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Fractional Moment Estimation of Linnik and Mittag-Leffler Parameters

T. J. KOZUBOWSKI
Department of Mathematics
University of Nevada
Reno, NV 89557, U.S.A.

Abstract—It is shown that Linnik and Mittag-Leffler distributions are geometric stable (GS). An estimation method for the parameters of Linnik and Mittag-Leffler distributions, based on fractional moments, is proposed. Algorithms for simulation of Linnik and Mittag-Leffler distributions, based on their representations as mixtures of Laplace and exponential distributions, are presented. The estimation procedure is validated on simulated data, and its potential applications are illustrated with S&P index data, that is shown to be consistent with a two-parameter Linnik model.

Keywords—Economics, Financial data, Geometric stable distribution, Heavy-tail distribution, Mixture, Random summation, Simulation, Stable distribution.

1. INTRODUCTION

Linnik, Mittag-Leffler, and geometric stable (GS) distributions have attracted the attention of numerous researchers in recent years. Each of the three distributions has been studied independently of the other two (see, e.g., [1–9] (Linnik); [10–13] (Mittag-Leffler); [14–21] (geometric stable)). We show a simple relation among these distributions: Linnik and Mittag-Leffler distributions are special cases of GS laws. Consequently, properties of these distributions follow from the general theory of GS laws.

GS distributions approximate (normalized) sums of i.i.d. random variables,

$$S_N = X_1 + \cdots + X_N, \tag{1}$$

where the number of terms has a geometric distribution with mean $1/p$ and $p \to 0$ (see [15]). As summation (1) naturally arises in numerous models in economics, insurance mathematics, reliability theory, and other fields (see, e.g., [22] and references therein), GS laws should have a wide range of applications. They should be particularly useful in modeling heavy-tailed phenomena resulting from a random number of independent innovations, as their tail probabilities are regularly varying at infinity (see [23,24]). Many financial data sets exhibit heavy tails, skewness, and peakedness, which are the characteristic features of GS distributions. Coupled with their

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stability properties (with respect to geometric summation) and domains of attraction, these properties make GS distributions well suited for modeling financial assets (see, e.g., [15,20]). However, the lack of explicit forms for densities and distribution functions of general GS distributions has handicapped their practical implementation.

We study the problem of parameter estimation of Linnik and Mittag-Leffler distributions. Using the theory of GS laws, we express fractional moments, $E|Y|^p$, in terms of the parameters of Linnik and Mittag-Leffler distributions. We then substitute sample factorial moments and solve the resulting equations for the parameters. We present generators for Linnik and Mittag-Leffler distributions, and validate the estimation procedure on simulated data. The estimation method is computationally simple, requires minimal implementation efforts, and provides accurate estimates even for small sample sizes. We illustrate the method on the S&P index data, showing consistency of the data with the Linnik hypothesis.

Here is the organization of our paper. In Section 2, we show that Linnik and Mittag-Leffler distributions are special cases of GS laws. In Section 3, we present generators of Linnik and Mittag-Leffler distributions, based on their recently discovered mixture representations. In Section 4, we describe our estimation procedure for Linnik and Mittag-Leffler parameters, and validate the method on simulated data. We illustrate the procedure on S&P data in Section 5 and conclude with Section 6 that contains tables.

2. GEOMETRIC STABLE, LINNIK, AND MITTAG-LEFFLER DISTRIBUTIONS

2.1. Geometric Stable Distributions

Geometric stable (GS) distributions are related to stable laws via the relation

$$\psi(t) = (1 - \log \phi(t))^{-1}, \tag{2}$$

where ψ and ϕ are GS and stable characteristic functions (ch.f.s), respectively (see [15]). The theory of GS distributions has been developed over the last decade (see, e.g., [14,15,17–21,24]). In one dimension, GS laws form a four parameter family given by the ch.f.,

$$\psi(t) = \left[1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t\right]^{-1}, \tag{3}$$

where

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \operatorname{sign}(x) \tan\left(\dfrac{\pi\alpha}{2}\right), & \text{if } \alpha \neq 1, \\ 1 + i\beta \dfrac{2}{\pi} \operatorname{sign}(x) \log|x|, & \text{if } \alpha = 1. \end{cases} \tag{4}$$

The parameter $\alpha \in (0,2]$ is the index of stability, determining the tail of the distribution, $\beta \in [-1,1]$ is the skewness parameter, and $\mu \in R$ and $\sigma \geq 0$ control the location and scale, respectively. We let $\mathrm{GS}_\alpha(\sigma,\beta,\mu)$ denote a GS distribution with ch.f. (3),(4). Strictly GS distributions, introduced in [14], correspond to strictly stable laws via (2), and have ch.f. (3) with $\alpha \neq 1$ and $\mu = 0$ or $\alpha = 1$ and $\beta = 0$.

2.2. Linnik Distributions

When $\beta = \mu = 0$, the ch.f. (3),(4) corresponds to a symmetric $\mathrm{GS}_\alpha(\sigma,0,0)$ law and takes a particularly simple form

$$\psi(t) = (1 + \sigma^\alpha |t|^\alpha)^{-1}. \tag{5}$$

Since its introduction in [1], it is known as the Linnik characteristic function. We let $L_{\alpha,\sigma}$ denote the Linnik distribution with ch.f. (5). It is interesting to note that the developments in the theory of Linnik distributions independently parallel those of GS laws (see, e.g., [2–4,7–9]).

Pakes [25] and Erdogan [5] studied a more general, nonsymmetric Linnik distribution, with ch.f.

$$\phi_\alpha^\theta(t) = \left(1 + \exp\left(-i\theta \operatorname{sign}(t)\right)|t|^\alpha\right)^{-1}, \tag{6}$$

where $\alpha \in (0,2]$ and $|\theta| \le \min(\pi\alpha/2, \pi - \pi\alpha/2)$. For $\theta = 0$, relation (6) yields the symmetric Linnik distribution (5).

When we compare (6) with (2) and recall the representation C of strictly stable distributions given in [26], it is clear that nonsymmetric Linnik distributions coincide with strictly GS laws. The following relations hold:

$$\phi_\alpha^\theta(t) \sim \mathrm{GS}_\alpha(\sigma,\beta,\mu), \qquad \text{where } \begin{cases} \sigma = 1, & \beta = 0, & \mu = 0, & \text{if } \alpha = 2, \\ \sigma = \cos^{1/\alpha}\theta, & \beta = 0, & \mu = \sin\theta, & \text{if } \alpha = 1, \\ \sigma = \cos^{1/\alpha}\theta, & \beta = \dfrac{\tan\theta}{\tan(\pi\alpha/2)}, & \mu = 0, & \text{if } \alpha \ne 1, 2. \end{cases} \tag{7}$$

REMARK. The multivariate Linnik distribution, introduced in [3] and studied in [8], is a GS law corresponding to a sub-Gaussian stable law through (2).

