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The statistics of the fractional moments: Is there any chance to "read quantitatively" any randomness?

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Abstract

The statistical meaning of higher $\Delta_N^{(p)}$ $(p = 1, 2, \ldots)$ and fractional $\Delta_N^{(p)}$ $(0<p<\infty)$ moments for an arbitrary random sequence of the length N has been found. The higher moments help to reduce the sequence analyzed to a finite set of k *statistically stable* parameters, keeping invariant the values of the first kth moments $\Delta_k^{(p)}$ $(p = 1, 2, \ldots, k)$. The conditions of statistical stability and proximity expressed in terms of higher moments $\Delta_N^{(p)} = \Delta_{N+k}^{(p)}$ $(p = 1, 2, \ldots, k)$ help to find k unknown stable points and predict possible future behavior of the random sequence analyzed. The generalized mean value (GMV)-function defined as $G_N^{(p)} = (\Delta_N^{(p)})^{1/p}$ is turned to be very effective in analysis of statistically close random sequences or containing large numbers of measured points $(N \gg 1)$. The approximate analytical expression for an arbitrary p value from the range $(-\infty<p<\infty)$ entering into $G_N^{(p)}$ has been found. It gives a possibility to transform *any* random sequence to the determined GMV curve and express *quantitatively* the reduced characteristics of any random sequence in terms of a 'universal' set of the fitting parameters defined by the determined GMV-function. Statistical proximity factor can be used for construction of *calibration curves*, when it is necessary to compare one random sequence with another one to respect of variations of some given external factor (small signal). The higher moments are easily generalized for the fractional and even complex moments. In turn, the GMV-function can be also generalized and then calculated for 2D and 3D random sequences. The approach developed in this paper is free from *any* model assumption and can be extremely helpful in comparison of different random sequences using for these purposes the 'unified' quantitative language based on introduction of the given set of fractional moments. The relationship between the value of the fractional moment and nonextensive parameter q entering into the definition of the nonextensive Tsallis entropy has been found. A possible model of statistical protection of plastic cards and other valuable documents demonstrating the effectiveness of the statistics of fractional moments has been considered. Some instructive examples in detection of superweak signals embedded into the basic random sequence $(S/N = 10^{-2}, 10^{-3})$ based on model and real data confirm the effectiveness of new approach and can serve a new basis for numerous practical applications. Analysis of dielectric spectroscopy data by means of fractional moments gives unique possibility to compare *quantitatively* each measurement with each other and express the influence of a neutral additive in terms of calibration curve *without* concrete knowledge of the corresponding fitting function.

Keywords: Higher moments; The generalized mean value function approach; Statistics of fractional and complex moments; Eigen-coordinates method

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1. Introduction

It is a common goal of any method developed or suggested in modern applied statistics is to improve the sensitivity and selectivity of the existing mathematical techniques by enhancing the signal-to-noise ratio, and reduce possible systematic errors associated with drift of possible instrumentation parameters. Another important problem is to elaborate a "universal" quantitative language for comparison of two arbitrary random sequences having in principle different number of points $N_1 \neq N_2$ with each other.

There are currently a number of signal-processing modern methods that are utilized to solve first problem, for example, Hurst analysis of fractal random samplings [1]. The powerful tool of wavelet-analysis, based on the generalization of the conventional Fourier transforms [2–5], has also been developed to extract weak signals from initial noise data. More recently, the stochastic dynamics of time correlations with discrete current time has been considered by Yulmetyev et al. [6–8]. It is necessary to remind also the fluctuation-noise spectroscopy (FNS) phenomenological method developed by Timashev with co-authors [9–11], which was turned out rather effective in detection of anomalous behavior of different random sequences as intermittency, the self-organized criticality, etc.

However, critical analysis of these current methods shows that we do not have a *universal* language for *quantitative* comparison of *arbitrary* random samplings that possess different statistical characteristics. Moreover, many of the aforementioned techniques are relatively insensitive to the detection of weak signals with amplitude (S) comparable with the level of noise (N) or much less in comparison of the level of noise ($S/N \ll 1$).

More recently, new statistical methods of signal detection have been developed based on the recognition of corresponding histograms [12,13] and fluctuation fork analysis [14,15]. These new approaches realise the possibility of a statistical method of detection of weak signals, which are completely hidden inside the random sampling analyzed.

In this paper, we are developing the statistics of higher (fractional) moments (SFM) that is turned out very effective in detection of *superweak* signal completely hidden in the random sequence analyzed, but nevertheless distorting the statistical behavior of the random sequence. The SFM method that is free from *any* model assumptions, contains conventional statistics (using only four integer moments) as a partial case and gives a unique possibility to compare arbitrary random sequences (having in principle different samplings) with each other and express their differences in terms of a universal set of quantitative parameters. These *random* differences (or distortions) can be expressed in terms of the given number of fractional moments and to be more precise in the form of the *determined* generalized mean value (GMV) function defined as $G_N^{(p)} = (\Delta_N^{(p)})^{1/p}$, where $\Delta_N^{(p)}$ is the absolute value of the moment of the pth order. The fitting parameters of the GMV-function found with the help of the ECs method can be used as a *universal quantitative* language for comparison of various random sequences with each other. The SFM method together with the GMV-function approach help to find a "universal" desired calibration curve, when some fitting parameters have monotonic behavior with respect to the influence of some external predominant factor. This factor slightly disturbs the pattern sequence, which it is supposed is free from an added signal. The external factor can be associated with concentration of an additive, temperature, humidity, etc. The practical realization of the GMV-function approach in the first time has been demonstrated recently in paper [16], where calibration curve designed by means of the GMV-approach allows to increase the sensitivity of the laboratory (near infra-red) NIR equipment more than one order of magnitude with respect to concentration of small additive. In general, statistics based on usage of higher moments are to all intents and purposes neglected in the field of natural sciences, though certain applications have been developed [17,18].

Here we want to demonstrate large potential possibilities of the statistics of the fractional moments (SFM) in *quantitative* comparison ("reading") of different random sequences. Especially, the effectiveness of the SFM will be demonstrated on example, *where* the GMV-function approach can lead to the transformation of *any* random sequence to a smooth function, which can be approximated with high accuracy by an analytical function containing a finite set of the reduced (fitting) parameters. These fitting parameters then serve as a specific 'fingerprint' of the random sequence and a possible *instrument* by which various random sequences with nonequal samplings (in general) may be compared with each other *quantitatively*. This simple approach is free from *any* model assumptions, and might therefore prove to be rather *universal* in different applications, providing an extraction of a hidden predominant factor among other perturbing factors that also modify the GMV-function, calculated in the space of fractional or complex (if it is necessary) moments. It is interesting to note that the fractional moments are closely associated with nonextensive parameter q, entering into definition of the Tsallis entropy. In this approach Tsallis entropy receives new interpretation and can be used also for comparison of stochastic characteristics of different random sequences.

The paper is organized into the following sections: In Section 2, the questions of stability of random sequences expressed in terms of higher moments are considered. In Section 3 the values of the fitting parameters of the generalized mean function are evaluated by means of the eigen-coordinates (ECs) method. Here the definition of *complex* moments and some generalizations of the GMV-function for 2D and 3D dimensions are considered. In Section 4 new relationship between fractional moments and nonextensive parameter q, entering into definition of the Tsallis entropy has been found. This relationship increases possibilities of nonextensive statistical mechanics in interpretation of stochastic properties of different random sequences. A possible scheme of statistical protection of valuable documents demonstrating the incredible sensitivity of the statistics of higher moments is described schematically in Section 5. Some model experiments demonstrating a power of the GMV-approach in extraction of superweak signals ($S/N = 10^{-2}, 10^{-3}$) and detection of statistical proximity of different distributions are shown in Section 6. Some problems based on real data analysis and related to construction of the desired calibration curve *without* knowledge of the fitting function are considered in Section 7. The final results are then collected in Section 8.

2. Evaluation of statistical stability of random sequences based on higher moments

We consider a segment of a random sequence as a discrete system of $j = 1, 2, \ldots, N$ points located in the band $y_{\min} \leqslant \{y_j\} \leqslant y_{\max}$. For the set of randomly located points N the average moment of the pth order is defined by expression:

$$\Delta_N^{(p)} = \frac{1}{N}\sum_{j=1}^{N} y_j^p. \tag{1}$$

The basic problem can be formulated as follows. Let us suppose that the chosen segment of a random sequence contains N points. To this set of points a new set of randomly distributed k points is added. One can formulate the following question: *to what system of equations these new set of added points should satisfy in order to keep invariant the values of the first k moments previously belonging to the initial segment*?

Mathematically, this condition can be written as

$$\Delta_k^{(p)} \equiv \frac{1}{k}\sum_{s=1}^{k} y_{N+s}^p = \frac{1}{N}\sum_{j=1}^{N} y_j^p \equiv \Delta_N^{(p)}, \tag{2}$$

where $p = 1, 2, \ldots, k$.

The solution of this problem could help in *prediction* of statistical stability of the initial segment considered. In the Mathematical Appendix condition (2) and its corresponding solution is considered in detail. Requirement (2) is equivalent to solution of the following nonlinear system of equations for the given set of stable k points:

$$\frac{y_{N+1}^p + y_{N+2}^p + \cdots + y_{N+k}^p}{k} = \Delta_N^{(p)}. \tag{3}$$

Here $p = 1, 2, \ldots, k$. With the usage of the Newton's recurrence relationships [19]:

$$pa_p + a_{p-1}S_1 + a_{p-2}S_2 + \cdots + a_0 S_p = 0, \quad a_0 = 1, \tag{4}$$

where

$$S_p \equiv \sum_{j=1}^{k} y_{N+j}^p.$$

System (3) can be reduced to the finding of the roots of the *characteristic* polynomial

$$y^k + a_1 y^{k-1} + \cdots + a_{k-1} y + a_k = 0 \tag{5}$$

for any $k \leqslant N$. For $k \leqslant 4$ this system of equations can be solved analytically [20], for $5 \leqslant k \leqslant 100$ this system can be solved numerically [21] by means of stable algorithms, for $k > 100$ the special numerical methods should be developed in order to increase the stability and accuracy of the numerical roots obtained.

