamond+\frac{\partial F}{\partial\mathfrak{z}_2}(Z)\mathfrak{h}_2+\frac{\partial F}{\partial\mathfrak{z}_2^\diamond}(Z)\mathfrak{h}_2^\diamond+\mathfrak{o}(H), \tag{4.5}$$

which is a hyperbolic reformulation of the real differentiability of a $\mathcal{C}^1$ bicomplex function.

Again, we can apply to the equation (4.5) a similar reasoning as made above. In particular, the definition of the hyperbolic differentiability of a bicomplex function is

$$F(Z+H)-F(Z)=\frac{\partial F}{\partial \mathfrak{z}_1}(Z)\mathfrak{h}_1+\frac{\partial F}{\partial \mathfrak{z}_2}(Z)\mathfrak{h}_2+\mathfrak{o}(H).$$

We are not aware of any other work where hyperbolic holomorphic mappings from $\mathbb{D}^2$ to $\mathbb{D}^2$ are dealt with.

4.2 Real Differentiability in Bicomplex Terms

Formula (4.1) as well as formulas (4.2), (4.4) and (4.5) express the real differentiability of a bicomplex function, although written in different languages: the first of them is in real language, the next two in complex ($\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$) language and the last is given in the hyperbolic one. Now we are going to see what the bicomplex language will give.

Let us first consider the bicomplex increment $H=h_1+\mathbf{j}h_2$, for which we have:

$$\begin{aligned} h_1 &= \frac{H+H^{\dagger}}{2}, \qquad & h_2 &= \frac{H-H^{\dagger}}{2\mathbf{j}},\\ \overline{h}_1 &= \frac{\overline{H}+H^{*}}{2}, \qquad & \overline{h}_2 &= \frac{\overline{H}-H^{*}}{2\mathbf{j}}. \end{aligned} \tag{4.6}$$

In this setup, the formula (4.2) becomes:

$$\begin{aligned} F(Z+H)-F(Z) = {} & \frac{1}{2}\left(\frac{\partial F}{\partial z_1}-\mathbf{j}\frac{\partial F}{\partial z_2}\right)(Z)\cdot H+\frac{1}{2}\left(\frac{\partial F}{\partial z_1}+\mathbf{j}\frac{\partial F}{\partial z_2}\right)(Z)\cdot H^{\dagger}\\ & +\frac{1}{2}\left(\frac{\partial F}{\partial \overline{z}_1}-\mathbf{j}\frac{\partial F}{\partial \overline{z}_2}\right)(Z)\cdot \overline{H}+\frac{1}{2}\left(\frac{\partial F}{\partial \overline{z}_1}-\mathbf{j}\frac{\partial F}{\partial \overline{z}_2}\right)(Z)\cdot H^{*}\\ & +\mathfrak{o}(H). \end{aligned} \tag{4.7}$$

We introduce the following bicomplex differential operators:

$$\begin{aligned} \frac{\partial}{\partial Z} &:= \frac{1}{2}\left(\frac{\partial}{\partial z_1}-\mathbf{j}\frac{\partial}{\partial z_2}\right), \qquad & \frac{\partial}{\partial Z^{\dagger}} &:= \frac{1}{2}\left(\frac{\partial}{\partial z_1}+\mathbf{j}\frac{\partial}{\partial z_2}\right),\\ \frac{\partial}{\partial \overline{Z}} &:= \frac{1}{2}\left(\frac{\partial}{\partial \overline{z}_1}-\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\right), \qquad & \frac{\partial}{\partial Z^{*}} &:= \frac{1}{2}\left(\frac{\partial}{\partial \overline{z}_1}+\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\right). \end{aligned} \tag{4.8}$$

We obtain the following intrinsic expression of the real differentiability of the bicomplex function F in terms of bicomplex differential operators and variables:

$$F(Z+H)-F(Z)=\frac{\partial F}{\partial Z}(Z)H+\frac{\partial F}{\partial Z^{\dagger}}(Z)H^{\dagger}+\frac{\partial F}{\partial \overline{Z}}(Z)\overline{H}+\frac{\partial F}{\partial Z^{*}}(Z)H^{*}+\mathfrak{o}(H). \tag{4.9}$$

While real differentiability uniquely defines the coefficients in (4.9), one may think that if we would have began with another writing of Z and H then formula (4.9) would be different, that is, other operators would have appeared in it. Direct computations however confirm that no matter in which form we write the functions and the variables (recall that the operators in (4.8) are given in terms of $\mathbb{C}(\mathbf{i})$ complex differential operators) the operators $\frac{\partial}{\partial Z}$, $\frac{\partial}{\partial Z^\dagger}$, $\frac{\partial}{\partial \overline{Z}}$, $\frac{\partial}{\partial Z^*}$ in the right-hand side of (4.9) are uniquely defined. But this requires to clarify what is meant by this "uniqueness". These are the same operators but only when acting on bicomplex functions, without taking into account any concrete intrinsic substructure in $\mathbb{BC}$. For instance, if the functions are considered $\mathbb{C}^2(\mathbf{i})$-valued or $\mathbb{C}^2(\mathbf{j})$-valued, then the operators are of course different; this is just because they act on objects of different nature. For this reason, one should be careful working with bicomplex functions and operators, having in mind all the time what is exactly the structure of $\mathbb{BC}$ which is of interest for a specific goal.

We give the table of writings of each operator:

$$
\begin{aligned}
\frac{\partial}{\partial Z} &= \frac{1}{2}\left(\frac{\partial}{\partial z_1} - \mathbf{j}\frac{\partial}{\partial z_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_1} - \mathbf{i}\frac{\partial}{\partial \zeta_2}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{z}_1} - \mathbf{i}\frac{\partial}{\partial \mathfrak{z}_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial w_1} + \mathbf{k}\frac{\partial}{\partial w_2}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial \omega_1} + \mathbf{k}\frac{\partial}{\partial \omega_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{w}_1} + \mathbf{j}\frac{\partial}{\partial \mathfrak{w}_2}\right), \\
\frac{\partial}{\partial Z^\dagger} &= \frac{1}{2}\left(\frac{\partial}{\partial z_1} + \mathbf{j}\frac{\partial}{\partial z_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_1^*} - \mathbf{i}\frac{\partial}{\partial \zeta_2^*}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{z}_1^\diamond} - \mathbf{i}\frac{\partial}{\partial \mathfrak{z}_2^\diamond}\right) = \frac{1}{2}\left(\frac{\partial}{\partial w_1} - \mathbf{k}\frac{\partial}{\partial w_2}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial \omega_1^*} - \mathbf{k}\frac{\partial}{\partial \omega_2^*}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{w}_1^\diamond} + \mathbf{j}\frac{\partial}{\partial \mathfrak{w}_2^\diamond}\right), \\
\frac{\partial}{\partial \overline{Z}} &= \frac{1}{2}\left(\frac{\partial}{\partial \overline{z}_1} - \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_1} + \mathbf{i}\frac{\partial}{\partial \zeta_2}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{z}_1^\diamond} + \mathbf{i}\frac{\partial}{\partial \mathfrak{z}_2^\diamond}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \overline{w}_1} - \mathbf{k}\frac{\partial}{\partial \overline{w}_2}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial \omega_1} - \mathbf{k}\frac{\partial}{\partial \omega_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{w}_1^\diamond} - \mathbf{j}\frac{\partial}{\partial \mathfrak{w}_2^\diamond}\right), \\
\frac{\partial}{\partial Z^*} &= \frac{1}{2}\left(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_1^*} + \mathbf{i}\frac{\partial}{\partial \zeta_2^*}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{z}_1} + \mathbf{i}\frac{\partial}{\partial \mathfrak{z}_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \overline{w}_1} + \mathbf{k}\frac{\partial}{\partial \overline{w}_2}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial \omega_1^*} + \mathbf{k}\frac{\partial}{\partial \omega_2^*}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{w}_1} - \mathbf{j}\frac{\partial}{\partial \mathfrak{w}_2}\right).
\end{aligned}
$$

