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Bicomplex Bergman and Bloch spaces

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Abstract In this article, we define the bicomplex weighted Bergman spaces on the bidisk and their associated weighted Bergman projections, where the respective Bergman kernels are determined. We study also the bicomplex Bergman projection onto the bicomplex Bloch space.

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1 Introduction

The Bergman space is a classic topic in the Complex Analysis. The last years has received a strong impetus (see [2,5,6,11,12]); on the other hand, in recent years, the theory of bicomplex holomorphic functions has consolidated its development (see [1,3,4,9] and references herein). This theory shows that it is quite adequate to deal with some analogous of the classical holomorphic functions spaces on the unit complex disk, but now defined in the bidisk. Precisely, in this article, we introduce the bicomplex weighted Bergman spaces $\mathbb{BC}\mathcal{A}^p_\alpha$ on the bidisk $\mathbb{U} \subset \mathbb{C}^2$. A previous work in this direction appears in [8] and [10]. We point out that a frequent tool in the theory of bicomplex holomorphic function is the so-called idempotent decomposition: Although it is ubiquitous in all this theory, it is a mistake to think—as we show in this paper—that everything in the theory can be reduced to the idempotent decomposition. In the preliminaries, we fix notations and some fundamental facts in bicomplex theory; also, we prove some results in the context of bicomplex numbers. In the third section, we define the bicomplex weighted Bergman spaces $\mathbb{BC}\mathcal{A}^p_\alpha$, we prove the decomposition (see Theorem 3.1)

$$\mathbb{BC}\mathcal{A}^p_\alpha = \mathcal{A}^p_\alpha \mathbf{e} + \mathcal{A}^p_\alpha \mathbf{e}^\dagger \ ,$$

and we determine their respective Bergman kernels [see (3.5)]:

$$K_{\mathbf{k},\alpha} = K_\alpha \mathbf{e} + K_\alpha \mathbf{e} \, .$$

The bicomplex weighted Bergman projection $\mathbf{P}_{\mathbf{k},\alpha}$ is factored with a slight modification (see Theorem 3.9).

We will see that the bicomplex Bloch space defined in the bidisk $\mathbb{U}$ can be splited in two classical Bloch spaces on the unit disk. Therefore, in the fourth section, we study the bicomplex Bergman projection onto the bicomplex Bloch space, see Theorem 4.1.

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2 Preliminaries

We present several common facts about bicomplex numbers and bicomplex holomorphic functions. We will be free to use results and notations of [9].

The set of bicomplex numbers $\mathbb{BC}$ is defined as

$$\mathbb{BC} := \{ z_1 + \mathbf{j} z_2 \ : \ z_1,\ z_2 \in \mathbb{C}(\mathbf{i}),\ \mathbf{j}^2 = -1\}.$$

Sum and product of bicomplex numbers are presented in the expected way. We write all the bicomplex numbers as $Z = z_1 + \mathbf{j} z_2$, with $z_l = x_l + \mathbf{i} y_l \in \mathbb{C}(\mathbf{i})$, in theirs $\mathbb{C}(\mathbf{i})$-idempotent form, that is:

$$Z = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^{\dagger}, \tag{2.1}$$

where

$$\beta_1 = z_1 - \mathbf{i} z_2 \qquad \text{and} \qquad \beta_2 = z_1 + \mathbf{i} z_2$$

and

$$\mathbf{e} := \frac{1 + \mathbf{ij}}{2} \qquad \text{and} \qquad \mathbf{e}^{\dagger} := \frac{1 - \mathbf{ij}}{2}.$$

Observe that $\mathbf{e}\,\mathbf{e}^{\dagger} = 0$; $1 = \mathbf{e} + \mathbf{e}^{\dagger}$ or more generally $\lambda = \lambda(\mathbf{e} + \mathbf{e}^{\dagger})$ with $\lambda \in \mathbb{C}(\mathbf{i})$.

In the special case that $Z = \beta_1 \mathbf{e} + \beta_2 \mathbf{e}^{\dagger}$ and $W = \gamma_1 \mathbf{e} + \gamma_2 \mathbf{e}^{\dagger}$, with $\beta_l,\ \gamma_l$ real numbers, we consider the partial order

$$W \preceq Z \quad \text{if and only if} \quad \gamma_l \leq \beta_l, \quad l = 1,\ 2.$$

There are several conjugations of bicomplex numbers; however, we consider here only the conjugation $Z^* = \overline{z_1} - \mathbf{j}\overline{z_2} = \overline{\beta_1}\mathbf{e} + \overline{\beta_2}\mathbf{e}^{\dagger}$. With this conjugation, we have:

$$\| Z \|_{\mathbf{k}}^2 = Z \cdot Z^* = |\beta_1|^2 \mathbf{e} + |\beta_2|^2 \mathbf{e}^{\dagger} \quad \text{and} \quad Z^{-1} = \frac{Z^*}{\| Z \|_{\mathbf{k}}^2},$$

where Z is not a zero-divisor. The hyperbolic or $\mathbf{k}$-norm of Z (we will say only norm) is defined as (see Sect. 2.7 in [9]):

$$\| Z \|_{\mathbf{k}} = \sqrt{Z \cdot Z^*} = |\beta_1| \mathbf{e} + |\beta_2| \mathbf{e}^{\dagger}.$$

More generally, from the definitions of logarithm and exponential bicomplex functions, we have for $\alpha \in \mathbb{R}$ that:

$$(\beta_1 \mathbf{e} + \beta_2 \mathbf{e}^{\dagger})^{\alpha} = \beta_1^{\alpha} \mathbf{e} + \beta_1^{\alpha} \mathbf{e}^{\dagger}, \qquad \text{with} \quad \beta_1,\ \beta_2 > 0.$$

In particular:

$$(1 - \| Z \|_{\mathbf{k}}^2)^{\alpha} = \left((1 - |\beta_1|^2)\mathbf{e} + (1 - |\beta_2|^2)\mathbf{e}^{\dagger}\right)^{\alpha} = (1 - |\beta_1|^2)^{\alpha} \mathbf{e} + (1 - |\beta_2|^2)^{\alpha} \mathbf{e}^{\dagger}. \tag{2.2}$$

Let $F : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ be a function defined on the domain Ω. The derivative $F'(Z_0)$ of the function F at a point $Z_0 \in \Omega$ is the limit, if it exists,

$$F'(Z_0) = \lim_{Z \to Z_0} \frac{F(Z) - F(Z_0)}{Z - Z_0},$$

such that $Z - Z_0$ is an invertible bicomplex number. If F is derivable for all $Z \in \Omega$, we say that it is a bicomplex holomorphic function in Ω. The following result is essential in this theory (see Theorem 7.6.3 in [1]):

Theorem 2.1 *Let $\Omega \subset \mathbb{BC}$ be a domain. A bicomplex function $F : \Omega \to \mathbb{BC}$ of class $\mathcal{C}^1$ with idempotent decomposition:*

$$F = G_1 \, \mathbf{e} + G_2 \, \mathbf{e}^{\dagger}$$

is $\mathbb{BC}$-holomorphic if and only if the following two conditions hold:

(a) The component G_1, seen as a $\mathbb{C}(\mathbf{i})$-valued function of the complex variables (β_1, β_2), is holomorphic; moreover, it does not depend on the variable β_2 and, thus, G_1 is a holomorphic function of the variable β_1.
(b) The component G_2, seen as a $\mathbb{C}(\mathbf{i})$-valued function of the complex variables (β_1, β_2), is holomorphic; moreover, it does not depend on the variable β_1 and, thus, G_2 is a holomorphic function of the variable β_2.

Its derivatives of any order are given by:

$$F^{(n)}(Z) = G_1^{(n)}(\beta_1)\mathbf{e} + G_2^{(n)}(\beta_2)\mathbf{e}^\dagger, \qquad n = 0,\ 1,\ 2.\dots \tag{2.3}$$

The rules of derivability are the usual ones.

The unit disk in $\mathbb{C}(\mathbf{i})$ will be denoted by U. The bidisk $\mathbb{U}$ in $\mathbb{BC}$ is the product-type domain:

$$\mathbb{U} := U(\mathbf{e} + \mathbf{e}^\dagger) = U_1\mathbf{e} + U_2\mathbf{e}^\dagger := \{\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger\ :\ \beta_1,\ \beta_2 \in U\}.$$

Note that such bidisk should be seen as the cartesian product $U \times U$ in the idempotent $\mathbb{C}^2(\mathbf{i})$.