It is instructive to give solutions for one and two added points.

For $k = 1$:

$$y_{N+1} = \Delta_N^{(1)}, \tag{6}$$

the added point coincides with arithmetic mean of the initial set.

For $k=2$:

$$y_{N+1}=\Delta_N^{(1)}+\sqrt{\Delta_N^{(2)}-(\Delta_N^{(1)})^2},$$
$$y_{N+2}=\Delta_N^{(1)}-\sqrt{\Delta_N^{(2)}-(\Delta_N^{(1)})^2}, \tag{7}$$

two added points are calculated as the values of the mean-squared deviation counted off its mean value. So, expressions (6) and (7) can be interpreted as *predictable* points keeping statistical stability of the initial random sequence.

For analytical analysis of the expressions corresponding to three and four points it is convenient to use relative moments counted off the first moment $\Delta_N^{(1)}$

$$M_p \equiv \frac{1}{N}\sum_{j=1}^{N} x_j^p \equiv \frac{1}{N}\sum_{j=1}^{N}(y_j-\Delta_N^{(1)})^p. \tag{8}$$

Based on the results obtained in the Mathematical Appendix it is easy to conclude that the system of Eq. (3) for this case is simplified and accepts the form

$$\frac{x_{N+1}^p+x_{N+2}^p+\cdots+x_{N+k}^p}{k}=M_p. \tag{9}$$

Taking into account the value $M_1 = 0$ it is easy to obtain the simplified expressions for the desired roots.

The case of three roots: $r=1,2,3$ ($k=3$):

$$y_{N+r}=\lambda u_r+\Delta_N^{(1)},\quad \lambda=\frac{2M_3}{3M_2},$$

$$u^3-bu-b=0,\quad b=\frac{27}{8\beta},\quad \beta=\frac{M_3^2}{M_2^3}. \tag{10}$$

The case of four roots: $r=1,2,3,4$ ($k=4$):

$$y_{N+r}=\lambda u_r+\Delta_N^{(1)},\quad \lambda=\frac{2M_3}{3M_2},$$

$$u^4-bu^2-bu-c=0,$$

$$b=\frac{9}{2\beta},\quad c=\frac{b^2}{4}(\gamma-2),$$
$$\beta=\frac{M_3^2}{M_2^3},\quad \gamma=\frac{M_4}{M_2^2}. \tag{11}$$

These expressions admit further analytical investigations because the corresponding equations for dimensionless variable u contains only one (b) for $k=3$ and two (b and c) parameters for $k=4$.

For $k>4$ the values of y_{N+k} are found numerically. So, the behavior of k points $(y_{N+1}, y_{N+2}, \ldots, y_{N+k})$ is becoming *completely predictable*, if these points satisfy the system of Eq. (3). In order to express all moments in the *same* units it is useful to introduce expression

$$G_N^{(p)}=\left[\frac{1}{N}\sum_{j=1}^{N} y_j^p\right]^{1/p} \equiv (\Delta_N^{(p)})^{1/p}. \tag{12}$$

The last expression serves the definition of the GMV function [19] depending on the current value of the moment p. Let us mark the basic properties of expression (12). For $p=-1$ it gives the *harmonic* mean

$$G_N^{(-1)}=\left[\frac{1}{N}\sum_{j=1}^{N} y_j^{-1}\right]^{-1} \equiv \frac{1}{(\Delta_N^{(-1)})}. \tag{13}$$

For $p=0$ expression (12) determines the *geometric* mean

$$\lim_{p\to 0} G_N^{(p)}=\exp\left[\frac{1}{N}\sum_{j=1}^{N}\ln(y_j)\right]=\left[\prod_{j=1}^{N} y_j\right]^{1/N} \tag{14}$$

and for $p=1$ it coincides with arithmetic mean. The limiting values are defined as

$$\lim_{p\to\infty} G_N^{(p)}=\max(y_1,y_2,\ldots,y_N)\equiv\max(y),$$

$$\lim_{p\to-\infty} G_N^{(p)}=\min(y_1,y_2,\ldots,y_N)\equiv\min(y). \tag{15}$$

The function $G_N^{(p)}$ for positive values of p is monotonic function, i.e.

$$G_N^{(t)}<G_N^{(s)} \quad \text{for } t<s \text{ and } (y_j\neq 0). \tag{16}$$

If the set of (y_j) is positive then expression (12) can be easily generalized for any real value of p. For this case we have generalization of the higher integer moments for real fractional moments. If the initial random sequence contains the negative values then one can dissect this sequence on two positive parts in accordance with relationships

$$y_j^{(p)}=\tfrac{1}{2}(y_j+|y_j|),\quad y_j^{(n)}=-\tfrac{1}{2}(y_j-|y_j|). \tag{17}$$

If it is necessary all zero points from (17) can be eliminated.

Before we considered the situation, when k added points were located on the right-hand side from initial sequence. Naturally one can define this situation as prediction of the statistical stability affected by subsequent (“future” points). But analysis of Eq. (3) shows that it admits an automatic *reduction* of the initial sequence to a set of k

statistically stable points located *inside* the random sequence considered. If we want to keep condition (3) again as the invariant condition of conservation of higher moments for two different samplings k and N, then one obtain

$$\frac{Y_1^p + Y_2^p + \cdots + Y_k^p}{k} = \Delta_N^{(p)}. \tag{18}$$

Eq. (18) is formally identical to (3) but it has one *principal* difference. It reduces the initial set of points $(y_1, y_2, \ldots, y_N)$ to a new set of the statistically stable points, located *inside* in the initial sequence. In particular, for $k = 1$ the reduced point coincides with arithmetic mean (6); for $k = 2$ the reduced points coincides with the values of standard deviations counted off the mean value (see expression (7)). So, for $k = 1, 2$ this approach recovers an 'intuitive' reduction of random sequences to a finite number of some quantitative parameters expressed in terms of the first and second moments. For $k = 3, 4$, Eqs. (10) and (11) recover other values of reduction parameters, which can be used as a natural definition of asymmetry (skewness) and sharpness (kurtosis) [18].

For $k > 4$ one can find the statistical differences between the compared random sequences if conventionally used the first and second moments are not sufficient for this purpose.

By analogy with this consideration we have a possibility in restoration of a "history" of initial sequence, i.e. the system of previous points located on the left-hand side from the initial random sequence. If one supposes that an initial sequence was generated by statistically stable set of points N, then unknown k points ($k < N$) entering into the previous ("historical") sequence preceding to the initial one can be found from the similar system of Eq. (18). In practice, it is difficult to expect the total coincidence of k additional points to condition

$$G_N^{(p)} \equiv [\Delta_N^{(p)}]^{1/p} = [\Delta_k^{(p)}]^{1/p} \equiv G_k^{(p)}, \tag{19}$$

especially for the cases when k is relatively large ($k > 100$). So for this case the special methods for comparison of two GMV-functions with arbitrary values of N and k are necessary.

3. The approximate expression for the GMV-function. Fractional and complex moments

For approximate solution of Eq. (19) when the value of $k > 100$ and for more complete comparison of two different GMV-functions $G_{N_1}^{(p)}$ and $G_{N_2}^{(p)}$ (when $N_{1,2} \gg 1$ are different) it is useful to find an analytical expression (the fitting function), which approximates this function with acceptable accuracy. In order to find this expression let us analyze the following function:

$$\begin{aligned} F_N^{(p)} &= \frac{1}{N}\sum_{j=1}^{N}\left(\frac{y_j}{y_{\max}}\right)^p \equiv \frac{1}{N}\sum_{j=1}^{N}(\theta_j)^p \\ &= \frac{1}{N}\sum_{j=1}^{N}\exp[p\ln(\theta_j)] \\ &\cong A_0 + \sum_{n=1}^{s} A_n \exp(-\lambda_n p). \end{aligned} \tag{20}$$

The parameter $\theta_j = (y_j/y_{\max}) < 1$ because of normalization of a random sequence on its maximal value. So, the last expression can be approximated by a finite number of exponential functions. The coefficient A_0 shows the contribution of the values y_j, which are close to its maximal value $(y_j/y_{\max}) \leqslant 1$; other coefficients A_n correspond to the contribution of random amplitudes y_j with the ratio $(y_j/y_{\max}) \ll 1$. Based on expression (20) it is easy to write the approximate expression for the GMV-function (12)

$$G_N^{(p)} = g\left[1 + \sum_{n=1}^{s} a_n \exp(-\lambda_n p)\right]^{1/p} \equiv y_{\max}(y_s(p))^{1/p}, \tag{21}$$

where $a_n = A_n/A_0$. Expression (21) allows in expression of any random sequence in terms of the reduced *quantitative* parameters, including also the area under the curve (AUC) $G_N^{(p)}$ and the value $y_{\max}$. For calculation of the required fitting parameters $(a_n,\ \lambda_n)$ for $n = 1, 2, \ldots, s$ one can use the ECs method, which can present function (20) in the form of straight lines and find true values of the fitting parameters corresponding to the *global* minimum. Not going into details of this effective method [22–25] one can write only the *basic linear relationship* (BLR) for internal function $y_s(p)$, containing three exponents ($s = 3$). Numerous test calculations show that in practical realization of expression (21) the number of different exponents usually does not exceed three or four. For example, the BLR for three-exponential function can be presented in the following form:

$$Y(p) = \sum_{q=1}^{6} C_q X_q(p). \tag{22}$$

Here

$$Y(p) = F_N^{(p)} - \langle \ldots \rangle,$$

$$X_1(p) = \int_{p_0=1}^{p} F_N^{(u)}\,\mathrm{d}u - \langle \ldots \rangle, \quad C_1 = \lambda_1 + \lambda_2 + \lambda_3,$$

