Hidden Markov Models (HMM)

ECE/CS 498 DS U/G

Lecture 14

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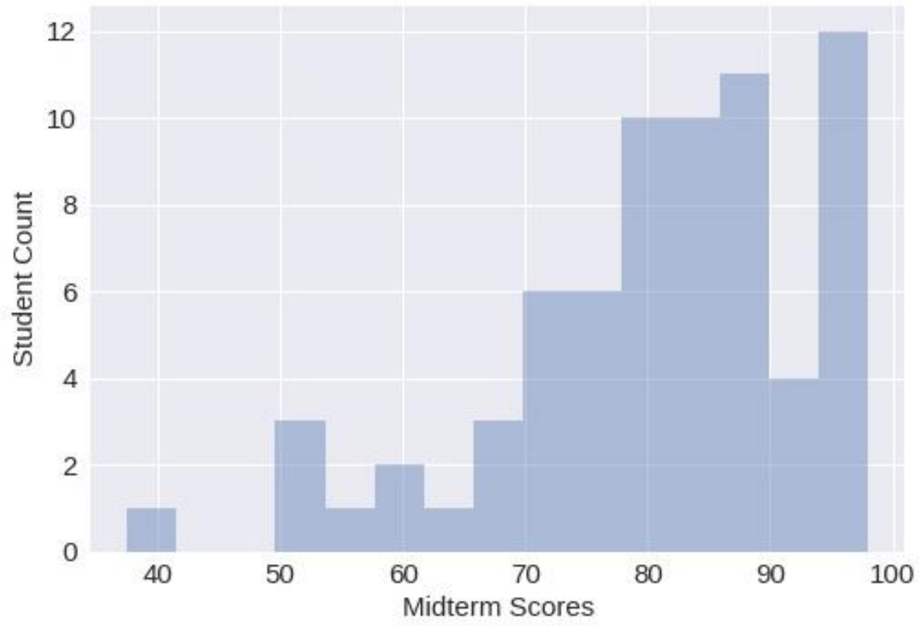
Announcements

- MP2 Checkpoint 3 due on Mar 27
- Midterm grades released on Compass2G
  - Please submit regrade request by Friday, Mar 15 5pm
- No discussion section on Friday, Mar 15
  - Additional office hours