Again, take a bicomplex function F and apply to it, say, the operator $\frac{\partial}{\partial Z^*}$; the resulting function $\frac{\partial F}{\partial Z^*}$ is a bicomplex function. But if we omit the bicomplex structure and consider F to be a mapping from $\mathbb{C}^2(\mathbf{i})$ to $\mathbb{C}^2(\mathbf{i})$ then such an F does understand already what the action of the operator

$$\frac{1}{2}\left(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\right) = \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial \overline{z}_1} & -\frac{\partial}{\partial \overline{z}_2} \\ \frac{\partial}{\partial \overline{z}_2} & \frac{\partial}{\partial \overline{z}_1} \end{pmatrix}$$

means, but it does not understand the action of the operator $\frac{1}{2}\left(\frac{\partial}{\partial \zeta_1^*} + \mathbf{i}\frac{\partial}{\partial \zeta_2^*}\right)$.

As a consequence of the previous discussion, we obtain the following result:

Theorem 4.3 *Given $F \in \mathcal{C}^1(\Omega, \mathbb{BC})$, if F is $\mathbb{BC}$-holomorphic then*

$$\frac{\partial F}{\partial Z^\dagger}(Z) = \frac{\partial F}{\partial \overline{Z}}(Z) = \frac{\partial F}{\partial Z^*}(Z) = 0. \tag{4.10}$$

holds on Ω.

Proof Since F is $\mathbb{BC}$ holomorphic, formula (3.9) holds for all $H \notin \mathfrak{S}_0$. But F is a $\mathcal{C}^1$ function, hence (4.9) holds as well for any $H \neq 0$, thus both formulas hold for non zero-divisors. Then (4.10) follows directly by recalling that both (3.9) and (4.9) are unique representations for a given function F, and by comparing them. $\square$

Remark 4.4 As we will show in Sect. 7, the converse of this result is true as well, but we need some additional steps before we can prove it.

In order to have more consistency with the previous reasonings of this section and in analogy with the cases of functions of real or complex variables, we introduce the following definition.

Definition 4.5 A bicomplex function $F \in \mathcal{C}^1(\Omega, \mathbb{BC})$ is called *bicomplex* ($\mathbb{BC}$-) *differentiable* in Ω if

$$F(Z+H) - F(Z) = A_Z \cdot H + \alpha(H)H \tag{4.11}$$

with $\alpha(H) \to 0$ when $H \to 0$ and A_Z a bicomplex constant.

Note that in this definition H is allowed to be a zero-divisor but taking H in (4.11) to be any non zero-divisor, we see from (3.9) that $\mathbb{BC}$ differentiability implies $\mathbb{BC}$ derivability. The reciprocal statement is more delicate and will not be treated immediately.

It turns out that Theorem 4.3 has many deep and far reaching consequences which we will discuss in the next section.

5 Various Interpretations and Consequences of Theorem 4.3

Take a $\mathbb{BC}$-holomorphic function F, which we write as $F = f_1 + \mathbf{j} f_2$ with the independent variable written as $Z = z_1 + \mathbf{j} z_2$. By Theorem 4.3, and Remark 4.4, this is equivalent to say that F verifies in Ω the system

$$\frac{\partial F}{\partial Z^\dagger} = \frac{\partial F}{\partial \overline{Z}} = \frac{\partial F}{\partial Z^*} = 0.$$

For the operators involved we use the appropriate representations from the table in the previous section. Using such a representation for the operators $\frac{\partial}{\partial \overline{Z}}$ and $\frac{\partial}{\partial Z^*}$ leads to the system

$$\frac{\partial F}{\partial \overline{z}_1} - \mathbf{j}\frac{\partial F}{\partial \overline{z}_2} = 0, \qquad \frac{\partial F}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial F}{\partial \overline{z}_2} = 0$$

implying that $\frac{\partial F}{\partial \overline{z}_1} = 0 = \frac{\partial F}{\partial \overline{z}_2}$. The latter is equivalent to the holomorphy, in the sense of complex functions of two $\mathbb{C}(\mathbf{i})$-complex variables, of the components f_1, f_2 of the function F. Thus F can be seen as a holomorphic mapping from $\Omega \subset \mathbb{C}^2(\mathbf{i}) \to \mathbb{C}^2(\mathbf{i})$. But we still have more information, since $\frac{\partial F}{\partial Z^\dagger} = 0$. Then $F = f_1 + \mathbf{j} f_2$ verifies

$$\begin{aligned}\frac{\partial F}{\partial Z^\dagger}(Z) &= \frac{1}{2}\left(\frac{\partial}{\partial z_1} + \mathbf{j}\frac{\partial}{\partial z_2}\right)(f_1 + \mathbf{j} f_2)(Z) \\ &= \frac{1}{2}\left(\left(\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2}(Z)\right) + \mathbf{j}\left(\frac{\partial f_2}{\partial z_1}(Z) + \frac{\partial f_1}{\partial z_2}(Z)\right)\right) = 0.\end{aligned}$$

Thus

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \qquad \frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1}, \tag{5.1}$$

that is, the complex partial derivatives of the holomorphic functions f_1, f_2 are not independent; they are tied with the Cauchy–Riemann type conditions (5.1). We recap the reasoning as

Proposition 5.1 *A function $F = f_1 + \mathbf{j} f_2 : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ of class $\mathcal{C}^1$ is $\mathbb{BC}$-holomorphic if and only if, seen as a mapping from $\Omega \subset \mathbb{C}^2(\mathbf{i}) \to \mathbb{C}^2(\mathbf{i})$, it is a holomorphic mapping with its components related by the Cauchy–Riemann type conditions* (5.1).