$$X_2(p) = \int_{p_0=1}^{p} (p-u) F_N^{(u)}\,\mathrm{d}u - \langle \ldots \rangle,$$

$$C_2 = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3), \tag{23a}$$

$$X_3(p) = \frac{1}{2}\int_{p_0=1}^{p} (p-u)^2 F_N^{(u)}\,\mathrm{d}u - \langle \ldots \rangle,$$

$$C_3 = \lambda_1\lambda_2\lambda_3,$$

$$X_4(p) = p^3 - \langle \ldots \rangle,$$

$$X_5(p) = p^2 - \langle \ldots \rangle,$$

$$X_6(p) = p - \langle \ldots \rangle. \tag{23b}$$

The constants C_4, C_5, C_6 contain unknown values of derivatives at initial point $y_3^{(r)}(p_0)(r = 3,2,1)$ and are not essential for calculation the desired roots $\{\lambda_r\}$ $(r = 3,2,1)$ by the linear least square method (LLSM). The symbol $\langle \ldots \rangle$ in (23) defines the corresponding arithmetic mean, which should be subtracted from the corresponding functions $Y(p)$, X_q (p) in order to provide the basic requirement $\langle \varepsilon \rangle = 0$ of the LLSM [26]. Unknown constants A_n $(n = 0,1,\ldots,s)$ are also found by the LLSM from relationship (20). We want to stress the following important advantages that could provide a wide application of the GMV-function for analysis of different random sequences:

(a) The approximate analytical expression (21) provides a 'universal' quantitative reduction of *any* random sequence to a set of parameters (A_n, λ_n), including also the AUC and $y_{\max}$ values.

(b) These fitting parameters allow in separation of the values (amplitudes) of a random sequence y_j onto the optimal statistical groups (clusters) n with parameters (A_n, λ_n) that correspond to the reduced description of the random sequence considered.

(c) In cases, when the volume of the sampling is large $(N > 100)$ this reduced presentation can be more informative with respect to an external factor then the numerical evaluation of the k roots from the system of Eq. (18). For this case one can fit each function $G_N^{(p)}$ and $G_k^{(p)}$ entering into expression (19) separately and then compare their proximity in terms of the fitting parameters (A_n, λ_n). The comparison of higher moments forming (in general) two different samplings $(k \neq N)$ is more precise and adequate. At $k = 1,2$ we obtain the conventional reduction expressed in terms of mean value and standard deviation. The "universal" description expressed in terms of the initial integer moments as Δ_1, Δ_2 can be *unsatisfactory* in most cases.

(d) Numerical calculations show that presentation $G_k^{(p)}$ with respect $G_N^{(p)}$ is very informative in comparison of one random sequence with another one. If two random sequences are statistically close to each other then this plot should give a straight line with the slope equaled 1 and intercept close to zero. Possible deviations from these values give a possibility to detect something "strange" (a presence of a signal or modification of the previous statistical behavior) in two random samplings compared. Examples of this convenient presentation are considered in Section 6. This simple analysis helps to detect self-similar (fractal) components, if there are located in the random sequence considered.

Let us suppose that there is a segment in two compared random sequences where the amplitudes approximately satisfy to condition

$$\frac{y_j}{u_m} \cong \lambda, \quad (\lambda \neq 1) \text{ and } j, m = l, l+1, \ldots, M. \tag{24a}$$

For this segment it is easy to notice that two GMV-functions are proportional to each other, i.e.:

$$G(y)_{M-l+1}^{(p)} = \lambda G(u)_{M-l+1}^{(p)}. \tag{24b}$$

Condition (24b) allows in detection of segments in two random sequences having self-similar (fractal) behavior. Here it is interesting to note that the self-similar behavior is identified in many real random sequences, if they are ordered into the so-called rank plot [15]. See below expression (31) that can be used as a "universal" fitting function for description of the recognized fractal behavior of different random sequences.

Eq. (24b) allows defining the statistical proximity of two different sequences compared. If two sequences having, respectively, $N_{1,2}$ points in the space of moments for some segment satisfy to condition ($\lambda \cong 1$ and b are arbitrary constants)

$$G_{N_1}^{(p)} = \lambda G_{N_2}^{(p)} + b, \tag{25}$$

then these sequences can be considered as statistically close each other.

When λ accepts an arbitrary value and $b \cong 0$ then the compared sequences can be considered as self-similar to each other.

When λ and b start to depend on parameter p then two random sequences compared can be considered as statistically different.

(e) The GMV-function approach allows in making a generalization of definition (12) for complex and any values of p. This two-dimensional generalization can be useful for more fine differentiations of one-dimensional random sequences. When the parameter p is described by a complex number $p=u+\mathrm{i}v$, then the GMV-function (12) is transformed into a surface

$$\begin{aligned} G_N^{(p)} &= \operatorname{Re} G_N^{(p)} + \mathrm{i}\operatorname{Im} G_N^{(p)} = \left[\frac{1}{N}\sum_{j=1}^{N} y_j^{u+\mathrm{i}v}\right]^{1/(u+\mathrm{i}v)} \\ &= [M_c(u,v)+\mathrm{i}M_s(u,v)]^{1/(u+\mathrm{i}v)} \\ &= \exp\left[\frac{1}{u+\mathrm{i}v}\ln\left(\frac{1}{N}\sum_{j=1}^{N}\exp[(u+\mathrm{i}v)\ln y_j]\right)\right]. \end{aligned} \tag{26}$$

After some algebraic transformations the real and imaginary parts of the GMV-function are defined by the following expressions:

$$M_c(u,v) = \frac{1}{N}\sum_{j=1}^{N} y_j^u \cos(v\ln(y_j)),$$

$$M_s(u,v) = \frac{1}{N}\sum_{j=1}^{N} y_j^u \sin(v\ln(y_j)), \tag{27}$$

$$\operatorname{Re} G_N^{(p)}(u,v) = \exp[F(u,v)]\cos[\Phi(u,v)],$$

$$\operatorname{Im} G_N^{(p)}(u,v) = \exp[F(u,v)]\sin[\Phi(u,v)],$$

$$F(u,v) = \frac{u\,\ln\left[\sqrt{G_1^2(u,v)+G_2^2(u,v)}\right]}{u^2+v^2} + \frac{v\cdot\varphi(u,v)}{u^2+v^2},$$

$$\Phi(u,v) = -\frac{v\,\ln\left[\sqrt{G_1^2(u,v)+G_2^2(u,v)}\right]}{u^2+v^2} + \frac{u\cdot\varphi(u,v)}{u^2+v^2}. \tag{28}$$

In turn, the functions $G_{1,2}(u,v)$ and their argument $\varphi(u,v)$ are determined by expressions:

$$G_1(u,v) = \ln\left[\frac{1}{N}\sum_{j=1}^{N}(y_j)^u\cos(v\,\ln(y_j))\right] = \ln(M_c(u,v)),$$

$$G_2(u,v) = \ln\left[\frac{1}{N}\sum_{j=1}^{N}(y_j)^u\sin(v\ln(y_j))\right] = \ln(M_s(u,v)),$$

$$\varphi(u,v) = a\tan\left[\frac{G_2(u,v)}{G_1(u,v)}\right]. \tag{29}$$

The last expressions need a special investigation in order to identify the cases when they could be the most suitable and informative.

(f) Definition (12) admits the further generalization for two-dimensional random sequences

$$G(p,q) = \left[\frac{1}{N_1N_2}\sum_{j_1=1}^{N_1}\sum_{j_2=1}^{N_2}(Y_{j_1,j_2})^{p+q}\right]^{1/(p+q)}. \tag{30a}$$

This surface can be analyzed similar to expression (12) if one replaces external parameter p for $p+q$. For random sequence having m different components by analogy with expression (30a) one can write the following expression:

$$G\left(\sum_{i=1}^{m}p_i\right) = \left[\frac{1}{N_1N_2\ldots N_m}\sum_{j_1=1}^{N_1}\sum_{j_2=1}^{N_2}\cdots\sum_{j_m=1}^{N_m}(Y_{j_1,j_2,\ldots,j_m})^{\sum_{i=1}^{m}p_i}\right]^{\left(\sum_{i=1}^{m}p_i\right)^{-1}}. \tag{30b}$$

In paper [15] we identified one 'universal' function that describes wide class of real random sequences. It describes the envelope of the ordered amplitudes determined as the rank plot and obtained for *detrended* random sequences having relatively large sampling volumes ($N\geqslant 1000$). In many real cases analyzed this envelope (or rank plot) is described by the function

$$y(t) = A_1t^{\nu_1} + A_2t^{\nu_2}. \tag{31}$$

The calculated fitting parameters of this function $(A_1(f), A_2(f), \nu_1(f)\nu_2(f))$ with respect to an external factor (f) can be analyzed in terms of integer moments (Eq. (5)) or with the help of the GMV-function (12). The SFM based on usage of higher (fractional) moments can be used for construction of *calibration curves*, which can show the variations of distinct *quantitative* parameters characterizing *any* random sequence (containing a trend) to respect of the desired *external* factor (concentration of an additive, value of the external field, temperature, pressure, pH-factor, etc.). We want to stress here again that these new approaches based on the calculation of higher moments do not use *any* model assumptions, based on traditional Gaussian statistics and its modifications. It is necessary to mark here that possible application of the SFM method depends on the ratio between identified number of additional points (k) and the volume of the sampling (N). If number of distinctive points is limited ($k/N \ll 1$) then the reduction based on calculation of the lower discrete moments in accordance with (6)–(11) is preferable. If it is necessary to compare large samplings ($k/N \leqslant 1$, $N \gg 1$) when many initial moments are close to each other then approach based on approximate analytical function (21) with subsequent calculation of the fitting parameters by the ECs method is more preferable. Sometimes for construction of the required calibration curves it is sufficient to use presentation (25) and then to take the AUC as a quantitative parameter calculated for the relative difference $(G_{N_2}^{(p)} - G_{N_1}^{(p)})/G_{N_1}^{(p)}$ Examples of this simplified approach in detection of the influence of an external factor are considered in Section 6.

