高等代数(2)课后题选讲

1. 设$M_n(F)$是数域F上全部n阶矩阵组成的向量空间,
$S=\{A\in M_n(F)\big|A^T=A\},T=\{A\in M_n(F)\big|A^T=-A\}$**.求证:**$M_n(F)=S\oplus T$.

[证] 显然$S+T\subseteq M_n(F)$.

对$\forall A\in M_n(F)$,取$B=\dfrac{1}{2}(A+A^T)\in S,C=\dfrac{1}{2}(A-A^T)\in T$,

则$A=B+C\in S+T$,即$M_n(F)\subseteq S+T$,故$M_n(F)=S+T$.

取$A\in S\bigcap T$,则$A^T=A=-A^T\Rightarrow A=O$,故$M_n(F)=S\oplus T$.

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2. 设$\overrightarrow{\alpha_i}=(a_{i_1},a_{i_2},\cdots,a_{i_n})\in F^n\ \ (i=1,\cdots,m)$线性无关.对每个$\overrightarrow{\alpha_i}$任填上$p$个数得到$F^{n+p}$的$m$个向量
$\overrightarrow{\beta_i}=(a_{i_1},\cdots,a_{i_n},b_{i_1},\cdots,b_{i_p})\ \ (i=1,\cdots,m)$**.求证:**$\left\{\overrightarrow{\beta_1},\cdots,\overrightarrow{\beta_m}\right\}$**线性无关.**

[证] 设$k_1\overrightarrow{\beta_1}+\cdots+k_m\overrightarrow{\beta_m}=\overrightarrow{0}$,即$\begin{cases}a_{1i}k_1+\cdots+a_{mi}k_m=0\ \ i=1,\cdots,n\\ b_{1j}k_1+\cdots+b_{mj}k_m=0\ \ j=1,\cdots,p\end{cases}$ ①.

欲证$\overrightarrow{\beta_1},\cdots,\overrightarrow{\beta_m}$线性无关,只需证明①只有零解.

因齐次线性方程组$k_1\overrightarrow{\alpha_{1i}}+\cdots+k_m\overrightarrow{\alpha_{mi}}=\overrightarrow{0}\ \ (i=1,\cdots,n)$ ②只有零解,

由①的解都是②的解,则①只有零解.

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3. 设向量$\overrightarrow{\beta}$可由$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_r}$线性表示,但不能由$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_{r-1}}$线性表示.求证:向量组①$\left\{\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_{r-1}},\overrightarrow{\alpha_r}\right\}$等价于向量组②$\left\{\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_{r-1}},\overrightarrow{\beta}\right\}$.

[证] 因$\overrightarrow{\beta}$可由①线性表示,而$\overrightarrow{\alpha_i}\ \ (i=1,\cdots,r-1)$可由①线性表示,则②可由①线性表示.

因$\overrightarrow{\alpha_i}=0\overrightarrow{\alpha_1}+\cdots+0\overrightarrow{\alpha_{i-1}}+1\overrightarrow{\alpha_i}+0\overrightarrow{\alpha_{i+1}}+\cdots+0\overrightarrow{\alpha_{r-1}}+0\overrightarrow{\beta}\ \ (i=1,\cdots,r-1)$,

则$\overrightarrow{\alpha_i}\ \ (i=1,\cdots,r-1)$可由②线性表示.

因$\overrightarrow{\beta}$可由①线性表示,则$\exists$不全为零的$k_1,\cdots,k_r\in F\ s.t.\ \overrightarrow{\beta}=k_1\overrightarrow{\alpha_1}+\cdots+k_r\overrightarrow{\alpha_r}$,

其中$k_r\neq 0$,否则$\overrightarrow{\beta}$可由$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_{r-1}}$线性表示,矛盾.

则$\overrightarrow{\alpha_r}=-\dfrac{k_1}{k_r}\overrightarrow{\alpha_1}-\dfrac{k_2}{k_r}\overrightarrow{\alpha_2}-\cdots-\dfrac{k_{r-1}}{k_r}\overrightarrow{\alpha_{r-1}}-\dfrac{1}{k_r}\overrightarrow{\beta}$,即$\overrightarrow{\alpha_r}$可由②线性表示,

进而①可由②线性表示,故①$\cong$②.

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4. 设S是数域F上所有对称矩阵构成的向量空间,求$\dim S$.

[解] 记$a_{ij}=a_{ji}=1$,其余元素都为0的n阶方阵为E_{ij}.

显然$E_{ij}^T=E_{ij}\in S$,且$\{E_{11},\cdots,E_{1n},E_{22},\cdots,E_{2n},\cdots,E_{nn}\}$线性无关.

显然S中任一对称矩阵可由它们线性表示,则该向量组是S的一个基,则$\dim S=\dfrac{n(n+1)}{2}$.

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5. 设W是n维向量空间V的一个子空间,且$0<\dim W<n$.求证:W在V中不止有一个余子空间.

[证] 设$\overrightarrow{w_1},\cdots,\overrightarrow{w_m}$是$W$的一组基,将其扩充为$V$的一组基$\overrightarrow{w_1},\cdots,\overrightarrow{w_m},\overrightarrow{v_1},\cdots,\overrightarrow{v_{n-m}}$.

显然$U=\mathscr{L}\left(\overrightarrow{v_1},\cdots,\overrightarrow{v_{n-m}}\right)$是$W$的补空间.

对$\forall$非零的$\overrightarrow{w}\in W,U'=\mathscr{L}\left(\overrightarrow{v_1}+\overrightarrow{w},\cdots,\overrightarrow{v_{n-m}}+\overrightarrow{w}\right)$是$W$的补空间.

若$U=U'$,则$\overrightarrow{w}+\overrightarrow{v_1}\in U$,而$\overrightarrow{v_1}\in U$,则$\overrightarrow{w}\in U$,矛盾.故$U\neq U'$.

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6. 设$\overrightarrow{\alpha_1}=(1,2,-1),\overrightarrow{\alpha_2}=(0,-1,3),\overrightarrow{\alpha_3}=(1,-1,0),\overrightarrow{\beta_1}=(2,1,5),\overrightarrow{\beta_2}=(-2,3,1),\overrightarrow{\beta_3}=(1,3,2)$.(1)求证:$\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right\}$和$\left\{\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right\}$都是$\mathbb{R}^3$的基;(2)求前者到后者的过渡矩阵.

[解] (1) 设$A=\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right\},B=\left\{\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right\}$.

因$A=\begin{vmatrix}1&0&1\\2&-1&-1\\-1&3&0\end{vmatrix}=8\neq0,B=\begin{vmatrix}2&-2&1\\1&3&3\\5&1&2\end{vmatrix}=-34\neq0$,

则$\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right\}$和$\left\{\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right\}$都线性无关,而$\dim\mathbb{R}^3=3$,则它们是$\mathbb{R}^3$的基.

(2) 因$\left(\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right)=\left(\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right)A,\left(\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right)=\left(\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right)B$,则$\left(\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right)=\left(\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right)A^{-1}B$,

其中$A^{-1}B=\begin{bmatrix}\frac{3}{8}&\frac{3}{8}&\frac{1}{8}\\\frac{1}{8}&\frac{1}{8}&\frac{3}{8}\\\frac{5}{8}&-\frac{3}{8}&-\frac{1}{8}\end{bmatrix}\begin{bmatrix}2&-2&1\\1&3&3\\5&1&2\end{bmatrix}=\begin{bmatrix}\frac{7}{4}&\frac{1}{2}&\frac{7}{4}\\\frac{9}{4}&\frac{1}{2}&\frac{5}{4}\\\frac{1}{4}&-\frac{5}{2}&-\frac{3}{4}\end{bmatrix}$.

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7. 设$\mathbb{R}^4$的一组基$\overrightarrow{\alpha_1}=(2,1,-1,1),\overrightarrow{\alpha_2}=(0,3,1,0),\overrightarrow{\alpha_3}=(5,3,2,1),\overrightarrow{\alpha_4}=(6,6,1,3)$.求$\mathbb{R}^4$中的一个非零向量使得它关于该基的坐标与关于标准基的坐标相同.

[解] 设从标准基到$\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}\right\}$的过渡矩阵$A=\begin{bmatrix}2&0&5&6\\1&3&3&6\\-1&1&2&1\\1&0&1&3\end{bmatrix}$.

设非零向量$(y_1,y_2,y_3,y_4)\in\mathbb{R}^4$满足要求,则$(y_1,y_2,y_3,y_4)^T=A(y_1,y_2,y_3,y_4)^T$,

进而$(A-I)(y_1,y_2,y_3,y_4)^T=\overrightarrow{0}$,即$\begin{bmatrix}1&0&5&6\\1&2&3&6\\-1&1&1&1\\1&0&1&2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

$\begin{bmatrix}1&0&5&6\\1&2&3&6\\-1&1&1&1\\1&0&1&2\end{bmatrix}\rightarrow\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$,即$\begin{cases}y_1+y_4=0\\y_2+y_4=0\\y_3+y_4=0\end{cases}$.取一组非零特解$(1,1,1,-1)$即满足要求.

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8. 设$f:V\rightarrow W$是向量空间V到W的一个同构映射,V_1是V的一个子空间.求证:$f(V_1)$是W的一个子空间.

[证] 因V_1是V的一个子空间,则$\overrightarrow{0}\in V_1$.因$f\left(\overrightarrow{0}\right)=\overrightarrow{0}$,则$\overrightarrow{0}\in f(V_1)$,即$f(V_1)\neq\varnothing$.

取$\overrightarrow{\alpha},\overrightarrow{\beta}\in f(V_1)$,则$\exists\overrightarrow{x},\overrightarrow{y}\in V_1\ s.t.\ f\left(\overrightarrow{x}\right)=\overrightarrow{\alpha},f\left(\overrightarrow{y}\right)=\overrightarrow{\beta}$.

对$\forall a,b\in F$,因V_1是V的子空间,则$a\overrightarrow{x}+b\overrightarrow{y}\in V_1$.

因f是同构映射,则$f\left(a\overrightarrow{x}+b\overrightarrow{y}\right)=af\left(\overrightarrow{x}\right)+bf\left(\overrightarrow{y}\right)=a\overrightarrow{\alpha}+b\overrightarrow{\beta}$,故$a\overrightarrow{\alpha}+b\overrightarrow{\beta}\in f(V_1)$,故证.

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9. 设A是一个m行的矩阵,$\mathrm{rank}\ A=r$.从A中任取s行,作一个s行的矩阵B.求证:$\mathrm{rank}\ B\geq r+s-m$.

[证] 设A的行向量组$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_m}$的极大线性无关组为$\overrightarrow{\alpha_{i_1}},\cdots,\overrightarrow{\alpha_{i_r}}$.

设B的行向量组为$\overrightarrow{\alpha_{j_1}},\cdots,\overrightarrow{\alpha_{j_s}}$,且$\mathrm{rank}\ B=t$.

将B的行向量组的极大无关组扩充为A的行向量组的极大无关组需补充$(r-t)$个线性无关的向量.

在A的行向量中,除$\overrightarrow{\alpha_{j_1}},\cdots,\overrightarrow{\alpha_{j_s}}$外的$(m-s)$个向量组成的向量组的极大无关组包含$(r-t)$个向量,

则$r-t\leq m-s$,即$t\geq r+s-m$.

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10. 求齐次线性方程组$\begin{cases}x_1+x_2+x_3+x_4+x_5=0\\3x_1+2x_2+x_3+x_4-3x_5=0\\5x_1+4x_2+3x_3+3x_4-x_5=0\\x_2+2x_3+2x_4+x_5=0\end{cases}$的通解.

[解] $\begin{bmatrix}1&1&1&1&1\\3&2&1&1&-3\\5&4&3&3&-1\\0&1&2&2&1\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1&1&1\\0&-1&-2&-2&-6\\0&-1&-2&-2&-6\\0&1&2&2&1\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1&1&1\\0&-1&-2&-2&-6\\0&0&0&0&0\\0&0&0&0&-5\end{bmatrix}$

$\rightarrow\begin{bmatrix}1&1&1&1&1\\0&1&2&2&6\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&-1&-1&0\\0&1&2&2&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$,

则$\begin{cases}x_1-x_3-x_4=0\\x_2+2x_3+2x_4=0\\x_5=0\end{cases}$,即$\begin{cases}x_1=x_3+x_4\\x_2=-2x_3-2x_4\\x_5=0\end{cases}$,其中$x_3,x_4$为自由变元.

取$x_3=1,x_4=0$和$x_3=0,x_4=1$,解得$\overrightarrow{\eta_1}=(1,-2,1,0,0)^T,\overrightarrow{\eta_2}=(1,-2,0,1,0)^T$.

