Chapter 8

Consequences of the Schwarz Lemma

8.1. The Schwarz Lemma in B

8.1.1. The familiar classical Schwarz lemma deals with functions defined in the open unit disc $U \subset \mathbb{C}$, and asserts the following:

(a) *If $f: U \to U$ is holomorphic, then $|f'(0)| < 1$, except when $f(\lambda) = c\lambda$ for some $c \in \mathbb{C}$ with $|c| = 1$.*
(b) *If also $f(0) = 0$, then $|f(\lambda)| < |\lambda|$ for every $\lambda \in U \backslash \{0\}$, except when $f(\lambda) = c\lambda$, as in (a).*

As we shall see, this implies a variety of analogous results in several variables. Our first example concerns holomorphic maps of one *balanced* region into another; a set $E \subset \mathbb{C}^n$ is said to be balanced if $\lambda z \in E$ whenever $z \in E$ and $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$. This terminology is customary in functional analysis. Balanced open sets in $\mathbb{C}^n$ are also known as *star-shaped circular regions*. Note that every balanced region is a neighborhood of the origin.

8.1.2. Theorem. *Suppose that*

(i) *Ω_1 and Ω_2 are balanced regions in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively,*
(ii) *Ω_2 is convex and bounded,*
(iii) *$F: \Omega_1 \to \Omega_2$ is holomorphic.*

Then

(a) *$F'(0)$ maps Ω_1 into Ω_2, and*
(b) *$F(r\Omega_1) \subset r\Omega_2$ $(0 < r \leq 1)$ if $F(0) = 0$.*

Recall that $F'(0)$ is a linear operator carrying $\mathbb{C}^n$ into $\mathbb{C}^m$; see §1.3.6.

Proof. The assumptions made on Ω_2 show that $\mathbb{C}^m$ may be regarded as a Banach space Y whose unit ball is Ω_2. The corresponding norm is

$$\|w\| = \inf\{c > 0 : c^{-1}w \in \Omega_2\}. \tag{1}$$

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Fix $z \in r\Omega_1$, where $0 < r \le 1$. Since Ω_1 is open, $z \in t\Omega_1$ for some $t < r$. Let L be a linear functional on Y, of norm 1. Then

$$(2) \qquad g(\lambda) = LF(\lambda t^{-1} z)$$

defines a holomorphic map g of U into U. By the chain rule,

$$(3) \qquad g'(0) = LF'(0)t^{-1}z.$$

Since $|g'(0)| \le 1$, by 8.1.1(a), and since this holds for every L of norm 1, the Hahn–Banach theorem implies that

$$(4) \qquad \|F'(0)t^{-1}z\| \le 1.$$

Thus $F'(0)z \in t\overline{\Omega}_2 \subset r\Omega_2$. This proves (a).

If also $F(0) = 0$ and g is given by (2), then $g(0) = 0$, hence $|g(\lambda)| \le |\lambda|$, and (b) follows by the same argument that gave (a).

Remark. If Ω_1 is also convex and bounded, then $\mathbb{C}^n$ is a Banach space X with unit ball Ω_1, and (a) asserts that $F'(0): X \to Y$ is a linear operator of norm at most 1. By analogy with the classical Schwarz lemma, one may ask whether F must then be linear whenever $\|F'(0)\| = 1$. This is not so when $n > 1$, even in the case $\Omega_1 = B_n$, $\Omega_2 = B_m$; we shall see this in §8.1.5. But the linearity of F does follow if $F'(0)$ is assumed to be an isometry:

8.1.3. Theorem. *If $F: B_n \to B_m$ is holomorphic and $F'(0)$ is an isometry of $\mathbb{C}^n$ into $\mathbb{C}^m$, then $F(z) = F'(0)z$ for all $z \in B_n$.*

Proof. Put $F'(0) = A$, $F(0) = a$, $G = \varphi_a \circ F$, where $\varphi_a \in \operatorname{Aut}(B_m)$ is as in §2.2.1. We claim that $a = 0$.

If $z \in B_n$ and $w = Az$, the chain rule gives

$$(1) \qquad G'(0)z = \varphi_a'(a)w.$$

By hypothesis, $|w| = |Az| = |z|$. By Theorem 8.1.2, $|G'(0)z| \le |z|$. Hence Theorem 2.2.2 shows that

$$(2) \qquad |s^{-2}Pw + s^{-1}Qw| \le |w| = |Pw + Qw|,$$

where $s = (1 - |a|^2)^{1/2}$ and $Pw \perp Qw$. This can only happen when $s = 1$, i.e., $a = 0$.

Thus $F(0) = 0$, hence $|F(z)| \le |z|$, by Theorem 8.1.2.

Pick $\zeta \in \mathbb{C}^n$, $|\zeta| = 1$, and define

$$(3) \qquad h(\lambda) = \langle F(\lambda\zeta), A\zeta\rangle \qquad (\lambda \in U).$$

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Then h is a holomorphic map of U into U with $h'(0) = |A\zeta|^2 = 1$, so that $h(\lambda) = \lambda$, or

$$(4) \qquad \langle \lambda^{-1} F(\lambda\zeta), A\zeta \rangle = 1 \qquad (0 < |\lambda| < 1).$$

Since $|F(\lambda\zeta)| \le |\lambda|$, the left side of (4) is the inner product of two vectors in B_m. This can only be 1 when the two vectors are equal (and have norm 1). Hence $F(\lambda\zeta) = \lambda A\zeta$, which gives the desired conclusion, since A is linear.

As in the case in one variable, part (b) of the Schwarz lemma can be generalized by applying automorphisms to both the domain and the range of F:

8.1.4. Theorem. *If $F: B_n \to B_m$ is holomorphic, $a \in B_n$, and $F(a) = b$, then*

$$(1) \qquad |\varphi_b(F(z))| \le |\varphi_a(z)| \qquad (z \in B_n).$$

Equivalently,

$$(2) \qquad \frac{|1 - \langle F(z), F(a)\rangle|^2}{(1 - |F(z)|^2)(1 - |F(a)|^2)} \le \frac{|1 - \langle z, a\rangle|^2}{(1 - |z|^2)(1 - |a|^2)}.$$

It is of course understood that $\varphi_a \in \mathrm{Aut}(B_n)$ and $\varphi_b \in \mathrm{Aut}(B_m)$; see §2.2.1. Assertion (1) can be stated in geometric terms: F maps each ellipsoid $E(a, \varepsilon)$ (see §2.2.7) into the ellipsoid $E(F(a), \varepsilon)$.

Proof. Since $\varphi_b \circ F \circ \varphi_a$ maps B_n into B_m and takes 0 to 0, Theorem 8.1.2 shows that

$$|\varphi_b(F(\varphi_a(z)))| \le |z|,$$

which gives (1) if z is replaced by $\varphi_a(z)$. If we square (1), subtract from 1, and apply the identity 2.2.2(iv), we obtain (2).

Note: If $m = n$ and $F \in \mathrm{Aut}(B_n)$, then equality holds in (2). To see this, apply (2) to F^{-1} as well as to F.

