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碩士學位論文

Department of Statistics

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Master Thesis

基於多重簽章之拜占庭共識演算法

Multi-Signature Byzantine Agreement

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國立臺灣大學碩士學位論文
口試委員會審定書

資訊系中碩士生學位論文之研究
Master's Thesis in Computer Science

本論文係王小明君（學號 R00000000）在國立臺灣大學資訊工程學系完成之碩士學位論文，於民國 104 年 7 月 31 日承下列考試委員審查通過及口試及格，特此證明

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摘要

測試

關鍵字：Sarmanov 分配

Abstract

The distributed ledger has been widely used in different scenarios after the Bitcoin turns out. Unlike the Proof of work (PoW) approaches, there ' s another mechanism that used to implement on the decentralize database or fault-tolerate system that have great potential to be transplanted to nowadays consortium blockchain systems. The research focus on the applying of consensus algorithm to financial organizations with the asynchronous network condition, byzantine attackers and permissioned policy in the system. We designed a novel algorithm that can tolerate at most no more than one third faulty nodes within the system, which is similar to the previous works but will have better performance base on our mathematical analysis and results of experiences.

Keywords: Sarmanov distribution

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2 Background

2.1 Copula

2.2 Sarmanov distribution

$$h_{Sarm}(\theta_1, \theta_2, \theta_3) = \prod_{j=1}^{3} h_j(\theta_j)$$
$$\times \left(1 + \sum_{1 \le j < k \le 3} \omega_{jk} \phi_j(\theta_j) \phi_k(\theta_k) + \omega_{123} \phi_1(\theta_1) \phi_2(\theta_2) \phi_3(\theta_3)\right) \quad (2.1)$$

$$\int_{\mathbb{R}} \phi_j(\theta_j) h_j(\theta_j) d\theta_j = 0, j = 1,\ 2,\ 3 \quad (2.2)$$

$$1 + \sum_{1 \le i < j \le 3} \omega_{jk} \phi_j(\theta_j) \phi_k(\theta_k) + \omega_{123} \phi_1(\theta_1) \phi_2(\theta_2) \phi_3(\theta_3) \ge 0, \theta_1, \theta_2, \theta_3 \in \mathbb{R}^3 \quad (2.3)$$

$$\Pr(\mathbf{Y}_i = \mathbf{y}) = \int_0^\infty \int_0^\infty \int_0^\infty \left(\prod_{j=1}^{3} e^{-E_{ij}\mu_{ij}\theta_j} \frac{(E_{ij}\mu_{ij}\theta_j)^{y_j}}{y_j!} \right) h_{Sarm}(\theta_1, \theta_2, \theta_3) d\theta_1 d\theta_2 d\theta_3$$

(2.4)

$$
\begin{aligned}
\Pr(\mathbf{Y}_i = \mathbf{y}) = \prod_{j=1}^{3} & \Pr(Y_{ij} = y_j) \\
& \left[1 + \sum_{1 \le j_1 < j_2 \le 3} \omega_{j_1 j_2} \prod_{k=1}^{2} \left(\left(\frac{\alpha_{jk} + E_{ij_k}\mu_{ij_k}}{\alpha_{jk} + E_{ij_k}\mu_{ij_k} + 1} \right)^{\alpha_{j_k} + y_{j_k}} - \left(\frac{\alpha_{j_k}}{\alpha_{j_k} + 1} \right)^{\alpha_{j_k}} \right) \right. \\
& \left. + \, \omega_{123} \prod_{j=1}^{3} \left(\left(\frac{\alpha_j + E_{ij}\mu_{ij}}{\alpha_j + E_{ij}\mu_{ij} + 1} \right)^{\alpha_j + y_j} - \left(\frac{\alpha_j}{\alpha_j + 1} \right)^{\alpha_j} \right) \right]
\end{aligned}
\tag{2.5}
$$

