國立政治大學商學院統計學系研究所

碩士學位論文

Department of Statistics
College of Commerce
National Chengchi University (NCCU)

Master Thesis

基於多重簽章之拜占庭共識演算法

Multi-Signature Byzantine Agreement

指導教授：黃佳慧 博士
Advisor: Chia-Hui Huang, Ph.D.
研究生: 陳盈諳
Ying-An Chen

中華民國 110 年 7 月
July, 2022

國立臺灣大學碩士學位論文
口試委員會審定書

資訊系中碩士生學位論文之研究
Master's Thesis in Computer Science

本論文係王小明君（學號 R00000000）在國立臺灣大學資訊工程學系完成之碩士學位論文，於民國 104 年 7 月 31 日承下列考試委員審查通過及口試及格，特此證明

口試委員：

______________________________（簽名）

（指導教授）

______________　______________

______________　______________

______________　______________

______________　______________

系主任　______________

誌謝

Acknowledgements

摘要

測試

關鍵字：Sarmanov 分配

Abstract

The distributed ledger has been widely used in different scenarios after the Bitcoin turns out. Unlike the Proof of work (PoW) approaches, there ' s another mechanism that used to implement on the decentralize database or fault-tolerate system that have great potential to be transplanted to nowadays consortium blockchain systems. The research focus on the applying of consensus algorithm to financial organizations with the asynchronous network condition, byzantine attackers and permissioned policy in the system. We designed a novel algorithm that can tolerate at most no more than one third faulty nodes within the system, which is similar to the previous works but will have better performance base on our mathematical analysis and results of experiences.

Keywords: Sarmanov distribution

Contents

選定某些性質相似的行政區的不同年份當作不同樣本（ex: 板橋在不同年視為不同樣本）

List of Figures

List of Tables

1 Introduction

2 Background

2.1 Copula

2.2 Sarmanov distribution

$$h_{Sarm}(\theta_1, \theta_2, \theta_3) = \prod_{j=1}^{3} h_j(\theta_j) \times \left(1 + \sum_{1 \le j < k \le 3} \omega_{jk}\phi_j(\theta_j)\phi_k(\theta_k) + \omega_{123}\phi_1(\theta_1)\phi_2(\theta_2)\phi_3(\theta_3)\right) \quad (2.1)$$

$$\int_{\mathbb{R}} \phi_j(\theta_j)h_j(\theta_j)d\theta_j = 0, j = 1,\ 2,\ 3 \quad (2.2)$$

$$1 + \sum_{1 \le i < j \le 3} \omega_{jk}\phi_j(\theta_j)\phi_k(\theta_k) + \omega_{123}\phi_1(\theta_1)\phi_2(\theta_2)\phi_3(\theta_3) \ge 0, \theta_1, \theta_2, \theta_3 \in \mathbb{R}^3 \quad (2.3)$$

$$\Pr(\mathbf{Y}_i = \mathbf{y}) = \int_0^\infty \int_0^\infty \int_0^\infty \left(\prod_{j=1}^{3} e^{-E_{ij}\mu_{ij}\theta_j} \frac{(E_{ij}\mu_{ij}\theta_j)^{y_j}}{y_j!}\right) h_{Sarm}(\theta_1, \theta_2, \theta_3)d\theta_1 d\theta_2 d\theta_3 \quad (2.4)$$

$$
\begin{aligned}
\Pr(\mathbf{Y}_i = \mathbf{y}) = & \prod_{j=1}^{3} \Pr(Y_{ij} = y_j) \\
& \Bigg[1 + \sum_{1 \le j_1 < j_2 \le 3} \omega_{j_1 j_2} \prod_{k=1}^{2} \left(\left(\frac{\alpha_{jk} + E_{ij_k}\mu_{ij_k}}{\alpha_{jk} + E_{ij_k}\mu_{ij_k} + 1} \right)^{\alpha_{j_k} + y_{j_k}} - \left(\frac{\alpha_{j_k}}{\alpha_{j_k} + 1} \right)^{\alpha_{j_k}} \right) \\
& + \omega_{123} \prod_{j=1}^{3} \left(\left(\frac{\alpha_j + E_{ij}\mu_{ij}}{\alpha_j + E_{ij}\mu_{ij} + 1} \right)^{\alpha_j + y_j} - \left(\frac{\alpha_j}{\alpha_j + 1} \right)^{\alpha_j} \right) \Bigg] \qquad (2.5)
\end{aligned}
$$

