CONTROL &INSTRUMENTATION ENGINEERING 3

Lab Report

S2269664

UNIVERSITY OF EDINBURGH

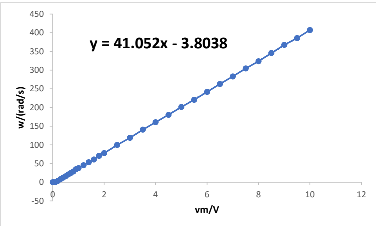
Static Characterisation of DC Motors

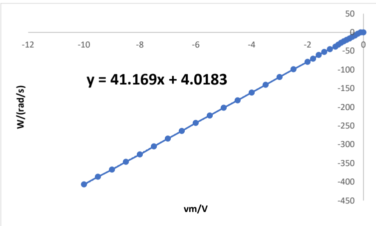
Exercise 2.3.1 *[4/100 marks]* Calculate and plot the gain of the motor vs. input voltage. What do you observe? Does it follow the behavior of the simplified dc motor given by Equation 2.1? ■

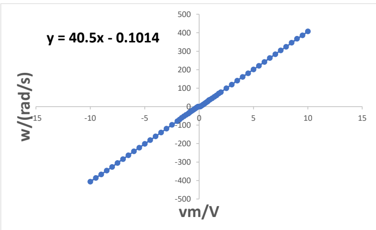
In this experiment, I am using finer increments for voltages near zero. From 0—1V and -1—0V, I increase/decrease the voltage by 0.1V each time. From 1—2V and -2—-1V, I change voltage by 0.2V each time, and from 2—10V and -10—-2V, I change voltage by 0.5V each time.

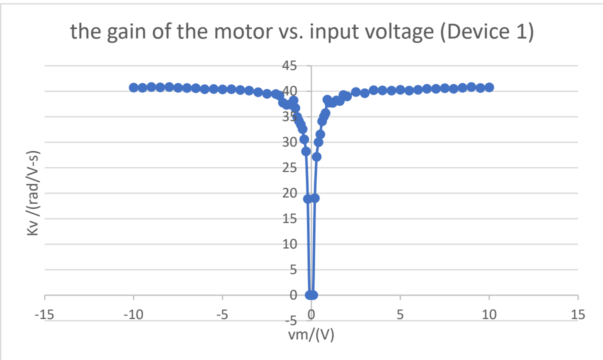
Every time when setting a new voltage, I allow the motor a few seconds to settle at the new speed before recording the measurement, however, the motor reacts quite quickly so I can carry out the experiment quickly.

Device 1:

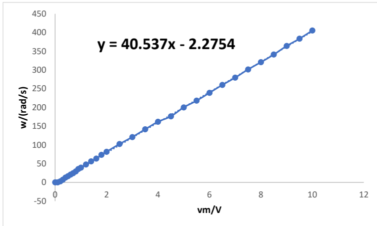


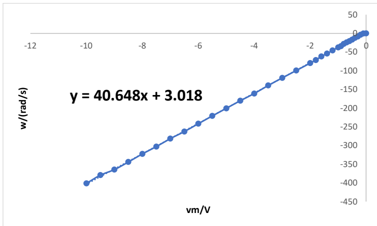


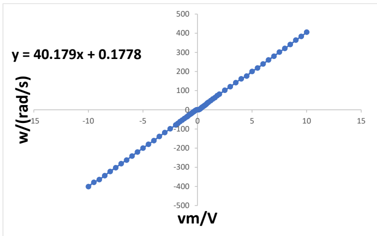


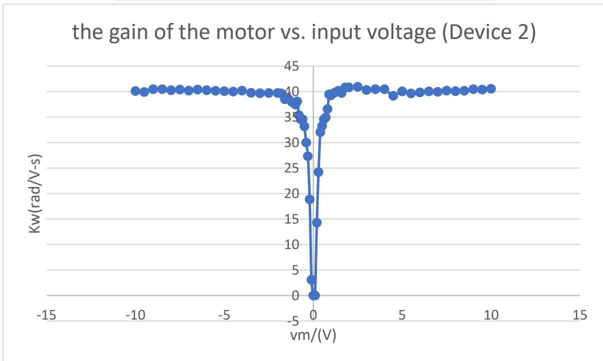


Device 2:









I plotted four graphs and analysed them: the relationship between angular velocity and input voltage at 0—10V, -10—0V, and -10—10V, and the relationship between the gain and the input voltage.

Analise the ω vs V graph:
The 0—10V, -10—0V graph is a bit far from a proportional function as the workbook shows us, and they have a larger K_w (41.05/41.17 for device 1 and 40.5/40.6 for device 2). However, the -10—10V graph is much closer to the ideal proportional function situation, and it has a lower K_w (40.5 for device 1 and 40.1 for device 2).

Analise the K_w vs V graph:
The value of K_w is stable when the input voltage is far from 0(-10—-2V&2—10V), but when the input voltage is close to 0, K_w decreases.

Basically, it follows the behaviour of the simplified DC motor given by Equation 2.1.

Exercise 2.3.2 *[4/100 marks]* Discuss the possible reasons for a potential non-perfect match of the response of the theoretical model with the one of the actual system. What are the possible causes of such errors and how would they affect the operation of a system using this motor? ■

The experimental results do not fit the theoretical model perfectly.

Analise the ω vs V graph:

The reason for the 0—10V, -10—0V situation is that when the input voltage is small, the motor will not move at all because of the static friction force, so when the input voltage equals +0.1 or -0.1, the motor speed is 0. And when the input voltage can generate the electromagnetic force that can move the motor, the sliding friction force is still considerable compared to the electromagnetic force, which will reduce the motor speed. As the input voltage increases, the sliding friction force will not increase, but the electromagnetic force increases, and the error caused by the friction will reduce. That is why the Kw in these situations will be larger, and there will be plus or minus a big number after it.

In the -10—10V situation, the error caused by friction is much smaller because both positive and negative condition cancels each other out. So, the non-perfect match may be caused by random error.

Analise the K_w vs V graph:

K_w near zero is significantly smaller because the ratio of friction to driving torque is greater. K_w also shakes slightly when V is much larger than zero because of random error.

Comparing 2 devices, the difference is the mass of the plates, and mass is related to their gravity, and because they are placed on the system vertically, it will make a difference to the pressure. Because they use the same base, we can assume that the coefficient of friction is the same. Thinking about the friction force function $f = \mu N$, the second device will be less influenced by friction under the same voltage. And we can tell this from the data, the difference between the -10—10V and 0—10V is smaller than device 1.

Moreover, when the measured value is small, the measurement error will be bigger.

Affects:

If the system can run the motor at a high voltage(V>2v or V<-2v), the motor will follow the equation.

If the input voltage is close to zero, the rotation motor will be hard to predict because it will not follow the equation, so, in this situation, we cannot use this motor.

Exercise 2.3.3 *[4/100 marks]* The encoder measuring the velocity of the spinner has a resolution of 500 CPR (counts or pulses per revolution), the value is stored in a 15-bit word and the sampling period is 0.05 s. **(a)** What is the maximum pulse count it measured during your experiment? **(b)** What is the maximum angular velocity that this setup can measure? **(c)** What is the maximum allowable sampling period for your system (rounded to the nearest 1/20th of a second)? ■

(a) Resolution: 500 counts per revolution

The value of count is stored in a 15-bits word, which means that the maximum count range is from 0 to $(2^{15} - 1)$.

Sampling period: 0.05s

To get count we need to find how many rounds the encoder has gone through. We can firstly get radius then divide it by 2π to get rounds.

The maximum angular velocity:

D1: ω_{max}=407.27rad/s

D2: ω_{max}=405.49rad/s

So, the max angle is

$$\text{D1: } \omega_{max} \times sampling\ period = 407.27rad/s \times 0.05s = 20.36rad$$

$$\text{D2: } \omega_{max} \times sampling\ period = 405.49rad/s \times 0.05s = 20.27rad$$

Divide it by 2π we have the maximum rounds

$$\text{D1: } max\ rounds = \frac{Max\ angle}{2\pi} = \frac{20.36}{2\pi} = 3.24$$

$$\text{D2: } max\ rounds = \frac{Max\ angle}{2\pi} = \frac{20.27}{2\pi} = 3.23$$

Then it multiplied by counts per round to get the maximum count

$$\text{D1: max count} = 3.24 \times 500 = 1620$$

$$\text{D2: max count} = 3.23 \times 500 = 1615$$

(b) Under the circumstance of the max velocity, it will use up all the storage space of 15 bits. So, the count value will reach $2^{15} - 1$, divide it by resolution, which is 500 CPR, we can get the max rounds $= \frac{2^{15}-1}{500} = 65.534.$

These rounds are executed in 0.05s (the sampling period), multiplied the rounds with 2π, max angle $= 65.5 \times 2\pi = 411.76\ rads.$

So, the maximum angular velocity

$$\text{maximum angular velocity} = \frac{411.76\ rads}{0.05s} = 8235.2 \text{ rad/s}$$

(c) The reason why the period can be extended is that the 15-bits space is not all used up, to get the largest sampling period, we need to assume all 15bits are fully used.