In other words, the theory of bicomplex holomorphic functions can be seen as a theory of a proper subset of holomorphic mappings in two complex variables. Each equation in Theorem 4.3 plays a different role: two of them together guarantee the holomorphy of the $\mathbb{C}(\mathbf{i})$ complex components and the third one provides the relation between them.

It is remarkable that the operator $\frac{\partial}{\partial Z^\dagger}$ arises in the works of Ryan [13] about complex Clifford analysis as a Cauchy–Riemann operator which is defined directly on holomorphic mappings with values in a complex Clifford algebra.

Next, take a bicomplex holomorphic function F in the form $F = g_1 + \mathbf{i} g_2$, where g_1 and g_2 take values in $\mathbb{C}(\mathbf{j})$ and we write now $Z = \zeta_1 + \mathbf{i}\zeta_2$, then the corresponding differential operators are:

$$\frac{\partial}{\partial Z^\dagger} = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_1^*} - \mathbf{i}\frac{\partial}{\partial \zeta_2^*}\right),$$

$$\frac{\partial}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_1} + \mathbf{i}\frac{\partial}{\partial \zeta_2}\right),$$
$$\frac{\partial}{\partial Z^*} = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_1^*} + \mathbf{i}\frac{\partial}{\partial \zeta_2^*}\right).$$

Using again Theorem 4.3 and Remark 4.4 we have that to be a bicomplex holomorphic function means for F that

$$\frac{\partial F}{\partial Z^\dagger} = \frac{1}{2}\left(\frac{\partial F}{\partial \zeta_1^*} - \mathbf{i}\frac{\partial F}{\partial \zeta_2^*}\right) = 0,$$
$$\frac{\partial F}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial F}{\partial \zeta_1} + \mathbf{i}\frac{\partial F}{\partial \zeta_2}\right) = 0,$$
$$\frac{\partial F}{\partial Z^*} = \frac{1}{2}\left(\frac{\partial F}{\partial \zeta_1^*} + \mathbf{i}\frac{\partial F}{\partial \zeta_2^*}\right) = 0.$$

The first and the third equations together give, again, that the components g_1 and g_2 of F are holomorphic functions of the $\mathbb{C}(\mathbf{j})$ complex variables ζ_1 and ζ_2, meanwhile the second equation gives:

$$\frac{\partial g_1}{\partial \zeta_1} = \frac{\partial g_2}{\partial \zeta_2} \qquad \frac{\partial g_1}{\partial \zeta_2} = -\frac{\partial g_2}{\partial \zeta_1}. \tag{5.2}$$

Proposition 5.2 *A function $F = g_1 + \mathbf{i}g_2 : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ of class $\mathcal{C}^1$ is $\mathbb{BC}$-holomorphic if and only if, seen as a mapping from $\Omega \subset \mathbb{C}^2(\mathbf{j}) \to \mathbb{C}^2(\mathbf{j})$, it is a holomorphic mapping with its components related by the Cauchy–Riemann type conditions* (5.2).

Finally, take a $\mathbb{BC}$-holomorphic function $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ in the form $F = \mathfrak{u}_1 + \mathbf{i}\mathfrak{u}_2$, where $\mathfrak{u}_1$, $\mathfrak{u}_2$ take values in $\mathbb{D}$ and write $Z = \mathfrak{z}_1 + \mathbf{i}\mathfrak{z}_2$, then the corresponding operators are

$$\frac{\partial}{\partial Z^\dagger} = \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{z}_1^\diamond} - \mathbf{i}\frac{\partial}{\partial \mathfrak{z}_2^\diamond}\right),$$
$$\frac{\partial}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{z}_1^\diamond} + \mathbf{i}\frac{\partial}{\partial \mathfrak{z}_2^\diamond}\right),$$
$$\frac{\partial}{\partial Z^*} = \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{z}_1} + \mathbf{i}\frac{\partial}{\partial \mathfrak{z}_2}\right).$$

Using again Theorem 4.3 and Remark 4.4 we have that to be a $\mathbb{BC}$-holomorphic function means for F that

$$\frac{\partial F}{\partial Z^\dagger} = \frac{1}{2}\left(\frac{\partial F}{\partial \mathfrak{z}_1^\diamond} - \mathbf{i}\frac{\partial F}{\partial \mathfrak{z}_2^\diamond}\right) = 0,$$
$$\frac{\partial F}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial F}{\partial \mathfrak{z}_1^\diamond} + \mathbf{i}\frac{\partial F}{\partial \mathfrak{z}_2^\diamond}\right) = 0,$$

$$\frac{\partial F}{\partial Z^*} = \frac{1}{2}\left(\frac{\partial F}{\partial \mathfrak{z}_1} + \mathbf{i}\frac{\partial F}{\partial \mathfrak{z}_2}\right) = 0.$$

The first and second equations together imply that the components $\mathfrak{u}_1$ and $\mathfrak{u}_2$ of F are holomorphic functions of hyperbolic variables $\mathfrak{z}_1$ and $\mathfrak{z}_2$, meanwhile the last equation gives the Cauchy–Riemann type conditions:

$$\frac{\partial \mathfrak{u}_1}{\partial \mathfrak{z}_1} = \frac{\partial \mathfrak{u}_2}{\partial \mathfrak{z}_2} \qquad \frac{\partial \mathfrak{u}_1}{\partial \mathfrak{z}_2} = -\frac{\partial \mathfrak{u}_2}{\partial \mathfrak{z}_1}. \tag{5.3}$$

They look exactly as their antecedent in one complex variable, but this is a totally different thing: we deal here with $\mathbb{D}$-valued functions of two hyperbolic variables and with hyperbolic partial derivatives.

Proposition 5.3 *A function $F = \mathfrak{u}_1 + \mathbf{i}\mathfrak{u}_2 : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ of class $\mathcal{C}^1$ is $\mathbb{BC}$-holomorphic if and only if, seen as a mapping from $\Omega \subset \mathbb{D}^2 \to \mathbb{D}^2$, it is a hyperbolic holomorphic mapping with its components related by the Cauchy–Riemann type conditions* (5.3).

