最短路径算法

6.图

优先级搜索

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通用算法

- 各种遍历算法的区别，仅在于选取顶点进行访问的次序

  广度/深度：优先访问与更早/更晚被发现的顶点相邻接者

  ...

- 不同的遍历算法，取决于顶点的选取策略

- 不同的顶点选取策略，取决于存放和提供顶点的数据结构——Bag

- 此类结构，为每个顶点v维护一个优先级数 priority(v)

  每个顶点都有初始优先级数；并可能随算法的推进而调整

- 通常的习惯是，优先级数越大/小，优先级越低/高

  特别地，priority(v) == INT_MAX，意味着v的优先级最低

统一框架

```
template <typename Tv, typename Te> //顶点类型、边类型
template <typename PU> //优先级更新器（函数对象）
void Graph<Tv, Te>::pfs( int s, PU prioUpdater ) { //PU的策略，因算法而异
   priority(s) = 0; status(s) = VISITED; parent(s) = -1; //起点s加至PFS树中
   while (1) { //将下一顶点和边加至PFS树中
      /* ... 依次引入n - 1个顶点（和n - 1条边） ... */
   } //while
} //如何推广至非连通图?
```

统一框架

```
while (1) { //依次引入n - 1个顶点（和n - 1条边）
   for ( int w = firstNbr(s); -1 < w; w = nextNbr(s, w) ) //对s各邻居w
      prioUpdater( this, s, w ); //更新顶点w的优先级及其父顶点
   for ( int shortest = INT_MAX, w = 0; w < n; w++ )
      if ( UNDISCOVERED == status(w) ) //从尚未加入遍历树的顶点中
         if ( shortest > priority(w) ) //选出下一个
            { shortest = priority(w); s = w; } //优先级最高的顶点s
   if ( VISITED == status(s) ) break; //直至所有顶点均已加入
   status(s) = VISITED; type( parent(s), s ) = TREE; //将s加入遍历树
} //while
```

复杂度

- 执行时间主要消耗于内、外两重循环；其中两个内循环前、后并列
- 前一内循环的累计执行时间：　若采用邻接矩阵，为$O(n^2)$；若采用邻接表，为$O(n + e)$

  后一循环中，优先级更新的次数呈算术级数变化{ n, n - 1, ..., 2, 1 }，累计为$O(n^2)$

  两项合计，为$O(n^2)$
- 后面将会看到：若采用优先队列，以上两项将分别是$O(elogn)$和$O(nlogn)$

  两项合计，为$O((e + n) * logn)$
- 这是很大的改进——尽管对于稠密图而言，反而是倒退　//已有接近于$O(e+nlogn)$的算法
- 基于这个统一框架，如何解决具体的应用问题...

6.图

最小支撑树

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最小 + 支撑 + 树

❖ 连通网络N = (V; E)的子图T = (V; F)

1）支撑/spanning

覆盖N中所有顶点

2）树/tree

连通且无环，|V| = |F| + 1

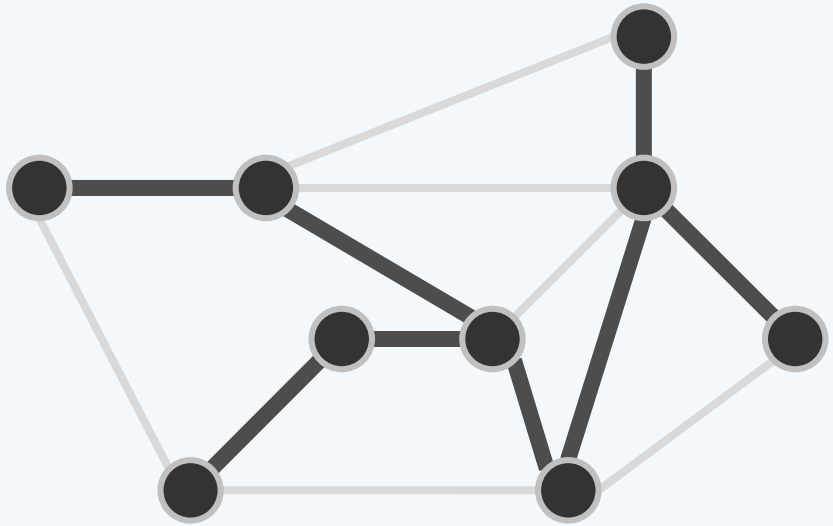
加边出单环，再删同环边即恢复为树

删边不连通，再加联接边即恢复为树

不难验证，同一网络的支撑树不唯一

3）最小/minimum

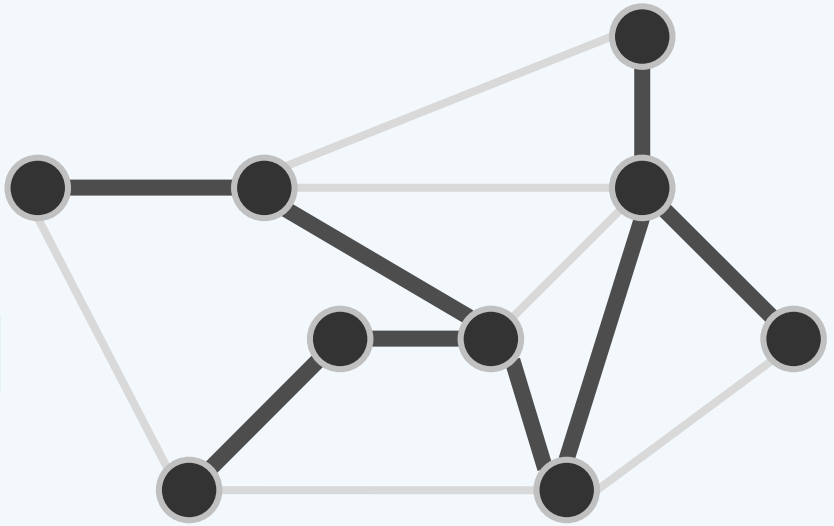
各边总权重wt(T) = $\sum_{e \in F}$ wt(e)达到最小



MST

❖ 谁感兴趣?

电信公司、网络设计师、VLSI布线算法设计师、...



❖ 为何重要?

应用中常见的共性问题，也是很多优化问题的基本模型

自身可有效计算

为许多NP问题提供足够好的近似解 //比如，Euclidean TSP

❖ 算法

Boruvka-1926, Jarnik-1930/Prim-1956, Kruskal-1956, ...