4. Relationship of the fractional moments with the nonextensive Tsallis entropy

Let us consider again a segment of a random sequence as a system of $j = 1, 2, \ldots, N$ points located in the band $y_{\min} \leqslant \{y_j\} \leqslant y_{\max}$. For these N randomly located points, the value of the *absolute* moment of the pth order one can define by more general expression

$$\Delta_w^{(p)} = \sum_{j=1}^{N} w_j y_j^p, \tag{32}$$

where values w_j determine a discrete set of a priori probabilities satisfying to normalization condition:

$$\sum_{j=1}^{N} w_j = 1. \tag{33}$$

The GMV-function determined earlier by expression (12) accepts the form

$$G_w^{(p)} = \left[\sum_{j=1}^{N} w_j y_j^p\right]^{1/p} \equiv (\Delta_w^{(p)})^{1/p}. \tag{34}$$

Now it becomes obvious *how* to obtain the reduced expression for the nonextensive Tsallis entropy [27–30]

$$S_q = -k_B \frac{1 - \sum_{j=1}^{m} w_j^q}{1 - q}. \tag{35}$$

Here w_j is a set of probabilities characterizing a part of some random system considered, the parameter q is interpreted as the nonextensive parameter, k_{B} is the Boltzmann constant.

In order to receive an elegant expression, which can be applicable for description of a random sampling of *any* nature it is necessary to specify an expression for probability related with some random sampling. Any random sampling can have positive and negative values of random amplitudes located as before in the band $y_{\min} \leqslant \{y_j\} \leqslant y_{\max}$. In the beginning we define two random *positive* sequences relatively the modulus value

$$y_j^+ = \tfrac{1}{2}(y_j + |y_j|), \quad y_j^- = -\tfrac{1}{2}(y_j - |y_j|). \tag{36}$$

Let us suppose that number of points for these two random sequences are defined correspondingly by two values $N_\pm$ satisfying to condition: $N_+ + N_- = N$. For these N randomly located points, a probability of appearing of random amplitude can be naturally defined by expressions:

$$w_j^+ = \frac{y_j^{(+)}}{S^{(+)}}, \quad w_j^- = \frac{y_j^-}{S^{(-)}}, \tag{37a}$$

where the sums $S^{(\pm)}$:

$$S^{(\pm)} = \sum_{j=1}^{N_\pm} y_j^{(\pm)} \tag{37b}$$

are chosen in such way in order to provide the normalization of each probability defined by (37a) to the unit value.

Let us consider now the expression figuring in (35). Taking into account the definition of a priori probabilities (37a) it can be rewritten in the form

$$F_q^{(s)} = \sum_{j=1}^{m} (w_j^{(s)})^q = \sum_{j=1}^{m} \left(\frac{y_j^{(s)}}{S^{(s)}}\right)^q. \tag{38}$$

Here index s defines the positive or negative sequence ($s = \pm$), respectively. Comparing the sum in the last expression with the definition of the generalized higher moments (32) with probabilities from definition (37a) one can establish the desired relationship between the nonextensive factor q and the value of the fractional moment of the order $q - 1$:

$$\begin{aligned} F_q^{(s)} &= \frac{1}{S_\pm^{q-1}} \sum_{j=1}^{N_\pm} \left(\frac{y_j^{(s)}}{S_\pm}\right) (y_j^{(s)})^{q-1} \\ &= \frac{1}{(S^{(s)})^{q-1}} \sum_{j=1}^{N_\pm} w_j^{(s)} (y_j^{(s)})^{q-1} \\ &= \frac{1}{(S^{(s)})^{q-1}} \Delta_{w^{(s)}}^{(q-1)} \equiv \Delta_{w^{(s)}}^{(q-1)}. \end{aligned} \tag{39}$$

The last expression in the right-hand side is correct for the *normalized* random sampling $y_j^{(s)} \Rightarrow y_j^{(s)}/S^{(s)}$. Using the same manipulations which lead to expression (20) we obtain approximately

$$F_q^{(s)} = \sum_{j=1}^{m} (w_j^{(s)})^q = \sum_{j=1}^{m} w_j \exp\left[-(q-1)\ln\left(\frac{1}{w_j^{(s)}}\right)\right]$$
$$= \sum_{j=1}^{m} w_j \exp\left[-(q-1)\ln\left(\frac{S_\pm}{y_j^{(s)}}\right)\right]$$
$$\cong A_0^{(s)} + \sum_{n=1}^{k} A_n^{(s)} \exp\left[-(q-1)\lambda_n^{(s)}\right]. \quad (40)$$

One can give the following interpretation of expression (40). Usually for characterizing of some random sampling, traditionally the mean value and the standard deviation are used. But these characteristics are *approximate* and correct only for the first *two* moments (7) with two added points. Here for characterizing of a random sampling we are using the *clusterization* hypothesis. In other words, we combine together the random amplitudes having approximately similar values of probabilities in the space of *fractional moments*. If a probability for some group of amplitudes close to one $w_j^{(s)} \cong 1$ then their contribution is reduced to the value $A_0^{(s)}$. In the opposite case, when for some group of random amplitudes the values of probabilities $w_j^{(s)} \cong 0$ then their contribution in expression (40) becomes negligible, i.e. $A_k^{(s)} \cong 0$. Other intermediate groups of statistically close clusters will be located between these limiting probabilities with amplitudes $A_n^{(s)}$ $(n = 1, 2, \ldots, k-1)$, determinedby the ECs method.

In other words, *any* random sequence in the space of the parameter q can be transformed approximately into analytical expression (40) and the nonextensive parameter q receives another interpretation as a value of the $(q-1)$ higher (fractional in general) moment. If all values $w_j^{(s)}$ in the last expression are strictly positive than the value of q is defined in the interval $(-\infty < q-1 < \infty)$. If for some units $w_j^{(s)} = 0$ (binary units, for example) then the values of q should be located in the interval $(1 < q < \infty)$. Taking into account relationship (40) expression (35) for the generalized nonextensive entropy accepts the following analytical form:

$$S_q^{(\pm)} = -k_B \frac{1 - \sum_{j=1}^{m} w_j^q}{1-q} = k_B \frac{1 - \Delta_{w_\pm}^{(q-1)}}{q-1}$$
$$\cong k_B \frac{1 - \left[A_0^{(\pm)} + \sum_{n=1}^{k} A_n^{(\pm)} \exp[-(q-1)\lambda_n^{(\pm)}]\right]}{q-1}. \quad (41)$$

The random sequence considered can be characterized now by a specific set of parameters $A_0^{(\pm)}, A_n^{(\pm)}, \lambda_n^{(\pm)}$. These parameters, as it follows from (41) are the *same* for the Tsallis entropy and the corresponding value of the fractional moment. The number of parameters n can be identified by the ECs method. So, the Tsallis entropy receives the extended interpretation and can be used for characterization of *any* random sampling. The parameter q is located in the interval $1 < q < \infty$. The limiting value of entropy (41) at $q = 1$ is determined by expression

$$S_1^{(\pm)} = -k_B \sum_{j=1}^{N} w_j \ln(w_j). \quad (42)$$

5. Statistical protection of valuable documents

As it is well known the reliable protection of banknotes, plastic cards, pictures, ancient manuscripts and other valuable documents constitutes a serious problem in the modern world. Not going into details of this problem we consider schematically a possible application of the SFM method for *statistical* protection of the documents, which are needed to be reliably protected. Let us suppose, for example, that Fig. 1 represents itself a surface of a plastic card and points (marked by crosses) are a set of special marks created by a device on the surface of this plastic card. The number of marked points (= 40) and location places should be kept as a confidential information. There is a potential "swindler", who wants to create a false card and thereby to mimic a real card. We admit that *number* of secret points is known for him. In this case, a special device 'tuned' for reading of the number of 'true' points together with their location recognizes easily the false card in the space of polynomial coefficients and the corresponding roots of the characteristic polynomial (5). They are shown in Figs. 2 and 3, correspondingly. It is interesting to note that the minimal order of characteristic polynomial, which detects the largest deviations

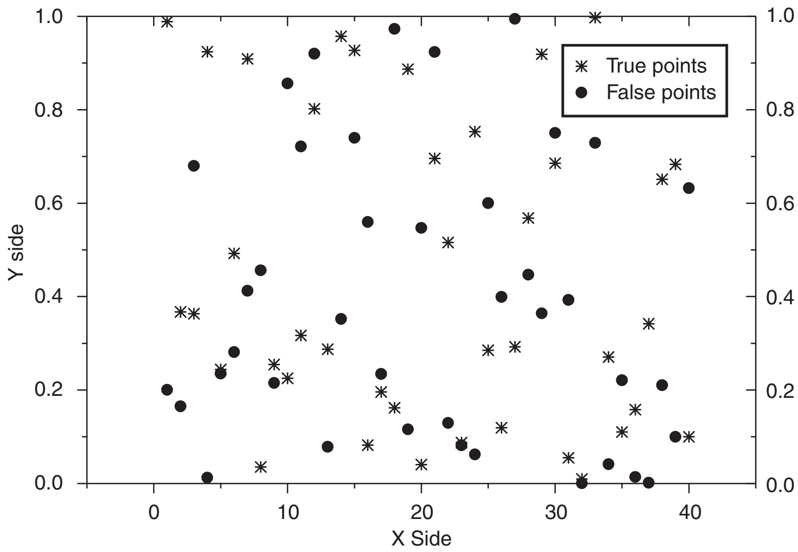

Fig. 1. The surface of a plastic card, containing 40 real points (marked by stars) and 40 imitated (false) points marked by black circles. Is it possible certainly to differentiate one set of 40 (real) points from the same number of imitated points?

