Hidden Markov Model and Viterbi Algorithm

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Many slides adapted from Michael Collins

Overview

- The Tagging Problem
- Generative models, and the noisy-channel model, for supervised learning
- Hidden Markov Model (HMM) taggers
  - Basic definitions
  - Parameter estimation
  - The Viterbi algorithm

Hidden Markov Models

- We have an input sentence $x = x_1, x_2, \ldots, x_n$ (x_i is the i'th word in the sentence)
- We have a tag sequence $y = y_1, y_2, \ldots, y_n$ (y_i is the i'th tag in the sentence)
- We'll use an HMM to define

$$p(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$

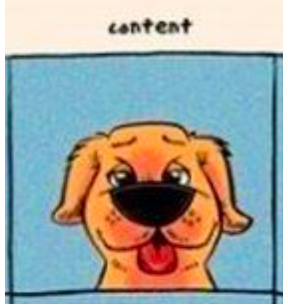
  for any sentence $x_1 \ldots x_n$ and tag sequence $y_1 \ldots y_n$ of the same length.
- Then the most likely tag sequence for x is

$$\arg\max_{y_1 \cdots y_n} p(x_1 \ldots x_n, y_1, y_2, \ldots, y_n)$$

Trigram Hidden Markov Models (Trigram HMMs)

For any sentence $x_1 \ldots x_n$ where $x_i \in \mathcal{V}$ for $i = 1 \ldots n$, and any tag sequence $y_1 \ldots y_{n+1}$ where $y_i \in \mathcal{S}$ for $i = 1 \ldots n$, and $y_{n+1} =$ STOP, the joint probability of the sentence and tag sequence is

$$p(x_1 \ldots x_n, y_1 \ldots y_{n+1}) = \prod_{i=1}^{n+1} q(y_i | y_{i-2}, y_{i-1}) \prod_{i=1}^{n} e(x_i | y_i)$$



where we have assumed that $x_0 = x_{-1} = *$.

Parameters of the model:

- $q(s|u, v)$ for any $s \in \mathcal{S} \cup \{\text{STOP}\}$, $u, v \in \mathcal{S} \cup \{*\}$

Trigram parameters

- $e(x|s)$ for any $s \in \mathcal{S}$, $x \in \mathcal{V}$

Emission parameters

An Example

If we have $n = 3$, $x_1 \ldots x_3$ equal to the sentence *the dog laughs*, and $y_1 \ldots y_4$ equal to the tag sequence $\texttt{D N V STOP}$, then

$$
\begin{aligned}
& p(x_1 \ldots x_n, y_1 \ldots y_{n+1}) \\
= \; & q(\texttt{D}|*, *) \times q(\texttt{N}|*, \texttt{D}) \times q(\texttt{V}|\texttt{D}, \texttt{N}) \times q(\texttt{STOP}|\texttt{N}, \texttt{V}) \\
& \times e(\textit{the}|\texttt{D}) \times e(\textit{dog}|\texttt{N}) \times e(\textit{laughs}|\texttt{V})
\end{aligned}
$$

- STOP is a special tag that terminates the sequence
- We take $y_0 = y_{-1} = $ *, where * is a special "padding" symbol

Why the Name?

$$p(x_1 \ldots x_n, y_1 \ldots y_n) \quad = \quad \underbrace{q(\textsf{STOP}|y_{n-1}, y_n) \prod_{j=1}^{n} q(y_j \mid y_{j-2}, y_{j-1})}_{\text{Markov Chain}}$$

$$\times \quad \underbrace{\prod_{j=1}^{n} e(x_j \mid y_j)}_{x_j\text{'s are observed}}$$

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Smoothed Estimation

$$
\begin{aligned}
q(\text{Vt} \mid \text{DT, JJ}) &= \lambda_1 \times \frac{\text{Count}(\text{Dt, JJ, Vt})}{\text{Count}(\text{Dt, JJ})} \\
&\quad + \lambda_2 \times \frac{\text{Count}(\text{JJ, Vt})}{\text{Count}(\text{JJ})} \\
&\quad + \lambda_3 \times \frac{\text{Count}(\text{Vt})}{\text{Count}()}
\end{aligned}
$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \text{and for all } i,\ \lambda_i \geq 0$$

$$e(\text{base} \mid \text{Vt}) = \frac{\text{Count}(\text{Vt, base})}{\text{Count}(\text{Vt})}$$

Dealing with Low-Frequency Words: An Example

Profits soared at Boeing Co. , easily topping forecasts on Wall Street , as their CEO Alan Mulally announced first quarter results .