2.3 Poisson Process

3 Model

3.1 Statistical Model

Let $N_{ij}(t)$ denote the number of counts with event types j for subject i that have occurred at time t. Assume that counts on a subject are collected at K time points $0 < t_1 < \ldots < t_K$, and define $N_{ij}(0) = 0$. Let $Y_{ijk} = N_{ij}(t_k) - N_{ij}(t_{k-1})$ be the number of counts that have occurred in the k period and for the subject i, and let $\mathbf{Y}_{ij} = (Y_{ij1}, \ldots, Y_{ijK})$. Denote $\mathbf{Z}_i = (1, Z_{i1}, \ldots, Z_{ip})'$ as a vector of covariates for subject i and $\beta_j = (\beta_{j0}, \beta_{j1}, \ldots, \beta_{jp})'$ as a vector of coefficients. We denote the observed data by $\mathbf{D}_i = \{Y_{ijk}, \mathbf{Z}_i, j = 1, 2, k = 1, 2, \ldots, K\}$ for subject i.

The number of counts for subject i have correlated structure between different event types. In our model, we reduce the trivariate Sarmanov distribution which is written as equation (2.1), we assume random effects θ_1, θ_2 are from bivariate Sarmanov distribution with the form

$$h_{Sarm}(\theta_1, \theta_2) = \prod_{j=1}^{2} h_j(\theta_j) \times (1 + \omega\phi_1(\theta_1)\phi_2(\theta_2)),$$

where $h_j(\theta_j)$ are the corresponding marginal distributions, ϕ_j are kernel functions, and ω

is a real number such that

$$1 + \omega\phi_1(\theta_1)\phi_2(\theta_2) > 0.$$

Conditioning on $\mathbf{Z}_i$ and θ_j, we assume $N_{ij}(\cdot)$ is a nonhomogeneous Poisson process with mean function

$$E\{N_{ij}(t_k) - N_{ij}(t_{k-1})|\theta_j, \mathbf{Z}_i\} = d\Lambda_{jk}\theta_j \exp\left(\beta_j^T \mathbf{Z}_i\right), \tag{3.1}$$

where $d\Lambda_j(t) = \int_{t_{k-1}}^{t_k} \lambda_j(t)dt$. The $\lambda_j(t)$ is an non-negative and unspecified function. Based on the model assumptions and nonhomogeneous Poisson process, we obtain the distribution of counts with the event type j in the K periods

$$\Pr(\mathbf{Y}_{ij} = \mathbf{y}_{ij}|\theta_j, \mathbf{Z}_i) = \prod_{k=1}^{K} \exp\left\{-d\Lambda_{jk}\theta_j \exp\left(\beta_j^T \mathbf{Z}_i\right)\right\} \frac{[d\Lambda_{jk}\theta_j \exp\left(\beta_j^T \mathbf{Z}_i\right)]^{y_{ijk}}}{y_{ijk}!}. \tag{3.2}$$

We consider the marginal distributions are Gamma(α_j, α_j) with mean 1 and variance $\frac{1}{\alpha_j}$ and the kernel functions $\phi_j(\theta_j) = e^{-\theta_j} - \left(\frac{\alpha_j}{\alpha_j+1}\right)^{\alpha_j}$ for the bivariate Sarmanov distribution. We can show that the joint probability function of the counts in K periods for subject i is

$$\begin{aligned}
&\Pr(\mathbf{Y}_{i1}=\mathbf{y}_{i1},\mathbf{Y}_{i2}=\mathbf{y}_{i2}|\mathbf{Z}_i)\\
&=\int_0^\infty\int_0^\infty\prod_{j=1}^{2}\left(\prod_{k=1}^{K}\exp\left\{-d\Lambda_{jk}\theta_j\exp\left(\beta_j^T\mathbf{Z}_i\right)\right\}\frac{[d\Lambda_{jk}\theta_j\exp\left(\beta_j^T\mathbf{Z}_i\right)]^{y_{ijk}}}{y_{ijk}!}\right)h_{Sarm}(\theta_1,\theta_2)d\theta_1 d\theta_2\\
&=\prod_{j=1}^{2}\left(\binom{y_{ij\cdot}}{y_{ij1},\ldots,y_{ijK}}\left(\prod_{k=1}^{K}p_{jk}^{y_{ijk}}\right)\frac{\Gamma(\alpha_j+y_{ij\cdot})}{\Gamma(\alpha_j)(y_{ij\cdot})!}\tau_{ijk}^{\alpha_j}(1-\tau_{ijk})^{y_{ij\cdot}}\right)\\
&\quad\times\left[1+\omega\prod_{j=1}^{2}\left(\left(\frac{\alpha_j+\exp\left(\beta_j^T\mathbf{Z}_i\right)d\Lambda_{j\cdot}}{\alpha_j+\exp\left(\beta_j^T\mathbf{Z}_i\right)d\Lambda_{j\cdot}+1}\right)^{\alpha_j+y_{ij\cdot}}-\left(\frac{\alpha_j}{\alpha_j+1}\right)^{\alpha_j}\right)\right] \qquad (3.3)
\end{aligned}$$