2.3 Poisson Process

3 Model

3.1 Statistical Model

Let $N_{ij}(t)$ denote the number of counts with event types j for subject i that have occurred at time t. Assume that counts on a subject are collected at K time points $0 < t_1 < \ldots < t_K$, and define $N_{ij}(0) = 0$. Let $Y_{ijk} = N_{ij}(t_k) - N_{ij}(t_{k-1})$ be the number of counts that have occurred in the k period and for the subject i, and let $\mathbf{Y}_{ij} = (Y_{ij1}, \ldots, Y_{ijK})$. Denote $\mathbf{Z}_i = (1, Z_{i1}, \ldots, Z_{ip})'$ as a vector of covariates for subject i and $\beta_j = (\beta_{j0}, \beta_{j1}, \ldots, \beta_{jp})'$ as a vector of coefficients. We denote the observed data by $\mathbf{D}_i = \{Y_{ijk}, \mathbf{Z}_i, j = 1, 2, k = 1, 2, \ldots, K\}$ for subject i.

The number of counts for subject i have correlated structure between different event types. In our model, we reduce the trivariate Sarmanov distribution which is written as equation (2.1), we assume random effects θ_1, θ_2 are from bivariate Sarmanov distribution with the form

$$h_{Sarm}(\theta_1, \theta_2) = \prod_{j=1}^{2} h_j(\theta_j) \times (1 + \omega\phi_1(\theta_1)\phi_2(\theta_2)),$$

where $h_j(\theta_j)$ are the corresponding marginal distributions, ϕ_j are kernel functions, and ω

is a real number such that

$$1 + \omega\phi_1(\theta_1)\phi_2(\theta_2) > 0. \tag{3.1}$$

Conditioning on $\mathbf{Z}_i$ and θ_j, we assume $N_{ij}(\cdot)$ is a nonhomogeneous Poisson process with mean function

$$E\{N_{ij}(t_k) - N_{ij}(t_{k-1})|\theta_j, \mathbf{Z}_i\} = d\Lambda_{jk}\theta_j \exp{(\beta_j^T \mathbf{Z}_i)}, \tag{3.2}$$

where $d\Lambda_j(t) = \int_{t_{k-1}}^{t_k} \lambda_j(t)dt$. The $\lambda_j(t)$ is an non-negative and unspecified function. Based on the model assumptions and nonhomogeneous Poisson process, we obtain the distribution of counts with the event type j in the K periods

$$\Pr(\mathbf{Y}_{ij} = \mathbf{y}_{ij}|\theta_j, \mathbf{Z}_i) = \prod_{k=1}^{K} \exp\left\{-d\Lambda_{jk}\theta_j \exp\left(\beta_j^T \mathbf{Z}_i\right)\right\} \frac{[d\Lambda_{jk}\theta_j \exp{(\beta_j^T \mathbf{Z}_i)}]^{y_{ijk}}}{y_{ijk}!}. \tag{3.3}$$

We consider the marginal distributions are Gamma(α_j, α_j) with mean 1 and variance $\frac{1}{\alpha_j}$ and the kernel functions $\phi_j(\theta_j) = e^{-\theta_j} - \left(\frac{\alpha_j}{\alpha_j+1}\right)^{\alpha_j}$ for the bivariate Sarmanov distribution. We can show that the joint probability function of the counts in K periods for subject i is