6 Second Order Complex and Hyperbolic Differential Operators

It is well known that complex holomorphic functions are tightly related with harmonic functions of two real variables which proved to be of a crucial importance for the theories of both classes of functions. On the general level, the same occurs with hyperholomorphic (synonymously monogenic, regular) functions of (real) Clifford analysis and the harmonic functions of the respective number of (real) variables. By this reason, both one complex variable theory and Clifford analysis are considered as refinements of the corresponding harmonic function theories. This relation is due to the following factorizations of the respective Laplace operators. If

$$\Delta_{\mathbb{R}^2} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \quad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathbf{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}\right),$$

then

$$\frac{\partial}{\partial \overline{z}} \circ \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \circ \frac{\partial}{\partial \overline{z}} = \frac{1}{4}\Delta_{\mathbb{R}^2}. \tag{6.1}$$

Similarly, if $\Delta_{\mathbb{R}^k}$ denotes the Laplace operator in $\mathbb{R}^k$, $\mathcal{D}_{CR}$ denotes the Cauchy–Riemann operator of Clifford analysis in $\mathbb{R}^{n+1}$, and $\mathcal{D}_{Dir}$ denotes the Dirac operator in $\mathbb{R}^n$, then

$$\begin{aligned} \mathcal{D}_{CR} \circ \overline{\mathcal{D}}_{CR} = \overline{\mathcal{D}}_{CR} \circ \mathcal{D}_{CR} &= \Delta_{\mathbb{R}^{n+1}}, \\ \mathcal{D}^2_{Dir} &= -\Delta_{\mathbb{R}^n}. \end{aligned}$$

The factorization (6.1) manifests the essence of the relation between complex holomorphic functions and harmonic functions. First of all note that the Laplace operator

$\Delta_{\mathbb{R}^2}$ acts initially on real-valued functions but its action extends onto complex valued functions component-wise: if $f = u + \mathbf{i}v$ then

$$\Delta_{\mathbb{R}^2}[f] := \Delta_{\mathbb{R}^2}[u] + \mathbf{i}\Delta_{\mathbb{R}^2}[v].$$

Thus the equality (6.1) holds on complex valued functions (of class $\mathcal{C}^2$, not just $\mathcal{C}^1$).

Let $f = u + \mathbf{i}v$ be a holomorphic function, that is, $\frac{\partial f}{\partial \bar{z}} = 0$, hence by (6.1) $\Delta_{\mathbb{R}^2}[f] = 4\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right) = 0$ and thus f is a harmonic function. Reciprocally, taking a real valued harmonic function u, consider $\frac{\partial u}{\partial z}$ (which is a formal derivative, that is, the result of the action of an operator, not the "honest" derivative of a holomorphic function). One has:

$$\frac{\partial}{\partial \bar{z}}\left[\frac{\partial u}{\partial z}\right] = \frac{1}{4}\Delta_{\mathbb{R}^2}[u] = 0,$$

which means that the complex-valued function $\frac{\partial u}{\partial z}$ generated by the harmonic function u is holomorphic.

We have shown above that the theory of bicomplex holomorphic functions has deep similarities with its one complex variable counterpart, so let us look for analogues of the harmonic functions for it, or equivalently, for analogues of the real Laplacian.

For bicomplex numbers, each of the sets $\mathbb{C}(\mathbf{i})$, $\mathbb{C}(\mathbf{j})$, and $\mathbb{D}$ plays, in a sense, a role quite similar to that of $\mathbb{R}$ for complex numbers, which hints at the following candidates for being the corresponding Laplacians:

$$\Delta_{\mathbb{C}^2(\mathbf{i})} := \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}; \tag{6.2}$$

$$\Delta_{\mathbb{C}^2(\mathbf{j})} := \frac{\partial^2}{\partial \zeta_1^2} + \frac{\partial^2}{\partial \zeta_2^2}; \tag{6.3}$$

$$\Delta_{\mathbb{D}} := \frac{\partial^2}{\partial \mathfrak{z}_1^2} + \frac{\partial^2}{\partial \mathfrak{z}_2^2}. \tag{6.4}$$

Operators (6.2) and (6.3) are called complex ($\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$ respectively) Laplacians and (6.4) is the hyperbolic Laplacian. The first of them acts on $\mathbb{C}(\mathbf{i})$-valued holomorphic functions of two complex variables z_1 and z_2; the second acts on $\mathbb{C}(\mathbf{j})$-valued holomorphic functions of the complex variables ζ_1 and ζ_2; and the third acts on $\mathbb{D}$-valued hyperbolic holomorphic functions of the hyperbolic variables $\mathfrak{z}_1$ and $\mathfrak{z}_2$.

Although we wrote in (6.2)–(6.4) the formal partial derivatives $\frac{\partial}{\partial z_1}$, etc., in fact we mean the authentic partial derivatives with respect to the corresponding variables $z_1, z_2, \zeta_1, \zeta_2, \mathfrak{z}_1, \mathfrak{z}_2$. Of course in this situation the formal and the authentic derivatives coincide but at the same time a confusion may arise, so that we rewrite (6.2)–(6.4) as

$$\Delta_{\mathbb{C}^2(\mathbf{i})} = \partial''_{z_1^2} + \partial''_{z_2^2}; \tag{6.5}$$

$$\Delta_{\mathbb{C}^2(\mathbf{j})} = \partial''_{\zeta_1^2} + \partial''_{\zeta_2^2}; \tag{6.6}$$

$$\Delta_{\mathbb{D}^2} = \partial''_{\mathfrak{z}_1^2} + \partial''_{\mathfrak{z}_2^2}. \tag{6.7}$$

This is in keeping with the notation we adopted where we wrote complex partial derivatives as f'_{z_1}, etc. Hence $\partial''_{z_1^2} f = f''_{z_1}$, etc.

The next step should be to extend any of the operators (6.5)–(6.7) onto bicomplex valued functions. It turns out that the form of writing of bicomplex numbers becomes important and it depends on the operator in (6.5)–(6.7). Indeed, for the operator $\Delta_{\mathbb{C}^2(\mathbf{i})}$ the bicomplex function F should be holomorphic in the sense of two complex variables and if we want the same for its components then we are forced to consider F as $F = f_1 + \mathbf{j} f_2$. Now we set

$$\Delta_{\mathbb{C}^2(\mathbf{i})}[F] := \partial''_{z_1^2}[f_1] + \mathbf{j}\partial''_{z_2^2}[f_2].$$

A similar reasoning holds for the other two operators: for $\Delta_{\mathbb{C}^2(\mathbf{j})}$ we take $F = \rho_1 + \mathbf{i}\rho_2$ and we set

$$\Delta_{\mathbb{C}^2(\mathbf{j})}[F] := \partial''_{\zeta_1^2}[\rho_1] + \mathbf{i}\partial''_{\zeta_2^2}[\rho_2];$$

for $\Delta_{\mathbb{D}^2}$ we take $F = \mathfrak{f}_1 + \mathbf{i}\mathfrak{f}_2$ and we set

$$\Delta_{\mathbb{D}^2}[F] := \partial''_{\mathfrak{z}_1^2}[\mathfrak{f}_1] + \mathbf{i}\partial''_{\mathfrak{z}_2^2}[\mathfrak{f}_2].$$