where

$$y_{ij\cdot}=\sum_{k=1}^{K}y_{ijk},\ d\Lambda_{j\cdot}=\sum_{k=1}^{K}d\Lambda_{jk},\ p_{jk}=\frac{d\Lambda_{jk}}{d\Lambda_{j\cdot}},\ \tau_{ijk}=\frac{\alpha_j}{\alpha_j+\exp\left(\beta_j^T\mathbf{Z}_i\right)d\Lambda_{j\cdot}}.$$

The joint pdf (3.3) combines the marginal components with correlated component. The marginal components are composed of a multinomial distribution and a negative binomial distribution with event type j.

3.2 Estimation and Asymptotic Properties

In section 3.1, we conduct the joint p.f. of the counts in the K periods under 2 different random effects for subject i. Assume the sample size is n. We denote the total data $\mathbf{D}=\{\mathbf{D}_i, i=1,\ldots,n\}$. We obtain the likelihood function

$$L(\boldsymbol{\alpha},\boldsymbol{\beta},d\boldsymbol{\Lambda},\omega|\mathbf{D})=\prod_{i=1}^{n}\Pr(\mathbf{Y}_{i1}=\mathbf{y}_{i1},\mathbf{Y}_{i2}=\mathbf{y}_{i2}|\mathbf{Z}_i),$$

where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$, $\boldsymbol{\beta}=(\beta_1,\beta_2)$, $d\boldsymbol{\Lambda}=(d\Lambda_{11},d\Lambda_{12},\ldots,d\Lambda_{1K},d\Lambda_{21},d\Lambda_{22},\ldots,d\Lambda_{2K})$.

Let $\boldsymbol{\Omega}=\{\boldsymbol{\alpha},\boldsymbol{\beta},d\boldsymbol{\Lambda},\omega\}$ be a set of parameters. The Log-likelihood function can be written as

$$l(\mathbf{\Omega}|\mathbf{D}) = \sum_{i=1}^{n} \log\left(\Pr\left(\mathbf{Y}_{i1} = \mathbf{y}_{i1}, \mathbf{Y}_{i2} = \mathbf{y}_{i2}|\mathbf{Z}_i\right)\right).$$

We obtain the score functions with respect to parameters by using partial derivative and let score function equal 0.

$$\frac{\partial}{\partial \mathbf{\Omega}} l(\Omega|\mathbf{D}) = 0$$

Then we solve these equations by software and obtain the maximum likelihood estimators $\hat{\mathbf{\Omega}} = \{\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, d\hat{\mathbf{\Lambda}}, \hat{\omega}\}$.

Let $\mathbf{\Omega}_0 = \{\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, d\mathbf{\Lambda}_0, \omega_0\}$ be the true parameter values. Based on properties of MLE and regular conditions, $\sqrt{n}\left(\hat{\mathbf{\Omega}} - \mathbf{\Omega}_0\right)$ is asymptotically normal with mean zero and variance-covariance matrix $\mathcal{I}_0^{-1}$,

$$\sqrt{n}\left(\hat{\mathbf{\Omega}} - \mathbf{\Omega}_0\right) \rightarrow N\left(0, \mathcal{I}_0^{-1}\right).$$

$\mathcal{I}_0$ is the ~~single observation~~ information matrix, ~~which is calculated by $E(n^{-1}\mathcal{I})$ with respect to parameters $\mathbf{\Omega}$, where $\mathcal{I}$ is the secondary derivative matrix of negative log-likelihood function to parameters.~~ The information matrix $\mathcal{I}_0$ can be consistently estimated by $n^{-1}\hat{\mathcal{I}}$, where $\hat{\mathcal{I}}$ is the second derivative matrix of negative log-likelihood function with respect to $\mathbf{\Omega}$ evalueated at $\hat{\mathbf{\Omega}}$.