故基础解系$\overrightarrow{\eta_1},\overrightarrow{\eta_2}$,通解$k_1\overrightarrow{\eta_1}+k_2\overrightarrow{\eta_2}$.

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11. 取定矩阵$A\in M_n(F)$,对$\forall X\in M_n(F)$,定义$\sigma(X)=AX-XA$.求证:(1)σ是$M_n(F)$到自身的线性映射;(2)对$\forall X,Y\in M_n(F)$,有$\sigma(XY)=\sigma(X)Y+X\sigma(Y)$.

[**证**] (1) 因$\sigma(X)=AX-XA\in M_n(F)$且唯一确定,则$\sigma$是$M_n(F)$到自身的线性映射.

对$\forall X,Y\in M_n(F),\forall a,b\in F$,

有$\sigma(aX+bY)=A(aX+bY)-(aX+bY)A=a(AX-XA)+b(AY-YA)=a\sigma(X)+b\sigma(Y)$.

(2) $\mathrm{LHS}=A(XY)-(XY)A=(AX-XA)Y+X(AY-YA)=\mathrm{RHS}$.

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12. 设F^4为数域F上的四元列向量空间.取$A=\begin{bmatrix}1&-1&5&-1\\1&1&-2&3\\3&-1&8&1\\1&3&-9&7\end{bmatrix}$.对$\forall\overrightarrow{\xi}\in F^4$,令$\sigma\left(\overrightarrow{\xi}\right)=A\overrightarrow{\xi}$.求线性映射$\sigma$的核空间和像空间的维数.

[**解**] $A\to\begin{bmatrix}1&-1&5&-1\\0&2&-7&4\\0&0&0&0\\0&0&0&0\end{bmatrix}$,则$\mathrm{rank}\,A=2$.设$A$的列向量组为$\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}\right\}$.

因$\mathrm{Ker}\sigma=\left\{A\overrightarrow{x}=\overrightarrow{0}\right\}$,其维数即$A\overrightarrow{x}=\overrightarrow{0}$的解空间的维数,即$n-\mathrm{rank}\,A=4-2=2$.

因$\mathrm{Im}\sigma=\left\{A\overrightarrow{x}\;\middle|\;\overrightarrow{x}\in F^4\right\}=\left\{k_1\overrightarrow{\alpha_1}+k_2\overrightarrow{\alpha_2}+k_3\overrightarrow{\alpha_3}+k_4\overrightarrow{\alpha_4}\;\middle|\;k_1,k_2,k_3,k_4\in F\right\}$.

$\dim(\mathrm{Im}\sigma)=\dim\mathscr{L}\left(\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}\right)=$向量组$\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}\right\}$的极大线性无关组所含元素个数

$=\mathrm{rank}\,A=2$.

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13. 设V和W都是数域F上的向量空间,$\dim V=n$.令σ是V和W的一个线性映射.选取V的一个基$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_s},\overrightarrow{\alpha_{s+1}},\cdots,\overrightarrow{\alpha_n}\ s.t.\ \overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_s}$是$\mathrm{Ker}\sigma$的一个基.求证:(1)$\sigma\left(\overrightarrow{\alpha_{s+1}},\cdots,\overrightarrow{\alpha_n}\right)$是$\mathrm{Im}\sigma$的一个基;(2)$\dim(\mathrm{Ker}\sigma)+\dim(\mathrm{Im}\sigma)=n$.

[**证**] (1) 因$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_s}$是$\mathrm{Ker}\sigma$的基,若$\overrightarrow{\eta'}\in\mathrm{Im}\sigma$,

则$\exists\overrightarrow{\eta}=\sum_{i=1}^n k_i\overrightarrow{\alpha_i}\in V\ s.t.\ \overrightarrow{\eta'}=\sigma\left(\overrightarrow{\eta}\right)=\sum_{i=s+1}^n k_i\sigma\left(\overrightarrow{\alpha_i}\right)$,

即$\mathrm{Im}\sigma$中的元素都可由$\sigma\left(\overrightarrow{\alpha_{s+1}}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)$线性表示,下证它们线性无关.

设$\sigma\left(\sum_{j=1}^{n-s}b_j\overrightarrow{\alpha_{s+j}}\right)=\overrightarrow{0}$,则$\sum_{j=1}^{n-s}b_j\overrightarrow{\alpha_{s+j}}\in\mathrm{Ker}\sigma$,

进而$\exists a_1,\cdots,a_s\ s.t.\ a_1\overrightarrow{\alpha_1}+\cdots+a_s\overrightarrow{\alpha_s}=\sum_{j=1}^{n-s}b_j\overrightarrow{\alpha_{s+j}}$,

则$a_1\overrightarrow{\alpha_1}+\cdots+a_s\overrightarrow{\alpha_s}-b_1\overrightarrow{\alpha_{s+1}}-\cdots-b_{n-s}\overrightarrow{\alpha_n}=\overrightarrow{0}$.

因$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}$是一个基,则它们线性无关,进而$b_1=\cdots=b_{n-s}$,即$\sigma\left(\overrightarrow{\alpha_{s+1}}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)$线性无关.

(2) 由(1)的证明即证.

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14. (1)设V是数域F上一有限维向量空间,σ是V的一个线性变换.求证下列三条件等价:①σ是满射;②$\mathrm{Ker}\sigma=\left\{\overrightarrow{0}\right\}$;③$\sigma$可逆;(2)$V$无限维时,条件①和②是否等价?

[**证**] (1) 设$\dim V=n$.

①⇒② 因σ是满射,则$\mathrm{Im}\sigma=V,\dim(\mathrm{Im}\sigma)=n$,进而$\dim(\mathrm{Ker}\sigma)=n-\dim(\mathrm{Im}\sigma)=0$,故$\mathrm{Ker}\sigma=\left\{\overrightarrow{0}\right\}$.

②⇒③ 因$\mathrm{Ker}\sigma=\left\{\overrightarrow{0}\right\}$,则$\dim(\mathrm{Ker}\sigma)=0$,进而$\dim(Im\sigma)=n-\dim(\mathrm{Ker}\sigma)=n$.

因$\mathrm{Im}\sigma\subseteq V,\dim v=n=dim(\mathrm{Im}\sigma)$,则$\mathrm{Im}\sigma=V$,即$\sigma$是满射.

下证σ是单射.若不然,$\exists\overrightarrow{x},\overrightarrow{y}\in V,\overrightarrow{x}\neq\overrightarrow{y}\ s.t.\ \sigma\left(\overrightarrow{x}\right)=\sigma\left(\overrightarrow{y}\right)$,则$\sigma\left(\overrightarrow{x}-\overrightarrow{y}\right)=\sigma\left(\overrightarrow{x}\right)-\sigma\left(\overrightarrow{y}\right)=\overrightarrow{0}$,

进而$\overrightarrow{x}-\overrightarrow{y}\in\mathrm{Ker}\sigma$.而$\overrightarrow{x}-\overrightarrow{y}\neq\overrightarrow{0}$,与$\mathrm{Ker}\sigma=\left\{\overrightarrow{0}\right\}$矛盾.故$\sigma$是双射,进而可逆.

③⇒① 因σ可逆,则σ是双射,进而是满射.

(2) 不等价.

反例I 在$[0,1]$上的可积函数集上定义$\sigma:\displaystyle\int_0^1 f(x)\mathrm{d}x$.

但$\displaystyle\int_0^1 f(x)\mathrm{d}x=0\nRightarrow f(x)=0$,如Riemann函数,故$\sigma$不是单射,但是满射.

反例II $\sigma:(x_1,x_2,\cdots)\mapsto(x_2,x_3,\cdots)$.

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15. 设$F^n=\{(x_1,\cdots,x_n)\big|x_i\in F\}$是数域$F$上的$n$维行空间.定义$\sigma(x_1,x_2,\cdots,x_n)=(0,x_1,\cdots,x_{n-1})$.(1)求证:$\sigma$是$F^n$的一个线性变换,且$\sigma^n=\theta$;(2)求$\mathrm{Ker}\sigma$和$\mathrm{Im}\sigma$的维数.

[**证**] (1) 设$\forall a,b\in F,\forall(x_1,\cdots,x_n),(y_1,\cdots,y_n)\in F^n$,有:

$\sigma(a(x_1,\cdots,x_n)+b(y_1,\cdots,y_n))=\sigma(ax_1+by_1,ax_2+by_2,\cdots,ax_n+by_n)$

$=(0,ax_1+by_1,ax_2+by_2,\cdots,ax_{n-1}+by_{n-1})=a(0,x_1,x_2,\cdots,x_{n-1})+b(0,y_1,y_2,\cdots,y_n)$

$=a\sigma(x_1,x_2,\cdots,x_n)+b\sigma(y_1,y_2,\cdots,n)$,则$\sigma$是$F^n$的线性变换.

设$\sigma^i(x_1,\cdots,x_n)=(\underbrace{0,\cdots,0}_{n个},x_i,\cdots,x_{n-i})\ \ (1\leq i\leq n-1)$,

则$\sigma^{i+1}(x_1,\cdots,x_n)=\sigma\cdot\sigma^i(x_1,\cdots,x_n)=\sigma(\underbrace{0,\cdots,0}_{i个},x_1,\cdots,x_{n-i})=\sigma(\underbrace{0,\cdots,0}_{(i+1)个},x_1,\cdots,x_{n-i-1})$.

故$\sigma^n(x_1,\cdots,x_n)=(\underbrace{0,\cdots,0}_{n个})$,即$\sigma^n=\theta$.

(2) $\mathrm{Ker}\sigma=\{(0,0,\cdots,x_1\big|x_i\in F)\},\mathrm{Im}\sigma=\{(0,x_1,\cdots,x_{n-1})\big|x_i\in F\}$,

则$\dim(\mathrm{Ker}\sigma)=1,\dim(\mathrm{Im}\sigma)=n-1$.

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16. 设σ是有限维向量空间V的一个线性变换,W是σ的一个不变子空间.求证:若σ可逆,则W在σ^{-1}下不变.

[证] 设$\sigma_1=\sigma\big|_W$,则它是W的一个线性变换.因σ可逆,则σ_1可逆,进而$\sigma_1(W)=W$.

因$\sigma_1^{-1}(W)=W$,则$\sigma^{-1}(W)=W$.

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17. 设σ和τ是向量空间V的线性变换,且$\sigma\tau=\tau\sigma$.求证:$\mathrm{Im}\sigma$和$\mathrm{Ker}\sigma$都在τ下不变.

[证] 对$\forall\overrightarrow{\varepsilon}\in\mathrm{Im}\sigma,\exists\overrightarrow{\xi}\in V\ s.t.\ \overrightarrow{\varepsilon}=\sigma\left(\overrightarrow{\xi}\right)$,则$\tau\left(\overrightarrow{\varepsilon}\right)=\tau\left(\sigma\left(\overrightarrow{\xi}\right)\right)=\sigma\left(\tau\left(\overrightarrow{\xi}\right)\right)\in\mathrm{Im}\sigma$.

对$\forall\overrightarrow{\xi}\in\mathrm{Ker}\sigma$,有$\sigma\left(\overrightarrow{\xi}\right)=\overrightarrow{0}$,则$\sigma\left(\tau\left(\overrightarrow{\xi}\right)\right)=\tau\left(\sigma\left(\overrightarrow{\xi}\right)\right)=\tau\left(\overrightarrow{0}\right)=\overrightarrow{0}$,即$\tau\left(\overrightarrow{\xi}\right)\in\mathrm{Ker}\sigma$.

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18. 设σ是数域F上的向量空间V的一个线性变换,且$\sigma^2=\sigma$.求证:(1)$\mathrm{Ker}\sigma=\left\{\overrightarrow{\xi}-\sigma\left(\overrightarrow{\xi}\right)\middle|\overrightarrow{\xi}\in V\right\}$;(2)$V=\mathrm{Ker}\sigma\oplus\mathrm{Im}\sigma$;(3)若$\tau$是$V$的一个线性变换,则$\mathrm{Ker}\sigma$和$\mathrm{Im}\sigma$都在$\tau$下不变的充要条件是:$\sigma\tau=\tau\sigma$.

[证] (1) 对$\forall\overrightarrow{x}\in\mathrm{Ker}\sigma$,有$\sigma\left(\overrightarrow{x}\right)=\overrightarrow{0}$,

则$\overrightarrow{x}-\sigma\left(\overrightarrow{x}\right)=\overrightarrow{x}\in\left\{\overrightarrow{\xi}-\sigma\left(\overrightarrow{\xi}\right)\middle|\overrightarrow{\xi}\in V\right\}$,即$\mathrm{Ker}\sigma\subseteq\left\{\overrightarrow{\xi}-\sigma\left(\overrightarrow{\xi}\right)\middle|\overrightarrow{\xi}\in V\right\}$.