Let $A,\ B,\ C,\ D \in \mathbb{BC}$ and define the Möbius transformation $T(Z) = \frac{AZ+B}{CZ+D}$ when $CZ + D$ is not a zero divisor. The idempotent form of the Möbius transformation T is:

$$T(\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger) = \frac{a_1\beta_1 + b_1}{c_1\beta_1 + d_1}\mathbf{e} + \frac{a_2\beta_2 + b_2}{c_2\beta_2 + d_2}\mathbf{e}^\dagger. \tag{2.4}$$

As an example of algebraic and derivative operativity, we prove the following result.

Lemma 2.2 *Let $A \in \mathbb{U}$ and define the bicomplex Möbius transformation $T : \mathbb{U} \to \mathbb{U}$ as:*

$$T(Z) = \Lambda\frac{A - Z}{1 - A^*Z}, \quad \textit{with} \quad \|\ \Lambda\ \|_{\mathbf{k}} = 1.$$

Then:

$$(1 - \|\ Z\ \|_{\mathbf{k}}^2)\ \|\ T'(Z)\ \|_{\mathbf{k}} = 1 - \|\ T(Z)\ \|_{\mathbf{k}}^2\ . \tag{2.5}$$

In particular $\|\ T(Z)\ \|_{\mathbf{k}} = 1$ if and only if $\|\ Z\ \|_{\mathbf{k}} = 1$, that is, if Z belongs to the distinguished boundary of $\mathbb{U}$.

Proof We calculate the derivative of T. Thus:

$$T'(Z) = \Lambda\frac{(1 - A^*Z)(-1) - (A - Z)(-A^*)}{(1 - A^*Z)^2} = \Lambda\frac{\|\ A\ \|_{\mathbf{k}}^2 - 1}{(1 - A^*Z)^2}.$$

It is enough to prove that

$$(1 - \|\ Z\ \|_{\mathbf{k}}^2)\ \|\|\ A\ \|_{\mathbf{k}}^2 - 1\ \|_{\mathbf{k}} = \|\ 1 - A^*Z\ \|_{\mathbf{k}}^2 - \|\ A - Z\ \|_{\mathbf{k}}^2,$$

and this equality is immediate. □

In the previous proof, we do not appeal to the idempotent form. Schwarz–Pick Lemma can be proved appealing to the idempotent form, that is, it follows from (2.1), (2.3) and the classical Schwarz–Pick Lemma.

Lemma 2.3 *Let $F : \mathbb{U} \to \mathbb{U}$ be a bicomplex holomorphic function. Then:*

$$\|\ F'(Z)\ \|_{\mathbf{k}} \preceq \frac{1 - \|\ F(Z)\ \|_{\mathbf{k}}^2}{1 - \|\ Z\ \|_{\mathbf{k}}^2} \quad \textit{for all } Z \in \mathbb{U}.$$

The equality occurs if and only if F is a Möbius transformation of the bidisk.

Let $\Omega \subset \mathbb{BC}$ be a domain and $F : \Omega \to \mathbb{BC}$ be a function. Then:

$$F(Z) = F_1(Z) + \mathbf{j}F_2(Z) = G_1(\beta_1, \beta_2)\mathbf{e} + G_2(\beta_1, \beta_2)\mathbf{e}^{\dagger}. \tag{2.6}$$

For $l = 1,\ 2$, we have the change of variables:

$$\beta_1 = x_1 + \mathbf{i}y_1 - \mathbf{i}(x_2 + \mathbf{i}y_2) = x_1 + y_2 + \mathbf{i}(y_1 - x_2) = \beta_{1,1} + \mathbf{i}\beta_{1,2}$$

$$\beta_2 = x_1 + \mathbf{i}y_1 + \mathbf{i}(x_2 + \mathbf{i}y_2) = x_1 - y_2 + \mathbf{i}(y_1 + x_2) = \beta_{2,1} + \mathbf{i}\beta_{2,2}.$$

The change basis matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

with determinant 4. In particular, if F is an integrable function:

$$\int_{\mathbb{U}} F(Z)\,\mathrm{d}V(Z) = \int_{\mathbb{U}} F_1(Z)\,\mathrm{d}V(Z) + \mathbf{j}\int_{\mathbb{U}} F_2(Z)\,\mathrm{d}V(Z)$$

$$= 4\int_{\mathbb{U}} G_1(\beta_1, \beta_2)\,\mathrm{d}V(\beta)\,\mathbf{e} + 4\int_{\mathbb{U}} G_2(\beta_1, \beta_2)\,\mathrm{d}V(\beta)\,\mathbf{e}^{\dagger}.$$

The normalized area measure on U_l will be denoted by dA. In terms of real coordinates, we have:

$$\mathrm{d}A(\beta_l) = \frac{1}{\pi}\,\mathrm{d}\beta_{l1}\mathrm{d}\beta_{l2}, \qquad \text{for} \qquad l = 1,\ 2.$$

Let $-1 < \alpha < \infty$. The bicomplex weighted measure $dV_\alpha(Z)$ is defined as:

$$\mathrm{d}V_\alpha(Z) = \frac{\alpha+1}{4}(1 - \parallel Z \parallel_{\mathbf{k}}^2)^\alpha\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2$$

$$= (\alpha+1)\left((1-|\beta_1|^2)^\alpha\,\mathbf{e} + (1-|\beta_2|^2)^\alpha\mathbf{e}^{\dagger}\right)\,\mathrm{d}\beta_{11}\,\mathrm{d}\beta_{12}\,\mathrm{d}\beta_{21}\,\mathrm{d}\beta_{22}$$

$$= \mathrm{d}A_\alpha(\beta_1)\mathrm{d}A(\beta_2)\mathbf{e} + \mathrm{d}A(\beta_1)\mathrm{d}A_\alpha(\beta_2)\mathbf{e}^{\dagger}\ , \tag{2.7}$$

where $dA_\alpha(\beta_l) = (\alpha+1)\left(1-|\beta_l|^2\right)^\alpha d\beta_{l1}d\beta_{l2}$, $l = 1,\ 2$ is the usual weighted measure.

Passing to polar coordinates, by definition of the Gamma function, we get the following.

Example 2.4 Let m and n be two nonnegative integers. Then, if $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^{\dagger}$, we have:

$$\int_{\mathbb{U}} Z^m Z^{*n}\,\mathrm{d}V_\alpha(Z) = \int_{\mathbb{U}} \beta_1^m\overline{\beta}_1^n\,\mathrm{d}A_\alpha(\beta_1)\mathrm{d}A(\beta_2)\mathbf{e} + \int_{\mathbb{U}} \beta_2^m\overline{\beta}_2^n\,\mathrm{d}A(\beta_1)\,\mathrm{d}A_\alpha(\beta_2)\mathbf{e}^{\dagger}$$

$$= \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{\Gamma(\alpha+2)\Gamma(n+1)}{\Gamma(\alpha+n+2)} & \text{if } m = n. \end{cases}$$

From this example, we get the following result (see [6] and [12]).

