$\sum_{k=0}^{\infty} p_k z^k$ where $p_k \geq 0$ and $\sum p_k = 1$. (T. E. Harris' book [Has63] begins with a very readable discussion of these ideas.) The coefficient p_k is to be interpreted as the probability that an individual will have k offspring and the k^{th} coefficient of φ_n is interpreted as the probability that there will be k individuals in the n^{th} generation. Questions concerning the eventual population size are related to the asymptotic behavior of the iterates of φ. We note that every probability generating function for a Galton–Watson process is an analytic function mapping D into itself, and if $p_0 + p_1 \neq 0$, φ satisfies the hypotheses of theorem 2.53. The bibliography of Harris [Has63] and K. Athreya and P. Ney [AtN72] are a guide to older work on branching processes.

In [Co81], Theorem 2.53 is used to study "fractional iterates" of φ, that is, a real semigroup of functions φ_t for $t \geq 0$, where $\varphi_1 = \varphi$ and $\varphi_{s+t} = \varphi_s \circ \varphi_t$. Since fractional iterates can be defined for linear fractional transformations, we try to define φ_t by $\Phi_t \circ \sigma = \sigma \circ \varphi_t$. But there are difficulties because, in general, σ is not univalent, and even if it were, σ^{-1} need not be defined on $\Phi_t(\sigma(D))$ unless $\sigma(D)$ meets some geometric condition. We do find, though, that for each z in D, $\varphi_t(z)$ can be defined for t sufficiently large. This is enough that we can find a meromorphic function on the disk that deserves to be called the infinitesimal generator of the semigroup. This point of view will become important in Section 7.7.

Exercise 2.4.2 was suggested by P. Bourdon. Much of this information is contained in Cowen's paper [Co84a]. The application of the model for iteration to the problem of commutation illustrates the use of the abstract formulation of the model and the resulting uniqueness. The problem of commutation of analytic functions mapping the disk to itself has also been studied by A. Shields [Shi64] and D. Behan [Beh73].

Iteration of functions for which $\varphi'(a) = 0$ is studied via Böttcher's functional equation, $f(\varphi(z)) = (f(z))^k$, in Cowen's paper [Co82].

2.5 The automorphisms of the ball

If $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ is an analytic map of an open set $\Omega \subset C^n$ into C^m, $\psi'(z) : C^n \to C^m$ is the linear operator which is represented (with respect to the standard basis on C^n) by the matrix (a_{jk}) where $a_{jk} = D_k\psi_j(z)$, for $1 \leq j \leq m$, $1 \leq k \leq n$ and $D_k = \frac{\partial}{\partial z_k}$. The derivative $\psi'(z)$ is the unique linear operator satisfying

$$\psi(z+h) - \psi(z) = \psi'(z)h + O(|h|^2)$$

for h near 0 in C^n. When $n = m$ the matrix (a_{jk}) is square and its determinant, the complex Jacobian, is denoted $J_\psi(z)$.

Before we can describe the automorphisms of B_N we need a several variable version of the Schwarz lemma, applicable to maps φ of B_N into B_N.

***THEOREM 2.69** (Schwarz Lemma in B_N)*
Suppose $\varphi : B_N \to B_N$ is analytic and $\varphi(0) = 0$. Then

(1) $|\varphi(z)| \leq |z|$ for all z in B_N and

(2) $\|\varphi'(0)\| \leq 1$.

PROOF For ζ, η any two points of ∂B_N define an analytic function of the disk D by

$$g(\lambda) = g_{\zeta,\eta}(\lambda) = \langle \varphi(\lambda\zeta), \eta \rangle$$

Note that g maps D into D and $g(0) = 0$. Thus the one variable Schwarz lemma implies that

$$|g(\lambda)| = |\langle \varphi(\lambda\zeta), \eta \rangle| \leq |\lambda|$$

for all ζ, η in ∂B_N. In particular, given z in B_N, write $z = r\zeta$ for some $r \geq 0$, ζ in ∂B_N and $\varphi(r\zeta) = s\eta$ for $s \geq 0$, η in ∂B_N. Thus $s \leq r$, that is $|\varphi(z)| \leq |z|$, giving (1).

Moreover, a computation shows that $g'(\lambda) = \langle \varphi'(\lambda\zeta)\zeta, \eta \rangle$ so again by the one variable Schwarz lemma we have

$$|g'(0)| = |\langle \varphi'(0)\zeta, \eta \rangle| \leq 1$$

for all ζ, η in ∂B_N. Thus $\|\varphi'(0)\| \leq 1$. ▮

Unlike the case $N = 1$ neither equality in (2) of Theorem 2.69, nor equality at some $z_0 \neq 0$ in (1) implies that φ is a linear map. To see this consider $\varphi(z_1, z_2) = (z_1 + \frac{1}{2}z_2^2, 0)$ mapping B_2 into B_2. Then $\varphi(z_1, 0) = (z_1, 0)$ and $\|\varphi'(0)\| = 1$, yet φ is not linear. Note that φ *is* a unitary map of the slice $[e_1]$ onto itself. Though we will not need it, we remark that this sort of partial linearity is a consequence of the equality cases of Theorem 2.69, see [Ab89b, p. 161] for the details. We will however have need of the next result, where we make the stronger assumption that $\varphi'(0)$ is an isometry of C^N into C^N.

PROPOSITION 2.70
Suppose $\varphi : B_N \to B_N$ is analytic with $\varphi(0) = 0$ and $\varphi'(0)$ unitary. Then $\varphi(z) = \varphi'(0)z$.

PROOF Fix ζ in ∂B_N and let $\eta = \varphi'(0)\zeta$. Define $g(\lambda) = \langle \varphi(\lambda\zeta), \eta \rangle$. Notice that $g'(\lambda) = \langle \varphi'(\lambda\zeta)\zeta, \eta \rangle$ and $g'(0) = 1$. By the Schwarz lemma in the disk $g(\lambda) = \lambda$, or $\langle \lambda^{-1}\varphi(\lambda\zeta), \eta \rangle = 1$. But

$$\frac{|\varphi(\lambda\zeta)|}{|\lambda|} \leq 1$$

so we must have $\varphi(\lambda\zeta) = \lambda\eta = \varphi'(0)(\lambda\zeta)$ for all $\lambda, |\lambda| < 1$. Since ζ is arbitrary this gives the conclusion. ▮

By an automorphism φ of B_N we mean a biholomorphic map of B_N onto B_N; that is, both φ and φ^{-1} are analytic maps of B_N onto B_N. It is true, but not obvious, that if φ maps B_N one-to-one and onto B_N then φ is an automorphism of B_N; see [Ru80, Theorem 15.1.8, p. 302].

We will first describe some automorphisms of B_N that are analogous to the disk automorphisms $(a - z)/(1 - \overline{a}z)$, for a in D. Let a be in B_N and set

$$P_a(z) = \frac{\langle z, a\rangle}{\langle a, a\rangle} a$$

so P_a is projection onto the subspace $[a]$ spanned by a, and $Q_a = I - P_a$, projection onto the orthogonal complement of $[a]$. To simplify notation write $s_a = \sqrt{1 - |a|^2}$. Define $\varphi_a(z)$ by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}$$

Clearly φ_a is analytic in $\overline{B_N}$, $\varphi_a(0) = a$ and $\varphi_a(a) = 0$. To obtain further properties of φ_a it is simplest to first consider the special case $a = (a_1, 0') \equiv (a_1, 0, \ldots, 0)$. Then one can easily compute the coordinate functions of φ_a to obtain

$$\varphi_a(z) = \varphi_a(z_1, z') = \left(\frac{a_1 - z_1}{1 - \overline{a_1} z_1}, \frac{-s_a z'}{1 - \overline{a_1} z_1}\right)$$

A direct computation will then verify that for $a = (a_1, 0')$

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1 - |a|^2)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)} \tag{2.5.1}$$

for any z, w in the closed ball. We will refer to Equation (2.5.1) as the fundamental identity for automorphisms; our next goal is to extend it to any a in B_N. Note that consequences of Equation (2.5.1) include (setting $z = w$)

(i) $\varphi_a(B_N) \subset B_N$

(ii) $\varphi_a(\partial B_N) \subset \partial B_N$.