8.1.5. Examples. (i) Suppose $f: B_n \to U$ is holomorphic. Then $f'(0)$ is the linear functional that takes $z \in B_n$ to

$$(1) \qquad \sum_{k=1}^{n} (D_k f)(0) z_k$$

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which lies in U, by Theorem 8.1.2 with $m = 1$. It follows that

$$\sum_{k=1}^{n} |(D_k f)(0)|^2 \le 1. \tag{2}$$

(ii) Suppose $F\colon U \to B_m$ is holomorphic, $F = (f_1, \ldots, f_m)$. Then $F'(0)$ is the linear map that takes $\lambda \in U$ to the vector

$$(f_1'(0)\lambda, \ldots, f_m'(0)\lambda) \tag{3}$$

in B_m, by Theorem 8.1.2 with $n = 1$. Hence

$$\sum_{i=1}^{m} |f_i'(0)|^2 \le 1. \tag{4}$$

(iii) As regards the remark that precedes Theorem 8.1.3, we shall now see that the extremal functions related to the Schwarz lemma need not be unique, even in the simplest case $\Omega_1 = B_2, \Omega_2 = U$.

The power series

$$1 - \sqrt{1-t} = \sum_{k=1}^{\infty} c_k t^k \qquad (|t| < 1) \tag{5}$$

has $c_k > 0$ for all k. Let the functions g_k be arbitrary members of $H^\infty(B_2)$, subject only to the inequality $\|g_k\|_\infty \le c_k$, and define

$$f(z, w) = z + w^2 g_1(z, w) + w^4 g_2(z, w) + w^6 g_3(z, w) + \cdots. \tag{6}$$

If $|z|^2 + |w|^2 < 1$, it follows that

$$|f(z, w)| \le |z| + 1 - \sqrt{1 - |w|^2} < 1. \tag{7}$$

Every f given by (6) *is thus a holomorphic map of B_2 into U.*

If $h \in H^\infty(B_2)$ and $\|h\|_\infty \le 1$, Theorem 8.1.2(a) implies that $\|h'(0)\| \le 1$. Equality holds for *every* f of the form (6), since $f'(0)e_1 = 1$ and $f'(0)e_2 = 0$.

If $h \in H^\infty(B_2)$, $\|h\|_\infty \le 1$, and $h(0, 0) = 0$, Theorem 8.1.2(b) implies that $|h(z, 0)| \le |z|$; again, equality holds for *every* f given by (6)

Simple examples of (6) are

$$z + \tfrac{1}{2}w^2 \quad \text{or} \quad z + 1 - \sqrt{1 - w^2}. \tag{8}$$

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8.2. Fixed-Point Sets in B

In Section 2.4 we saw that the fixed-point sets of automorphisms of B are affine. Theorem 8.2.3 will show that this property is shared by all holomorphic maps of B into B. But we first consider a somewhat more general situation.

8.2.1. Definition. Let Ω be a balanced, convex, bounded region in $\mathbb{C}^n$. As pointed out in the proof of Theorem 8.1.2, $\mathbb{C}^n$ may then be regarded as a Banach space X whose unit ball is Ω. We say that Ω is *strictly convex* if to every linear functional L on X, with $\|L\| = 1$, corresponds just one $z \in \bar{\Omega}$ (the closure of Ω) such that $Lz = 1$.

Evidently, B is strictly convex.

8.2.2. Theorem (Rudin [13]). *Let Ω be a balanced, bounded, strictly convex region in $\mathbb{C}^n$. If $F: \Omega \to \Omega$ is holomorphic and $F(0) = 0$, then F and the linear operator $F'(0)$ fix the same points of Ω.*

Proof. Let X be the Banach space whose unit ball is Ω. We shall use $\|\cdot\|$ for the norm in X, for the corresponding norms of linear functionals on X, and for the norms of linear operators on X.

Put $F'(0) = A$. By Theorem 8.1.2,

$$(1) \qquad \|A\| \leq 1 \quad \text{and} \quad \|F(z)\| \leq \|z\| \qquad (z \in \Omega).$$

Fix $z \in \Omega$, $z = ru$, where $0 < r < 1$, $\|u\| = 1$. By the Hahn–Banach theorem, there is a linear functional L on X with

$$(2) \qquad \|L\| = 1, \qquad Lu = 1.$$

Put

$$(3) \qquad g(\lambda) = LF(\lambda u) \qquad (\lambda \in U).$$

Then $g: U \to U$ is holomorphic, $g(0) = 0$, and $g'(0) = LAu$.

If $F(z) = z$, then $g(r) = L(ru) = r$, hence $g(\lambda) = \lambda$ for all λ, hence $g'(0) = 1$. Thus $LAu = 1$. The strict convexity of Ω, combined with (2), implies now that $Au = u$. Hence $Az = z$.

Conversely, assume $Az = z$. Then $g'(0) = Lu = 1$, hence $g(r) = r$, or

$$(4) \qquad L(r^{-1}F(ru)) = 1.$$

By (1), $\|r^{-1}F(ru)\| \leq 1$. The strict convexity of Ω, combined with (2) and (4), implies now that $r^{-1}F(ru) = u$, hence $F(z) = z$.

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8.2.3. Theorem (Rudin [13]). *If $F: B \to B$ is holomorphic, then the fixed-point set E of F is affine.*

Proof. Suppose $a \in E$, and let E_a denote the fixed point set of $\varphi_a \circ F \circ \varphi_a$. Then $0 \in E_a$, and Theorem 8.2.2 implies that E_a is affine. Since $E = \varphi_a(E_a)$, it follows from Proposition 2.4.2 that E is affine.

8.2.4. Holomorphic Retracts. A map $F: B \to B$ is said to be a *retraction* of B if $F(F(z)) = F(z)$ for every $z \in B$. The range of F is then exactly its fixed-point set. A *holomorphic retract* of B is, by definition, the range of some holomorphic retraction of B.

Theorem 8.2.3 thus has the following corollary.

Corollary (Suffridge [1]). *The holomorphic retracts of B are exactly the affine subsets of B.*

Indeed, if $E \subset B$ is affine and $a \in E$, then $\varphi_a(E) = B \cap Y$, where Y is a subspace of $\mathbb{C}^n$. Let P be the orthogonal projection of $\mathbb{C}^n$ onto Y. Then $\varphi_a P \varphi_a$ is a holomorphic retraction of B onto E. The converse follows from Theorem 8.2.3.

Although the holomorphic retracts of B are thus very simple, there exist very complicated holomorphic retractions. For example, let $f \in H^\infty(B_2)$ be any one of the functions described by 8.1.5(6), and put $F(z, w) = (f(z, w), 0)$. Since $f(z, 0) = z$, F retracts B onto the set $\{(z, 0): |z| < 1\}$.

8.3. An Extension Problem

8.3.1. Statement of the Problem. Suppose $1 \leq n < m$, and let $\Phi: B_n \to B_m$ be holomorphic. Let us say that Φ has the *norm-preserving H^∞ extension property* (or property $(*)$, for brevity) if the following is true:

$(*)$ To every $f \in H^\infty(B_n)$ corresponds a $g \in H^\infty(B_m)$ such that
(a) $g \circ \Phi = f$, and
(b) $\|g\|_\infty = \|f\|_\infty$.

The problem is: *Which Φ have property* $(*)$?

The reason for calling this an extension problem is quite simple. Clearly, $(*)$ implies that Φ is one-to-one. Every $f \in H^\infty(B_n)$ corresponds therefore to a function $\tilde{f}$ on $\Phi(B_n)$ such that $\tilde{f} \circ \Phi = f$, and any g that satisfies (a) is an extension of $\tilde{f}$. The requirement (b) is of course extremely strong, and one should expect that only very special Φ's can satisfy it. Theorem 8.3.2 confirms this expectation.

If Φ has property $(*)$ one sees very easily that $\psi \circ \Phi$ has property $(*)$ for every $\psi \in \mathrm{Aut}(B_m)$. Theorem 8.3.2 implies therefore that every Φ with property $(*)$ has affine range.