The analogues of the formula (6.1) arise if one uses the corresponding operators from the table in Sect. 4; more exactly the operators should be taken in an appropriate form. For instance, if we want to use the operators $\frac{\partial}{\partial Z^\dagger}$ and $\frac{\partial}{\partial Z}$ in the form

$$\frac{\partial}{\partial Z^\dagger} = \frac{1}{2}\left(\frac{\partial}{\partial z_1} + \mathbf{j}\frac{\partial}{\partial z_2}\right), \qquad \frac{\partial}{\partial Z} = \frac{1}{2}\left(\frac{\partial}{\partial z_1} - \mathbf{j}\frac{\partial}{\partial z_2}\right)$$

then we are forced to write the variable as $Z = z_1 + \mathbf{j} z_2$ and the function F as $F = f_1 + \mathbf{j} f_2$. With this understanding we have that

$$\Delta_{\mathbb{C}^2(\mathbf{i})} = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} = 4\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}, \tag{6.8}$$

where the operators act on $\mathbb{BC}$-valued functions holomorphic in the sense of the complex variables z_1, z_2. Hence, the theory of $\mathbb{BC}$-holomorphic functions can (should?) be seen as the function theory for $\mathbb{C}(\mathbf{i})$-complex Laplacian.

Analogously to what was done above, the factorization allows us to establish direct relations between $\mathbb{BC}$-holomorphic functions and complex harmonic functions, that is, null solutions to the operator $\Delta_{\mathbb{C}^2(\mathbf{i})}$. Indeed, let $F = f_1 + \mathbf{j} f_2$ be a $\mathbb{BC}$-holomorphic function, that is, it is automatically holomorphic in the sense of two complex variables, but also $\frac{\partial F}{\partial Z^\dagger} = 0$. Hence

$$\Delta_{\mathbb{C}^2(\mathbf{i})}[F] = 4\frac{\partial}{\partial Z}\left(\frac{\partial F}{\partial Z^\dagger}\right) = 0,$$

and thus F is a $\mathbb{BC}$-valued complex harmonic function. Reciprocally, taking a $\mathbb{C}(\mathbf{i})$-valued complex harmonic function f_1, consider the $\mathbb{BC}$-valued function $\frac{\partial}{\partial Z}[f_1] = \frac{\partial f_1}{\partial Z}$ (which is a formal operation on a holomorphic function of two complex variables, not the bicomplex derivative of a $\mathbb{BC}$-holomorphic function). One has:

$$\frac{\partial}{\partial Z^\dagger}\left[\frac{\partial f_1}{\partial Z}\right] = \frac{1}{4}\Delta_{\mathbb{C}^2(\mathbf{i})}[f_1] = 0,$$

which means that the $\mathbb{BC}$-valued function $\frac{\partial f_1}{\partial Z}$ generated by the complex harmonic function f_1 is $\mathbb{BC}$-holomorphic.

The operators in (6.2) and (6.3) are dealt with in exactly the same way, although now another first order operator enters into the game. In case of $Z = \zeta_1 + \mathbf{j}\zeta_2$ we take

$$\frac{\partial}{\partial Z} = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_1} - \mathbf{i}\frac{\partial}{\partial \zeta_2}\right), \qquad \frac{\partial}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial}{\partial \zeta_1} + \mathbf{i}\frac{\partial}{\partial \zeta_2}\right),$$

arriving at

$$\Delta_{\mathbb{C}^2(\mathbf{j})} = \frac{\partial^2}{\partial \zeta_1^2} + \frac{\partial^2}{\partial \zeta_2^2} = 4\frac{\partial}{\partial Z}\frac{\partial}{\partial \overline{Z}}; \tag{6.9}$$

in the case of $Z = \mathfrak{z}_1 + \mathbf{i}\mathfrak{z}_2$ we take

$$\frac{\partial}{\partial Z} = \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{z}_1} - \mathbf{i}\frac{\partial}{\partial \mathfrak{z}_2}\right), \qquad \frac{\partial}{\partial Z^*} = \frac{1}{2}\left(\frac{\partial}{\partial \mathfrak{z}_1} + \mathbf{i}\frac{\partial}{\partial \mathfrak{z}_2}\right),$$

arriving at

$$\Delta_{\mathbb{D}} = \frac{\partial^2}{\partial \mathfrak{z}_1^2} + \frac{\partial^2}{\partial \mathfrak{z}_2^2} = 4\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^*}. \tag{6.10}$$

So, we have provided each of the operators (6.2)–(6.4) with an adequate function theory. There is a fine point here: it is, in a sense, one and the same theory of $\mathbb{BC}$-holomorphic functions, but where different aspects of it are taken into account. This "common" function theory allows us to realize the similarities and the differences between the three operators (6.2)–(6.4). Indeed, (6.2) and (6.3) look identical from the viewpoint of classical complex analysis, but viewing them from the bicomplex perspective reveals subtle differences between them. At the same time, (6.4) seems to be quite different with any of (6.2) and (6.3) but bicomplex functions, again, show that there exists a deep underlying unity between all the three of them.

The direct relations between the null solutions of any of the operators $\Delta_{\mathbb{C}^2(\mathbf{j})}$ and $\Delta_{\mathbb{D}^2}$ and the $\mathbb{BC}$-holomorphic function theory is established in the same way as we did this for the operator $\Delta_{\mathbb{C}^2(\mathbf{i})}$.

Another well known second order operator, in real case, is the wave operator, and one can be tempted to look at some of its complex analogues:

$$\Box_{\mathbb{C}^2(\mathbf{i})} := \frac{\partial^2}{\partial w_1^2} - \frac{\partial^2}{\partial w_2^2}, \tag{6.11}$$

$$\Box_{\mathbb{C}^2(\mathbf{j})} := \frac{\partial^2}{\partial \omega_1^2} - \frac{\partial^2}{\partial \omega_2^2}. \tag{6.12}$$