Lemma 2.5 *For any $-1 < \alpha < \infty$ and any real η, then:*

$$\int_{\mathbb{U}} \frac{1- \parallel W \parallel_{\mathbf{k}}^{\alpha}}{\parallel 1 - ZW^* \parallel_{\mathbf{k}}^{2+\alpha+\eta}} \, dV(W) \approx \begin{cases} 1 & \text{if } \eta < 0, \\ \ln \dfrac{1}{1- \parallel Z \parallel_{\mathbf{k}}} & \text{if } \eta = 0, \\ \dfrac{1}{(1- \parallel Z \parallel_{\mathbf{k}}^2)^{\eta}} & \text{if } \eta > 0. \end{cases}$$

Proof Let $\lambda = \frac{2+\alpha+\eta}{2}$, and then, by Example 2.4:

$$\int_{\mathbb{U}} \frac{1- \parallel W \parallel_{\mathbf{k}}^{\alpha}}{\parallel 1 - ZW^* \parallel_{\mathbf{k}}^{2+\alpha+\eta}} \, \mathrm{d}V(W)$$

$$= \int_{\mathbb{U}} \sum_{m,n=0}^{\infty} \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)} \frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} (ZW^*)^n (Z^*W)^m (1- \parallel W \parallel_{\mathbf{k}}^2)^{\alpha} \, \mathrm{d}V(W)$$

$$= \sum_{m,n=0}^{\infty} \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)} \frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} Z^n Z^{*m} \int_{\mathbb{U}} W^{*n} W^m (1- \parallel W \parallel_{\mathbf{k}}^2)^{\alpha} \, \mathrm{d}V(W)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda)^2}{n!^2 \Gamma(\lambda)^2} |Z|_{\mathbf{k}}^{2n} \int_{\mathbb{U}} (1- \parallel W \parallel_{\mathbf{k}}^2)^{\alpha} \parallel W \parallel_{\mathbf{k}}^{2n} \; \mathrm{d}V(W)$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\lambda)^2} \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda)^2}{n!\Gamma(n+\alpha+2)} \parallel Z \parallel_{\mathbf{k}}^{2n} .$$

By Stirling's formula and the definition of λ:

$$\frac{\Gamma(n+\lambda)^2}{n!\Gamma(n+\alpha+2)} \approx (n+1)^{\eta-1}, \qquad \text{as } n \to \infty.$$

The result follows from this estimation (see [12]). □

3 Bicomplex Bergman Spaces $\mathbb{BC}\mathcal{A}_{\alpha}^{p}$

For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted bicomplex Bergman space $\mathbb{BC}\mathcal{A}_{\alpha}^{p} = \mathbb{BC}\mathcal{A}_{\alpha}^{p}(\mathbb{U})$ of the bidisk $\mathbb{U}$ is the space of bicomplex holomorphic functions F in the complete space $L_{\mathbf{k}}^{p}(\mathbb{U}, dV_{\alpha}(Z))$; that is, $F : \mathbb{U} \to \mathbb{BC}$ is a bicomplex holomorphic function and satisfies:

$$\int_{\mathbb{U}} \parallel F(Z) \parallel_{\mathbf{k}}^{p} \; \mathrm{d}V_{\alpha}(Z) \prec \infty.$$

Theorem 3.1 *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Then,*

$$\mathbb{BC}\mathcal{A}_{\alpha}^{p} = \mathbb{BC}\mathcal{A}_{\alpha}^{p}(\mathbb{U}) = \mathcal{A}_{\alpha}^{p}(\mathbf{i})\,\mathbf{e} + \mathcal{A}_{\alpha}^{p}(\mathbf{i})\,\mathbf{e}^{\dagger} .$$

Moreover, if $F(Z) = G_1(\beta_1)\,\mathbf{e} + G_2(\beta_2)\,\mathbf{e}^{\dagger}$ is a $\mathbb{BC}$-holomorphic function in $\mathbb{BC}\mathcal{A}_{\alpha}^{p}$, then:

$$\int_{\mathbb{U}} \parallel F(Z) \parallel_{\mathbf{k}}^{p} \, dV_{\alpha}(Z) = \int_{U_1} |G_1(\beta_1)|^p \, dA_{\alpha}(\beta_1)\mathbf{e} + \int_{U_2} |G_2(\beta_2)|^p \, dA_{\alpha}(\beta_2)\mathbf{e}^{\dagger}; \tag{3.1}$$

that is:

$$\parallel F \parallel_{\mathbf{k},p,\alpha} = \parallel G_1 \parallel_{p,\alpha} \mathbf{e} + \parallel G_2 \parallel_{p,\alpha} \; \mathbf{e}^{\dagger}.$$

Proof From the definition of logarithm and exponential bicomplex functions and (2.2), we can rewrite the integral expression as:

$$
\begin{aligned}
&\int_{\mathbb{U}} \| F(Z) \|_{\mathbf{k}}^{p} \, \mathrm{d}V_{\alpha}(Z) \\
&\quad = \int_{\mathbb{U}} \left(|G(\beta_1)|^p \, \mathbf{e} + |G(\beta_2)|^p \mathbf{e}^{\dagger}\right) \left(\mathrm{d}A_{\alpha}(\beta_1)\mathrm{d}A(\beta_2)\, \mathbf{e} + \mathrm{d}A(\beta_1)\mathrm{d}A_{\alpha}(\beta_2)\, \mathbf{e}^{\dagger}\right) \\
&\quad = \int_{\mathbb{U}} |G(\beta_1)|^p \mathrm{d}A_{\alpha}(\beta_1)\mathrm{d}A(\beta_2)\, \mathbf{e} + \int_{\mathbb{U}} |G(\beta_2)|^p \mathrm{d}A(\beta_1)\mathrm{d}A_{\alpha}(\beta_2)\, \mathbf{e}^{\dagger} \\
&\quad = \int_{U_1} |G(\beta_1)|^p \mathrm{d}A_{\alpha}(\beta_1) \int_{U_2} \mathrm{d}A(\beta_2)\, \mathbf{e} + \int_{U_1} \mathrm{d}A(\beta_1) \int_{U_2} |G(\beta_2)|^p \mathrm{d}A_{\alpha}(\beta_2)\, \mathbf{e}^{\dagger} \\
&\quad = \int_{U_1} |G(\beta_1)|^p \mathrm{d}A_{\alpha}(\beta_1)\, \mathbf{e} + \int_{U_2} |G(\beta_2)|^p \mathrm{d}A_{\alpha}(\beta_2)\, \mathbf{e}^{\dagger}.
\end{aligned}
$$

Then:

$$
\begin{aligned}
\| F \|_{\mathbf{k},p,\alpha} &:= \left\| \int_{\mathbb{U}} \| F(Z) \|_{\mathbf{k}}^{p} \, \mathrm{d}V_{\alpha}(Z) \right\|_{\mathbf{k}}^{\frac{1}{p}} \\
&= \left[\int_{U_1} |G(\beta_1)|^p \mathrm{d}A_{\alpha}(\beta_1)\right]^{\frac{1}{p}} \mathbf{e} + \left[\int_{U_2} |G(\beta_2)|^p \mathrm{d}A_{\alpha}(\beta_2)\right]^{\frac{1}{p}} \mathbf{e}^{\dagger} \\
&:= \| G_1 \|_{p,\alpha} \, \mathbf{e} + \| G_2 \|_{p,\alpha} \, \mathbf{e}^{\dagger}.
\end{aligned}
$$

□

The decomposition of Corollary 2.1 is very useful. The following results are an immediate consequence of the usual theory of Bergman spaces.

Theorem 3.2 *Suppose* $0 < p < \infty$, $-1 < \alpha < \infty$, *and that* K *is a compact subset of* $\mathbb{U}$. *If* $F : \mathbb{U} \to \mathbb{BC}$ *is a bicomplex holomorphic function, then there exists a positive constant* $C = C(n, K, p, \alpha)$, *such that:*

$$
\sup \left\{ \| F^{(n)}(Z) \|_{\mathbf{k}} : \ Z \in \mathbb{U} \right\} \preceq C \, \| F \|_{k,p,\alpha} \, .
$$

In particular, every point evaluation in $\mathbb{U}$ *is a bounded linear functional on* $\mathbb{BC}\mathcal{A}_{\alpha}^{p}$.

Proof Let $l = 1,\ 2$. We can suppose that $K = K_1\mathbf{e} + K_2\mathbf{e}^{\dagger}$, where $K_l \subset U_l$ is a compact set. It is well known (see [12]) that there exists $C_l = C(n, K_l, p, \alpha)$ such that if $F = G_1\,\mathbf{e} + G_2\,\mathbf{e}^{\dagger}$, then:

$$
\sup \left\{ |G_l^{(n)}(\beta_l)| \ : \ \beta \in U_l \right\} \leq C_l \, \| G_l \|_{p,\alpha};
$$

so the result follows, since $F^{(n)} = G_1^{(n)}\,\mathbf{e} + G_2^{(n)}\,\mathbf{e}^{\dagger}$. □

A similar result to the previous one was proved in [7].