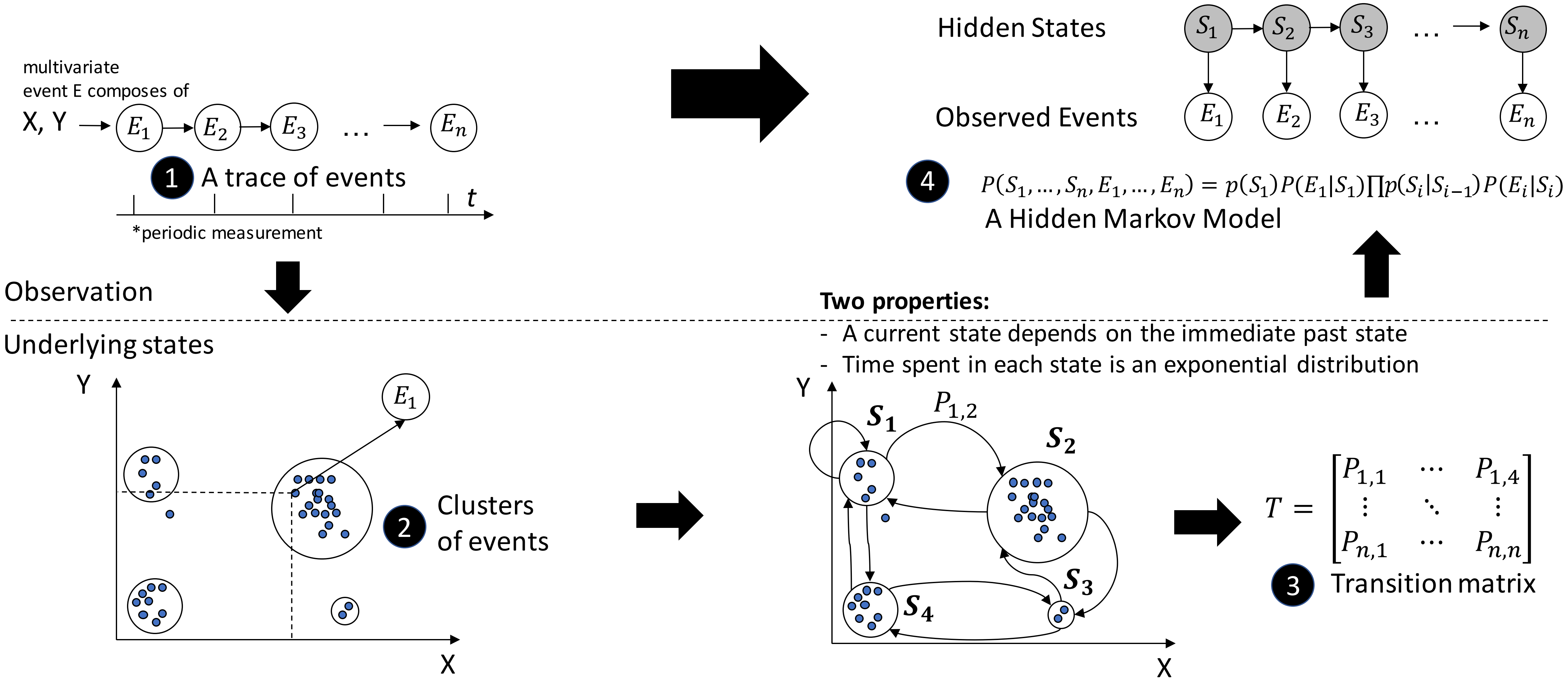


Mean	Std	Median	Max	Min
80.70	12.81	83.25	98.00	37.50

Midterm summary

- Calculating variance: with some students forgetting to take denominator into consideration
- Pay attention to the relative X and Y scale for the PCA problem, and keep in mind what do PC1 and PC2 capture for the data points as an entity
- Make sure that you are familiar with how to project original data points onto the new axis defined by PCA
- Learn how to marginalize and factorize the joint distribution in a Bayes Network
- In Mixture Models, $p(x|c)$ is the probability density function of variable x given source c
- In Problem 5 Bonus question, maximizing L wrt λ_1 required setting dL/d λ_1 = 0 and then using differentiation by parts

From a trace of events to a Hidden Markov Model



Markov Model

- Consider a system which can occupy one of *N* discrete *states* or *categories*

$$x_t \in \{1, 2, \ldots, N\} \longrightarrow \text{state at time } t$$

- We are interested in *stochastic* systems, in which state evolution is random
- Any *joint* distribution can be factored into a series of *conditional* distributions:

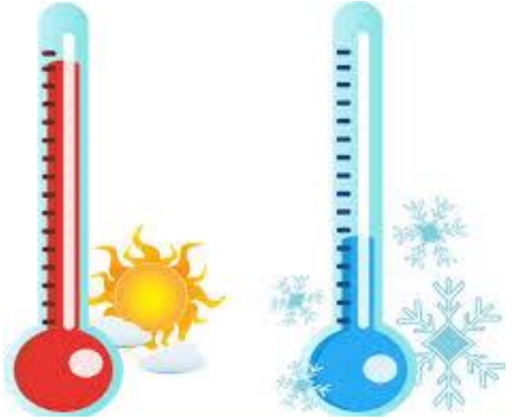
$$p(x_0, x_1, \ldots, x_T) = p(x_0) \prod_{t=1}^{T} p(x_t \mid x_0, \ldots, x_{t-1})$$

- For a *Markov* process, the next state depends only on the current state:

$$p(x_{t+1} \mid x_0, \ldots, x_t) = p(x_{t+1} \mid x_t)$$

HMM Motivating Example: Paleontological Temperature Model

- Want to determine the average temperature at a particular place on earth over a sequence of years in the distant past
- Only annual average temperatures -- hot **(H)** and cold (**C**)
  - Probability of a hot year followed by another hot year is 0.7, and the probability of a cold year followed by another cold year is 0.6, independent of the temperature in prior years
- Correlation between the size of tree growth rings and temperature
  - Three different ring sizes, small (**T**), medium (**D**), and large (**L**)
- Assume that probability values from current period held in paleontological period too
- Determine the most likely temperature state in past years
  - Can't directly observe the temperature in the past
  - We can observe the size of tree rings – can this information be used?