They do not produce great novelties, the same $\mathbb{BC}$-holomorphic function theory provides all the necessary. This can be obtained in two ways. First of all, the simple holomorphic changes of variables

$$(w_1, w_2) \mapsto (z_1, -\mathbf{i}z_2), \qquad (\omega_1, \omega_2) \mapsto (\zeta_1, -\mathbf{j}\zeta_2),$$

turn the operator $\Box_{\mathbb{C}^2(\mathbf{i})}$ into $\Delta_{\mathbb{C}^2(\mathbf{i})}$ and the operator $\Box_{\mathbb{C}^2(\mathbf{j})}$ into $\Delta_{\mathbb{C}^2(\mathbf{j})}$. But they can be factorized directly:

$$\begin{aligned}\Box_{\mathbb{C}^2(\mathbf{i})} &= \left(\frac{\partial}{\partial w_1} + \mathbf{k}\frac{\partial}{\partial w_2}\right) \cdot \left(\frac{\partial}{\partial w_1} - \mathbf{k}\frac{\partial}{\partial w_2}\right) \\ &= 4\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger};\end{aligned}$$

compare with (6.8). Although formally they are the same operators, they are employed in other forms taken in the table in Sect. 4. In the same mode,

$$\begin{aligned}\Box_{\mathbb{C}^2(\mathbf{j})} &= \left(\frac{\partial}{\partial \omega_1} + \mathbf{k}\frac{\partial}{\partial \omega_2}\right) \cdot \left(\frac{\partial}{\partial \omega_1} - \mathbf{k}\frac{\partial}{\partial \omega_2}\right) \\ &= 4\frac{\partial}{\partial Z}\frac{\partial}{\partial \overline{\overline{Z}}},\end{aligned}$$

where we take, again, another form of writing the operators involved. Somewhat paradoxically, the same bicomplex "tricks" do not work for the "hyperbolic wave operator"

$$\frac{\partial^2}{\partial \mathfrak{w}_1^2} - \frac{\partial^2}{\partial \mathfrak{w}_2^2}$$

which can neither be factorized directly nor reduced to (6.4); this is because in the hyperbolic world there is only one hyperbolic-type imaginary unit and there are no complex-type imaginary units.

One can consider another hyperbolic Laplacian:

$$\frac{\partial^2}{\partial \mathfrak{w}_1^2} + \frac{\partial^2}{\partial \mathfrak{w}_2^2}$$

but one applies the change of variables

$$(\mathfrak{w}_1, \mathfrak{w}_2) \mapsto (\mathfrak{z}_1, -\mathbf{k}\mathfrak{z}_2)$$

reducing it to (6.4).

7 Bicomplex Holomorphy and Bicomplex Derivability: The Case of Idempotent Representation

Take a bicomplex function $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ with Ω being a domain. We write all the bicomplex numbers involved in $\mathbb{C}(\mathbf{i})$ idempotent form, for instance,

$$\begin{aligned} Z &= \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger = (\ell_1 + \mathbf{i} m_1)\mathbf{e} + (\ell_2 + \mathbf{i} m_2)\mathbf{e}^\dagger, \\ F(Z) &= G_1(Z)\mathbf{e} + G_2(Z)\mathbf{e}^\dagger, \\ H &= \eta_1 \mathbf{e} + \eta_2 \mathbf{e}^\dagger = (u_1 + \mathbf{i} v_1)\mathbf{e} + (u_2 + \mathbf{i} v_2)\mathbf{e}^\dagger. \end{aligned}$$

Let us introduce the sets

$$\Omega_1 := \{\beta_1 \mid \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger \in \Omega\} \subset \mathbb{C}(\mathbf{i})$$

and

$$\Omega_2 := \{\beta_2 \mid \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^\dagger \in \Omega\} \subset \mathbb{C}(\mathbf{i}).$$

It is direct to prove that Ω_1 and Ω_2 are domains in $\mathbb{C}(\mathbf{i})$.

We assume that $F \in \mathcal{C}^1(\Omega)$ where the real partial derivatives are taken with respect to the "idempotent real variables": ℓ_1, m_1, ℓ_2, m_2. The condition $F \in \mathcal{C}^1(\Omega)$ ensures the real differentiability of F in Ω:

$$\begin{aligned} F(Z + H) - F(Z) = &\frac{\partial F}{\partial \ell_1}(Z) \cdot u_1 + \frac{\partial F}{\partial m_1}(Z) \cdot v_1 \\ &+ \frac{\partial F}{\partial \ell_2}(Z) \cdot u_2 + \frac{\partial F}{\partial m_2}(Z) \cdot v_2 + \mathfrak{o}(H) \end{aligned} \tag{7.1}$$

for $H \to 0$. We are going to follow the line of Sect. 4 so we omit many details. First of all let us translate formula (7.1) in $\mathbb{C}(\mathbf{i})$ complex language: since

$$\begin{aligned} u_1 &= \frac{1}{2}(\eta_1 + \overline{\eta}_1); \qquad u_2 = \frac{1}{2}(\eta_2 + \overline{\eta}_2); \\ v_1 &= \frac{\mathbf{i}}{2}(\overline{\eta}_1 - \eta_1); \qquad v_2 = \frac{\mathbf{i}}{2}(\overline{\eta}_2 - \eta_2), \end{aligned}$$

then

$$F(Z + H) - F(Z) = \frac{\partial F}{\partial \ell_1}(Z) \cdot \frac{1}{2}(\eta_1 + \overline{\eta}_1) + \frac{\partial F}{\partial m_1}(Z) \cdot \frac{\mathbf{i}}{2}(\overline{\eta}_1 - \eta_1)$$

$$
\begin{aligned}
&+\frac{\partial F}{\partial \ell_2}(Z)\cdot\frac{1}{2}(\eta_2+\overline{\eta}_2)+\frac{\partial F}{\partial m_2}(Z)\cdot\frac{\mathbf{i}}{2}(\overline{\eta}_2-\eta_2)+\mathfrak{o}(H)\\
&=\eta_1\frac{1}{2}\left(\frac{\partial F}{\partial \ell_1}(Z)-\mathbf{i}\frac{\partial F}{\partial m_1}(Z)\right)+\overline{\eta}_1\frac{1}{2}\left(\frac{\partial F}{\partial \ell_1}(Z)+\mathbf{i}\frac{\partial F}{\partial m_1}(Z)\right)\\
&+\eta_2\frac{1}{2}\left(\frac{\partial F}{\partial \ell_2}(Z)-\mathbf{i}\frac{\partial F}{\partial m_2}(Z)\right)+\overline{\eta}_2\frac{1}{2}\left(\frac{\partial F}{\partial \ell_2}(Z)+\mathbf{i}\frac{\partial F}{\partial m_2}(Z)\right)+\mathfrak{o}(H)\\
&=:\eta_1\frac{\partial F}{\partial \beta_1}(Z)+\overline{\eta}_1\frac{\partial F}{\partial \overline{\beta}_1}(Z)+\eta_2\frac{\partial F}{\partial \beta_2}(Z)+\overline{\eta}_2\frac{\partial F}{\partial \overline{\beta}_2}(Z)+\mathfrak{o}(H).
\end{aligned}
$$

Remark 7.1 Note that the above calculations show that, as it is well known, the bicomplex function F of class $\mathcal{C}^1$, seen as a mapping from $\mathbb{C}^2(\mathbf{i})\to\mathbb{C}^2(\mathbf{i})$ is holomorphic with respect to β_q $(q=1,2)$ if and only if $\frac{\partial F}{\partial \overline{\beta}_q}(Z)=0$ in Ω. Note that if F is a bicomplex function, and we express it in cartesian coordinates, it turns out that F is $\mathbb{BC}$-holomorphic if and only if its components are holomorphic as functions of two complex variables and satisfy a Cauchy–Riemann type relation between them. As we will show later, this is definitely not the case when we express F in the idempotent representation. In this case, $\mathbb{BC}$-holomorphy will be equivalent to the requirement that each component is a holomorphic function of a single complex variable and there are no relations between the components.