2.3. Mittag-Leffler Distributions

For $0 < \alpha \le 1$ and $\sigma = 1$, the probability distribution on $(0,\infty)$ with Laplace transform

$$f_{\alpha,\sigma}(s) = \left(1 + \sigma^\alpha s^\alpha\right)^{-1}, \qquad s \ge 0, \tag{8}$$

and denoted $\mathrm{ML}_{\alpha,\sigma}$, is called Mittag-Leffler (see, e.g., [10–12]). It is a generalization of exponential distribution, to which it reduces for $\alpha = 1$. Mittag-Leffler distributions appear in physics literature in connection with the relaxation phenomena (see, e.g., [13] and references therein).

Clearly, for $\alpha = 1$, formula (8) is the Laplace transform of an exponential distribution with mean σ, while for $0 < \alpha < 1$, formula (8) is the Laplace transform of a $\mathrm{GS}_\alpha(\sigma[\cos(\pi\alpha/2)]^{1/\alpha}, 1, 0)$ distribution. Thus, we have the following relation between $\mathrm{ML}_{\alpha,\sigma}$ and $\mathrm{GS}_\alpha(\sigma,\beta,\mu)$ distributions:

$$\mathrm{ML}_{\alpha,\sigma} = \begin{cases} \mathrm{GS}_\alpha(0,1,\sigma), & \text{if } \alpha = 1, \\ \mathrm{GS}_\alpha\left(\sigma\left[\cos\left(\dfrac{\pi\alpha}{2}\right)\right]^{1/\alpha}, 1, 0\right), & \text{if } 0 < \alpha < 1. \end{cases} \tag{9}$$

Note that $\sigma Y \sim \mathrm{ML}_{\alpha,\sigma}$ if $Y \sim \mathrm{ML}_{\alpha,1}$ (σ is a scale parameter).

3. REPRESENTATION AND SIMULATION

In this section, we present algorithms for simulating Linnik and Mittag-Leffler distributions. Although both distributions have representations relating them to stable laws, such representations are not well suited for simulations. The generators we discuss below are based on alternative mixture representations, recently derived in [9,27] for Linnik distributions, and in [28,29] for Mittag-Leffler distributions.

3.1. Mixture Representations

For $0 < \rho < 1$, consider the Cauchy density

$$f_\rho(x) = \frac{\sin(\pi\rho)}{\pi\left[(x + \cos(\pi\rho))^2 + \sin^2(\pi\rho)\right]}, \qquad x \in R. \tag{10}$$

Note that $\int_0^\infty f_\rho(x)\,dx = \rho$. Thus, the function

$$g_\rho(x) = \frac{1}{\rho} f_\rho(x) = \frac{\sin(\pi\rho)}{\pi\rho\left[x^2 + 2x\cos(\pi\rho) + 1\right]} \tag{11}$$

is a density on $(0,\infty)$. Let W be a nonnegative r.v. with density (11). The following representation was proved in [9], and also in [27] for the case $1 < \alpha < \alpha' = 2$.

THEOREM 3.1. *Let $0 < \alpha < \alpha' \le 2$ and $\rho = \alpha/\alpha' < 1$. Let W be a nonnegative r.v. with density (11), and let $Y_{\alpha',\sigma}$ be a Linnik r.v. with ch.f. (5), independent of W. Then, a Linnik r.v. $Y_{\alpha,\sigma}$ with ch.f. (5) admits the representation*

$$Y_{\alpha,\sigma} \stackrel{d}{=} Y_{\alpha',\sigma} \cdot W^{1/\alpha}. \tag{12}$$

Pakes [28], and then independently Kozubowski [29], proved a similar representation for Mittag-Leffler distributions.

THEOREM 3.2. *Let $0 < \alpha < \alpha' \le 1$ and $\rho = \alpha/\alpha' < 1$. Let W be a nonnegative r.v. with density (11), and let $M_{\alpha',\sigma}$ be a Mittag-Leffler r.v. (9), independent of W. Then, a Mittag-Leffler r.v. $M_{\alpha,\sigma}$ admits the representation*

$$M_{\alpha,\sigma} \stackrel{d}{=} M_{\alpha',\sigma} \cdot W^{1/\alpha}. \tag{13}$$

The representations given in Theorems 3.1 and 3.2 can be written in terms of densities. In particular, taking the largest values of α' leads to the expressions for Linnik and Mittag-Leffler densities and distribution functions that are particularly convenient for their numerical approximation. We skip the elementary derivations of the following formulas.

PROPOSITION 3.1. *For any $0 < \alpha < 2$, the density and distribution functions of an $L_{\alpha,1}$ distribution have the representations*

$$f_L(x) = \frac{\sin(\pi\alpha/2)}{\pi} \int_0^\infty \frac{y^\alpha \exp(-xy)}{y^{2\alpha} + 1 + 2y^\alpha \cos(\pi\alpha/2)}\, dy, \qquad x > 0, \tag{14}$$

and

$$F_L(x) = 1 - \frac{\sin(\pi\alpha/2)}{\pi} \int_0^\infty \frac{y^{\alpha-1} \exp(-xy)}{y^{2\alpha} + 1 + 2y^\alpha \cos(\pi\alpha/2)}\, dy, \qquad x > 0, \tag{15}$$

respectively.

PROPOSITION 3.2. *For any $0 < \alpha < 1$, the density and distribution functions of an $\mathrm{ML}_{\alpha,1}$ distribution have the representations*

$$f_{\mathrm{ML}}(x) = \frac{\sin\pi\alpha}{\pi} \int_0^\infty \frac{y^\alpha \exp(-xy)}{y^{2\alpha} + 1 + 2y^\alpha \cos\pi\alpha}\, dy, \qquad x > 0, \tag{16}$$

and

$$F_{\mathrm{ML}}(x) = 1 - \frac{\sin\pi\alpha}{\pi} \int_0^\infty \frac{y^{\alpha-1} \exp(-xy)}{y^{2\alpha} + 1 + 2y^\alpha \cos\pi\alpha}\, dy, \qquad x > 0, \tag{17}$$

respectively.

Figure 1 contains selected densities of Linnik and Mittag-Leffler distributions, calculated from the above representations via Monte Carlo integration.

REMARK 1. Note that by symmetry of $L_{\alpha,\sigma}$ distribution, we have $f_L(x) = f_L(-x)$ and $F_L(x) = 1 - F_L(-x)$ for $x < 0$. The densities are not defined at zero if $\alpha < 1$.

REMARK 2. In case $1 < \alpha < 2$, relations (14) and (15) follow from Theorem 2.1 of [21].

It is interesting to note how Linnik and Mittag-Leffler distributions are interrelated via stable laws. Let $S_\alpha(\sigma, \beta, \mu)$ denote the stable law corresponding to ch.f. (3),(4) via (2) (see, e.g., [30]). Then, every $L_{\alpha,\sigma}$ distribution is a scale mixture of stable distributions $S_{\alpha'}(m^{1/\alpha'}\sigma, 0, 0)$, where m has an $\mathrm{ML}_{\alpha/\alpha',1}$ distribution.