The verification of Equation (2.5.1) for arbitrary a in B_N will follow from the next result.

LEMMA 2.71

If U is a unitary map, then $\varphi_{Ua} = U\varphi_a U^{-1}$, for any a in B_N.

PROOF This is Exercise 2.5.1. ▮

Returning to Equation (2.5.1), let a be in B_N and find U unitary such that $Ua = (c, 0')$. Then

$$\begin{aligned} 1 - \langle \varphi_{Ua}(z), \varphi_{Ua}(w)\rangle &= 1 - \langle U\varphi_a U^{-1}(z), U\varphi_a U^{-1}(w)\rangle \\ &= 1 - \langle \varphi_a U^{-1}(z), \varphi_a U^{-1}(w)\rangle \end{aligned}$$

Now replace z by Uz and w by Uw and use the fact that we have verified Equation (2.5.1) for $Ua = (c, 0')$. Thus Equation (2.5.1) holds for any a in B_N.

Next we compute, *for the special case* $a = (a_1, 0')$, the matrix of the derivative map $\varphi_a'(z)$, using the description of the coordinate functions of φ_a. Since the (j,k) entry of $\varphi_a'(z)$ is $D_k\varphi_j(z)$ we see that

$$\begin{aligned}\varphi_a'(0) &= \operatorname{diag}\{-s_a^2, -s_a, \ldots, -s_a\}\\ \varphi_a'(a) &= \operatorname{diag}\{-1/s_a^2, -1/s_a, \ldots, -1/s_a\}\end{aligned}$$

and hence

$$\varphi_a'(a)\varphi_a'(0) = \varphi_a'(0)\varphi_a'(a) = I \tag{2.5.2}$$

Lemma 2.71 can be used again to extend Equation (2.5.2) from $a = (a_1, 0')$ to arbitrary a in B_N. The details are left to the reader.

For any a in B_N the analytic map $\varphi_a \circ \varphi_a$ of B_N into B_N satisfies

(i) $\varphi_a \circ \varphi_a(0) = 0$

(ii) $(\varphi_a \circ \varphi_a)'(0) = I$

Proposition 2.70 implies that $\varphi_a \circ \varphi_a(z) = z$ for all z in B_N. Hence φ_a is an automorphism of B_N, with $\varphi_a^{-1} = \varphi_a$. We can now give a complete description of the group of all automorphisms of B_N, $\operatorname{Aut}(B_N)$. This description should be compared with the description of the automorphisms of the unit disk given in Section 2.3.

THEOREM 2.72

Let ψ be an automorphism of B_N with $\psi^{-1}(0) = a$. There exists a unitary map U so that $\psi = U\varphi_a$.

PROOF The map $\psi \circ \varphi_a$ is an automorphism of B_N fixing 0. We wish to show $\psi \circ \varphi_a$ is unitary. Write $\tau = \psi \circ \varphi_a$ and consider, for θ real,

$$\lambda(z) = \tau^{-1}(e^{-i\theta}\tau(e^{i\theta}z))$$

Check that $\lambda(0) = 0$ and $\lambda'(0) = I$. Thus, as before, an appeal to Proposition 2.70 implies $\lambda(z) = z$ and therefore $\tau(e^{i\theta}z) = e^{i\theta}\tau(z)$, for every real θ. If we write out the homogeneous expansion of τ, that is write $\tau(z) = Lz + \sum_{s=2}^{\infty} F_s(z)$ where L is linear and the components of F_s are homogeneous polynomials of degree s, we see that the homogeneity of τ, $\tau(e^{i\theta}z) = e^{i\theta}\tau(z)$, forces $F_s \equiv 0$ for $s \geq 2$, and we are done, since a linear map which is an automorphism of B_N is unitary. ∎

With this classification theorem we can extend the fundamental identity (2.5.1) to all ψ in $\operatorname{Aut}(B_N)$. Set $\psi^{-1}(0) = a$. Then for z, w in $\overline{B_N}$

$$1 - \langle \psi(z), \psi(w) \rangle = \frac{(1 - |a|^2)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)} \tag{2.5.3}$$

We will next study the fixed point sets of automorphisms of B_N. Our first result deals with the automorphisms φ_a.

THEOREM 2.73

For $a \neq 0$ the only point of $\overline{B_N}$ which is fixed by φ_a is $a/(1+s_a)$ where $s_a = \sqrt{1-|a|^2}$.

PROOF If z is a fixed point of φ_a then

$$a - P_a z - s_a Q_a z = (1 - \langle z, a\rangle) z$$

Applying Q_a to both sides of this equation

$$-s_a Q_a(z) = (1 - \langle z, a\rangle) Q_a(z)$$

which can only hold if $Q_a(z) = 0$ since $\mathrm{Re}\,(1 - \langle z, a\rangle) \geq 0$. Thus any fixed point of $\varphi_a(z)$ is of the form λa for λ complex. Using the definition of φ_a we see that if $\varphi_a(\lambda a) = \lambda a$ then $\lambda^2 |a|^2 - 2\lambda + 1 = 0$ which has solutions

$$\lambda = \frac{1 \pm s_a}{|a|^2} = \frac{1}{1 \pm s_a}$$

Only $\lambda = 1/(1+s_a)$ gives a point in $\overline{B_N}$; moreover $a/(1+s_a)$ is in B_N. ▮

Of course for a unitary map U, the fixed point set of U in $\overline{B_N}$ is the intersection of $\overline{B_N}$ with a (complex) subspace of C^N, either $\{0\}$ or the eigenspace of U corresponding to the eigenvalue 1. By an ***affine subset of*** B_N we mean the intersection of B_N with a translate of a subspace of C^N; its dimension is the dimension of this subspace. An affine subset of C^N is simply a translate of a subspace of C^N. It is easy to see that a set $E \subset C^N$ is affine if and only if whenever $z_1, \ldots, z_n$ in E and $\lambda_1, \ldots, \lambda_n$ in C satisfy $\sum \lambda_j = 1$, then $\sum \lambda_j z_j$ is in E. Write $A[z_1, \ldots, z_n]$ for the affine set generated by $z_1, z_2, \ldots, z_n$; that is, $A[z_1, \ldots, z_n] = \{\sum \lambda_j z_j : \sum \lambda_j = 1\}$. Suppose $z = \sum \lambda_j z_j$ where $\sum \lambda_j = 1$ and $\langle z, a\rangle \neq 1$. Then $\varphi_a(z) = \sum \mu_j \varphi_a(z_j)$ where $\mu_j = \lambda_j(1 - \langle z_j, a\rangle)/(1 - \langle z, a\rangle)$ so that $\sum \mu_j = 1$. This says

$$\varphi_a\{A[z_1, \ldots, z_n] \setminus V_a\} \subset A[\varphi_a(z_1), \ldots, \varphi_a(z_n)] \tag{2.5.4}$$

where V_a is the singularity set for φ_a; that is, $V_a = \{z : \langle z, a\rangle = 1\}$. This observation is the key step in proving the next result.

THEOREM 2.74

The image under ψ in $Aut(B_N)$ of an affine subset E of $\overline{B_N}$ is an affine subset of $\overline{B_N}$.