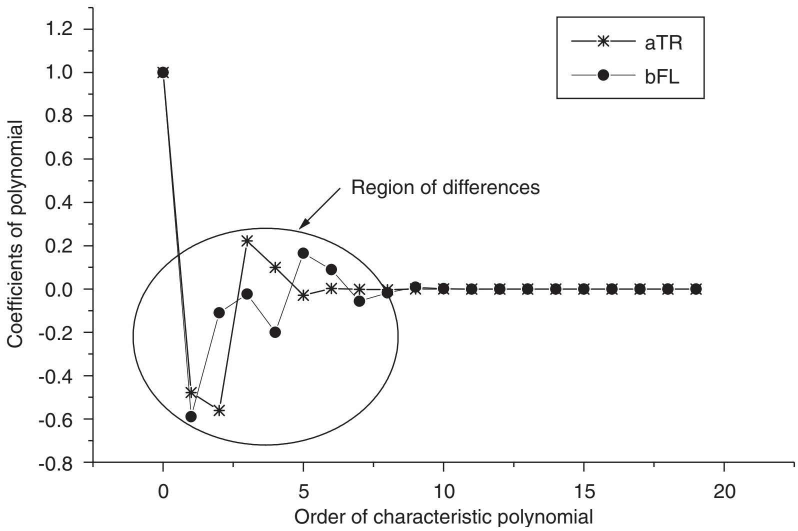

Fig. 2. This figure demonstrates the difference between two characteristic polynomials corresponding to different set of coefficients. The order of polynomial in this case equals 19.

depends on the statistical properties of the initial random sequence and this confidential (secret) parameter can be used also for additional protection. In this case it equals 19. Let us imagine the situation when a potential swindler is able to know the *location* of all secret points except *one*. This possible situation is presented in Fig. 4. Is it possible to detect small deviations in this case? Answer is also positive as it follows from consideration of Figs. 5 and 6. Again, there is an optimal order of characteristic polynomial, which depends on the statistical properties of the initial sequence. In this model case it equals 20. As model calculations show the appearance of the complex-conjugated roots makes the detection procedure *more sensitive* in comparison with polynomials, having only real roots. If this statistical sequence is not a constant and changed in time and the detector 'tuned' for reading of a true plastic card 'knows' the secret order of the random sequences activated for recognition then this statistical protection (from our point of view) makes impossible to create a false

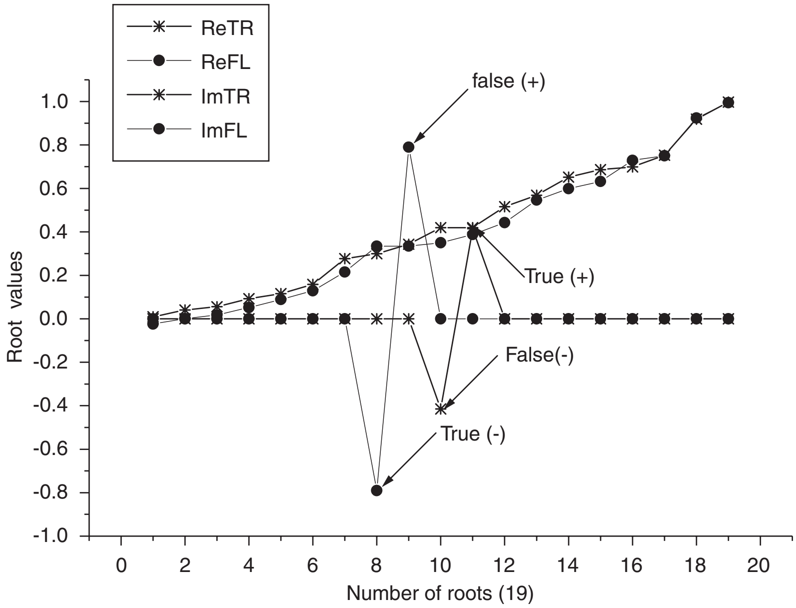

Fig. 3. The differentiation between two sets of random points (Fig. 1) shown on 19 different roots of characteristic polynomial. For differentiation one can use the complex roots of the polynomials, which have distinctive values. They are shown by arrows. Probably, complex roots appeared in the result of numerical calculation of the corresponding roots, but they can be accepted because the numerical program for their finding was the same.

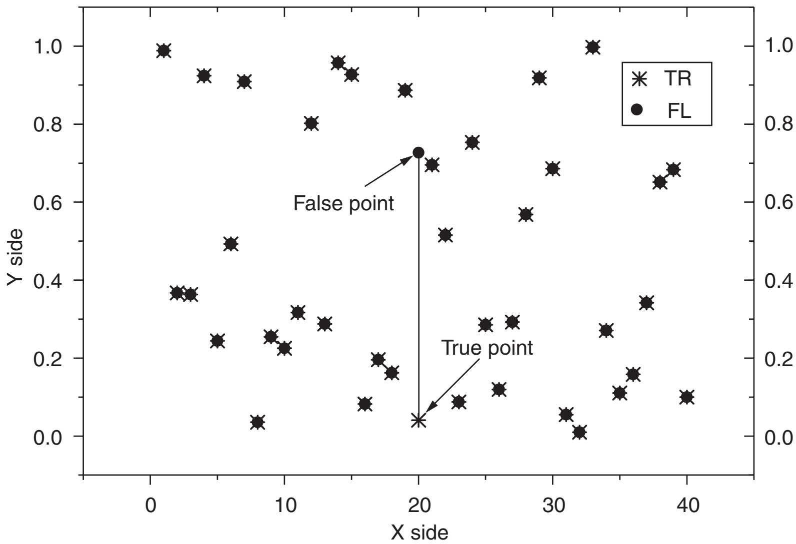

Fig. 4. An attempt of a potential swindler in imitation of all real points except one shown by arrows. Is it possible to differentiate the real set of points from the false set of points, when the difference 'has a price' of one false (strange) point?

plastic card or banknote. This simple idea in reality needs a modification related to the fact that true plastic card is *not* ideal but nevertheless it can be modified with real requirements and used successfully alongside with other measures for protection of valuable documents.

It is interesting to compare the first approach based on the calculation of the desired roots of the characteristic polynomial (5) with GMV-function approach based on the calculation of the fitting parameters of function (28). We realized this procedure for random sequences imitating the real

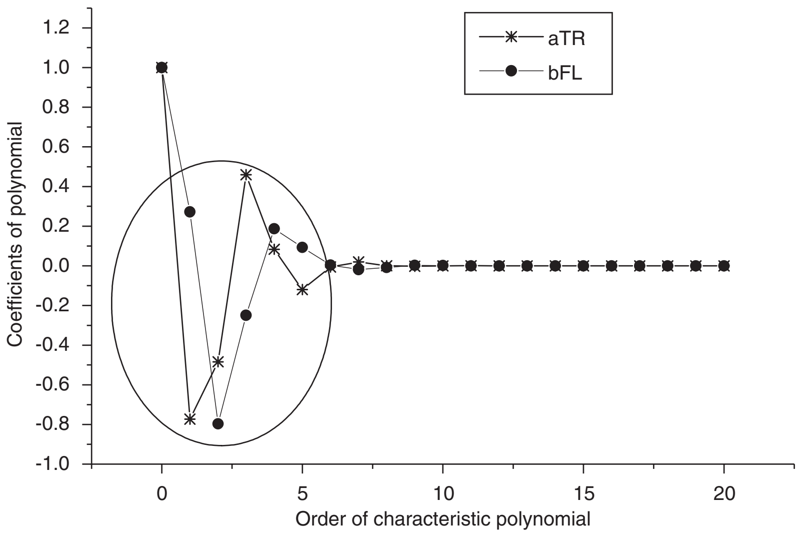

Fig. 5. This figure demonstrates the difference between two set of random points having the difference in one point. It is interesting to note the coefficients of characteristic polynomial of the 20th order helps to differentiate this difference and express the initial difference in terms of seven different points! The order of the characteristic polynomial is found empirically.

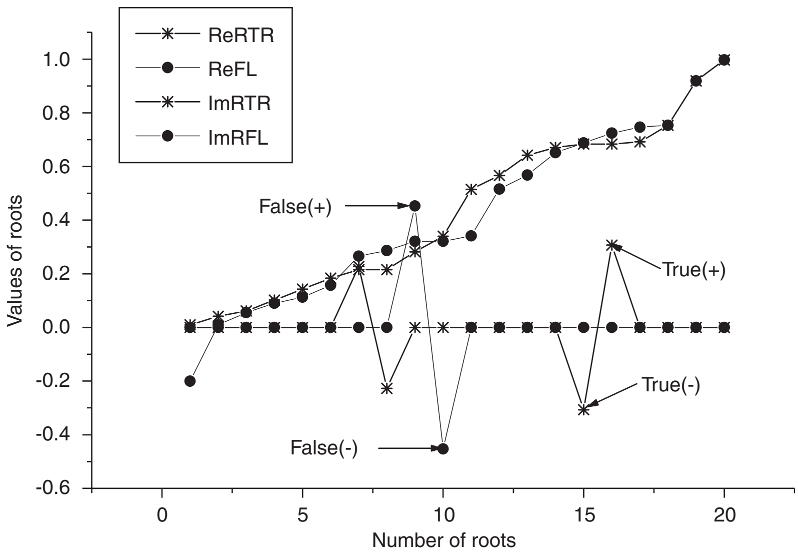

Fig. 6. The picture of differentiation expressed in the terms of calculated roots of the characteristic polynomial. Again, the imaginary parts of the corresponding roots are more sensitive to this possible deviation and can be expressed by number of points exceeding one!

and false plastic cards, presented by Fig. 4. We deliberately chose the second situation when the difference between these two random sequences is expressed only by one "strange" point. The results of the fitting of the corresponding function $G_N^{(p)}$ by a linear combination containing three exponents are shown in Fig. 7.