Theorem 3.3 *For every* $0 < p < \infty$ *and* $-1 < \alpha < \infty$, *the weighted Bergman space* $\mathbb{BC}\mathcal{A}_{\alpha}^{p}$ *is closed in* $L_{\mathbf{k}}^{p}(\mathbb{U}, dV_{\alpha})$ *and thus complete.*

Proof Let $\{F_n = G_{1,n}\mathbf{e} + G_{2,n}\mathbf{e}^{\dagger}\}$ be a sequence in $\mathbb{BC}\mathcal{A}_{\alpha}^{p}$ and assume that $F_n \to F$ in $L_{\mathbf{k}}^{p}(\mathbb{U}, dV_{\alpha})$. In principle, $F(Z) = G_1(\beta_1, \beta_2)\mathbf{e} + G_2(\beta_1, \beta_2)\mathbf{e}^{\dagger}$. Since $\{F_n\}$ is a Cauchy sequence in $L_{\mathbf{k}}^{p}(\mathbb{U}, dV_{\alpha})$, by Theorem 3.2, it converges uniformly on every compact subset of $\mathbb{U}$ precisely to F. Since $G_{1,n},\ G_{2,n}$ converge uniformly on every compact set of $\mathbb{U}$, their limits are holomorphic functions. Then, $F(Z) = G_1(\beta_1)\mathbf{e} + G_2(\beta_2)\mathbf{e}^{\dagger}$ and F is a bicomplex holomorphic function and belongs to $\mathbb{BC}\mathcal{A}_{\alpha}^{p}$. □

By standard approximation, we obtain the next result.

Proposition 3.4 *Let $F : \mathbb{U} \to \mathbb{BC}$ be a bicomplex holomorphic function on $\mathbb{U}$ and $0 < r < 1$. Let F_r be the dilated function defined by $F_r(Z) = F(rZ)$, $Z \in \mathbb{U}$. Then:*

- *For every $F \in \mathbb{BC}\mathcal{A}^p_\alpha$, we have $\| F_r - F \|_{k,p,\alpha} \to 0$ as $r \to 1^-$.*
- *For every $F \in \mathbb{BC}\mathcal{A}^p_\alpha$, there exists a sequence $\{P_n\}$ of polynomials, such that $\| F - P_n \|_{k,p,\alpha} \to 0$ as $n \to \infty$.*

We now consider de case $p = 2$. By Theorem 3.3, the bicomplex Bergman space $\mathbb{BC}\mathcal{A}^2_\alpha$ is complete and we will see that is a Hilbert space. Define for $n = 0, 1, 2, \ldots$:

$$E_n(Z) = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} Z^n, \qquad Z \in \mathbb{U}. \tag{3.2}$$

Lemma 3.5 *The set $\{E_n(Z)\}$ is orthonormal in $L^2_{\mathbf{k}}(\mathbb{U}, \mathbb{BC})$. In particular is a basis of the space $\mathbb{BC}\mathcal{A}^2_\alpha$.*

Proof We prove that $\{E_n(Z)\}$ is an orthonormal set in $L^2_{\mathbf{k}}(\mathbb{U}, \mathbb{BC})$ according to the following inner product:

$$\begin{aligned}
\langle E_m(Z), E_n(Z)\rangle_{\mathbf{k},\alpha} &= \int_{\mathbb{U}} E_m(Z) E_n(Z)^* \, \mathrm{d}V_\alpha(Z) \\
&= \int_{\mathbb{U}} \sqrt{\frac{\Gamma(m+2+\alpha)}{m!\Gamma(2+\alpha)}} Z^m \left(\sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} Z^n \right)^* \mathrm{d}V_\alpha(Z) \\
&= \sqrt{\frac{\Gamma(m+2+\alpha)}{m!\Gamma(2+\alpha)} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} \int_{\mathbb{U}} Z^m {Z^*}^n \, \mathrm{d}V_\alpha(Z) \\
&= \delta_{mn},
\end{aligned}$$

where we applied in the last equality Example 2.4. Since the set of polynomials is dense in $\mathbb{BC}\mathcal{A}^2_\alpha$, we conclude that $\{E_n(Z)\}$ is an orthonormal basis for $\mathbb{BC}\mathcal{A}^2_\alpha$. □

In particular, if:

$$F(Z) = \sum_{n=0}^{\infty} A_n Z^n \qquad \text{and} \qquad G(Z) = \sum_{n=0}^{\infty} B_n Z^n$$

are two functions in $\mathbb{BC}\mathcal{A}^2_\alpha$, with

$$A_n = a_{1n}\mathbf{e} + a_{2n}\mathbf{e}^\dagger, \qquad B_n = b_{1n}\mathbf{e} + b_{2n}\mathbf{e}^\dagger,$$

we can rewrite:

$$F(Z) = \sum_{n=0}^{\infty} \left(a_{1n}\beta_1^n \mathbf{e} + a_{2n}\beta_2^n \mathbf{e}^\dagger \right) \qquad \text{and} \qquad G(Z) = \sum_{n=0}^{\infty} \left(b_{1n}\beta_1^n \mathbf{e} + b_{2n}\beta_2^n \mathbf{e}^\dagger \right).$$

Thus, performing similar computations as were made in Lemma 3.5, we have:

$$\langle F(Z), G(Z)\rangle_{\mathbf{k},\alpha} = \sum_{n=0}^{\infty} \frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} \left(a_{1n}\overline{b_{1n}}\mathbf{e} + a_{2n}\overline{b_{2n}}\mathbf{e}^\dagger \right).$$

In the special case $F = G$, we have:

$$\| F \|^2_{\mathbf{k},\alpha} = \sum_{n=0}^{\infty} \frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} \| A_n \|^2_{\mathbf{k}}.$$

Corollary 3.6 *The bicomplex holomorphic function*

$$F(Z)=\sum_{n=0}^{\infty} A_n Z^n$$

belongs to the Bergman space $\mathbb{BC}\mathcal{A}^2_\alpha$ *if and only if:*

$$\| F \|^2_{\mathbf{k},\alpha}=\sum_{n=0}^{\infty}\frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} \| A_n \|^2_{\mathbf{k}} \prec \infty.$$

Proposition 3.7 *Let* $\lambda > 0$ *and* $Z,\ W \in \mathbb{U}$. *Then:*

$$\frac{1}{(1-ZW^*)^{\lambda}}=\sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} Z^n W^{*n}. \tag{3.3}$$

Proof Let $Z=\beta_1\mathbf{e}+\beta_2\mathbf{e}^{\dagger}$ and $W=\gamma_1\mathbf{e}+\gamma_2\mathbf{e}^{\dagger}$; then, $1-ZW^*=(1-\beta_1\overline{\gamma_1})\mathbf{e}+(1-\beta_2\overline{\gamma_2})\mathbf{e}^{\dagger}$. Thus:

$$\frac{1}{(1-ZW^*)^{\lambda}}=\frac{1}{(1-\beta_1\overline{\gamma_1})^{\lambda}}\mathbf{e}+\frac{1}{(1-\beta_2\overline{\gamma_2})^{\lambda}}\mathbf{e}^{\dagger}. \tag{3.4}$$

Now, using the usual power series for the generalized binomial, we get:

$$\begin{aligned}\frac{1}{(1-ZW^*)^{\lambda}} &= \sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)}\beta_1^n\overline{\gamma_1^n}\,\mathbf{e}+\sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)}\beta_2^n\overline{\gamma_2^n}\,\mathbf{e}^{\dagger}\\ &= \sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)}\left(\beta_1\,\mathbf{e}+\beta_2\,\mathbf{e}^{\dagger}\right)^n\left(\overline{\gamma_1}\,\mathbf{e}+\overline{\gamma_2}\,\mathbf{e}^{\dagger}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} Z^n W^{*n}.\end{aligned}$$

□

Theorem 3.8 *For* $-1<\alpha<\infty$, *let* $\mathbf{P}_{\mathbf{k},\alpha}$ *be the orthogonal projection from* $L^2_{\mathbf{k}}(\mathbb{U}, dV_\alpha)$ *onto* $\mathbb{BC}\mathcal{A}^2_\alpha$. *Then:*

$$\mathbf{P}_{\mathbf{k},\alpha}F(Z)=\int_{\mathbb{U}}\frac{F(W)\,dV_\alpha(W)}{(1-ZW^*)^{2+\alpha}}$$

for all $F\in L^2_{\mathbf{k}}(\mathbb{U}, dV_\alpha)$ *and* $Z\in\mathbb{U}$.