$$
\begin{aligned}
&\Pr(\mathbf{Y}_{i1}=\mathbf{y}_{i1},\mathbf{Y}_{i2}=\mathbf{y}_{i2}|\mathbf{Z}_i)\\
=&\int_0^\infty\int_0^\infty\prod_{j=1}^{2}\left(\prod_{k=1}^{K}\exp\left\{-d\Lambda_{jk}\theta_j\exp\left(\beta_j^T\mathbf{Z}_i\right)\right\}\frac{[d\Lambda_{jk}\theta_j\exp\left(\beta_j^T\mathbf{Z}_i\right)]^{y_{ijk}}}{y_{ijk}!}\right)h_{Sarm}(\theta_1,\theta_2)d\theta_1 d\theta_2\\
=&\prod_{j=1}^{2}\left(\binom{y_{ij\cdot}}{y_{ij1},\dots,y_{ijK}}\left(\prod_{k=1}^{K}p_{jk}^{y_{ijk}}\right)\frac{\Gamma(\alpha_j+y_{ij\cdot})}{\Gamma(\alpha_j)(y_{ij\cdot})!}\tau_{ijk}^{\alpha_j}(1-\tau_{ijk})^{y_{ij\cdot}}\right)\\
&\times\left[1+\omega\prod_{j=1}^{2}\left(\left(\frac{\alpha_j+\exp\left(\beta_j^T\mathbf{Z}_i\right)d\Lambda_{j\cdot}}{\alpha_j+\exp\left(\beta_j^T\mathbf{Z}_i\right)d\Lambda_{j\cdot}+1}\right)^{\alpha_j+y_{ij\cdot}}-\left(\frac{\alpha_j}{\alpha_j+1}\right)^{\alpha_j}\right)\right] \qquad (3.4)
\end{aligned}
$$

where

$$
y_{ij\cdot}=\sum_{k=1}^{K}y_{ijk},\ d\Lambda_{j\cdot}=\sum_{k=1}^{K}d\Lambda_{jk},\ p_{jk}=\frac{d\Lambda_{jk}}{d\Lambda_{j\cdot}},\ \tau_{ijk}=\frac{\alpha_j}{\alpha_j+\exp\left(\beta_j^T\mathbf{Z}_i\right)d\Lambda_{j\cdot}}.
$$

The joint pdf (3.4) combines the marginal components with correlated component. The marginal components are composed of a multinomial distribution and a negative binomial distribution with event type j.

3.2 Estimation and Asymptotic Properties

In section 3.1, we conduct the joint p.f. of the counts in the K periods under 2 different random effects for subject i. Assume the sample size is n. We denote the total data $\mathbf{D}=\{\mathbf{D}_i, i=1,\dots,n\}$. We obtain the likelihood function

$$
L(\boldsymbol{\alpha},\boldsymbol{\beta},d\boldsymbol{\Lambda},\omega|\mathbf{D})=\prod_{i=1}^{n}\Pr(\mathbf{Y}_{i1}=\mathbf{y}_{i1},\mathbf{Y}_{i2}=\mathbf{y}_{i2}|\mathbf{Z}_i),
$$

where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$, $\boldsymbol{\beta}=(\beta_1,\beta_2)$, $d\boldsymbol{\Lambda}=(d\Lambda_{11},d\Lambda_{12},\dots,d\Lambda_{1K},d\Lambda_{21},d\Lambda_{22},\dots,d\Lambda_{2K})$.

Let $\boldsymbol{\Omega}=\{\boldsymbol{\alpha},\boldsymbol{\beta},d\boldsymbol{\Lambda},\omega\}$ be a set of parameters. The Log-likelihood function can be written as

$$l(\mathbf{\Omega}|\mathbf{D}) = \sum_{i=1}^{n} \log\left(\Pr\left(\mathbf{Y}_{i1} = \mathbf{y}_{i1}, \mathbf{Y}_{i2} = \mathbf{y}_{i2} | \mathbf{Z}_i\right)\right).$$

We obtain the score functions with respect to parameters by using partial derivative and let score function equal 0.

$$\frac{\partial}{\partial \mathbf{\Omega}} l(\Omega|\mathbf{D}) = 0$$

Then we solve these equations by software and obtain the maximum likelihood estimators $\hat{\mathbf{\Omega}} = \{\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, d\hat{\mathbf{\Lambda}}, \hat{\omega}\}$.

