Problem 1.

a) $E[\bar{X}] = E[\frac{1}{n}\sum_{i=1}^{n} X_i]$

$= \frac{1}{n} E[\sum_{i=1}^{n} X_i]$

$= \frac{1}{n}\sum_{i=1}^{n} E[X_i]$

$\because X_i \sim N(\mu, \sigma^2)$

$\therefore E[X_i] = \mu$

$\therefore E[\bar{x}] = \frac{1}{n}\cdot n\mu = \mu \Rightarrow \bar{X}$ is an unbiased estimator of μ

b) $E[S^2] = E[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2]$

$= \frac{1}{n-1} E[\sum_{i=1}^{n}(X_i-\bar{X})^2]$

$= \frac{1}{n-1} E[\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n}\bar{X}^2 - \sum_{i=1}^{n} 2X_i\bar{X}]$

$= \frac{1}{n-1}[\sum_{i=1}^{n} E[X_i^2] + E[n\bar{X}^2 - 2n\bar{X}^2]]$

$= \frac{1}{n-1}[\sum_{i=1}^{n} E[X_i^2] - E[n\bar{X}^2]]$

$= \frac{n}{n-1}\cdot(\frac{1}{n}\sum E(X_i^2) - E[\bar{X}^2])$ ①

$E[X_i^2] = Var[X_i] + E^2[X_i] = \sigma^2 + \mu^2$

$E[\bar{X}^2] = Var[\bar{X}] + E^2[\bar{X}] = \frac{\sigma^2}{n} + \mu^2$

$\therefore$ ① $= \frac{n}{n-1}(\sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2)$

$= \frac{n}{n-1}\cdot\sigma^2\cdot\frac{n-1}{n}$

$= \sigma^2$

$\therefore E[S^2] = \sigma^2 \Rightarrow S^2$ is an unbiased estimator of σ^2

c) $\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{\bar{y}})^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i}[(y_{ij}-\bar{y}_i) + (\bar{y}_i - \bar{\bar{y}})]^2$

$= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{y}_i-\bar{\bar{y}})^2 + \underbrace{2\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)(\bar{y}_i-\bar{\bar{y}})}_{①}$

To prove the partitioning of the total variability

$\Rightarrow$ to prove that ① $= 0$:

$\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)(\bar{y}_i-\bar{\bar{y}}) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}\bar{y}_i - y_{ij}\bar{\bar{y}} - \bar{y}_i^2 + \bar{y}_i\bar{\bar{y}}) = \sum_{i=1}^{k}\bar{y}_i\cdot n_i\bar{y}_i - \bar{\bar{y}}\cdot n\bar{\bar{y}} - \sum_{i=1}^{k} n_i\bar{y}_i^2 + \sum_{i=1}^{n}\bar{\bar{y}}n_i\bar{y}_i$

$= \sum_{i=1}^{k} n_i\bar{y}_i^2 - \sum_{i=1}^{k} n_i\bar{y}_i^2 - n\bar{\bar{y}}^2 + \bar{\bar{y}}\sum_{i=1}^{k} n_i\bar{y}_i$

$= -n\bar{\bar{y}}^2 + \bar{\bar{y}}\cdot n\cdot\bar{\bar{y}}$

$= 0 \quad \square \quad \Rightarrow \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{\bar{y}})^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{y}_i-\bar{\bar{y}})^2$