若$\overrightarrow{x}\in\left\{\overrightarrow{\xi}-\sigma\left(\overrightarrow{\xi}\right)\middle|\overrightarrow{\xi}\in V\right\}$,则$\exists\overrightarrow{\xi}\in V\ s.t.\ \overrightarrow{x}=\overrightarrow{\xi}-\sigma\left(\overrightarrow{\xi}\right)$,

进而$\sigma\left(\overrightarrow{x}\right)=\sigma\left(\overrightarrow{\xi}-\sigma\left(\overrightarrow{\xi}\right)\right)=\sigma\left(\overrightarrow{\xi}\right)-\sigma^2\left(\overrightarrow{\xi}\right)=\sigma\left(\overrightarrow{\xi}\right)-\sigma\left(\overrightarrow{\xi}\right)=\overrightarrow{0}$,

即$\overrightarrow{x}\in\mathrm{Ker}\sigma$,则$\left\{\overrightarrow{\xi}-\sigma\left(\overrightarrow{\xi}\right)\middle|\overrightarrow{\xi}\in V\right\}\subseteq\mathrm{Ker}\sigma$,故证.

(2) ①先证$V=\mathrm{Ker}\sigma+\mathrm{Im}\sigma$.

对$\forall\overrightarrow{x}\in V$,注意到$\overrightarrow{x}=\left[\overrightarrow{x}-\sigma\left(\overrightarrow{x}\right)\right]+\sigma\left(\overrightarrow{x}\right)$,

因$\overrightarrow{x}-\sigma\left(\overrightarrow{x}\right)\in\mathrm{Ker}\sigma,\sigma\left(\overrightarrow{x}\right)\in\mathrm{Im}\sigma$.则$V\subseteq\mathrm{Ker}\sigma+\mathrm{Im}\sigma$.

因σ是V的线性变换,则$\mathrm{Ker}\sigma+\mathrm{Im}\sigma\subseteq V$,故$V=\mathrm{Ker}\sigma+\mathrm{Im}\sigma$.

②再证$\mathrm{Ker}\sigma\bigcap\mathrm{Im}\sigma=\left\{\overrightarrow{0}\right\}$.

取$\overrightarrow{x}\in\mathrm{Ker}\sigma\bigcap\mathrm{Im}\sigma$,则$\sigma\left(\overrightarrow{x}\right)=\overrightarrow{0}$,且$\exists\overrightarrow{y}\in V\ s.t.\ \sigma\left(\overrightarrow{y}\right)=\overrightarrow{x}$.

因$\sigma^2\left(\overrightarrow{y}\right)=\sigma\left(\sigma\left(\overrightarrow{y}\right)\right)=\sigma\left(\overrightarrow{x}\right)=\overrightarrow{0}$,又$\sigma^2\left(\overrightarrow{y}\right)=\sigma\left(\overrightarrow{y}\right)=\overrightarrow{x}$,故$\overrightarrow{x}=\overrightarrow{0}$.

(3) (充) 见**17**.

(必) 对$\forall\overrightarrow{\alpha}\in V$,因$\overrightarrow{\alpha}=\left[\overrightarrow{\alpha}-\sigma\left(\overrightarrow{\alpha}\right)\right]+\sigma\left(\overrightarrow{\alpha}\right)$,

则$\sigma\tau\left(\overrightarrow{\alpha}\right)=\sigma\tau\left(\left[\overrightarrow{\alpha}-\sigma\left(\overrightarrow{\alpha}\right)\right]+\sigma\left(\overrightarrow{\alpha}\right)\right)=\sigma\left(\tau\left(\overrightarrow{\alpha}-\sigma\left(\overrightarrow{\alpha}\right)\right)\right)+\sigma\left(\tau\left(\sigma\left(\overrightarrow{\alpha}\right)\right)\right)$.

因$\overrightarrow{\alpha}-\sigma\left(\overrightarrow{\alpha}\right)\in\mathrm{Ker}\sigma$,而$\mathrm{Ker}\sigma$在$\tau$下不变,则$\tau\left(\overrightarrow{\alpha}-\sigma\left(\overrightarrow{\alpha}\right)\right)\in\mathrm{Ker}\sigma$,进而$\sigma\left(\tau\left(\overrightarrow{\alpha}-\sigma\left(\overrightarrow{\alpha}\right)\right)\right)=\overrightarrow{0}$.

因$\mathrm{Im}\sigma$在τ下不变,则$\tau\left(\sigma\left(\overrightarrow{\alpha}\right)\right)\in\mathrm{Im}\sigma$,进而$\exists\overrightarrow{\beta}\in V\ s.t.\ \tau\left(\sigma\left(\overrightarrow{\alpha}\right)\right)=\sigma\left(\overrightarrow{\beta}\right)$.

故$\sigma\tau\left(\overrightarrow{\alpha}\right)=\sigma\left(\tau\left(\sigma\left(\overrightarrow{\alpha}\right)\right)\right)=\sigma\left(\sigma\left(\overrightarrow{\beta}\right)\right)=\sigma\left(\overrightarrow{\beta}\right)=\tau\left(\sigma\left(\overrightarrow{\alpha}\right)\right)=\tau\sigma\left(\overrightarrow{\alpha}\right)$.

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19. 求下列矩阵在$\mathbb{R}^3$上的特征根和特征向量:(1)$A_1=\begin{bmatrix}3&-2&0\\-1&3&-1\\-5&7&-1\end{bmatrix}$**;(2)**$A_2=\begin{bmatrix}4&-5&7\\1&-4&9\\-4&0&5\end{bmatrix}$**;(3)** $A_3=\begin{bmatrix}3&6&6\\0&2&0\\-3&-12&-6\end{bmatrix}$.

[解I] (1) $|\lambda I-A_1|=\begin{vmatrix}\lambda-3&2&0\\1&\lambda-3&1\\5&-7&\lambda+1\end{vmatrix}=(\lambda-1)(\lambda-2)^2$,$\mathbb{R}$上的特征根$1$、$2$(二重).

属于特征值1的特征向量是齐次线性方程组$[I-A_1]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$的解,

解得$(x_1,x_2,x_3)^T=(a,a,a)^T\ \ (a\in\mathbb{R},a\neq 0)$.

属于特征值2的特征向量是齐次线性方程组$[2I-A_1]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$,

解得$(x_1,x_2,x_3)^T=(2b,b,-b)^T\ \ (b\in\mathbb{R},b\neq 0)$.

(2) $|\lambda I-A_2|=\begin{vmatrix}\lambda-4&5&-7\\-1&\lambda+4&-9\\4&0&\lambda-5\end{vmatrix}=(x-1)(x^2-4x+13)$,$\mathbb{R}$上的特征根$1$.

属于特征根1的特征向量是齐次线性方程组$[I-A_2]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$的解,

解得$(x_1,x_2,x_3)^T=(a,2a,a)^T\ \ (a\in\mathbb{R},a\neq 0)$.

(3) $|\lambda I-A_3|=\begin{vmatrix}\lambda-3&-6&-6\\0&\lambda-2&0\\3&12&\lambda+6\end{vmatrix}=x(x-2)(x+3)$,$\mathbb{R}$上的特征根$0,2,-3$.

属于特征根0的特征向量是齐次线性方程组$[-A_3]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$的解,

解得$(x_1,x_2,x_3)^T=(2a,0,-a)^T\ \ (a\in\mathbb{R},a\neq 0)$.

属于特征根2的特征向量是齐次线性方程组$[2I-A_3]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$的解,

解得$(x_1,x_2,x_3)^T=(12b,-5b,3b)^T\ \ (b\in\mathbb{R},b\neq 0)$.

属于特征根-3的特征向量是齐次线性方程组$[-3I-A_3]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$的解,

解得$(x_1,x_2,x_3)^T=(c,0,-c)^T\ \ (c\in\mathbb{R},c\neq 0)$.

[解II] 快速求特征根.

(1) $A_1=\begin{bmatrix}3&-2&0\\-1&3&-1\\-5&7&-1\end{bmatrix},\det A_1=4=2\times 2\times 1,\operatorname{tr}A_1=5=2+2+1$,故特征根为$2,2,1.$

(3) $A_3=\begin{bmatrix}3&6&6\\0&2&0\\-3&-12&-6\end{bmatrix},\det A_3=0,\operatorname{tr}A_3=-1.$有一特征根$0.$

特征多项式$|\lambda I-A_3|=\begin{vmatrix}\lambda-3&-6&-6\\0&\lambda-2&0\\3&12&\lambda+6\end{vmatrix}.$

$\dfrac{\text{row2}}{\text{row1}}$得:$\dfrac{0}{\lambda-3}=\dfrac{\lambda-2}{-6}=\dfrac{0}{-6}\Rightarrow\lambda=2.$

$\dfrac{\text{row3}}{\text{row1}}$得:$\dfrac{3}{\lambda-3}=\dfrac{12}{-6}=\dfrac{\lambda+6}{-6}\Rightarrow\lambda\in\varnothing.$

$\dfrac{\text{row2}}{\text{row3}}$得:$\dfrac{0}{3}=\dfrac{\lambda-2}{12}=\dfrac{0}{\lambda+6}\Rightarrow\lambda=2.$

最后一个特征根$=-1-2=-3$,故特征根$0,2,-3.$

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20. 设A是$\mathbb{C}$上的n阶矩阵.(1)求证:$\exists$可逆矩阵$T\in M_n(\mathbb{C})\ s.t.T^{-1}AT=\begin{bmatrix}\lambda_1&b_{12}&\cdots&b_{1n}\\0&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&\ddots&\vdots\\0&b_{n2}&\cdots&b_{nn}\end{bmatrix}$;(2)求证:$A$相似于上三角矩阵$\begin{bmatrix}\lambda_1&*&\cdots&*\\&\lambda_2&\cdots&*\\&&\ddots&\vdots\\&&&\lambda_n\end{bmatrix}$.

[**证**] (1) 因σ是$\mathbb{C}$上的n维向量空间V的一个线性变换,且它关于给定基的矩阵为A,

则A在$\mathbb{C}$上有特征根,记作λ_1,设其对应的特征向量为$\overrightarrow{\xi}$.

将$\overrightarrow{\xi}$扩充为V的一组基$\overrightarrow{\xi},\overrightarrow{\alpha_2},\cdots,\overrightarrow{\alpha_n}$,则$A\overrightarrow{\alpha_2}$可用$\overrightarrow{\xi},\overrightarrow{\alpha_2},\cdots,\overrightarrow{\alpha_n}$线性表示.

记$A\overrightarrow{\alpha_2}=b_{12}\overrightarrow{\xi}+b_{22}\overrightarrow{\alpha_2}+\cdots+b+n2\overrightarrow{\alpha_n}$,同理$A\overrightarrow{\alpha_k}=b_{1k}\overrightarrow{\xi}+b_{2k}\overrightarrow{\alpha_2}+\cdots+b+nk\overrightarrow{\alpha_n}\ \ (2\le k\le n).$

设$A\left(\overrightarrow{\xi},\overrightarrow{\alpha_2},\cdots,\overrightarrow{\alpha_n}\right)=\left(\overrightarrow{\xi},\overrightarrow{\alpha_2},\cdots,\overrightarrow{\alpha_n}\right)\begin{bmatrix}\lambda_1&b_{12}&\cdots&b_{1n}\\0&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&\ddots&\vdots\\0&b_{n2}&\cdots&b_{nn}\end{bmatrix}.$

设$T=\left(\overrightarrow{\xi},\overrightarrow{\alpha_2},\cdots,\overrightarrow{\alpha_n}\right)$,因$\overrightarrow{\xi},\overrightarrow{\alpha_2},\cdots,\overrightarrow{\alpha_n}$是$V$的一组基,则它们线性无关,进而$T$可逆.

故$T^{-1}AT=\begin{bmatrix}\lambda_1&b_{12}&\cdots&b_{1n}\\0&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&\ddots&\vdots\\0&b_{n2}&\cdots&b_{nn}\end{bmatrix}.$

(2) $n=1$时显成立.设$\mathbb{C}$上的任一$(n-1)$阶矩阵与一上三角矩阵相似.

因$A\in M_n(\mathbb{C})$,由(1)知:$\exists$可逆矩阵$T\in M_n(\mathbb{C})\ s.t.\ T^{-1}AT=\begin{bmatrix}\lambda_1 & b_{12} & \cdots & b_{1n}\\ 0 & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & b_{n2} & \cdots & b_{nn}\end{bmatrix}$.