Karger-Klein-Tarjan-1995, Chazelle-2000

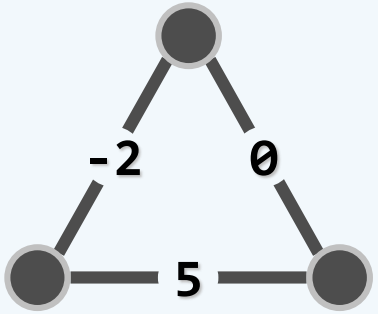
...是否存在$O(n + e)$算法?

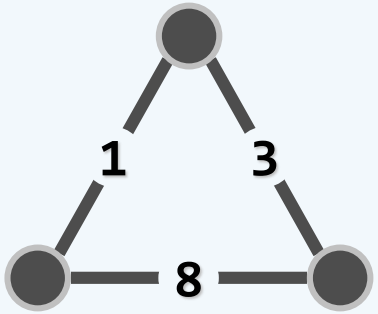
退化

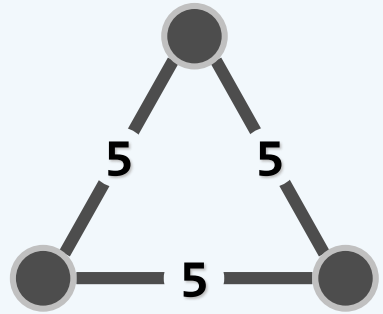
- 权值必须是正数？

  允许为零，会有什么影响？

  允许为负数呢？







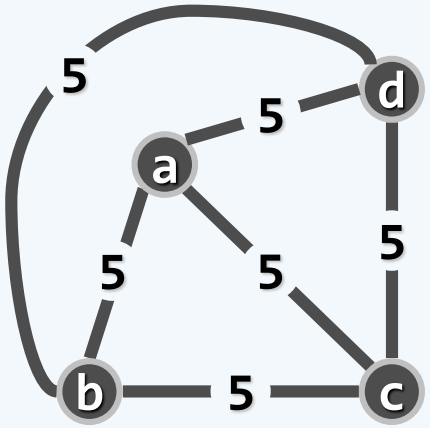
- 所有支撑树所含的边数，必然相等

  故可统一调整：increase(1 - findMin())

- The minimum?

  A minimal　　同一网络N可能有多棵MST

  The minimum!　可强制消除歧义...

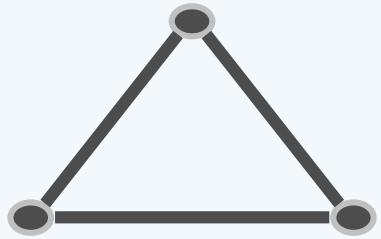


- 合成数 (composite number) : (w(u, v), min(u, v), max(u, v))

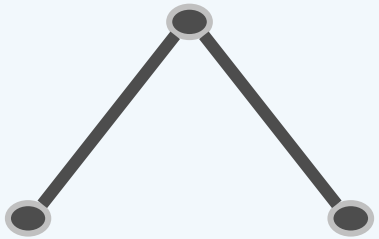
  5ab < 5ac < 5ad < 5bc < 5bd < 5cd

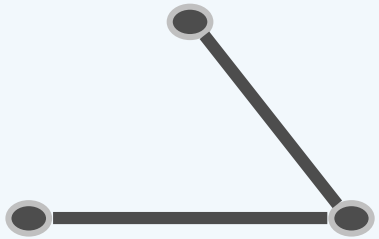
蛮力算法

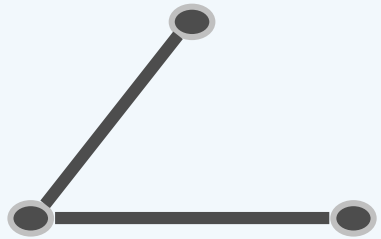
- 枚举出N的所有支撑树，从中找出代价最小者
- 这一策略是否可行，取决于...
- n个互异顶点组成的图，可能有多少棵支撑树?



n = 1	1
n = 2	1
n = 3	3
n = 4	16
...	...







- Cayley公式：联接n个互异顶点的树共有n^{n-2}棵；或等价地，完全图K_n有n^{n-2}棵支撑树
- 如何高效地构造MST呢？后续算法部分讲述

6.图

Dijkstra算法

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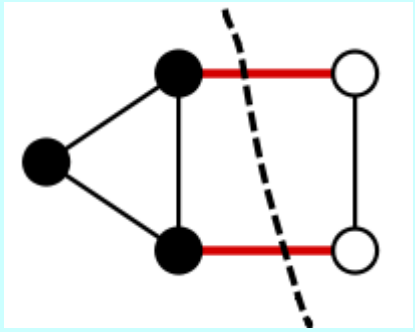
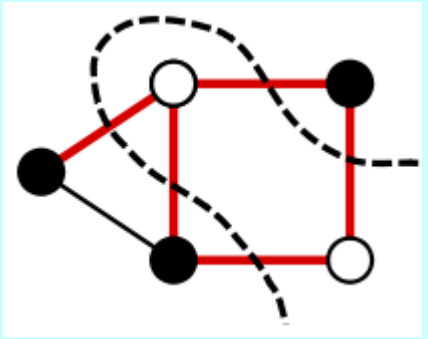
预备知识

- In graph theory, a cut （割） is a partition of the vertices of a graph into two disjoint subsets.
- Any cut determines a cut-set, the set of edges that have one endpoint in each subset of the partition. These edges are said to cross the cut.
- 图G=(V,E)中，V的任一非平凡子集U及其补集V\U都构成G的一个割。记作（U: V\U）。若边(u,v)满足u∈ U，v∈ V\U，则称跨越边(crossing edge)。