THEOREM 3.3. *Let $0 < \alpha < \alpha' \le 2$ and $\rho = \alpha/\alpha' < 1$. Let $X_{\alpha',\sigma} \sim S_{\alpha'}(\sigma, 0, 0)$ and $M_{\rho,1} \sim \mathrm{ML}_{\rho,1}$ be independent. Then, $Y_{\alpha,\sigma} \sim L_{\alpha,\sigma}$ admits the representation*

$$Y_{\alpha,\sigma} \stackrel{d}{=} M_{\rho,1}^{1/\alpha'} X_{\alpha',\sigma}. \tag{18}$$

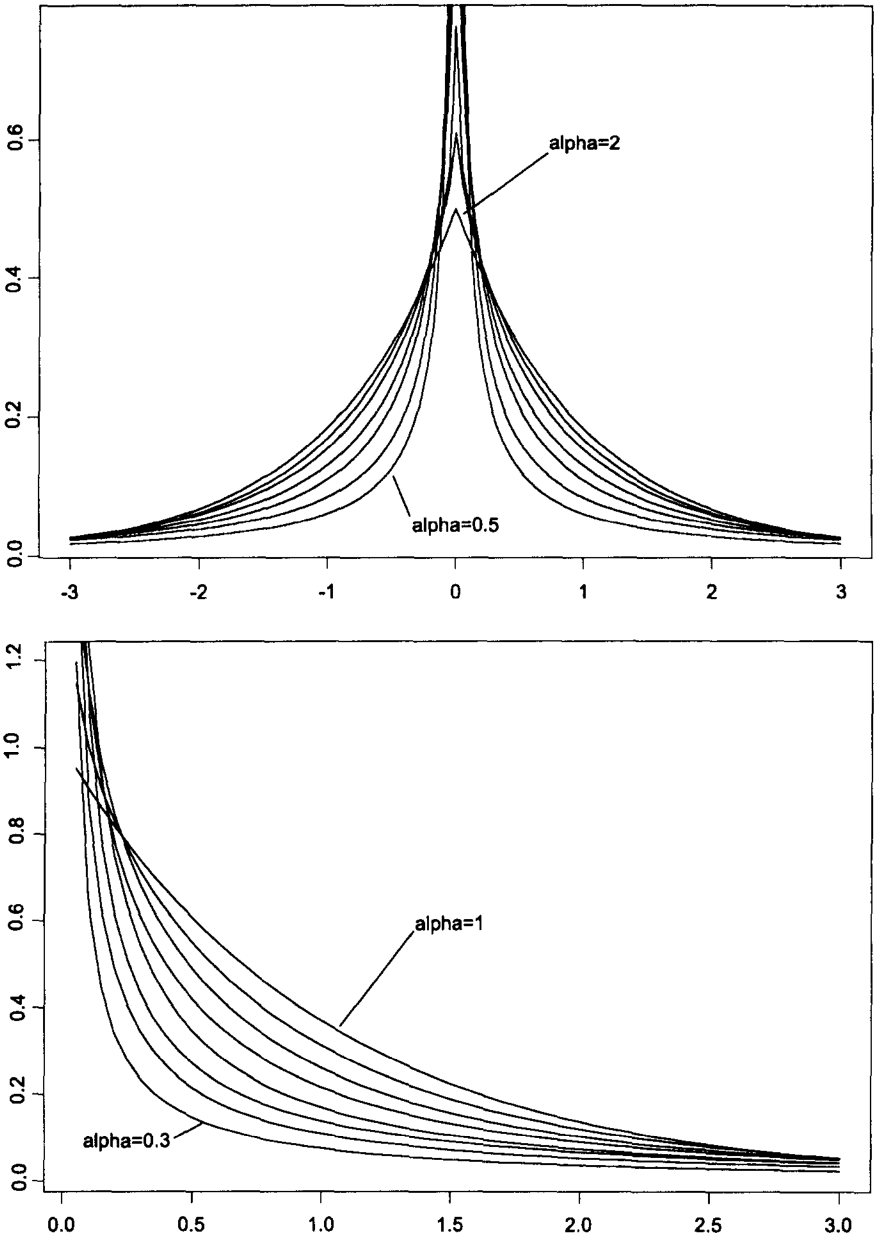


Figure 1. Densities of Linnik (top) and Mittag-Leffler (bottom) distributions, with $\sigma = 1$ and αs equal to 0.5, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00 (Linnik) and 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 (Mittag-Leffler).

PROOF. Let $L_{\alpha,\sigma} = \mathrm{GS}_\alpha(\sigma, 0, 0)$ denote Linnik distribution with ch.f. (5). Consider $0 < \alpha < \alpha' \leq 2$, $\rho = \alpha/\alpha'$, and $Y_{\alpha,\sigma} \sim L_{\alpha,\sigma}$. Then, $Y_{\alpha,\sigma} \stackrel{d}{=} Z^{1/\alpha} X_{\alpha,\sigma}$, where Z is standard exponential, $X_{\alpha,\sigma} \sim S_\alpha(\sigma, 0, 0)$, and Z and $X_{\alpha,\sigma}$ are independent (see [2]). By Proposition 1.3.1 in [30], we also have $X_{\alpha,\sigma} \stackrel{d}{=} A^{1/\alpha'} X_{\alpha',\sigma}$, where $X_{\alpha',\sigma} \sim S_{\alpha'}(\sigma, 0, 0)$ and $A \sim S_\rho([\cos(\pi\rho/2]^{1/\rho}, 1, 0)$, where $X_{\alpha',\sigma}$ and A are independent. Thus,

$$Y_{\alpha,\sigma} \stackrel{d}{=} Z^{1/\alpha} X_{\alpha,\sigma} \stackrel{d}{=} Z^{1/\alpha} A^{1/\alpha'} X_{\alpha',\sigma} \stackrel{d}{=} \left[Z^{1/\rho} A\right]^{1/\alpha'} X_{\alpha',\sigma}. \tag{19}$$

Finally, note that $M_{\rho,1} \stackrel{d}{=} Z^{1/\rho} A \sim \mathrm{ML}_{\rho,1}$ (see, e.g., [20, Corollary 4.2]) to obtain (18) and conclude the proof. ∎

REMARK. Note that taking $\alpha' = 2$, shows that $L_{\alpha,\sigma}$ is a scale mixture of normal distributions, and consequently, is conditionally Gaussian.