由归纳假设,$\exists$可逆矩阵$S\in M_{n-1}(\mathbb{C})\ s.t.\ S^{-1}\begin{bmatrix}b_{22} & \cdots & b_{2n}\\ \vdots & \ddots & \vdots\\ b_{n2} & \cdots & b_{nn}\end{bmatrix}S=\begin{bmatrix}\lambda_2 & \cdots & *\\ & \ddots & \vdots\\ & & \lambda_n\end{bmatrix}$.

故$\begin{bmatrix}1 & 0\\ 0 & S^{-1}\end{bmatrix}T^{-1}AT\begin{bmatrix}1 & 0\\ 0 & S\end{bmatrix}=\begin{bmatrix}\lambda_1 & * & \cdots & *\\ & \lambda_2 & \cdots & *\\ & & \ddots & \vdots\\ & & & \lambda_n\end{bmatrix}$.

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21. 将$A=\begin{bmatrix}3 & 6 & 6\\ 0 & 2 & 0\\ -3 & -12 & -6\end{bmatrix}$在$\mathbb{R}$上对角化,并求出过渡矩阵$T$.

[解] 易求得特征根$0,2,-3$,它们对应的特征向量分别是$(2,0,-1)^T,(12,-5,3)^T,(1,0,-1)^T$.

故A可对角化,且$T=\begin{bmatrix}2 & 12 & 1\\ 0 & -5 & 0\\ -1 & 3 & -1\end{bmatrix}$,则$T^{-1}=\dfrac{1}{5}\begin{bmatrix}5 & 15 & 5\\ 0 & -1 & 0\\ -5 & -18 & -10\end{bmatrix}$,且$T^{-1}AT=\begin{bmatrix}0 & & \\ & 2 & \\ & & -3\end{bmatrix}$.

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22. 设$A=\begin{bmatrix}4 & 6 & 0\\ -3 & -5 & 0\\ -3 & -6 & 1\end{bmatrix}$.求$A^{10}$.

[解] $|\lambda I-A|=\begin{vmatrix}\lambda-4 & -6 & 0\\ 3 & \lambda+5 & 0\\ 3 & 6 & \lambda-1\end{vmatrix}$.

快速求特征根$\det A=-2=(-2)\times 1\times 1,\mathrm{tr}\,A=0=-2+1+1$.

属于特征根-2的特征向量$(1,-1,-1)^T$.

属于特征根1的线性无关的特征向量$(0,1,0)^T,(0,1,1)^T$.

故$\exists$可逆矩阵$T=\begin{bmatrix}0 & 0 & 1\\ 1 & 1 & -1\\ 0 & 1 & -1\end{bmatrix}\ s.t.\ T^{-1}AT=\begin{bmatrix}1 & & \\ & 1 & \\ & & -2\end{bmatrix}$.

故$A^{10}=T\begin{bmatrix}1^{10} & & \\ & 1^{10} & \\ & & (-2)^{10}\end{bmatrix}T^{-1}=\begin{bmatrix}-1022 & -2046 & 0\\ 1023 & 2047 & 0\\ 1023 & 2046 & 1\end{bmatrix}$.

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23. 设σ是数域F上的n维向量空间V的一个对合变换,即$\sigma^2=\iota$,其中ι是单位变换.求证:(1)σ的本征值只能为± 1;(2) $V=V_1\oplus V_{-1}$,其中V_1、V_{-1}分别是σ的属于特征值1、-1的本征子空间.

[**证**] (1) 设σ的本征值为λ,其对应的特征向量为$\overrightarrow{\alpha}$,则$\sigma\left(\overrightarrow{\alpha}\right)=\lambda\overrightarrow{\alpha},\sigma^2\left(\overrightarrow{\alpha}\right)=\lambda^2\overrightarrow{\alpha}$.

因$\sigma^2=\iota$,则$\lambda^2\overrightarrow{\alpha}=\overrightarrow{\alpha}$,即$(\lambda^2-1)\overrightarrow{\alpha}=\overrightarrow{0}$.因$\overrightarrow{\alpha}$为特征向量,则它非零,进而$\lambda=\pm 1$.

(2) 取$\overrightarrow{\alpha}\in V$,注意到$\overrightarrow{\alpha}=\dfrac{\overrightarrow{\alpha}+\sigma\left(\overrightarrow{\alpha}\right)}{2}+\dfrac{\overrightarrow{\alpha}-\sigma\left(\overrightarrow{\alpha}\right)}{2}$,

因$\sigma\left(\dfrac{\overrightarrow{\alpha}+\sigma\left(\overrightarrow{\alpha}\right)}{2}\right)=\dfrac{\sigma\left(\overrightarrow{\alpha}\right)+\overrightarrow{\alpha}}{2}=1\cdot\dfrac{\sigma\left(\overrightarrow{\alpha}\right)+\overrightarrow{\alpha}}{2}$,

则$\dfrac{\overrightarrow{\alpha}+\sigma\left(\overrightarrow{\alpha}\right)}{2}\in V_1$,同理$\dfrac{\overrightarrow{\alpha}-\sigma\left(\overrightarrow{\alpha}\right)}{2}\in V_{-1}$,故$V=V_1+V_{-1}$.

取$\overrightarrow{\beta}=V_1\bigcap V_{-1}$,则$\sigma\left(\overrightarrow{\beta}\right)=\overrightarrow{\beta},\sigma\left(\overrightarrow{\beta}\right)=-\overrightarrow{\beta}\Rightarrow\overrightarrow{\beta}=\overrightarrow{0}$,故$V=V_1\oplus V_{-1}$.

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24. 设$\overrightarrow{\alpha_1}=(0,2,1,0),\overrightarrow{\alpha_2}=(1,-1,0,0),\overrightarrow{\alpha_3}=(1,2,0,-1),\overrightarrow{\alpha_4}=(1,0,0,1)$是$\mathbb{R}^4$的一组基.对其施行正交化方法,求出$\mathbb{R}^4$的一组规范正交基.

[**解I**] Schmit正交化方法.

$$\overrightarrow{\beta}_1=\overrightarrow{\alpha_1}=(0,2,1,0).$$

因$\left\langle\overrightarrow{\alpha}_2,\overrightarrow{\beta}_1\right\rangle=-2,\left\langle\overrightarrow{\beta}_1,\overrightarrow{\beta}_1\right\rangle=5$,

$$\overrightarrow{\beta}_2=\overrightarrow{\alpha}_2-\frac{\left\langle\overrightarrow{\alpha}_2,\overrightarrow{\beta}_1\right\rangle}{\left\langle\overrightarrow{\beta}_1,\overrightarrow{\beta}_1\right\rangle}\overrightarrow{\beta}_1=(1,-1,0,0)+\frac{2}{5}(0,2,1,0)=\left(1,-\frac{1}{5},\frac{2}{5},0\right).$$

因$\left\langle\overrightarrow{\alpha}_3,\overrightarrow{\beta}_1\right\rangle=4,\left\langle\overrightarrow{\alpha}_3,\overrightarrow{\beta}_2\right\rangle=\dfrac{3}{5},\left\langle\overrightarrow{\beta}_2,\overrightarrow{\beta}_2\right\rangle=\dfrac{6}{5}$,

$$\overrightarrow{\beta}_3=\overrightarrow{\alpha}_3-\frac{\left\langle\overrightarrow{\alpha}_3,\overrightarrow{\beta}_1\right\rangle}{\left\langle\overrightarrow{\beta}_1,\overrightarrow{\beta}_1\right\rangle}\overrightarrow{\beta}_1-\frac{\left\langle\overrightarrow{\alpha}_3,\overrightarrow{\beta}_2\right\rangle}{\left\langle\overrightarrow{\beta}_2,\overrightarrow{\beta}_2\right\rangle}\overrightarrow{\beta}_2$$

$$=(1,2,0,-1)-\frac{4}{5}(0,2,1,0)-\frac{1}{2}\left(1,-\frac{1}{5},\frac{2}{5},0\right)=\left(\frac{1}{2},\frac{1}{2},-1,-1\right).$$

因$\left\langle\overrightarrow{\alpha}_4,\overrightarrow{\beta}_1\right\rangle=0,\left\langle\overrightarrow{\alpha}_4,\overrightarrow{\beta}_2\right\rangle=1,\left\langle\overrightarrow{\alpha}_4,\overrightarrow{\beta}_3\right\rangle=-\dfrac{1}{2},\left\langle\overrightarrow{\beta}_3,\overrightarrow{\beta}_3\right\rangle=\dfrac{5}{2}$,

$$\overrightarrow{\beta}_4=\overrightarrow{\alpha}_4-\frac{\left\langle\overrightarrow{\alpha}_4,\overrightarrow{\beta}_1\right\rangle}{\left\langle\overrightarrow{\beta}_1,\overrightarrow{\beta}_1\right\rangle}\overrightarrow{\beta}_1-\frac{\left\langle\overrightarrow{\alpha}_4,\overrightarrow{\beta}_2\right\rangle}{\left\langle\overrightarrow{\beta}_2,\overrightarrow{\beta}_2\right\rangle}\overrightarrow{\beta}_1-\frac{\left\langle\overrightarrow{\alpha}_4,\overrightarrow{\beta}_3\right\rangle}{\left\langle\overrightarrow{\beta}_3,\overrightarrow{\beta}_3\right\rangle}\overrightarrow{\beta}_1$$

$$=(1,0,0,1)-\frac{5}{6}\left(1,-\frac{1}{5},\frac{2}{5},0\right)+\frac{1}{5}\left(\frac{1}{2},\frac{1}{2},-1,-1\right)=\left(\frac{4}{15},\frac{4}{15},-\frac{8}{15},\frac{4}{5}\right).$$

$$\vec{\gamma_1}=\frac{\vec{\beta_1}}{\left|\vec{\beta_1}\right|}=\left(0,\frac{2}{\sqrt5},\frac{1}{\sqrt5},0\right),\vec{\gamma_2}=\frac{\vec{\beta_2}}{\left|\vec{\beta_2}\right|}=\left(\frac{5}{\sqrt{30}},-\frac{1}{\sqrt{30}},\frac{2}{\sqrt{30}},0\right),$$

$$\vec{\gamma_3}=\frac{\vec{\beta_3}}{\left|\vec{\beta_3}\right|}=\left(\frac{1}{\sqrt{10}},\frac{1}{\sqrt{10}},-\frac{2}{\sqrt{10}},-\frac{2}{\sqrt{10}}\right),\vec{\gamma_4}=\frac{\vec{\beta_4}}{\left|\vec{\beta_4}\right|}=\left(\frac{1}{\sqrt{15}},\frac{1}{\sqrt{15}},-\frac{2}{\sqrt{15}},\frac{3}{\sqrt{15}}\right).$$

[解II] 快速正交化.$\vec{\alpha_1}=(0,2,1,0),\vec{\alpha_2}=(1,-1,0,0),\vec{\alpha_3}=(1,2,0,-1),\vec{\alpha_4}=(1,0,0,1)$.

注意到$\vec{\alpha_1}\perp\vec{\alpha_4}$,固定$\vec{\alpha_1},\vec{\alpha_4}$为$\vec{\beta_1},\vec{\beta_2}$,

为使得$\vec{\beta_3}$与$\vec{\beta_1}$正交,$\vec{\beta_3}$有形式$(?,a,-2a,?)$.为使得$\vec{\beta_3}$与$\vec{\beta_2}$正交,$\vec{\beta_3}$有形式$(b,?,?,-b)$.

取$\vec{\beta_3}=(1,1,-2,-1)$.

设$\vec{\beta_4}=(y,x,-2x,-y)$,它与$\vec{\beta_3}$正交,则$y+x+4x+y=0\Rightarrow 2y=\dfrac{5}{4}x$.

取$x=8$,则$y=10$,此时$\vec{\beta_4}=(10,8,-16,-10)$,约分后取$\vec{\beta_4}=(5,4,-8,-5)$.

再将它们分别单位化即可.

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25. 求证:$\mathbb{R}^3$中的向量(x_0,y_0,z_0)到平面$W=\left\{(x,y,z)\in\mathbb{R}^3\middle|ax+by+cz=0\right\}$的最短距离为$\dfrac{|ax_0+by_0+cz_0|}{\sqrt{a^2+b^2+c^2}}$.

[证] 设$\vec{\beta_1}=(x_1,y_1,z_1)$是$\vec{\beta_0}=(x_0,y_0,z_0)$在$W$上的正投影,

则$\left|\vec{\beta_0}-\vec{\beta_1}\right|$是$\vec{\beta_0}$到$W$的最短距离,且$\vec{\beta_0}-\vec{\beta_1}\in W^\perp$.

设$\vec{n}=(a,b,c)$,则$\vec{n}\in W^\perp$.因W是平面,则a,b,c不全为零,且$\dim W=2,W\oplus W^\perp=\mathbb{R}^3$,

进而$\dim W^\perp=n-\dim W=1$.