The maximum angle this storage can measure is calculated in section (b)---411.76 rads, and the biggest angular velocity in this experiment is 407.27rad/s, so the max period is

$$\text{D1: max period} = \frac{411.76 \text{ rad}}{407.27 rad/s} = 1.01 = 1.00\text{s}$$

$$\text{D2: max period} = \frac{411.76 \text{ rad}}{405.49 rad/s} = 1.02 = 1.00\text{s}$$

(rounded to the nearest 1/20th of a second)

Summary: the relationship among the values could be represented as: $\frac{\omega \times T}{2\pi} \times$ CPR = count.

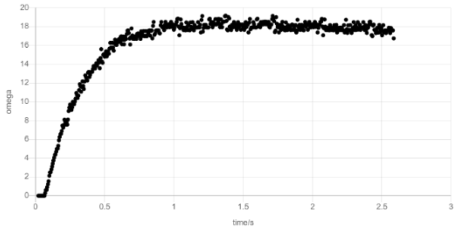
Characterisation of 1st Order Systems

Exercise 3.5.1 *[4/100 marks]* Provide a detailed description of the experimental process as well as tables and figures of all the results. ■

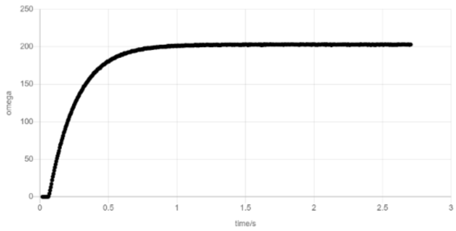
Experimental procedure:

1. Open the remote lab, get device 1, open the Voltage (open loop) option, select Input mode:' Step', and open the graph tool from the menu.
2. Set the step size to 0.5 volts. Run the system and record the data until the output reaches steady state.
3. Download and save the CSV file and the graph. Calculate the gain K and time constant τ of the system. Put the result in the chart.
   1) Calculate K:
      Determine the u_{max} and the $u_{min}$of the input signal.
      $u_{min} = 0V$
      $u_{max} = 0.5V\ or\ 5V$
      $\Delta u = u_{max} - u_{min}$
      Determine the y_{ss} and the $y_0$of the input signal.
      y_{ss}: the steady-state value of output. Since y_{ss} is unstable, I calculated the average of 30 steady-state values as the final value.
      $y_0 = 0\text{rad/s}$
      $\Delta y = y_{ss} - y_0$
      Steady-state gain: $K = \frac{\Delta y}{\Delta u}$
   2) Calculate τ:
      τ is the time it takes for the system to reach 1-1/e = 63.2% of the difference between y_{ss} and y_0.
      t_0: starting time
      t_1: $y(t_1) = 0.632\Delta y + y_0$
      $\tau = t_1 - t_0$
4. Repeat step 2,3 for a 5-volt step voltage.
5. Carry out the experiment for device 2.

Device 1:

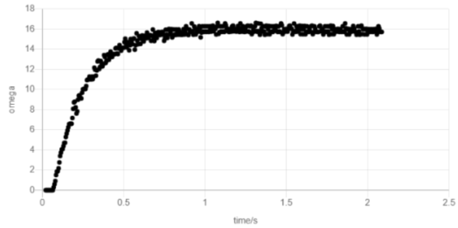


0.5V

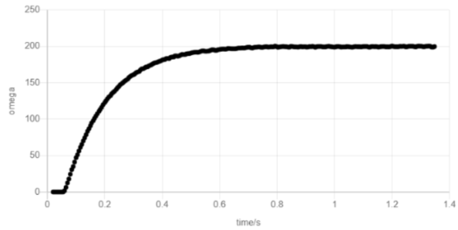


5V

Device 2:



0.5V



5V

Disk Info		0.5 V step		5 V step	
No.	Details	K/ (rad/V-s)	τ/s	K/ (rad/V-s)	τ/s
36(D1)	d:55mm t:20mm m:127g	34.08	0.224	40.48	0.21
40(D2)	d:60mm t:10mm m:75g	32.92	0.21	39.94	0.16

Exercise 3.5.2 *[4/100 marks]* Derive the transfer function of the system for each disk. ■

$$\text{Transfer function: } \frac{\Omega_m(s)}{V_m(s)} = \frac{K}{\tau s + 1}$$

Device 1 (No. 36) 0.5V:

$$\left.\begin{array}{l}\Delta x = 0.5\text{V} \\ \Delta y = 17.04\end{array}\right\} \Rightarrow K = \frac{\Delta y}{\Delta x} = 34.08$$

$$y_0 = 0$$

$$y(t_1) = \left(1 - \frac{1}{e}\right)\Delta y + y_0 = 0.632 \times 17.04 = 10.77$$

$$\left.\begin{array}{l}t_1 = 0.294\text{s} \\ t_0 = 0.07\text{s}\end{array}\right\} \Rightarrow \tau = t_1 - t_0 = 0.294 - 0.07 = 0.224\text{s}$$

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{34.08}{0.224s + 1}$$

Device 1 (No. 36) 5V:

$$\left.\begin{array}{l}\Delta x = 5\text{V} \\ \Delta y = 202.39\end{array}\right\} \Rightarrow K = \frac{\Delta y}{\Delta x} = 40.48$$

$$y_0 = 0$$

$$y(t_1) = \left(1 - \frac{1}{e}\right)\Delta y + y_0 = 0.632 \times 202.39 = 127.91$$

$$\left.\begin{array}{l}t_1 = 0.275\text{ s} \\ t_0 = 0.065\text{s}\end{array}\right\} \Rightarrow \tau = t_1 - t_0 = 0.275 - 0.065 = 0.21\text{s}$$

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{40.08}{0.21s + 1}$$

Device 2 (No. 40) 0.5V:

$$\left.\begin{array}{l}\Delta x = 0.5\text{V} \\ \Delta y = 16.46\end{array}\right\} \Rightarrow K = \frac{\Delta y}{\Delta x} = 32.92$$

$$y_0 = 0$$

$$y(t_1) = \left(1 - \frac{1}{e}\right)\Delta y + y_0 = 0.632 \times 16.46 = 10.40$$

$$\left.\begin{array}{l}t_1 = 0.26\text{s} \\ t_0 = 0.05\text{s}\end{array}\right\} \Rightarrow \tau = t_1 - t_0 = 0.26 - 0.05 = 0.21\text{s}$$

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{32.92}{0.21s + 1}$$

Device 2 (No. 40) 5V:

$$\left.\begin{array}{l}\Delta x = 5\text{V} \\ \Delta y = 199.69\end{array}\right\} \Rightarrow K = \frac{\Delta y}{\Delta x} = 39.94$$

$$y_0 = 0$$

$$y(t_1) = \left(1 - \frac{1}{e}\right)\Delta y + y_0 = 0.632 \times 199.69 = 126.20$$

$$\left.\begin{array}{l}t_1 = 0.21\text{s} \\ t_0 = 0.05\text{s}\end{array}\right\} \Rightarrow \tau = t_1 - t_0 = 0.21 - 0.05 = 0.16\text{s}$$

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{39.94}{0.16s + 1}$$

Exercise 3.5.3 *[4/100 marks]* **(a)** The step response, and hence the experimentally derived transfer function for a 0.5 V input step differs to the one obtained with a 5 V step input. Discuss the possible reasons for this discrepancy. **(b)** What is the impact of the physical size of the rotating mass on the transient and steady state response of the system? ■

(a) For the same device, the friction is the same. A larger input voltage will generate larger driving torque, which relatively will make the impact of the friction smaller. So, with less interference of friction, the rotating speed will rise faster (shorter rise time and smaller τ).
The input voltage also affects the steady-state gain. A larger input voltage will decrease the effect of friction. So, in the 5V situation, K is closer to the ideal value we calculated in the ***Static Characterization of DC Motors*** (which means bigger input voltage will cause bigger K). While in the 0.5V case, the experiment error will increase because friction is relatively affecting the system more. And in chapter 2, when input is 0.5V, K in none-linear zone while when input is 0.5V, K in linear zone, which agrees with this experiment.

(b) For cylindrical objects, greater mass and radius of section result in greater moment of inertia.

D1: $J1 = \frac{1}{2} \times 127g \times \left(\frac{55mm}{2}\right)^2 = 480.22gcm^2$ | **D2:** $J2 = \frac{1}{2} \times 75g \times \left(\frac{60mm}{2}\right)^2 = 337.50gcm^2$

After calculation, device 1 has larger momentum of inertia.

Transient state:

The same input voltage will cause the same driving torque, based on the relation among momentum of Inertia, torque and angular acceleration ($\tau = I\alpha$), disc with less inertia have larger angular acceleration and shorter rising time (smaller τ).

Steady state:

The steady state gain K is quite similar between different devices. Under 0.5V and 5V input, K of device 1 is slightly larger than K of device 2. Then I increased the input voltage to 8V and found that $K1_{0.8v} < K2_{0.8v}$. And according to the calculation in ***Validation using first principles***, K under different circumstances should remain the same. So, this difference should be caused by random error and maybe there are more factors affecting the system than we are considering now.

Exercise 3.5.4 *[5/100 marks]* The powertrain of an electric vehicle includes its traction batteries, the power electronic drive and the electric motor. How do an EV's shape, size and mass affect both the design requirements of each component of the powertrain as well as the dynamic behaviour of the EV under cruise control? ■

The shape of the EV should be streamlined to reduce air friction, proper shape can also lead to lift force, which can improve EV's cruise performance.