3.2. Simulation

Mixture representations of Theorems 3.1 and 3.2 are useful for generating random variates from Linnik and Mittag-Leffler distributions. First, consider Linnik distribution. Let $Y_{\alpha,\sigma} \sim L_{\alpha,\sigma}$, with $\alpha < 2$, be Linnik with ch.f. (5). Kotz and Ostrovskii [9] noted that taking $\alpha' = 2$ in (12) produces $Y_{2,\sigma}$ with Laplace distribution, which is easy to generate. Note that $\rho = \alpha/2 < 1$ when $\alpha' = 2$. The generation of W is straightforward. Simple calculation yields the distribution function of W:

$$F_W(x) = \frac{1}{\pi\rho}\left[\arctan\left(\frac{x}{\sin\pi\rho} + \cot\pi\rho\right) - \frac{\pi}{2}\right] + 1, \tag{20}$$

as well as its inverse,

$$F_W^{-1}(x) = \sin(\pi\rho)\cot\left(\pi\rho(1-x)\right) - \cos(\pi\rho). \tag{21}$$

To obtain a random variate W, use the inversion method (see [31]). Generate uniform $[0,1]$ variate U, and return $W = F_W^{-1}(U)$. Here is the algorithm for generating Linnik variates.

A Linnik $L_{\alpha,\sigma}$ generator

Generate random variate Z from $L_{2,1}$ distribution (standard Laplace with location 0 and scale 1)
Generate uniform $[0,1]$ variate U, independent of Z
Set $\rho \leftarrow \alpha/2$
Set $W \leftarrow \sin(\pi\rho)\cot(\pi\rho U) - \cos(\pi\rho)$
Set $Y \leftarrow \sigma \cdot Z \cdot W^{1/\alpha}$
RETURN Y

Next, we turn to simulation of Mittag-Leffler distributions (9). The procedure is based on representation (13) with $\alpha' = 1$, which shows that a Mittag-Leffler distribution is a scale mixture of exponential distributions. Here is the algorithm for generating Mittag-Leffler random variates.

A Mittag-Leffler $\mathrm{ML}_{\alpha,\sigma}$ generator

Generate random variate Z from $\mathrm{ML}_{1,1}$ distribution (standard exponential)
Generate uniform $[0,1]$ variate U, independent of Z
Set $\rho \leftarrow \alpha$
Set $W \leftarrow \sin(\pi\rho)\cot(\pi\rho U) - \cos(\pi\rho)$
Set $Y \leftarrow \sigma \cdot Z \cdot W^{1/\alpha}$
RETURN Y

To illustrate the algorithms, we simulated Linnik and Mittag-Leffler distributions, and compared the resulting histograms with their theoretical densities (Figure 2).

4. ESTIMATION

In this section, we develop procedures for estimating the parameters of Linnik and Mittag-Leffler distributions. We adapt the approach based on fractional moments, used by Nikias and Shao [32] for estimating stable parameters.

4.1. Linnik Distribution

Let $Y_1, \dots, Y_n$ be a random sample from a Linnik $L_{\alpha,\sigma}$ distribution. For $0 < p < \alpha$, let $e(p) = E|Y_1|^p$ denote the p^{th} absolute moment of $L_{\alpha,\sigma}$. The following formula for $e(p)$ follows from Proposition 5.3 in [23].

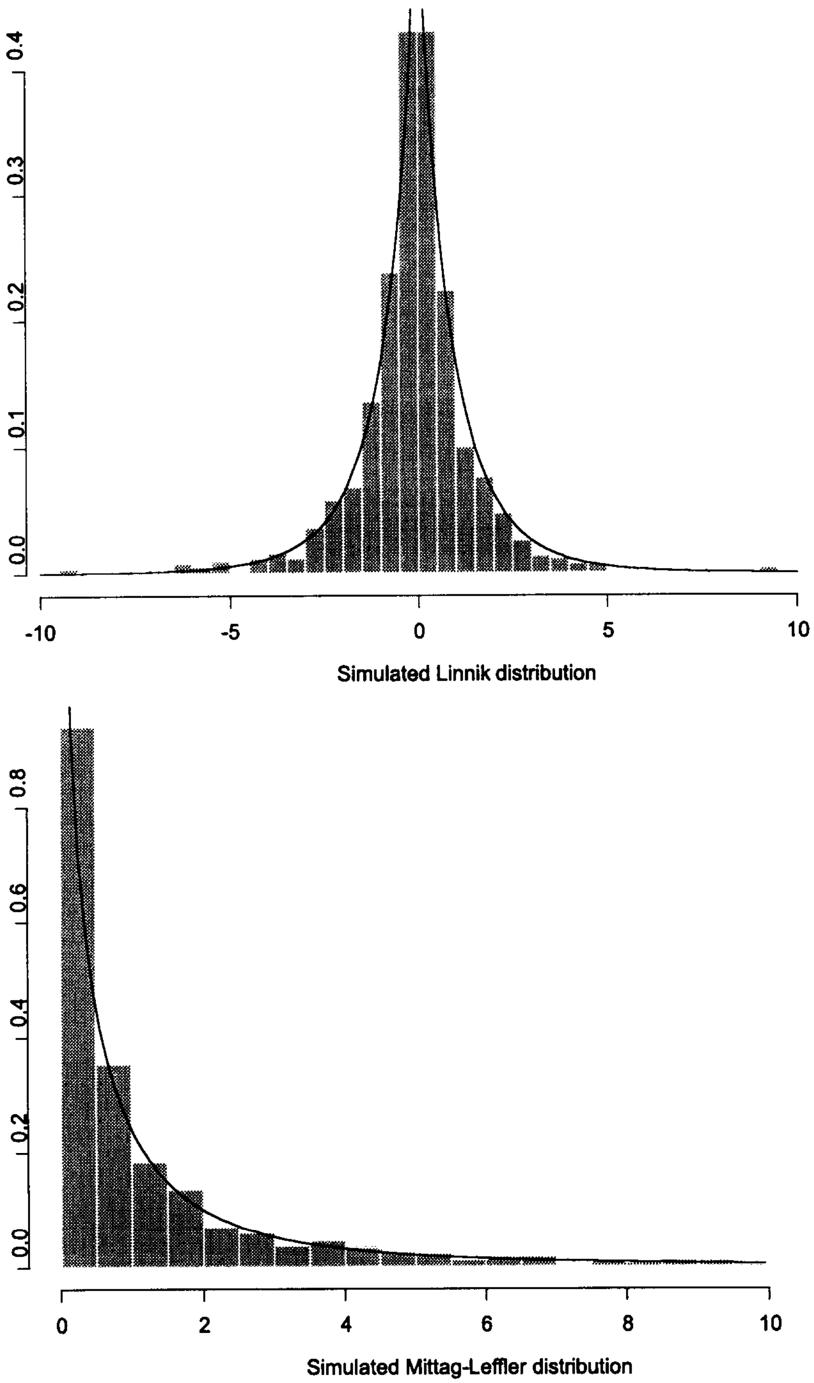


Figure 2. Theoretical densities and simulated samples of size $n = 1000$ from a symmetric Linnik distribution (top), where $\alpha = 1.75$, $\sigma = 1$, and from a Mittag-Leffler distribution (bottom), with $\alpha = 0.75$, $\sigma = 1$.