因$\vec{n}\neq\vec{0}$,则它是$W^\perp$的一个基,进而$\exists k\in\mathbb{R}\ s.t.\ \vec{\beta_0}-\vec{\beta_1}=k\vec{n}$.

因$\left\langle\vec{\beta_0}-\vec{\beta_1},\vec{n}\right\rangle=\left\langle k\vec{n},\vec{n}\right\rangle=k\left|\vec{n}\right|^2$,

$$\left\langle\vec{\beta_0}-\vec{\beta_1},\vec{n}\right\rangle=\left\langle\vec{\beta_0},\vec{n}\right\rangle-\left\langle\vec{\beta_1},\vec{n}\right\rangle=\left\langle\vec{\beta_0},\vec{n}\right\rangle=ax_0+by_0+cz_0,\text{则}k=\frac{ax_0+by_0+cz_0}{\left|\vec{n}\right|^2}.$$

故$\left|\vec{\beta_0}-\vec{\beta_1}\right|=\left|k\vec{n}\right|=|k|\left|\vec{n}\right|=\dfrac{|ax_0+by_0+cz_0|}{\left|\vec{n}\right|^2}\left|\vec{n}\right|=\dfrac{|ax_0+by_0+cz_0|}{\sqrt{a^2+b^2+c^2}}$.

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26. 设正交矩阵U.求证:(1)$\det U=1$或-1;(2)U的特征根的模为1;(3)若λ是U的一个特征根,则$\dfrac{1}{\lambda}$也是U的一个特征根;(4)U的伴随矩阵U^*是正交矩阵.

[**证**] (1) $(\det U)^2=(\det U^T)(\det U)=\det\left(U^TU\right)=\det I=1.$

(2) 设$\overrightarrow{\eta}$是U关于特征根λ的特征向量,则$U\overrightarrow{\eta}=\lambda\overrightarrow{\eta}.$

两边左乘U^T得:$U^TU\overrightarrow{\eta}=U^T\lambda\overrightarrow{\eta}=\lambda U^T\overrightarrow{\eta}=\lambda^2\overrightarrow{\eta}.$

两边左乘$\overrightarrow{\eta}^T$得:$\overrightarrow{\eta}^TU^TU\overrightarrow{\eta}=\lambda^2\overrightarrow{\eta}^T\overrightarrow{\eta}$,即$(\lambda^2-1)\overrightarrow{\eta}^T\overrightarrow{\eta}=\overrightarrow{0}.$

因$\overrightarrow{\eta}$是特征向量,则它非负,进而$\left|\overrightarrow{\eta}\right|=1.$

(3) 设$\overrightarrow{\eta}$是U关于特征根λ的特征向量,则$U\overrightarrow{\eta}=\lambda\overrightarrow{\eta}$.两边左乘$U^T$得:$\overrightarrow{\eta}=\lambda U^T\overrightarrow{\eta}.$

由(2)知:$\lambda\neq 0$,则$U^T\overrightarrow{\eta}=\dfrac{1}{\lambda}\overrightarrow{\eta}$,即$\dfrac{1}{\lambda}$是$U^T$的特征根.由$U$与$U^T$特征根相同即证.

(4) 因$U^T=U^{-1},\det U=\pm 1$,则$U^*=(\det U)U^{-1}=\pm U^T$,

进而$(U^*)^TU^*=(\pm U)(\pm U)^T=UU^T=I.$

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27. 求证:$\mathbb{R}^n$中两正交变换之积是正交变换.

[**证**] 设σ和τ是$\mathbb{R}^n$的两正交变换,$\left\{\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}\right\}$是$\mathbb{R}^n$的一个规范正交基,

且σ和τ在该基下的矩阵为正交矩阵A、B.

$(AB)(AB)^T=(AB)(B^TA^T)=A(BB^T)A=AA^T=I.$

$(AB)^T(AB)=(B^TA^T)(AB)=B^T(A^TA)B=B^TB=I.$

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28. 求证:$\mathbb{R}^n$中的正交变换之逆是正交变换.

[**证**] 设σ是$\mathbb{R}^n$的正交变换,$\left\{\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}\right\}$是$\mathbb{R}^n$的一个规范正交基,且$\sigma$在该基下的矩阵为正交矩阵$A$.

由$AA^T=A^TA=I$知:$A^{-1}=A^T.$

$(A^{-1})(A^{-1})^T=A^T(A^T)^T=A^TA=I,(A^{-1})^T(A^{-1})=(A^T)^TA^T=AA^T=I,$

则A^{-1}是正交矩阵,进而σ可逆,其逆在该基下的矩阵为A^{-1},故σ^{-1}是正交变换.

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29. 设σ是欧氏空间V到自身的线性映射,且对$\forall\overrightarrow{\xi},\overrightarrow{\eta}$,有$\left\langle\sigma\left(\overrightarrow{\xi}\right),\sigma\left(\overrightarrow{\eta}\right)\right\rangle=\left\langle\overrightarrow{\xi},\overrightarrow{\eta}\right\rangle$.求证:$\sigma$是$V$的一个线性变换,进而是正交变换.

[**证**] 因$\left\langle\sigma\left(a\overrightarrow{\xi}\right)-a\sigma\left(\overrightarrow{\xi}\right),\sigma\left(a\overrightarrow{\xi}\right)-a\sigma\left(\overrightarrow{\xi}\right)\right\rangle$

$=\left\langle\sigma\left(a\overrightarrow{\xi}\right),\sigma\left(a\overrightarrow{\xi}\right)\right\rangle-\left\langle\sigma\left(a\overrightarrow{\xi}\right),a\sigma\left(\overrightarrow{\xi}\right)\right\rangle-\left\langle a\sigma\left(\overrightarrow{\xi}\right),\sigma\left(a\overrightarrow{\xi}\right)\right\rangle+\left\langle a\sigma\left(\overrightarrow{\xi}\right),a\sigma\left(\overrightarrow{\xi}\right)\right\rangle$

$=2a^2\left\langle\overrightarrow{\xi},\overrightarrow{\xi}\right\rangle-2a^2\left\langle\overrightarrow{\xi},\overrightarrow{\xi}\right\rangle=0$,则$\sigma\left(a\overrightarrow{\xi}\right)-a\sigma\left(\overrightarrow{\xi}\right)=\overrightarrow{0}$,即$\sigma\left(a\overrightarrow{\xi}\right)=a\sigma\left(\overrightarrow{\xi}\right).$

因$\left\langle\sigma\left(\overrightarrow{\xi}+\overrightarrow{\eta}\right)-\sigma\left(\overrightarrow{\xi}\right)-\sigma\left(\overrightarrow{\eta}\right),\sigma\left(\overrightarrow{\xi}+\overrightarrow{\eta}\right)-\sigma\left(\overrightarrow{\xi}\right)-\sigma\left(\overrightarrow{\eta}\right)\right\rangle$

$$= \left\langle \overrightarrow{\xi}+\overrightarrow{\eta}, \overrightarrow{\xi}+\overrightarrow{\eta} \right\rangle - \left\langle \overrightarrow{\xi}, \overrightarrow{\xi}+\overrightarrow{\eta} \right\rangle - \left\langle \overrightarrow{\eta}, \overrightarrow{\xi}+\overrightarrow{\eta} \right\rangle - \left\langle \overrightarrow{\xi}+\overrightarrow{\eta}, \overrightarrow{\xi} \right\rangle$$

$$+ \left\langle \overrightarrow{\xi}, \overrightarrow{\xi} \right\rangle + \left\langle \overrightarrow{\eta}, \overrightarrow{\xi} \right\rangle - \left\langle \overrightarrow{\xi}+\overrightarrow{\eta}, \overrightarrow{\eta} \right\rangle + \left\langle \overrightarrow{\xi}, \overrightarrow{\eta} \right\rangle + \left\langle \overrightarrow{\eta}, \overrightarrow{\eta} \right\rangle = 0,\text{则}\sigma\left(\overrightarrow{\xi}+\overrightarrow{\eta}\right) = \sigma\left(\overrightarrow{\xi}\right) + \sigma\left(\overrightarrow{\eta}\right).$$

故σ是V的一个线性变换,又$\left\langle \sigma\left(\overrightarrow{\xi}\right), \sigma\left(\overrightarrow{\eta}\right) \right\rangle = \left\langle \overrightarrow{\xi}, \overrightarrow{\eta} \right\rangle$,故$\sigma$是正交变换.

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30. 设三阶实矩阵U的行列式为1.求证:(1)U有一特征根1;(2)U的特征多项式有形状 $f(x) = x^3 - tx^2 + tx - 1\ \ (-1 \le t \le 3)$.

[**证**] (1) 由**26**知:U的特征根的模为1,且$\det U = 1$,则U的三特征根之积为1.

因U是实矩阵,则U有一实特征根.若U的三特征根都不为1,则实特征根为-1.

(i) 若三个特征根为-1,则$\det U = -1$,矛盾.

(ii) 若另两特征根是一对共轭复数,则$\det U = -1$,矛盾.

综上,U至少有一特征根为1.

(2) V_3的任一正交变换在某一规范正交基下的矩阵为下列三种形式之一:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}.$$

因U是正交矩阵,则它与上述三矩阵之一相似.又U有特征根1,则只能与前两个矩阵相似.

因相似矩阵的特征多项式相同,

$$\begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = \lambda^3 - \lambda^2 - \lambda + 1,$$

$$\begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-\cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \lambda-\cos\varphi \end{vmatrix} = (\lambda-1)[(\lambda-\cos\varphi)^2 + (\sin\varphi)^2]$$

$$= \lambda^3 - (1+2\cos\varphi)\lambda^2 + (1+2\cos\varphi)\lambda - 1,\text{其中}1+2\cos\varphi \in [-1,3].$$

综上,U的特征多项式有形状$f(x) = x^3 - tx^2 + tx - 1\ \ (-1 \le t \le 3)$.

---

31. 设σ是n维欧氏空间V的一个对称变换,且$\sigma^2 = \sigma$.求证:$\exists V$的一个规范正交基 $s.t.\ \sigma$关于该基的矩阵有形状 $\operatorname{diag}\{1,\cdots,1,0,\cdots,0\}$.

[**证**] 取V的一个规范正交基$\left\{\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}\right\}$ $s.t.\ \sigma$在该基下的矩阵A是实对称矩阵.

因A是实对称矩阵,则$\exists$正交矩阵$U\ s.t.\ U^TAU$是对角矩阵,且主对角元为A的特征值.

因$\sigma^2 = \sigma$,则$A^2 = A$.设$\overrightarrow{x} \neq \overrightarrow{0}$是$A$属于特征根$\lambda$的特征向量,则$A\overrightarrow{x} = \lambda\overrightarrow{x}$.

因$A\overrightarrow{x} = A^2\overrightarrow{x} = A\left(\lambda\overrightarrow{x}\right) = \lambda A\overrightarrow{x} = \lambda^2\overrightarrow{x}$,而$A\overrightarrow{x} = \lambda\overrightarrow{x}$,则$\lambda^2\overrightarrow{x} = \lambda\overrightarrow{x}$,即$(\lambda^2 - \lambda)\overrightarrow{x} = \overrightarrow{0}$.

因$\overrightarrow{x} \neq \overrightarrow{0}$,则$\lambda = 0, 1$.

取$\left(\overrightarrow{\beta_1},\cdots,\overrightarrow{\beta_n}\right) = \left(\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}\right)U$,因$U$是正交矩阵,则$\left\{\overrightarrow{\beta_1},\cdots,\overrightarrow{\beta_n}\right\}$是规范正交基,

且σ在该基下的矩阵U^TAU有形状$\text{diag}\{1,\cdots,1,0,\cdots,0\}$.

---

32. 求一正交矩阵$U\ s.t.\ U^TAU$是对角矩阵,其中$A=\begin{bmatrix}11&2&-8\\2&2&10\\-8&10&5\end{bmatrix}$.

[解] $|\lambda I-A|=(\lambda+9)(\lambda-9)(\lambda-18)$,则$A$的特征根$-9,9,18$.

A的属于$-9,9,18$的特征向量分别是$\overrightarrow{\xi_1}=(1,-2,2),\overrightarrow{\xi_2}=(2,\ 2,\ 1),\overrightarrow{\xi_3}=(2,-1,2)$.

因A是对称矩阵,则$\overrightarrow{\xi_1},\overrightarrow{\xi_2},\overrightarrow{\xi_3}$两两正交,将它们分别单位化得:

$\overrightarrow{\eta_1}=\dfrac{1}{3}(1,-2,2),\overrightarrow{\eta_2}=\dfrac{1}{3}(2,2,1),\overrightarrow{\eta_1}=\dfrac{1}{3}(2,-1,-2)$.