The size of EV should be as small as possible when acceptable. A smaller size can lead to better aerodynamics behaviour and faster reacting time.

The mass of the EV should be lighter. Smaller weight will reduce acceleration time and make the car react faster under the same motor power change. The car will accelerate, decelerate, and turn faster. Moreover, as the load increases, the current in the motor will be higher, which will cause more electricity consumption. Smaller mass can also lead to a faster maximum speed.

Validation using first principles

Exercise 4.3.1 *[6/100 marks]* Using appropriate mathematical tools such as the Laplace transform, derive the transfer function of the complete system, between the input voltage and the angular velocity, for each disk you tested. Show your calculations.

Total equivalent moment of inertia:

$$J_{eq} = J_m + J_h + J_d$$

$$J_m = 41.7\text{gcm}^2 = constant$$
$$J_h = 10\text{gcm}^2 = constant$$
$$J_d = \frac{1}{2}MR^2$$

$$J_{d1} = \frac{1}{2} \times 127g \times \left(\frac{55mm}{2}\right)^2 = 480.22gcm^2$$
$$J_{d2} = \frac{1}{2} \times 75g \times \left(\frac{60mm}{2}\right)^2 = 337.50gcm^2$$

From Kirchhoff's Voltage Law:

$$\left.\begin{aligned} v_m(t) - R_m i_m(t) - L_m \frac{di_m(t)}{dt} - e_b(t) = 0 \\ e_b(t) = K_m \omega_m(t) \end{aligned}\right\} \Rightarrow v_m(t) - R_m i_m(t) - L_m \frac{di_m(t)}{dt} - K_m \omega_m(t) = 0$$

$$\left.\begin{aligned} &v_m(t) - R_m i_m(t) - L_m \frac{di_m(t)}{dt} - K_m \omega_m(t) = 0 \\ &\text{The motor inductance } L_m \text{ is small so it can be ignored.} \end{aligned}\right\} \Rightarrow v_m(t) - R_m i_m(t) - K_m \omega_m(t) = 0$$

$$\left.\begin{aligned} i_m(t) = \frac{v_m(t) - K_m \omega_m(t)}{R_m} \\ J_{eq} \frac{d\omega_m}{dt} = \tau_m(t) = K_t i_m(t) \end{aligned}\right\} \Rightarrow K_t \cdot \frac{v_m(t) - K_m \omega_m(t)}{R_m} = J_{eq} \frac{d\omega_m}{dt}$$

$$\frac{K_t}{R_m} \cdot V_m(t) - \frac{K_t K_m}{R_m} \cdot \omega_m(t) = J_{eq} \cdot \frac{d\omega_m}{dt}$$

According to the Laplace transform:

$$V_m(t) \Rightarrow V_m(s)$$
$$\omega_m(t) \Rightarrow \omega_m(s)$$
$$\frac{d\omega_m(t)}{dt} \Rightarrow s \cdot \omega_m(s)$$

$$\frac{K_t}{R_m} \cdot V_m(s) - \frac{K_t K_m}{R_m} \cdot \omega_m(s) = J_{eq} \cdot s \cdot \omega_m(s)$$
$$\frac{K_t}{R_m} \cdot V_m(s) = \left(\frac{K_t K_m}{R_m} + J_{eq} \cdot s\right) \omega_m(s)$$

$$\boldsymbol{\frac{\omega_m(s)}{V_m(s)} = \frac{\frac{K_t}{R_m}}{\frac{K_t K_m}{R_m} + J_{eq} \cdot s}}$$

	M	$D = 2R$	$J_d = \frac{1}{2}MR^2$	J_m	J_h	J_{eq}
Device 1	$127g$	$55mm$	$480.22gcm^2$	41.7gcm^2	10gcm^2	531.92gcm^2 $5.3192 \times 10^{-5}\text{kgm}^2$
Device 2	$75g$	$60mm$	$337.50gcm^2$			389.20gcm^2 $3.8920 \times 10^{-5}\text{kgm}^2$

$$\frac{K_t}{R_m} = \frac{0.0243\text{Nm/A}}{2.08\text{Ohm}} = 1.1683 \times 10^{-2} Nm/V$$

$$\frac{K_t \cdot K_m}{R_\text{m}} = \frac{0.0243\text{Nm/A} \cdot 0.0242\text{Vs/rad}}{2.08\text{Ohm}} = 2.8272 \times 10^{-4} Nms/rad$$

$$\text{Device 1: } \frac{1.1683 \times 10^{-2}}{2.8272 \times 10^{-4} + 5.3192 \times 10^{-5}\text{s}} = \frac{41.324}{0.1881\text{s} + 1}$$

$$\text{Device 2: } \frac{1.1683 \times 10^{-2}}{2.8272 \times 10^{-4} + 3.8920 \times 10^{-5}\text{s}} = \frac{41.324}{0.1377\text{s} + 1}$$

Exercise 4.3.2 *[4/100 marks]* Do these transfer functions match the ones derived in Section 3? If not, can you suggest any reasons for this mismatch? ■

$$\frac{K}{\tau s+1}=\frac{\frac{1}{K_m}}{\frac{R_m J_{eq}}{K_t K_m}s+1} \qquad K=\frac{1}{K_m}\ ;\ \tau=\frac{R_m J_{eq}}{K_t K_m}$$

The theoretical transfer functions calculated here do not match the transfer functions from the experiment perfectly.

Firstly, the theoretical τ is smaller than the experimental τ, which means that in theory, the system should react faster. The reason is that friction plays an important role in real world, but in theory friction is not taken into consideration.

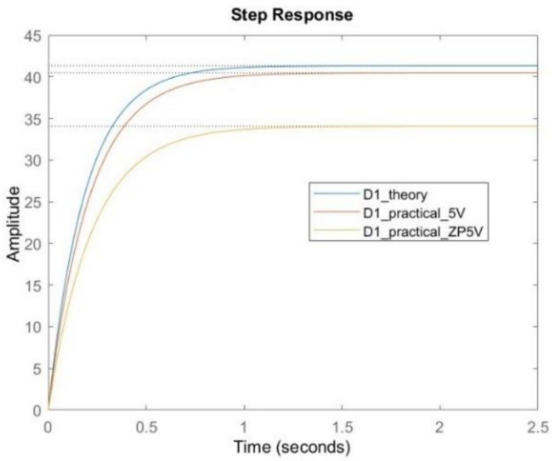
Secondly, the theoretical K is larger than the experimental K, which means that the motor in theory spins faster (same device and the same input voltage). The reason is also the friction. In reality the device has to overcome frictional resistance, so the gain K will be smaller in experiment.

If compare the behaviour under different voltage, 5V have less error than 0.5V. that is because in this case, the ratio of friction to driving force is greater and this will affect the system more.

Exercise 4.3.3 *[4/100 marks]* Using the functions `tf` and `step` in Matlab, simulate and discuss the step response of the transfer functions you derived here. How do they compare with the experimental results? ■

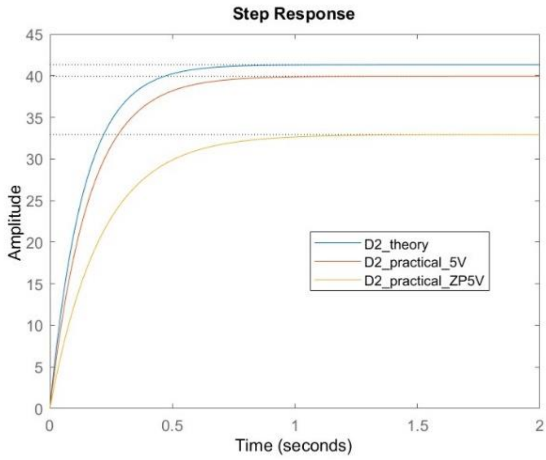
Device 1

```
D1_theory = tf(41.324,[0.1881,1]);
D1_practical_5V = tf(40.48,[0.21,1]);
D1_practical_ZP5V = tf(34.08,[0.224,1]);
step(D1_theory,D1_practical_5V,D1_practical_ZP5V)
```



Device 2

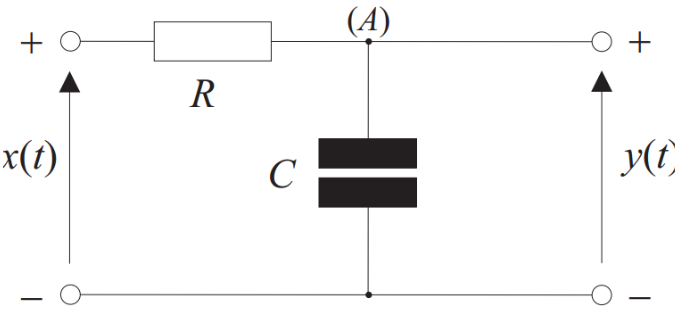
```
D2_theory = tf(41.324,[0.1377,1]);
D2_practical_5V = tf(39.94,[0.16,1]);
D2_practical_ZP5V = tf(32.92,[0.21,1]);
step(D2_theory,D2_practical_5V,D2_practical_ZP5V)
```



From the diagram, the motor's behaviour under 5V input is very close to the theoretical behaviour, while there are a lot of difference under 0.5V input. Under 0.5V input, it takes longer time for the motor to reach the steady state and the angular velocity is less. This is because of larger influence of friction.

