

# Learning From Data

## Lecture 4: Generative Learning Algorithms

Xiangxiang Xu  
xuxx14@mails.tsinghua.edu.cn

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# Today's Lecture

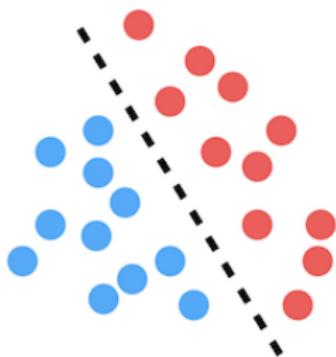
## Supervised Learning (Part II)

- ▶ Discriminative & Generative Models
- ▶ Gaussian Discriminant Analysis
- ▶ Naïve Bayes

## Two Learning Approaches

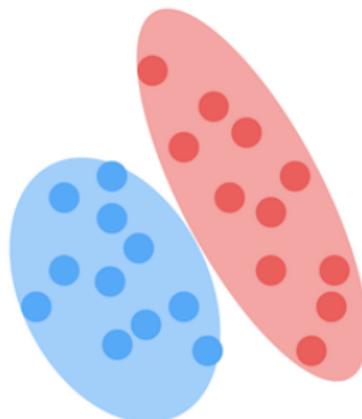
Classify input data  $x$  into two classes  $y \in \{0, 1\}$

**Discriminative**



Discriminate between  
classes of data points

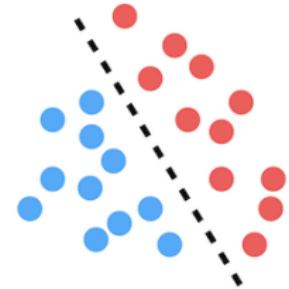
**Generative**



Model the underlying distri-  
bution of the data

## Discriminative Learning Algorithms

A class of learning algorithms that try to learn the **conditional probability**  $p(y|x)$  directly or learn mappings directly from  $\mathcal{X}$  to  $\mathcal{Y}$ .

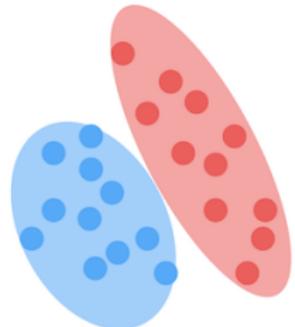


- ▶ e.g. linear regression, logistic regression, k-Nearest Neighbors

...

## Generative Learning Algorithms

A class of learning algorithms that model the **joint probability**  $p(x, y)$ .



- ▶ Equivalently, generative algorithms model  $p(x|y)$  and  $p(y)$
- ▶  $p(y)$  is called the **class prior**
- ▶ Learned models are transformed to  $p(y|x)$  later to classify data using Bayes' rule

## Bayes Rule

The posterior distribution on  $y$  given  $x$ :

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)}$$

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Make predictions in a generative model:

$$\begin{aligned}\operatorname{argmax}_y p(y|x) &= \operatorname{argmax}_y \frac{p(x|y)p(y)}{p(x)} \\ &= \operatorname{argmax}_y p(x|y)p(y)\end{aligned}$$

No need to calculate  $p(x)$ .

# Generative Models

Generative classification algorithms:

- ▶ Continuous input: Gaussian Discriminant Analysis
- ▶ Discrete input: Naïve Bayes

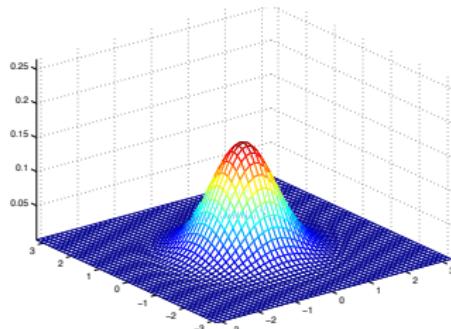
# Multivariate Normal Distribution

**Multivariate normal (or multivariate Gaussian) distribution**  
 $N(\mu, \Sigma)$

- ▶  $\mu \in \mathbb{R}^n$  is the mean vector
- ▶  $\Sigma \in \mathbb{R}^{n \times n}$  is the covariance matrix.  $\Sigma$  is symmetric and PSD