取$U=\dfrac{1}{3}\begin{bmatrix}1&2&2\\-2&2&-1\\2&1&-2\end{bmatrix}$,则$U^TAU=\text{diag}\{-9,9,18\}$.

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33. 求证:非奇异的对称矩阵合同于其逆.

[证] $A^TA^{-1}A=A^T=A$.

---

34. 求可逆矩阵$P\ s.t.\ P^TAP$是对角矩阵,其中$A=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}$.

[解] $$\begin{bmatrix}A\\I\end{bmatrix}=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow[\text{row1+=row2}]{\text{col1+=col2}}\begin{bmatrix}2&1&2&2\\1&0&1&1\\2&1&0&1\\2&1&1&0\\1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow[\substack{\text{row2}-=\frac{1}{2}\text{row1}\\ \text{row3,4}-=\text{row1}}]{\substack{\text{col2}-=\frac{1}{2}\text{col1}\\ \text{col3,4}-=\text{col1}}}\begin{bmatrix}2&0&0&0\\0&-\frac{1}{2}&0&0\\0&0&-2&-1\\0&0&-1&-2\\1&-\frac{1}{2}&-1&-1\\1&\frac{1}{2}&-1&-1\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\xrightarrow[\text{row4}-=\frac{1}{2}\text{row3}]{\text{col4}-=\frac{1}{2}\text{col3}}\begin{bmatrix}2&0&0&0\\0&-\frac{1}{2}&0&0\\0&0&-2&0\\0&0&0&-2\\1&-\frac{1}{2}&-1&-\frac{1}{2}\\1&\frac{1}{2}&-1&-\frac{1}{2}\\0&0&1&-\frac{1}{2}\\0&0&0&1\end{bmatrix},\text{故}P=\begin{bmatrix}1&-\frac{1}{2}&-1&-\frac{1}{2}\\1&\frac{1}{2}&-1&-\frac{1}{2}\\0&0&1&-\frac{1}{2}\\0&0&0&1\end{bmatrix}.$$

---

35. 设S是$\mathbb{C}$上的n阶对称矩阵.求证:$\exists\mathbb{C}$上的矩阵$A\ s.t.\ S=A^TA$.

[**证**] 设$\operatorname{rank} S=r$,则$\exists$可逆矩阵$P\ s.t.\ P^TSP=\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}$,即$S$合同于$\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}$.

因$\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}=\begin{bmatrix}I_r & O\\ O & O\end{bmatrix},\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}^T=\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}$,

则$S=\left[P^{-1}\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}\right]\left[\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}(P^T)^{-1}\right]$.

取$A=\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}(P^T)^{-1}$,则$A^T=P^{-1}\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}$,进而$S=A^TA$.

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36. 设$\mathbb{R}$上的矩阵$A=\begin{bmatrix}5 & 4 & 3\\ 4 & 5 & 3\\ 3 & 3 & 2\end{bmatrix},B=\begin{bmatrix}4 & 0 & -6\\ 0 & 1 & 0\\ -6 & 0 & 9\end{bmatrix}$.(1)求证:$A$在$\mathbb{R}$上合同于$B$;(2)求可逆矩阵$P\ s.t.\ P^TAP=B$.

[**解**] (1) $|\lambda I-A|=\lambda(\lambda-1)(\lambda-11)$,则$A$有特征根$0,1,11$,进而$A$的秩和符号差都为$2$.

$|\lambda I-B|=\lambda(\lambda-1)(\lambda-13)$,则$B$有特征根$0,1,13$,进而$B$的秩和符号差都为$2$.

因A与B的秩和符号差都相等,故A合同于B.

(2) A属于$0,1,11$的特征向量分别为

$$\overrightarrow{\alpha_1}=(a,a,-3a),\overrightarrow{\alpha_2}=(b,-b,0),\overrightarrow{\alpha_3}=(3c,3c,2c)\ \ (a,b,c\in\mathbb{R};a,b,c\neq 0).$$

取$\overrightarrow{\beta_1}=(1,1,-3),\overrightarrow{\beta_2}=(1,-1,0),\overrightarrow{\beta_3}=(3,3,2)$,将它们单位化得:

$$\overrightarrow{\gamma_1}=\left(\frac{1}{\sqrt{11}},\frac{1}{\sqrt{11}},-\frac{3}{\sqrt{11}}\right),\overrightarrow{\gamma_2}=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right),\overrightarrow{\gamma_3}=\left(\frac{3}{\sqrt{22}},\frac{3}{\sqrt{22}},\frac{2}{\sqrt{22}}\right).$$

取$T_1=\begin{bmatrix}\dfrac{1}{\sqrt{11}} & \dfrac{1}{\sqrt{2}} & \dfrac{3}{\sqrt{22}}\\ \dfrac{1}{\sqrt{11}} & -\dfrac{1}{\sqrt{2}} & \dfrac{3}{\sqrt{22}}\\ -\dfrac{3}{\sqrt{11}} & 0 & \dfrac{2}{\sqrt{22}}\end{bmatrix}$,则$T_1^TAT_1=\operatorname{diag}\{0,1,11\}$.

同理$\exists$可逆矩阵$T_2\ s.t.\ T_2^TAT_2=\operatorname{diag}\{0,1,13\}$.

取$P=T_1\begin{bmatrix}0 & & \\ & 1 & \\ & & \dfrac{1}{\sqrt{11}}\end{bmatrix}\begin{bmatrix}0 & & \\ & 1 & \\ & & \sqrt{13}\end{bmatrix}T_2^{-1}$,则$P^TAP=B$.

---

37. 判定下列实二次型是否正定:(1)$10x_1^2-2x_2^2+3x_3^2+4x_1x_2+4x_1x_3$;(2)$5x_1^2+x_2^2+5x_3^2+4x_1x_2-8x_1x_3-4x_2x_3$.

[**解**] (1) 二次型的矩阵$A=\begin{bmatrix}10 & 2 & 2\\ 2 & -2 & 0\\ 2 & 0 & 3\end{bmatrix}$,其一阶主子式$A_1=10>0$,

二阶主子式$\begin{vmatrix}10 & 2\\ 2 & -2\end{vmatrix}=-24<0$,故不正定.

(2) 二次型的矩阵$A=\begin{bmatrix}5&2&-4\\2&2&-2\\-4&-2&5\end{bmatrix}$,其一阶主子式$A_1=5>0$,

二阶主子式$\begin{vmatrix}5&2\\2&2\end{vmatrix}=6>0$,三阶主子式$\begin{vmatrix}5&2&-4\\2&2&-2\\-4&-2&5\end{vmatrix}=10>0$,故正定.

---

38. λ取何值时,实二次型$\lambda(x_1^2+x_2^2+x_3^2)+2x_1x_2-2x_2x_3-2x_1x_3+x_4^2$正定.

[解] 矩阵$A=\begin{bmatrix}\lambda&1&-1&0\\1&\lambda&-1&0\\-1&-1&\lambda&0\\0&0&0&1\end{bmatrix}$的各阶主子式$>0$,则$\begin{cases}\lambda>0\\\lambda^2-1>0\\\lambda^3-3\lambda+2>0\end{cases}\Rightarrow\lambda>1$.

---

39. 对任一实对称矩阵A,求证:$\exists$足够大的$t\in\mathbb{R}\ s.t.\ tI+A$正定.

[证] 记$tI+A$的$k\ \ (k=1,\cdots,n)$阶主子式为D_k,则它是关于t的k次首一多项式.

设$D_1(t)=D_2(t)=\cdots=D_n(t)=0$的实数解分别为$t_1,t_2,\cdots,t_s$.取$t_0=\max\{t_1,\cdots,t_s\}$.

下证$t>t_0$时,有$D_k(t)>0\ \ (k=1,\cdots,n)$.若不然,$\exists t>t_0\ s.t.\ D_k(t)\leq0$.

因$D_k(t)\to+\infty\ \ (t\to+\infty)$,由介值定理:$\exists t'\geq t\ s.t.\ D_k(t')=0$,

则$t'\in\{t_1,\cdots,t_s\}$且$t'>t_0$,矛盾.

故$t>t_0$时,以$tI+A$为矩阵的实二次型正定,进而$tI+A$正定.

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40. 设三阶实对称矩阵A的特征值$\lambda_1=6,\lambda_2=\lambda_3=3$,其中$\lambda_1=6$对应的特征向量为$\overrightarrow{\alpha_1}=(1,1,1)^T$.求$A$.

[解I] 因A是三阶实对称矩阵,则$\exists$正交矩阵$Q\ s.t.\ Q^TAQ=\begin{bmatrix}6&&\\&3&\\&&3\end{bmatrix}$,则$Q^T(A-3I)Q=\begin{bmatrix}3&&\\&0&\\&&0\end{bmatrix}$.

设$Q=\left[\overrightarrow{\xi_1},\overrightarrow{\xi_2},\overrightarrow{\xi_3}\right]$,则$\overrightarrow{\xi_1}$是$A-3I$的属于特征值3的单位特征向量,

进而它是A的属于特征值6的单位特征向量,则$\overrightarrow{\xi_1}=\dfrac{\overrightarrow{\alpha_1}}{\left|\overrightarrow{\alpha_1}\right|}=\dfrac{1}{\sqrt{3}}(1,1,1)^T$.

$$A-3I=Q\begin{bmatrix}3&&\\&0&\\&&0\end{bmatrix}Q^T=\left[\overrightarrow{\xi_1},\overrightarrow{\xi_2},\overrightarrow{\xi_3}\right]\begin{bmatrix}3&&\\&0&\\&&0\end{bmatrix}\begin{bmatrix}\overrightarrow{\xi_1}^T\\\overrightarrow{\xi_2}^T\\\overrightarrow{\xi_3}^T\end{bmatrix}=\left[3\overrightarrow{\xi_1},\overrightarrow{0},\overrightarrow{0}\right]\begin{bmatrix}\overrightarrow{\xi_1}^T\\\overrightarrow{\xi_2}^T\\\overrightarrow{\xi_3}^T\end{bmatrix}$$

$=3\overrightarrow{\xi_1}\overrightarrow{\xi_1}^T=3\cdot\dfrac{1}{\sqrt{3}}(1,1,1)^T\cdot\dfrac{1}{\sqrt{3}}(1,1,1)=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$,故$A=\begin{bmatrix}4&1&1\\1&4&1\\1&1&4\end{bmatrix}$.

[解II] 设A的属于特征值3的特征向量$\overrightarrow{\alpha_2}=(x_1,x_2,x_3)^T,\overrightarrow{\alpha_3}=(y_1,y_2,y_3)^T$.

由实对称矩阵属于不同特征值的特征向量相互正交知:$\left\langle\overrightarrow{\alpha_1},\overrightarrow{\alpha_2}\right\rangle=\left\langle\overrightarrow{\alpha_1},\overrightarrow{\alpha_3}\right\rangle=0$,

即$x_1+x_2+x_3=0,y_1+y_2+y_3=0$,即$\overrightarrow{\alpha_2}$和$\overrightarrow{\alpha_3}$是齐次线性方程组$x_1+x_2+x_3=0$的线性无关的解.

取$\overrightarrow{\alpha_2}=(1,1,-2)^T$.为使得$\left\langle\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right\rangle=0$,则$y_1+y_2-2y_3=0$.

由$\begin{cases}y_1+y_2+y_3=0\\y_1+y_2-2y_3=0\end{cases}$,解得$\overrightarrow{\alpha_3}=(a,-a,0)^T\ \ (a\in\mathbb{R},a\neq 0)$,不妨取$\overrightarrow{\alpha_3}=(1,-1,0)$.

令$\overrightarrow{\beta_1}=\dfrac{1}{\sqrt{3}}(1,1,1)^T,\overrightarrow{\beta_2}=\dfrac{1}{\sqrt{6}}(1,1,-2),\overrightarrow{\beta_3}=\dfrac{1}{\sqrt{2}}(1,-1,0)$.

取$U=\left(\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right)$,则它是正交矩阵,且$Q^TAT=\begin{bmatrix}6&&\\&3&\\&&3\end{bmatrix}$.

故

$$A=Q\begin{bmatrix}6&&\\&3&\\&&3\end{bmatrix}Q^T=\begin{bmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{3}}&-\frac{2}{\sqrt{6}}&0\end{bmatrix}\begin{bmatrix}6&&\\&3&\\&&3\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}&-\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{2}}&-\frac{2}{\sqrt{2}}&0\end{bmatrix}=\begin{bmatrix}4&1&1\\1&4&1\\1&1&4\end{bmatrix}.$$

---

41. 求A^n,其中:(1)$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$;(2)$A=\begin{bmatrix}3&4\\4&-3\end{bmatrix}$;(3)$A=\begin{bmatrix}0&1&-2\\1&0&1\\2&1&0\end{bmatrix}$.

