# CURM Quaternion Stuff

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**Introduction and Background:**

I will first introduce the definition of $\mathbb{H}$, the field of quaternions as to build towards a quaternion-valued metric space. Each element $h \in \mathbb{H}$ is defined as $h = a + bi + cj + dk$ where $a$, $b$, $c$, $d \in \mathbb{R}$ and $i$, $j$, $k$ are symbols following the equality, $i^2 = j^2 = k^2 = ijk = -1$. These symbols are imaginary much like $i$ is the imaginary component of complex numbers. There are many surprisingly pragmatic applications of quaternions in physics and programming. I even found mention of so-called 'bi-quaternions' in special relativity. The norm of a quaternion is defined as $|h| = \sqrt{a^2 + b^2 + c^2 + d^2}$. The trouble with proving Theorem 1 in Dr. Thompson's paper comes in the extension of Denjoy-Wolff due to the lack of a simple derivative to work with. We have to use left and right derivatives because quaternions are non-commutative. Getting a Cauchy's Integral Formula will be quite difficult, but I think it might be possible.

**Partial Ordering of $\mathbb{H}$**

In order to have a quaternion-valued metric space we must first have some kind of ordering to the set $\mathbb{H}$. This comes in the paper titled *Fixed Point Theorems in Quaternion-Valued Metric Spaces*. The partial order $\preceq$ on $\mathbb{H}$ is given as follows, $h_1 \preceq h_2 \iff Re(h_1) \leq R(h_2) \wedge Im_i(h_1) \leq Im_i(h_2) \wedge Im_j(h_1) \leq Im_j(h_2) \wedge Im_k(h_1) \leq Im_k(h_2)$.

**Quaternion-Valued Metric Space**

Let $S$ be a nonempty set. $d_{\mathbb{H}}$ is a quaternion valued metric on $S$, and $(S, d_{\mathbb{H}})$ is a quaternion-valued metric space if and only if these three properties hold:

(1) $0 \preceq d_{\mathbb{H}}(x, y)$ for all $x$, $y \in S$ and $d_{\mathbb{H}}(x, y) = 0$ if and only if $x = y$,

(2) $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(y, x)$ for all $x$, $y \in S$,

(3) $d_{\mathbb{H}}(x, y) \preceq d_{\mathbb{H}}(x, z) + d_{\mathbb{H}}(z, y)$ for all $x$, $y$, $z \in S$.

**Theorem 1:** *Suppose $\phi : \mathbb{D} \to \mathbb{D}$ is analytic and continuous on $\partial\mathbb{D}$. If the Denjoy-Wolff point $a$ of $\phi$ is in $\mathbb{D}$, then $\phi_n \to a$ uniformly if and only if there is $N > 0$ such that $\phi_N(\overline{\mathbb{D}}) \subset \mathbb{D}$.*

*Proof.* The direction that is trivial due to Denjoy-Wolff in the given context is no longer easy in this new quaternion-valued setting. This gives me a feeling of much unease. Nonetheless, I will accomplish the other direction by a neat trick with inequalities.

Let $M$ be the minimum distance along any basis direction between $a$ and the unit ball. For a given real number $\epsilon > 0$, let $q = \frac{\epsilon}{2} + i\frac{\epsilon}{2} + j\frac{\epsilon}{2} + k\frac{\epsilon}{2}$. For $\epsilon = \frac{M}{2}$, there exists $N > 0$ such that $|d_{\mathbb{H}}(\phi_N(h),\, a)| < |q| = \epsilon$, $\forall h \in \mathbb{D}$. Assume for contradiction that $\phi_N(h_1) = h_2$, $|h_1| = |h_2| = 1$. Then, since $\phi_N$ is continuous there exists $\delta > 0$ such that $|d_{\mathbb{H}}(h_1,\, h)| < |q| < \delta \implies |d_{\mathbb{H}}(h_2,\, \phi_N(h))| < |q| < \epsilon$. Thus, $M \leq |d_{\mathbb{H}}(h_2,\, a)| \leq |d_{\mathbb{H}}(h_2,\, \phi_N(h)))| + |d_{\mathbb{H}}(\phi_N(h),\, a)| < 2|q| < 2\epsilon = M$. This is not true, so the proof is valid by contradiction. $\square$

Disclaimer: this technique feels very hand-wavy and I definitely need to put more details in.