This important result shows that sequences (having initially only one deviated point!) can be differentiated and, hence, the GMV-function can be used as *very sensitive* detector for comparison of statistically close random sequences of *any* nature.

6. Model experiments

6.1. Detection of a predominant factor (superweak signal)

The problem of detection of a superweak signal, when the amplitude of a possible signal (S) is less

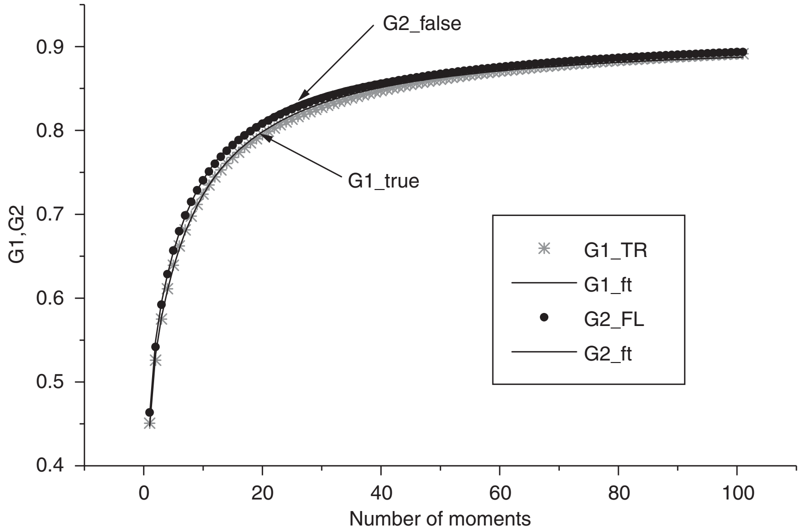

Fig. 7. The generalized averaged values and their fitting functions calculated for random sequences depicted in Fig. 4. So, there is a unique possibility to differentiate very close random sequences, having only one different point. The fitting function contains three exponents and the fitting parameters are the following (in parenthesis we are giving the corresponding fitting parameters for the false function): [$A_1 = 0.5127$ (0.4886), $A_2 = 0.1740$ (0.1765), $A_3 = 0.0631$ (0.0876), $\lambda_1 = -0.6983$ (-0.6959), $\lambda_2 = -0.2546$ (-0.2548), $\lambda_3 = -0.0921$ (-0.0920), Std dev $= 0.003$ (0.0022)]. The last parameter shows the absolute value of difference expressed in the terms of unbiased mean-squared deviation. All these parameters can serve as quantitative measure for comparison of two close sequences depicted in Fig. 4. In addition, we are giving the values of the Pearson's correlation coefficients (PCC) calculated for the generalized average function and its fitting replica. PCC $= 0.9997$ (0.9998).

than the amplitude of a noise (N) ($S/N < 1$) plays an important role for the whole signal processing. Not going to discuss this problem in detail here we want to show only how to use the GMV-function for detection of distortions evoked by an influence of a small predominant factor. Let us consider a model noise having uniform distribution with random amplitudes located in the interval $nin_j \in [0, 1]$. Number of points $j = 1, 2, \ldots, N$ ($N = 400$), the variable x_j is located in the interval $[0, 10.0]$. To this initial nose we add a signal with a small amplitude $a \in [0.01 - 0.1]$ expressed by the function $\sin^2(\Omega \cdot x_j)$ ($\Omega = 1$). The mathematical scheme of this numerical experiment can be expressed as

$$ns_j = nin_j(1 + a\sin^2(\Omega x_j)). \tag{43}$$

Because the amplitude of the added signal is very small it is not visually detected as it can be seen from Fig. 8a. The problem can be formulated as follows. Is it possible to detect an influence of a small predominant factor which is considered as a signal $-\sin^2(\Omega x_j)$ and then to express it in terms of some increasing parameter with respect to the value of amplitude a? In order to find the desired influence (which can be defined as calibration curve) we realized the following procedure.

1. In accordance with definition (12) the GMV-functions corresponding to initial noise and noise affected by a signal are calculated. The most convenient presentation for possible visual detection of a predominant factor is dependence of $G_{s,N}^{(p)}$ (containing a signal) against $G_{0,N}^{(p)}$ (GMV-function for a segment without signal) calculated in the space of moments. Fig. 8b shows a monotonic tendency of deviation of the values $G_{s,N}^{(p)}$ from the function $G_{0,N}^{(p)}$ corresponding to a pure noise.

2. In order to calculate the desired calibration curve we calculated in the space of fractional moments the AUC, formed as a relative difference

$$D(p) = \frac{G_{s,N}^{(p)} - G_{0,N}^{(p)}}{G_{0,N}^{(p)}}. \tag{44}$$

The value of the AUC calculated for the function $D(p)$ is shown in Fig. 8(c). This curve reflecting a monotonic behavior of a predominant factor can be used as a calibration curve.

In practice is it difficult to expect that initial noise and noise distorted by a presence of a small signal will be the same. So in the beginning it is necessary to compare two initial random sequences and then "suspicious" sequence distorted by some external

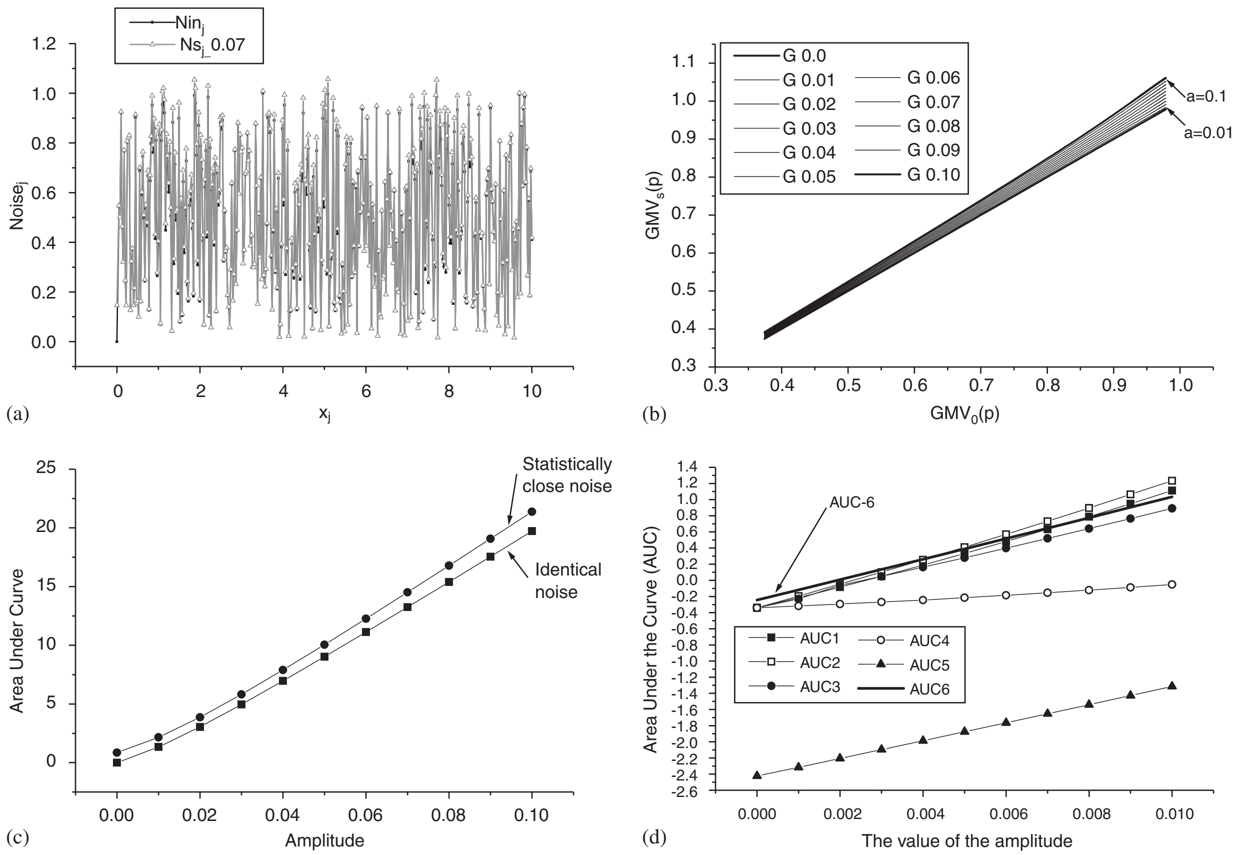

Fig. 8. (a) This figure shows two random sequences. One (marked by grey color) contains a signal with amplitude $a = 0.07$. Visually a presence of the signal cannot be detected. Special methods of detection of the presence of the signal are necessary. The presence of the signal can be detected with the help of GMV-function, calculated in the space of higher (fractional) moments. (b) This figure shows the first step of detection of a presence of a small signal. Possible distortions evoked by a presence of a signal can be detected on the plot $G_{Ns}^{(p)}$ against $G_{0N}^{(p)}$. The basic plot $G_{0N}^{(p)}/G_{0N}^{(p)}$ forms a straight line with the slope equaled one and intercept equaled zero. If two different random samplings can be presented in the form of straight lines, which are close to each other then they can be defined as statistically close to each other. Possible small signal causes the distortions of these parameters. If the external factor (amplitude of the signal applied) causes monotonic distortions then the desired calibration curve can be calculated. In this case monotonic distortions are located above the basic plot and are deviated monotonically with increasing of the value of the amplitude. (c) This figure shows the final result of the detection of a small signal. The sensitive factor as the area under the curve (AUC) calculated for the difference $(G_{sN}^{(p)} - G_{0N}^{(p)})/G_{0N}^{(p)}$ helps to construct the desired calibration curve. The low curve corresponds to ideal situation when the initial noise in the absence of signal is the same. The upper curve corresponds more realistic case, when the initial noises are different but corresponds the same (uniform in this case) distribution. (d) Different calibration curves obtained from expression (45). The curve AUC-1 corresponds at $(nin_2(x_j) \equiv \exp[(x_j - \langle x \rangle)^2/\sigma] \equiv \Phi(x_j) \equiv 1)$; the curve AUC-2 $(nin_2(x_j) \equiv \exp[(x_j - \langle x \rangle)^2/\sigma] \equiv 1,\ \Phi(x_j)$ corresponds to the uniform distribution): the curve AUC-3 $(nin_2(x_j) \equiv \Phi(x_j) \equiv 1,\ \sigma = \langle x \rangle^2)$, the curve AUC-4 $(nin_2(x_j) \equiv \Phi(x_j) \equiv 1,\ \sigma = 1)$; the curve AUC-5 $(\exp[(x_j - \langle x \rangle)^2/\sigma] \equiv \Phi(x_j) \equiv 1$, $nin_2(x_j)$ corresponds to the normal distribution); the last curve AUC-6 stressed by bold line corresponds to the same set of parameters as the curve AUC-5, but the number of points in ten times more, $N = 5000$. For all calculated curves $\Omega = 1$, $N = 500$ expect the last curve.