4 Simulations

In this chapter, ~~two~~ simulations are conducted to study the performance of the proposed model and estimation method. We use the R software to generate data and study the performance with different sample sizes by observing four indexes: Bias means the average of difference between true values and estimated values; SD means the simulation standard deviation of estimated values; SE is the average of standard error estimated based on the inverse information matrix, and CP represents the proportion of the 95% confidence intervals based on the asymptotic normality of estimated values that covered the true values of parameters.

4.1 Generate Data

Before generating data, we should generate random effects from bivariate Sarmanov distribution first. We use the conditional distribution method described in Nelsen (2006) to generate random effects θ_1, θ_2. The method is based on a conditional distribution of a random vector (U_1, U_2), which have uniform marginal distributions and cdf it's C ~~of bivariate Sarmanov distribution~~, t~~he cdf can be written as~~

as follows

given in the following

$$C(u_1, u_2) = u_1 u_2 + \omega \int_0^{u_1} \phi_1(F_1^{-1}(s))ds \int_0^{u_2} \phi_2(F_2^{-1}(t))dt, 0 \leq u_1,\ u_2 \leq 1, \tag{4.1}$$

Gamma(\
alpha_j,
\alpha_j)

where F_1^{-1}, F_2^{-1} are inverse function of marginals ~~of bivariate Sarmanov distribution~~, ~~and assume they exist.~~ The conditional distribution can be expressed as $C_{u_1}(u_2) = \Pr(U_2 \leq$
u2 condition on u1
$u_2|U_1 = u_1)$. Note that

$$
\begin{aligned}
C_{u_1}(u_2) &= \cancel{\lim_{\Delta u_1 \to 0^+} \frac{C(u_1 + \Delta u_1, u_2) - C(u_1, u_2)}{\Delta u_1}} = \frac{\partial C(u_1, u_2)}{\partial u_1} \\
&= u_2 + \omega \phi_1(F_1^{-1}(u_1)) \int_0^{u_2} \phi_s(F_2^{-1}(t))dt. \qquad (4.2)
\end{aligned}
$$

The following steps are the procedure for generating random effects:

1. Generate two independent random numbers u_1 and z from uniform distribution $U(0, 1)$.

2. Set $C_{u_1}(u_2) = z$, where C_{u_1} is the conditional distribution function. We obtain
4.2放在這邊
the u_2 by solving the equation, then obtain the pair $(u_1,\ u_2)$ from cdf of bivariate Sarmanov distribution.

3. Assume the inverse functions of F_1, F_2 exist, we obtain the pair $(\theta_1,\ \theta_2)$ ~~from bivariate Sarmanov distribution~~ ~~by solving the~~ two equations $F_1(\theta_1) = u_1$ and
with
$F_2(\theta_2) = u_2$.

We consider the covariates $\mathbf{Z}_i = (1,\ Z_{i1},\ Z_{i2})$, where Z_{i1} follows a standard normal
i先不要寫
distribution, and Z_{i2} follows a Bernoulli distribution with probability $p = 0.5$. For sim-
然後生成了i筆資料
plicity, we assume both observed time points of event types 1 and 2 are $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, $t_4 = 4$, $t_5 = 5$, which divide timeline into five equally-spaced periods.
calculate
We assign different functions $\lambda_j(t)$ to ~~generate~~ $d\Lambda_{jk}$ and set the true values of coefficients
在不同case裡面固定lambda_1 lambda_2
β_j. Then we can use the equation (3.2) to generate the data $D_i = \{Y_{ijk}, \mathbf{Z}_i,\ j = 1, 2,\ k = 1, 2, \ldots, 5\}$ for subject i.

+ sample size的選擇

sample size x2
beta fixed
omega x2

4.2 Result

In the first case, we set the true values of parameters $\alpha_1 = 3$, $\alpha_2 = 2$, $\beta_1 = (0.5, 0.5, -0.5)$, $\beta_2 = (0.5, -0.5, 0.5)$, $\omega = 0.5$, and assign the $\lambda_1(t) = 0.05$, $\lambda_2(t) = 0.05t$ to generate $d\Lambda_{jk}$. Table 1 shows the results with different sample sizes under fixed parameters. The bias of estimates are small, the average of standard errors close to the simulation standard deviations, and the coverage probabilities of the 95% confidence intervals reach the level we expect.

