X-Ray Fokker-Planck Equation

Fundamentals

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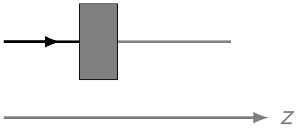
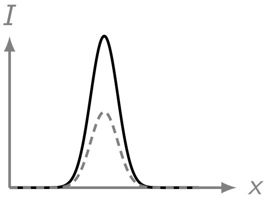
Conventional X-Ray Imaging

[American Society for Surgery of the Hand, 2015]

Conventional X-Ray Imaging

Attenuation of the X-Ray beam





adapted from [Morgan et al., 2019]

[Bech et al., 2013]

X-Ray Imaging: Complementary Modalities

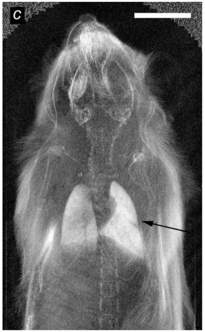
Attenuation Image

[Bech et al., 2013]

Phase-Contrast Image

[Bech et al., 2013]

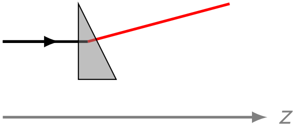
Dark-Field Image

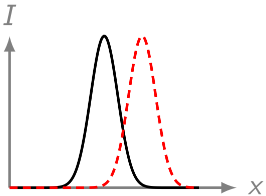


[Bech et al., 2013]

Complementary Modalities

Phase Shift caused by soft tissues



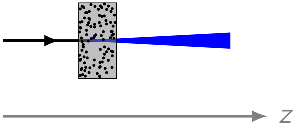


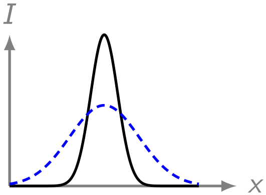
adapted from [Morgan et al., 2019]

[Bech et al., 2013]

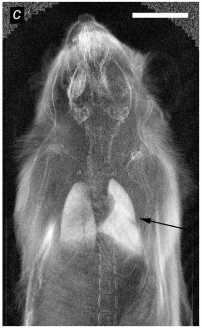
Complementary Modalities

Dark-Field Signal resulting from unresolved, sub-pixel structures





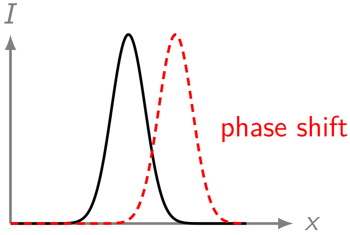
adapted from [Morgan et al., 2019]

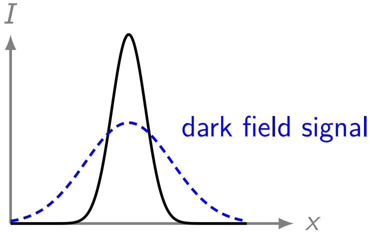


[Bech et al., 2013]

Complementary Modalities

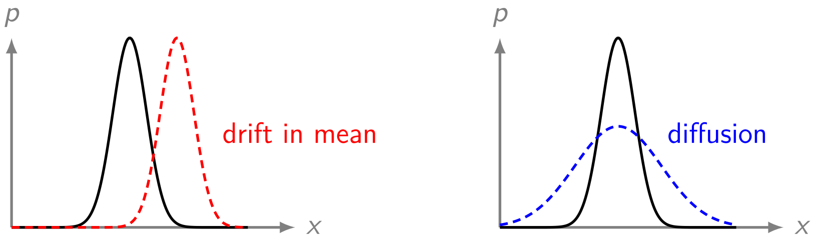
- detector can only measure intensities
- mathematical model required to extract **phase shift** and **dark-field signal**





Complementary Modalities

- integrated intensity conserved post sample
- analogous to a **probability distribution**

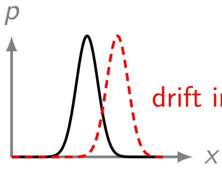


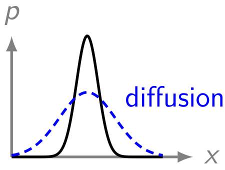
Fokker-Planck Equation

Time-evolution of probability distribution $p(x,t)$:

$$\partial_t p = \boxed{-\partial_x \left[D_1 p\right]} + \boxed{\partial_x^2 \left[D_2 p\right]}$$

- drift velocity $D_1(x,t)$
- diffusion coefficient $D_2(x,t)$





Fokker-Planck Equation

Applications:

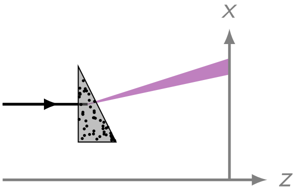
- Brownian motion
- hydrodynamics
- electron and photon transport in biological tissues
- gaseous micro-flows
- droplet nucleation
- **X-ray imaging**

X-Ray Fokker-Planck Equation

Modification from **time-evolution** $\partial_t p$ to **intensity change** $\partial_z I$:

$$\partial_t p = \boxed{-\partial_x \left[D_1 p\right]} + \boxed{\partial_x^2 \left[D_2 p\right]} \qquad \longrightarrow \qquad \partial_z I = \boxed{-\frac{1}{k}\partial_x \left[I \partial_x \phi\right]} + \boxed{\partial_x^2 \left[D(z) I\right]}$$

- t: change over propagation distance z
- p: beam intensity I
- D_1: phase shift $\frac{1}{k}\partial_x \phi$
- D_2: “linear” diffusion coefficient $D(z)$



X-Ray Fokker-Planck Equation

1-dimensional detector signal:

$$\partial_z I = \boxed{-\frac{1}{k}\partial_x \left[I \partial_x \phi\right]} + \boxed{\partial_x^2 \left[D(z) I\right]}$$

2-dimensional detector signal:

$$\partial_z I = \boxed{-\frac{1}{k}\nabla_\perp \left[I \nabla_\perp \phi\right]} + \boxed{\nabla_\perp^2 \left[D(z) I\right]}$$

- Transport-of-intensity equation
- diffusion term

Transport-of-Intensity Equation

Homogeneous **Helmholtz-Equation** for a monochromatic wave ψ_ω:

$$\left(\nabla^2 + k^2\right) \psi_\omega = 0$$

Paraxial Approximation:

$$\psi_\omega = \Phi e^{ikz}$$

- $\Phi\left(x, y, z\right)$: complex envelope with slow z-variation such that $\partial_z^2 \Phi = 0$
- e^{ikz}: oscillatory plane wave behaviour

Paraxial Wave Equation:

$$\left(2ik\partial_z^2 + \nabla_\perp^2\right)\Phi = 0$$

Projection Approximation:

$$\Phi = \sqrt{I}e^{i\phi}$$

- no diffraction / scattering **within** the (thin) sample
- $\sqrt{I(x,y,z)}$: amplitude
- $\phi(x,y,z)$: phase

$$\partial_z I = -\frac{1}{k}\nabla_\perp\left[I\nabla_\perp\phi\right]$$

X-Ray Fokker-Planck Equation

Transport-of-Intensity Equation

$$\partial_z I = \boxed{-\frac{1}{k}\nabla_\perp \left[I \nabla_\perp \phi\right]}$$

- **energy conservation** under *coherent* x-ray energy transport

Diffusion Term

$$\partial_z I = \boxed{\nabla_\perp^2 \left[D(z) I\right]}$$

- **energy conservation** under *diffusive* x-ray energy transport

Constant (Noether) Charge: integrated intensity

$$\mathfrak{N} = \iint I\left(x, y, z\right) \mathrm{d}x \, \mathrm{d}y$$

Transport-of-Intensity Equation

$$\partial_z I = \boxed{-\frac{1}{k}\nabla_\perp \left[I \nabla_\perp \phi\right]}$$

Diffusion Term

$$\partial_z I = \boxed{\nabla_\perp^2 \left[D(z) I\right]}$$

Conserved (Noether) Current:

transverse Poynting vector

$$\mathbf{J}_\perp^{(1)} \equiv \frac{1}{k} I \nabla_\perp \phi$$

Fick's first law of diffusion

$$\mathbf{J}_\perp^{(2)} \equiv -\nabla_\perp \left[D(z) I\right]$$

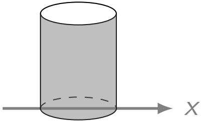
Continuity Equation:

$$\partial_z I = -\nabla_\perp \left(\mathbf{J}_\perp^{(1)} + \mathbf{J}_\perp^{(2)}\right) = -\nabla_\perp \mathbf{J}_\perp$$