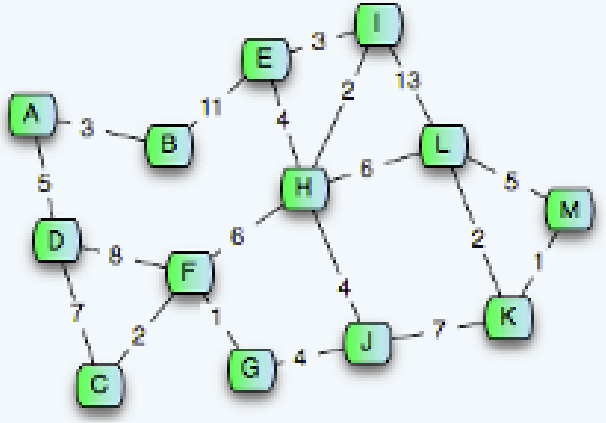


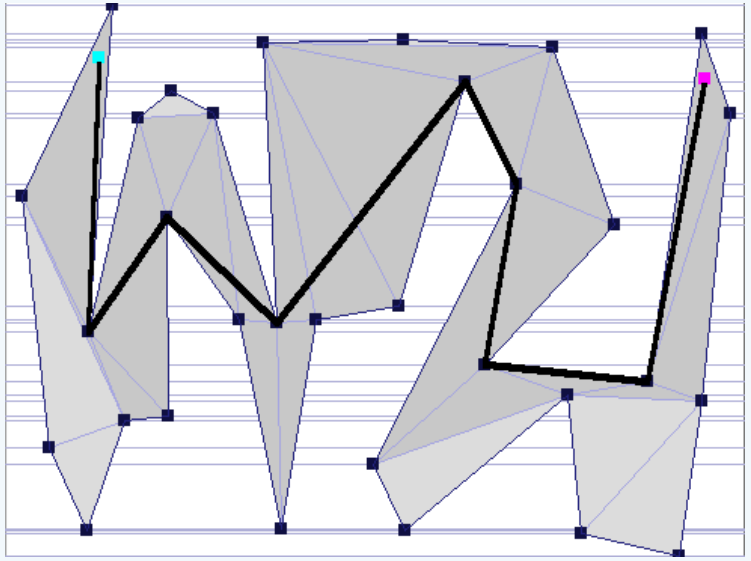


问题描述

- 给定：连通有向图G及其中的顶点u和v
  找到：从u到v的最短路径及其长度
- 旅游者：最经济的出行路线
  路由器：最快地将数据包传送到目标位置
  路径规划：多边形区域内的自主机器人

……



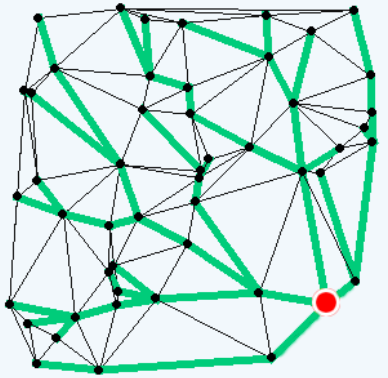


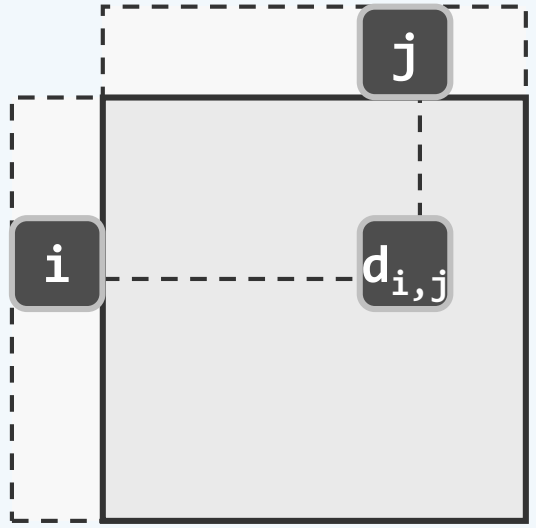
问题分类

- 按照图的类型
  - 无权图/等权图：BFS
  - 带权有向图 //负权值呢?
- 单源点到各顶点的最短路径

  Single-source shortest paths
  - 给定顶点x，计算x到其余各个顶点的最短路径及长度
  - E. Dijkstra, 1959
- 所有顶点对之间的最短路径

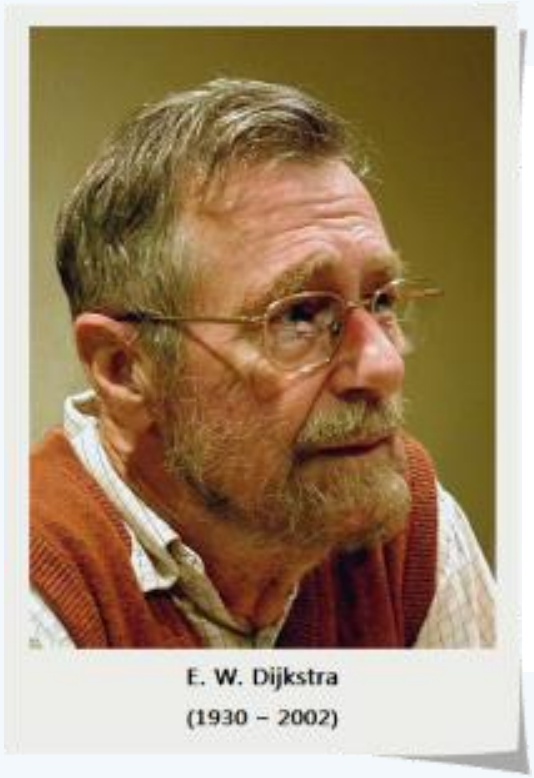
  All shortest paths
  - 找出每对顶点i和j之间的最短路径及长度
  - Floyd-Warshall, 1962



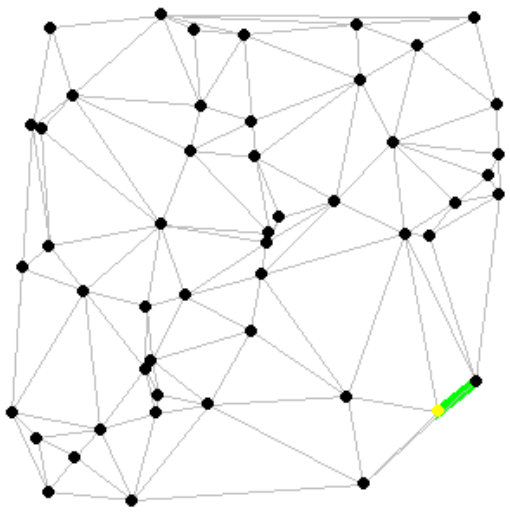


E. W. Dijkstra

❖ Turing Award, 1972



E. W. Dijkstra
(1930 – 2002)



Numerische Mathematik 1, 269—271 (1959)

A Note on Two Problems in Connexion with Graphs

By

E. W. DIJKSTRA

We consider n points (nodes), some or all pairs of which are connected by a branch; the length of each branch is given. We restrict ourselves to the case where at least one path exists between any two nodes. We now consider two problems.