H C

T D L

Tree ring size

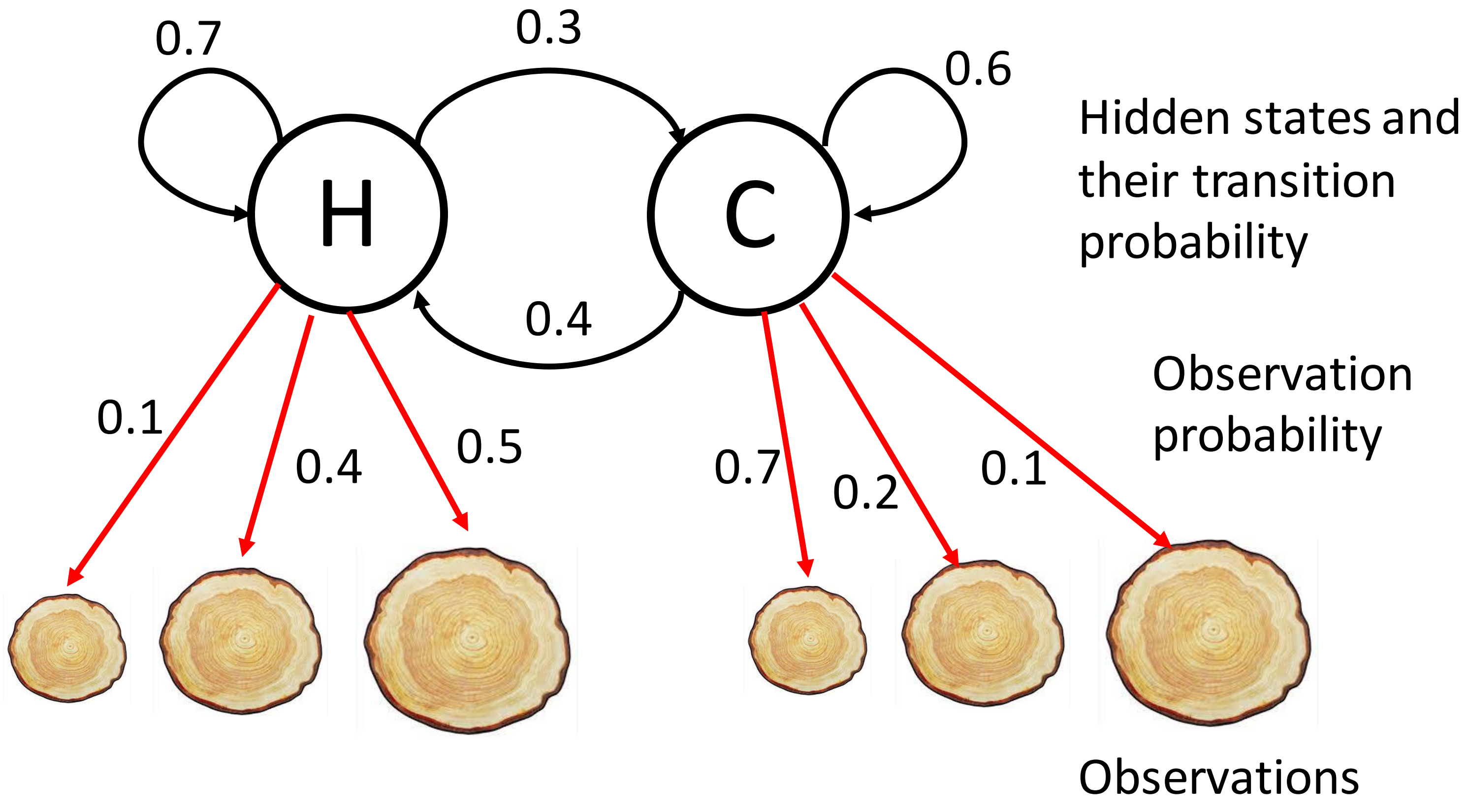
Source: http://www.math.unl.edu/~sdunbar1/ProbabilityTheory/Lessons/HiddenMarkovModels/Examples/examples.html

Paleontological Temperature Model

- State space of hidden states: $S = \{H, C\}$
- State space of observations: $E = \{T, D, L\}$

- Transition probability matrix: A
- Observation Matrix: B
- Initial distribution for the hidden states: π

Given by an oracle



$$\begin{array}{c c} & \begin{matrix} H & C \end{matrix} \\ \begin{matrix} H \\ C \end{matrix} & \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

A

$$\begin{array}{c c} & \begin{matrix} T & D & L \end{matrix} \\ \begin{matrix} H \\ C \end{matrix} & \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.7 & 0.2 & 0.1 \end{bmatrix} \end{array}$$

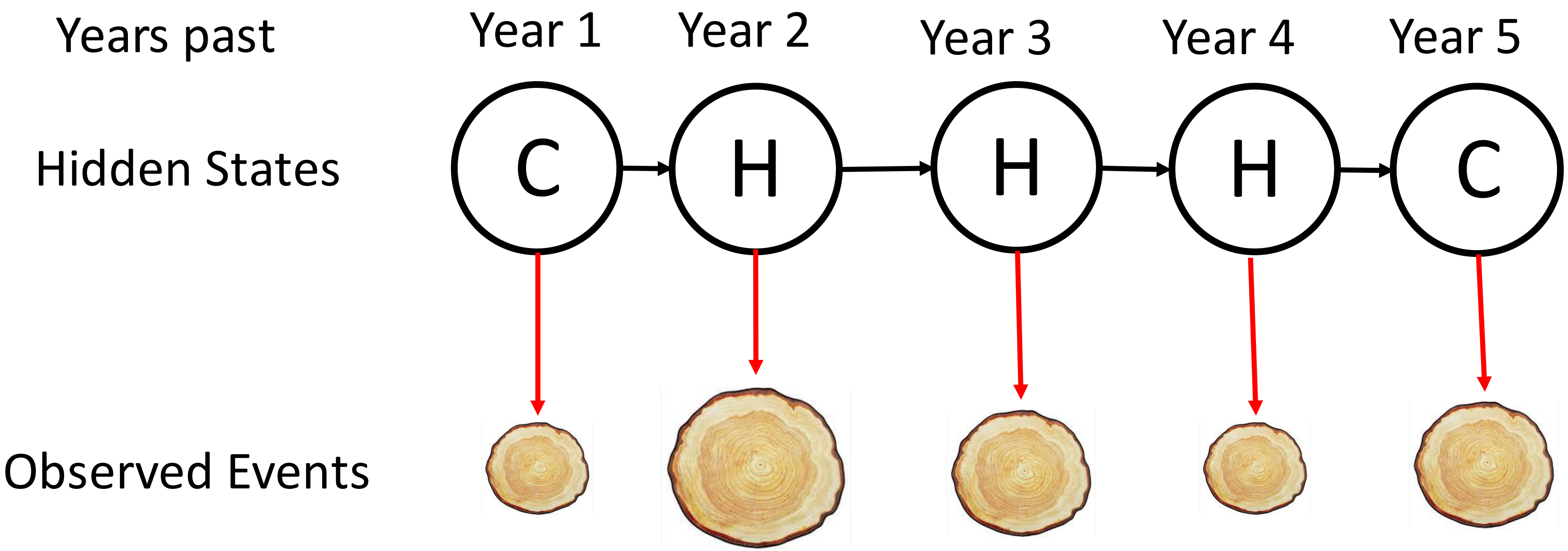
B

$$\begin{array}{c} \begin{matrix} H & C \end{matrix} \\ [0.5 \; 0.5] \end{array}$$