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8.3.2. Theorem. *For a holomorphic map $\Phi: B_n \to B_m$ with $\Phi(0) = 0$, the following are equivalent:*

(i) *Φ has property $(*)$.*
(ii) *Φ is a linear isometry.*
(iii) *There is a multiplicative linear operator*

$$E: H^\infty(B_n) \to H^\infty(B_m)$$

such that $(Ef) \circ \Phi = f$ for every $f \in H^\infty(B_n)$.

Proof. Assume (i). Pick $\zeta \in \mathbb{C}^n$, $|\zeta| = 1$, and put $f(z) = \langle z, \zeta \rangle$. Then $f \in H^\infty(B_n)$, $\|f\|_\infty = 1$. Hence there is a $g \in H^\infty(B_m)$, with $\|g\|_\infty = 1$, such that $g(\Phi(z)) = \langle z, \zeta \rangle$. With $z = \lambda\zeta$, this becomes

$$(1) \qquad g(\Phi(\lambda\zeta)) = \lambda \qquad (\lambda \in U).$$

Since $\Phi(0) = 0$, differentiation of (1) gives

$$(2) \qquad g'(0)\Phi'(0)\zeta = 1.$$

By Theorem 8.1.2, $\Phi'(0)\zeta \in \bar{B}_m$ and $g'(0)$ is a linear functional on $\mathbb{C}^m$, of norm at most 1. Hence (2) implies that $\Phi'(0)\zeta$ is a unit vector in $\mathbb{C}^m$ for every unit vector ζ in $\mathbb{C}^n$. This says that $\Phi'(0)$ is an isometry, hence $\Phi(z) = \Phi'(0)z$, by Theorem 8.1.3. Thus (i) implies (ii).

If (ii) holds, then $\Phi(z) = Az$, where A is a linear isometry of $\mathbb{C}^n$ onto a subspace Y of $\mathbb{C}^m$. Let P be the orthogonal projection of $\mathbb{C}^m$ onto Y, and define

$$(3) \qquad (Ef)(w) = f(A^{-1}Pw) \qquad (w \in B_m)$$

for all $f \in H^\infty(B_n)$. (Note that A^{-1} is linear and well-defined on the range of P, and that P maps B_m onto $Y \cap B_m$.) It is clear that E is linear and multiplicative; also, $(Ef) \circ \Phi = f$, because $A^{-1}P\Phi(z) = z$. Thus (ii) implies (iii).

Finally, assume (iii). Since E is multiplicative, $Ef = E(f \cdot 1) = (Ef) \cdot (E1)$, hence $E1 = 1$. (Note that $Ef \equiv 0$ implies $f \equiv 0$.) If $fg = 1$, it follows that $(Ef) \cdot (Eg) = E(fg) = 1$. Thus Ef is invertible in $H^\infty(B_m)$ whenever f is invertible in $H^\infty(B_n)$. It follows that the sets $f(B_n)$ and $(Ef)(B_m)$ have the same closures in $\mathbb{C}$. In particular, $\|Ef\|_\infty = \|f\|_\infty$. Thus (iii) implies (i).

Note: The only $f \in H^\infty(B_n)$ that were needed to prove the implication (i) $\to$ (ii) were the linear functions $\langle z, \zeta \rangle$.

8.3.3. This problem has been treated by Stanton [1] with finite Riemann surfaces in place of B_n.

If one drops condition (b) in §8.3.1, one obtains what is usually called the H^∞ extension problem. Henkin [5] and Adachi [1] have studied this in

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strictly pseudoconvex domains. With polydiscs in place of balls, extension problems of this type occur in Chapter 7 of Rudin [1].

For extension theorems in the context of Bergman spaces and Hardy spaces, we refer to Amar [1], [3], and to Cumenge [1]. Extension theorems with C^∞-data were investigated by Elgueta [1].

8.4. The Lindelöf–Čirka Theorem

The classical theorem of Lindelöf which Čirka extended to several variables concerns the limit of a function $f \in H^\infty(U)$ at a single boundary point. It is thus not a theorem of Fatou type. Although Lindelöf's theorem is an elementary consequence of the maximum modulus principle, it does not seem to appear in the standard elementary texts. For this reason, a proof is included here.

8.4.1. Theorem (Lindelöf [1]). *Suppose $f \in H^\infty(U)$ and $\gamma: [0, 1) \to U$ is a continuous curve such that $\gamma(t) \to 1$ as $t \to 1$. If*

$$(1) \qquad \lim_{t\to 1} f(\gamma(t)) = L$$

exists, then f has nontangential limit L at the point 1.

Note that there is no restriction on the manner in which $\gamma(t)$ tends to 1, except that $\gamma(t)$ must lie in U for all $t < 1$.

Proof. Without loss of generality, assume $\|f\|_\infty = 1$ and $L = 0$. Let Σ be the strip defined by $|\mathrm{Re}\, z| < 1$. Let φ be a conformal map of U onto Σ, with $\varphi(0) = 0$, such that, setting $\Gamma = \varphi \circ \gamma$, we have $\mathrm{Im}\, \Gamma(t) \to +\infty$ as $t \to 1$. Replace f by $F = f \circ \varphi^{-1}$. Then $F \in H^\infty(\Sigma)$, $|F| \le 1$, $F(\Gamma(t)) \to 0$ as $t \to 1$. Given $\delta \in (0, 1)$, we have to prove that $F(x + iy) \to 0$ as $y \to +\infty$, uniformly in $|x| \le 1 - \delta$.

Fix ε, $0 < \varepsilon < 1$. Choose any $y > \mathrm{Im}\, \Gamma(0)$, so large that $|F(\Gamma(t))| < \varepsilon$ whenever $\mathrm{Im}\, \Gamma(t) \ge y$. We claim that then

$$(2) \qquad |F(x + iy)| \le \varepsilon^{\delta/4} \quad \text{if} \quad |x| \le 1 - \delta.$$

The theorem follows obviously from (2).

To prove (2), assume $y = 0$, without loss of generality (by a vertical translation of Σ), choose t_0 so that $\mathrm{Im}\, \Gamma(t_0) = 0$ but $\mathrm{Im}\, \Gamma(t) > 0$ if $t_0 < t < 1$, let $E = \{\Gamma(t): t_0 \le t < 1\}$, and let $\overline{E}$ be the reflection of E in the real axis. Then $E \cup \overline{E}$ intersects the real axis in a unique point x_0.

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Assume $x_0 < x \leq 1 - \delta$. Define

$$G_\eta(z) = \frac{F(z)\overline{F(\bar{z})}\varepsilon^{(1+z)/2}}{1 + \eta(1 + z)} \qquad (z \in \Sigma) \tag{3}$$

where η is a positive parameter. Then $G_\eta \in H^\infty(\Sigma)$. On E, $|F(z)| < \varepsilon$; on $\bar{E}$, $|\overline{F(\bar{z})}| < \varepsilon$; hence $|G_\eta| < \varepsilon$ on $E \cup \bar{E}$. On the right edge of Σ, the boundary values of $|G_\eta|$ are $< \varepsilon$. When $|\mathrm{Im}\, z|$ is sufficiently large, then $|G_\eta(z)| < \varepsilon$ because of the denominator in (3). These facts imply that $|G_\eta(x)| < \varepsilon$, by the maximum modulus principle, applied to G_η in the component of $\Sigma \backslash (E \cup \bar{E})$ that contains x. Letting $\eta \to 0$, we obtain therefore

$$|F(x)|^2 \leq \varepsilon \cdot \varepsilon^{-(1+x)/2} = \varepsilon^{(1-x)/2} \leq \varepsilon^{\delta/2}$$

since $1 - x \geq \delta$.