THEOREM 4.1. *Let $Y \sim L_{\alpha,\sigma}$ with $0 < \alpha \leq 2$. Then, for every $0 < p < \alpha$, we have*

$$e(p) = E|Y|^p = \frac{p(1-p)\sigma^p \pi}{\alpha\Gamma(2-p)\sin(\pi p/\alpha)\cos(\pi p/2)}. \tag{22}$$

REMARK. In case $p = 1$, we set $(1-p)/\cos(\pi p/2)$ equal to $2/\pi$, which is its limiting value when $p \to 1$.

Formula (22) leads to the method of moments estimators for α and σ. Namely, we choose two values of p, say p_1 and p_2, replace $e(p_k)$ in (22) with its sample counterpart $\hat{e}(p_k) =$

$(1/n)\sum |Y_i|^{p_k}$, $k = 1,2$, and solve the resulting equations for α and σ. Although α can in general take any value in $(0,2]$, we consider $1 < \alpha \le 2$, which is a likely situation with financial return data (the distribution has a finite mean, but may have an infinite variance). Further, we use $p_1 = 1/2$ and $p_2 = 1$, so that (22) yields two equations:

$$\hat{e}\left(\frac{1}{2}\right) = \frac{1}{n}\sum |Y_i|^{1/2} = \sqrt{\frac{\pi\sigma}{2}}\,\frac{1}{\alpha\sin(\pi/2\alpha)} \tag{23}$$

and

$$\hat{e}(1) = \frac{1}{n}\sum |Y_i| = \frac{2\sigma}{\alpha\sin(\pi/\alpha)}. \tag{24}$$

We eliminate σ from (23) and (24) by squaring both sides of (23) and dividing the two sides of the resulting equation into the corresponding sides of equation (24). This results in the following equation:

$$\frac{\hat{e}(1)}{(\hat{e}(1/2))^2} = \frac{4\alpha\sin^2(\pi/2\alpha)}{\pi\sin(\pi/\alpha)}. \tag{25}$$

The following lemma shows that the RHS of (25) is strictly decreasing in α.

LEMMA 4.1. *The function*

$$g(\alpha) = \frac{(4/\pi)\,\alpha\sin^2(\pi/2\alpha)}{\sin(\pi/\alpha)} \tag{26}$$

is strictly decreasing on $(1,2]$, *with* $g(1^+) = \infty$ *and* $g(2) = 4/\pi$.

PROOF. Note that the derivative of g,

$$\frac{d}{d\alpha}\,g(\alpha) = \frac{4}{\pi}\left(\frac{\sin(\pi/2\alpha)}{\sin(\pi/\alpha)}\right)^2\left(\sin\frac{\pi}{\alpha} - \frac{\pi}{\alpha}\right), \tag{27}$$

is negative, since $\sin(\pi/\alpha) < (\pi/\alpha)$ for $\alpha \in (1,2]$. This proves monotonicity. The values $g(1^+) = \infty$ and $g(2) = 4/\pi$ are easily obtained. ∎

Since the function g is well behaved, finding $\hat{\alpha}$, the numerical solution of (25), is straightforward. Substituting $\hat{\alpha}$ into (23) and (24), we solve the resulting equations for $\hat{\sigma}_1$ and $\hat{\sigma}_2$:

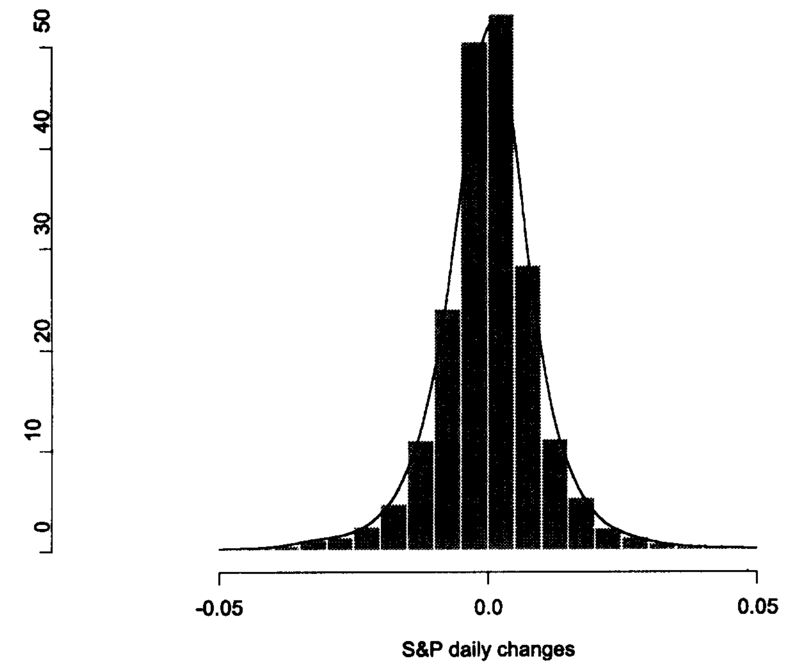
$$\hat{\sigma}_1 = \frac{2}{\pi}\,\hat{\alpha}^2\sin^2\frac{\pi}{2\hat{\alpha}}\left[\hat{e}\left(\frac{1}{2}\right)\right]^2, \tag{28}$$

$$\hat{\sigma}_2 = \frac{1}{2}\,\hat{\alpha}\sin\frac{\pi}{\hat{\alpha}}\,\hat{e}(1). \tag{29}$$

Finally, we compute the average, $\hat{\sigma} = (\hat{\sigma}_1 + \hat{\sigma}_2)/2$. We have checked the above estimation procedure on simulated data. We simulated ten random samples from $L_{\alpha,\sigma}$, each of size $n = 50000$. We used $\sigma = 10$ and $\alpha = 1 + 0.1k$, where $k = 1,\dots,10$. For each data set, the estimation procedure described above was applied to subsamples of the first m values in the set, with $m = 100$, 1000, 10000, 25000, and 50000. The results of the simulation experiment are presented in Table 1 of Section 6. Overall, the estimators perform well on simulated data. The results are most accurate when α is close to 2, and generally improve as n increases. Remarkably, the procedure yields satisfactory results even for sample sizes as small as 100.

4.2. Mittag-Leffler Distribution

The method of estimating α and σ of the Mittag-Leffler distribution (9) is analogous to that of Linnik distribution. Let $e(p) = E|Y|^p$, where $0 < p < \alpha$, denote the p^{th} absolute moment of $Y \sim \mathrm{ML}_{\alpha,\sigma}$. The following formula for $e(p)$ was derived in [11].