Exercise 4.3.4 *[4/100 marks]* Assuming that the gain K of the servo system is equal to 1, can you design and derive the transfer function of an appropriate resistor-capacitor (RC) circuit that can mimic the behaviour of your system? If the capacitor size is $47\mu F$, what should the resistor value be in order to get the same response as the one you got with each of the two disks? ■

In this case, a low pass RC circuit can mimic the behaviour of the motor system.



Applying Kirchhoff's current law at point A:

$$\frac{x(t) - y(t)}{R} = C\frac{dy(t)}{dt}$$

Writing the differential equation in standard form:

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Applying the Laplace transform:

$$\frac{dy(t)}{d(t)} \Rightarrow S \cdot Y(S)$$

$$y(t) \Rightarrow Y(S)$$

$$x(t) \Rightarrow X(s)$$

$$RC \cdot SY(S) + Y(S) = X(S)$$

$$\frac{Y(S)}{X(S)} = \frac{1}{RC \cdot S + 1}$$

Device 1

$R_1C = \tau_1 \quad R_1 \cdot 47 \times 10^{-6} = 0.1881 \quad R_1 = 4002\Omega$

Device 1

$R_2C = \tau_2 \quad R_2 \cdot 47 \times 10^{-6} = 0.1377 \quad R_2 = 2930\Omega$

Characterisation of 2nd Order Systems

Exercise 5.6.1 *[4/100 marks]* Based on the analysis shown in Sections 5.2-5.3 and using your measurements obtained in this experiment, calculate the underdamped natural frequency and damping ratio for each disk, and derive the corresponding transfer function. ■

Experimental functions:

$$\text{P.O.} = \frac{100(y_{max} - R_0)}{R_0}$$

$$t_p = t_{max} - t_0$$

Theoretical functions:

$$\text{P.O.} = 100 \times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

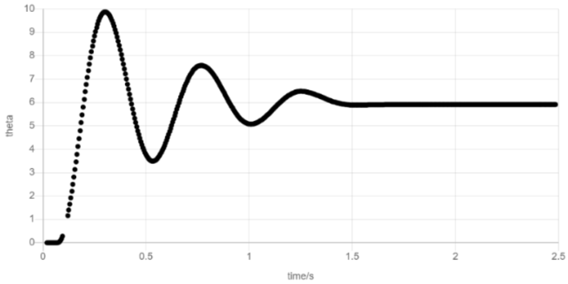
Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Step Response measurements for the 2nd order system.

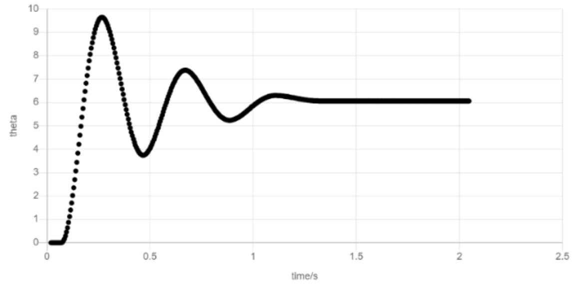
Disk Info				Measurements		Calculations	
No.	Details			t_p/s	$P.O.$	ζ	ω_n/Hz
	Diameter/mm	Thickness/mm	Mass/g				
D1 (36)	55	20	127	0.225	64.50%	0.138	14.10
D2 (40)	60	10	75	0.190	60.67%	0.157	16.74

Device 1:



$$y_{max} = 9.87\ rad$$
$$R_0 = 6\ rad$$
$$\text{P.O.} = \frac{100(9.87-6)}{6} = 64.50\%$$
$$64.50\% = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$
$$\zeta = 0.138$$
$$t_{max} = 0.3\ s$$
$$t_0 = 0.075\ s$$
$$t_p = 0.3 - 0.075 = 0.225\ \text{s}$$
$$0.225 = \frac{\pi}{\omega_n\sqrt{1-0.138^2}}$$
$$\omega_n = 14.10$$
$$\frac{Y(s)}{R(s)} = \frac{14.10^2}{s^2 + 2\cdot 0.138\cdot 14.10s + 14.10^2}$$
$$= \frac{198.8}{s^2 + 3.892s + 198.8}$$

Device 2:



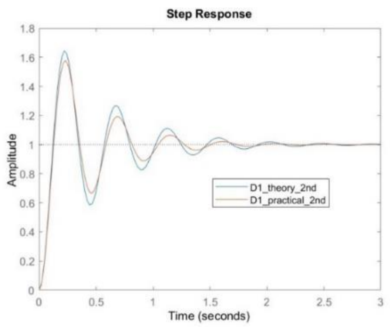
$$y_{max} = 9.64\ rad$$
$$R_0 = 6\ rad$$
$$\text{P.O.} = \frac{100(9.64-6)}{6} = 60.67\%$$
$$60.67\% = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$
$$\zeta = 0.157$$
$$t_{max} = 0.265\ s$$
$$t_0 = 0.075\ s$$
$$t_p = 0.265 - 0.075 = 0.190\ \text{s}$$
$$0.190 = \frac{\pi}{\omega_n\sqrt{1-0.157^2}}$$
$$\omega_n = 16.74$$
$$\frac{Y(s)}{R(s)} = \frac{16.74^2}{s^2 + 2\cdot 0.157\cdot 16.74s + 16.74^2}$$
$$= \frac{280.2}{s^2 + 5.256s + 280.2}$$

Exercise 5.6.2 *[6/100 marks]* In Chapter 3 you derived experimentally the transfer functions of the 1st order angular velocity systems with two different disks. Using these, and the analysis shown in Section 5.4, derive the transfer function of the 2nd order angular displacement system. Do they match with the ones you calculated in Exercise 5.6.1? Discuss. ■

$$G(s) = \frac{\Theta_m(s)}{\Theta_r(s)} = \frac{C(s)P(s)}{1+C(s)P(s)} = \frac{\frac{K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}}$$

Device 1

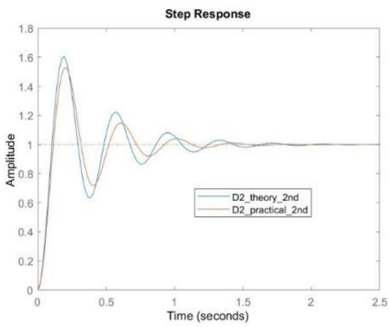
$$K = 40.48\ \text{rad/V} - \text{s}$$
$$\tau = 0.21\ s$$
$$G(s) = \frac{\frac{40.48}{0.21}}{s^2 + \frac{1}{0.21}s + \frac{40.48}{0.21}} = \frac{192.8}{s^2 + 4.762s + 192.8}$$




```
D1_theory_2nd = tf(198.8,[1,3.892,198.8]);
D1_practical_2nd = tf(192.8,[1,4.762,192.8]);
step(D1_theory_2nd,D1_practical_2nd)
```

Device 2

$$K = 39.94\ \text{rad/V} - \text{s}$$
$$\tau = 0.16\ s$$
$$G(s) = \frac{\frac{39.94}{0.16}}{s^2 + \frac{1}{0.16}s + \frac{39.94}{0.16}} = \frac{249.6}{s^2 + 6.250s + 249.6}$$




```
D2_theory_2nd = tf(280.2,[1,5.256,280.2]);
D2_practical_2nd = tf(249.6,[1,6.250,249.6]);
step(D2_theory_2nd,D2_practical_2nd)
```

Analyse:

The theoretical TF is similar to the experimental one, but there are still some differences.

Comparing the theoretical transfer function with the experimental transfer function, the molecular term and the zero-order term of s: ω_n^2 (related to natural frequency) of the theoretical equation are more significant than the experimental equation. However, the system's natural frequency is not affected by external factors and is only related to the system itself. This difference should be caused by systematic error. The first-order term of s of the theoretical equation: $2\zeta\omega_n$ is smaller than that of the experimental equation, which shows that the theoretical damping ratio is less than the actual damping ratio because friction will increase damping ratio and make the system reach the steady-state faster, which is not considered in the theoretical derivation.

Comparing the graphs, theoretical graph has higher overshoot, which is related to smaller damping ratio (caused by not considering friction). Moreover, the theoretical graph has a phase lead than the experimental graph, however, if it is an underdamped system, the phase which is related to natural frequency should be the same. This should be caused by system error.