Problem 1. Construct the tree of minimum total length between the n nodes. (A tree is a graph with one and only one path between every two nodes.)

In the course of the construction that we present here, the branches are subdivided into three sets:

I. the branches definitely assigned to the tree under construction (they will form a subtree);

II. the branches from which the next branch to be added to set I, will be selected;

III. the remaining branches (rejected or not yet considered).

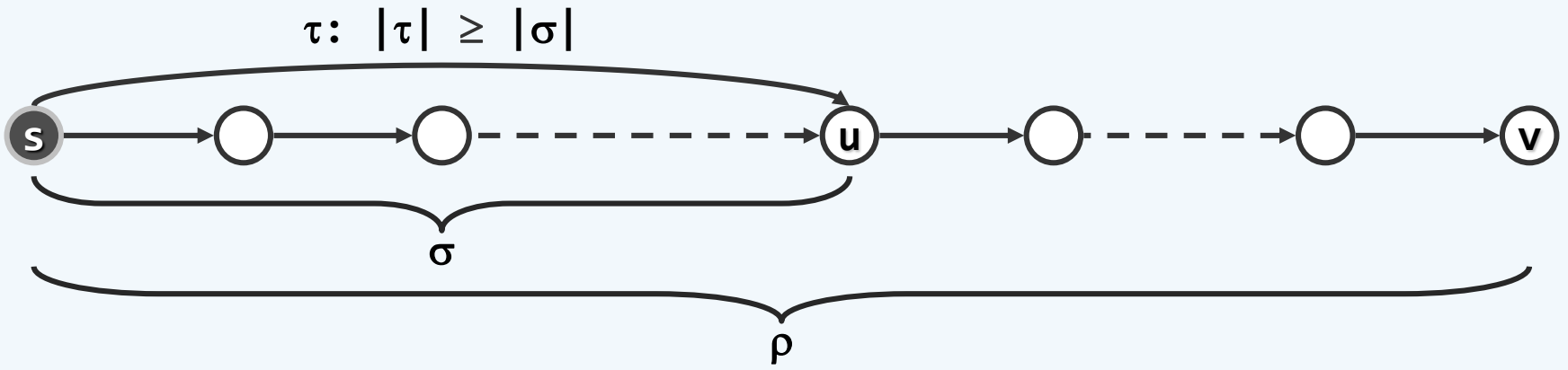
SPT

1)连通图中，s到每个顶点都有（至少）一条最短路径

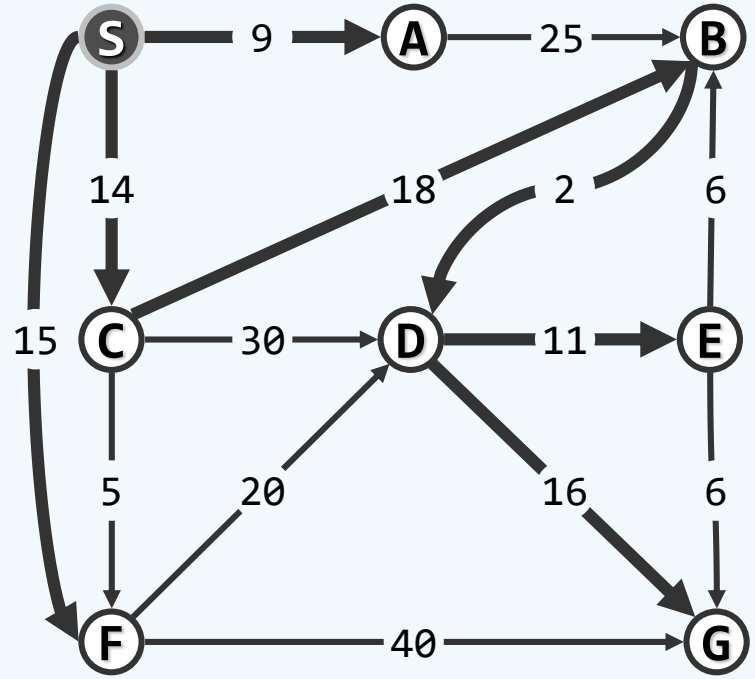
//非退化假设：每个顶点对应的最短路径唯一

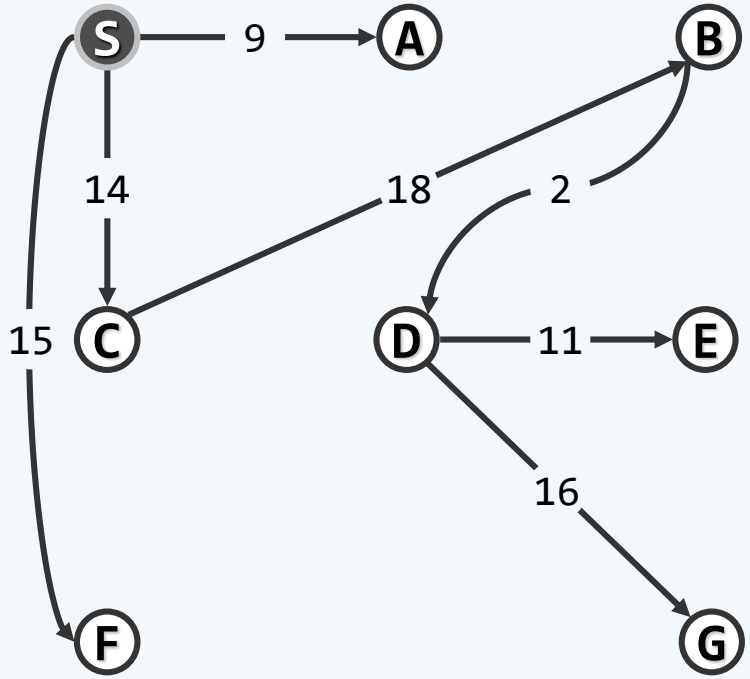
2)就同一起点s而言，任何最短路径的前缀，也是一条最短路径

3)就同一起点s而言，所有最短路径的并，不含回路

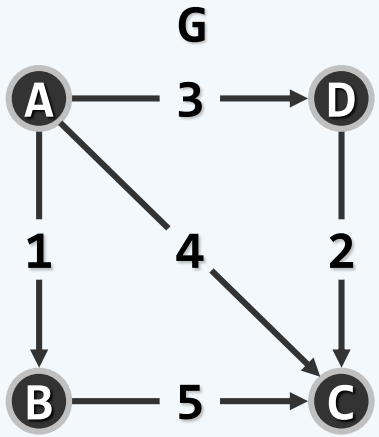


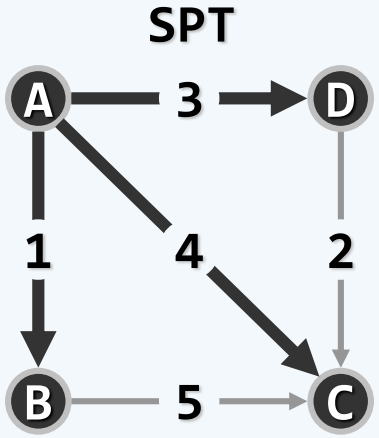
❖ 因此，所有最短路径的并，构成一棵树 (shortest path tree)

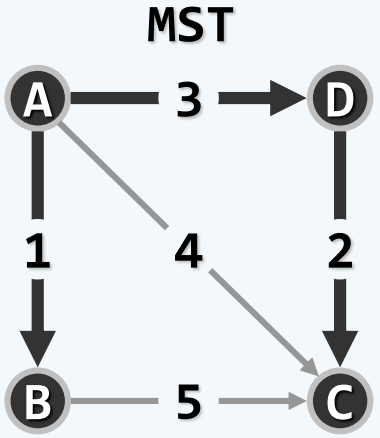




SPT ≠ MST







u_1

- 按照到s的最短距离，对其余的顶点排序

  $dist(s, u_1) \le dist(s, u_2) \le ... \le dist(s, u_{n-1})$

- 最短距离最短者u_1 = ?