factor. As an initial noise we chose the noise obeying to uniform distribution. The calibration curve calculated for this case is shown in Fig. 8c, which is shifted up in comparison with the ideal case, when the initial noise of two initial random sequences is identical to each other.

In the same manner we considered the cases when the amplitude is in 10 times less, located in the interval $(0.001, 0.002, \ldots, 0.01)$ and affected by another external factor. All these possible situations mathematically can be expressed as

$$ns(x_j) = nin_1(x_j) + a\sin^2(\Omega x_j + \Phi(x_j)) \times \exp\left[-\frac{(x_j - \langle x \rangle)^2}{\sigma}\right] nin_2(x_j). \tag{45}$$

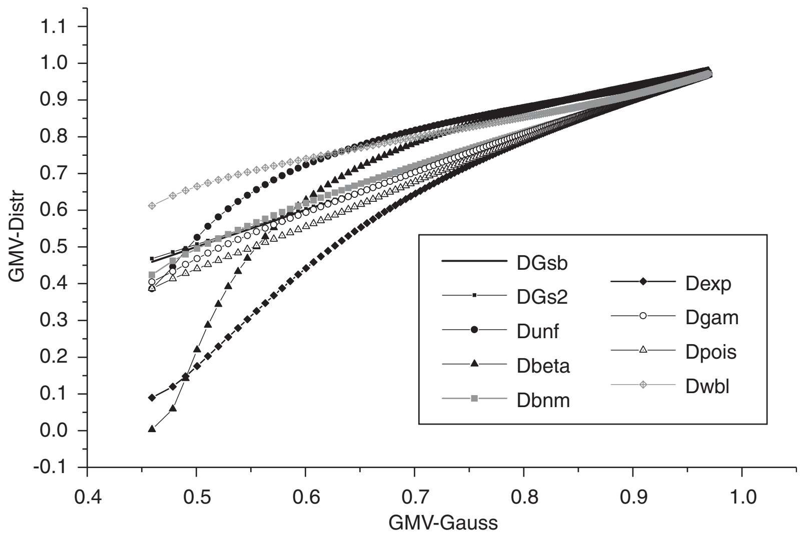

Fig. 9. This figure illustrates the observation that different random sequences having different distributions can be visually differentiated from each other if one presents them in the space of fractional moments. Here all random sequences corresponding to different distributions are normalized to the interval [0, 1]. The fractional moments are calculated in the interval [0.1–200] with step $h_p = 0.4$. The notations used in the frame to this figure are the following: (“DGsb”, “DGs2”—corresponds to the normal distribution with average value 0.5 and dispersion equaled one; “Dunf” corresponds to the uniform distribution generated in the interval [0.01, 1]; “Dbeta” correspond to beta-distribution with parameters $a = 0.1$ and $b = 0.5$; “Dbnm” corresponds to the binomial distribution with $m = 50$ and $p = 0.5$. One can notice that this distribution is very close to the normal one. “Dexp” corresponds to the exponential distribution with parameter $e = 50$; “Dgam” corresponds to the gamma-distribution with parameter $s = 50$; “Dpois” corresponds to the Poisson’s distribution with $s = 25$ and finally the “Dwbl” corresponds to the Weibull distribution with parameter $w = 15$. One can conclude that each distribution has own statistical “drawing”.

Here $nin_{1,2}(x_j)$ and $\Phi(x_j)$ are different random functions, the exponential function characterizes the degree of location of the small signal. The calculations realized with function (45) shows that in spite of the fact that amplitude is very small $a \in [10^{-3}, 10^{-2}]$ it is possible to calculate the calibration curve at different influence of other factors as Ω, σ and at arbitrary choice of the functions $nin_{1,2}(x_j)$ and $\Phi(x_j)$. The corresponding calibration curves are shown in Fig. 8d with necessary comments given by the figure caption. These calculations show that the localization of the signal (depending on the value of σ) and monotonic behavior of the hidden factor (expressed in terms of the value a) are important conditions in calculation of the desired calculation curve. This observation should be taken into account in treatment of real data, where it is necessary “to catch” an influence of an external factor which is small and completely hidden inside the random sequence analyzed.

6.2. The detection of statistical proximity

The statistics of the fractional moments expressed in terms of the GMV-function gives a unique possibility to compare different distributions and determine their statistical proximity. If for two random sequences part of the moments ($p = 0, 1, \ldots, k$) are very close to each other then we determine them as statistically close to each other (see definition (25)). For example, Fig. 8b demonstrates statistically close random sequences. If the distributions of random points are different, one can expect that their “statistical drawings” expressed in terms of the GMV-functions should be different. Fig. 9 illustrates this observation expressed as a plot $G^{(p)}_{\mathrm{ND}}$ (calculated for normal distribution) against the GMV-functions, corresponding to different distributions. As one can see from this figure the behavior of initial moments belonging to different distributions is really different. This preliminary analysis can be used for detection of a small predominant factor in cases, when the factor can modify the initial distribution.

7. Real data treatment

Foetal Calf Serum (Sigma Aldrich) with glucose concentrations of 100, 10, 1, 0.1, 0.01, 0.001 and 0.0001 mM were prepared. The dielectric properties of the serum samples were measured at 25 °C between the frequencies, $2.10^8-2.10^{10}$ Hz using a Hewlett Packard 85070M dielectric probe and 8720D Network Analyzer. Twenty repeat measurements were carried out for *each* glucose concentration. Typical complex permittivity curves Re(ε) and Im(ε) for pure Serum are shown in Fig. 10a. The problem can be formulated as follows: is it possible to calculate the desired calibration curve *without* knowledge of the fitting function, which enables to describe the measured data? This problem can be solved in terms of the SFM method at condition that an additive (glucose in our case) shifts the initial curve monotonically and does not interact chemically or physically with basic substance (serum). In order to see these visual differences it is convenient to calculate the second relative moments for the values Re(ε) and Im(ε):

$$\Phi(\mathrm{Re}(\varepsilon)) = \left[\frac{1}{M}\sum_{m=1}^{M}(\mathrm{Re}(\varepsilon)_m - \langle \mathrm{Re}(\varepsilon)\rangle)^2\right]^{1/2},$$

$$\Phi(\mathrm{Im}(\varepsilon)) = \left[\frac{1}{M}\sum_{m=1}^{M}(\mathrm{Im}(\varepsilon)_m - \langle \mathrm{Im}(\varepsilon)\rangle)^2\right]^{1/2},$$

$$\langle f\rangle = \frac{1}{M}\sum_{m=1}^{M} f_m. \tag{46}$$

They are calculated in accordance with expressions (46), where index m denotes a current measurement, index M corresponds to the total number of repeat measurements ($M = 20$ in our case).

Figs. 10b and c show the calculated values of relative fluctuations. Their visual analysis shows that these functions are located *monotonically* with respect of different concentrations of glucose and hence can be separated. Considering each function as a random sequence and realizing the same treatment procedure described in Section 6 one can calculate the desired calibration curve. For this case it is sufficient to calculate the AUC for the relative difference $D(p)$ defined by (44). The corresponding calibration curves calculated for real and imaginary parts of the relative second moments are given in Fig. 10d. One can notice that these curves are more or less monotonic but they cannot be fit by a straight line, because the effect of separation given by Figs. 10b and c is *not uniform*.

The SFM method gives a new possibility to verify a stability and statistical proximity of the different experimental equipment used and to present a "similar" measurement as unique action and then express this action *quantitatively* in terms of AUC calculated again with the usage of expression (44) for each measurement. These plots could give some additional information in analysis of different measurements. In particular, one can evaluate more precisely in the space of moments the uncertainty region and consider the distribution of deviations obtained during whole experiment. The corresponding plots for pure serum and the concentration of glucose ($c = 10^2$) calculated for each measurement ($m = 1, 2, \ldots, 20$) are presented in Fig. 10e.

8. Results and discussion

In this section we summarize in brief the basics of the statistics fractional moments (SFM) that can find wide applications in quantitative comparison of different random sequences.