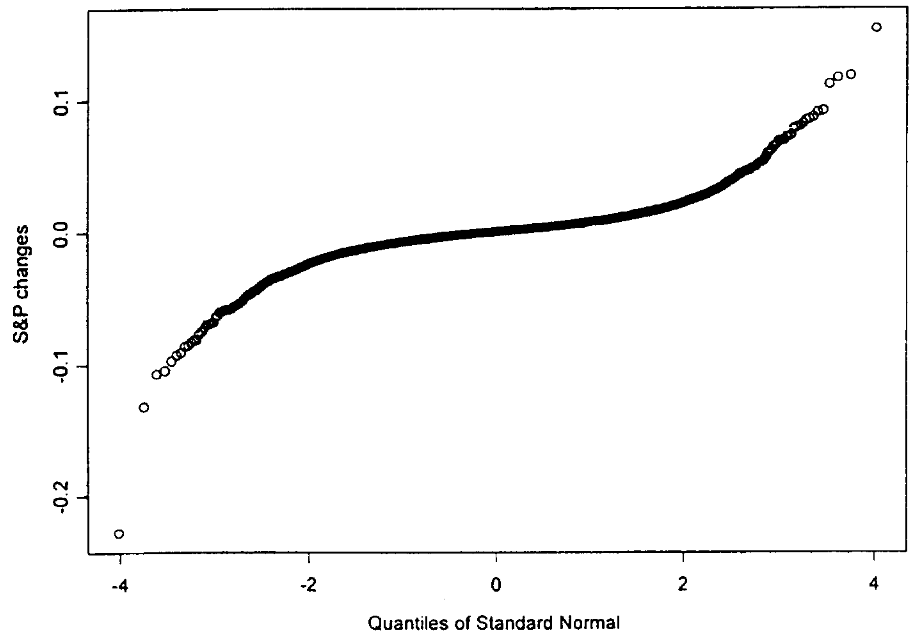


Figure 3. A histogram and smoothed density of S&P data set (top), and its quantile plot with normal distribution (bottom).

THEOREM 4.2. *Let $Y \sim \mathrm{ML}_{\alpha,\sigma}$ with $0 < \alpha \leq 1$. Then, for every $0 < p < \alpha$, we have*

$$e(p) = E|Y|^p = \frac{p\sigma^p \pi}{\alpha\Gamma(1-p)\sin(\pi p/\alpha)}. \tag{30}$$

Consider a random sample, $Y_1, \ldots, Y_n$, from an $\mathrm{ML}_{\alpha,\sigma}$ distribution. Formula (30) leads to the method of moments estimators for α and σ. As before, choose two values of p, say p_1 and p_2, replace $e(p_k)$ in (30) with its sample counterpart $\hat{e}(p_k) = (1/n)\sum |Y_i|^{p_k}$, $k = 1, 2$, and solve the resulting equations for α and σ. To illustrate the procedure, we assume that $0.5 < \alpha \leq 1$, and choose $p_1 = 1/2$ and $p_2 = 1/4$. Then, equation (30) produces

$$\hat{e}\left(\frac{1}{2}\right) = \frac{1}{n}\sum |Y_i|^{1/2} = \frac{\sqrt{\pi\sigma}}{2\alpha\sin(\pi/2\alpha)} \tag{31}$$

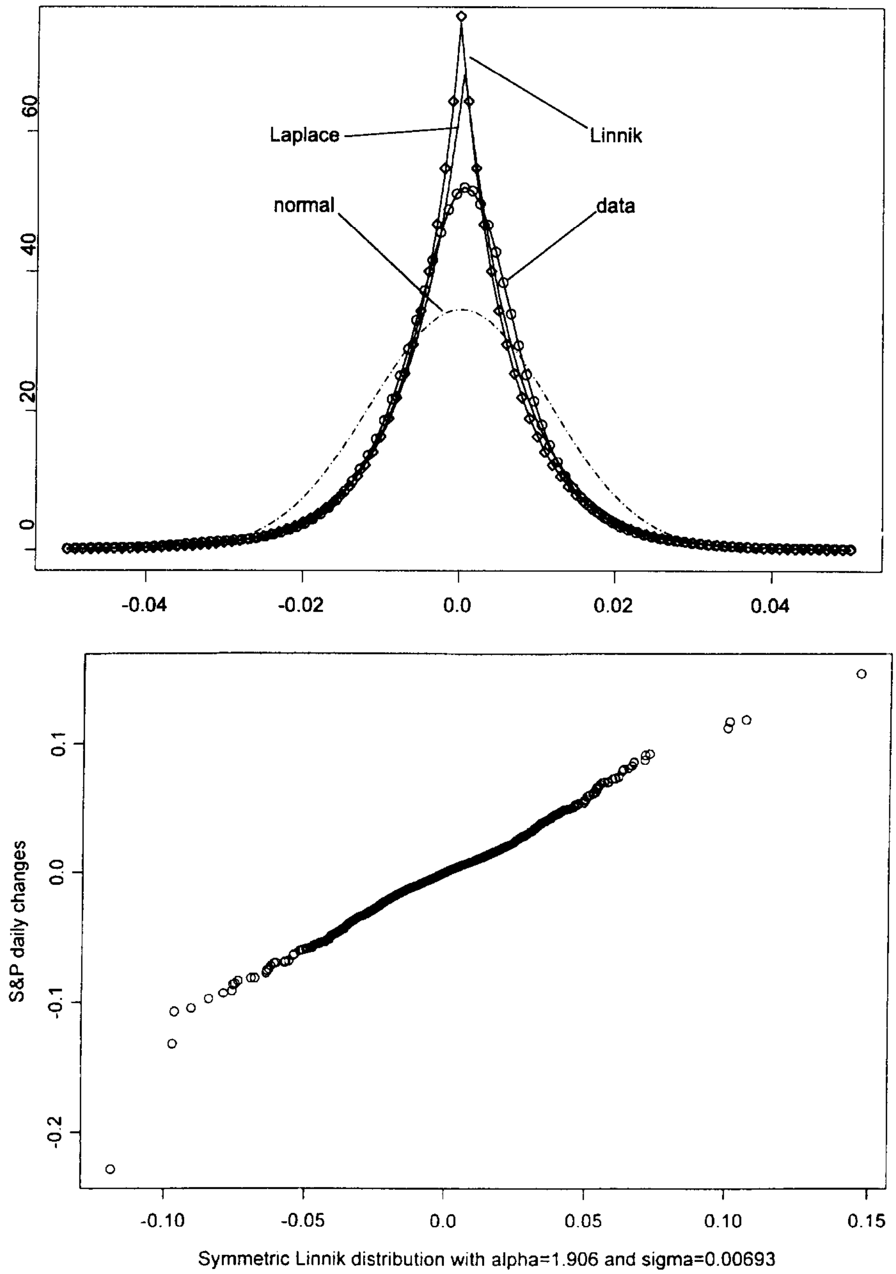


Figure 4. Diagnostics for fitting S&P data set. The top graph shows the smoothed data density (obtained via kernel density estimation), and the exact densities of fitted Linnik, Laplace, and normal distributions. The bottom graph is a quantile plot of S&P daily changes data with fitted Linnik distribution, where $\alpha = 1.906$ and $\sigma = 0.00696$.