- 沿任一最短路径，各顶点到s的最短距离单调变化

- u_1必与s直接相联

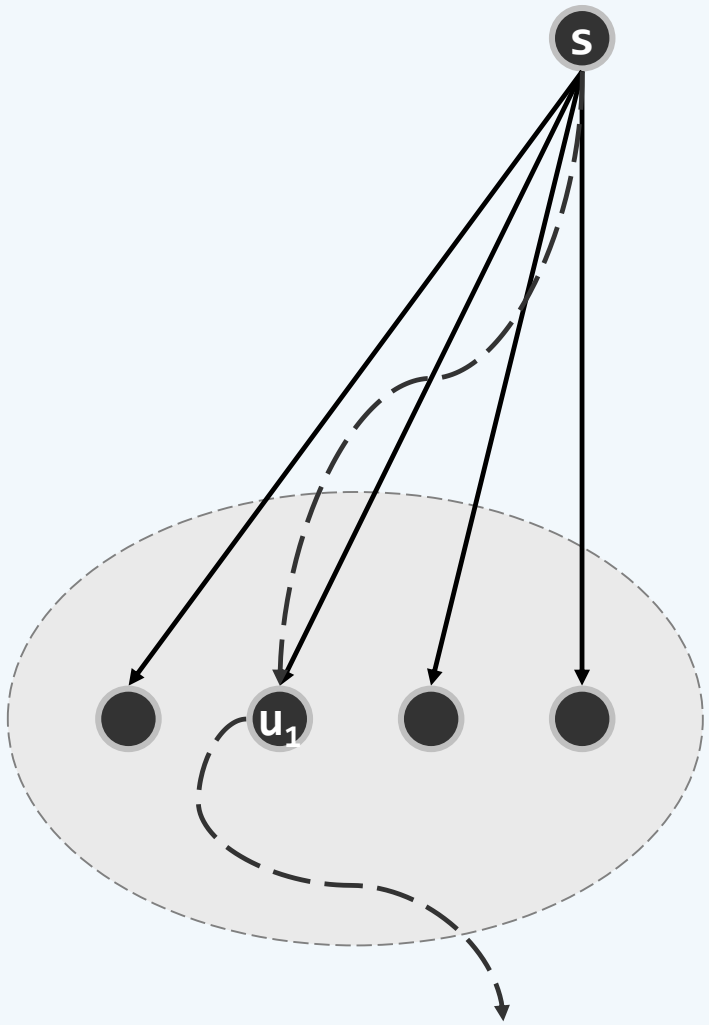
  $dist(s, s_1) = w(s, s_1) < \infty$

- $\forall u \ne s$,

  $w(s, u) < \infty$ 仅当 $w(s, u_1) \le w(s, u)$

- 为找到u_1，只需

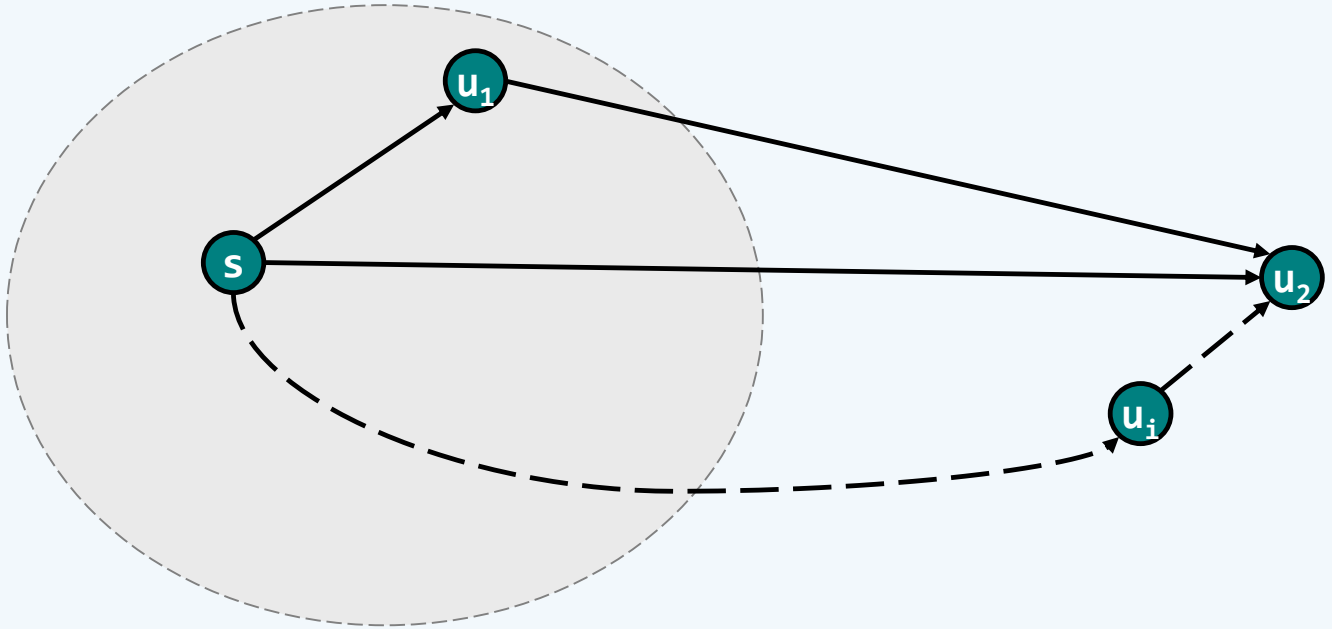
  在与s关联的各顶点中，找到对应边权值最小者



u_2

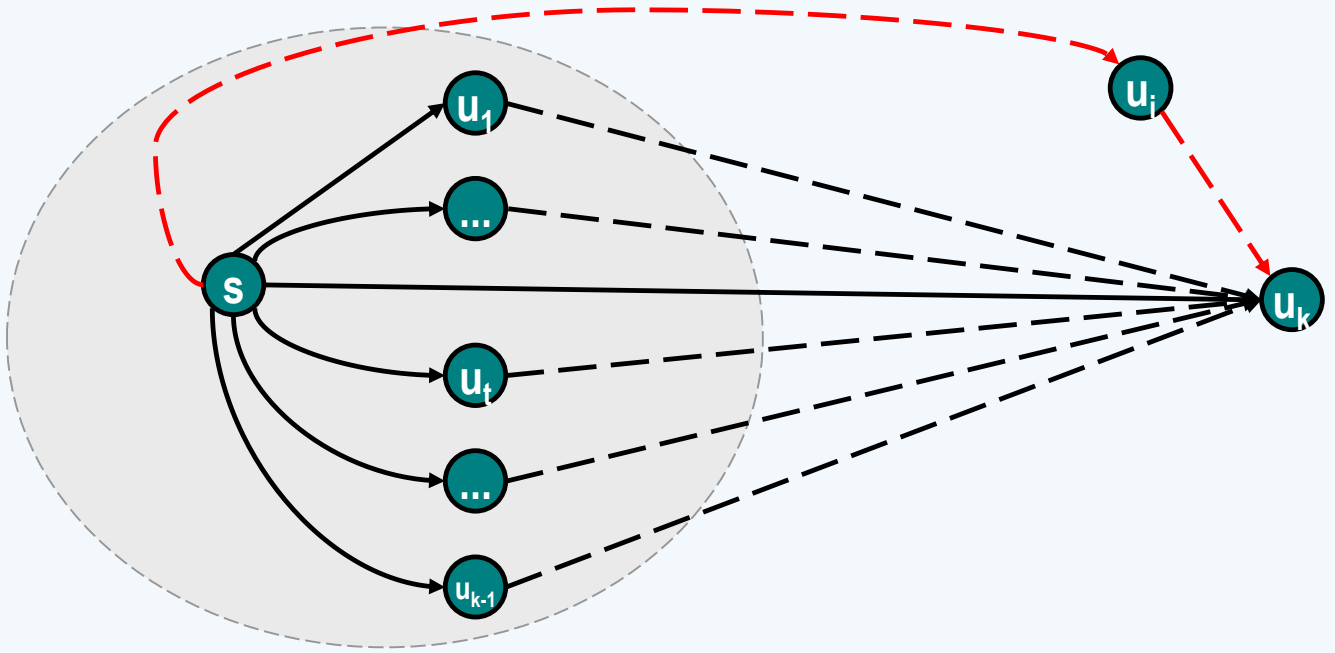
❖ 最短距离次小的顶点u_2 = ?

❖ dist(s, u_2) = min{ w(s, u_2), dist(s, u_1) + w(u_1, u_2) }