Exercise 5.6.3 *[4/100 marks]* Similarly, calculate the peak time and percentage overshoot from the transfer functions you calculated in Exercise 5.6.2. How much do they differ from the ones you measured in this experiment? ■

$$Comparing\ \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\ \ with\ \ G(s) = \frac{\frac{K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}}$$

$$\omega_n = \sqrt{\frac{K}{\tau}}$$

$$\zeta = \frac{1}{2 \cdot \tau \cdot \omega_n} = \frac{1}{2 \cdot \sqrt{K\tau}}$$

$$\text{P.O.} = 100 \times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Device 1:

$$\omega_n = \sqrt{\frac{40.48}{0.21}} = 13.88$$

$$\zeta = \frac{1}{2 \cdot \sqrt{40.28 \cdot 0.21}} = 0.1719$$

$$\text{P.O.} = 100 \times e^{-\frac{\pi 0.1719}{\sqrt{1-0.1719^2}}} = 57.80\%$$

Error of P.O.: $57.80 - 64.50 = -6.70$

$$\left|\frac{57.80 - 64.50}{64.50}\right| \times 100\% = 10.39\%$$

$$t_p = \frac{\pi}{13.88 \cdot \sqrt{1 - 0.1719^2}} = 0.2300\ s$$

Error of t_p: $0.2300 - 0.2250 = 0.0050s$

$$\left|\frac{0.2300 - 0.2250}{0.2250}\right| \times 100\% = 2.222\%$$

Device 2 :

$$\omega_n = \sqrt{\frac{39.94}{0.16}} = 15.80$$

$$\zeta = \frac{1}{2 \cdot \sqrt{39.94 \cdot 0.16}} = 0.1978$$

$$\text{P.O.} = 100 \times e^{-\frac{\pi 0.1978}{\sqrt{1-0.1978^2}}} = 53.05\%$$

Error of P.O.: $53.05 - 60.67 = -7.62$

$$\left|\frac{53.05 - 60.67}{60.67}\right| \times 100\% = 12.56\%$$

$$t_p = \frac{\pi}{15.80 \cdot \sqrt{1 - 0.1978^2}} = 0.2028\ s$$

Error of t_p: $0.2028 - 0.1900 = 0.0128s$

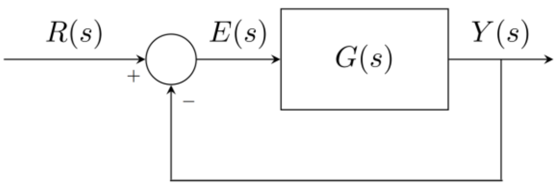
$$\left|\frac{0.2028 - 0.1900}{0.1900}\right| \times 100\% = 6.737\%$$

Analise:

The calculated P.O. is smaller, and the calculated peak time is longer. This is because the values of ω_n and ζ are different in different situations.

The percentage error of P.O. is greater than that of peak time.

Exercise 5.6.4 *[6/100 marks]* For the transfer functions you derived in Exercise 5.6.1, use the final value theorem to calculate the steady state error for both disks, for the step, and the two ramp inputs. Do they agree with your measurements? ■



$$E(s) = R(s) - Y(s)$$
$$Y(s) = R(s)G(s)$$
$$E(s) = R(s) - R(s)G(s)$$
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$
$$\therefore E(s) = \left(1 - \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) R(s)$$

Steady state error $\Rightarrow \quad t \to \infty$

According to the final value theorem, when $t \to \infty, s \to 0$.

Steady state error for ramp and step input applied to the 2^{nd} order system

Disk Info				Steady state error e_{ss}					
No.	Details			Ramp: 1rad/s		Ramp: 5rad/s		Step: 6rad	
	D/mm	T/mm	M/g	experiment	theory	experiment	theory	experiment	theory
D1 (36)	55	20	127	0.17	0.0196	0.53	0.0979	0.09	0
D2 (40)	60	10	75	0.22	0.0191	0.55	0.0957	−0.06	0

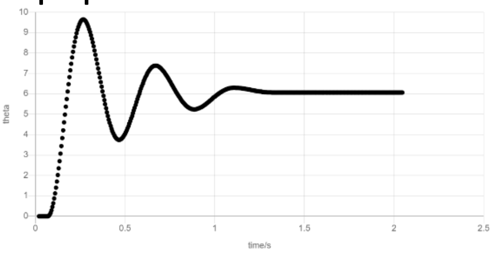
Step

$$\text{s} - \text{domain input: } R(s) = \frac{A}{s}$$

$$e_{ss} = \lim_{s\to 0} s \cdot E(s) = \lim_{s\to 0} s \cdot \frac{A}{s} \cdot \left(1 - \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$
$$= \lim_{s\to 0} A \cdot \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$
$$= A \cdot 0$$
$$= 0$$

Device 1

Step input to 6 radians:



Command: 6

Steady state angle: 5.91

Steady state drive: 0.09

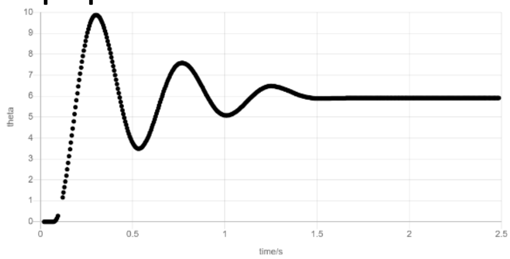
Steady state error: 0.09

$$experimental\ e_{ss} = 6 - 5.91 = 0.09$$

$$\left|\frac{5.91 - 6}{6}\right| \times 100\% = 1.5\%$$

Device 2

Step input to 6 radians:



Command: 6

Steady state angle: 6.06

Steady state drive: -0.06

Steady state error: -0.06

$$experimental\ e_{ss} = 6 - 6.06 = -0.06$$

$$\left|\frac{6.06 - 6}{6}\right| \times 100\% = 1\%$$

According to the experimental data, the steady-state errors for step input are 0.09 and -0.06 and the ratios are 1.5% and 1%, which is close to $theoretical\ e_{ss}$: 0. The difference should be caused by random error. In this case, the theoretical e_{ss} agrees with the experimental e_{ss}.

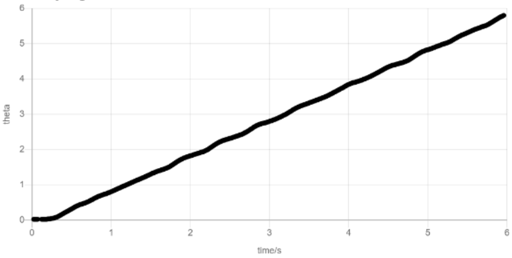
Ramp

$$\text{s}-\text{domain input: } R(s)=\frac{A}{s^2}$$

$$e_{ss}=\lim_{s\to 0} s\cdot E_{(s)}=\lim_{s\to 0} s\cdot\frac{A}{s^2}\cdot\left(1-\frac{\omega_n^2}{s^2+2\zeta\omega_n S+\omega_n^2}\right)$$
$$=\lim_{s\to 0} A\cdot\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$
$$=\frac{2A\zeta}{\omega_n}$$

Device 1

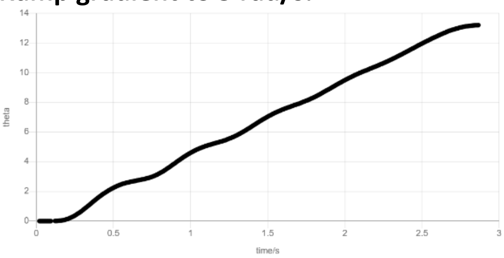
Ramp gradient to 1 rad/s:



Angle at 6 sec = 5.83
Command at 6 sec = 5.97
Drive at 6 sec = 0.14
Error at 6 sec = 0.14
angle in theory: 6

$$experimental\ e_{ss}=6-5.83\ =\ 0.17$$
$$\zeta=0.138$$
$$\omega_n=14.10$$
$$theoretical\ e_{ss}=\frac{2\cdot A\cdot\zeta}{\omega_n}=\frac{2\cdot 1\cdot 0.138}{14.10}\ =\ 0.0196$$

Ramp gradient to 5 rad/s:

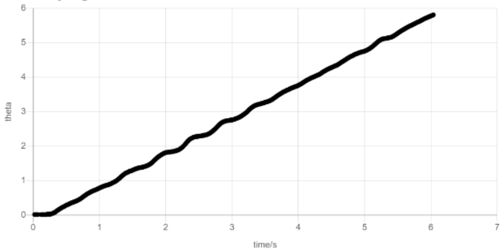


Angle at 2.5 sec = 11.97
Command at 2.5 sec = 12.16
Drive at 2.5 sec = 0.19
Error at 2.5 sec = 0.19
Angle in theory: 12.5

$$experimental\ e_{ss}=12.5-11.97\ =\ 0.53$$
$$\zeta=0.138$$
$$\omega_n=14.10$$
$$theoretical\ e_{ss}=\frac{2\cdot A\cdot\zeta}{\omega_n}=\frac{2\cdot 5\cdot 0.138}{14.10}\ =\ 0.0979$$

Device 2

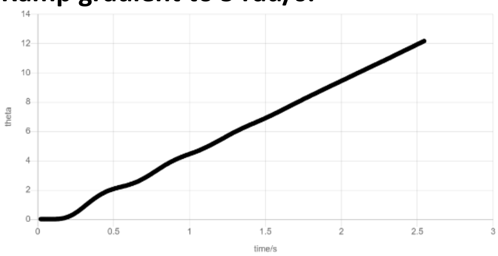
Ramp gradient to 1 rad/s:



Angle at 6 sec = 5.78
Command at 6 sec = 5.96
Drive at 6 sec = 0.18
Error at 6 sec = 0.18
angle in theory: 6

$$experimental\ e_{ss}=6-5.78\ =\ 0.22$$
$$\zeta=0.157$$
$$\omega_n=16.74$$
$$theoretical\ e_{ss}=\frac{2\cdot A\cdot\zeta}{\omega_n}=\frac{2\cdot 1\cdot 0.157}{16.74}\ =\ 0.0191$$