and

$$\hat{e}\left(\frac{1}{4}\right) = \frac{1}{n}\sum |Y_i|^{1/4} = \frac{\pi\sigma^{1/4}}{4\alpha\Gamma(3/4)\sin(\pi/4\alpha)}. \tag{32}$$

We eliminate σ from (31) and (32) by squaring both sides of (32) and dividing the two sides of the resulting equation into the corresponding sides of equation (31). This results in the following equation, for α:

$$\frac{\hat{e}(1/2)}{(\hat{e}(1/4))^2} = \frac{\Gamma^2(3/4)}{\sqrt{\pi}}\, g(2\alpha), \tag{33}$$

with the function g as in (26). By Lemma 4.1, the RHS of (33) is strictly decreasing on $(1/2, 1]$, so that obtaining $\hat{\alpha}$, the numerical solution of (33), is straightforward. We then substitute $\hat{\alpha}$ into (31) and (32) and solve the resulting equations for $\hat{\sigma}_1$ and $\hat{\sigma}_2$, obtaining

$$\hat{\sigma}_1 = 4\hat{\alpha}^2 \sin^2 \frac{\pi}{2\hat{\alpha}} \frac{[\hat{e}(1/2)]^2}{\pi}, \tag{34}$$

$$\hat{\sigma}_2 = \left[4\hat{\alpha} \sin \frac{\pi}{4\hat{\alpha}} \Gamma\left(\frac{3}{4}\right) \frac{\hat{e}(1/4)}{\pi}\right]^4. \tag{35}$$

Finally, we average the two estimates, $\hat{\sigma} = (\hat{\sigma}_1 + \hat{\sigma}_2)/2$. We have checked the procedure on simulated data. We simulated ten random samples from $\mathrm{ML}_{\alpha,\sigma}$, each of size $n = 50000$. We used $\sigma = 10$ and $\alpha = 0.5 + 0.05k$, where $k = 1, \dots, 10$. For each data set, we applied the estimation procedure described above to subsamples of the first m values in the set, with $m = 100$, 1000, 10000, 25000, and 50000. The results of the simulation experiment are presented in Table 2 of Section 6. Like in the Linnik case, the estimators perform quite well on simulated data, and are most accurate when α is close to 1. Although the results generally improve as n increases, they are quite satisfactory even for small sample sizes.

5. APPLICATIONS

To illustrate the potential of geometric stable distributions in modeling financial data, we examined daily figures for the S&P index over the period January 3, 1928 to May 30, 1990. We considered the usual logarithmic changes, $y_t = \log x_t - \log x_{t-1}$, where x_t represents the value of the S&P index on day t. There were $n = 16736$ data values after the transformation. A histogram of the data set is given in Figure 3. It appears to be symmetric at zero, where it has a high peak. A quantile plot with the standard normal, also presented in Figure 3, reveals heavy tails: the S&P data set is clearly not normal. The comparison with Figure 1 suggests fitting a symmetric Linnik distribution to the S&P data. The estimation procedure described in Section 4 produced the Linnik parameters $\hat{\alpha} = 1.906$ and $\hat{\sigma} = 0.00693$. We also fitted the special case of Laplace distribution, obtaining the values of 0.000461 and 0.00728 for the location and scale, respectively, from the standard maximum likelihood estimation procedure. For the normal, the maximum likelihood estimation of location and scale produced the values of 0.00018 and 0.0115, respectively. The density plots of Figure 4 show a poor normal fit, as expected, and reasonably good Linnik and Laplace fits. In addition, the quantile plot of the S&P data with simulated Linnik distribution, shown in Figure 4, shows only a slight departure from a straight line. We conclude that the S&P data is consistent with the Linnik model.

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APPENDIX

Table 1. Fractional moments estimators for the parameters of Linnik $L_{\alpha,\sigma}$ distribution.

α, σ	$n = 100$	$n = 1000$	$n = 10000$	$n = 25000$	$n = 50000$
$\alpha = 2$ $\sigma = 10$	$\hat{\alpha} = 2.000$ $\hat{\sigma} = 9.109$	$\hat{\alpha} = 2.000$ $\hat{\sigma} = 9.276$	$\hat{\alpha} = 1.999$ $\hat{\sigma} = 9.910$	$\hat{\alpha} = 2.000$ $\hat{\sigma} = 10.016$	$\hat{\alpha} = 2.000$ $\hat{\sigma} = 9.960$
$\alpha = 1.9$ $\sigma = 10$	$\hat{\alpha} = 1.806$ $\hat{\sigma} = 8.652$	$\hat{\alpha} = 1.884$ $\hat{\sigma} = 10.128$	$\hat{\alpha} = 1.896$ $\hat{\sigma} = 9.896$	$\hat{\alpha} = 1.902$ $\hat{\sigma} = 9.971$	$\hat{\alpha} = 1.909$ $\hat{\sigma} = 10.009$
$\alpha = 1.8$ $\sigma = 10$	$\hat{\alpha} = 1.700$ $\hat{\sigma} = 8.469$	$\hat{\alpha} = 1.862$ $\hat{\sigma} = 9.992$	$\hat{\alpha} = 1.809$ $\hat{\sigma} = 9.935$	$\hat{\alpha} = 1.822$ $\hat{\sigma} = 10.041$	$\hat{\alpha} = 1.814$ $\hat{\sigma} = 10.054$
$\alpha = 1.7$ $\sigma = 10$	$\hat{\alpha} = 1.958$ $\hat{\sigma} = 9.660$	$\hat{\alpha} = 1.819$ $\hat{\sigma} = 9.801$	$\hat{\alpha} = 1.738$ $\hat{\sigma} = 10.164$	$\hat{\alpha} = 1.727$ $\hat{\sigma} = 10.023$	$\hat{\alpha} = 1.721$ $\hat{\sigma} = 10.072$
$\alpha = 1.6$ $\sigma = 10$	$\hat{\alpha} = 1.766$ $\hat{\sigma} = 9.491$	$\hat{\alpha} = 1.675$ $\hat{\sigma} = 10.203$	$\hat{\alpha} = 1.635$ $\hat{\sigma} = 10.268$	$\hat{\alpha} = 1.659$ $\hat{\sigma} = 10.234$	$\hat{\alpha} = 1.637$ $\hat{\sigma} = 10.139$
$\alpha = 1.5$ $\sigma = 10$	$\hat{\alpha} = 1.562$ $\hat{\sigma} = 9.626$	$\hat{\alpha} = 1.619$ $\hat{\sigma} = 10.738$	$\hat{\alpha} = 1.404$ $\hat{\sigma} = 9.558$	$\hat{\alpha} = 1.427$ $\hat{\sigma} = 9.540$	$\hat{\alpha} = 1.449$ $\hat{\sigma} = 9.745$
$\alpha = 1.4$ $\sigma = 10$	$\hat{\alpha} = 1.351$ $\hat{\sigma} = 10.257$	$\hat{\alpha} = 1.540$ $\hat{\sigma} = 10.614$	$\hat{\alpha} = 1.478$ $\hat{\sigma} = 10.397$	$\hat{\alpha} = 1.414$ $\hat{\sigma} = 10.239$	$\hat{\alpha} = 1.423$ $\hat{\sigma} = 10.212$
$\alpha = 1.3$ $\sigma = 10$	$\hat{\alpha} = 1.537$ $\hat{\sigma} = 9.669$	$\hat{\alpha} = 1.465$ $\hat{\sigma} = 10.652$	$\hat{\alpha} = 1.412$ $\hat{\sigma} = 10.768$	$\hat{\alpha} = 1.397$ $\hat{\sigma} = 10.641$	$\hat{\alpha} = 1.362$ $\hat{\sigma} = 10.364$
$\alpha = 1.2$ $\sigma = 10$	$\hat{\alpha} = 1.235$ $\hat{\sigma} = 9.795$	$\hat{\alpha} = 1.440$ $\hat{\sigma} = 10.982$	$\hat{\alpha} = 1.255$ $\hat{\sigma} = 10.304$	$\hat{\alpha} = 1.283$ $\hat{\sigma} = 10.722$	$\hat{\alpha} = 1.289$ $\hat{\sigma} = 10.769$
$\alpha = 1.1$ $\sigma = 10$	$\hat{\alpha} = 1.396$ $\hat{\sigma} = 8.629$	$\hat{\alpha} = 1.320$ $\hat{\sigma} = 11.893$	$\hat{\alpha} = 1.209$ $\hat{\sigma} = 11.459$	$\hat{\alpha} = 1.180$ $\hat{\sigma} = 11.137$	$\hat{\alpha} = 1.180$ $\hat{\sigma} = 11.131$