u_k

- u_3 = ?, u_4 = ?, ..., u_k = ?
- 三角不等式: dist(s, v) <= dist(s, u) + w(u, v)
- 若记u_0 = s, 则有: dist(s, u_k) = min{ dist(s, u_i) + w(u_i, u_k) | 0 ≤ i < k }
- 算法?



算法

- 从$T_1 = (\{v_1\}; \varnothing)$开始，逐步构造$T_2$、$T_3$、...、$T_n$，其中

  $v_1 = s$

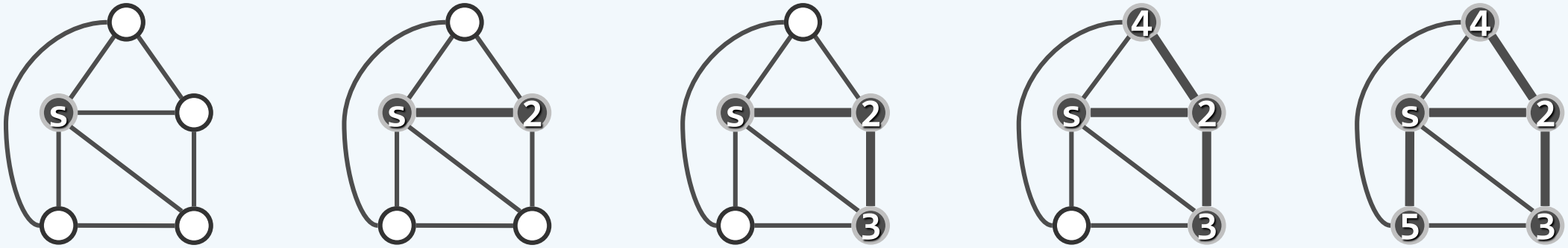
  $T_k = (V_k; E_k)$，$|V_k| = k$，$|E_k| = k-1$，$V_k \subset V_{k+1}$