[解] (1) $|\lambda I-A|=\lambda^2-1$,则$A^2-I=O\Rightarrow A^2=I$.

$n=2k$时,$A^{2k}=(A^2)^k=I^k=I;n=2k+1$时,$A^{2k+1}=A^{2k}A=IA=A$.

(2) $|\lambda I-A|=\lambda^2-25$,则$A^2-25I=O\Rightarrow A^2=25I$.

$n=2k$时,$A^{2k}=(A^2)^k=(25I)^k=25^kI;n=2k+1$时,$A^{2k+1}=A^{2k+1}A=25^kIA=25^kA$.

(3) $|\lambda I-A|=\lambda^3+2\lambda$,则$A^3+2A=O\Rightarrow A^3=-2A$.两边右乘$A^{m-3}\ \ (m\geq 3)$得:
$A^m=-2A^{m-2}\ \ (m\geq 3)$.

$n=2k$时,$A^{2k}=-2A^{2k-2}=\cdots=(-2)^{k-1}A^2;$

$n=2k+1$时,$A^{2k+1}=A^{2k}A=(-2)^{k-1}A^3=(-2)^kA$.

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42. 设$A=\begin{bmatrix}1&0&1\\0&2&0\\1&0&1\end{bmatrix}$,求$A^n-2A^{n-1}$和$A^n$.

[解] $|\lambda I-A|=\lambda(\lambda-2)(\lambda-2)$,则$A(A-2I)(A-2I)=O$,即$(A^2-2A)A=(A^2-2A)2I$,

亦即$A^3-2A^2=2(A^2-2A)$.两边同右乘$A^{m-3}\ \ (m\geq 3)$得:$A^m-2A^{m-1}=2(A^{m-1}-2A^{m-2})$.

初始条件$A^2=\begin{bmatrix}2&0&2\\0&4&0\\2&0&2\end{bmatrix}=2A$,则$A^2-2A=O$.

令$a_m=A^m-2A^{m-1}$,则$a_m=2a_{m-1}\ \ (m\geq 3)$,进而
$a_n=2a_{n-1}=\cdots=2^{n-2}a_2=2^{n-2}(A^2-2A)=O$,

即$a_n=O\ \ (n\geq 2)$. *注意$m\geq 3$,故不能递归到a_1

$A^n=2A^{n-1}=\cdots=2^{n-1}A.$

---

43. 设方阵A相似于方阵B.判断正误:(1)对相同的特征值λ,两矩阵有相同的特征向量;(2)A与B都与同一对角矩阵相似.

[解] (1) 错.如矩阵$A=\begin{bmatrix}4&1\\1&4\end{bmatrix}$相似于$B=\begin{bmatrix}3&0\\0&5\end{bmatrix}$,特征值$\lambda_1=3,\lambda_2=5.$

A和B属于特征值$\lambda=3$的特征向量分别为$(1,-1)^T$和$(3,-5)^T$.

(2) 错.如$A=\begin{bmatrix}1&2\\0&1\end{bmatrix}$不可相似对角化.对矩阵$B=\begin{bmatrix}5&8\\-2&-3\end{bmatrix}$,$\exists P=\begin{bmatrix}1&1\\1&2\end{bmatrix}\ s.t.\ P^{-1}AP=B.$

或Jordan块不可对角化,但可通过左乘和右乘一个可逆矩阵构造一个与其相似的矩阵.

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44. 设三阶矩阵A.若A^{-1}的特征值为$1,2,3$,求A^*的迹.

[引理] 设A的特征值为λ,则$\lambda\overrightarrow{\alpha}=A\overrightarrow{\alpha}$,两边左乘$A^*$得:$\lambda A^*\overrightarrow{\alpha}=A^*A\overrightarrow{\alpha}=|A|\overrightarrow{\alpha}\Rightarrow A^*$的特征值为$\dfrac{|A|}{\lambda}$.

$A^*A\overrightarrow{\alpha}=|A|\overrightarrow{\alpha}$,两边同除以$|A|$得:$A^{-1}$的特征值为$\dfrac{1}{\lambda}$.

[解] A的特征值为$1,\dfrac{1}{2},\dfrac{1}{3}$,则$|A|=\dfrac{1}{6}$,进而$A^*$的特征值为$\dfrac{\frac{1}{6}}{1},\dfrac{\frac{1}{6}}{\frac{1}{2}},\dfrac{\frac{1}{6}}{\frac{1}{3}}$,即$\dfrac{1}{6},\dfrac{1}{3},\dfrac{1}{2}$,故$\mathrm{tr}\ A=1.$

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45. 设$\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}$是数域$K$上的向量空间$V$的一组基.已知线性变换$\sigma$在该基下的矩阵$A=\begin{bmatrix}1&0&2&1\\-1&2&1&3\\1&2&5&5\\2&-2&1&-2\end{bmatrix}$.

求$\mathrm{Ker}\sigma$和$\mathrm{Im}\sigma$.

[解] 设$\overrightarrow{\xi}=x_1\overrightarrow{\alpha_1}+x_2\overrightarrow{\alpha_2}+x_3\overrightarrow{\alpha_3}+x_4\overrightarrow{\alpha_4}\in\mathrm{Ker}\sigma$,

则$A\overrightarrow{\xi}=A\left[\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}\right]\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right]=\left[A\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}\right]\right]\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}\right]A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\overrightarrow{0}$,

这表明:$\mathrm{Ker}\sigma$中的所有向量在$\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}$下的坐标构成$A\overrightarrow{X}=\overrightarrow{0}$的一个解空间.

因$A\rightarrow\begin{bmatrix}1&0&2&1\\0&2&3&4\\0&0&0&0\\0&0&0&0\end{bmatrix}$,解得$A\overrightarrow{X}=\overrightarrow{0}$的基础解系为$\overrightarrow{\beta_1}=\begin{bmatrix}-2\\1\\2\\-2\end{bmatrix}$,$\overrightarrow{\beta_2}=\begin{bmatrix}1\\2\\0\\-1\end{bmatrix}$,

取$\overrightarrow{\gamma_1}=-2\overrightarrow{\alpha_1}+\overrightarrow{\alpha_2}+2\overrightarrow{\alpha_3}-2\overrightarrow{\alpha_4},\overrightarrow{\gamma_2}=\overrightarrow{\alpha_1}+2\overrightarrow{\alpha_2}-\overrightarrow{\alpha_4}$,则$\mathrm{Ker}\sigma=\mathscr{L}\left(\overrightarrow{\gamma_1},\overrightarrow{\gamma_2}\right)$.

设$\overrightarrow{\eta}=y_1\overrightarrow{\alpha_1}+y_2\overrightarrow{\alpha_2}+y_3\overrightarrow{\alpha_3}+y_4\overrightarrow{\alpha_4}\in\mathrm{Im}\sigma$,则$A\overrightarrow{\alpha}=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}\right]A\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}$,

这表明:$\mathrm{Im}\sigma$中所有向量在$\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3},\overrightarrow{\alpha_4}$下的坐标构成$A$的列空间.

因$\operatorname{rank} A=2$,取A的极大线性无关组$\overrightarrow{\delta_1}=\begin{bmatrix}1\\-1\\1\\2\end{bmatrix},\overrightarrow{\delta_2}=\begin{bmatrix}0\\2\\2\\-2\end{bmatrix}$,则$A$的列空间为$\mathscr{L}\left(\overrightarrow{\delta_1},\overrightarrow{\delta_2}\right)$.

取$\overrightarrow{\chi_1}=\overrightarrow{\alpha_1}-\overrightarrow{\alpha_2}+\overrightarrow{\alpha_3}+2\overrightarrow{\alpha_4},\overrightarrow{\chi_2}=2\overrightarrow{\alpha_2}+2\overrightarrow{\alpha_3}-2\overrightarrow{\alpha_4}$,则$\operatorname{Im}\sigma=\mathscr{L}\left(\overrightarrow{\chi_1},\overrightarrow{\chi_2}\right)$.

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46. 设线性变换σ关于基$\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2}\right\}$的矩阵为$\begin{bmatrix}a&b\\c&d\end{bmatrix}$,求$\sigma$关于基$\left\{3\overrightarrow{\alpha_2},\overrightarrow{\alpha_1}\right\}$的矩阵.

[解] 因$\sigma\left(\overrightarrow{\alpha_1},\overrightarrow{\alpha_2}\right)=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2}\right]\begin{bmatrix}a&b\\c&d\end{bmatrix}$,则$\sigma\left(\overrightarrow{\alpha_1}\right)=a\overrightarrow{\alpha_1}+c\overrightarrow{\alpha_2},\sigma\left(\overrightarrow{\alpha_2}\right)=b\overrightarrow{\alpha_1}+d\overrightarrow{\alpha_2}$,

进而$\sigma\left(3\overrightarrow{\alpha_2}\right)=d\left(3\overrightarrow{\alpha_2}\right)+3b\overrightarrow{\alpha_1},\sigma\left(\overrightarrow{\alpha_1}\right)=\dfrac{c}{3}\left(3\overrightarrow{\alpha_2}\right)+a\overrightarrow{\alpha_1}$.

故σ关于基$\left\{3\overrightarrow{\alpha_2},\overrightarrow{\alpha_1}\right\}$的矩阵为$\begin{bmatrix}d&\dfrac{c}{3}\\3b&a\end{bmatrix}$.

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47. 设数域F上的三维向量空间V的线性变换σ关于基$\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right\}$的矩阵为$A=\begin{bmatrix}15&-11&5\\20&-15&8\\8&-7&6\end{bmatrix}$.设 $\overrightarrow{\beta_1}=2\overrightarrow{\alpha_1}+3\overrightarrow{\alpha_2}+\overrightarrow{\alpha_3},\overrightarrow{\beta_2}=3\overrightarrow{\alpha_1}+4\overrightarrow{\alpha_2}+\overrightarrow{\alpha_3},\overrightarrow{\beta_3}=\overrightarrow{\alpha_1}+2\overrightarrow{\alpha_2}+\overrightarrow{\alpha_3}$.(1)求$\sigma$关于基$\left\{\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right\}$的矩阵;(3)设 $\overrightarrow{\xi}=2\overrightarrow{\alpha_1}+\overrightarrow{\alpha_2}-\overrightarrow{\alpha_3}$,求$\sigma\left(\overrightarrow{\xi}\right)$关于基$\left\{\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right\}$的坐标.

[解] (1) 设$T=\begin{bmatrix}2&3&1\\3&4&2\\1&1&2\end{bmatrix}$,

则$\left[\sigma\left(\overrightarrow{\alpha_1}\right),\sigma\left(\overrightarrow{\alpha_2}\right),\sigma\left(\overrightarrow{\alpha_3}\right)\right]=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]A,\left[\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right]=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]T$.

进而σ关于基$\left\{\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right\}$的矩阵

$$T^{-1}AT=\begin{bmatrix}-6&5&-2\\4&-3&1\\1&-1&1\end{bmatrix}\begin{bmatrix}15&-11&5\\20&-15&8\\0&-7&6\end{bmatrix}\begin{bmatrix}2&3&1\\3&4&2\\1&1&2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}.$$

(2) $\sigma\left(\overrightarrow{\xi}\right)=\left[\sigma\left(\overrightarrow{\alpha_1}\right),\sigma\left(\overrightarrow{\alpha_2}\right),\sigma\left(\overrightarrow{\alpha_3}\right)\right]\begin{bmatrix}2\\1\\-1\end{bmatrix}=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]A\begin{bmatrix}2\\1\\-1\end{bmatrix}=\left[\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right]T^{-1}A\begin{bmatrix}2\\1\\-1\end{bmatrix}$,

故$\sigma\left(\overrightarrow{\xi}\right)$关于基$\left\{\overrightarrow{\beta_1},\overrightarrow{\beta_2},\overrightarrow{\beta_3}\right\}$的坐标为$T^{-1}A\begin{bmatrix}2\\1\\-1\end{bmatrix}=\begin{bmatrix}-5\\8\\0\end{bmatrix}$.

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48. 求证:两矩阵的特征值相等是它们相似的必要条件.

[**证**] (必要性) 设矩阵A相似于B,则$\exists$可逆矩阵$P\ s.t.\ P^{-1}AP=B$.

$$|\lambda I-B|=|P^{-1}(\lambda I)P-P^{-1}AP|=|P^{-1}(\lambda I-A)P|=|P^{-1}||\lambda I-A||P|=|\lambda I-A|,$$

即A与B的特征多项式相等,进而特征值相等.

(充分性) $A=\begin{bmatrix}0&0\\0&0\end{bmatrix}$和$B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$的特征值都为0,但它们不相似.