Density function:

$$p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right)$$



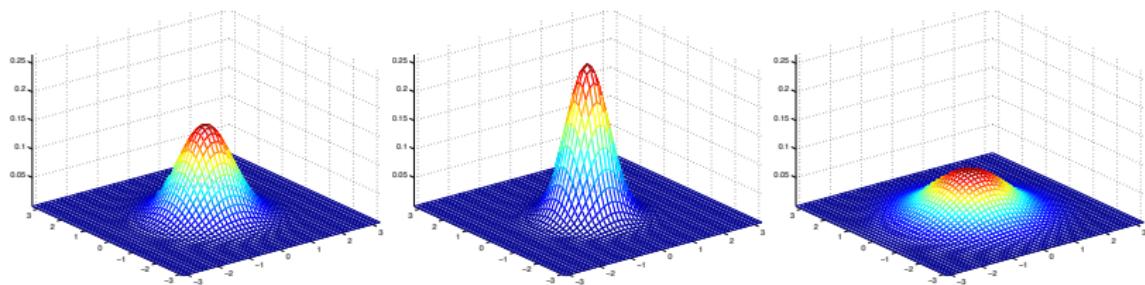
## Multivariate Normal Distribution

Let  $X \in \mathbb{R}^n$  be a random vector. If  $X \sim N(\mu, \Sigma)$ ,

$$\mathbb{E}[X] = \int p(x; \mu, \Sigma) dx = \mu$$

$$\text{Cov}(X) = \mathbb{E} \left[ (X - \mathbb{E}[X])(X - \mathbb{E}[X])^T \right] = \Sigma$$

# Gaussian Discriminative Analysis



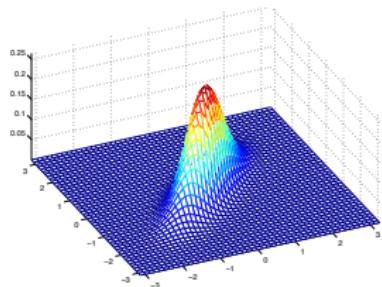
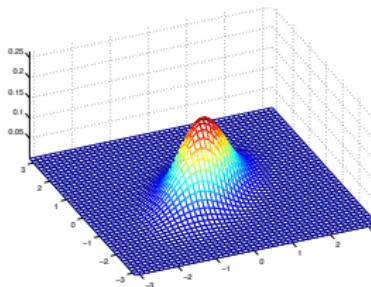
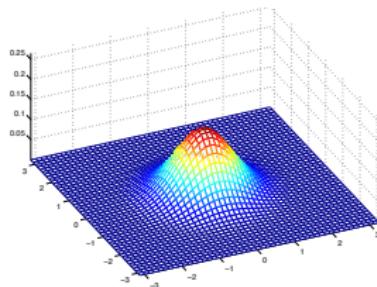
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Diagonal entries of  $\Sigma$  controls the “spread” of the distribution

# Gaussian Discriminative Analysis



$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

The distribution is no longer oriented along the axes when off-diagonal entries of  $\Sigma$  are non-zero.

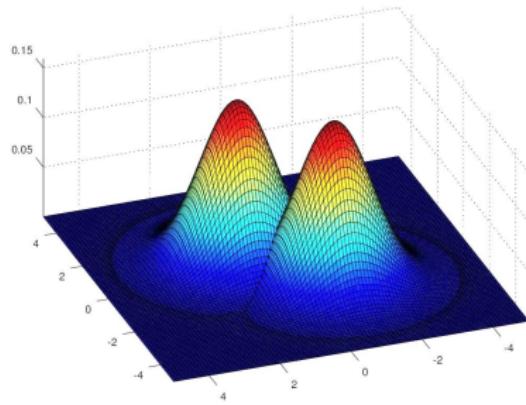
# Gaussian Discriminant Analysis (GDA) Model