Ramp gradient to 5 rad/s:



Angle at 2.5 sec = 11.95
Command at 2.5 sec = 12.2
Drive at 2.5 sec = 0.25
Error at 2.5 sec = 0.25
Angle in theory: 12.5

$$experimental\ e_{ss}=12.5-11.95\ =\ 0.55$$
$$\zeta=0.157$$
$$\omega_n=16.74$$
$$theoretical\ e_{ss}=\frac{2\cdot A\cdot\zeta}{\omega_n}=\frac{2\cdot 5\cdot 0.157}{16.74}\ =\ 0.0957$$

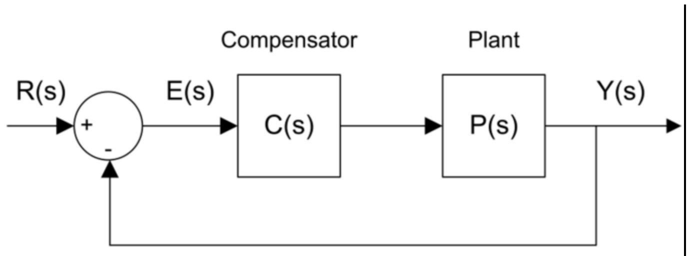
For ramp input under 1rad/s, the experimental e_{ss} is 10 times the theoretical e_{ss}.

For ramp input under 5rad/s, the experimental e_{ss} is 5 times the theoretical e_{ss}.

In this case, the theoretical e_{ss} does not agree with the experimental e_{ss}. Because the friction is slowing the spinning down. The less the driving power, the more influence the friction have on the system.

PID Controller Design (based on device 1)

Exercise 6.2.1 *[5/100 marks]* In previous exercises you showed that the spinning disk angular position control system of our lab can be approximated by a unity feedback, 2nd order transfer function. Assuming a pure proportional controller with a transfer function $C(s) = K_p$, make an assessment of the stability of the system (show your calculations).



$$P(s) = \frac{K}{s(\tau s + 1)}$$

$$C(s) = K_p$$

$$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{K_p \cdot \frac{K}{s(\tau s+1)}}{1 + K_p \cdot \frac{K}{s(\tau s+1)}}$$

$$= \frac{\frac{K_p K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K_p K}{\tau}}$$

The condition for the system to be stable is: The real part of all the poles must be negative.

Based on the Routh-Hurwitz criterion, stability is determined by the poles (roots) of the characteristic equation:

$$s^2 + \frac{1}{\tau}s + \frac{K_p K}{\tau}$$

$$a_2 = 1$$

$$a_1 = \frac{1}{\tau}$$

$$a_0 = \frac{K_p K}{\tau}$$

Routh-Hurwitz array:

s^2	a_2	a_0
s^1	a_1	0
s^0	b_1	0

s^2	1	$\frac{K_p K}{\tau}$
s^1	$\frac{1}{\tau}$	0
s^0	$\frac{K_p K}{\tau}$	0

$$b_1 = \frac{a_1 a_0 - 0 \times a_2}{a_1} = a_0$$

a_0, a_1 and a_2 must all have the same sign if the system is stable.

$$1 > 0$$

$$\frac{1}{\tau} > 0 \Rightarrow \tau > 0$$

$$\frac{K_p K}{\tau} > 0 \Rightarrow K_p K > 0$$

In this case, $K > 0$ and $\tau > 0$. So, if $K_p > 0$, the system is stable, otherwise, the system is unstable.

Exercise 6.2.2 *[5/100 marks]* In Exercise 5.6.4 you derived the steady state system error for a 2nd order system subjected to a step input. Based on this, and on your answer to Exercise 6.2.1, select an appropriate controller configuration (P, PI, PD or PID) and explain your decision.

	$C(s)$	$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)}$	$e_{ss} = \lim\limits_{s\to 0} s[1 - G(s)]R(s)$
P	K_p	$\frac{\frac{K_p K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K_p K}{\tau}}$	0
PI	$K_p + \frac{K_i}{s}$	$\frac{KK_p s + KK_i}{\tau s^3 + s^2 + KK_p s + KK_i}$	$\lim\limits_{s\to 0} A \cdot \frac{\tau s^3 + s^2}{\tau s^3 + s^2 + KK_p s + KK_i} = 0$
PD	$K_p + K_d \cdot s$	$\frac{KK_d s + KK_p}{\tau s^2 + (KK_d + 1)s + KK_p}$	$\lim\limits_{s\to 0} A \cdot \frac{\tau s^2 + s}{\tau s^2 + (KK_d + 1)s + KK_p} = 0$
PID	$K_p + \frac{K_i}{s} + K_d \cdot s$	$\frac{KK_d s^2 + KK_p s + KK_i}{\tau s^3 + (KK_d + 1)s^2 + KK_p s + KK_i}$	$\lim\limits_{s\to 0} A \cdot \frac{\tau s^3 + s^2}{\tau s^3 + (KK_d + 1)s^2 + KK_p s + KK_i} = 0$

I calculated the steady state error for all four occasions, and the steady state errors are all 0. And by carefully choosing the parameters all four control systems can be stable.

After checking the step response behaviour of the system in chapter 5 which is using only P and $K_p = 1$, we can see that the steady state error is close to 0, and the response is quite oscillatory.

Analise how K_p, K_i and K_d affecting the system:

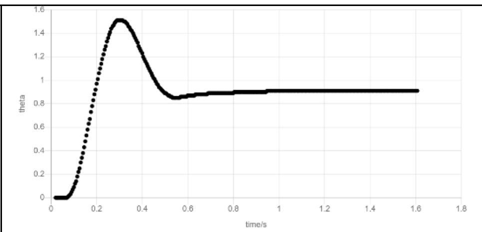
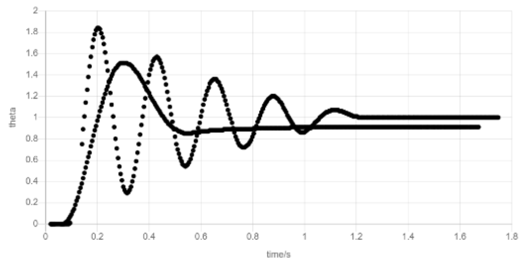
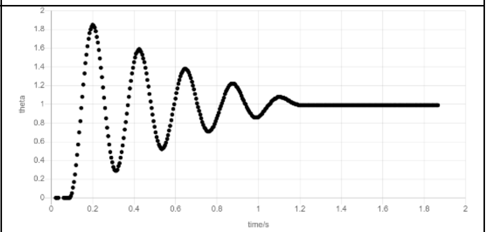
K_p	Increasing the proportional coefficient P will speed up the system's response, improve the control sensitivity, and reduce the steady-state error. However, the overshoot of the system will increase if P is too large.
K_i	Integral control can eliminate the steady-state error to improve control accuracy. Because the accumulation of integral output is gradual, its effect always lags the error change, so it alone is not easy to make the control system stable. Therefore, integral control is often combined with proportional control. In this way, the system can react rapidly and eliminate steady-state error. However, a large integral coefficient will worsen the system's stability, resulting in oscillation and overshoot. For control system with large inertia, integral control should be avoided.
K_d	Differential control can adjust system speed, reduce overshoot and oscillation, and improve stability. Therefore, the proportional + differential (PD) controller can improve the system's dynamic characteristics for the controlled object with large inertia or lag. For constant error input, differential control has no control at all. Therefore, the differential control law cannot be used alone.

In this case, there is already no steady state error, and the oscillation is considerable, so integral control is unnecessary. The system has a big overshoot, so differential control can be used to improve stability.

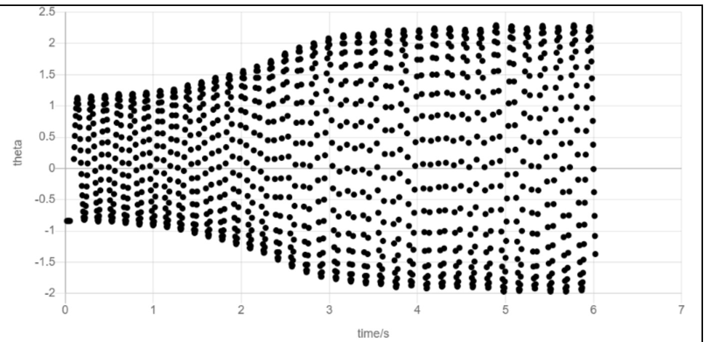
In conclusion, I choose PD controller.

Exercise 6.3.1 *[5/100 marks]* Describe the experimental process you followed in Section 6.2 and include tables and representative figures of the results. It is important that your answer illustrates clearly your understanding of the process and your interpretation of the results. ■

I tested the response with different K_p(step input = 1rad):

$K_p = 1$ $K_i = 0$ $K_d = 0$		 Compare 2 graphs in 1 picture
$K_p = 4$ $K_i = 0$ $K_d = 0$		

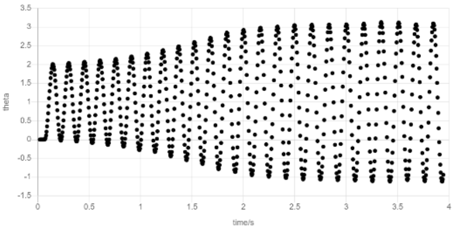
The response is now more oscillatory.