Application to Forward Problems

Phase-Contrast Image

$$\partial_z I = \boxed{-\tfrac{1}{k}\partial_x \left[I \partial_x \phi\right]}$$



sample

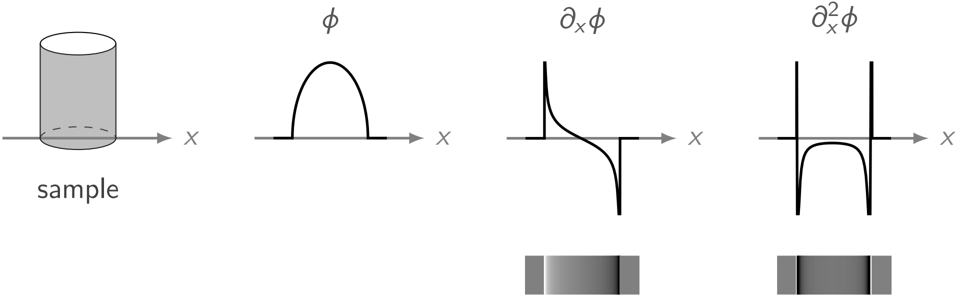
Transport-of-intensity equation: **finite difference**

$$I(z = \Delta) = I(z = 0) - \frac{\Delta}{k}\left(\partial_x I \cdot \partial_x \phi + I \cdot \partial_x^2 \phi\right)$$

- intensity at small distance Δ from sample

Phase-Contrast Image

$$I(z=\Delta) = I(z=0) - \frac{\Delta}{k}\left(\partial_x I \cdot \boxed{\partial_x \phi} + I \cdot \boxed{\partial_x^2 \phi}\right)$$

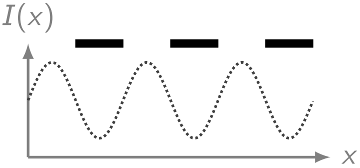


Phase Effects

$$I(z=\Delta) = I(z=0) - \frac{\Delta}{k}\left(\partial_x I \cdot \partial_x \phi + I \cdot \partial_x^2 \phi\right)$$

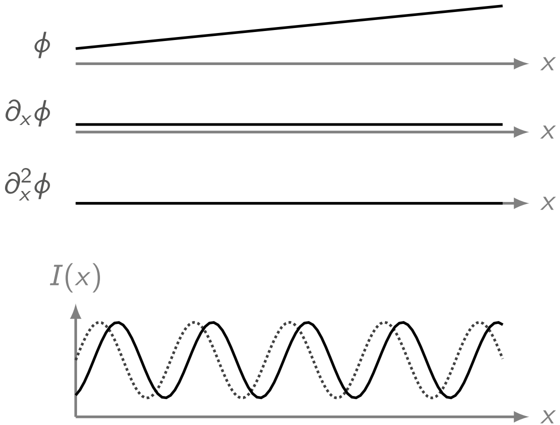
Detector image: **sinusoidal illumination**

$$I(x) = a \sin\left(\frac{x}{p}\right) + b$$



- **prism effects**
- **focusing effects**
- **edge effects**

Prism Effect



$$\partial_x \phi > 0$$

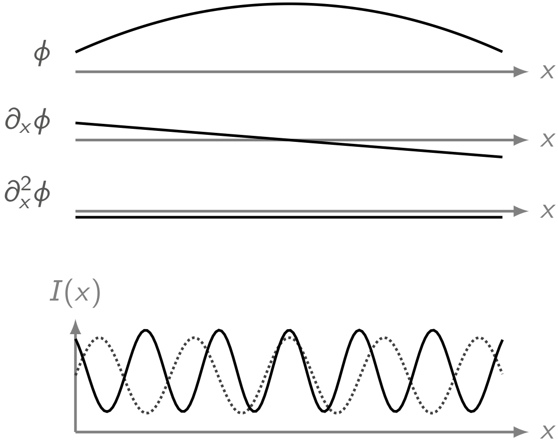
- Shift $\rightarrow$

$$\partial_x^2 = 0$$

- Mean unchanged
- Period unchanged

adapted from [Morgan and Paganin, 2019]

Focusing Effect



$\partial_x \phi > 0$

- Shift $\rightarrow$

$\partial_x \phi < 0$

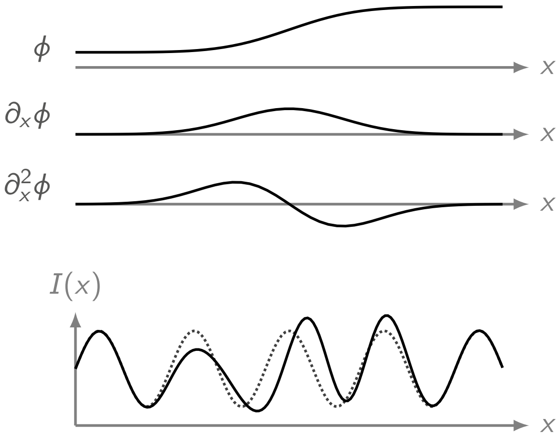
- Shift $\leftarrow$

$\partial_x^2 < 0$

- Mean $\uparrow$
- Period $\downarrow$

adapted from [Morgan and Paganin, 2019]

