CURM Quaternion Stuff

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Introduction and Background:

I will first introduce the definition of $\mathbb{H}$, the field of quaternions as to build towards a quaternion-valued metric space. Each element $h \in \mathbb{H}$ is defined as $h = a + bi + cj + dk$ where a, b, c, $d \in \mathbb{R}$ and i, j, k are symbols following the equality, $i^2 = j^2 = k^2 = ijk = -1$. These symbols are imaginary much like i is the imaginary component of complex numbers. There are many surprisingly pragmatic applications of quaternions in physics and programming. I even found mention of so-called 'bi-quaternions' in special relativity.

The norm of a quaternion is defined as $|h| = \sqrt{a^2 + b^2 + c^2 + d^2}$. The distance $d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H}$ is defined as the quaternion valued function $d_{\mathbb{H}}(p,q) = |a_0 - b_0| + i|a_1 - b_1| + j|a_2 - b_2| + k|a_3 - b_3|$, where p, $q \in \mathbb{H}$ are $p = a_0 + ia_1 + ja_2 + ka_3$ and $q = b_0 + ib_1 + jb_2 + kb_3$. The trouble with proving Theorem 1 in Dr. Thompson's paper comes in the extension of Denjoy-Wolff due to the lack of a simple derivative to work with. We have to use left and right derivatives because quaternions are non-commutative. Getting a Cauchy's Integral Formula will be quite difficult, but I think it might be possible.

Partial Ordering of $\mathbb{H}$

In order to have a quaternion-valued metric space we must first have some kind of ordering to the set $\mathbb{H}$. This comes in the paper titled *Fixed Point Theorems in Quaternion-Valued Metric Spaces*. The partial order $\preceq$ on $\mathbb{H}$ is given as follows, $(h_1 \preceq h_2) \iff [Re(h_1) \leq R(h_2)] \wedge [Im_i(h_1) \leq Im_i(h_2)] \wedge [Im_j(h_1) \leq Im_j(h_2)] \wedge [Im_k(h_1) \leq Im_k(h_2)]$.

Quaternion-Valued Metric Space

Let S be a nonempty set. $d_{\mathbb{H}}$ is a quaternion valued metric on S, and $(S, d_{\mathbb{H}})$ is a quaternion-valued metric space if and only if these three properties hold:

(1) $0 \preceq d_{\mathbb{H}}(x, y)$ for all x, $y \in S$ and $d_{\mathbb{H}}(x, y) = 0$ if and only if $x = y$,
(2) $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(y, x)$ for all x, $y \in S$,
(3) $d_{\mathbb{H}}(x, y) \preceq d_{\mathbb{H}}(x, z) + d_{\mathbb{H}}(z, y)$ for all x, y, $z \in S$.

Theorem 1: *Suppose $\phi : \mathbb{D} \to \mathbb{D}$ is analytic and continuous on $\partial\mathbb{D}$. If the Denjoy-Wolff point a of ϕ is in $\mathbb{D}$, then $\phi_n \to a$ uniformly if and only if there is $N > 0$ such that $\phi_N(\overline{\mathbb{D}}) \subset \mathbb{D}$.*

Proof. The direction that is trivial due to Denjoy-Wolff in the given context is no longer easy in this new quaternion-valued setting. This gives me a feeling of much unease. Nonetheless, I will accomplish the other direction by a neat trick with inequalities.

Let M be the minimum distance along any basis direction between a and the unit ball. This unit ball is denoted by $\mathbb{D} = \{h : |d_{\mathbb{H}}(h, 0)| < 1\}$. For a given real number $\epsilon > 0$, let $q = \frac{\epsilon}{2} + i\frac{\epsilon}{2} + j\frac{\epsilon}{2} + k\frac{\epsilon}{2}$. For $\epsilon = \frac{M}{2}$, there exists $N > 0$ such that $|d_{\mathbb{H}}(\phi_N(h),\, a)| < |q| = \epsilon$, $\forall h \in \mathbb{D}$. Assume for contradiction that $\phi_N(h_1) = h_2$, $|h_1| = |h_2| = 1$. Then, since ϕ_N is continuous there exists $\delta > 0$ such that $|d_{\mathbb{H}}(h_1,\, h)| < |q| < \delta \implies |d_{\mathbb{H}}(h_2,\, \phi_N(h))| < |q| < \epsilon$. Thus, $M \leq |d_{\mathbb{H}}(h_2,\, a)| \leq |d_{\mathbb{H}}(h_2,\, \phi_N(h)))| + |d_{\mathbb{H}}(\phi_N(h),\, a)| < 2|q| < 2\epsilon = M$. This is not true, so the proof is valid by contradiction.

$\square$

This technique is essentially a component-wise $\epsilon - \delta$ proof from real analysis.