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49. 设A是n阶实对称矩阵,P是n阶可逆矩阵.已知n维列向量$\overrightarrow{\alpha}$是A属于特征值λ的特征向量,求矩阵$(P^{-1}AP)^T$属于特征值λ的特征向量.

[**解**] $A\overrightarrow{\alpha}=\lambda\overrightarrow{\alpha},(P^{-1}AP)^T=P^TA(P^T)^{-1}$,两边右乘$P^T\overrightarrow{\alpha}$得:

$(P^{-1}AP)^T\left(P^T\overrightarrow{\alpha}\right)=P^TA\left[(P^T)^{-1}P^T\right]\overrightarrow{\alpha}=P^TA\overrightarrow{\alpha}=\lambda\left(P^T\overrightarrow{\alpha}\right)$,故答案为$P^T\overrightarrow{\alpha}$.

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50. 设$\mathbb{R}$上的n维向量空间V的线性变换σ关于基$\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right\}$的矩阵$A=\begin{bmatrix}3&3&2\\1&1&-2\\-3&-1&0\end{bmatrix}$.求$\sigma$的本征值及其对应的本征向量.

[**解**] $|\lambda I-A|=(x-4)(x^2+4)$,本征值$\lambda=4$.

$(4I-A)\overrightarrow{X}=\overrightarrow{0}\Rightarrow\overrightarrow{X}=(a,a,-a)\ \ (a\in\mathbb{R},a\neq 0)$,

故σ属于本征值4的本征向量为$a\overrightarrow{\alpha_1}+a\overrightarrow{\alpha_2}+a\overrightarrow{\alpha_3}\ \ (a\in\mathbb{R},a\neq 0)$.

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51. 定义$\mathbb{R}^3$上的线性变换$\sigma(x_1,x_2,x_3)=(2x_1-x_2,x_2+x_3,x_1)$.(1)求$\sigma$在标准基下的矩阵;(2)设$\overrightarrow{\alpha}=(1,0,-2)$,求$\sigma\left(\overrightarrow{\alpha}\right)$在基$\overrightarrow{\alpha_1}=(2,0,1),\overrightarrow{\alpha_2}=(0,-1,1),\overrightarrow{\alpha_3}=(-1,0,2)$下的坐标;(3)证明$\sigma$可逆,并求$\sigma^{-1}$.

[**解I**] (1) $\sigma\left(\overrightarrow{\varepsilon_1}\right)=(2,0,1),\sigma\left(\overrightarrow{\varepsilon_2}\right)=(-1,1,0),\sigma\left(\overrightarrow{\varepsilon_3}\right)=(0,1,0)$,

则$\sigma\left(\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right)=\left[\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right]\begin{bmatrix}2&-1&0\\0&1&1\\1&0&0\end{bmatrix}$.

(2) $\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]=\left[\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right]\begin{bmatrix}2&0&-1\\0&-1&0\\1&1&2\end{bmatrix}$,

$$\sigma\left(\overrightarrow{\alpha}\right)=\sigma\left[\left[\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right]\begin{bmatrix}1\\0\\-2\end{bmatrix}\right]=\sigma\left(\left[\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right]\right)\begin{bmatrix}1\\0\\-2\end{bmatrix}$$

$$=\left[\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right]\begin{bmatrix}2&-1&0\\0&1&1\\1&0&0\end{bmatrix}\begin{bmatrix}1\\0\\-2\end{bmatrix}=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]\begin{bmatrix}2&0&-1\\0&-1&0\\1&1&2\end{bmatrix}^{-1}\begin{bmatrix}2&-1&0\\0&1&1\\1&0&0\end{bmatrix}\begin{bmatrix}1\\0\\-2\end{bmatrix}$$

$$=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]\begin{bmatrix}\frac{3}{5}\\2\\-\frac{4}{5}\end{bmatrix}.$$

(3) 因σ关于标准基的矩阵$\begin{bmatrix}2&-1&0\\0&1&1\\1&0&0\end{bmatrix}$可逆,则$\sigma$可逆.对$\forall\overrightarrow{\beta}=(x_1,x_2,x_3)\in\mathbb{R}^3$,

$$\sigma^{-1}\left(\overrightarrow{\beta}\right)=\sigma^{-1}\left(\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\left[\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right]\begin{bmatrix}2&-1&0\\0&1&1\\1&0&0\end{bmatrix}^{-1}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$=\left[\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right]\begin{bmatrix}2&0&-1\\0&-1&0\\1&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\left[\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right]\begin{bmatrix}2x_1-3\\-x_2\\x_1+x_2+2x_3\end{bmatrix}=(2x_1-3,-x_2,x_1+x_2+2x_3).$$

[**解II**] (2) $\sigma\left(\overrightarrow{\alpha}\right)=\sigma(1,0,-2)=(2,-2,1)=\left[\overrightarrow{\varepsilon_1},\overrightarrow{\varepsilon_2},\overrightarrow{\varepsilon_3}\right]\begin{bmatrix}2\\-2\\1\end{bmatrix}$

$$=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]\begin{bmatrix}2&0&-1\\0&-1&0\\1&1&2\end{bmatrix}^{-1}\begin{bmatrix}2\\-2\\1\end{bmatrix}=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]\begin{bmatrix}\frac{2}{5}&\frac{1}{5}&\frac{1}{5}\\0&-1&0\\-\frac{1}{5}&\frac{2}{5}&\frac{2}{5}\end{bmatrix}^{-1}\begin{bmatrix}2\\-2\\1\end{bmatrix}$$

$$=\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]^{-1}\begin{bmatrix}\frac{3}{5}\\2\\-\frac{4}{5}\end{bmatrix}.$$

[**解III**] (2) $\sigma\left(\overrightarrow{\alpha}\right)=(2,-2,1)$.

设$\sigma\left(\overrightarrow{\alpha}\right)$在基$\left\{\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right\}$下的坐标为$(x_1,x_2,x_3)$,则$\left[\overrightarrow{\alpha_1},\overrightarrow{\alpha_2},\overrightarrow{\alpha_3}\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\sigma\left(\overrightarrow{\alpha}\right)=(2,-2,1)$.

解方程组$\left[\overrightarrow{\alpha_1}^T,\overrightarrow{\alpha_2}^T,\overrightarrow{\alpha_3}^T\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-2\\1\end{bmatrix}$,即$\begin{bmatrix}2&0&-1\\0&-1&0\\1&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-2\\1\end{bmatrix}$,解得$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\frac{3}{5}\\2\\-\frac{4}{5}\end{bmatrix}$.

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52. 设σ是数域F上n维向量空间V的线性变换,$\left\{\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}\right\}$是$V$的一个基.求证:$\left\{\sigma\left(\overrightarrow{\alpha_1}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)\right\}$是$V$的基的充要条件是:$\sigma$可逆.

[**证**] (充分性) 若$k_1\sigma\left(\overrightarrow{\alpha_1}\right)+\cdots+k_n\sigma\left(\overrightarrow{\alpha_n}\right)=\overrightarrow{0}$,两边作用$\sigma^{-1}$得:$k_1\overrightarrow{\alpha_1}+\cdots+k_n\overrightarrow{\alpha_n}=\overrightarrow{0}$.

这表明:$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}$线性无关,又$\dim V=n$,则$\left\{\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}\right\}$是$V$的一个基.

(必要性) 因$\left\{\sigma\left(\overrightarrow{\alpha_1}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)\right\}$是$V$的一个基,则$\dim(\mathrm{Im}\sigma)=n$,进而$\dim(\mathrm{Ker}\sigma)=0$,

即$\mathrm{Ker}\sigma=\left\{\overrightarrow{0}\right\}$,则$\sigma$是双射,进而可逆.

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53. 设σ是线性空间的可逆线性变换,$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}$是线性无关组.求证:$\sigma\left(\overrightarrow{\alpha_1}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)$线性无关.

[**证**] $k_1\sigma\left(\overrightarrow{\alpha_1}\right)+\cdots+k_n\sigma\left(\overrightarrow{\alpha_n}\right)=\sigma\left(k_1\overrightarrow{\alpha_1}+\cdots+k_n\overrightarrow{\alpha_n}\right)=\overrightarrow{0}$,

两边作用σ^{-1}得:$k_1\overrightarrow{\alpha_1}+\cdots+k_n\overrightarrow{\alpha_n}=\overrightarrow{0}$.

因$\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}$线性无关,则$k_1=\cdots=k_n=0$,故$\sigma\left(\overrightarrow{\alpha_1}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)$线性无关.

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54. 设σ是数域F上的n维向量空间V的一个线性变换,$\{\overrightarrow{\alpha_1},\cdots,\overrightarrow{\alpha_n}\}$是$V$的一个基.求证: $\mathrm{Im}\sigma=\mathscr{L}\left(\sigma\left(\overrightarrow{\alpha_1}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)\right)$.

[**证**] 取$\overrightarrow{\beta}\in\mathrm{Im}\sigma$,则$\exists\overrightarrow{\gamma}\in V\ s.t.\ \sigma\left(\overrightarrow{\gamma}\right)=\overrightarrow{\beta}$.设$\overrightarrow{\gamma}=\sum_{i=1}^n k_i\overrightarrow{\alpha_i}$,则$\sigma\left(\overrightarrow{\gamma}\right)=\sum_{i=1}^n k_i\sigma\left(\overrightarrow{\alpha_i}\right)$,

即$\overrightarrow{\beta}=\sum_{i=1}^n k_i\sigma\left(\overrightarrow{\alpha_i}\right)$,这表明:$\mathrm{Im}\sigma\subseteq\mathscr{L}\left(\sigma\left(\overrightarrow{\alpha_1}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)\right)$.

取$\overrightarrow{\beta}\in\mathscr{L}\left(\sigma\left(\overrightarrow{\alpha_1}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)\right)$,则$\overrightarrow{\beta}=\sum_{i=1}^n k_i\sigma\left(\overrightarrow{\alpha_i}\right)=\sigma\left(\sum_{i=1}^n k_i\overrightarrow{\alpha_i}\right)$,

即$\overrightarrow{\beta}\in\mathrm{Im}\sigma$,这表明:$\mathscr{L}\left(\sigma\left(\overrightarrow{\alpha_1}\right),\cdots,\sigma\left(\overrightarrow{\alpha_n}\right)\right)\subseteq\mathrm{Im}\sigma$,故证.

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55. 求证:不同的线性变换在同一组基下的矩阵一定不同.

[**解**] 设σ和τ是n维向量V的两个线性变换,且它们在标准基下的矩阵都为A,

即$\sigma\left(\overrightarrow{\varepsilon_1},\cdots,\overrightarrow{\varepsilon_n}\right)=\tau\left(\overrightarrow{\varepsilon_1},\cdots,\overrightarrow{\varepsilon_n}\right)=\left(\overrightarrow{\varepsilon_1},\cdots,\overrightarrow{\varepsilon_n}\right)A$.

取$\overrightarrow{\alpha}=\sum_{i=1}^n k_i\overrightarrow{\varepsilon_i}$,则$\sigma\left(\overrightarrow{\alpha}\right)=\sigma\left(k_1\overrightarrow{\varepsilon_1}+\cdots+k_n\overrightarrow{\varepsilon_n}\right)=k_1\sigma\left(\overrightarrow{\varepsilon_1}\right)+\cdots+k_n\sigma\left(\overrightarrow{\varepsilon_n}\right)$

$$=\left(\sigma\left(\overrightarrow{\varepsilon_1}\right),\cdots,\sigma\left(\overrightarrow{\varepsilon_n}\right)\right)\begin{bmatrix}k_1\\\vdots\\k_n\end{bmatrix}=\left(\overrightarrow{\varepsilon_1},\cdots,\overrightarrow{\varepsilon_n}\right)A\begin{bmatrix}k_1\\\vdots\\k_n\end{bmatrix}.$$

$\tau\left(\overrightarrow{\alpha}\right)=\tau\left(k_1\overrightarrow{\varepsilon_1}+\cdots+k_n\overrightarrow{\varepsilon_n}\right)=k_1\tau\left(\overrightarrow{\varepsilon_1}\right)+\cdots+k_n\tau\left(\overrightarrow{\varepsilon_n}\right)$

$$=\left(\tau\left(\overrightarrow{\varepsilon_1}\right),\cdots,\tau\left(\overrightarrow{\varepsilon_n}\right)\right)\begin{bmatrix}k_1\\\vdots\\k_n\end{bmatrix}=\left(\overrightarrow{\varepsilon_1},\cdots,\overrightarrow{\varepsilon_n}\right)A\begin{bmatrix}k_1\\\vdots\\k_n\end{bmatrix}=\sigma\left(\overrightarrow{\alpha}\right).$$

由$\overrightarrow{\alpha}$的任意性知:σ和τ是同一线性变换,故不同线性变换在同一组基下的矩阵一定不同.