π

Paleontological Temperature Model

Example sequence with 5 observations



Determine the sequence of hidden states

Hidden Markov Models

Model

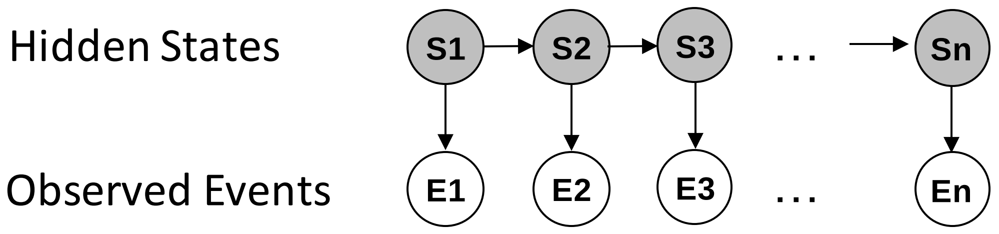
- Set of hidden states $\boldsymbol{S} = \{\boldsymbol{\sigma_1}, \dots, \boldsymbol{\sigma_N}\}$
- Set of observable events $\boldsymbol{E} = \{\boldsymbol{\epsilon_1}, \dots, \boldsymbol{\epsilon_M}\}$
- Transition probability matrix $\boldsymbol{A}$
- Observation matrix $\boldsymbol{B}$
- Initial distribution of hidden states $\boldsymbol{\pi}$

Model assumptions

- An observation depends on its hidden state
- A state variable only depends on the immediate previous state (Markov assumption)
- The future observations and the past observations are conditionally independent given the current hidden state

Advantages:

- HMM can model sequential nature of input data (future depends on the past)
- HMM has a linear-chain structure that clearly separates system state and observed events.



A Hidden Markov model on observed events and system states

$$P(S_1, \dots, S_n, E_1, \dots, E_n) = P(S_1)P(E_1|S_1)\prod_{i=2}^{n} P(S_i|S_{i-1})P(E_i|S_i)$$

Inference question – Paleontological Temperature

Given the sequence of 5 observations T, L, D, T, D and the model (A, B, π), how do we choose a corresponding state sequence $S_1, S_2, \ldots, S_n$ which is optimal in some meaningful sense (i.e., best explains the observations) where $S_t \in \{H, C\}$?

A simpler question: Given the sequence of 5 observations T, L, D, T, D and the model (A, B, π), which of the two is more probable eg., $S_3 = H$ or $S_3 = C$?