PROOF By Theorem 2.72 we may write $\psi = U\varphi_a$ for some unitary U. Thus it is enough to show $\varphi_a(E)$ is affine. Now $E = A[z_1, \ldots, z_m] \cap \overline{B_N}$ for some

$z_1, \ldots, z_m$ and $V_a \cap \overline{B_N} = \emptyset$, so Equation (2.5.4) shows

$$\varphi_a(E) \subset A[\varphi_a(z_1), \ldots, \varphi_a(z_m)] \cap \overline{B_N}$$

Since $\varphi_a(\varphi_a(z)) = z$ this also implies

$$\varphi_a\{A[\varphi_a(z_1), \ldots, \varphi_a(z_m)] \cap \overline{B_N}\} \subset A[z_1, \ldots, z_m] \cap \overline{B_N} = E$$

Thus $E = \varphi_a\{A[\varphi_a(z_1), \ldots, \varphi_a(z_m)] \cap \overline{B_N}\}$ and $\varphi_a(E)$ is the affine subset $A[\varphi_a(z_1), \ldots, \varphi_a(z_m)] \cap \overline{B_N}$. ∎

COROLLARY 2.75
If ψ in Aut(B_N) has a fixed point in B_N then the fixed point set of ψ in the open ball B_N is affine.

PROOF If ψ fixes a in B_N, consider $\varphi_a \circ \psi \circ \varphi_a$ which fixes 0. By Theorem 2.72, this automorphism is a unitary map whose fixed point set is a subspace V of C^N. It is easy to see that the fixed point set of ψ in B_N is $\varphi_a(V) \cap B_N$; by the last theorem this is an affine set. ∎

Perhaps surprisingly, this last result is actually true in much greater generality.

THEOREM 2.76
Let $\varphi : B_N \to B_N$ be analytic. Then the fixed point set of φ in B_N is affine or empty.

We will not give the proof of this result here, but instead refer the reader to [Ru80, p. 165] where it is shown that if $\varphi(0) = 0$ then φ and $\varphi'(0)$ have the same fixed points in B_N. The theorem then follows from the previous result for automorphisms, and the fact that automorphisms take affine sets to affine sets.

Exercises

2.5.1 Prove Lemma 2.71. (Hint: The goal is to show $\varphi_{Ua}(Uz) = U\varphi_a(z)$. Note that $s_a = s_{Ua}$.)

2.5.2 Use Theorem 2.69 and the fundamental identity, Equation (2.5.3), to prove the Schwarz–Pick Theorem for B_N: If $\varphi : B_N \to B_N$ is analytic and a in B_N then

$$\frac{|1 - \langle \varphi(z), \varphi(a) \rangle|^2}{(1 - |\varphi(z)|^2)(1 - |\varphi(a)|^2)} \leq \frac{|1 - \langle z, a \rangle|^2}{(1 - |z|^2)(1 - |a|^2)}$$

2.5.3 For a in B_N and $0 < r < 1$ set $\mathcal{E}(a, r) = \varphi_a(rB_N)$.

(a) Show $U\mathcal{E}(a, r) = \mathcal{E}(Ua, r)$.

(b) Show that if $a = (a_1, 0')$ then $\mathcal{E}(a,r)$ is an ellipsoid and identify its center $c = (c_1, 0')$. Also show that $\mathcal{E}(a,r) \cap [e_1]$ is a disk of radius $\sim r(1-|a|^2)$ while $\mathcal{E}(a,r) \cap \{z_1 = c_1\}$ is a ball of radius $\sim r\sqrt{1-|a|^2}$ for r small. By virtue of (a) these observations give the shape of an arbitrary $\mathcal{E}(a,r)$.

(c) Interpret Exercise 2.5.2 in terms of these ellipsoids.

Notes

Early contributors to the study of the Schwarz Lemma in several variables include K. Reinhardt, C. Carathéodory and H. Cartan; see [Di89] for a discussion of some of the history of and references for generalizations of the Schwarz Lemma.

The automorphisms in B_2 were first described by H. Poincaré in [Poi07]. The discussion of $\mathrm{Aut}(B_N)$ given here was greatly influenced by the exposition in [Ru80].

Theorem 2.76 is due to M. Hervé [He63]; see also [Ru78].

2.6 Julia–Carathéodory theory in the ball

In this section we will discuss the analogues of the ideas of Section 2.3 for the ball. As much as possible our treatment here will parallel the treatment in the disk, although a few new technical details will be needed. For ζ in ∂B_N we will continue to use the notation $d(\zeta)$ for $\liminf_{z\to\zeta}(1-|\varphi(z)|)/(1-|z|)$ where φ is an analytic map of the ball B_N into itself. Julia's Lemma (Lemma 2.41) generalizes to the following:

***LEMMA 2.77** (Julia's Lemma in B_N)*
Suppose ζ is in ∂B_N with $d(\zeta) < \infty$. Suppose $a_n \to \zeta$ satisfies

$$\lim_{n\to\infty} \frac{1-|\varphi(a_n)|}{1-|a_n|} = d(\zeta)$$

and $\lim_{n\to\infty} \varphi(a_n) = \eta$ where η is in ∂B_N. Then for every z in B_N

$$\frac{|1-\langle \varphi(z), \eta\rangle|^2}{1-|\varphi(z)|^2} \le d(\zeta)\frac{|1-\langle z, \zeta\rangle|^2}{1-|z|^2}$$

PROOF By the Schwarz–Pick Theorem in B_N (Exercise 2.5.2) we have

$$\frac{|1-\langle \varphi(z), \varphi(a_n)\rangle|^2}{(1-|\varphi(z)|^2)(1-|\varphi(a_n)|^2)} \le \frac{|1-\langle z, a_n\rangle|^2}{(1-|z|^2)(1-|a_n|^2)}$$

or

$$\frac{|1-\langle \varphi(z), \varphi(a_n)\rangle|^2}{1-|\varphi(z)|^2} \le \frac{1-|\varphi(a_n)|^2}{1-|a_n|^2}\frac{|1-\langle z, a_n\rangle|^2}{1-|z|^2}$$

Letting $n \to \infty$ gives the conclusion. ∎

As in the one variable case, this lemma has an appealing geometric interpretation. Set $E(k,\zeta) = \{z \in B_N : |1 - \langle z, \zeta\rangle|^2 \leq k(1 - |z|^2)\}$. In the special case $\zeta = e_1$ a computation shows that this is equivalent to

$$\left|z_1 - \frac{1}{1+k}\right|^2 + \frac{k}{1+k}|z'|^2 \leq \left(\frac{k}{1+k}\right)^2$$

which is an ellipsoid tangent at e_1 with center $\frac{1}{1+k}e_1$. Its intersection with $[e_1]$ is a disk of radius $\frac{k}{1+k}$,while its intersection with $z_1 = \frac{1}{1+k}$ is a ball of radius $\sqrt{\frac{k}{1+k}}$. Because unitary maps preserve inner products, $U(E(k,\zeta)) = E(k, U\zeta)$ for any unitary U. Thus in general $E(k,\zeta)$ is an ellipsoid internally tangent to the unit sphere at ζ with center $\frac{1}{1+k}\zeta$. Julia's Lemma says φ maps each ellipsoid $E(k,\zeta)$ into the corresponding ellipsoid $E(d(\zeta)k, \eta)$.