Proof Let $\{E_n(Z)\}$ be the orthonormal basis of $\mathbb{BC}\mathcal{A}^2_\alpha$ given in (3.2). Then, for every $F\in L^2_{\mathbf{k}}(\mathbb{U}, \mathrm{d}V_\alpha)$, we have:

$$\mathbf{P}_{\mathbf{k},\alpha}F(Z)=\sum_{n=0}^{\infty}\langle\mathbf{P}_{\mathbf{k},\alpha}F, E_n\rangle_{\mathbf{k},\alpha}E_n(Z), \qquad \text{for every } Z\in\mathbb{U},$$

and the series converges uniformly on every compact subset of $\mathbb{U}$. Since a projection is selfadjoint, then:

$$\begin{aligned}\langle\mathbf{P}_{\mathbf{k},\alpha}F, E_n\rangle_{\mathbf{k}\alpha} &= \langle F, \mathbf{P}_{\mathbf{k},\alpha}E_n\rangle_{\mathbf{k},\alpha}=\langle F, E_n\rangle_{\mathbf{k},\alpha}\\ &= \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}}\int_{\mathbb{U}}F(W)(W^*)^n\,\mathrm{d}V_\alpha(W).\end{aligned}$$

Thus, we have by (3.3) with $\lambda = 2 + \alpha$:

$$\begin{aligned}
\mathbf{P}_{\mathbf{k},\alpha} F(Z) &= \sum_{n=0}^{\infty} \langle \mathbf{P}_{\mathbf{k},\alpha} F, E_n \rangle_{\mathbf{k},\alpha} E_n(Z) \\
&= \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} \int_{\mathbb{U}} F(W)(W^*)^n \, \mathrm{d}V_\alpha(W) \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} Z^n \\
&= \int_{\mathbb{U}} F(W) \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} (ZW^*)^n \, \mathrm{d}V_\alpha(W) \\
&= \int_{\mathbb{U}} \frac{F(W) \, \mathrm{d}V_\alpha(W)}{(1-ZW^*)^{2+\alpha}}.
\end{aligned}$$

The interchange of integration and summation is justified, because for each fixed $Z \in \mathbb{U}$, the series (3.3) converges uniformly in $W \in \mathbb{U}$. □

The operators $\mathbf{P}_{\mathbf{k},\alpha}$ are the weighted Bergman projections on $\mathbb{U}$ and the functions

$$K_{\mathbf{k},\alpha}(Z, W) = \frac{1}{(1-ZW^*)^{2+\alpha}}, \qquad Z,\ W \in \mathbb{U}$$

are the weighted Bergman kernels of the bidisk.

The decomposition formula (3.4), with $\lambda = \alpha + 2$, can be written as:

$$K_{\mathbf{k},\alpha}(Z, W) = K_\alpha(\beta_1, \gamma_1)\mathbf{e} + K_\alpha(\beta_2, \gamma_2)\mathbf{e}^\dagger; \tag{3.5}$$

that is, the Bergman kernels are decomposed as factors of usual weighted Bergman kernels.

Theorem 3.9 *For $-1 < \alpha < \infty$, let $\mathbf{P}_{\mathbf{k},\alpha}$ be the weighted Bergman projection from $L^2_{\mathbf{k}}(\mathbb{U}, dV_\alpha)$ onto $\mathbb{BC}\mathcal{A}^2_\alpha$. Then:*

$$\mathbf{P}_{\mathbf{k},\alpha} F(Z) = \mathbf{P}_\alpha \widetilde{G}_1(\beta_1)\, \mathbf{e} + \mathbf{P}_\alpha \widetilde{G}_2(\beta_2)\, \mathbf{e}^\dagger$$

for all $F \in L^2_{\mathbf{k}}(\mathbb{U}, dV_\alpha)$ and $Z = \beta_1\, \mathbf{e} + \beta_2\, \mathbf{e}^\dagger \in \mathbb{U}$, where $\mathbf{P}_\alpha$ is the usual Bergman weighted projection and

$$\widetilde{G}_1(\gamma_1) = \int_{U_2} G_1(\gamma_1, \gamma_2)\, dA(\gamma_2) \quad \text{and} \quad \widetilde{G}_2(\gamma_2) = \int_{U_1} G_2(\gamma_1, \gamma_2)\, dA(\gamma_1)\,.$$

Proof From the decomposition formulas (2.6), (3.4), and (2.7), we obtain:

$$\begin{aligned}
\mathbf{P}_{\mathbf{k},\alpha} F(Z) &= \int_{\mathbb{U}} \frac{G_1(\gamma_1, \gamma_2)}{(1-\beta_1\overline{\gamma_1})^{2+\alpha}} \mathrm{d}A_\alpha(\gamma_1)\mathrm{d}A(\gamma_2)\, \mathbf{e} + \int_{\mathbb{U}} \frac{G_2(\gamma_1, \gamma_2)}{(1-\beta_2\overline{\gamma_2})^{2+\alpha}} \mathrm{d}A(\gamma_1)\mathrm{d}A_\alpha(\gamma_2)\, \mathbf{e}^\dagger \\
&= \int_{U_1} \frac{\int_{U_2} G_1(\gamma_1, \gamma_2)\, \mathrm{d}A(\gamma_2)}{(1-\beta_1\overline{\gamma_1})^{2+\alpha}} \mathrm{d}A_\alpha(\gamma_1)\, \mathbf{e} + \int_{U_2} \frac{\int_{U_1} G_2(\gamma_1, \gamma_2)\, \mathrm{d}A(\gamma_1)}{(1-\beta_2\overline{\gamma_2})^{2+\alpha}} \mathrm{d}A_\alpha(\gamma_2)\, \mathbf{e}^\dagger \\
&= \mathbf{P}_\alpha \widetilde{G}_1(\beta_1)\, \mathbf{e} + \mathbf{P}_\alpha \widetilde{G}_2(\beta_2)\, \mathbf{e}^\dagger.
\end{aligned}$$

□

If F is a function in $\mathbb{BC}\mathcal{A}^2_\alpha$, then $\mathbf{P}_{\mathbf{k},\alpha} F = F$, so that:

$$F(Z) = \int_{\mathbb{U}} \frac{F(W)\, \mathrm{d}V_\alpha(W)}{(1-ZW^*)^{2+\alpha}} \qquad Z \in \mathbb{U}.$$

Since this is a pointwise formula and $\mathbb{BC}\mathcal{A}^2_\alpha$ is dense in $\mathbb{BC}\mathcal{A}^1_\alpha$, we obtain the following corollary.

Corollary 3.10 *If F is a function in $\mathbb{BC}\mathcal{A}^1_\alpha$, then:*

$$F(Z)=\int_{\mathbb{U}}\frac{F(W)\,dV_\alpha(W)}{(1-ZW^*)^{2+\alpha}},\qquad Z\in\mathbb{U}$$

and the integral converges uniformly for Z in every compact subset of $\mathbb{U}$.

This corollary shows a reproducing formula. The Bergman kernels are special types of reproducing kernels.