If $-1 + \delta \leq x \leq x_0$, replace $1 + z$ by $1 - z$ in (3); this leads to the same conclusion.

Thus (2) holds, and the proof is complete.

8.4.2. Remark. We stated in Lindelöf's theorem in the disc U but proved it in the strip Σ. Other conformal maps will of course transfer the theorem to other regions in $\mathbb{C}$.

For example, let $\Pi_\alpha = \{z = re^{i\theta}: r > 0, |\theta| < \alpha\}$. If $f \in H^\infty(\Pi_\alpha)$, $f \to L$ along *some* curve γ_0 in Π_α that approaches 0, and $\beta < \alpha$, then f tends to L along *every* curve γ that approaches 0 within Π_β.

8.4.3. Approach Curves in B. A curve in B that approaches a point $\zeta \in S$ will be called a *ζ-curve*. More precisely, a ζ-curve is a continuous map $\Gamma: [0, 1) \to B$ such that $\Gamma(t) \to \zeta$ as $t \to 1$. Usually, however, it will not be necessary to refer to any parametrization.

With each ζ-curve Γ we associate its orthogonal projection

$$\gamma = \langle \Gamma, \zeta \rangle \zeta \tag{1}$$

into the complex line through 0 and ζ. Then $(\Gamma - \gamma) \perp \gamma$, so that

$$|\Gamma - \gamma|^2 + |\gamma|^2 = |\Gamma|^2. \tag{2}$$

Since $|\Gamma| < 1$, (2) implies

$$\frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} < 1. \tag{3}$$

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As ζ-curve Γ is said to be *special* if

$$\lim_{t\to 1}\frac{|\Gamma(t)-\gamma(t)|^2}{1-|\gamma(t)|^2}=0 \tag{4}$$

and is said to be *restricted* if it satisfies both (4) and

$$\frac{|\gamma(t)-\zeta|}{1-|\gamma(t)|}\le A \qquad (0\le t<1) \tag{5}$$

for some $A<\infty$.

The restricted ζ-curves Γ are thus the special ones whose projection γ is nontangential.

There is a simple relation between restricted ζ-curves and the Korányi regions $D_\alpha(\zeta)$. Recall that $z\in D_\alpha(\zeta)$ precisely when

$$|1-\langle z,\zeta\rangle|<\frac{\alpha}{2}(1-|z|^2). \tag{6}$$

Assume that Γ satisfies (5), and also (3), but with some $c<1$ in place of 1. Then (2) leads to

$$1-|\Gamma|^2=1-|\gamma|^2-|\Gamma-\gamma|^2>(1-c)(1-|\gamma|^2)$$

and (5) shows that

$$|1-\langle\Gamma,\zeta\rangle|=|\langle\zeta-\gamma,\zeta\rangle|\le|\zeta-\gamma|\le A(1-|\gamma|).$$

Thus

$$\frac{|1-\langle\Gamma,\zeta\rangle|}{1-|\Gamma|^2}<\frac{A}{(1-c)(1+|\gamma|)} \tag{7}$$

which tends to $A/2(1-c)$ as $t\to 1$.

We conclude: If $\alpha>A/(1-c)$, then Γ lies in $D_\alpha(\zeta)$ eventually; that is to say, $\Gamma(t)\in D_\alpha(\zeta)$ for all t that are sufficiently close to 1.

If (3) is replaced by (4), the above holds for arbitrarily small c. Thus:

Every restricted ζ-curve Γ satisfying (5) *lies eventually in $D_\alpha(\zeta)$, for all $\alpha>A$.*

Conversely, every ζ-curve Γ that lies in $D_\alpha(\zeta)$ satisfies (5) with $A=\alpha$.

We shall say that a function $f\colon B\to\mathbb{C}$ has *restricted K-limit L* at ζ if $\lim f(\Gamma(t))=L$ as $t\to 1$, for every restricted ζ-curve Γ.

The preceding discussion shows that this happens whenever f has a K-limit at ζ. However, an $f\in H^\infty(B)$ may have a restricted K-limit at a point ζ,

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without having a K-limit at ζ. The simplest example of this is probably given by the function

$$(8) \qquad f(z, w) = \frac{w^2}{1 - z^2}$$

which is in $H^\infty(B_2)$, has restricted K-limit 0 at $(1, 0)$, but fails to have a K-limit there, since

$$(9) \qquad f(t, c\sqrt{1 - t^2}) = c^2 \qquad (0 \le t < 1)$$

for every $c \in U$. The expansion of (8), namely

$$(10) \qquad f(z, w) = \sum_{k=0}^{\infty} z^{2k} w^2$$

is a very simple example of a power series that converges absolutely at every point of S although the convergence is not uniform.

8.4.4. Theorem (Čirka [1]). *Suppose $f \in H^\infty(B)$, $\zeta \in S$, Γ_0 is a special ζ-curve, and*

$$(1) \qquad \lim_{t \to 1} f(\Gamma_0(t)) = L.$$

Then f has restricted K-limit L at ζ.

Proof. Let Γ be any special ζ-curve. Fix $t \in [0, 1)$ for the moment. Since $(\Gamma - \gamma) \perp \gamma$, the point $(1 - \lambda)\gamma(t) + \lambda\Gamma(t)$ lies in B whenever $|\gamma|^2 + |\lambda|^2|\Gamma - \gamma|^2 < 1$, i.e., whenever $|\lambda| < R = R(t)$, where

$$(2) \qquad R^2 = \frac{1 - |\gamma|^2}{|\Gamma - \gamma|^2}.$$

By 8.4.3(3), $R > 1$. Since Γ is special, $R(t) \to \infty$ as $t \to 1$.

If $|\lambda| < R$ we can define

$$(3) \qquad g(\lambda) = f((1 - \lambda)\gamma(t) + \lambda\Gamma(t)).$$

The Schwarz lemma, applied to $g(\lambda) - g(0)$ in the disc $\{|\lambda| < R\}$, shows that

$$(4) \qquad |g(1) - g(0)| \le \frac{2\|f\|_\infty}{R(t)}.$$

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Since $R(t) \to \infty$, we conclude from (3) and (4) that

$$\lim_{t \to 1} \{f(\Gamma(t)) - f(\gamma(t))\} = 0. \tag{5}$$

We now apply (5) to the given curve Γ_0 and to an arbitrary restricted ζ-curve Γ. By (1) and (5), $f(\gamma_0(t)) \to L$. Since γ is nontangential, Lindelöf's theorem (applied in the disc $\{\lambda\zeta : \lambda \in U\}$) shows that $f(\gamma(t)) \to L$. Hence $f(\Gamma(t)) \to L$, by (5), and the proof is complete.

8.4.5. Asymptotic Values. If f is a function in B, Γ is a ζ-curve, and $f(z)$ tends to L as z tends to ζ along Γ, then L is said to be an *asymptotic value* of f at ζ.

Lindelöf's theorem implies that no $f \in H^\infty(U)$ can have more than one asymptotic value at any boundary point. This is false if U is replaced by B; the function f mentioned at the end of §8.4.3 has every c with $|c| \le 1$ as an asymptotic value at $(1, 0)$, even though $|f| \le 1$. But Čirka's theorem shows that we still have uniqueness if we restrict ourselves to special ζ-curves:

If $f \in H^\infty(B)$, $\zeta \in S$, and f tends to L_1 and L_2 along special ζ-curves Γ_1 and Γ_2, then $L_1 = L_2$.