General Inference question

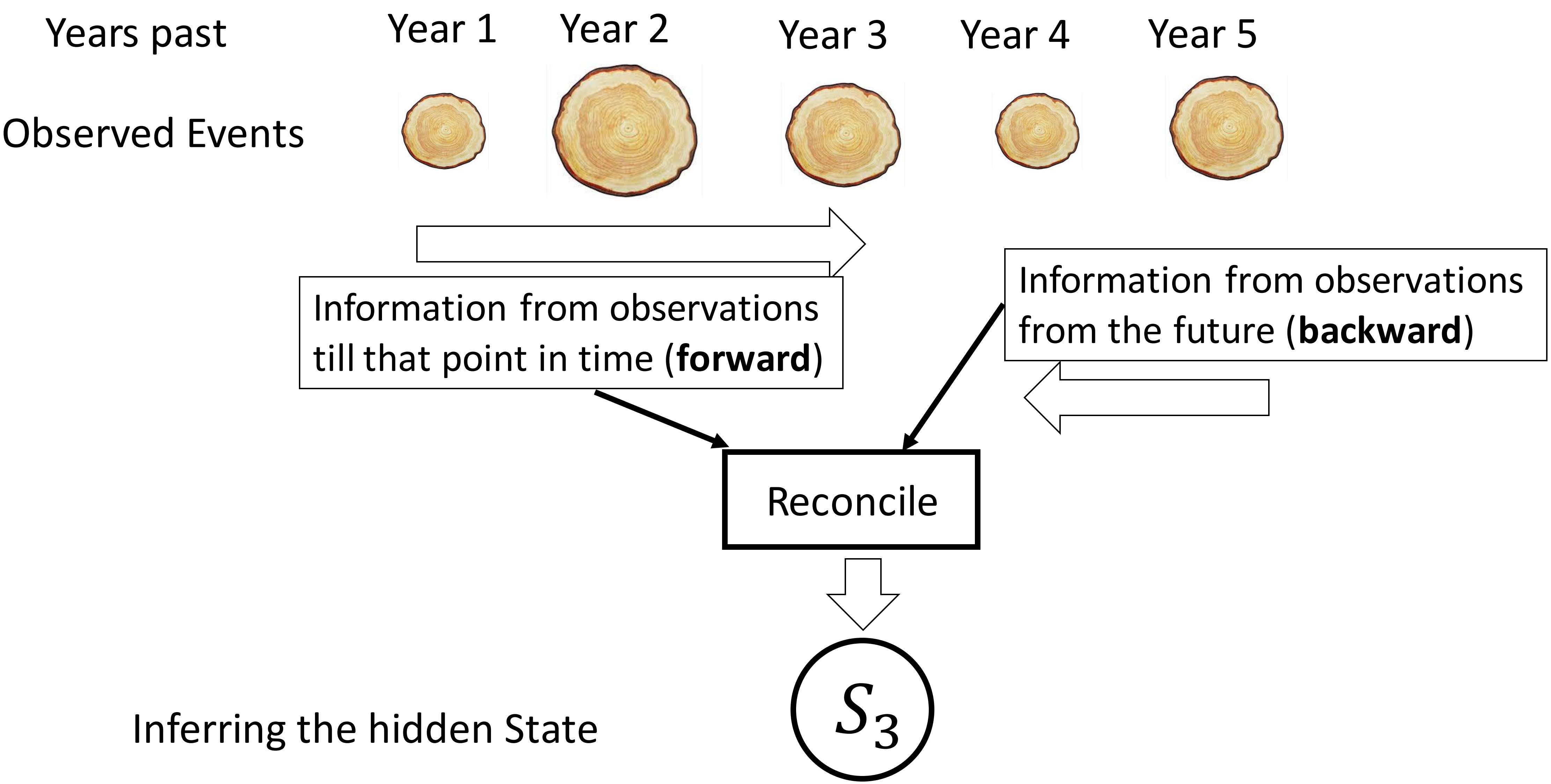
Given the sequence of n observations $E_1, E_2, \dots, E_n$, and the model (A, B, π), how do we choose a corresponding state sequence $S_1, S_2, \dots, S_n$ which is optimal in some meaningful sense (i.e., best explains the observations)?

A simpler question: Given the sequence of n observations $E_1, E_2, \dots, E_n$, and the model (A, B, π), what is the most probable state S_t at $t \in \{1, \dots, n\}$?

$$\underset{j \in \{1,\dots,N\}}{\operatorname{argmax}} P(S_t = \sigma_j | E_1, E_2, \dots, E_n)$$

$$S = \{\sigma_1, \dots, \sigma_N\}$$

Intuition behind solution



Breaking down the inference question

$$P(S_t|E_1,E_2,\dots,E_n) = \frac{P(S_t,E_1,\dots,E_n)}{P(E_1,\dots,E_n)} = \frac{P(S_t,E_1,\dots,E_t,E_{t+1},\dots,E_n)}{P(E_1,\dots,E_n)}$$

Bayes rule

$$= \frac{P(E_{t+1},\dots,E_n\,|S_t,E_1,\dots,E_t)\,P(S_t,E_1,\dots,E_t)}{P(E_1,\dots,E_n)}$$

Bayes rule

$$= P(E_{t+1},\dots,E_n\,|S_t,E_1,\dots,E_t)\,P(S_t|E_1,\dots,E_t)\frac{P(E_1,\dots,E_t)}{P(E_1,\dots,E_n)}$$

Markov property

$$= \frac{P(E_{t+1},\dots,E_n\,|S_t)\,P(S_t|E_1,\dots,E_t)}{P(E_{t+1},\dots,E_n|E_1,\dots,E_t)}$$

Breaking down the inference question

$$P(S_t|E_1, E_2, \dots, E_n) = \frac{P(E_{t+1}, \dots, E_n\ |S_t)\ P(S_t|E_1, \dots, E_t)}{P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)}$$

$P(S_t|E_1, \dots, E_t)$:
Probability of hidden state at time t given observation up to time t (**Forwards algorithm**)

$P(E_{t+1}, \dots, E_n\ |S_t)$:
Probability of the future observed sequence given the hidden state at time t (**Backwards algorithm**)

$P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)$:
Does not depend on the hidden state (will not affect the maximization because it is just a scaling factor)

Forwards algorithm: Paleontological Temperature

Want to calculate $P(S_t|E_1, \dots, E_t)$

- Let us calculate it for $t = 2$
- In the example, $E_1 = T, E_2 = L$
- Find $P(S_2 = H|E_1 = T, E_2 = L)$?

$$P(S_2 = H|E_1 = T, E_2 = L) = \frac{P(S_2 = H, E_1 = T, E_2 = L)}{P(E_1 = T, E_2 = L)}$$

Adding hidden state S_1

$$= \frac{\sum_{s\in\{H,C\}} P(S_2 = H, E_1 = T, E_2 = L, S_1 = s)}{P(E_1 = T, E_2 = L)}$$

Forwards algorithm: Paleontological Temperature

$$\frac{\sum_{s\in\{H,C\}} P(S_2 = H, E_1 = T, E_2 = L, S_1 = s)}{P(E_1 = T, E_2 = L)}$$

