Introduction to Calculus, Math 1100, Fall 2012, Bob Palais

Solutions for Exam 1, Chapters 1 and 2

Limits and Derivatives

Show all your work on the exam for full credit.

1. (20 points) Find the equation of the line l that is perpendicular to the segment joining the points $A = (3, 0)$ and $B = (0, 4)$ and that contains the midpoint M of the segment. (In other words, the line l is the perpendicular bisector of the segment $\overline{AB}$.)

The midpoint M of $\overline{AB}$ is

$$M = (\frac{3+0}{2}, \frac{0+4}{2}) = (\frac{3}{2}, 2).$$

The equation of any line containing M is

$$y - 2 = m(x - \frac{3}{2})$$

where m is the slope of the line. The slope of the line containing A and B is

$$m_{\overline{AB}} = \frac{4-0}{0-3} = -\frac{4}{3}.$$

The slope of any line perpendicular to line is the negative reciprocal of $m_{\overline{AB}}$,

$$m_{\overline{AB}^{\perp}} = \frac{3}{4}.$$

So the equation of the line l is

$$y - 2 = \frac{3}{4}(x - \frac{3}{2}).$$

Bonus) (5 points) Find the distance from the point A to the line l. (Hint: There is an easier way and a harder way to find that distance.)

The distance from A to l means the distance from A to the closest point on l to A. This closest point occurs at a point P where the line AP is perpendicular to the line l. In this case, $P = M$. In other words, the distance from either endpoint of a segment $\overline{AB}$ to the perpendicular bisector of the segment is the same as the distance from the endpoints to their midpoint.

So either compute the distance between A and M, or realize that it will be half the distance from A to B, the hypotenuse of a $3-4-5$ triangle. Either way, the answer is

$$\frac{5}{2} = \sqrt{(3 - \frac{3}{2})^2 + (0-2)^2}.$$

2. (20 points) Find the derivative of each of the following functions:

a) (10 points) Use the form provided by filling in the blanks.

$$f(x) = 9 + 8x - 7\frac{x^2}{2} + 5\frac{x^3}{6}$$

As a consequence of the power rule and the addition and scalar multiplication rules, differentiation moves the coefficient of $\frac{x^n}{n!}$ to the coefficient of $\frac{x^{n-1}}{(n-1)!}$ and antidifferentiation moves it to the coefficient of $\frac{x^{n+1}}{(n+1)!}$

$$f'(x) = 8 - 7x + 5\frac{x^2}{2} + 0\frac{x^3}{6}$$

b) (10 points)

$$g(t) = (1-t)(1+t+t^2)$$

$$g(t) = (1-t)(1+t+t^2) = 1 - t^3.$$

This is the factorization we have seen for geometric series. Distribute either way to check:

$$g(t) = (1-t)(1+t+t^2) = (1-t) + (t-t^2) + (t^2-t^3) = 1 - t^3$$

or

$$g(t) = (1-t)(1+t+t^2) = (1+t+t^2) - (t+t^2+t^3) = 1 - t^3.$$

Either way,

$$g'(t) = -3t^2.$$

By the product rule instead,

$$g'(t) = (-1)(1+t+t^2) + (1-t)(1+2t) = -1 - t - t^2 + 1 + t - 2t^2 = -3t^2.$$

Bonus. (5 points) Find the general form of an antiderivative of $f(x)$ and $g(x)$ from a) and b):

$$\int f(x)\ dx = \underline{\quad\quad} + \underline{\quad\quad}\ x + \underline{\quad\quad}\ \frac{x^2}{2} + \underline{\quad\quad}\ \frac{x^3}{6} + \underline{\quad\quad}\ \frac{x^4}{24}$$

Using the simplified form above,

$$\int g(x)\ dt = t - \frac{t^4}{4} + C.$$