Dealing with Low-Frequency Words

A common method is as follows:

- **Step 1**: Split vocabulary into two sets

  Frequent words $=$ words occurring $\geq$ 5 times in training
  Low frequency words $=$ all other words

- **Step 2**: Map low frequency words into a small, finite set, depending on prefixes, suffixes etc.

Dealing with Low-Frequency Words: An Example

[Bikel et. al 1999] **(named-entity recognition)**

Word class	Example	Intuition
twoDigitNum	90	Two digit year
fourDigitNum	1990	Four digit year
containsDigitAndAlpha	A8956-67	Product code
containsDigitAndDash	09-96	Date
containsDigitAndSlash	11/9/89	Date
containsDigitAndComma	23,000.00	Monetary amount
containsDigitAndPeriod	1.00	Monetary amount, percentage
othernum	456789	Other number
allCaps	BBN	Organization
capPeriod	M.	Person name initial
firstWord	first word of sentence	no useful capitalization information
initCap	Sally	Capitalized word
lowercase	can	Uncapitalized word
other	,	Punctuation marks, all other words

Dealing with Low-Frequency Words: An Example

Profits/NA soared/NA at/NA Boeing/SC Co./CC ,/NA easily/NA topping/NA forecasts/NA on/NA Wall/SL Street/CL ,/NA as/NA their/NA CEO/NA Alan/SP Mulally/CP announced/NA first/NA quarter/NA results/NA ./NA

$$\Downarrow$$

`firstword`/NA soared/NA at/NA `initCap`/SC Co./CC ,/NA easily/NA `lowercase`/NA forecasts/NA on/NA `initCap`/SL Street/CL ,/NA as/NA their/NA CEO/NA Alan/SP `initCap`/CP announced/NA first/NA quarter/NA results/NA ./NA

NA	= No entity
SC	= Start Company
CC	= Continue Company
SL	= Start Location
CL	= Continue Location

...

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The Viterbi Algorithm

Problem: for an input $x_1 \ldots x_n$, find

$$\arg\max_{y_1 \cdots y_{n+1}} p(x_1 \ldots x_n, y_1 \ldots y_{n+1})$$

where the $\arg\max$ is taken over all sequences $y_1 \ldots y_{n+1}$ such that $y_i \in \mathcal{S}$ for $i = 1 \ldots n$, and $y_{n+1} =$ STOP.

We assume that p again takes the form

$$p(x_1 \ldots x_n, y_1 \ldots y_{n+1}) = \prod_{i=1}^{n+1} q(y_i | y_{i-2}, y_{i-1}) \prod_{i=1}^{n} e(x_i | y_i)$$

Recall that we have assumed in this definition that $y_0 = y_{-1} = *$, and $y_{n+1} =$ STOP.

Brute Force Search is Hopelessly Inefficient

Problem: for an input $x_1 \ldots x_n$, find

$$\arg\max_{y_1 \ldots y_{n+1}} p(x_1 \ldots x_n, y_1 \ldots y_{n+1})$$

where the $\arg\max$ is taken over all sequences $y_1 \ldots y_{n+1}$ such that $y_i \in \mathcal{S}$ for $i = 1 \ldots n$, and $y_{n+1} =$ STOP.

The Viterbi Algorithm

Andrew Viterbi, 1967

- Define n to be the length of the sentence
- Define S_k for $k = -1 \ldots n$ to be the set of possible tags at position k:

$$S_{-1} = S_0 = \{*\}$$
$$S_k = S \quad \text{for } k \in \{1 \ldots n\}$$

- Define

$$r(y_{-1}, y_0, y_1, \ldots, y_k) = \prod_{i=1}^{k} q(y_i | y_{i-2}, y_{i-1}) \prod_{i=1}^{k} e(x_i | y_i)$$