Theorem 3.11 *Let $\mathbf{P}_{\mathbf{k},\alpha}$ be the Bergman projection.*

(a) Let $m,\ n$ be two none negative integers, such that $m+n>-2$ and $\rho+\alpha>-1$. Then:

$$\mathbf{P}_{\mathbf{k},\alpha}(1-\parallel Z\parallel^2_{\mathbf{k}})^\rho Z^m Z^{*n}=\begin{cases}0 & \text{if } m<n,\\ (\alpha+1)\frac{\Gamma(m-n+\alpha+2)\Gamma(\alpha+\rho+1)\Gamma(m+1)}{(m-n)!\Gamma(2+\alpha)\Gamma(\alpha+\rho+m+2)}Z^{m-n} & \text{if } m\geq n.\end{cases}$$

(b) Let ρ be a non negative integer and $F(Z)=\sum_{l=2\rho+1}^{\infty}A_lZ^l$. Then:

$$\mathbf{P}_{\mathbf{k},\alpha}\frac{(1-\parallel Z\parallel^2_{\mathbf{k}})^\rho}{Z^{*\rho}}F^{(\rho)}(Z)=(\alpha+1)\frac{\Gamma(\rho+\alpha+1)}{\Gamma(\alpha+2)}\sum_{l=2\rho+1}^{\infty}A_lZ^l\ ;$$

in other words:

$$\mathbf{P}_{\mathbf{k},\alpha}\frac{\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\rho+\alpha+1)}\frac{(1-\parallel Z\parallel^2_{\mathbf{k}})^\rho}{Z^{*\rho}}F^{(\rho)}(Z)=\sum_{l=2\rho+1}^{\infty}A_lZ^l\ .$$

Proof (a) By (3.3) and Example (2.4), we have for $m-n\geq 0$:

$$\mathbf{P}_{\mathbf{k},\alpha}(1-\parallel Z\parallel^2_{\mathbf{k}})^\rho Z^m Z^{*n}=\int_{\mathbb{U}}\frac{(1-\parallel W\parallel^2_{\mathbf{k}})^\rho W^mW^{*n}}{(1-ZW^*)^{2+\alpha}}\,\mathrm{d}V_\alpha(W)$$

$$=\int_{\mathbb{U}}(1-\parallel W\parallel^2_{\mathbf{k}})^\rho W^mW^{*n}\sum_{l=0}^{\infty}\frac{\Gamma(l+2+\alpha)}{l!\Gamma(2+\alpha)}(ZW^*)^l\,\mathrm{d}V_\alpha(W)$$

$$=(\alpha+1)\sum_{l=0}^{\infty}\frac{\Gamma(l+2+\alpha)}{l!\Gamma(2+\alpha)}Z^l\int_{\mathbb{U}}(1-\parallel W\parallel^2_{\mathbf{k}})^{\rho+\alpha}W^mW^{*n+l}\,\mathrm{d}V(W)$$

$$=(\alpha+1)\frac{\Gamma(m-n+\alpha+2)\Gamma(\alpha+\rho+1)\Gamma(m+1)}{(m-n)!\Gamma(2+\alpha)\Gamma(\alpha+\rho+m+2)}Z^{m-n}\ .$$

(b) Observe that:

$$F^\rho(Z)=\sum_{l=2\rho+1}^{\infty}l(l-1)\cdots(l-\rho+1)A_lZ^{l-\rho}.$$

Then:

$$
\begin{aligned}
\mathbf{P}_{\mathbf{k},\alpha}\frac{(1-\parallel Z\parallel_{\mathbf{k}}^2)^{\rho}}{Z^{*\rho}}F^{(\rho)}(Z) &= (\alpha+1)\int_{\mathbb{U}}\sum_{n=0}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}\frac{(1-\parallel W\parallel_{\mathbf{k}})^{\rho+\alpha}}{W^{*\rho}}(ZW^*)^n\\
&\quad\cdot\sum_{l=2\rho+1}^{\infty}l(l-1)\cdots(l-\rho+1)A_l W^{l-\rho}\,dA(W)\\
&= (\alpha+1)\sum_{n=0}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}Z^n\sum_{l=2\rho+1}^{\infty}l(l-1)\cdots(l-\rho+1)A_l\\
&\quad\cdot\lim_{t\to 1^-}\int_0^t\int_0^{2\pi}(1-r^2)^{\rho+\alpha}r^{n-k+l-k+1}e^{i\theta(l-\rho+\rho-n)}\frac{\mathrm{d}\theta\,\mathrm{d}r}{\pi}\\
&= (\alpha+1)\sum_{n=2\rho+1}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}Z^n n(n-1)\cdots(n-\rho+1)A_l\\
&\quad\cdot\int_0^1(1-u)^{\rho+\alpha}u^{n-\rho}\,\mathrm{d}u\\
&= (\alpha+1)\frac{\Gamma(\rho+\alpha+1)}{\Gamma(\alpha+2)}\sum_{n=2\rho+1}^{\infty}A_n Z^n\ .
\end{aligned}
$$

□

From the previous result, we see that the projection $\mathbf{P}_{\mathbf{k},\alpha}: L^p_{\mathbf{k}}(\mathbb{U}, dV_\alpha)\to\mathbb{BC}\mathcal{A}^1_\alpha$ is onto, and moreover, there exist an infinity number of preimages.

Corollary 3.12 *Let $F(Z)=\sum_{n=0}^{\infty}A_nZ^n$ be a bicomplex holomorphic function with $F\in L^1_{\mathbf{k}}(\mathbb{U}, dV_\alpha)$. Let:*

$$
\begin{aligned}
E(Z) &= \sum_{n=0}^{2\rho+1}\frac{\Gamma(n+\alpha+3)}{(\alpha+1)\Gamma(n+\alpha+2)}(1-\parallel Z\parallel_{\mathbf{k}}^2)A_nZ^n\\
&\quad+\frac{\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+\rho+1)}\frac{(1-\parallel Z\parallel_{\mathbf{k}}^2)^{\rho}}{Z^{*\rho}}\sum_{n=2\rho+1}^{\infty}A_n n(n-1)\cdots(n-\rho+1)Z^{n-\rho}.
\end{aligned}
$$

Then:

$$
\mathbf{P}_{\mathbf{k},\alpha}E(Z)=F(Z).
$$

Theorem 3.9 permits us to apply directly some results of the weighted Bergman spaces.

Theorem 3.13 *Suppose $-1<\alpha,\ \eta<\infty$ and $1\le p<\infty$. Then, $\mathbf{P}_{\mathbf{k},\eta}$ is a bounded projection from $L^p_{\mathbf{k}}(\mathbb{U}, dV_\alpha)$ onto $\mathbb{BC}\mathcal{A}^p_\alpha$ if and only if $\alpha+1<(\eta+1)p$.*

Proof We will use the fact that the statement of this theorem is true in the case of one complex variable (see Theorem 1.10 in [6]). Let $\alpha,\ \beta$ and p as in the statement. We need to prove that if $F\in L^p_{\mathbf{k}}(\mathbb{U}, \mathrm{d}V_\alpha)$, there exists $C>0$, such that:

$$
\parallel\mathbf{P}_{\mathbf{k},\eta}F\parallel_{\mathbf{k},p,\alpha}\le C\parallel F\parallel_{\mathbf{k},p,\alpha}\ .
$$

We analyse only the first component, and thus, there exists $C_1>0$, such that:

$$
\begin{aligned}
\left(\int_{\mathbb{U}}|\mathbf{P}_\eta\widetilde{G}_1(\beta_1)|^p\,\mathrm{d}A_\alpha(\beta_1)\mathrm{d}A(\beta_2)\right)^{\frac{1}{p}} &= \left(\int_{U_1}|\mathbf{P}_\eta\widetilde{G}_1(\beta_1)|^p\,\mathrm{d}A_\alpha(\beta_1)\int_{U_2}\mathrm{d}A(\beta_2)\right)^{\frac{1}{p}}\\
&\le\left(C_1\int_{U_1}|\widetilde{G}_1(\beta_1)|^p\,\mathrm{d}A_\alpha(\beta_1)\right)^{\frac{1}{p}}
\end{aligned}
$$

$$\leq C_1^{\frac{1}{p}} \left(\int_{U_1} \left| \int_{U_2} G_1(\beta_1, \gamma_2)\, \mathrm{d}A(\gamma_2) \right|^p \mathrm{d}A_\alpha(\beta_1) \right)^{\frac{1}{p}}$$

$$\leq C_1^{\frac{1}{p}} \left(\int_{U_1} \left(\int_{U_2} |G_1(\beta_1, \gamma_2)|\, \mathrm{d}A(\gamma_2) \right)^p \mathrm{d}A_\alpha(\beta_1) \right)^{\frac{1}{p}}$$

$$\leq C_1^{\frac{1}{p}} \left(\int_{U_1} \int_{U_2} |G_1(\beta_1, \gamma_2)|^p \, \mathrm{d}A(\gamma_2)\, \mathrm{d}A_\alpha(\beta_1) \right)^{\frac{1}{p}}$$

$$= C_1^{\frac{1}{p}} \left(\int_{\mathbb{U}} |G_1(\beta_1, \beta_2)|^p \, \mathrm{d}A_\alpha(\beta_1)\, \mathrm{d}A(\beta_2) \right)^{\frac{1}{p}};$$

as $1 \leq p < \infty$, we have applied the Jensen's inequality in the last inequality. A similar result is true for the second component, so we finish the proof. □

Proposition 3.14 *Suppose* $1 \leq p < \infty$, $-1 < \alpha < \infty$ *and that* n *is a positive integer. Then, a bicomplex holomorphic function* F *in* $\mathbb{U}$ *belongs to* $\mathbb{BC}\mathcal{A}^p_\alpha$ *if and only if the function* $(1- \parallel Z \parallel^2_{\mathbf{k}})^n F^{(n)}$ *is in* $L^p_{\mathbf{k}}(\mathbb{U}, dV_\alpha)$.