In the second case, we set the true parameters $\alpha_1 = 3$, $\alpha_2 = 5$, $\beta_1 = (1, 0.5, -0.5)$, $\beta_2 = (-0.5, -0.5, 1)$, $\omega = 0.8$, and we assign the more complex $\lambda_j(t)$ functions $\lambda_1(t) = 0.04t^2$, $\lambda_2(t) = 0.08t^3$. Table 2 shows the similar performances of the proposed model and estimation method. The bias of estimates are small, the average of standard errors close to the simulation standard deviations, and the coverage probabilities of the 95% confidence intervals reach the level we expect.

交通事故資料
一個行政區，行政區人口、面積不能差異太大
以年當作time period (k = 1, 2, 3..., 12)
type j (j = 1, 2) -> ex: car, motor

Table 1: Simulations results under the true parameters of case 1

拆開sample size成兩個表格

表格重排

	n=300				n=500			
Parameter	**Bias**	**SD**	**SE**	**CP**	**Bias**	**SD**	**SE**	**CP**
$\alpha_1 = 3$	0.0140	0.2551	0.2602	0.944	0.0030	0.0882	0.0866	0.946
$\beta_{10} = 0.5$	-0.0070	0.0867	0.0880	0.934	0.0002	0.0246	0.0245	0.938
$\beta_{11} = 0.5$	-0.0017	0.0873	0.0880	0.925	0.0000	0.0277	0.0280	0.949
$\beta_{12} = -0.5$	-0.0021	0.0771	0.0764	0.959	-0.0009	0.0382	0.0381	0.932
$d\Lambda_{11} = 0.05$	0.0002	0.0043	0.0043	0.943	0.0001	0.0023	0.0023	0.917
$d\Lambda_{12} = 0.05$	0.0001	0.0045	0.0043	0.941	0.0000	0.0012	0.0012	0.940
$d\Lambda_{13} = 0.05$	0.0000	0.0044	0.0043	0.944	0.0001	0.0023	0.0023	0.914
$d\Lambda_{14} = 0.05$	0.0002	0.0044	0.0043	0.948	0.0000	0.0023	0.0023	0.900
$d\Lambda_{15} = 0.05$	0.0002	0.0043	0.0043	0.954	0.0000	0.0024	0.0023	0.907
$\alpha_2 = 2$	0.0075	0.1717	0.1738	0.94	0.0065	0.0926	0.0924	0.904
$\beta_{20} = 0.5$	-0.0005	0.0698	0.0702	0.966	0.0006	0.0265	0.0263	0.946
$\beta_{21} = -0.5$	-0.0066	0.0942	0.0958	0.936	-0.0010	0.0440	0.0429	0.919
$\beta_{22} = 0.5$	-0.0084	0.1069	0.1059	0.919	0.0011	0.0278	0.0280	0.936
$d\Lambda_{21} = 0.025$	0.0000	0.0022	0.0022	0.945	0.0000	0.0012	0.0012	0.922
$d\Lambda_{22} = 0.075$	0.0003	0.0066	0.0065	0.937	-0.0001	0.0035	0.0035	0.921
$d\Lambda_{23} = 0.125$	0.0004	0.0106	0.0108	0.937	0.0002	0.0047	0.0047	0.937
$d\Lambda_{24} = 0.175$	0.0005	0.0150	0.0152	0.942	0.0005	0.0071	0.0071	0.922
$d\Lambda_{25} = 0.225$	0.0017	0.0200	0.0195	0.952	0.0002	0.0093	0.0091	0.931
$\omega = 0.5$	-0.0040	0.0701	0.0703	0.962	-0.0002	0.0265	0.0263	0.941