Let $\mathbf{\Omega}_0 = \{\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, d\mathbf{\Lambda}_0, \omega_0\}$ be the true parameter values. Based on properties of MLE and regular conditions, $\sqrt{n}\left(\hat{\mathbf{\Omega}} - \mathbf{\Omega}_0\right)$ is asymptotically normal with mean zero and variance-covariance matrix $\mathcal{I}_0^{-1}$,

$$\sqrt{n}\left(\hat{\mathbf{\Omega}} - \mathbf{\Omega}_0\right) \to N\left(0, \mathcal{I}_0^{-1}\right).$$

$\mathcal{I}_0$ is the information matrix. The information matrix $\mathcal{I}_0$ can be consistently estimated by $n^{-1}\hat{\mathcal{I}}$, where $\hat{\mathcal{I}}$ is the second derivative matrix of negative log-likelihood function with respect to $\mathbf{\Omega}$ evalueated at $\hat{\mathbf{\Omega}}$.

4 Simulations

In this chapter, simulations are conducted to study the performance of the proposed model and estimation method. We use the R software to generate data and study the performance with different sample sizes by observing four indexes: Bias means the average of difference between true values and estimated values; SD means the simulation standard deviation of estimated values; SE is the average of standard error estimated based on the inverse information matrix, and CP represents the proportion of the 95% confidence intervals based on the asymptotic normality of estimated values that covered the true values of parameters. It is difficult to conduct the secondary derivative of log-likelihood, so we use bootstrap to calculate the average of standard error.

second

4.1 Generate Data

Before generating data, we should generate random effects from bivariate Sarmanov distribution first. We use the conditional distribution method described in Nelsen (2006) to generate random effects θ_1, θ_2. The method is based on a conditional distribution of a random vector (U_1, U_2), which has uniform marginal distributions and it's cdf C given in the following

$$C(u_1, u_2) = u_1 u_2 + \omega \int_0^{u_1} \phi_1(F_1^{-1}(s))ds \int_0^{u_2} \phi_2(F_2^{-1}(t))dt, 0 \leq u_1,\, u_2 \leq 1, \quad (4.1)$$

where F_1^{-1}, F_2^{-1} are inverse function of Gamma(α_j, α_j) marginals with mean 1 and variance $\frac{1}{\alpha_j}$. repectively
j = 1, 2

The following steps are the procedure for generating random effects:

1. Generate two independent random numbers u_1 and z from uniform distribution $U(0, 1)$.

2. Set $C_{u_1}(u_2) = z$, where $C_{u_1}(u_2)$ is the conditional distribution function

$$\begin{aligned} C_{u_1}(u_2) = \Pr(U_2 \le u_2 | U_1 = u_1) &= \frac{\partial C(u_1, u_2)}{\partial u_1} \\ &= u_2 + \omega \phi_1(F_1^{-1}(u_1)) \int_0^{u_2} \phi_s(F_2^{-1}(t)) dt. \end{aligned}$$

   We obtain the u_2 by solving the equation, then ~~obtain~~ the pair (u_1, u_2) ~~from~~ follows cdf of bivariate Sarmanov distribution.

3. Assume F_1^{-1}, F_2^{-1} exist, we obtain the pair (θ_1, θ_2) with $\theta_1 = F^{-1}(u_1)$ and $\theta_2 = F_2^{-1}(u_2)$.

We consider the covariates $\mathbf{Z} = (1, Z_1, Z_2)$, where Z_1 follows a standard normal distribution, and Z_2 follows a Bernoulli distribution with probability $p = 0.5$, then we generate n subjects. For simplicity, we assume both observed time points of event types 1 and 2 are $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, $t_4 = 4$, $t_5 = 5$, which divide timeline into five equally-spaced periods. We assign functions $\lambda_1(t) = 0.05t^3$ and $\lambda_2(t) = 0.1t^2$ to calculate $d\Lambda_{jk}$ and set the true values of coefficients β_j. ~~The parameter ω~~ the boundary of parameter ω is constrained by equation (3.1). Then we can use the equation (3.3) to generate the observed data $\mathbf{D}_i = \{Y_{ijk}, \mathbf{Z}_i,\ j = 1, 2,\ k = 1, 2, \ldots, 5\}$ for subject i, $i = 1, \ldots, n$, which are ~~assume~~ assumed to be independent and identically distributed, and we consider the sample size~~s~~ n are 300 and 500.