Then I increased K_p to 8 and the system is performing a sustained oscillation, so the ultimate gain: $K_U = K_p = 8$.	
I measured the time over 10 oscillation periods and then dividing that time by 10. The calculated ultimate period: $T_U = 0.167s$.	

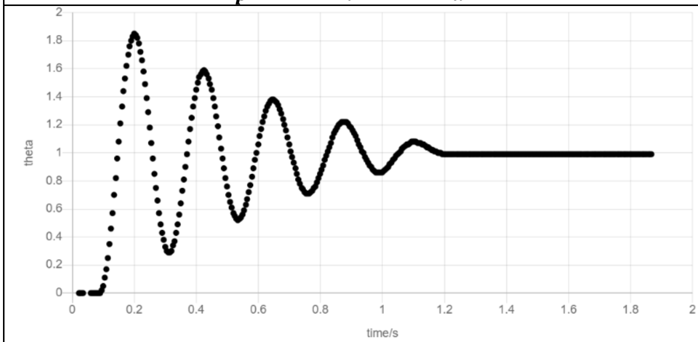
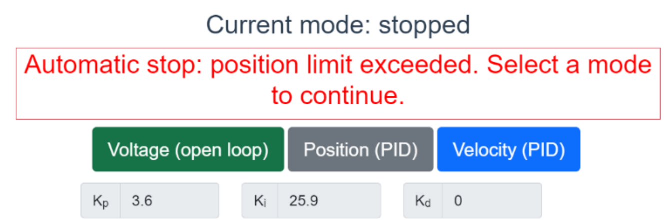
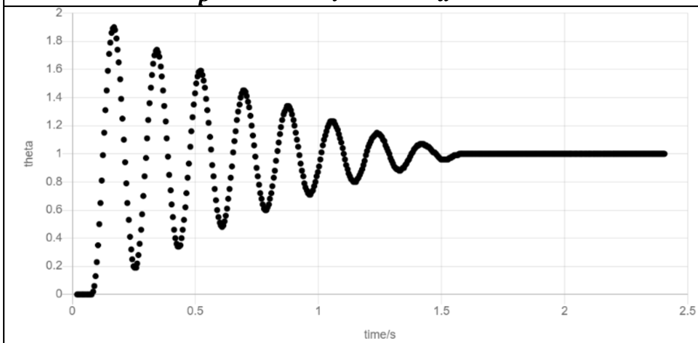
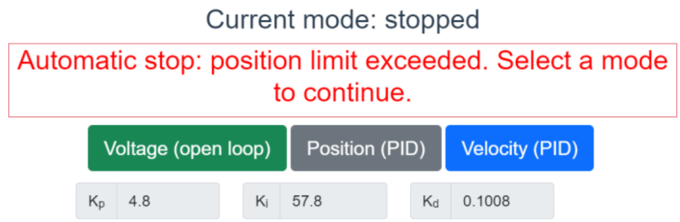
According to Ziegler-Nichols Ultimate Gain method

Type	K_p	T_i	K_i	T_d	K_d	
P	$0.5K_U = 4$	∞	0	0	0	$T_i = \frac{K_p}{K_i}$
PI	$0.45K_U = 3.6$	$T_U/1.2 = 0.139$	$K_i = \frac{K_p}{T_i} = 25.9$	0	0	
PD	$0.8K_U = 6.4$	∞	0	$T_U/8 = 0.021$	$K_d = K_p \cdot T_d = 0.1344$	$T_d = \frac{K_d}{K_p}$
PID	$0.6K_U = 4.8$	$T_U/2 = 0.083$	$K_i = \frac{K_p}{T_i} = 57.8$	$T_U/8 = 0.021$	$K_d = K_p \cdot T_d = 0.1008$	

After running the experiment once again the outcome is the same, $K_U = 8$ and $T_U = 0.167s$. So, the Z-N table remains the same.



Response from original parameters calculated from Z-N method:

<table>
<tr><td colspan="2">Response from original Z-N method</td></tr>
<tr><td>P: $\boldsymbol{K_p} = 4$; $\boldsymbol{K_i} = 0$; $\boldsymbol{K_d} = 0$</td><td>PI: $\boldsymbol{K_p} = 3.6$; $\boldsymbol{K_i} = 25.9$; $\boldsymbol{K_d} = 0$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>PD: $\boldsymbol{K_p} = 6.4$; $\boldsymbol{K_i} = 0$; $\boldsymbol{K_d} = 0.1344$</td><td>PID: $\boldsymbol{K_p} = 4.8$; $\boldsymbol{K_i} = 57.8$; $\boldsymbol{K_d} = 0.1008$</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

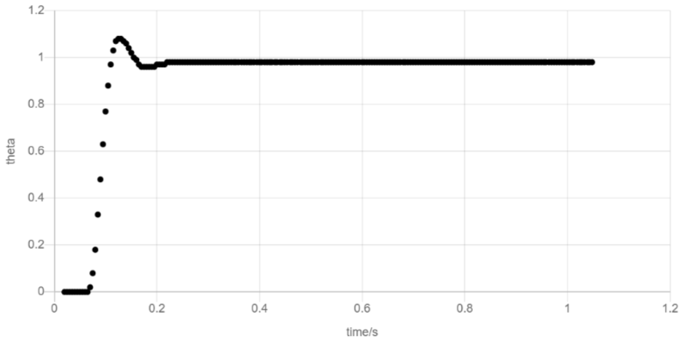
None of these controllers meets the standards (maximum overshoot<10% & fast rise time).

P and PD controllers have high overshoot and a long settle time.

PI and PID controllers make the angle increases so fast that the system is over its limit.

Among these responses, PD controller is the most promising one. So, I tried to modify the value of $\boldsymbol{K_p}$ and $\boldsymbol{K_d}$ to get the perfect response.

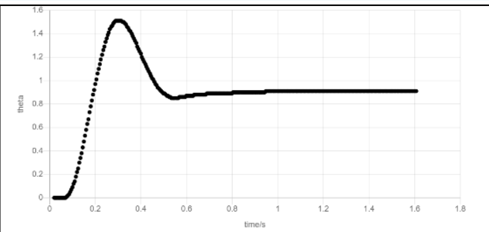
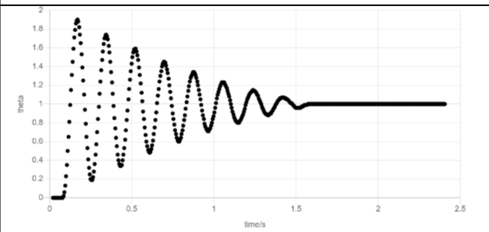
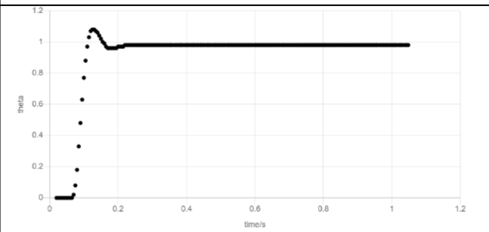
After trying different parameters, I found that when $\boldsymbol{K_p} = 3$ and $\boldsymbol{K_d} = 4$, the system has the best response.



Settling time is when the system settles within a percentage $\boldsymbol{\delta}$ of the final value. Settling time criteria: $\boldsymbol{\delta} = 5\%$	Rise time is the time for the step response to rise from 10% to 90% of the final value.	$Command = 1\text{rad}$ $Steady\ state\ value = 0.98\text{rad}$ $Steady\ state\ error = 0.02\text{rad}$ $Peak\ value = 1.08\text{rad}$ $P.O. = 8\% < 10\%$
$t_0 = 0.07\text{s}$ $t_{\delta=5\%} = 0.14\text{s}$ $Settling\ time = 0.07\text{s}$	$t_{10\%} = 0.075\text{s}$ $t_{90\%} = 0.105\text{s}$ $Rise\ time = 0.03\text{s}$	

The system now has a maximum overshoot of less than 10%, the setting time and the rise time are short, and the oscillation is also tiny, which is good.

Exercise 6.3.2 *[4/100 marks]* Compare the experimental step response curve that you obtained for the position control experiments between the original, uncontrolled system ($K_p = 1$, $T_i = inf$ and $T_d = 0$), the system with the controller parameters straight out of the Ziegler-Nichols table, and your final, optimised controller. Comment on the achieved improvement of the steady state error value, oscillatory behavior (stability) and speed of response (rise time). ■

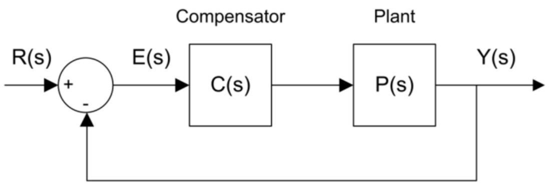
Original system		$Steady\ state\ error = 0.09\text{rad}$ It has some steady state error. $Steability: bad$ It only has one oscillation, but this oscillation is huge. $Rise\ time = 0.1\text{s}$ The rise time is the longest.
Z-N controller (PD)		$Steady\ state\ error = 0$ It has no steady state error. $Steability: bad$ It has multiple oscillation with large amplitude. $Rise\ time = 0.03\text{s}$ The rise time is shorter.
Optimised controller		$Steady\ state\ error = 0.02\text{rad}$ It has an acceptable steady state error. $Steability: good$ It only has one tiny oscillation. $Rise\ time = 0.03\text{s}$ The rise time is shorter.