Table 2: Simulations results under the true parameters of case 2

	n=300				n=500			
Parameter	**Bias**	**SD**	**SE**	**CP**	**Bias**	**SD**	**SE**	**CP**
$\alpha_1 = 3$	0.0149	0.2303	0.2257	0.951	0.0065	0.1204	0.1215	0.928
$\beta_{10} = 1$	-0.0050	0.0639	0.0639	0.957	-0.0003	0.0212	0.0212	0.927
$\beta_{11} = 0.5$	0.0010	0.0713	0.0702	0.955	0.0016	0.0341	0.0351	0.916
$\beta_{12} = -0.5$	-0.0062	0.0948	0.0958	0.933	-0.0019	0.0384	0.0381	0.926
$d\Lambda_{11} = 0.0133$	0.0000	0.0010	0.0010	0.953	0.0000	0.0007	0.0007	0.912
$d\Lambda_{12} = 0.0933$	0.0006	0.0082	0.0081	0.935	0.0000	0.0031	0.0032	0.945
$d\Lambda_{13} = 0.2533$	0.0014	0.0258	0.0250	0.926	0.0003	0.0119	0.0117	0.920
$d\Lambda_{14} = 0.4933$	0.0022	0.0422	0.0428	0.942	-0.0001	0.0228	0.0228	0.930
$d\Lambda_{15} = 0.8133$	0.0048	0.0748	0.0754	0.932	0.0005	0.0375	0.0376	0.934
$\alpha_2 = 5$	0.0213	0.4295	0.4333	0.927	0.0094	0.2019	0.2022	0.926
$\beta_{20} = -0.5$	-0.0014	0.0943	0.0957	0.933	-0.0034	0.0471	0.0476	0.926
$\beta_{21} = -0.5$	-0.0063	0.0954	0.0957	0.936	-0.0011	0.0468	0.0477	0.918
$\beta_{22} = 1$	-0.0049	0.0967	0.0963	0.923	-0.0012	0.0318	0.0319	0.930
$d\Lambda_{21} = 0.02$	0.0001	0.0018	0.0017	0.953	0.0000	0.0006	0.0006	0.937
$d\Lambda_{22} = 0.30$	0.0011	0.0269	0.0261	0.935	0.0001	0.0102	0.0104	0.922
$d\Lambda_{23} = 1.30$	0.0124	0.1143	0.1128	0.937	0.0011	0.0445	0.0450	0.925
$d\Lambda_{24} = 3.50$	0.0199	0.2892	0.2836	0.949	0.0042	0.1457	0.1416	0.942
$d\Lambda_{25} = 7.38$	0.0415	0.6494	0.6402	0.925	0.0088	0.3070	0.2985	0.929
$\omega = 0.8$	-0.0066	0.0782	0.0782	0.957	-0.0006	0.0309	0.0311	0.935

5 Conclusion

測試用

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Figures and Tables

Table A.1: MLE with $\lambda_1(t) = 0.05$ and $\lambda_2(t) = 0.05t$

	n=200				n=400			
Parameter	**Bias**	**SD**	**SE**	**CP**	**Bias**	**SD**	**SE**	**CP**
$\alpha_1 = 3$	0.0081	0.2979	0.2959	0.981	-0.0006	0.1017	0.1040	0.955
$\beta_{10} = 0.5$	-0.0086	0.1124	0.1131	0.979	-0.0011	0.0242	0.0245	0.970
$\beta_{11} = 0.5$	0.0023	0.1210	0.1238	0.966	-0.0012	0.0279	0.0280	0.970
$\beta_{12} = -0.5$	-0.0106	0.1303	0.1300	0.956	-0.0028	0.0476	0.0477	0.943
$d\Lambda_{11} = 0.05$	0.0006	0.0080	0.0079	0.906	0.0000	0.0023	0.0023	0.952
$d\Lambda_{12} = 0.05$	0.0005	0.0083	0.0080	0.913	0.0000	0.0011	0.0012	0.964
$d\Lambda_{13} = 0.05$	0.0006	0.0074	0.0073	0.921	-0.0001	0.0023	0.0023	0.938
$d\Lambda_{14} = 0.05$	0.0010	0.0081	0.0080	0.904	0.0001	0.0023	0.0023	0.950
$d\Lambda_{15} = 0.05$	0.0008	0.0080	0.0079	0.923	0.0000	0.0023	0.0023	0.953
$\alpha_2 = 2$	0.0053	0.1941	0.1913	0.978	0.0029	0.0918	0.0924	0.955
$\beta_{20} = 0.5$	-0.0077	0.1063	0.1059	0.970	-0.0014	0.0263	0.0263	0.957
$\beta_{21} = -0.5$	-0.0078	0.1258	0.1252	0.962	0.0001	0.0418	0.0429	0.941
$\beta_{22} = 0.5$	-0.0075	0.1168	0.1131	0.971	0.0023	0.0278	0.0280	0.958
$d\Lambda_{21} = 0.025$	0.0003	0.0040	0.0040	0.909	0.0000	0.0011	0.0012	0.948
$d\Lambda_{22} = 0.075$	0.0016	0.0120	0.0118	0.903	0.0001	0.0034	0.0035	0.965
$d\Lambda_{23} = 0.125$	0.0008	0.0195	0.0197	0.919	-0.0001	0.0046	0.0047	0.949
$d\Lambda_{24} = 0.175$	0.0017	0.0271	0.0275	0.922	0.0004	0.0071	0.0071	0.954
$d\Lambda_{25} = 0.225$	0.0012	0.0341	0.0341	0.914	0.0003	0.0091	0.0091	0.951
$\omega = 0.5$	-0.0029	0.1046	0.1058	0.971	-0.0015	0.0265	0.0262	0.953