- Define a dynamic programming table

$$\pi(k, u, v) = \text{maximum probability of a tag sequence ending in tags } u, v \text{ at position } k$$

that is,
$\pi(k, u, v) = \max_{\langle y_{-1}, y_0, y_1, \ldots, y_k \rangle : y_{k-1} = u, y_k = v} r(y_{-1}, y_0, y_1 \ldots y_k)$

An Example

$$\pi(k, u, v) \quad = \quad \text{maximum probability of a tag sequence ending in tags } u, v \text{ at position } k$$

The man saw the dog with the telescope

A Recursive Definition

Base case:

$$\pi(0, *, *) = 1$$

Recursive definition:
For any $k \in \{1 \dots n\}$, for any $u \in \mathcal{S}_{k-1}$ and $v \in \mathcal{S}_k$:

$$\pi(k, u, v) = \max_{w \in \mathcal{S}_{k-2}} \left(\pi(k-1, w, u) \times q(v|w, u) \times e(x_k|v)\right)$$

Justification for the Recursive Definition

For any $k \in \{1 \ldots n\}$, for any $u \in \mathcal{S}_{k-1}$ and $v \in \mathcal{S}_k$:

$$\pi(k, u, v) = \max_{w \in \mathcal{S}_{k-2}} \left(\pi(k-1, w, u) \times q(v|w, u) \times e(x_k|v)\right)$$

The man saw the dog with the telescope

The Viterbi Algorithm

Input: a sentence $x_1 \ldots x_n$, parameters $q(s|u, v)$ and $e(x|s)$.

Initialization: Set $\pi(0, *, *) = 1$

Definition: $\mathcal{S}_{-1} = \mathcal{S}_0 = \{*\}$, $\mathcal{S}_k = \mathcal{S}$ for $k \in \{1 \ldots n\}$

Algorithm:

- For $k = 1 \ldots n$,
  - For $u \in \mathcal{S}_{k-1}$, $v \in \mathcal{S}_k$,

    $$\pi(k, u, v) = \max_{w \in \mathcal{S}_{k-2}} \left(\pi(k-1, w, u) \times q(v|w, u) \times e(x_k|v) \right)$$

- **Return** $\max_{u \in \mathcal{S}_{n-1}, v \in \mathcal{S}_n} \left(\pi(n, u, v) \times q(\mathsf{STOP}|u, v) \right)$

The Viterbi Algorithm with Backpointers

Input: a sentence $x_1 \ldots x_n$, parameters $q(s|u,v)$ and $e(x|s)$.

Initialization: Set $\pi(0, *, *) = 1$

Definition: $\mathcal{S}_{-1} = \mathcal{S}_0 = \{*\}$, $\mathcal{S}_k = \mathcal{S}$ for $k \in \{1 \ldots n\}$
Algorithm:

- For $k = 1 \ldots n$,
  - For $u \in \mathcal{S}_{k-1}$, $v \in \mathcal{S}_k$,

$$\begin{aligned} \pi(k,u,v) &= \max_{w \in \mathcal{S}_{k-2}} \left(\pi(k-1,w,u) \times q(v|w,u) \times e(x_k|v)\right) \\ bp(k,u,v) &= \arg\max_{w \in \mathcal{S}_{k-2}} \left(\pi(k-1,w,u) \times q(v|w,u) \times e(x_k|v)\right) \end{aligned}$$

- Set $(y_{n-1}, y_n) = \arg\max_{(u,v)} \left(\pi(n,u,v) \times q(\text{STOP}|u,v)\right)$
- For $k = (n-2) \ldots 1$, $y_k = bp(k+2, y_{k+1}, y_{k+2})$
- **Return** the tag sequence $y_1 \ldots y_n$

The Viterbi Algorithm: Running Time

- $O(n|\mathcal{S}|^3)$ time to calculate $q(s|u,v) \times e(x_k|s)$ for all k, s, u, v.
- $n|\mathcal{S}|^2$ entries in π to be filled in.
- $O(|\mathcal{S}|)$ time to fill in one entry
- $\Rightarrow O(n|\mathcal{S}|^3)$ time in total

Pros and Cons

- Hidden markov model taggers are very simple to train (just need to compile counts from the training corpus)
- Perform relatively well (over 90% performance on named entity recognition)
- Main difficulty is modeling

$$e(word \mid tag)$$

can be very difficult if "words" are complex