8.4.6. Example (Nagel–Rudin [2]). Here is an example that is a bit more ambitious than the one given at the end of §8.4.3. It exhibits a function $f \in H^\infty(B_2)$ whose restricted K-limit is 0 at every point on the circle $\{(e^{i\theta}, 0): -\pi \le \theta \le \pi\}$ but which has no K-limit at any of these points.

To do this, pick positive integers n_j and corresponding radii $r_j = 1 - 1/n_j$, so that $n_1 = 2$,

$$n_j > 10(n_1 + \cdots + n_{j-1}) \qquad (j = 2, 3, 4, \ldots) \tag{1}$$

and

$$n_j \exp\left\{-\frac{n_j}{n_k}\right\} < j^{-2} \qquad (1 \le k < j). \tag{2}$$

Define

$$f(z, w) = w^2 g(z) = w^2 \sum_{j=1}^{\infty} n_j z^{n_j}. \tag{3}$$

Since $n_j|z|^{n_j} < 2(n_j - n_{j-1})|z|^{n_j} < 2\sum |z|^m$, where the sum extends over all m with $n_{j-1} < m \le n_j$, it follows that $|g(z)| < 2/(1 - |z|)$, hence $|f(z, w)| < 4$ in B_2. Thus $f \in H^\infty(B_2)$.

Since $f(z, 0) = 0$, Čirka's theorem implies that f has restricted K-limit 0 at all points $(z, 0)$ with $|z| = 1$.

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For $k \geq 2$, $\frac{1}{4} \leq (1 - 1/k)^k < 1/e$. If $|z| = r_p$, it follows from (1), (2), (3) that

$$|g(z)| > \frac{n_p}{4} - \frac{n_p}{10} - \sum_{p+1}^{\infty} j^{-2}. \tag{4}$$

Thus $|g(z)| > n_p/20 = 1/(20(1 - r_p))$ as soon as n_p is large enough, and therefore

$$|f(r_p e^{i\theta}, c\sqrt{1 - r_p^2})| > \frac{|c|^2}{20} \tag{5}$$

if $|c| < 1$. The points at which f is evaluated in (5) lie in $D_\alpha(e^{i\theta}, 0)$ when $\alpha > 2/(1 - |c|^2)$. Hence f has no K-limit at $(e^{i\theta}, 0)$.

8.4.7. Example. Fix a constant $c > \frac{1}{2}$ and define f in B_2 by

$$f(z, w) = (1 - z)^{-c} w. \tag{1}$$

Then $f \notin H^\infty(B_2)$, but $f \in H^p(B_2)$ for all $p < 4/(2c - 1)$. If $\frac{1}{2} < \delta < c$ and

$$\Gamma(t) = (t, (1 - t)^\delta) \qquad (0 \leq t < 1) \tag{2}$$

then Γ tends to $(1, 0)$ restrictedly, and

$$f(\Gamma(t)) = (1 - t)^{\delta - c} \to \infty. \tag{3}$$

Since $f(z, 0) = 0$, we see that f has no restricted K-limit at $(1, 0)$.

Take a point $(a, b) \in S$, $a \neq 1$, and consider the rectilinear path

$$\Gamma(t) = (t + (1 - t)a, (1 - t)b) \tag{4}$$

from (a, b) to $(1, 0)$. On this path,

$$f(\Gamma(t)) = \frac{b}{(1 - a)^c} \cdot (1 - t)^{1-c}. \tag{5}$$

When $c < 1$, this tends to 0 as $t \to 1$. Thus all "rectilinear limits" of f at $(1, 0)$ are 0. In fact, $f(z, w) \to 0$ as $(z, w) \to (1, 0)$ within any cone in B whose vertex is at $(1, 0)$, although (as we saw above) the restricted K-limit of f does not exist there.

When $c = 1$, then $f(z, w) = w/(1 - z)$, and (5) shows that f is constant on each of the lines (4). All rectilinear limits of f exist therefore at $(1, 0)$, but they are not equal. In fact, they cover $\mathbb{C}$.

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By Čirka's theorem, no $f \in H^\infty(B)$ can behave in this way.

The following version of Čirka's theorem will be used in Section 8.5. It will be clear from the proof that the hypotheses could be varied considerably, but it seems best to stick to a simple statement.

8.4.8. Theorem. *Suppose $f \in H(B)$, $\zeta \in S$, f is bounded in every region $D_\alpha(\zeta)$, and the radial limit of f exists at ζ. Then the restricted K-limit of f exists at ζ.*

Proof. Let $\gamma_0(t) = t\zeta$, $0 \le t < 1$, and let Γ be any restricted ζ-curve, with projection γ as in §8.4.3. Then γ is nontangential. Lindelöf's theorem shows therefore that f has the same limit along γ and γ_0. It is thus enough to prove that

$$\lim_{t\to 1}\{f(\Gamma(t)) - f(\gamma(t))\} = 0. \tag{1}$$

We saw in §8.4.3 that $\Gamma(t) \in D_\alpha(\zeta)$ eventually, for some α. Choose $\beta > \alpha$. A slight modification of the proof of Theorem 8.4.4 shows that $(1 - \lambda)\gamma + \lambda\Gamma \in D_\beta$ whenever $|\lambda| < R = R(t)$, where

$$R^2 = \frac{1 - |\gamma|^2 - (2/\beta)|1 - \gamma|}{|\Gamma - \gamma|^2}. \tag{2}$$

If $\Gamma(t) \in D_\alpha$, then $|1 - \gamma| < (\alpha/2)(1 - |\gamma|^2)$, so that

$$R^2 > \frac{\beta - \alpha}{\beta}\cdot\frac{1 - |\gamma|^2}{|\Gamma - \gamma|^2} \tag{3}$$

which tends to ∞ as $t \to 1$, since Γ is special.

Now define $g(\lambda) = f((1 - \lambda)\gamma + \lambda\Gamma)$ for $|\lambda| < R$, use the fact that f is bounded in D_β, and estimate $g(1) - g(0)$ by the Schwarz lemma, as in the proof of Theorem 8.4.4. This leads to (1).

8.5. The Julia–Carathéodory Theorem

8.5.1. In the present section, the following one-variable facts will be generalized to holomorphic maps from one ball into another:

Suppose $f: U \to U$ is holomorphic. If there is some sequence $\{z_i\}$ in U, with $z_i \to 1$ and $f(z_i) \to 1$, along which

$$\frac{1 - |f(z_i)|}{1 - |z_i|}$$

is bounded, then f maps each circular disc in U that has 1 in its boundary into a disc of the same sort. This (in a more quantitative form) is Julia's

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theorem. Carathéodory added that $f'(z)$ then has a nontangential positive finite limit at $z = 1$.

Full details of this may be found in vol. 2 of Carathéodory's book [3].

The generalizations to several variables will be proved directly, without any reference to the theorems just mentioned. In fact, if one takes $n = 1$ and $m = 1$, the proofs that follow are the classical ones.

8.5.2. The Setting. Throughout this section, m and n will be fixed, F will be a holomorphic map of B_n into B_m, ζ will be a fixed boundary point of B_n, and we define

(1) $$L = \liminf_{z \to \zeta} \frac{1 - |F(z)|^2}{1 - |z|^2}.$$

The basic assumption we make is that $L < \infty$.

There is then a sequence $\{a_i\}$ in B_n that converges to ζ, such that

(2) $$\lim_{i \to \infty} \frac{1 - |F(a_i)|^2}{1 - |a_i|^2} = L,$$

and such that $F(a_i)$ converges to some boundary point of B_m. By unitary transformations we may choose coordinates so that $\zeta = e_1$ and $F(a_i)$ converges to e_1. (The symbol e_1 is here used with two meanings; it designates the first element in the standard basis of $\mathbb{C}^n$ as well as $\mathbb{C}^m$. It is unlikely that this will cause any confusion.)