Table A.2: MLE with $\lambda_1(t) = 0.04t^2$ and $\lambda_2(t) = 0.08t^3$

	n=200				n=400			
Parameter	**Bias**	**SD**	**SE**	**CP**	**Bias**	**SD**	**SE**	**CP**
$\alpha_1 = 3$	0.0016	0.4322	0.4369	0.938	0.0019	0.1710	0.1736	0.942
$\beta_{10} = 1$	-0.0022	0.0537	0.0532	0.991	-0.0005	0.0257	0.0255	0.965
$\beta_{11} = 0.5$	-0.0039	0.0711	0.0702	0.990	0.0004	0.0355	0.0350	0.954
$\beta_{12} = -0.5$	-0.0068	0.0980	0.0958	0.978	0.0000	0.0463	0.0477	0.955
$d\Lambda_{11} = 0.0133$	0.0001	0.0020	0.0019	0.926	0.0001	0.0008	0.0008	0.946
$d\Lambda_{12} = 0.0933$	0.0002	0.0109	0.0108	0.966	0.0000	0.0032	0.0032	0.952
$d\Lambda_{13} = 0.2533$	0.0042	0.0366	0.0368	0.934	0.0006	0.0148	0.0146	0.916
$d\Lambda_{14} = 0.4933$	0.0061	0.0731	0.0718	0.932	0.0000	0.0222	0.0228	0.952
$d\Lambda_{15} = 0.8133$	0.0054	0.1179	0.1185	0.939	0.0028	0.0375	0.0376	0.947
$\alpha_2 = 5$	0.0137	0.5693	0.5806	0.964	-0.0045	0.2289	0.2309	0.948
$\beta_{20} = -0.5$	-0.0037	0.0968	0.0957	0.975	-0.0018	0.0566	0.0572	0.928
$\beta_{21} = -0.5$	-0.0058	0.1147	0.1153	0.954	-0.0036	0.0479	0.0476	0.947
$\beta_{22} = 1$	-0.0024	0.0951	0.0962	0.981	-0.0007	0.0486	0.0489	0.952
$d\Lambda_{21} = 0.02$	0.0004	0.0030	0.0029	0.923	0.0000	0.0007	0.0007	0.967
$d\Lambda_{22} = 0.30$	0.0027	0.0348	0.0348	0.965	0.0002	0.0124	0.0121	0.959
$d\Lambda_{23} = 1.30$	0.0132	0.1478	0.1509	0.970	0.0005	0.0731	0.0752	0.918
$d\Lambda_{24} = 3.50$	0.0333	0.4148	0.4057	0.964	0.0121	0.2048	0.2023	0.932
$d\Lambda_{25} = 7.38$	0.0394	0.8449	0.8572	0.968	0.0043	0.4229	0.4265	0.945
$\omega = 0.8$	-0.0084	0.1309	0.1311	0.961	-0.0020	0.0469	0.0468	0.944