Proof We will use again the fact that this theorem is true in the case of one complex variable. By (2.2) and (2.3), we have:

$$(1- \parallel Z \parallel^2_{\mathbf{k}})^n F^{(n)}(Z) = (1 - |\beta_1|^2)^n G_1^{(n)}(\beta_1)\, \mathbf{e} + (1 - |\beta_2|^2)^n G_2^{(n)}(\beta_2)\, \mathbf{e}^\dagger.$$

Since each $G_l \in \mathcal{A}^p_\alpha(\mathbf{i})$, $l = 1,\ 2$ by (3.1), we conclude the proof. □

4 Bicomplex Bloch space

A bicomplex holomorphic function F in $\mathbb{U}$ is said to be in the bicomplex Bloch space $\mathcal{B}_{\mathbb{BC}}$ if:

$$\parallel F \parallel_{\mathcal{B}_{\mathbb{BC}}} = \sup\{(1- \parallel Z \parallel^2_{\mathbf{k}}) \parallel F'(Z) \parallel_{\mathbf{k}} :\ Z \in \mathbb{U}\} \prec \infty.$$

By (2.5), the $\mathbf{k}$-seminorm $\|\cdot\|_{\mathcal{B}_{\mathbb{BC}}}$ is Möbius invariant. The little (vanishing) bicomplex Bloch space $\mathcal{B}_{\mathbb{BC},0}$ is the closed subspace of $\mathcal{B}_{\mathbb{BC}}$ consisting of functions F with:

$$\lim_{\|Z\|_k \to 1^-} (1- \parallel Z \parallel^2_{\mathbf{k}}) \parallel F'(Z) \parallel_{\mathbf{k}} = 0.$$

The $\mathbf{k}$-norm in the bicomplex Bloch space $\mathcal{B}_{\mathbb{BC}}$ is defined as:

$$\parallel F \parallel = \parallel F(0) \parallel_{\mathbf{k}} + \parallel F \parallel_{\mathcal{B}_{\mathbb{BC}}}.$$

With this $\mathbf{k}$-norm, the bicomplex Bloch space is complete.

By idempotent decomposition it is immediate that:

$$\mathcal{B}_{\mathbb{BC}} = \mathcal{B}\mathbf{e} + \mathcal{B}\mathbf{e}^\dagger \quad \text{and} \quad \mathcal{B}_{\mathbb{BC},0} = \mathcal{B}_0\mathbf{e} + \mathcal{B}_0\mathbf{e}^\dagger\ ,$$

where $\mathcal{B}$ and $\mathcal{B}_0$ are the one-dimensional complex Bloch spaces. If F is a bicomplex holomorphic function in $\mathbb{U}$ with $\parallel F \parallel_{\mathbf{k},\infty} \prec \infty$, then by Schwarz–Pick Lemma 2.3:

$$(1- \parallel Z \parallel^2_{\mathbf{k}}) \parallel F'(Z) \parallel_{\mathbf{k}} \prec 1- \parallel F(Z) \parallel^2_{\mathbf{k}}, \qquad \text{for all } Z \in \mathbb{U}.$$

It follows that $\mathbb{BC}^\infty \subset \mathcal{B}_{\mathbb{BC}}$ with $\parallel F \parallel_{\mathcal{B}_{\mathbb{BC}}} \prec \parallel F(Z) \parallel_{\mathbf{k},\infty}$.

Let $C(\overline{\mathbb{U}})$ be the space of continuous functions on the closed bidisk $\overline{\mathbb{U}}$ and $C_0(\overline{\mathbb{U}})$ be the subspace of $C(\overline{\mathbb{U}})$ consisting of functions vanishing on the set $\partial\mathbb{D}$, in particular on $\partial U_1 \times \partial U_2$. It is clear that $C(\overline{\mathbb{U}})$ and $C_0(\overline{\mathbb{U}})$ are closed subspaces of $L^\infty_{\mathbf{k}}(\mathbb{U})$. There is an interesting interplay between the Bloch space and the Bergman projection, as the following result shows.

Theorem 4.1 *Suppose* $-1 < \alpha < \infty$ *and that* $\mathbf{P}_{\mathbf{k},\alpha}$ *is the corresponding weighted Bergman projection. Then:*

- $\mathbf{P}_{\mathbf{k},\alpha}$ *maps* $L^\infty_{\mathbf{k}}(\mathbb{U})$ *boundedly onto* $\mathcal{B}_{\mathbb{BC}}$.

- $\mathbf{P}_{\mathbf{k},\alpha}$ *maps* $C(\overline{\mathbb{U}})$ *boundedly onto* $\mathcal{B}_{\mathbb{BC},0}$.
- $\mathbf{P}_{\mathbf{k},\alpha}$ *maps* $C(\mathbb{U})$ *boundedly onto* $\mathcal{B}_{\mathbb{BC},0}$.

Proof Suppose $\Psi \in L^{\infty}_{\mathbf{k}}(\mathbb{U})$ and $F = \mathbf{P}_{\mathbf{k},\alpha}\Psi$, and thus:

$$F(Z) = (\alpha+1)\int_{\mathbb{U}} \frac{(1-\parallel W \parallel^2_{\mathbf{k}})^{\alpha}\,\Psi(W)}{(1-ZW^*)^{2+\alpha}}\,\mathrm{d}A(W), \quad Z \in \mathbb{U}.$$

Derivating under the integral, we have:

$$F'(Z) = (\alpha+1)(\alpha+2)\int_{\mathbb{U}} \frac{(1-\parallel W \parallel^2_{\mathbf{k}})^{\alpha}\,W^*\Psi(W)}{(1-ZW^*)^{3+\alpha}}\,\mathrm{d}A(W), \quad Z \in \mathbb{U}$$

and

$$\parallel F'(Z) \parallel_{\mathbf{k}} \preceq (\alpha+1)(\alpha+2)\int_{\mathbb{U}} \frac{(1-\parallel W \parallel^2_{\mathbf{k}})^{\alpha}\parallel W^*\Psi(W)\parallel_{\mathbf{k}}}{\parallel 1-ZW^* \parallel^{3+\alpha}_{\mathbf{k}}}\,\mathrm{d}A(W)$$

$$\preceq (\alpha+1)(\alpha+2)\parallel \Psi \parallel_{\mathbf{k},\infty}\int_{\mathbb{U}} \frac{(1-\parallel W \parallel^2_{\mathbf{k}})^{\alpha}}{\parallel 1-ZW^* \parallel^{3+\alpha}_{\mathbf{k}}}\,\mathrm{d}A(W).$$

Then, by Lemma 2.5, there exists $0 \preceq C$, such that:

$$\parallel F \parallel \preceq C \parallel \Psi \parallel_{\mathbf{k},\infty}.$$

