PoGaIN Supplementary Material

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I. Maximum Likelihood Derivation

A. Poisson-Noise Modeling

Let us denote the observed noisy image as y and the ground-truth noise-free image as x. Then, the Poisson-Gaussian model takes the form of the following equation

$$y = \frac{1}{a}\alpha + \beta, \quad \alpha \sim \mathcal{P}(ax), \quad \beta \sim \mathcal{N}(0, b^2). \tag{1}$$

Using the linearity property of expectation, we can compute the expected value

$$\mathbb{E}[y] = \frac{1}{a}\mathbb{E}[\alpha] = \frac{1}{a}ax = x. \tag{2}$$

Further, the variance has the following expression

$$\mathbb{V}[y] = \mathbb{E}\left[\left(\frac{1}{a}\alpha + \beta\right)^2\right] - x^2 = \frac{1}{a^2}\mathbb{E}[\alpha^2] + b^2 - x^2. \tag{3}$$

Given that $\mathbb{E}[\alpha^2] = ax + a^2x^2$, we have

$$\mathbb{V}[y] = \frac{x}{a} + x^2 + b^2 - x^2 = \frac{x}{a} + b^2. \tag{4}$$

B. Likelihood Function of Single-Pixel Image

From the definition of the probability mass function (PMF) of a Poisson random variable α, we get

$$\mathbb{P}[\alpha = k] = \frac{e^{-ax}(ax)^k}{k!}, \quad k \geq 0. \tag{5}$$

From the relation between the probability density function (PDF) and the PMF of discrete random variable established with the Dirac delta function, i.e. $f_X(t) = \sum_{k\in\mathbb{Z}} \mathbb{P}[X = k]\delta(t-k)$, we can derive that

$$f_\alpha(t|a,x) = \sum_{k=0}^{\infty} \frac{e^{-ax}(ax)^k}{k!}\delta(t-k). \tag{6}$$

Let us define $\alpha' = \frac{1}{a}\alpha$. Then, the cumulative distribution function (CDF) of this random variable α' has the following form

$$F_{\alpha'}(t) = \mathbb{P}[\alpha' \leq t] = \mathbb{P}[\alpha \leq at] = F_\alpha(at). \tag{7}$$

By taking the derivative of Equation (7), the PDF of α' can be found

$$f_{\alpha'}(t) = \frac{dF_{\alpha'}(t)}{dt} = \frac{dF_\alpha(at)}{dt} = af_\alpha(at). \tag{8}$$

Hence, by combining Equations (6) and (8), the likelihood function of α', which consists of the first part of the noise model, can be derived

$$\begin{aligned} f_{\alpha'}(t|a,x) &= a\sum_{k=0}^{\infty} \frac{e^{-ax}(ax)^k}{k!} \underbrace{\delta(at-k)}_{=\frac{1}{a}\delta(t-\frac{k}{a})} \\ &= \sum_{k=0}^{\infty} \frac{e^{-ax}(ax)^k}{k!}\delta(t-k/a). \end{aligned} \tag{9}$$

On the other hand, the likelihood function of a Gaussian random variable β with 0 mean is defined as

$$f_\beta(t|b) = \frac{1}{b\sqrt{2\pi}}e^{-t^2/2b^2}. \tag{10}$$

We then combine those equations and find the likelihood function of y. Since we know that α' and β are independent of each other, we have that

$$\begin{aligned} \mathcal{L}(y|a,b,x) &= (f_{\alpha'} * f_\beta)(y|a,b,x) \\ &= \sum_{k=0}^{\infty} \frac{(ax)^k}{k!b\sqrt{2\pi}} \exp\left(-ax - \frac{(y-k/a)^2}{2b^2}\right). \end{aligned} \tag{11}$$

C. Maximum Likelihood Solution for Single-Pixel Image

As derived, the maximum likelihood solution for a single-pixel image is the following

$$\begin{aligned} \hat{a}, \hat{b} &= \arg\max_{a,b} \mathcal{L}(y|a,b,x) \\ &= \arg\max_{a,b} \sum_{k=0}^{\infty} \frac{(ax)^k}{k!b\sqrt{2\pi}} \exp\left(-ax - \frac{(y-k/a)^2}{2b^2}\right). \end{aligned} \tag{12}$$

D. Likelihood Function of Multi-Pixel Image

We represent images as vectors of pixels, like y_n and x_n where $n \in \mathbb{N}$ is the index of single pixels. Hence, using this notation we obtain

$$\mathcal{L}(y_n|a,b,x_n) = \sum_{k=0}^{\infty} \frac{(ax_n)^k}{k!b\sqrt{2\pi}} \exp\left(-ax_n - \frac{(y_n-k/a)^2}{2b^2}\right). \tag{13}$$

Given x, i.e., the vector of all x_n, we can see that y_n and $y_{n'}$ are independent $\forall n \neq n'$. Therefore, we have

$$\begin{aligned} \mathcal{L}(y|a,b,x) = \prod_n \sum_{k=0}^{\infty} &\frac{(ax_n)^k}{k!b\sqrt{2\pi}} \\ &\exp\left(-ax_n - \frac{(y_n-k/a)^2}{2b^2}\right). \end{aligned} \tag{14}$$