Bayes rule

$$= \frac{\sum_{s\in\{H,C\}} P(E_2 = L|S_2 = H, E_1 = T, S_1 = s)P(S_2 = H, E_1 = T, S_1 = s)}{P(E_1 = T, E_2 = L)}$$

Markov property

Bayes rule

$$= \frac{\sum_{s\in\{H,C\}} P(E_2 = L|S_2 = H)P(S_2 = H| E_1 = T, S_1 = s)P(S_1 = s|E_1 = T)P(E_1 = T)}{P(E_1 = T, E_2 = L)}$$

Markov property

Bayes rule

$$= \frac{\sum_{s\in\{H,C\}} P(E_2 = L|S_2 = H)P(S_2 = H| S_1 = s)P(S_1 = s|E_1 = T)}{P(E_2 = L|E_1 = T)}$$

Forwards algorithm: Paleontological Temperature

Hidden state given all observations up to that point

Observation probability

Transition probability

Hidden state given all observations up to that point

$$P(S_2 = H|E_1 = T, E_2 = L) = \frac{P(E_2 = L|S_2 = H)\sum_{s\in\{H,C\}} P(S_2 = H|\, S_1 = s)P(S_1 = s|E_1 = T)}{P(E_2 = L|E_1 = T)}$$

Define: $\alpha_t(i) = P(S_t = \sigma_i|E_1, E_2, \ldots, E_t)$ and $Z_t = P(E_t|E_1, \ldots E_{t-1})$

Above equation can be written as,

$$\boxed{\alpha_2(H)} = \frac{1}{Z_2} P(E_2 = L|S_2 = H) \sum_{s\in\{H,C\}} P(S_2 = H|\, S_1 = s)\boxed{\alpha_1(s)}$$

Where, $\quad Z_2 = \alpha_2(H) + \alpha_2(C)$

Recursion

Forwards algorithm: General Expression

Define: $\alpha_t(j) = P(S_t = \sigma_j | E_1, E_2, \dots, E_t)$ and $Z_t = P(E_t | E_1, \dots E_{t-1})$

In general,

$$\alpha_t(j) = \frac{1}{Z_t} P\big(E_t \big| S_t = \sigma_j\big) \sum_{i=1}^{N} P(S_t = \sigma_j \mid S_{t-1} = \sigma_i)\, \alpha_{t-1}(i) \qquad Z_t = \sum_{j=1}^{N} \alpha_t(j)$$

Transition probability a_{ij}

Above equation can be written as a matrix for all j,

$$\begin{bmatrix} \alpha_t(1) \\ \vdots \\ \alpha_t(j) \\ \vdots \\ \alpha_t(N) \end{bmatrix} \propto \begin{bmatrix} P(E_t|S_t = \sigma_1) \\ \vdots \\ P(E_t|S_t = \sigma_j) \\ \vdots \\ P(E_t|S_t = \sigma_N) \end{bmatrix} \odot \begin{bmatrix} a_{11} & \dots & \dots & \dots & a_{N1} \\ \vdots & \ddots & \dots & \dots & \vdots \\ a_{1j} & \dots & a_{ij} & \dots & a_{Nj} \\ \vdots & \dots & \dots & \ddots & \dots \\ a_{1N} & \dots & \dots & \dots & a_{NN} \end{bmatrix} \begin{bmatrix} \alpha_{t-1}(1) \\ \vdots \\ \alpha_{t-1}(i) \\ \vdots \\ \alpha_{t-1}(N) \end{bmatrix}$$

$\odot$ Represents elementwise product (Hadamard product)

$$\alpha_t \propto b_t \odot (A^T \alpha_{t-1})$$

b_t is the column of the observation matrix B corresponding to E_t

Forwards Algorithm: Paleontological Temperature

For observations T, L, D, T, L

$P(S_2|E_1 = T, E_2 = L)$ is,

$$\begin{bmatrix}\alpha_2(H)\\ \alpha_2(C)\end{bmatrix} \propto \begin{bmatrix}0.5\\ 0.1\end{bmatrix} \odot \left(\begin{bmatrix}0.7 & 0.4\\ 0.3 & 0.6\end{bmatrix}\begin{bmatrix}\alpha_1(H)\\ \alpha_1(C)\end{bmatrix}\right)$$

$$\begin{array}{c c} & \begin{array}{cc} H & C \end{array} \\ \begin{array}{c} H \\ C \end{array} & \begin{bmatrix}0.7 & 0.3\\ 0.4 & 0.6\end{bmatrix}\end{array}$$

Transition probability matrix

$$\begin{array}{c c} & \begin{array}{ccc} T & D & L \end{array} \\ \begin{array}{c} H \\ C \end{array} & \begin{bmatrix}0.1 & 0.4 & 0.5\\ 0.7 & 0.2 & 0.1\end{bmatrix}\end{array}$$