- 由以上分析，为由T_k构造T_{k+1}，只需

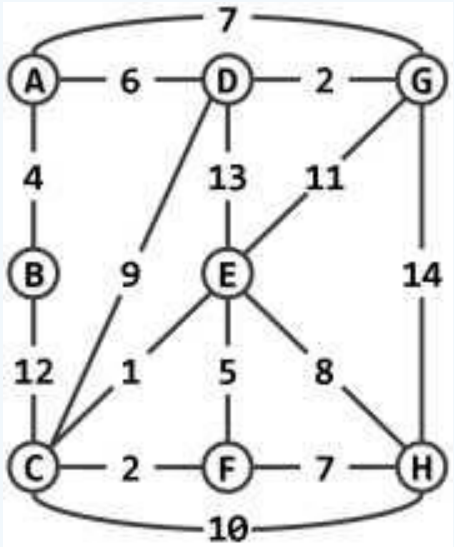
  将$(V_k : V \backslash V_k)$视作原图的一个割

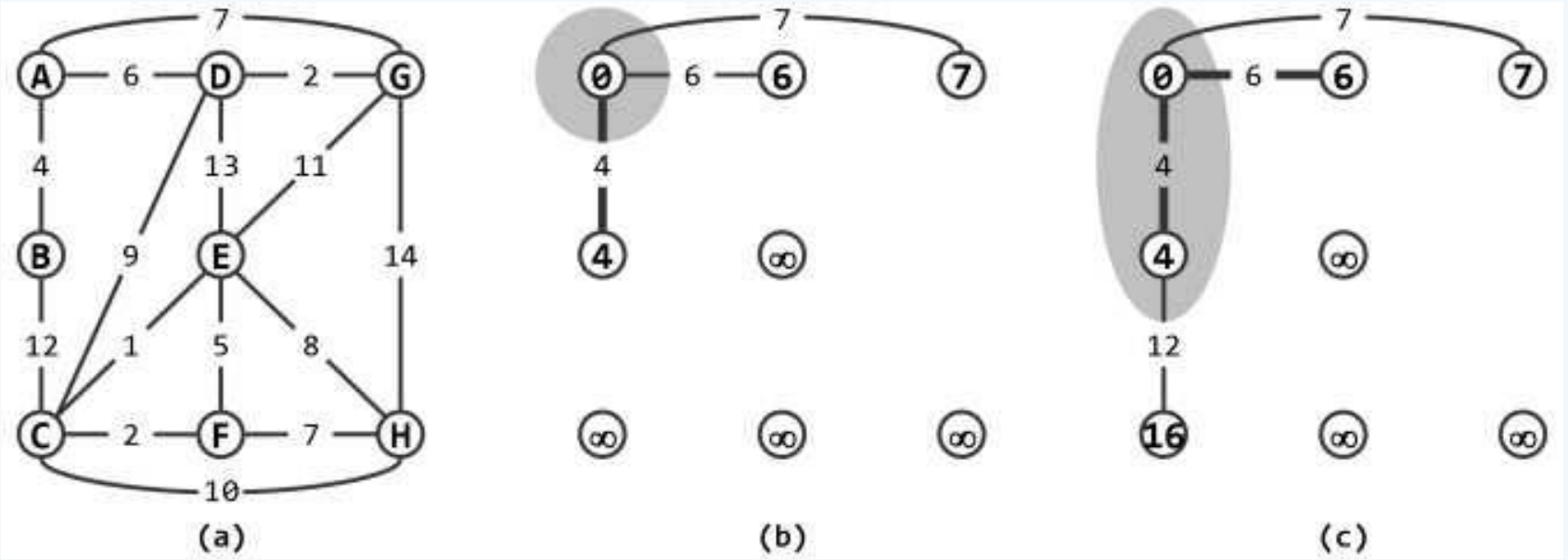
  在该割的所有**跨边**中，找出**极近者**$e_k = (v_k, u_k)$（u_k到s距离**极近**）