One important technical complication for the Julia–Carathéodory theory in B_N arises from the appropriate analogue for nontangential approach regions when $N > 1$. For many of the basic function theory results in B_N, approach within a Koranyi approach region replaces nontangential approach. These regions are defined for $\alpha > 1$ by

$$\Gamma(\zeta, \alpha) = \{z \in B_N : |1 - \langle z, \zeta\rangle| < \frac{\alpha}{2}(1 - |z|^2)\}$$

Setting $N = 1$ we see that, since

$$\frac{\alpha}{2}(1 - |z|^2) \sim \alpha(1 - |z|)$$

for $|z|$ near 1, in essence, this gives our previous definition for a nontangential approach region in the disk, and we will use the same notation. However, when $N > 1$ these regions have a perhaps unexpected geometric property. Since $U\Gamma(\zeta, \alpha) = \Gamma(U\zeta, \alpha)$, we can concentrate on understanding $\Gamma(e_1, \alpha)$. An easy computation shows that while the intersection of $\Gamma(e_1, \alpha)$ with the complex line through e_1 is a standard nontangential approach region in the disk $B_N \cap [e_1]$, in other directions $\Gamma(e_1, \alpha)$ permits tangential approach to ∂B_N. In particular, the intersection of $\Gamma(e_1, \alpha)$ with $\{z : \text{Im}\, z_1 = 0\}$ is the ball

$$(x_1 - 1/\alpha)^2 + |z'|^2 < (1 - 1/\alpha)^2$$

containing e_1 in its boundary. We say f has ***admissible limit at*** ζ if it has a limit $f^*(\zeta)$ along every curve lying in some Koranyi region $\Gamma(\zeta, \alpha)$.

In the definition of a Koranyi approach region one is comparing the distance (in the Euclidean metric) from z to $\zeta + T^C(\zeta)$, where $T^C(\zeta)$ is the maximal complex subspace of the tangent space to ∂B_N at ζ, with the distance from z to ∂B_N. In one variable this point of view degenerates, since the maximal complex subspace of the tangent space to ∂D at any point is just $\{0\}$, and the comparison then becomes simply that of the distances from z to ζ and from z to ∂D.

In the Julia–Carathéodory Theorem in the disk (Theorem 2.44), we made use of the fact that an analytic function in D that is bounded in every $\Gamma(\zeta, \alpha)$ and has radial limit at ζ in fact has nontangential limit at ζ. Since the analogue of this using Koranyi regions is false when $N > 1$, we need to use a smaller collection of approach sets.

DEFINITION 2.78 *We say f has* **restricted limit at** *ζ in ∂B_N if f has limit $f^*(\zeta)$ along every curve $\Lambda(t)$ approaching ζ that satisfies*

$$\text{(1)}\ \lim_{t\to 1} \frac{|\Lambda(t) - \lambda(t)|^2}{1 - |\lambda(t)|^2} = 0$$

$$\text{and (2)}\ \frac{|\lambda(t) - \zeta|}{1 - |\lambda(t)|} \leq M < \infty \text{ for } 0 \leq t < 1$$

where $\lambda(t)$ is the orthogonal projection of $\Lambda(t)$ into the complex line $[\zeta]$ through 0 and ζ:

$$\lambda(t) = \langle \Lambda(t), \zeta \rangle \zeta$$

The meaning of (2) is that the projection $\lambda(t)$ lies in a nontangential approach region in the copy of the unit disk lying in $[\zeta]$. The notion of restricted limit arises because the full analogue of Lindelöf's Theorem, with admissible limits replacing nontangential limits, does not hold for $N > 1$. The following theorem, a version of Čirka's Theorem, provides an adequate substitute. A proof can be found in [Ru80, Theorem 8.4.8, p. 174].

THEOREM 2.79
Suppose f is analytic in B_N and bounded in every approach region $\Gamma(\zeta, \alpha)$. If $lim_{r\to 1} f(r\zeta)$ exists, then f has restricted limit at ζ.

DEFINITION 2.80 *We say $\varphi : B_N \to B_N$ has* **finite angular derivative** *at ζ in ∂B_N if there exists η in ∂B_N so that*

$$\frac{\langle \eta - \varphi(z), \eta \rangle}{\langle \zeta - z, \zeta \rangle}$$

has finite restricted limit at ζ.

The following result is the Julia–Carathéodory Theorem in the ball. When η is in ∂B_N we use the notation φ_η for the coordinate of φ in the η–direction, i.e. $\varphi_\eta(z) = \langle \varphi(z), \eta \rangle$.

THEOREM 2.81 ***(Julia–Carathéodory Theorem in B_N)***
For $\varphi : B_N \to B_N$ analytic and ζ in ∂B_N, the following are equivalent:

(1) $d(\zeta) = \liminf_{z\to\zeta}(1 - |\varphi(z)|)/(1 - |z|) < \infty$ *where the limit is taken as z approaches ζ unrestrictedly in B_N.*

(2) *φ has finite angular derivative at ζ.*

(3) *φ has restricted limit η at ζ, where $|\eta| = 1$ and $D_\zeta \varphi_\eta(z) = \langle \varphi'(z)\zeta, \eta\rangle$ has finite restricted limit at ζ.*

Moreover, when these conditions hold, $D_\zeta \varphi_\eta(z)$ has restricted limit $d(\zeta)$ at ζ.

The proof of this theorem parallels the proof of Theorem 2.44 in the disk and requires the following extension of Lemma 2.45 to Koranyi approach regions in B_N.

LEMMA 2.82
Let $1 < \alpha < \beta$. There exists $\delta = \delta(\alpha, \beta) > 0$ so that if (z_1, z') is in $\Gamma(e_1, \alpha)$ then $(z_1 + \lambda, z')$ is in $\Gamma(e_1, \beta)$ for all λ with $|\lambda| \le \delta|1 - z_1|$.

PROOF If (z_1, z') is in $\Gamma(e_1, \alpha)$ then

$$1 - |z_1|^2 - |z'|^2 \ge \frac{2}{\alpha}|1 - z_1|$$

If $|\lambda| < 1$ we have

$$1 - |z_1 + \lambda|^2 \ge 1 - |z_1|^2 - 2|\lambda| - |\lambda|^2 \ge 1 - |z_1|^2 - 3|\lambda|$$

so that

$$1 - |z_1 + \lambda|^2 - |z'|^2 \ge \frac{2}{\alpha}|1 - z_1| - 3|\lambda|$$

and hence if $|\lambda| \le \delta|1 - z_1|$ for some positive δ less than $1/2$

$$1 - |z_1 + \lambda|^2 - |z'|^2 \ge (\frac{2}{\alpha} - 3\delta)|1 - z_1|$$

But

$$\frac{2}{\beta}|1 - z_1 - \lambda| \le \frac{2}{\beta}(|1 - z_1| + |\lambda|) \le \frac{2}{\beta}(1 + \delta)|1 - z_1|$$

and $(z_1 + \lambda, z')$ will lie in $\Gamma(e_1, \beta)$ provided

$$\frac{2}{\alpha} - 3\delta \ge \frac{2}{\beta}(1 + \delta) \qquad (2.6.1)$$

since $(z_1 + \lambda, z')$ is in $\Gamma(e_1, \beta)$ if

$$\frac{2}{\beta}|1 - z_1 - \lambda| < 1 - |z_1 + \lambda|^2 - |z'|^2$$

Inequality (2.6.1) holds provided δ is less than the positive number

$$\frac{2/\alpha - 2/\beta}{3 + 2/\beta}$$

the minimum of this value and $1/2$ is the δ we seek. ∎

PROOF (of Theorem 2.81) We leave it as an exercise to check that it is sufficient to prove the theorem in the case $\zeta = \eta = e_1$, which will simplify notation.