4.2 Result

4.2.1 Results under different sample size n

We compare the results with sample size $n = 300$, $n = 500$ with fixed true parameters $\alpha_1 = 5$, $\alpha_2 = 2$, $\beta_1 = (0.5, 0.5, -0.5)$, $\beta_2 = (0.5, -0.5, 0.5)$, $\omega = 0.5$. Based on the table 1 and table 2, we observe the bias reduce with increasing sample size n. The average of standard errors close to the simulation standard deviations. The average of standard errors with sample size 500 are smaller than those with sample size 300, and simulation standard deviations are the same. The coverage probabilities of 95% confidence intervals achieve the level. As stated above, we may obtain better estimates by increasing the sample size, the improvement of simulation standard deviations and average of standard errors are especially obvious.

Table 1: Simulation results with $\omega = 0.5$ and $n = 300$ 標題太擠

parameter	Parameter true value	Bias	SD	SE	CP
ω	0.5	-0.0046	0.0616	0.0609	0.974
α_1	5	0.0246	0.5055	0.5031	0.941
β_{10}	0.5	0.0001	0.0618	0.0609	0.968
β_{11}	0.5	-0.0051	0.0596	0.0610	0.956
β_{12}	-0.5	-0.0003	0.0634	0.0663	0.966
$d\Lambda_{11}$	0.0125	0.0001	0.0013	0.0013	0.911
$d\Lambda_{12}$	0.1875	0.0005	0.0186	0.0189	0.933
$d\Lambda_{13}$	0.8125	-0.0037	0.0812	0.0817	0.938
$d\Lambda_{14}$	2.1875	0.0184	0.2235	0.2197	0.940
$d\Lambda_{15}$	4.6125	0.0103	0.4566	0.4631	0.928
α_2	2	0.0022	0.1956	0.2013	0.903
β_{20}	0.5	0.0007	0.0596	0.0607	0.978
β_{21}	-0.5	-0.0003	0.0848	0.0830	0.961
β_{22}	0.5	0.0013	0.0631	0.0610	0.967
$d\Lambda_{21}$	0.0333	0.0002	0.0034	0.0034	0.918
$d\Lambda_{22}$	0.2333	0.0001	0.0189	0.0188	0.949
$d\Lambda_{23}$	0.6333	0.0009	0.0625	0.0637	0.925
$d\Lambda_{24}$	1.2333	0.0060	0.1224	0.1243	0.945
$d\Lambda_{25}$	2.0333	0.0018	0.2071	0.2051	0.931

Table 2: Simulation results with $\omega = 0.5$ and $n = 500$

	Parameter	Bias	SD	SE	CP
ω	0.5	-0.0009	0.0301	0.0303	0.944
α_1	5	0.0230	0.2563	0.2505	0.938
β_{10}	0.5	0.0008	0.0308	0.0303	0.930
β_{11}	0.5	-0.0005	0.0304	0.0304	0.939
β_{12}	-0.5	-0.0022	0.0330	0.0330	0.931
$d\Lambda_{11}$	0.0125	0.0000	0.0006	0.0006	0.909
$d\Lambda_{12}$	0.1875	0.0004	0.0076	0.0075	0.929
$d\Lambda_{13}$	0.8125	-0.0002	0.0401	0.0407	0.908
$d\Lambda_{14}$	2.1875	0.0025	0.1124	0.1094	0.916
$d\Lambda_{15}$	4.6125	0.0104	0.2244	0.2309	0.918
α_2	2	0.0038	0.1013	0.1001	0.919
β_{20}	0.5	-0.0013	0.0311	0.0304	0.936
β_{21}	-0.5	0.0001	0.0509	0.0495	0.937
β_{22}	0.5	0.0008	0.0303	0.0303	0.930
$d\Lambda_{21}$	0.0333	0.0001	0.0017	0.0017	0.906
$d\Lambda_{22}$	0.2333	0.0002	0.0092	0.0093	0.936
$d\Lambda_{23}$	0.6333	-0.0002	0.0319	0.0317	0.915
$d\Lambda_{24}$	1.2333	0.0013	0.0623	0.0617	0.918
$d\Lambda_{25}$	2.0333	0.0088	0.0992	0.1019	0.912

Table 3 and table 4 show the similar results like table 1 and table 2. The bias of estimates are negligible, and reduce with the increasing sample size. The average of standard errors close to the simulation standard deviations, and which with sample size 500 are obviously smaller than which with sample size 300. The coverage probabilities of the 95% confidence intervals reach the level we expect. As stated above, we can conclude that we may obtain better estimates when sample size increases.