  令$T_{k+1} = (V_{k+1}; E_{k+1}) = (V_k \cup \{u_k\}; E_k \cup \{e_k\})$

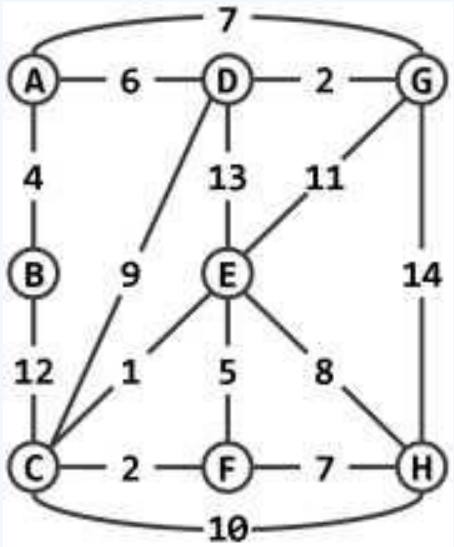


实例





实例


A
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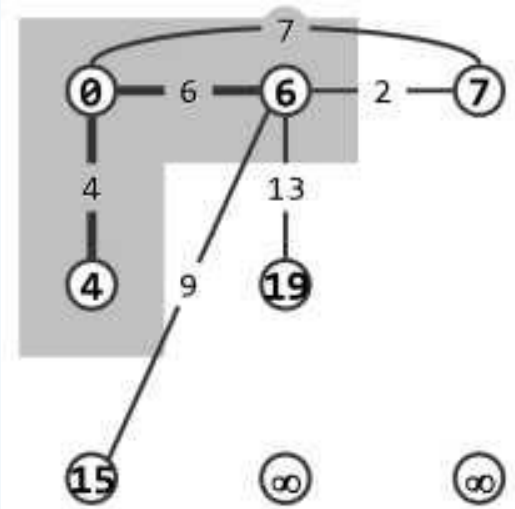
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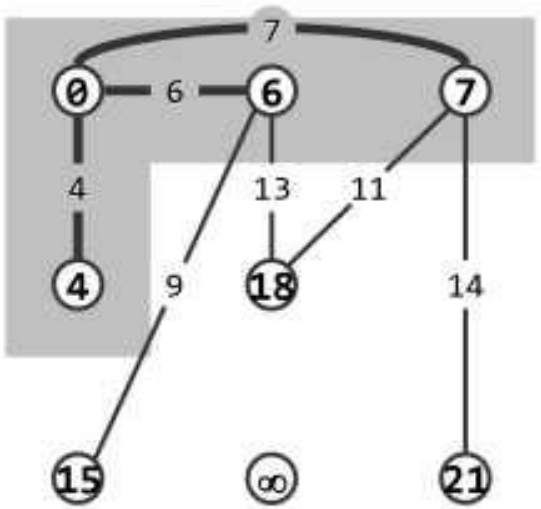
(d)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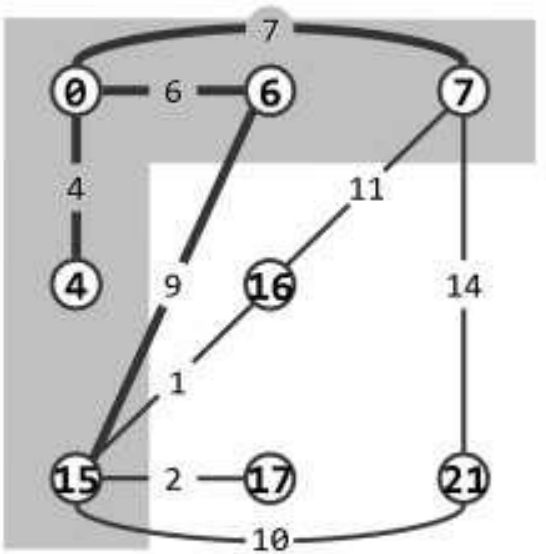
(e)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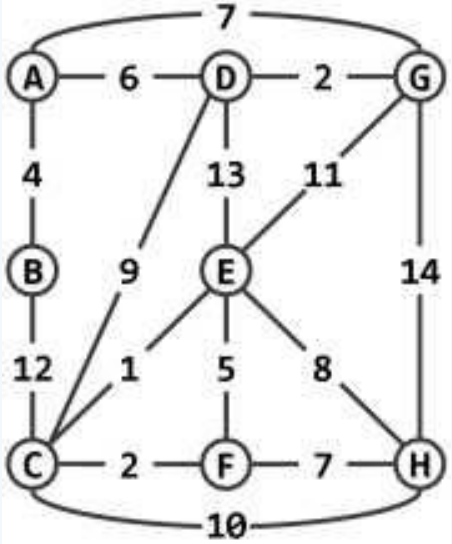
(f)

实例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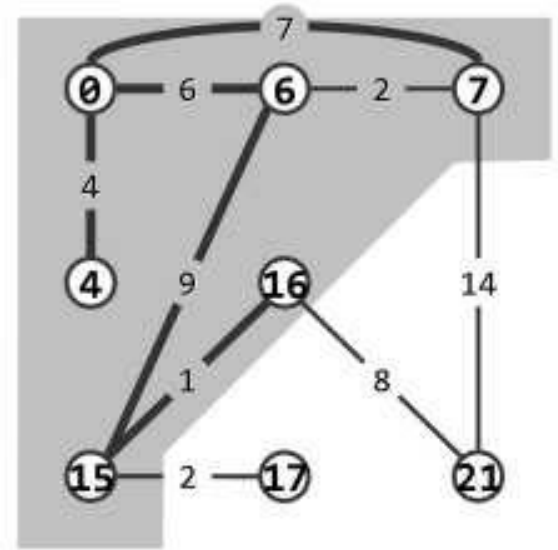
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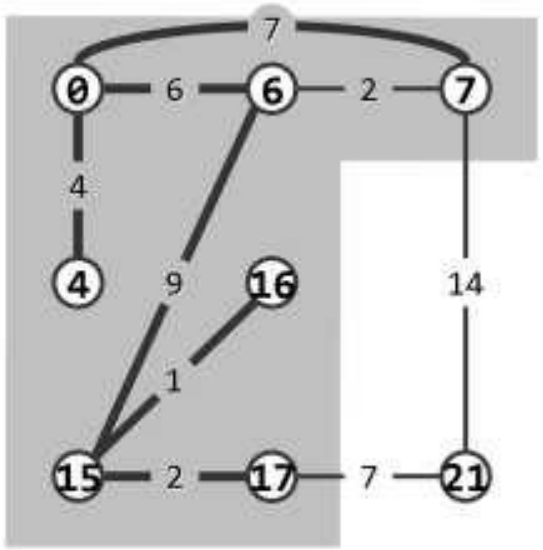
(g)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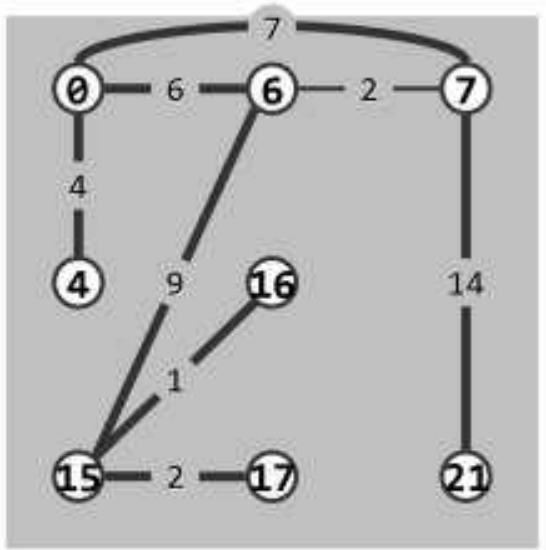
(h)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(i)

实现

- 对于V_k外各顶点v，令priority(v) = v到s的距离

  于是套用优先级遍历算法框架...

- 为将T_k扩充至T_{k+1}，可以

  选出优先级最高的跨边e_k及其对应顶点u_k，并将其加入T_k

  随后，更新V_{k+1}外所有顶点的优先级（数）

- 注意，优先级数随后可能改变（降低）的顶点，必与u_k邻接

- 因此，只需枚举u_k的每一邻接顶点v，并取

  $$priority(v) = \min(priority(v), priority(u_k) + |u_k, v|)$$

- 为此，需按照prioUpdater()模式，编写一个优先级（数）更新器...

实现

❖ g->pfs(0, DijkstraPU<char, int>()); //从顶点0出发，启动Dijkstra算法

❖ template <typename Tv, typename Te> //顶点类型、边类型

```
struct DijkstraPU { //Dijkstra算法的顶点优先级更新器
   virtual void operator()( Graph<Tv, Te>* g, int uk, int v ) {
      if ( UNDISCOVERED != g->status(v) ) return;
   // 对uk的每个尚未被发现的邻居v，按Dijkstra策略做松弛
      if ( g->priority(v) > g->priority(uk) + g->weight(uk, v) ) {
         g->priority(v) = g->priority(uk) + g->weight(uk, v);
         g->parent(v)   = uk;
      }
   }
};
```