For the implication (1) $\Rightarrow$ (2), assume $d \equiv d(e_1) < \infty$; our goal is to show $(1 - \varphi_1(z))/(1 - z_1)$ has restricted limit at e_1. By Julia's Lemma for the ball (Lemma 2.77)

$$\frac{|1 - \varphi_1(z)|^2}{1 - |\varphi(z)|^2} \leq d\frac{|1 - z_1|^2}{1 - |z|^2}$$

For $0 < r < 1$ we have

$$\begin{aligned}
\frac{1 - |\varphi(re_1)|}{1 - r}\frac{1 + r}{1 + |\varphi(re_1)|} &\leq \frac{1 - |\varphi_1(re_1)|}{1 - r}\frac{1 + r}{1 + |\varphi_1(re_1)|} \\
&\leq \frac{|1 - \varphi_1(re_1)|^2}{1 - |\varphi_1(re_1)|^2}\frac{1 - r^2}{(1 - r)^2} \\
&\leq \frac{|1 - \varphi_1(re_1)|^2}{1 - |\varphi(re_1)|^2}\frac{1 - r^2}{(1 - r)^2} \leq d
\end{aligned}$$

from which it follows (using (1)) that

$$\lim_{r \to 1} \frac{1 - |\varphi(re_1)|}{1 - r} = d$$

and

$$\lim_{r \to 1} \frac{1 - |\varphi_1(re_1)|}{1 - r} = d$$

and

$$\lim_{r \to 1} \frac{|1 - \varphi_1(re_1)|}{1 - r} = d$$

so that

$$\lim_{r \to 1} \frac{1 - \varphi_1(re_1)}{1 - r} = d$$

exactly as in the proof of Theorem 2.44. According to Theorem 2.79, restricted convergence will follow from this radial convergence if we can show that $(1 - \varphi_1(z))/(1 - z_1)$ is bounded in every $\Gamma(e_1, \alpha)$. To this end fix $\Gamma(e_1, \alpha)$ and z in $\Gamma(e_1, \alpha)$. Set $|1 - z_1| = \delta$ so that

$$\frac{|1 - z_1|^2}{1 - |z|^2} = \frac{|1 - z_1|}{1 - |z|^2}\delta \leq \alpha\delta/2$$

and by Julia's Lemma

$$\frac{|1-\varphi_1(z)|^2}{1-|\varphi(z)|^2} \leq \alpha\delta d/2$$

or

$$\frac{|1-\varphi_1(z)|}{|1-z_1|} \leq \frac{\alpha\delta d(1-|\varphi_1(z)|^2)}{2\delta|1-\varphi_1(z)|} \leq \alpha d$$

as desired.

Next we turn to (2) $\Rightarrow$ (3). The existence of a finite angular derivative at e_1 clearly implies that φ has restricted limit of norm 1 at e_1; our normalization is that the limit is e_1. We also need to show that $D_1\varphi_1$ has finite restricted limit at e_1. Fix $1 < \alpha < \beta$ and let $\delta = \delta(\alpha, \beta)$ be the δ of Lemma 2.82 so that (z_1, z') in $\Gamma(e_1, \alpha)$ implies $(z_1 + \lambda, z')$ is in $\Gamma(e_1, \beta)$ for all λ with $|\lambda| \leq \delta|1-z_1|$. As in the proof of Theorem 2.44, the Cauchy integral formula applied to $D_1\varphi_1 - 1$ yields, for $r = r(z) = \delta(\alpha, \beta)|1-z_1|$,

$$\begin{aligned} D_1\varphi_1(z) = D_1(\varphi_1 - 1)(z) &= \int_0^{2\pi} \frac{(\varphi_1 - 1)(z_1 + re^{i\theta}, z')}{re^{i\theta}} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \left(\frac{\varphi_1(z_1 + re^{i\theta}, z') - 1}{z_1 + re^{i\theta} - 1} \right) \left(\frac{z_1 + re^{i\theta} - 1}{re^{i\theta}} \right) \frac{d\theta}{2\pi} \qquad (2.6.2) \end{aligned}$$

If (2) holds, then $d(e_1) < \infty$ and the above argument shows that $(1-\varphi_1(z))/(1-z_1)$ is bounded in every Koranyi approach region. Since $(z_1 + re^{i\theta}, z')$ is in $\Gamma(e_1, \beta)$, the first factor in the integrand of Equation (2.6.2) is bounded as θ varies over $(0, 2\pi)$ and (z_1, z') varies in $\Gamma(e_1, \alpha)$. The second factor in the integrand is bounded by $1+\delta^{-1}$. It is easy to see that $(t+re^{i\theta}, 0)$ approaches e_1 nontangentially as t approaches 1, where $r = r(te_1) = (\delta)(1-t)$. Thus, if $z = te_1$, the first factor in the integrand of Equation (2.6.2) converges to the restricted limit of $(1-\varphi_1)/(1-z_1)$ as t approaches 1, while the second factor converges to $1+\delta^{-1}e^{-i\theta}$. The dominated convergence theorem implies $D_1\varphi_1$ has radial limit equal to the restricted limit of $(1-\varphi_1)/(1-z_1)$ at e_1; since $D_1\varphi_1$ is also bounded in every Koranyi region, we may use Theorem 2.79 to conclude this is in fact the restricted limit of $D_1\varphi_1$ at e_1.

We finish by showing that (3) implies (1). Since $D_1\varphi_1(re_1)$ has a finite limit as $r \to 1$ there exists a finite number M so that

$$|1-\varphi_1(re_1)| = \left| \int_r^1 D_1\varphi_1(te_1)dt \right| \leq M(1-r)$$

Thus

$$\frac{1-|\varphi(re_1)|}{1-r} \leq \frac{|1-\varphi_1(re_1)|}{1-r} \leq M$$

and $d(\zeta) < \infty$. ∎

A version of the Denjoy–Wolff Theorem (Theorem 2.51) continues to hold in the ball B_N when $N > 1$. We state it here in the form that we will find most useful later on.

THEOREM 2.83 (Denjoy–Wolff Theorem in B_N)
If φ is an analytic map of the ball into itself with no fixed points in B_N, then there is a point ζ of norm 1 so that the iterates φ_n of φ converge to ζ uniformly on compact subsets of B_N.

The point ζ will be referred to as the Denjoy–Wolff point of φ. We will prove this theorem in a sequence of steps. Curiously, the most difficult step will be a verification of the theorem in the special case that φ is an automorphism of B_N.

Since the analytic maps of B_N into itself form a normal family, every sequence of iterates of some $\varphi : B_N \to B_N$ will have a convergent subsequence. A simple observation about the subsequential limits of $\{\varphi_n\}$ is contained in the next lemma.

LEMMA 2.84
If $\psi : B_N \to \overline{B_N}$ is analytic, then either $\psi(B_N) \subset B_N$ or $\psi(z) \equiv \zeta$, for some ζ in ∂B_N and all z in B_N.