1. The values of higher moments that keep invariant the length of the initial sequence (condition (2)) helps to understand the stability and proximity of different random series expressed quantitatively in terms of integer ($k \ll N$) or fractional moments ($k \cong N \gg 1$). This condition in general is given by definition (25).
2. The solutions of Eq. (3) for relatively small k have unique sensitivity and help to detect one (!) 'strange' point in two initial sequences compared. An example of such application is considered in Section 5 and can be used for statistical protection of plastic cards and other valuable surfaces.
3. The GMV-function determined by expression (12) helps to transform a random sequence into determined function (21) and the fitting parameters of this function helps to compare *quantitatively* two arbitrary random sequences with each other that undoubtedly will find a wide practical application in construction of different calibration curves and in experimental data treatment procedure.
4. The higher integer moments can be easily generalized for the *fractional* or even *complex* moments that increase the limits of this new statistics and sensitivity of the SFM method at whole.
5. The SFM is closely related with nonextensive Tsallis entropy (41) that extends this definition and helps to express the stochastic properties of any random series in terms of fractional moments.
6. Instructive examples considered in this paper should convince a potential reader in

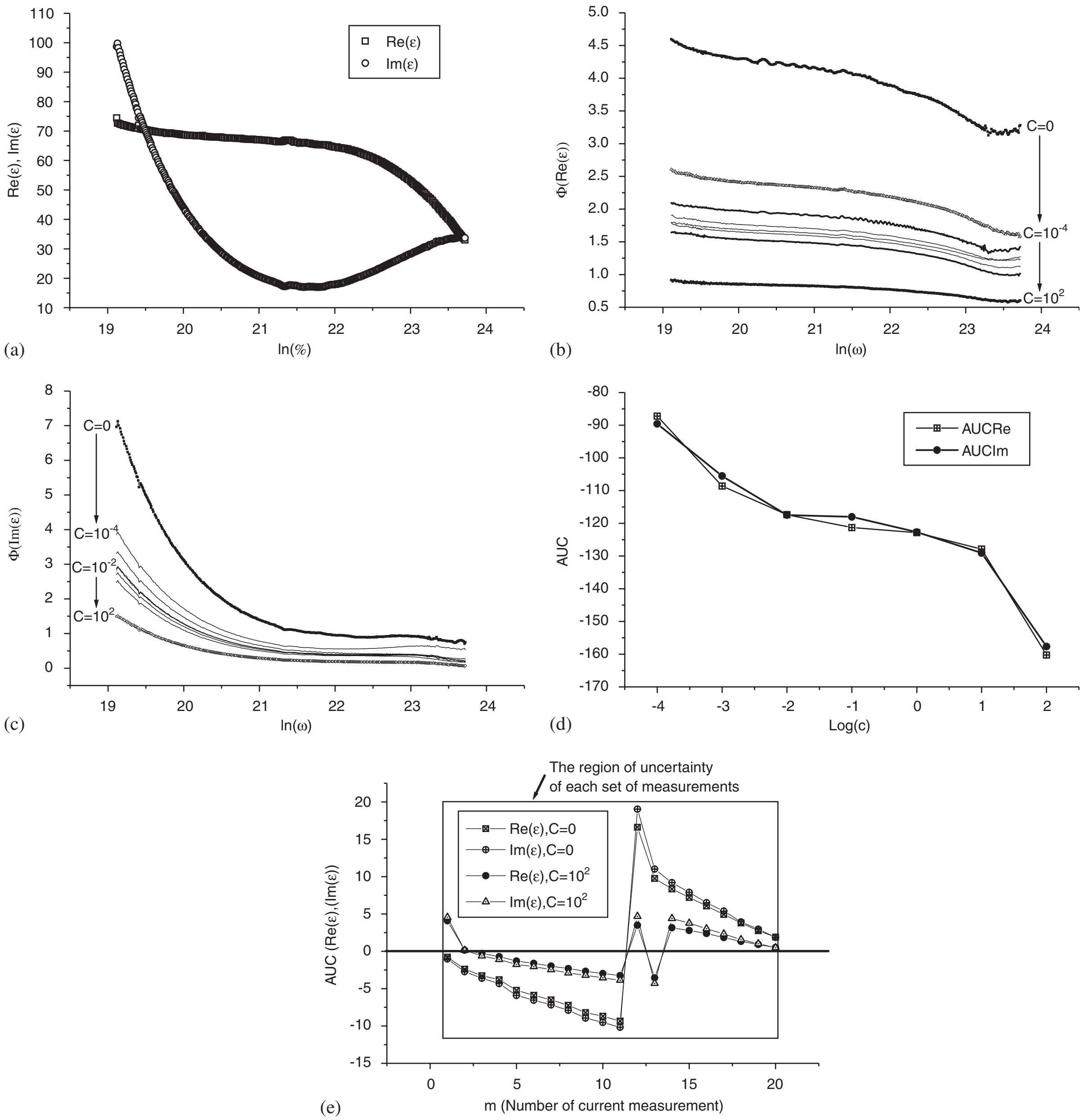

Fig. 10. (a) Typical curves for complex permittivity measured for pure calf serum at room temperature ($T = 25\,^{\circ}C$). These curves (if the fitting function is not known) are considered as random sequences. Other curves corresponding to different concentration of glucose cannot be depicted in this figure because they are very close to each other. (b) Set of curves corresponding to the relative second moments calculated for $Re(\varepsilon)$. The concentration of glucose monotonically increasing from upper curve to the bottom curve. One can notice that this increasing is not uniform. (c) Set of curves corresponding to the relative second moments calculated for $Im(\varepsilon)$. As for the plots depicted in panel (b) The concentration of glucose monotonically increasing from upper curve to the bottom curve. One can notice that this increasing is not uniform also. (d) Two calibration curves calculated for the curves depicted in panels (b) and (c). The curves cannot be fitted by straight lines because of nonuniform behavior of the corresponding curves presented in these figures. (e) Analysis of each set of identical measurements with the help of GMV-function helps to find a region of uncertainty of the equipment used and other additional information related to stability of each device involved in the process of measurement.

advantages of new statistics that enables to solve new problems as differentiation of statistical stability, *quantitative* comparison of different randomness one can help in solution of a basic problem as forecasting of different temporal sequences.

7. A new language of presentation randomness in terms of new fitting parameters (a_n, λ_n) entering into expression (21) opens a new possibilities of reconsideration and further substitution of an approximate language of statistical distributions widely used in the conventional statistics as the common instrument. Activity in this direction merits a special research.

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Appendix A. (Mathematical) The deduction of a system of Eqs. (3) and (15) and their possible generalizations

Let us consider at first the case $k = 1$. For this case we have

$$\begin{aligned}\Delta_N^{(1)} &= \frac{1}{N}(y_1 + y_2 + \cdots + y_N)\\ &= \frac{N-1}{N}\Delta_{N-1}^{(1)} + \frac{1}{N}y_N\\ &= \Delta_{N-1}^{(1)} + \frac{1}{N}(y_N - \Delta_{N-1}^{(1)}).\end{aligned} \quad \text{(A.1)}$$

If we require the coincidence

$$\Delta_N^{(1)} = \Delta_{N-1}^{(1)}, \quad \text{(A.2)}$$

then we have

$$y_N = \Delta_{N-1}^{(1)}. \quad \text{(A.3)}$$

For $k = 2$ we have

$$\begin{aligned}\Delta_N^{(1)} &= \frac{1}{N}(y_1 + y_2 + \cdots + y_N)\\ &= \frac{N-2}{N}\Delta_{N-2}^{(1)} + \frac{1}{N}(y_{N-1} + y_N)\\ &= \Delta_{N-2}^{(1)} + \frac{1}{N}(y_{N-1} + y_N - 2\Delta_{N-2}^{(1)}).\end{aligned} \quad \text{(A.4)}$$

Replacing y_j for y_j^2 we have by analogy with (A.4)

$$\begin{aligned}\Delta_N^{(2)} &= \frac{1}{N}(y_1^2 + y_2^2 + \cdots + y_N^2)\\ &= \frac{N-2}{N}\Delta_{N-2}^{(1)} + \frac{1}{N}(y_{N-1}^2 + y_N^2)\\ &= \Delta_{N-2}^{(2)} + \frac{1}{N}(y_{N-1}^2 + y_N^2 - 2\Delta_{N-2}^{(2)}).\end{aligned} \quad \text{(A.5)}$$

From the condition $\Delta_N^{(p)} = \Delta_{N-2}^{(p)}$, $p = 1, 2$ we have

$$\frac{y_{N-1} + y_N}{2} = \Delta_{N-2}^{(1)}, \quad \frac{y_{N-1}^2 + y_N^2}{2} = \Delta_{N-2}^{(2)}. \quad \text{(A.6)}$$

For an arbitrary k the simple manipulations lead to the following relationship:

$$\begin{aligned}\Delta_N^{(p)} &= \frac{1}{N}(y_1^p + y_2^p + \cdots + y_N^p) = \frac{N-k}{N}\Delta_{N-k}^{(p)}\\ &\quad + \frac{1}{N}(y_{N-k}^p + y_{N-k+1}^p + \cdots + y_N^p)\\ &= \Delta_{N-k}^{(p)} + \frac{1}{N}(y_{N-k}^p + y_{N-k+1}^p\\ &\quad + \cdots + y_N^p - k\Delta_{N-k}^{(p)}), \quad p = 1, 2\ldots, k.\end{aligned} \quad \text{(A.7)}$$

From the condition $\Delta_N^{(p)} = \Delta_{N-k}^{(p)}$, $p = 1, 2, \ldots, k$ we obtain

$$\frac{y_{N-k+1}^{(p)} + \cdots + y_N^{(p)}}{k} = \Delta_{N-k}^{(p)}, \quad p = 1, 2\ldots, k. \quad \text{(A.8)}$$

This relationship represents the system of Eqs. (15) for a set of k points located *inside* N points. For the set of points k located *outside* of the given random sequence it is necessary to replace $N - k \to N$. These simple calculations admit the following generalization. It is possible to replace the absolute value of a random variable $y_j^p \to (y_j - a)^p$, where a is an arbitrary constant. When $a = \mu \equiv \Delta_N^{(1)}$, the absolute value of the moment coincides with relative value of the moment defined by relationship (8).

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