Let $f_1, \ldots, f_m$ be the components of F.

The Schwarz lemma (Theorem 8.1.4) states that

(3) $$\frac{|1 - \langle F(z), F(a_i)\rangle|^2}{1 - |F(z)|^2} \le \frac{1 - |F(a_i)|^2}{1 - |a_i|^2} \cdot \frac{|1 - \langle z, a_i\rangle|^2}{1 - |z|^2}$$

for all $z \in B_n$. As $i \to \infty$, $\langle z, a_i\rangle \to z_1$ and $\langle F(z), F(a_i)\rangle \to f_1(z)$. Hence (2) and (3) yield

8.5.3. Julia's Theorem. *Under the above hypotheses*

(1) $$\frac{|1 - f_1(z)|^2}{1 - |F(z)|^2} \le L \frac{|1 - z_1|^2}{1 - |z|^2} \qquad (z \in B_n).$$

One incidental consequence of (1) is that $L > 0$.

The inequality (1) has an appealing geometric interpretation that involves ellipsoids: For $0 < c < 1$, let E_c be the set of all $z \in B_n$ that satisfy

(2) $$\frac{|1 - z_1|^2}{1 - |z|^2} < \frac{c}{1 - c}.$$

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Writing $z = (z_1, z')$ in the usual way, a little computing shows that (2) is the same as

$$\text{(3)} \qquad \frac{|z_1 - (1-c)|^2}{c^2} + \frac{|z'|^2}{c} < 1.$$

Thus E_c is an ellipsoid in B_n that has e_1 as a boundary point, has its center at $(1-c)e_1$, has radius c in the e_1-plane, and has radius $\sqrt{c}$ in the directions orthogonal to e_1.

If $\gamma/(1-\gamma) = Lc/(1-c)$ and if E_γ denotes the corresponding ellipsoid in B_m, then it follows from (1) and (2) *that F maps E_c into E_γ, where*

$$\text{(4)} \qquad \gamma = \frac{Lc}{1 + Lc - c}.$$

Let us now add an inessential assumption that will simplify the statements of some inequalities, namely: $F(0) = 0$. Then $|F(z)| \le |z|$ (Theorem 8.1.2), hence $L \ge 1$, and thus (4) implies the simpler statement $\gamma \le Lc$.

This proves the first part of the following geometric version of Julia's theorem:

8.5.4. Theorem. *If F is as in* §8.5.2 *and if also* $F(0) = 0$, *then*

(i) $F(E_c) \subset E_{Lc}$ *when* $0 < c < 1/L$, *and*
(ii) $F(D_\alpha) \subset D_{\alpha\sqrt{L}}$ *for all* $\alpha > 1$.

To prove the assertion about the Korányi regions $D_\alpha = D_\alpha(e_1)$, simply multiply the inequalities 8.5.3(1) and

$$\frac{1}{1 - |F(z)|^2} \le \frac{1}{1 - |z|^2}.$$

Then take square roots, to obtain

$$\frac{|1 - f_1(z)|}{1 - |F(z)|^2} \le \sqrt{L} \cdot \frac{|1 - z_1|}{1 - |z|^2} < \alpha\sqrt{L}$$

if $z \in D_\alpha$.

We shall need the following relation between D_α and D_β.

8.5.5. Lemma. *Suppose* $1 < \alpha < \beta$, $\delta = \frac{1}{3}(1/\alpha - 1/\beta)$, *and* $z = (z_1, z') \in D_\alpha$.

(i) *If* $|\lambda| \le \delta|1 - z_1|$ *then* $(z_1 + \lambda, z') \in D_\beta$.
(ii) *If* $|w'| \le \delta|1 - z_1|^{1/2}$ *then* $(z_1, z' + w') \in D_\beta$.

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Proof. The condition that $z \in D_\alpha$ can be written in the form

$$(1) \qquad |z'|^2 < 1 - |z_1|^2 - \frac{2}{\alpha}|1 - z_1|$$

in which z_1 and z' are separated.

Since $|z_1| < 1$, $|\lambda| < 1$, $\beta > 1$, and $5\delta + 2/\beta < 2/\alpha$, we have

$$\begin{aligned}|z_1 + \lambda|^2 + \frac{2}{\beta}|1 - z_1 - \lambda| &< |z_1|^2 + 5|\lambda| + \frac{2}{\beta}|1 - z_1| \\ &< |z_1|^2 + \frac{2}{\alpha}|1 - z_1| < 1 - |z'|^2,\end{aligned}$$

which proves (i). Since $2|z'| < 3|1 - z_1|^{1/2}$ for all $z \in B$, we have

$$\begin{aligned}|z' + w'|^2 &\le |z'|^2 + (3\delta + \delta^2)|1 - z_1| \\ &< 1 - |z_1|^2 + \left(4\delta - \frac{2}{\alpha}\right)|1 - z_1| \\ &< 1 - |z_1|^2 - \frac{2}{\beta}|1 - z_1|,\end{aligned}$$

which proves (ii).

We are now ready for the generalization of Carathéodory's theorem.

Recall that $D_1, \ldots, D_n$ denote the partial derivatives with respect to $z_1, \ldots, z_n$.

8.5.6. Theorem. *Suppose $F = (f_1, \ldots, f_m)$ is a holomorphic map of B_n into B_m, $F(0) = 0$,*

$$(1) \qquad L = \liminf_{z \to e_1} \frac{1 - |F(z)|^2}{1 - |z|^2} < \infty,$$

and $F(a_i) \to e_1$ for some sequence $\{a_i\}$ in B_n such that $a_i \to e_1$ and

$$(2) \qquad \lim_{i \to \infty} \frac{1 - |F(a_i)|^2}{1 - |a_i|^2} = L.$$

Suppose $2 \le j \le m$ and $2 \le k \le n$.

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The following functions are then bounded in every region $D_\alpha(e_1)$:

(i) $(1 - f_1(z))/(1 - z_1)$
(ii) $(D_1 f_1)(z)$
(iii) $f_j(z)/(1 - z_1)^{1/2}$
(iv) $(1 - z_1)^{1/2}(D_1 f_j)(z)$
(v) $(D_k f_1)(z)/(1 - z_1)^{1/2}$
(vi) $(D_k f_j)(z)$.

Moreover, the functions (i), (ii) *have restricted K-limit L at* e_1 *and the functions* (iii), (iv), (v) *have restricted K-limit* 0 *at* e_1.

Corollary. *In the case* $m = n$, *the Jacobian* JF *of* F *is bounded in every region* $D_\alpha(e_1)$.

Because of its length, the proof will be divided into several steps.

Step 1. Radial Behavior. We shall first prove that

$$\lim_{x \to 1} \frac{1 - f_1(xe_1)}{1 - x} = L \tag{3}$$

and

$$\lim_{x \to 1} \frac{f_j(x)}{(1 - x)^{1/2}} = 0 \qquad (2 \le j \le m), \tag{4}$$

where it is understood that $0 < x < 1$.