PROOF If ψ takes a point z_0 in B_N to a point ζ in ∂B_N, consider $G(z) \equiv (1 + \langle z, \zeta\rangle)/2$. Then G is in the ball algebra of functions analytic in B_N and continuous on $\overline{B_N}$, $G(\zeta) = 1$ and $|G(z)| < 1$ for $z \neq \zeta$ in $\overline{B_N}$. Apply the maximum modulus theorem (for analytic functions on B_N, see [Ru80, p. 5]) to $G \circ \psi$, noting that $G \circ \psi(z_0) = 1$ while $|G \circ \psi| \le 1$ on B_N, to conclude that $G \circ \psi$ is identically 1 and hence $\psi(z) \equiv \zeta$. ∎

We get our candidate for the limit of the iterates of an arbitrary fixed point free φ by a method reminiscent of the proof of Theorem 2.48: Choose positive numbers r_n increasing to 1 and find, using the Brouwer Fixed Point Theorem, fixed points a_n of the maps $r_n\varphi : r_n\overline{B_N} \to r_n\overline{B_N}$. Passing to a subsequence if necessary we may assume the a_n's have a limit ζ in $\overline{B_N}$. Just as in the proof of Theorem 2.48 in the disk, since φ has no fixed points in the open ball, ζ must be in the boundary of B_N. Without loss of generality we may suppose it to be e_1. Setting $f_n = r_n\varphi$ and considering f_n as a map of B_N into B_N, the Schwarz-Pick Theorem for the ball (Exercise 2.5.2) gives

$$\frac{|1 - \langle f_n(a_n), f_n(w)\rangle|^2}{(1 - |f_n(a_n)|^2)(1 - |f_n(w)|^2)} \le \frac{|1 - \langle a_n, w\rangle|^2}{(1 - |a_n|^2)(1 - |w|^2)}$$

for all w in B_N. Since $f_n(a_n) = a_n$ this gives

$$\frac{|1 - \langle a_n, f_n(w)\rangle|^2}{1 - |f_n(w)|^2} \le \frac{|1 - \langle a_n, w\rangle|^2}{1 - |w|^2}$$

Letting $n \to \infty$ and recalling that $a_n \to e_1$ and $f_n(w) \to \varphi(w)$ we obtain

$$\frac{|1-\varphi_1(w)|^2}{1-|\varphi(w)|^2} \le \frac{|1-w_1|^2}{1-|w|^2}$$

We summarize this in geometric language:

PROPOSITION 2.85

If φ is a analytic map of the unit ball into itself with no fixed points in B_N, there is a unique ζ in ∂B_N so that each ellipsoid $E(k,\zeta)$ is mapped into itself by φ and every iterate of φ.

PROOF All that remains to be shown is the uniqueness of ζ and the argument follows the proof of the one variable case, Theorem 2.48. Suppose ζ_1 and ζ_2 both have the stated property. Choose k_1 and k_2 so that $\overline{E(k_1,\zeta_1)}$ and $\overline{E(k_2,\zeta_2)}$ are tangent to each other at w in B_N. Then $\varphi(w)$ is in $\overline{E(k_1,\zeta_1)} \cap \overline{E(k_2,\zeta_2)} = \{w\}$, contradicting the hypothesis that φ is fixed point free in B_N. ∎

We denote the mapping $\psi(z) \equiv \zeta$ of Proposition 2.85 by $\zeta(\varphi)$.

COROLLARY 2.86

If φ has no interior fixed points, then the only constant map which can appear as a subsequential limit of iterates of φ is $\zeta(\varphi)$.

PROOF The proof is identical to the proof of Lemma 2.49. ∎

For non-constant maps which are subsequential limits of $\{\varphi_n\}$ we have the following analogue of Lemma 2.50 in the disk. An idempotent is a map ψ with $\psi \circ \psi = \psi$. Idempotents are also called retractions.

PROPOSITION 2.87

If $\{\varphi_n\}$ has a non-constant limit, then there is a non-constant idempotent among the subsequential limits of $\{\varphi_n\}$.

PROOF Suppose $\varphi_{n_i} \to \tau$ where τ is not a constant map. Necessarily $\tau(B_N) \subset B_N$, by Lemma 2.84. Set $m_i = n_{i+1} - n_i$ and choose a convergent subsequence of $\{\varphi_{m_i}\}$, say $\varphi_{m_{i_k}} \to \psi$. On the one hand $\varphi_{m_{i_k}} \circ \varphi_{n_{i_k}} \to \psi \circ \tau$, while also $\varphi_{m_i} \circ \varphi_{n_i} = \varphi_{n_i+1} \to \tau$. So $\psi \circ \tau = \tau$, or ψ is the identity on the range of τ, which consists of more than one point. So, using Theorem 2.76, the fixed point set of ψ is an affine subset A of B_N with dimension at least 1. Clearly the range of ψ contains A; if this containment is proper the same argument, now applied to ψ instead of τ, produces another subsequential limit of $\{\varphi_n\}$ which is the identity on an affine subset A' of dimension strictly greater than the dimension of A. Since this process can only be repeated a finite number of times there must be a subsequential limit whose range in B_N is its fixed point set. This is an idempotent, and since the dimension of its fixed point set is at least one, it is non-constant. ∎

Next we verify Theorem 2.83 when φ is an automorphism. Automorphisms of B_N with no interior fixed points will fix either exactly one point or exactly two points in ∂B_N (see Exercise 2.6.3 and Exercise 2.6.5). Those with two boundary fixed points are easy to handle.

PROPOSITION 2.88

Let φ be an automorphism of B_N which fixes precisely two points of ∂B_N. Then φ_n converges to one of these fixed points.

PROOF Suppose φ fixes ζ_1 and ζ_2 in the boundary. Consider the complex line L through ζ_1 and ζ_2. Since automorphisms take affine sets to affine sets (Theorem 2.74), φ maps $L \cap B_N$ onto $L \cap B_N$. The Denjoy–Wolff Theorem in one variable implies that the iterates of φ restricted to $L \cap B_N$ converge to one of the fixed points, say ζ_1. By Lemma 2.84 every convergent subsequence of $\{\varphi_n\}$ must converge to ζ_1. This implies $\varphi_n \to \zeta_1$ since the analytic self-maps of B_N form a normal family. ∎

Clearly the limit identified in this last proposition must be the map $\zeta(\varphi)$ identified in Proposition 2.85. An alternate proof of Proposition 2.88 can be given using the result of Exercise 2.6.5.

The one boundary fixed point automorphisms require a more complicated analysis. It is conceptually simpler to discuss these automorphisms in an unbounded realization of B_N, the Siegel upper half space

$$\Omega = \{(w_1, w') \in C^N : \operatorname{Im} w_1 > |w'|^2 = |w_2|^2 + \cdots |w_N|^2\}$$

The Cayley transform $\Phi(z) = i(e_1 + z)/(1 - z_1)$, defined for $z_1 \neq 1$, is a biholomorphic map of B_N onto Ω that extends (with $\Phi(e_1) = \infty$) to a homeomorphism of $\overline{B_N}$ onto $\Omega \cup \partial\Omega \cup \{\infty\}$, the one point compactification of the closure of Ω. The automorphisms of B_N that fix only e_1 correspond, under conjugation by Φ, to automorphisms of Ω fixing only $\{\infty\}$. An example of a class of such automorphisms are the Heisenberg translations:

DEFINITION 2.89 *For each (b_1, b') in $\partial\Omega$ set*

$$h_b(w_1, w') = (w_1 + b_1 + 2i\langle w', b'\rangle, w' + b')$$

These "translations" are called the Heisenberg translations of Ω; they form a subgroup of Aut(Ω) and for each $b \neq 0$, h_b fixes ∞ only. The corresponding map in the disk, $\Phi^{-1} \circ h_b \circ \Phi$ will be called a Heisenberg translation of B_N.

Unlike the situation in one variable, these "translations" are *not* the complete set of all automorphisms fixing only e_1. For example, if λ_j is a complex number of modulus 1 for $j = 2, 3, \ldots, N$ and $b \neq 0$ is real, the map $(w_1, w') \to (w_1 + b, \lambda_2 w_2, \ldots, \lambda_N w_N)$ is an automorphism of Ω fixing ∞ only but is not a Heisenberg translation. Notice that this mapping fixes *as a set* the image under Φ of the complex line through 0 and e_1; that is the set $\{(w_1, w') \in \Omega : w' = 0\}$. That this is no accident is the content of the next theorem. We say φ ***fixes*** A ***as a set*** if $\varphi(A) \subset A$.