We emphasize that now we are considering the identification between $\mathbb{BC}$ and $\mathbb{C}^2(\mathbf{i})$

$$
Z=\beta_1\mathbf{e}+\beta_2\mathbf{e}^\dagger\longleftrightarrow(\beta_1,\beta_2)\in\mathbb{C}^2(\mathbf{i}),
$$

where however the basis in $\mathbb{C}^2(\mathbf{i})$ is not the canonical basis $\{1,\mathbf{j}\}$, but rather the idempotent basis $\{\mathbf{e},\mathbf{e}^\dagger\}$. For the next step recall the formulas

$$
\begin{aligned}
H&=\eta_1\mathbf{e}+\eta_2\mathbf{e}^\dagger, & H^\dagger&=\eta_2\mathbf{e}+\eta_1\mathbf{e}^\dagger,\\
\overline{H}&=\overline{\eta}_2\mathbf{e}+\overline{\eta}_1\mathbf{e}^\dagger, & H^*&=\overline{\eta}_1\mathbf{e}+\overline{\eta}_2\mathbf{e}^\dagger,
\end{aligned}
$$

which imply that

$$
\begin{aligned}
\eta_1&=H\mathbf{e}+H^\dagger\mathbf{e}^\dagger, & \overline{\eta}_1&=H^*\mathbf{e}+\overline{H}\mathbf{e}^\dagger,\\
\eta_2&=H^\dagger\mathbf{e}+H\mathbf{e}^\dagger, & \overline{\eta}_2&=\overline{H}\mathbf{e}+H^*\mathbf{e}^\dagger.
\end{aligned}
$$

The condition of real differentiability after substitutions becomes:

$$
\begin{aligned}
&F(Z+H)-F(Z)\\
&=H\left(\frac{\partial F}{\partial \beta_1}(Z)\mathbf{e}+\frac{\partial F}{\partial \beta_2}(Z)\mathbf{e}^\dagger\right)+H^\dagger\left(\frac{\partial F}{\partial \beta_2}(Z)\mathbf{e}+\frac{\partial F}{\partial \beta_1}(Z)\mathbf{e}^\dagger\right)\\
&+\overline{H}\left(\frac{\partial F}{\partial \overline{\beta}_2}(Z)\mathbf{e}+\frac{\partial F}{\partial \overline{\beta}_1}(Z)\mathbf{e}^\dagger\right)+H^*\left(\frac{\partial F}{\partial \overline{\beta}_1}(Z)\mathbf{e}+\frac{\partial F}{\partial \overline{\beta}_2}(Z)\mathbf{e}^\dagger\right)+\mathfrak{o}(H). \qquad (7.2)
\end{aligned}
$$

Note that the expressions in the parentheses are not, yet, the idempotent forms of anything, since the coefficients of $\mathbf{e}$ and $\mathbf{e}^\dagger$ are bicomplex numbers, not $\mathbb{C}(\mathbf{i})$ complex numbers. Thus we are required to make one more step. Using the formula $F = G_1\mathbf{e} + G_2\mathbf{e}^\dagger$ we arrive at

$$\begin{aligned}
&F(Z+H) - F(Z)\\
&= H\left(\frac{\partial G_1}{\partial \beta_1}(Z)\mathbf{e} + \frac{\partial G_2}{\partial \beta_2}Z\mathbf{e}^\dagger\right) + H^\dagger\left(\frac{\partial G_1}{\partial \beta_2}(Z)\mathbf{e} + \frac{\partial G_2}{\partial \beta_1}Z\mathbf{e}^\dagger\right)\\
&\quad + \overline{H}\left(\frac{\partial G_1}{\partial \overline{\beta}_2}(Z)\mathbf{e} + \frac{\partial G_2}{\partial \overline{\beta}_1}(Z)\mathbf{e}^\dagger\right) + H^*\left(\frac{\partial G_1}{\partial \overline{\beta}_1}(Z)\mathbf{e} + \frac{\partial G_2}{\partial \overline{\beta}_2}(Z)\mathbf{e}^\dagger\right) + \mathfrak{o}(H).
\end{aligned} \tag{7.3}$$

This formula is valid for any F in $\mathcal{C}^1(\Omega)$, and let us analyze how $\mathbb{BC}$-holomorphic functions are singled out in $\mathcal{C}^1$.

Theorem 7.2 *The $\mathcal{C}^1$ function F is $\mathbb{BC}$-holomorphic if and only if the three coefficients of $H^\dagger$, $\overline{H}$ and H^* are all zero for any Z in Ω.*

Proof The *if* direction follows as in Theorem 4.3. Specifically, since F is $\mathbb{BC}$-holomorphic, formula (3.9) holds for all $H \notin \mathfrak{S}_0$. But F is a $\mathcal{C}^1$ function, hence (7.3) holds as well for any $H \neq 0$, thus both formulas hold for non zero-divisors. Then the result follows directly by recalling that both (3.9) and (7.3) are unique representations for a given function F, and by comparing them. In order to prove the *only if*, it is helpful to write explicitly the meaning of the vanishing of these coefficients, namely:

$$\begin{aligned}
&\frac{\partial G_1}{\partial \beta_2}(Z)\mathbf{e} + \frac{\partial G_2}{\partial \beta_1}(Z)\mathbf{e}^\dagger = 0,\\
&\frac{\partial G_1}{\partial \overline{\beta}_2}(Z)\mathbf{e} + \frac{\partial G_2}{\partial \overline{\beta}_1}(Z)\mathbf{e}^\dagger = 0,\\
&\frac{\partial G_1}{\partial \overline{\beta}_1}(Z)\mathbf{e} + \frac{\partial G_2}{\partial \overline{\beta}_2}(Z)\mathbf{e}^\dagger = 0.
\end{aligned} \tag{7.4}$$

Now note that the second and the third equations, because of the independence of $\mathbf{e}$ and $\mathbf{e}^\dagger$, impose that G_1 and G_2 are $\mathbb{C}(\mathbf{i})$ valued holomorphic functions of the complex variables β_1, β_2 and thus they have authentic complex partial derivatives. What is more, the first equation in (7.4) says that one of the partial derivatives of each G_1 and G_2 is identically zero: $\frac{\partial G_1}{\partial \beta_2}(Z) = 0$, $\frac{\partial G_2}{\partial \beta_1}(Z) = 0$ for any $Z \in \Omega$. Hence G_1 is a holomorphic function of the single variable $\beta_1 \in \Omega_1$ and G_2 is a holomorphic function of the single variable $\beta_2 \in \Omega_2$. We now want to show that these equations imply that F is $\mathbb{BC}$-holomorphic. But in fact, because of these equations we have that for any invertible H there holds:

$$\frac{F(Z+H) - F(Z)}{H} = \frac{G_1(\beta_1 + \eta_1) - G_1(\beta_1)}{\eta_1}\mathbf{e} + \frac{G_2(\beta_2 + \eta_2) - G_2(\beta_2)}{\eta_2}\mathbf{e}^\dagger,$$

where Z is an arbitrary point in Ω.