Suppose that actually $1 - x < 1/L$. Put $1 - x = 2c$. Then xe_1 is a boundary point of E_c. By Theorem 8.5.4, $F(xe_1)$ lies in the closure of E_{Lc}. Since $2Lc < 1$, it follows that $|F(xe_1)| \ge 1 - 2Lc$, which is the same as

$$1 - |F(xe_1)| \le L(1 - x). \tag{5}$$

Since $F(0) = 0$, $1 + |F(x)| \le 1 + x$. Hence (5) implies

$$\frac{1 - |F(xe_1)|^2}{1 - x^2} \le \frac{1 - |F(xe_1)|}{1 - x} \le L. \tag{6}$$

By the definition of L as the lower limit (1), it follows from (6) that

$$\lim_{x \to 1} \frac{1 - |F(xe_1)|^2}{1 - x^2} = L. \tag{7}$$

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To simplify the notation, we now write $w = w(x)$ for $f_1(xe_1)$. By (6) and Theorem 8.5.3,

$$\frac{|1-w|^2}{(1-x)^2} \le L \cdot \frac{1-|F(xe_1)|^2}{1-x^2} \le L^2. \tag{8}$$

Since $1 - |F(xe_1)| \le 1 - |w| \le |1-w|$, we conclude from (6), (7), and (8) that

$$\lim_{x\to 1} \frac{1-|w(x)|}{1-x} = \lim_{x\to 1} \frac{|1-w(x)|}{1-x} = L. \tag{9}$$

The ratio of the two numerators in (9) converges therefore to 1 as $x \to 1$. This implies that also

$$\lim_{x\to 1} \frac{1-w(x)}{1-|w(x)|} = 1, \tag{10}$$

and (3) is thus a consequence of (9).

Since $w(x) \to 1$ as $x \to 1$, (3) is the same as

$$\lim_{x\to 1} \frac{1-|f_1(xe_1)|^2}{1-x^2} = L. \tag{11}$$

Now (4) follows from (7) and (11), because

$$|F|^2 = |f_1|^2 + \cdots + |f_m|^2. \tag{12}$$

Step 2. The Functions (i) *and* (iii). Fix $\alpha > 1$, and assume $z \in D_\alpha(e_1)$ is so close to e_1 that $Lc < 1$ if $c = (\alpha/2)|1 - z_1|$.

Then $|1 - z_1|^2 = (2c/\alpha)|1 - z_1| < c(1 - |z|^2)$. Since $c < c/(1-c)$, it follows that $z \in E_c$ (see 8.5.3(2)), hence $F(z) \in E_{Lc}$, and therefore

$$|1 - f_1(z)| < 2Lc = \alpha L|1 - z_1|. \tag{13}$$

Since (13) holds for every $z \in D_\alpha(e_1)$ that is sufficiently close to e_1, we conclude that the function $(1 - f_1)/(1 - z_1)$ is bounded in every $D_\alpha(e_1)$; by (3) and Theorem 8.4.8, its restricted K-limit at e_1 is L.

If $2 \le j \le m$, the inclusion $F(z) \in E_{Lc}$ shows that

$$|f_j(z)|^2 < Lc = \tfrac{1}{2}\alpha L|1 - z_1|. \tag{14}$$

Hence $f_j(z)/(1 - z_1)^{1/2}$ is bounded in every $D_\alpha(e_1)$, and its restricted K-limit at e_1 is 0, because of (4) and Theorem 8.4.8.

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Step 3. The Functions (ii) *and* (iv). These involve differentiation with respect to z_1. Suppose $1 < \alpha < \beta$, choose δ as in Lemma 8.5.5, let $z \in D_\alpha$, and put

(15) $$r = r(z) = \delta|1 - z_1|.$$

Then $(z_1 + \lambda, z') \in D_\beta$ for all λ with $|\lambda| \le r$. By the Cauchy formula,

(16) $$(D_1 f_1)(z) = \frac{1}{2\pi i}\int_{|\lambda|=r} f_1(z_1 + \lambda, z')\lambda^{-2}\, d\lambda.$$

The integral is unchanged if f_1 is replaced by $f_1 - 1$. Do this, then multiply and divide the integrand by $z_1 + \lambda - 1$, and put $\lambda = re^{i\theta}$, to obtain

(17) $$(D_1 f_1)(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - f_1(z_1 + re^{i\theta}, z')}{1 - (z_1 + re^{i\theta})} \cdot \left\{1 - \frac{1 - z_1}{re^{i\theta}}\right\} d\theta.$$

The first factor in the integrand is bounded, by Step 2, since $(z_1 + re^{i\theta}, z') \in D_\beta(e_1)$. The second factor is at most $1 + 1/\delta$, by (15). We conclude that $D_1 f_1$ is bounded in $D_\alpha(e_1)$.

When $z = xe_1$ in (17), then the second factor in the integrand is $1 - \delta^{-1}e^{-i\theta}$, and the first factor converges boundedly to L as $x \to 1$, since $x + r(x)e^{i\theta} \to 1$ nontangentially, for every θ, by (15). Hence $(D_1 f_1)(xe_1) \to L$ as $x \to 1$, by the dominated convergence theorem. Another application of Theorem 8.4.8 shows now that $D_1 f_1$ has restricted K-limit L at e_1.

If $2 \le j \le m$, a similar application of the Cauchy formula gives

(18) $$(D_1 f_j)(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_j(z_1 + re^{i\theta}, z')}{(1 - z_1 - re^{i\theta})^{1/2}} \cdot \frac{(1 - z_1 - re^{i\theta})^{1/2}}{re^{i\theta}}\, d\theta,$$

from which it follows exactly as above (using Step 2 and Theorem 8.4.8) that $(1 - z_1)^{1/2}(D_1 f_j)(z)$ is bounded in $D_\alpha(e_1)$ and that its restricted K-limit at e_1 is 0.

Step 4. The Functions (v) *and* (vi). These involve differentiation with respect to z_k for $2 \le k \le n$. Without loss of generality, take $k = 2$.

Suppose $1 < \alpha < \beta$, choose δ as in Lemma 8.5.5, let $z \in D_\alpha(e_1)$, and put

(19) $$\rho = \rho(z) = \delta|1 - z_1|^{1/2}.$$

Then $(z_1, z' + w') \in D_\beta(e_1)$ for all w' with $|w'| \le \rho$. If we apply the Cauchy formula as in Step 3, we obtain

(20) $$\frac{(D_2 f_1)(z)}{(1 - z_1)^{1/2}} = -\frac{(1 - z_1)^{1/2}}{\rho(z)} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - f_1(z_1, z_2 + \rho e^{i\theta}, \ldots)}{1 - z_1}\, e^{-i\theta}\, d\theta$$

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and, for $j \geq 2$,

$$(21)\quad (D_2 f_j)(z) = \frac{(1 - z_1)^{1/2}}{\rho(z)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_j(z_1, z_2 + \rho e^{i\theta}, \ldots)}{(1 - z_1)^{1/2}} e^{-i\theta}\, d\theta.$$

The integrands are bounded, by the bounds of (i) and (iii) in $D_\beta(e_1)$. In view of (19), the left sides of (20) and (21) are therefore bounded in $D_\alpha(e_1)$.