THEOREM 2.90

An automorphism of B_N which fixes e_1 only is either a Heisenberg translation of B_N or fixes as a set some non-empty proper affine subset of B_N.

Before we can prove this theorem we need to establish some other results about the automorphisms of B_N fixing e_1, or equivalently, the automorphisms of Ω fixing ∞. We will find it convenient to transfer back and forth between the ball and the upper half space by means of the Cayley transform Φ and temporarily adopt the convention of using lower case letters for automorphisms of B_N and the corresponding upper case letters for the associated automorphisms in Ω obtained by composition on the right and left by Φ and Φ^{-1} respectively.

LEMMA 2.91

Given any (a_1, a') in Ω with $\operatorname{Im} a_1 - |a'|^2 = 1$ there exists a Heisenberg translation h_b so that $h_b(a_1, a') = (i, 0')$.

PROOF Write $a_1 = c + i(1 + |a'|^2)$ with c real. Set $b = (-c + i|a'|^2, -a')$ so that b is in $\partial\Omega$. The Heisenberg translation h_b has the desired property. ∎

LEMMA 2.92

Suppose g in Aut(Ω) fixes ∞ only. Then for every $w = (w_1, w')$ in Ω

$$\operatorname{Im} g_1(w) - |g'(w)|^2 = \operatorname{Im} w_1 - |w'|^2$$

where $g = (g_1, g_2, \ldots, g_N) = (g_1, g')$.

PROOF Set $g(i, 0') = (a_1, a')$ and let $t = \operatorname{Im} a_1 - |a'|^2$, a positive quantity since (a_1, a') is in Ω. For $s > 0$ define the ***non-isotropic dilation*** in Aut(Ω) by

$$\delta_s(w_1, w') = (s^2 w_1, s w') \tag{2.6.3}$$

If $s \neq 1$ the fixed point set of δ_s is $\{0, \infty\}$. With $s = 1/\sqrt{t}$ we may use Lemma 2.91 to find a Heisenberg translation h_b so that $h_b \circ \delta_s \circ g$ fixes $(i, 0')$ and ∞. The corresponding automorphism of B_N fixes e_1 and 0, hence is unitary. Moreover, it must fix as a set the orthogonal complement of the complex line through 0 and e_1.

This implies

$$h_b \circ \delta_s \circ g(w_1, w') = (w_1, Uw')$$

for some unitary U on C^{N-1}. Equivalently

$$g(w_1, w') = \delta_{\sqrt{t}} \circ h_b^{-1}(w_1, Uw') = (t(w_1 - \overline{b_1} - 2i\langle Uw', b'\rangle), \sqrt{t}(Uw' - b'))$$

We claim that the hypothesis on g shows that t must be 1. If not, we may solve $\sqrt{t}(Uw' - b') = w'$ since $(U - t^{-1/2}I)$ is non-singular. Denote its solution by v'. Set $v_1 = \alpha + i|v'|^2$ for real α to be specified. We have (v_1, v') is in $\partial\Omega$ and therefore $g(v_1, v')$ is in $\partial\Omega$ since g is an automorphism. By our choice of v'

$$g(v_1, v') = (t(\alpha + i|v|^2 - \overline{b_1} - 2i\langle Uv', b'\rangle), v')$$

Since this is a point of the boundary of Ω we must have

$$\operatorname{Im}(t(\alpha + i|v'|^2 - \overline{b_1} - 2i\langle Uv', b'\rangle)) = |v'|^2 = \operatorname{Im}(\alpha + i|v'|^2)$$

Thus (v_1, v') will be a fixed point of g if α real is chosen to satisfy

$$\operatorname{Re}(t(\alpha + i|v'|^2 - \overline{b_1} - 2i\langle Uv', b'\rangle)) = \alpha$$

or

$$t\alpha + t\operatorname{Re}(-b_1 - 2i\langle Uv', b'\rangle) = \alpha$$

If $t \neq 1$ this may be solved for real α and then the fixed point set of g will contain more than one point, a contradiction. Thus t must be 1 and therefore

$$g(w_1, w') = (w_1 - \overline{b_1} - 2i\langle Uw', b'\rangle, Uw' - b')$$

from which it follows by direct calculation that

$$\operatorname{Im} g_1(w) - |g'(w)|^2 = \operatorname{Im} w_1 - |w'|^2$$

for all (w_1, w') in Ω. ∎

Note that the proof of Lemma 2.92 shows that any automorphism g of Ω fixing ∞ only can be written as $g(w_1, w') = h_c(w_1, Uw')$ for some unitary U on C^{N-1} and Heisenberg translation h_c. If $\Phi(z) = w$, where Φ is the Cayley transform, then $\operatorname{Im} w_1 - |w'|^2 = (1 - |z|^2)/(|1 - z_1|^2)$, so the geometric meaning of Lemma 2.92 is as follows: if an automorphism g of Ω fixes ∞ only, the corresponding automorphism $G = \Phi^{-1} \circ g \circ \Phi$ in $\operatorname{Aut}(B_N)$ maps the boundary of each ellipsoid $E(k, e_1)$ into (and therefore onto) itself. This should be compared with the result of Exercise 2.3.7 for parabolic automorphisms in the disk.

PROOF (of Theorem 2.90) Let G be the given automorphism of B_N and let g be the corresponding automorphism of Ω. Set $g(i, 0') = (a_1, a')$. By Lemma 2.92 we know $\operatorname{Im} a_1 - |a'|^2 = 1$ so we may apply Lemma 2.91 to find a Heisenberg translation h_b so that $h_b \circ g$ fixes both ∞ and $(i, 0')$. As in the proof of Lemma 2.92 the corresponding ball automorphism $F = H_b \circ G$ is unitary and has the form $F(z_1, z') = (z_1, Uz')$ for some unitary U. Since

$$F \circ \Phi^{-1}(w_1, w') = \left(\frac{w_1 - i}{w_1 + i}, \frac{2}{w_1 + i}Uw'\right) = \Phi^{-1} \circ F(w_1, w')$$

on all points of C^N with first coordinate not equal to $-i$, we have $f(w_1, w') = (w_1, Uw')$ where $f = \Phi \circ F \circ \Phi^{-1}$ and (w_1, w') is in Ω.

At this point we distinguish two cases. If every eigenvalue of U is 1 then U, and hence F, is the identity and our original automorphism G is the Heisenberg translation H_b^{-1}.

On the other hand, if U has an eigenvalue $e^{i\theta} \neq 1$ find $0 \neq \Lambda = (\lambda_2, \lambda_3, \ldots, \lambda_N)$ so that $\Lambda(U) = e^{i\theta}\Lambda$ where (U) is the matrix of the operator U with respect to the standard basis on C^{N-1}. Recall that $g = \Phi \circ G \circ \Phi^{-1} = \Phi \circ H_b^{-1} \circ F \circ \Phi^{-1} = h_b^{-1} \circ f$ where $h_b^{-1} = h_{\widetilde{b}}, \widetilde{b} = (-\overline{b_1}, -b')$. Let V be the column vector $(-b_2, \ldots, -b_N)^t$ so that $\Lambda V = \sum_{j=2}^{N} -b_j\lambda_j$. Consider the set

$$\mathcal{A} = \left\{(w_1, \ldots, w_N) \in \Omega : \sum_{j=2}^{N} \lambda_j w_j = \frac{\Lambda V}{1 - e^{i\theta}}\right\}$$

which is a non-empty proper affine subset of Ω. We claim that g fixes $\mathcal{A}$ as a set. To see this, let $(w_1, w_2, \ldots, w_N)$ be in $\mathcal{A}$. Now $g(w_1, w') = h_b^{-1} \circ f(w_1, w') = h_b^{-1}(w_1, Uw')$. Writing $W' = (w_2, \ldots, w_N)^t$ we see that the last $N - 1$ coordinates of $h_b^{-1}(w_1, Uw')$ are $((U)W' + V)^t$. Since

$$\Lambda((U)W' + V) = e^{i\theta}\Lambda W' + \Lambda V = \frac{e^{i\theta}\Lambda V}{(1 - e^{i\theta})} + \Lambda V = \frac{\Lambda V}{1 - e^{i\theta}}$$

and $g(w_1, w')$ is in $\mathcal{A}$ as desired. Since Φ preserves affine sets (Exercise 2.6.4) the corresponding map G on B_N fixes as a set some non-empty, proper, affine subset of B_N. ▮

We can now prove Theorem 2.83 in the case that φ is an automorphism fixing one point of ∂B_N. We may normalize so that the fixed point is e_1.