Next, assume $\Psi \in C(\overline{\mathbb{U}})$. We need to prove that $F = \mathbf{P}_{\mathbf{k},\alpha}\Psi \in \mathcal{B}_{\mathbb{BC},0}$. Consider functions of the form

$$\Psi_{l,m} = Z^l Z^{*m}, \qquad l,\ m = 0,\ 1,\ 2, \ldots$$

By the Stone–Weierstrass Theorem, the function Ψ can be uniformly approximated on $\mathbb{U}$ by finite linear combinations of these functions. Now:

$$\mathbf{P}_{\mathbf{k},\alpha}\Psi_{l,m} = (\alpha+1)\int_{\mathbb{U}} \frac{(1-\parallel W \parallel^2_{k})^{\alpha}}{(1-ZW^*)^{2+\alpha}} W^l W^{*m}\,\mathrm{d}V(Z)$$

$$= (\alpha+1)\int_{\mathbb{U}} (1-\parallel W \parallel^2_{\mathbf{k}})^{\alpha} W^l W^{*m} \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} Z^n W^{*n}\,\mathrm{d}V(W)$$

$$= (\alpha+1)\sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} Z^n \int_{\mathbb{U}} (1-\parallel W \parallel^2_{\mathbf{k}})^{\alpha} W^l W^{*m+n}\,\mathrm{d}V(W)$$

$$= \frac{\Gamma(l+\alpha+2-m)l!}{\Gamma(l+\alpha+2)(l-m)!} Z^{l-m}$$

with $l \geq m$. Therefore, $\mathbf{P}_{\mathbf{k},\alpha}\Psi_{l,m} \in \mathcal{B}_{\mathbb{BC},0}$. Since $\mathbf{P}_{\mathbf{k},\alpha}$ maps $L^{\infty}_{\mathbf{k}}(\mathbb{U})$ boundedly into $\mathcal{B}_{\mathbb{B}}\mathbb{C}$, and $\mathcal{B}_{\mathbb{BC},0}$ is closed in $\mathcal{B}_{\mathbb{B}}\mathbb{C}$, we conclude that $\mathbf{P}_{\mathbf{k},\alpha}$ maps $C(\overline{\mathbb{U}})$ boundedly into $\mathcal{B}_{\mathbb{BC},0}$.

We now prove that the quoted projections are onto. Let $F \in \mathcal{B}_{\mathbb{B}}\mathbb{C}$ with development:

$$F(Z) = A_0 + A_1 Z + A_2 Z^2 + \sum_{n=3}^{\infty} A_n Z^n, \qquad Z \in \mathbb{U}.$$

Define:

$$E(Z) = (1-\parallel Z \parallel^2_{\mathbf{k}})\left(\frac{\alpha+2}{\alpha+1}A_0 + \frac{\alpha+3}{\alpha+1}A_1 Z + \frac{\alpha+4}{\alpha+1}A_2 Z^2 + \frac{1}{\alpha+1}\frac{\sum_{n=3}^{\infty} n A_n Z^{n-1}}{Z^*}\right).$$

The function $\frac{1}{\alpha+1}\frac{\sum_{n=3}^{\infty} nA_n Z^{n-1}}{Z^*} \in C(\mathbb{U})$. Applying linearity of the Bergman projection and Theorem 3.11 to get:

$$\mathbf{P}_{\mathbf{k},\alpha} E(Z) = F(Z).$$

Thus, it is clear that $g \in C_0(\mathbb{U})$ if F is in the little Bloch space and $\mathbf{P}_{\mathbf{k},\alpha}$ maps $L^{\infty}_{\mathbf{k}}(\overline{\mathbb{U}})$ onto $\mathcal{B}_{\mathbb{B}}\mathbb{C}$ and it maps $C_0(\mathbb{U})$ (and, hence, $C(\overline{\mathbb{U}})$) onto $\mathcal{B}_{\mathbb{BC},0}$. □

We induce the Poincaré metric in $\mathbb{U}$ in the following form. For $Z \in \mathbb{U}$, define the Möbius transformation:

$$\varphi_Z : \mathbb{U} \to \mathbb{U} \quad \text{by} \quad \varphi_Z(W) = \frac{Z - W}{1 - Z^* W}.$$

The Poincaré metric $d_{\mathbf{k}}$ on $\mathbb{U}$ is defined by:

$$d_{\mathbf{k}}(Z, W) = \frac{1}{2} \ln \frac{1+ \parallel \varphi_Z(W) \parallel_{\mathbf{k}}}{1- \parallel \varphi_Z(W) \parallel_{\mathbf{k}}}.$$

By (2.4), we can translate many properties of the usual hyperbolic metric in the unit disk. For example, the metric $d_{\mathbf{k}}$ is Möbius invariant.

Lemma 4.2 *The infinitesimal distance element for the metric $d_{\mathbf{k}}$ is*

$$\frac{\parallel dZ \parallel_{\mathbf{k}}}{1- \parallel Z \parallel_{\mathbf{k}}^2}.$$

Proof If $a \to 0$, then $\ln \frac{1+a}{1-a} \approx \frac{2a}{1-a}$. Thus, if $W \to Z$ and $W - Z$ is not a zero divisor:

$$\frac{d_{\mathbf{k}}(Z, W)}{\parallel Z - W \parallel_{\mathbf{k}}} \approx \frac{\parallel Z - W \parallel_{\mathbf{k}}}{\parallel 1 - Z^* W \parallel_{\mathbf{k}} - \parallel Z - W \parallel_{\mathbf{k}}} \cdot \frac{1}{\parallel Z - W \parallel_{\mathbf{k}}}$$

$$\overset{W \to Z}{\mapsto} \frac{1}{\parallel 1 - Z^* W \parallel_{\mathbf{k}} - \parallel Z - W \parallel_{\mathbf{k}}} = \frac{1}{1- \parallel Z \parallel_{\mathbf{k}}^2}.$$

This concludes the proof. □

A precise relationship between the Bloch space and the Bergman metric is obtained in the following result

Theorem 4.3 *Let $F : \mathbb{U} \to \mathbb{BC}$ be a bicomplex holomorphic function. Then, F belongs to the bicomplex Bloch space if and only if there exists $0 \prec C$, such that:*

$$\parallel F(Z) - F(W) \parallel_{\mathbf{k}} \preceq C d_{\mathbf{k}}(Z, W) \quad \text{for all} \quad Z,\ W \in \mathbb{U}.$$

Proof As F is a bicomplex holomorphic function in $\mathbb{U}$, then:

$$F(Z) - F(0) = Z \int_0^1 F'(tZ)\, \mathrm{d}t \qquad \text{for all } Z \in \mathbb{U}.$$

If F is in the Bloch space, we have:

$$\parallel F(Z) - F(0) \parallel_{\mathbf{k}} \preceq \parallel Z \parallel_{\mathbf{k}} \int_0^1 \parallel F'(tZ) \parallel_{\mathbf{k}} dt \preceq \parallel F \parallel_{\mathcal{B}_{\mathbb{BC}}} \int_0^1 \frac{\parallel Z \parallel_{\mathbf{k}} \ \mathrm{d}t}{1 - t^2 \parallel Z \parallel_{\mathbf{k}}^2}$$

$$\preceq \frac{1}{2} \parallel F \parallel_{\mathcal{B}_{\mathbb{BC}}} \ln \frac{1+ \parallel Z \parallel_{\mathbf{k}}}{1- \parallel Z \parallel_{\mathbf{k}}} = \parallel F \parallel_{\mathcal{B}_{\mathbb{BC}}} d_{\mathbf{k}}(Z, 0)$$

for all $Z \in \mathbb{U}$. By the Möbius invariance of $\| \cdot \|_{\mathcal{B}_{\mathbb{BC}}}$ and $d_{\mathbf{k}}$ and replacing F by $F \circ \varphi_Z$ and Z by $\varphi_Z(W)$:

$$\| (F \circ \varphi_Z)(\varphi_Z(W)) - F \circ \varphi_Z(0) \|_{\mathbf{k}} \preceq \| F \circ \varphi \|_{\mathcal{B}_{\mathbb{BC}}} d_{\mathbf{k}}(\varphi_Z(W), 0)$$

$$\| F(W) - F(Z) \|_{\mathbf{k}} \preceq \| F \|_{\mathcal{B}_{\mathbb{BC}}} d_{\mathbf{k}}(\varphi_Z(\varphi_Z(W)), \varphi_Z(0))$$

$$\| F(W) - F(Z) \|_{\mathbf{k}} \preceq \| F \|_{\mathcal{B}_{\mathbb{BC}}} d_{\mathbf{k}}(W, Z)$$

If there exists $0 \preceq C$, such that if $W - Z$ is not a zero divisor:

$$\frac{\| F(W) - F(Z) \|_{\mathbf{k}}}{\| W - Z \|_{\mathbf{k}}} \frac{\| W - Z \|_{\mathbf{k}}}{d_{\mathbf{k}}(W, Z)} = \frac{\| F(W) - F(Z) \|_{\mathbf{k}}}{d_{\mathbf{k}}(W, Z)} \preceq C,$$

then by the proof of Lemma 4.2, we get the result when $W \to Z$. □

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