Now, by the properties of G_1 and G_2 we deduce that the right hand side has, for $\mathfrak{S} \not\ni H \to 0$, the limit $G_1'(\beta_1)\mathbf{e} + G_2'(\beta_2)\mathbf{e}^\dagger$, which concludes the proof: the limit in the left hand-side exists also for any $Z \in \Omega$ with $\mathfrak{S} \not\ni H \to 0$ and it coincides with the derivative $F'(Z)$ making F $\mathbb{BC}$-holomorphic in Ω. □

As a matter of fact, the proof allows to make a more precise characterization of $\mathcal{C}^1$ functions which are $\mathbb{BC}$-holomorphic.

Theorem 7.3 *A bicomplex function $F = G_1\mathbf{e} + G_2\mathbf{e}^\dagger : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ of class $\mathcal{C}^1$ is $\mathbb{BC}$-holomorphic if and only if the following two conditions hold:*

(I) *The component G_1, seen as a $\mathbb{C}(\mathbf{i})$ valued function of two complex variables (β_1, β_2) is holomorphic; what is more, it does not depend on the variable β_2 and thus G_1 is a holomorphic function of the variable β_1.*

(II) *The component G_2, seen as a $\mathbb{C}(\mathbf{i})$ valued function of two complex variables (β_1, β_2) is holomorphic; what is more, it does not depend on the variable β_1 and thus G_2 is a holomorphic function of the variable β_2.*

Remark 7.4 The functions G_1 and G_2 are independent in the sense that there are no Cauchy–Riemann type conditions relating them.

We are in a position now to prove that the converse to Theorem 4.3 is true as well.

Theorem 7.5 *Given $F \in \mathcal{C}^1(\Omega, \mathbb{BC})$, then condition (4.10) implies that F is $\mathbb{BC}$-holomorphic.*

Proof If (4.10) holds then a direct computation shows that all the three formulas in (7.4) are true, and by Theorem 7.2 F is $\mathbb{BC}$-holomorphic. □

Corollary 7.6 *Let F be a $\mathbb{BC}$-holomorphic function in Ω, then F is of the form $F(Z) = G_1(\beta_1)\mathbf{e} + G_2(\beta_2)\mathbf{e}^\dagger$ with $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \Omega$ and its derivative is given by*

$$F'(Z) = G_1'(\beta_1)\mathbf{e} + G_2'(\beta_2)\mathbf{e}^\dagger.$$

Taking into account the relations between β_1, β_2 and the cartesian components z_1, z_2, we have also that

$$\begin{aligned} F'(z_1 + \mathbf{j}z_2) &= G_1'(z_1 - \mathbf{i}z_2)\mathbf{e} + G_2'(z_1 + \mathbf{i}z_2)\mathbf{e}^\dagger \\ &= G_1'(Z\mathbf{e} + Z^\dagger\mathbf{e}^\dagger)\mathbf{e} + G_2'(Z\mathbf{e} + Z^\dagger\mathbf{e}^\dagger)\mathbf{e}^\dagger. \end{aligned}$$

Remark 7.7 Although formula (7.3) is quite similar to formula (4.9) its consequences for the function F are paradoxically different: meanwhile formula (4.9) has allowed to conclude that the cartesian components f_1, f_2 are holomorphic functions of two complex variables which are not independent, formula (7.3) explains us that the idempotent components G_1, G_2 are usual holomorphic functions of one complex variable which are, besides, independent.

Remark 7.8 We have proved that if F is $\mathbb{BC}$-holomorphic in Ω then for any $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \Omega$ it is of the form

$$F(Z) = G_1(\beta_1)\mathbf{e} + G_2(\beta_2)\mathbf{e}^\dagger.$$

But the right hand-side of the latter is well-defined on the wider set $\tilde{\Omega} := \Omega_1 \cdot \mathbf{e} + \Omega_2 \cdot \mathbf{e}^{\dagger} \supset \Omega$ (in general, this inclusion is proper). What is more, by Theorem 7.3 the function $\tilde{F}$ defined by

$$\tilde{F}(Z) := G_1(\beta_1)\mathbf{e} + G_2(\beta_2)\mathbf{e}^{\dagger}, \qquad Z \in \tilde{\Omega},$$

is $\mathbb{BC}$-holomorphic in $\tilde{\Omega}$. Since $\tilde{F}\big|_{\Omega} \equiv F$ we see that, unlike what happens in the complex case, not every domain in $\mathbb{BC}$ is a domain of $\mathbb{BC}$-holomorphy: every function which is $\mathbb{BC}$-holomorphic in a domain Ω extends $\mathbb{BC}$-holomorphically up to the minimal set of the form $X_1 \cdot \mathbf{e} + X_2 \cdot \mathbf{e}^{\dagger}$ containing Ω. One can compare this with [14].

Remark 7.9 The same analysis can be done for the idempotent representation with $\mathbb{C}(\mathbf{j})$ coefficients.

Remark 7.10 Recall that in Sect. 3 we departed from the cartesian representation of bicomplex numbers and we investigated many properties of derivable bicomplex functions, in particular such functions proved to have complex partial derivatives with respect to z_1, z_2. This approach fails immediately when one tries to apply it to the case of idempotent representation: the matter is that the definition of the derivative excludes precisely the values of H which are necessary for the complex partial derivatives with respect to β_1, β_2. But in the proof of Theorem 4.3 we have proved, as a matter of fact, that such partial derivatives of a $\mathbb{BC}$-holomorphic functions do exist and, moreover, $\frac{\partial F}{\partial \beta_1}(Z) = G_1'(\beta_1) \cdot \mathbf{e}$ and $\frac{\partial F}{\partial \beta_2}(Z) = G_2'(\beta_2) \cdot \mathbf{e}^{\dagger}$.

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