Edge Effect



$\partial_x < 0$

- Shift $\rightarrow$

$\partial_x^2 \phi > 0$

- Mean $\downarrow$
- Period $\uparrow$

$\partial_x^2 \phi < 0$

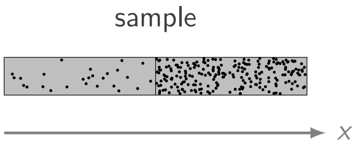
- Mean $\uparrow$
- Period $\downarrow$

adapted from [Morgan and Paganin, 2019]

Dark-Field Effects

Scattering Edge

$$\partial_z I = \boxed{\partial_x^2 \left[D(z)I\right]}$$

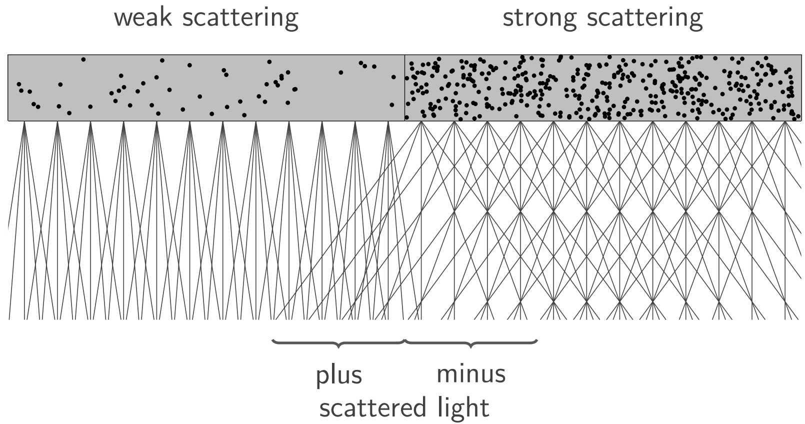


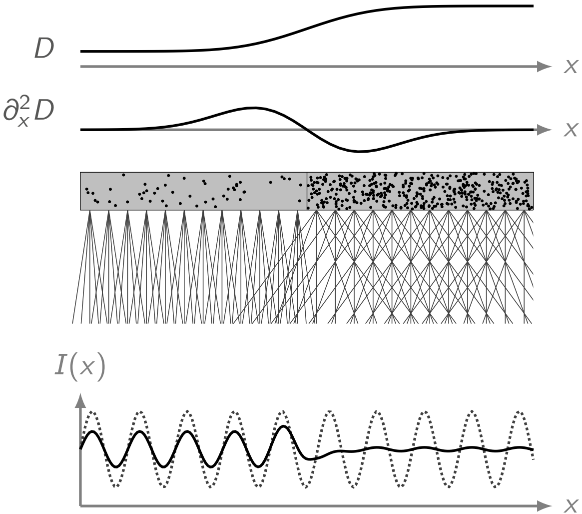
Diffusion equation: **finite difference**

$$I(z=\Delta) = I(z=0) + \Delta\left(\partial_x^2 \left[D(z)I\right]\right)$$

- intensity at small distance Δ from sample

Scattering Edge





$D > 0$

- Visibility $\downarrow$

$\partial_x^2 = 0$

- Mean unchanged

$\partial_x^2 > 0$

- Mean $\uparrow$

$\partial_x^2 < 0$

- Mean $\downarrow$

adapted from [Morgan and Paganin, 2019]

Thank you for your attention!

References I

American Society for Surgery of the Hand (2015).

Bech, M., Tapfer, A., Velroyen, A., Yaroshenko, A., Pauwels, B., Hostens, J., Bruyndonckx, P., Sasov, A., and Pfeiffer, F. (2013).
In-vivo dark-field and phase-contrast x-ray imaging.
Scientific Reports, 3(1):3209.

References II

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In Muller, B. and Wang, G., editors, *Proceedings of SPIE*, volume 11113 of *Proceedings of SPIE - The International Society for Optical Engineering*, United States of America. SPIE - International Society for Optical Engineering. Developments in X-ray Tomography 2019 ; Conference date: 11-08-2019 Through 15-08-2019.

References III

Morgan, K. S. and Paganin, D. M. (2019).
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