Observation matrix

Similarly, $P(S_3|E_1 = T, E_2 = L,\ E_3 = D)$ is,

$$\begin{bmatrix}\alpha_3(H)\\ \alpha_3(C)\end{bmatrix} \propto \begin{bmatrix}0.4\\ 0.2\end{bmatrix} \odot \left(\begin{bmatrix}0.7 & 0.4\\ 0.3 & 0.6\end{bmatrix}\begin{bmatrix}\alpha_2(H)\\ \alpha_2(C)\end{bmatrix}\right)$$

Forwards Algorithm

1. Input: (A, B, π) and observed sequence $E_1, \ldots, E_n$
2. $[\alpha_1, Z_1]$ = normalize($b_1 \odot \pi$)
3. **for** $t = 2{:}\,n$ **do**
   $[\alpha_t, Z_t]$ = normalize($b_t \odot (A^T \alpha_{t-1})$)
4. return $\alpha_1, \ldots, \alpha_n$ and $\log\big(P(E_1, \ldots, E_n)\big) = \sum_t \log(Z_t)$

5. Subroutine: [v, Z] = normalize(u): $Z = \sum_j u_j$; $v_j = u_j / Z$;

Breaking down the inference question

$$P(S_t|E_1, E_2, \dots, E_n) = \frac{P(E_{t+1}, \dots, E_n\ |S_t)\ P(S_t|E_1, \dots, E_t)}{P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)}$$

$P(S_t|E_1, \dots, E_t)$:
Probability of hidden state at time t given observation up to time t (**Forwards algorithm**)

$P(E_{t+1}, \dots, E_n\ |S_t)$:
Probability of the future observed sequence given the hidden state at time t (**Backwards algorithm**)

$P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)$:
Does not depend on the hidden state (will not affect the maximization because it is just a scaling factor)

Backwards Algorithm (similar to Forwards Algo.)

Calculate $P(E_{t+1},\ldots,E_n \mid S_t)$

Define: $\beta_t(j) = P(E_{t+1},\ldots,E_n \mid S_t = \sigma_j)$

Include S_t to use information from the one-step future

$$\boxed{\beta_{t-1}(j)} = P(E_t,\ldots,E_n \mid S_{t-1} = \sigma_j) = \sum_{i=1}^{N} P(S_t = \sigma_i, E_t,\ldots,E_n \mid S_{t-1} = \sigma_j)$$

Chain rule

$$= \sum_{i=1}^{N} P(E_{t+1},\ldots,E_n \mid S_{t-1} = \sigma_j, S_t = \sigma_i, E_t)\, P(E_t \mid S_{t-1} = \sigma_j, S_t = \sigma_i) P(S_t = \sigma_i \mid S_{t-1} = \sigma_j)$$

Markov property

$$= \sum_{i=1}^{N} P(E_{t+1},\ldots,E_n \mid S_t = \sigma_i)\, P(E_t \mid S_t = \sigma_i) P(S_t = \sigma_i \mid S_{t-1} = \sigma_j)$$

By definition of $\beta_t(j)$

$$= \sum_{i=1}^{N} \boxed{\beta_t(i)} P(E_t \mid S_t = \sigma_i) P(S_t = \sigma_i \mid S_{t-1} = \sigma_j)$$

Emission probability

Transition probability

In matrix form, we get,

$$\beta_{t-1} = A(b_t \odot \beta_t)$$

$$\beta_t = \begin{bmatrix} \beta_t(1) \\ \vdots \\ \beta_t(N) \end{bmatrix}$$

Backwards Algorithm

1. Input: (A, B, π) and observed sequence $E_1, \dots, E_n$
2. $\beta_n = 1$; // initialize $\beta_n(j)$ to 1 for all states σ_j
3. **for** $t = n - 1 : 1$ **do**
   $\beta_{t-1} = A(b_t \odot \beta_t)$
4. return $\beta_1, \dots, \beta_n$

Breaking down the inference question

$$P(S_t|E_1, E_2, \dots, E_n) = \frac{P(E_{t+1}, \dots, E_n\ |S_t)\ P(S_t|E_1, \dots, E_t)}{P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)}$$

$P(S_t|E_1, \dots, E_t)$:
Probability of hidden state at time t given observation up to time t (**Forwards algorithm**)

$P(E_{t+1}, \dots, E_n\ |S_t)$:
Probability of the future observed sequence given the hidden state at time t (**Backwards algorithm**)

$P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)$:
Does not depend on the hidden state (will not affect the maximization because it is just a scaling factor)

Inference – using Forwards-Backwards expressions

$$P(S_t|E_1, E_2, \dots, E_n) = \frac{P(E_{t+1}, \dots, E_n\,|S_t)\,P(S_t|E_1, \dots, E_t)}{P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)}$$

For $S_t = \sigma_j$ and $\gamma_t(j) = P\left(S_t = \sigma_j \middle| E_1, E_2, \dots, E_n\right)$, the above equation is:

$$P\left(S_t = \sigma_j \middle| E_1, E_2, \dots, E_n\right) = \frac{P(E_{t+1}, \dots, E_n\,|S_t = \sigma_j)\,P(S_t = \sigma_j|E_1, \dots, E_t)}{P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)}$$

$$\gamma_t(j) = \frac{\beta_t(j)\alpha_t(j)}{P(E_{t+1}, \dots, E_n|E_1, \dots, E_t)} = \frac{\beta_t(j)\alpha_t(j)}{\sum_{i=1}^{N} \beta_t(j)\alpha_t(j)}$$