Table 3: Simulation results with $\omega = 1$ and $n = 300$

	Parameter	Bias	SD	SE	CP
ω	1	0.0014	0.0376	0.0368	0.983
α_1	5	0.0299	0.4987	0.5035	0.911
β_{10}	0.5	-0.0015	0.0607	0.0609	0.967
β_{11}	0.5	-0.0031	0.0620	0.0609	0.954
β_{12}	-0.5	-0.0057	0.0664	0.0663	0.967
$d\Lambda_{11}$	0.0125	0.0001	0.0012	0.0013	0.930
$d\Lambda_{12}$	0.1875	0.0007	0.0189	0.0189	0.918
$d\Lambda_{13}$	0.8125	0.0073	0.0819	0.0818	0.932
$d\Lambda_{14}$	2.1875	0.0180	0.2222	0.2205	0.927
$d\Lambda_{15}$	4.6125	-0.0136	0.4711	0.4641	0.923
α_2	2	0.0159	0.2010	0.2015	0.924
β_{20}	0.5	-0.0034	0.0630	0.0608	0.966
β_{21}	-0.5	-0.0051	0.0818	0.0830	0.952
β_{22}	0.5	-0.0043	0.0612	0.0610	0.967
$d\Lambda_{21}$	0.0333	0.0003	0.0032	0.0034	0.924
$d\Lambda_{22}$	0.2333	0.0000	0.0182	0.0188	0.932
$d\Lambda_{23}$	0.6333	0.0035	0.0654	0.0637	0.922
$d\Lambda_{24}$	1.2333	0.0064	0.1207	0.1241	0.915
$d\Lambda_{25}$	2.0333	0.0120	0.2125	0.2047	0.931

Table 4: Simulation results with $\omega = 1$ and $n = 500$

	Parameter	Bias	SD	SE	CP
ω	1	-0.0004	0.0183	0.0184	0.946
α_1	5	-0.0031	0.2585	0.2503	0.934
β_{10}	0.5	-0.0006	0.0308	0.0303	0.945
β_{11}	0.5	-0.0006	0.0243	0.0243	0.954
β_{12}	-0.5	0.0011	0.0314	0.0330	0.937
$d\Lambda_{11}$	0.0125	0.0000	0.0005	0.0005	0.914
$d\Lambda_{12}$	0.1875	0.0000	0.0074	0.0075	0.902
$d\Lambda_{13}$	0.8125	0.0021	0.0407	0.0407	0.907
$d\Lambda_{14}$	2.1875	0.0063	0.1063	0.1096	0.911
$d\Lambda_{15}$	4.6125	0.0122	0.2317	0.2307	0.907
α_2	2	0.0049	0.0976	0.1000	0.913
β_{20}	0.5	-0.0011	0.0299	0.0304	0.939
β_{21}	-0.5	0.0011	0.0501	0.0496	0.906
β_{22}	0.5	-0.0001	0.0300	0.0304	0.917
$d\Lambda_{21}$	0.0333	0.0000	0.0016	0.0017	0.914
$d\Lambda_{22}$	0.2333	0.0004	0.0092	0.0094	0.940
$d\Lambda_{23}$	0.6333	0.0000	0.0310	0.0317	0.911
$d\Lambda_{24}$	1.2333	0.0007	0.0497	0.0493	0.933
$d\Lambda_{25}$	2.0333	0.0015	0.0848	0.0814	0.922

4.2.2 Results with different ω

In this subsection, we set the parameters $\alpha_1 = 5$, $\alpha_2 = 2$, $\beta_1 = (0.5, 0.5, -0.5)$, $\beta_2 = (0.5, -0.5, 0.5)$, and the sample size $n = 300$. We study the simulation performance with different parameter ω, the selected values of ω are 0.5, 1, 2, and 3. Table 1, table 3, table 5, table 6 show that the bias of estimates are negligible with different ω, the average of standard errors and the standard deviations don't fluctuate obviously, and the coverage probabilities achieve the level we desired. As stated above, we obtain similar results with different ω, which means the bias, average of standard errors, and standard deviations don't change obviously with the ω we chose.