Table 2. Fractional moments estimators for the parameters of Mittag-Leffler $ML_{\alpha,\sigma}$ distribution.

α, σ	$n = 100$	$n = 1000$	$n = 10000$	$n = 25000$	$n = 50000$
$\alpha = 1$ $\sigma = 10$	$\hat{\alpha} = 1.000$ $\hat{\sigma} = 8.634$	$\hat{\alpha} = 0.996$ $\hat{\sigma} = 10.401$	$\hat{\alpha} = 1.000$ $\hat{\sigma} = 10.212$	$\hat{\alpha} = 1.000$ $\hat{\sigma} = 10.031$	$\hat{\alpha} = 1.000$ $\hat{\sigma} = 10.048$
$\alpha = 0.95$ $\sigma = 10$	$\hat{\alpha} = 0.984$ $\hat{\sigma} = 10.439$	$\hat{\alpha} = 0.972$ $\hat{\sigma} = 10.433$	$\hat{\alpha} = 0.958$ $\hat{\sigma} = 10.071$	$\hat{\alpha} = 0.956$ $\hat{\sigma} = 10.014$	$\hat{\alpha} = 0.952$ $\hat{\sigma} = 9.996$
$\alpha = 0.90$ $\sigma = 10$	$\hat{\alpha} = 0.907$ $\hat{\sigma} = 11.478$	$\hat{\alpha} = 0.915$ $\hat{\sigma} = 10.082$	$\hat{\alpha} = 0.906$ $\hat{\sigma} = 9.961$	$\hat{\alpha} = 0.906$ $\hat{\sigma} = 9.965$	$\hat{\alpha} = 0.899$ $\hat{\sigma} = 9.970$
$\alpha = 0.85$ $\sigma = 10$	$\hat{\alpha} = 0.778$ $\hat{\sigma} = 10.533$	$\hat{\alpha} = 0.748$ $\hat{\sigma} = 8.776$	$\hat{\alpha} = 0.830$ $\hat{\sigma} = 9.876$	$\hat{\alpha} = 0.844$ $\hat{\sigma} = 9.900$	$\hat{\alpha} = 0.853$ $\hat{\sigma} = 10.093$
$\alpha = 0.80$ $\sigma = 10$	$\hat{\alpha} = 0.865$ $\hat{\sigma} = 11.899$	$\hat{\alpha} = 0.811$ $\hat{\sigma} = 10.656$	$\hat{\alpha} = 0.794$ $\hat{\sigma} = 10.095$	$\hat{\alpha} = 0.800$ $\hat{\sigma} = 10.139$	$\hat{\alpha} = 0.790$ $\hat{\sigma} = 9.974$
$\alpha = 0.75$ $\sigma = 10$	$\hat{\alpha} = 0.841$ $\hat{\sigma} = 12.658$	$\hat{\alpha} = 0.815$ $\hat{\sigma} = 10.606$	$\hat{\alpha} = 0.740$ $\hat{\sigma} = 9.787$	$\hat{\alpha} = 0.739$ $\hat{\sigma} = 9.848$	$\hat{\alpha} = 0.744$ $\hat{\sigma} = 9.769$
$\alpha = 0.70$ $\sigma = 10$	$\hat{\alpha} = 0.751$ $\hat{\sigma} = 15.866$	$\hat{\alpha} = 0.720$ $\hat{\sigma} = 11.499$	$\hat{\alpha} = 0.702$ $\hat{\sigma} = 10.143$	$\hat{\alpha} = 0.715$ $\hat{\sigma} = 10.312$	$\hat{\alpha} = 0.719$ $\hat{\sigma} = 10.297$
$\alpha = 0.65$ $\sigma = 10$	$\hat{\alpha} = 0.741$ $\hat{\sigma} = 9.741$	$\hat{\alpha} = 0.706$ $\hat{\sigma} = 10.876$	$\hat{\alpha} = 0.693$ $\hat{\sigma} = 11.257$	$\hat{\alpha} = 0.660$ $\hat{\sigma} = 10.166$	$\hat{\alpha} = 0.657$ $\hat{\sigma} = 10.126$
$\alpha = 0.60$ $\sigma = 10$	$\hat{\alpha} = 0.692$ $\hat{\sigma} = 13.818$	$\hat{\alpha} = 0.665$ $\hat{\sigma} = 12.299$	$\hat{\alpha} = 0.637$ $\hat{\sigma} = 12.024$	$\hat{\alpha} = 0.640$ $\hat{\sigma} = 11.995$	$\hat{\alpha} = 0.628$ $\hat{\sigma} = 11.399$
$\alpha = 0.55$ $\sigma = 10$	$\hat{\alpha} = 0.526$ $\hat{\sigma} = 31.132$	$\hat{\alpha} = 0.580$ $\hat{\sigma} = 12.732$	$\hat{\alpha} = 0.608$ $\hat{\sigma} = 12.539$	$\hat{\alpha} = 0.597$ $\hat{\sigma} = 12.420$	$\hat{\alpha} = 0.598$ $\hat{\sigma} = 12.364$