Theorem of total probability

$$\boxed{\gamma_t(j) \propto \beta_t(j)\alpha_t(j)}$$

Inference: Most likely state

- Forwards-backwards algorithm gives $P(S_t = \sigma_j | E_1, \dots, E_n)$ for all j
- Find the individually most likely state at time t given all observations

$$S_t^* = \underset{j \in \{1,..,N\}}{\operatorname{argmax}} \gamma_t(j)$$

Optimality of inference

- In the inference problem we attempt to uncover the hidden part of HMM, i.e., find the “correct” state sequence
- It is impossible to find the “correct” state sequence (solution)
- Use optimality criterion to find the “best” possible solution
- Several reasonable criteria exist and is a strong function of the intended application
  - Most likely state given observations
    - Application in finding average statistics, expected number of correct states
    - Solved using Forwards-Backwards algorithm
  - Single best sequence that maximises probability of observed events
    - Application in continuous speech recognition
    - Solved using Viterbi algorithm

HMM Security Example

- Suppose you are a security expert monitoring the NCSA system
- By monitoring the system events, you want to say whether the system is safe or not
  - System's safety is a hidden state
  - Events are observed
  - Events are related to the safety of the system
- Is the system safe?
  - **HMM** to the rescue!

Security Example: Transition Matrix

Transition matrix (A)

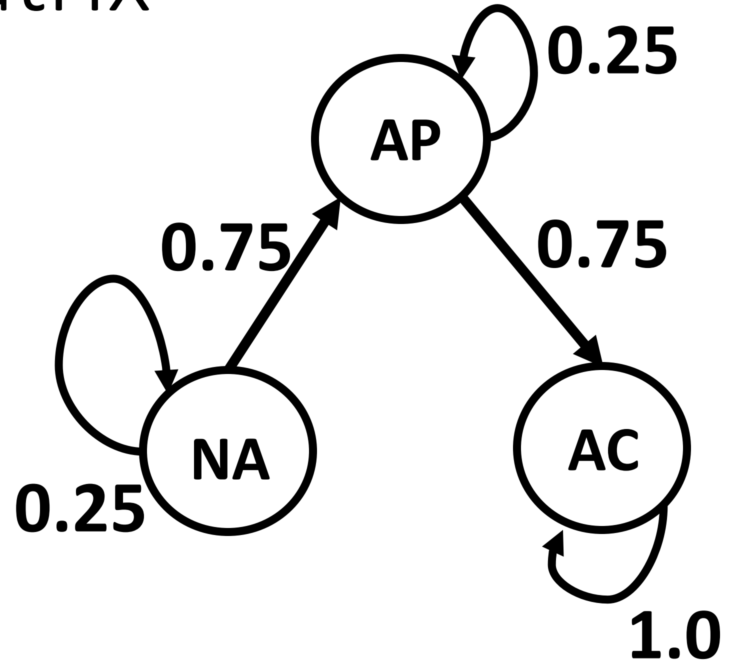
The system has three distinct security states –

(a) No Attack **(NA)**,

(b) Attack in Progress **(AP),** and

(c) Attack Complete **(AC).**

- Every hour, the system is being attacked by attackers coordinating together around the world and trying to compromise the system.
- The system states always transition from **NA to AP** and **AP to AC.**
- An attacker is successful in changing the state of the system with probability of 0.75 and fails with a probability of 0.25.
- If the attack fails, the system stays in its current state.
- If the system state reaches **AC** the attack is complete, and the system stays in that state.



$$A = \begin{array}{c} \\ \textbf{NA} \\ \textbf{AP} \\ \textbf{AC} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{NA} & \textbf{AP} & \textbf{AC} \end{array} \\ \begin{pmatrix} 0.25 & 0.75 & 0 \\ 0 & 0.25 & 0.75 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Transition Probability Matrix

Security Example: Emission matrix and initial distribution

Observation matrix (B)

- Your monitoring system reports two types of events
  - Port Scan **(PS)**
  - Software Installation **(SI)**
- Monitors are always accurate and works. Attackers cannot compromise the monitors. Every hour, we get information from the monitors if the attackers are trying to do **PS or SI.**

$$\mathbf{B} = \begin{array}{c c} & \begin{array}{cc} \textbf{PS} & \textbf{PI} \end{array} \\ \begin{array}{c} \textbf{NA} \\ \textbf{AP} \\ \textbf{AC} \end{array} & \begin{pmatrix} P_{PS|NA} & P_{PI|NA} \\ P_{PS|AP} & P_{PI|AP} \\ P_{PS|AC} & P_{PI|AC} \end{pmatrix} \end{array}$$

Observation Matrix

Initial distribution (π)

- We have no idea about the initial state of the system.

$$\pi_0 = \begin{array}{c} \textbf{NA} \\ \textbf{AP} \\ \textbf{AC} \end{array} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

Initial state distribution/prior

Resources

Rabiner’s (excellent) paper:
https://www.ece.ucsb.edu/Faculty/Rabiner/ece259/Reprints/tutorial%20on%20hmm%20and%20applications.pdf