THEOREM 2.93
Suppose φ is an automorphism of B_N fixing e_1 only. Then $\varphi_n \to e_1$, uniformly on compact subsets of B_N.

PROOF If φ is a Heisenberg translation $\Phi^{-1} \circ h_b \circ \Phi$ then the result is immediate from the easy observation that $(h_b)_n \to \infty$. The remainder of the proof is an inductive argument.

By Theorem 2.51 the result holds for $N = 1$. We assume it holds for $k < N$ and that φ is not a Heisenberg translation. Then by Theorem 2.90 φ fixes as a set some affine subset $\mathcal{A}$ of B_N with dimension k, $1 \leq k < N$. The restriction $\widetilde{\varphi}$ of φ to $\mathcal{A} \cong B_k$ may be considered as an automorphism of B_k fixing e_1 only (it is easy to check from the description of $\mathcal{A}$ in the proof of Theorem 2.90 that e_1 is in $\partial\mathcal{A}$). By induction, $\widetilde{\varphi}_n \to e_1$ and then by Lemma 2.84, $\varphi_n \to e_1$. ∎

A final observation before we prove Theorem 2.83:

LEMMA 2.94
If some subsequence of $\{\varphi_n\}$ converges to the identity map, then φ must be an automorphism.

PROOF Suppose $\varphi_{n_i} \to I$. By passing to a subsequence if necessary we may assume $\varphi_{n_i-1} \to \tau$. Then $\varphi_{n_i-1} \circ \varphi$ converges to $\tau \circ \varphi$ and also to I, so we must have $\tau \circ \varphi = I$. In particular, $\tau(B_N) \subset B_N$ and therefore $\varphi_{n_i} = \varphi \circ \varphi_{n_i-1} \to \varphi \circ \tau$ which implies that $\varphi \circ \tau = I$ as well. ∎

PROOF (of Theorem 2.83.) We have already established the result for an automorphism with no fixed point in B_N. Suppose now that φ is an arbitrary fixed point free self map of B_N. If every subsequential limit of $\{\varphi_n\}$ is constant, then $\varphi_n \to \zeta(\varphi)$ by Corollary 2.86, and we are done.

If there is a non-constant subsequential limit, then there is a non-constant idempotent ψ so that for some $\{n_i\}$, $\varphi_{n_i} \to \psi$. Let $\mathcal{A}$ be the fixed point set of ψ, an affine set (by Theorem 2.76) of dimension at least 1.

We claim that φ maps $\mathcal{A}$ into $\mathcal{A}$. To see this choose z_0 in $\mathcal{A}$. Since $\varphi_{n_i}(z_0) \to \psi(z_0) = z_0$ we have $\varphi(\varphi_{n_i}(z_0)) \to \varphi(z_0)$. But

$$\varphi(\varphi_{n_i}(z_0) = \varphi_{n_i}(\varphi(z_0)) \to \psi(\varphi(z_0))$$

so $\psi(\varphi(z_0)) = \varphi(z_0)$ and $\varphi(z_0)$ is in $\mathcal{A}$. Moreover, since φ_{n_i} restricted to $\mathcal{A}$ converges to the identity on $\mathcal{A}$, Lemma 2.94 (with B replaces by $\mathcal{A}$) implies that φ restricted to $\mathcal{A}$ is an automorphism of $\mathcal{A}$, and is clearly a fixed point free automorphism. Thus by the result already established for automorphisms, the sequence of restrictions of φ_{n_i} to $\mathcal{A}$ converges to a constant, contradicting $\varphi_{n_i} \to \psi$. We conclude that all subsequential limits of $\{\varphi_n\}$ must be constant and we are done. ∎

Exercises

2.6.1 Verify that the normalizations $\eta = \zeta = e_1$ cause no loss of generality in the proof of Theorem 2.81

2.6.2 Suppose that φ is an analytic map of B_N into B_N with finite angular derivative at e_1 and $\varphi(e_1) = e_1$. Show that for $j = 2, 3, \ldots, N$

(a)

$$\lim_{r\to 1} \frac{\varphi_j(re_1)}{(1-r)^{1/2}} = 0$$

(b)

$$\frac{\varphi_j(z)}{(1-z_1)^{1/2}}$$

is bounded in every region $\Gamma(e_1, \alpha)$.

Hence

$$\frac{\varphi_j(z)}{(1-z_1)^{1/2}}$$

has restricted limit 0 at e_1.

2.6.3 If φ is an automorphism of B_N fixing at least three distinct points of ∂B_N then φ has a fixed point in B_N.

Hints: Suppose $\zeta_1, \zeta_2, \zeta_3$ are fixed by φ. Use the fundamental identity for automorphisms, Equation (2.5.3), to show $\langle \zeta_1, a\rangle = \langle \zeta_2, a\rangle$ where $a = \varphi^{-1}(0)$. Then show that φ fixes $(\zeta_1 + \zeta_2)/2$.

2.6.4 The Cayley transform Φ sends affine sets in B_N to affine sets in Ω.

2.6.5 Recall that for $0 < s \neq 1$, the non-isotropic dilation $\delta_s(w_1, w') = (s^2 w_1, sw')$ is an automorphism of Ω fixing 0 and ∞ only.

(a) Show that an arbitrary g in $\text{Aut}(\Omega)$ fixing 0 and ∞ only has the form

$$g(w_1, w') = (s^2 w_1, sUw')$$

where s is positive and not equal to 1 and U is unitary on C^{N-1}.

(b) Show that if φ is an automorphism of the ball fixing ζ_1, ζ_2 in the boundary of the ball and no other points then φ is conjugate to a map g as in (a); i.e.

$$\varphi = \psi^{-1} \circ g \circ \psi$$

for some biholomorphic map ψ of B_N onto Ω.

2.6.6 This exercise explores the nature of the subsequential limits of $\{\varphi_n\}$ when φ has fixed point(s) in B_N.

(a) Show that if there is a constant function $g(z) \equiv z_0$, z_0 in B_N among the subsequential limits of $\{\varphi_n\}$, then in fact $\varphi_n \to g$.

(b) Show that if case (a) does not occur, then φ must act as an automorphism on some affine subset of B_N of dimension at least 1.

(c) If φ is not an automorphism, then every subsequential limit of $\{\varphi_n\}$ is degenerate in the sense that its range is contained in an affine subset of B_N of dimension less than N.

2.6.7 Give an example to show that the fixed point set of a subsequential limit of $\{\varphi_n\}$ may be strictly larger than the fixed point set of φ.

Notes

Parts of the Julia–Carathéodory Theorem in the ball are due to M. Hervé [He63]. An expanded version of the theorem was given by W. Rudin and the treatment given here follows [Ru80, §8.5] very closely. There are other extensions of Theorem 2.81 that give