Table 5: Simulation results with $\omega = 2$ and $n = 300$

	Parameter	Bias	SD	SE	CP
ω	2	-0.0002	0.0135	0.0135	0.991
α_1	5	0.0155	0.5053	0.5034	0.917
β_{10}	0.5	-0.0033	0.0619	0.0609	0.971
β_{11}	0.5	-0.0035	0.0600	0.0610	0.975
β_{12}	-0.5	0.0004	0.0862	0.0833	0.946
$d\Lambda_{11}$	0.0125	0.0001	0.0013	0.0013	0.928
$d\Lambda_{12}$	0.1875	0.0017	0.0190	0.0189	0.926
$d\Lambda_{13}$	0.8125	0.0027	0.0809	0.0819	0.911
$d\Lambda_{14}$	2.1875	0.0113	0.2216	0.2206	0.919
$d\Lambda_{15}$	4.6125	0.0320	0.4633	0.4646	0.932
α_2	2	0.0201	0.2085	0.2010	0.928
β_{20}	0.5	-0.0014	0.0608	0.0610	0.979
β_{21}	-0.5	-0.0080	0.0831	0.0830	0.950
β_{22}	0.5	-0.0054	0.0606	0.0609	0.964
$d\Lambda_{21}$	0.0333	0.0002	0.0033	0.0034	0.938
$d\Lambda_{22}$	0.2333	0.0022	0.0242	0.0235	0.911
$d\Lambda_{23}$	0.6333	0.0007	0.0631	0.0638	0.924
$d\Lambda_{24}$	1.2333	0.0033	0.1198	0.1240	0.944
$d\Lambda_{25}$	2.0333	0.0169	0.2225	0.2048	0.926

Table 6: Simulation results with $\omega = 3$ and $n = 300$

	Parameter	**Bias**	**SD**	**SE**	**CP**
ω	3	0.0000	0.0050	0.0049	0.986
α_1	5	0.0274	0.4772	0.5033	0.908
β_{10}	0.5	-0.0038	0.0606	0.0609	0.965
β_{11}	0.5	-0.0034	0.0613	0.0609	0.973
β_{12}	-0.5	0.0004	0.0656	0.0663	0.962
$d\Lambda_{11}$	0.0125	0.0000	0.0013	0.0013	0.924
$d\Lambda_{12}$	0.1875	0.0018	0.0188	0.0188	0.925
$d\Lambda_{13}$	0.8125	0.0018	0.0812	0.0819	0.922
$d\Lambda_{14}$	2.1875	0.0088	0.2315	0.2200	0.934
$d\Lambda_{15}$	4.6125	0.0442	0.4817	0.4656	0.932
α_2	2	0.0256	0.2126	0.2013	0.912
β_{20}	0.5	-0.0019	0.0607	0.0610	0.964
β_{21}	-0.5	-0.0009	0.0825	0.0831	0.940
β_{22}	0.5	-0.0025	0.0595	0.0609	0.966
$d\Lambda_{21}$	0.0333	0.0002	0.0035	0.0034	0.927
$d\Lambda_{22}$	0.2333	0.0010	0.0197	0.0188	0.946
$d\Lambda_{23}$	0.6333	0.0055	0.0654	0.0638	0.931
$d\Lambda_{24}$	1.2333	-0.0012	0.1180	0.1243	0.902
$d\Lambda_{25}$	2.0333	0.0120	0.2064	0.2050	0.922

5 Conclusion

測試用

References

Bahraoui, Z., Bolancé, C., Pelican, E., & Vernic, R. (2015). On the bivariate sarmanov distribution and copula. an application on insurance data using truncated marginal distributions. *Statistics and Operations Research Transactions, SORT*, *39*(2), 209–230.

Bolancé, C., & Vernic, R. (2019). Multivariate count data generalized linear models: Three approaches based on the sarmanov distribution. *Insurance: Mathematics and Economics*, *85*, 89–103.

Kotz, S., Balakrishnan, N., & Johnson, N. L. (2004). *Continuous multivariate distributions, volume 1: Models and applications* (Vol. 1). John Wiley & Sons.

Nelsen, R. B. (2006). *An introduction to copulas*. Springer Science & Business Media.

Figures and Tables