To finish, we have to prove that the left side of (20) has restricted K-limit 0 at e_1. By Theorem 8.4.8 it is enough to prove this for the radial limit. Moreover, it involves now no loss of generality to assume $n = 2$, $m = 1$, in which case $f_1 = F$. Writing (z, w) in place of (z_1, z_2), we can expand F in the form

$$(22)\qquad F(z, w) = f(z) + 2w(1 - z)^{1/2} g(z) + \sum_{j=2}^{\infty} g_j(z) w^j.$$

Then $(D_2 F)(z, 0)/(1 - z)^{1/2} = 2g(z)$. It is therefore enough to show that

$$(23)\qquad g(x) \to 0 \quad \text{as} \quad x \nearrow 1.$$

We know that $(1 - f(z))/(1 - z) \to L$ as $z \to 1$ nontangentially, and that g is nontangentially bounded at 1, and we make one further reduction:

If $|\sum_0^\infty c_k w^k| < 1$ *in a certain disc with center at* 0, *then also* $|c_0 + \frac{1}{2}c_1 w| < 1$ *in this same disc.*

This is so because $c_0 + \frac{1}{2}c_1 w$ is the arithmetic mean of the first two partial sums of the power series. If we apply this to (22), we see that (23) is a consequence of the following proposition (in which there is some redundancy in the hypotheses):

8.5.7. Proposition. *Suppose* $h: B_2 \to U$ *has the form*

$$(1)\qquad h(z, w) = f(z) + w(1 - z)^{1/2} g(z)$$

where $f, g \in H(U)$, $(1 - f(z))/(1 - z)$ *has finite nontangential limit* L *at* $z = 1$, *and* g *is nontangentially bounded at* 1. *Then*

$$(2)\qquad g(x) \to 0 \quad \textit{as} \quad x \nearrow 1.$$

Proof. Choose $\varepsilon > 0$, put $c = L^2/\varepsilon^2$, let z tend to 1 along the line $z = x + ic(1 - x)$. Then $1 - z = (1 - ic)(1 - x)$, hence

$$(3)\qquad |1 - z| \geq c(1 - x),$$

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and also $1 - |z|^2 > 1 - x$ if $c^2/(1 + c^2) < x < 1$, an assumption that will be made in the rest of this proof. Note that

$$f(z) = 1 - (L + o(1))(1 - ic)(1 - x) \tag{4}$$

so that

$$\operatorname{Re} f(z) = 1 - (L + o(1))(1 - x). \tag{5}$$

Associate with every z under consideration a $w \in \mathbb{C}$ with $|w|^2 = 1 - |z|^2 > 1 - x$, whose argument is so chosen that

$$w(1 - z)^{1/2}g(z) = |w(1 - z)^{1/2}g(z)| \geq c^{1/2}(1 - x)|g(z)|, \tag{6}$$

by (3). Hence, by (5) and (6),

$$1 \geq \operatorname{Re} h(z, w) \geq 1 + \{c^{1/2}|g(z)| - L - o(1)\}(1 - x). \tag{7}$$

Consequently,

$$\limsup_{x \to 1} |g(x + ic(1 - x))| \leq Lc^{-1/2} = \varepsilon. \tag{8}$$

The same estimate holds on the line $z = x - ic(1 - x)$. Since $g(z)$ is bounded as $z \to 1$ between these two lines, it follows that

$$\limsup_{x \to 1} |g(x)| \leq \varepsilon, \tag{9}$$

which proves (2), since ε was arbitrary.

8.5.8. Examples. We shall now show that the conclusions of Theorem 8.5.6 are optimal. The numbers (i) through (vi) will refer to Theorem 8.5.6.

The first two examples will use the function

$$g(z) = \exp\left\{-\frac{\pi}{2} - i\log(1 - z)\right\} \qquad (z \in U). \tag{1}$$

Note that $|g| < 1$ in U, and that

$$g'(z) = \frac{ig(z)}{1 - z}. \tag{2}$$

As $z \to 1$, $g(z)$ spirals around the origin without approaching it.

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First Example. Take $n = m = 2$, define $F: B_2 \to B_2$ by

$$F(z, w) = (z, wg(z)). \tag{3}$$

The hypotheses of Theorem 8.5.6 hold, with $L = 1$. Since $D_1 f_1 = 1$ and $D_2 f_1 = 0$, we have

$$(JF)(z, w) = (D_2 f_2)(z, w) = g(z). \tag{4}$$

Therefore the radial limit of $D_2 f_2$ and of JF does not exist at e_1.

This dealt with (vi). As regards (iv),

$$(1 - z)^{1/2}(D_1 f_2)(z, w) = \frac{iw}{(1 - z)^{1/2}} \cdot g(z). \tag{5}$$

This has no K-limit at e_1, although its restricted K-limit is 0. We see also that the boundedness assertion made about (iv) becomes false if the exponent $\frac{1}{2}$ is replaced by any smaller one.

Second Example. Take $n = 2$, $m = 1$, put

$$F(z, w) = z + \tfrac{1}{2}w^2 g(z). \tag{6}$$

Example 8.1.5 shows that F maps B_2 into U. The hypotheses of Theorem 8.5.6 hold again with $L = 1$. Since $F = f_1$,we now have

$$\frac{1 - f_1(z, w)}{1 - z} = 1 - \frac{w^2}{2(1 - z)} \cdot g(z), \tag{7}$$

$$(D_1 f_1)(z, w) = 1 + \frac{iw^2}{2(1 - z)} \cdot g(z), \tag{8}$$

$$\frac{(D_2 f_1)(z, w)}{(1 - z)^{1/2}} = \frac{w}{(1 - z)^{1/2}} \cdot g(z). \tag{9}$$

Hence (i), (ii), *and* (v) *need have no K-limit at e_1, and the boundedness assertion made about* (v) *becomes false if $\frac{1}{2}$ is replaced by any larger exponent.*

Third Example. This will show that the exponent $\frac{1}{2}$ is best possible in (iii).

Take $n = 1$, $m = 2$. Pick $\varepsilon > 0$, put

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} |\theta|^{1+\varepsilon}\, d\theta, \tag{10}$$

and note that h lies in the disc algebra, that $h(1) = 0$, and that $\operatorname{Re} h(z) > 0$ for all other $z \in \overline{U}$.

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Put $c = 1/2\pi^{1+\varepsilon}$ and define $F = (f_1, f_2)$ by

$$f_1(z) = ze^{-\mathrm{ch}(z)}, \qquad f_2(z) = c^{1/2}(1-z)^{(1+\varepsilon)/2}z. \tag{11}$$

Let $u = \mathrm{Re}[\mathrm{ch}]$. Since $|1 - e^{i\theta}| \le |\theta|$ if $|\theta| \le \pi$, we have $|f_2|^2 \le u$ on the unit circle, hence $|f_2|^2 \le u$ in U, because $|f_2|^2$ is subharmonic. Therefore

$$|f_1|^2 + |f_2|^2 \le u + e^{-2u} < 1 \tag{12}$$

in U; the last inequality holds because $0 < u < \frac{1}{2}$ by our choice of c.

Thus F maps U into B_2 and $F(0) = 0$.

To show that F satisfies the other hypothesis of Theorem 8.5.6, it is enough to show that $f_1'(x)$ has a finite limit as $x \nearrow 1$, since then $(1 - |F(x)|^2)/(1 - x^2)$ is bounded. By (10),

$$h'(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}}{(e^{i\theta} - x)^2}|\theta|^{1+\varepsilon}\, d\theta. \tag{13}$$

Since $|e^{i\theta} - x| \ge \sin|\theta/2|$ if $0 < x < 1$, $|\theta| \le \pi$, the dominated convergence theorem leads from (13) to

$$\lim_{x\to 1} h'(x) = \frac{-1}{2\pi}\int_{-\pi}^{\pi} \frac{|\theta|^{1+\varepsilon}}{1 - \cos\theta}\, d\theta \tag{14}$$

which is finite because $\varepsilon > 0$. By (14) and (11), $\lim f'_1(x)$ is also finite. (Ahern and Clark [1] have proved much more general theorems about derivatives of functions of the form (10).)

By (11), $f_2(x)/(1 - x)^{(1/2)+\varepsilon}$ is unbounded as $x \nearrow 1$. *The boundedness assertion concerning* (iii) *becomes therefore false if* $\frac{1}{2}$ *is replaced by any larger exponent.*

Finally, we note that the map F defined by (3) furnishes an example in which the function (iii) has no K-limit at e_1.

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