The original system with a low Kp has some steady-state error and longer rise time, while in the other two systems where Kp is more significant, the steady-state error is much smaller, and the reaction (rise time) is faster.

The optimised controller with a large Kd is much more stable than the other two with Kp almost zero.

Through this process, I learned that the Z-N method does not always provide the best parameters for the controller. Nevertheless, it can be a good start for finding the best control model when the transfer function is unclear. In this case, we know the transfer function, so it is slightly different.

Exercise 6.3.3 *[4/100 marks]* Derive the transfer function of the complete closed loop system with your selected controller.



$$C_{(s)} = K_p + \frac{K_i}{s} + K_d \cdot s$$

$$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{\frac{K}{s(\tau s + 1)}\left(K_p + \frac{K_i}{s} + K_d \cdot s\right)}{1 + \frac{K}{s(\tau s + 1)}\left(K_p + \frac{K_i}{s} + K_d \cdot s\right)}$$

$$G(s) = \frac{KK_d \cdot s^2 + KK_p \cdot s + KK_i}{\tau \cdot s^3 + (KK_d + 1)s^2 + KK_p \cdot s + KK_i}$$

Apply $K_p = 3$,$K_i = 0,\ K_d = 4,\ \tau = 0.21, K = 40.48$:

$$G(s) = \frac{KK_d \cdot s + KK_p}{\tau \cdot s^2 + (KK_d + 1) \cdot s + KK_p} = \frac{161.92s + 121.44}{0.21s^2 + 162.92s + 121.44}$$

Exercise 6.3.4 *[5/100 marks]* If your selected controller is not the full PID, include the missing components (integral and/or derivative). Using the parameter values of your controller as determined by the experiment, calculate the range of the integral gain K_i that would result in a stable system. (e.g., if you selected a PD controller and your values for K_p and K_d were 10 and 0.5 respectively, use these, and determine the range of K_i for a stable system).

$$E(s) = R(s) - R(s)G(s) = [1 - G(s)]R(s)$$

$$R(s) = \frac{A}{s}$$

$$e_{ss} = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} A \cdot \frac{\tau s^3 + s^2}{\tau \cdot s^3 + (KK_d + 1)s^2 + KK_p \cdot s + KK_i} = 0$$

Analise stability using Routh-Hurwitz criterion:

$$a_3 = \tau$$

$$a_2 = KK_d + 1$$

$$a_1 = KK_p$$

$$a_0 = KK_i$$

s^3	a_3	a_1
s^2	a_2	a_0
s^1	b_2	b_0
s^0	c_2	

Routh-Hurwitz array:

s^3	τ	KK_p
s^2	$KK_d + 1$	KK_i
s^1	$\frac{KK_p(KK_d + 1) - \tau KK_i}{KK_d + 1}$	0
s^0	KK_i	

$$b_2 = \frac{-1}{a_2}\begin{vmatrix} a_3 & a_1 \\ a_2 & a_0 \end{vmatrix} = \frac{a_2 a_1 - a_3 a_0}{a_2} = \frac{KK_p(KK_d + 1) - \tau KK_i}{KK_d + 1}$$

$$b_0 = \frac{-1}{a_2}\begin{vmatrix} a_3 & 0 \\ a_2 & 0 \end{vmatrix} = 0$$

$$c_2 = \frac{-1}{b_2}\begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} = \frac{a_0 b_2 - 0 \times a_2}{b_2} = a_0$$

$$\tau > 0$$

$$K \cdot K_d + 1 > 0$$

$$\frac{KK_p(KK_d + 1) - \tau KK_i}{KK_d + 1} > 0$$

$$K \cdot K_i > 0$$

$$K \cdot K_i > 0$$

$$K_i < \frac{K \cdot K_p \cdot K_d\ +\ K_p}{\tau}$$

$$\because K > 0\ and\ KK_i > 0$$

$$\therefore K_i > 0$$

$$K_i < \frac{40.48 \times 3 \times 4\ +\ 3}{0.21}$$

$$K_i < 2327.43$$

The range of the K_i is $0 < K_i < 2327.43$

Exercise 6.3.5 *[5/100 marks]* It was shown in the PID controller design process that when it comes to selecting values for the P, I and D parameters there are compromises that may have to be made, related to the required response characteristics, in terms of its steady state error, maximum overshoot, rise time and settling time. Discuss your observations and explain how a decision can be made for a real-world application. ■

Increasing K_p will decrease $\boldsymbol{e_{ss}}$ and $\boldsymbol{t_r}$, increase $\boldsymbol{y_m}$ and make small change to $\boldsymbol{t_s}$. Increasing K_i will eliminate $\boldsymbol{e_{ss}}$, decrease $\boldsymbol{t_r}$ and increase $\boldsymbol{y_m}$ and $\boldsymbol{t_s}$. Increasing K_d will decrease $\boldsymbol{y_m}$ and $\boldsymbol{t_s}$, make small change to $\boldsymbol{t_r}$ and not influence $\boldsymbol{e_{ss}}$.	Steady state error: $\boldsymbol{e_{ss}}$ Maximum overshoot: $\boldsymbol{y_m}$ Rise time: $\boldsymbol{t_r}$ Settling time: $\boldsymbol{t_s}$

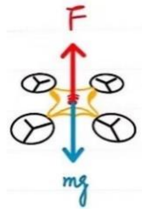
The original system has a small steady-state error and a significant overshoot, so the integral controller is not involved. In the PD controller system,

Increasing Kp will increase overshoot while increasing Kd can decrease overshoot. So, when finding the best controller, I usually need to increase/decrease these two parameters together to make overshoot under control.

However, Kp and Kd cannot be too small. Because in this case, steady-state error and the rise time will increase.

Kp and Kd cannot be too large as well. In this case, although the number of oscillations does not change, its amplitude increases, causing longer settling time.

A real-world application:



Drone altitude control

Use PID to control the motor rotation speed of the propeller.

For example, if we want to control the drone to fly from a height of 0 meters to 100 meters, the input value(goal) will be 100, the initial height is 0, so the initial error is 100.

A certain propeller speed will provide a particular lift. Suppose when the propeller speed is 100rpm, the drone can hover. When the lift is balanced with gravity, the drone will hover, and when the lift is greater than gravity, the drone will rise.

We first only use P for control first.

If P is 1, multiply the initial error 100 by P we can get the speed of the propeller is 100rpm. At this time, the drone hovers at 0 meters. If P is 2, then at the beginning, the propeller speed is two multiplied by 100, and the speed is 200rpm, which is greater than 100. The drone rises, and when it rises to 50 meters, the error is 100-50=50 meters; multiplied by P, the propeller speed is 100, which is just hovering.

If the altitude goes higher, the error decreases, the multiplication of P and error is less than 100, and the drone descends. In this case, the drone will hover at 50 meters, and the system has a steady-state error of 50m. Increasing the value of P reduces the steady-state error but cannot eliminate it. For example, adding P to 100 will still cause an error of 1m.

To eliminate the static error, the integral control I is added. I is the integral of the error over time. If the error is not 0, the I term will increase and change the motor speed. When the static error is 0, the P term will be 0, and the I term will become a fixed value. So, the system can maintain the ideal output state.

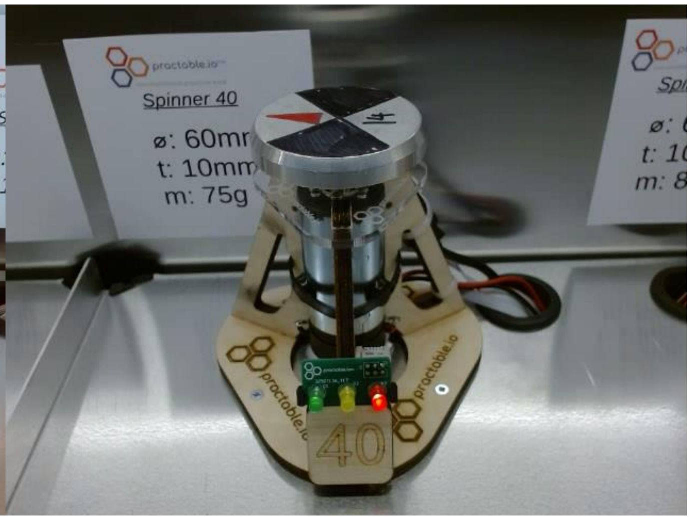
However, in the beginning, the propeller's rotating speed is tremendous due to the significant error, and the inertia will cause a large overshoot. In order to avoid this, we introduce the D term, which is the derivation of the error change. When the error decreases too fast, there will be a more significant negative value caused by D to slow down the error variation. Vice versa. This way, we can reduce overshoot.

The PID can achieve adequate control of the height of the drone.

Appendix

Device information

Device 1



Device 2