Given parameters  $\phi, \mu_0, \mu_1, \Sigma$ ,

$$y \sim \text{Bernoulli}(\phi)$$

$$x|y=0 \sim \mathcal{N}(\mu_0, \Sigma)$$

$$x|y=1 \sim \mathcal{N}(\mu_1, \Sigma)$$



Probability density functions:

$$p(y) = \phi^y(1 - \phi)^{1-y}$$

$$p(x|y=0) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu_0)^T \Sigma^{-1} (x - \mu_0)\right)$$

$$p(x|y=1) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu_1)^T \Sigma^{-1} (x - \mu_1)\right)$$

Log likelihood of the data:

$$\begin{aligned} I(\phi, \mu_0, \mu_1, \Sigma) &= \log \prod_{i=1}^m p(x^{(i)}, y^{(i)}; \phi, \mu_0, \mu_1, \Sigma) \\ &= \log \prod_{i=1}^m p(x^{(i)}|y^{(i)}; \mu_0, \mu_1, \Sigma) p(y^{(i)}; \phi) \end{aligned}$$

Maximum likelihood estimate of the parameters:

$$\phi = \frac{1}{m} \sum_{i=1}^m \mathbf{1}\{y^{(i)} = 1\}$$

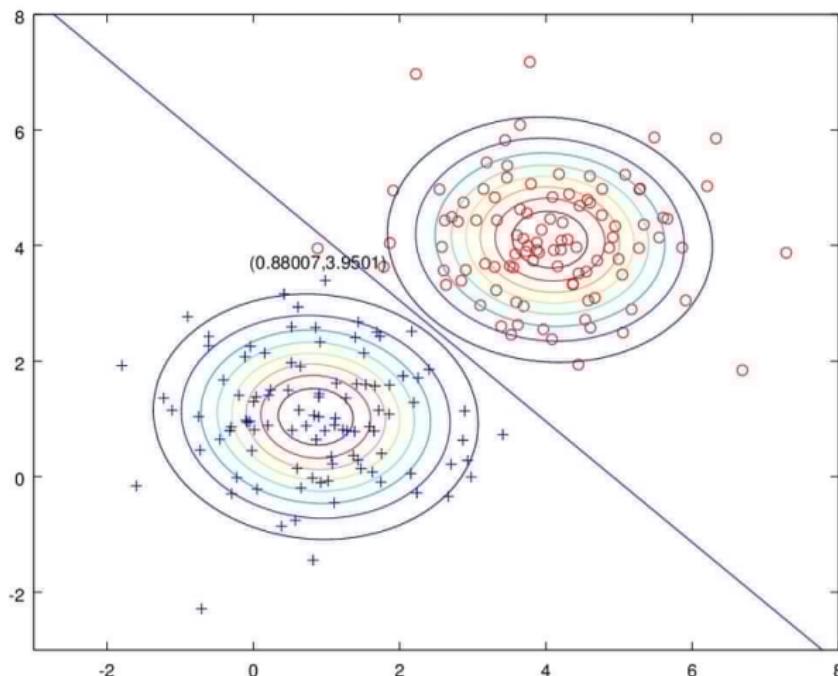
$$\mu_b = \frac{\sum_{i=1}^m \mathbf{1}\{y^{(i)} = b\} x^{(i)}}{\sum_{i=1}^m \mathbf{1}\{y^{(i)} = b\}} \text{ for } b = 0, 1$$

$$\Sigma = \frac{1}{m} \sum_{i=1}^m (x^{(i)} - \mu_{y^{(i)}})(x^{(i)} - \mu_{y^{(i)}})^T$$

## Maximum likelihood estimation of GDA

GDA finds a linear decision boundary at which

$$p(y = 1|x) = p(y = 0|x) = 0.5$$



## GDA and Logistic Regression

$p(y = 1|x; \phi, \mu_0, \mu_1, \Sigma)$  can be written in the form:

$$p(y = 1|x; \phi, \Sigma, \mu_0, \mu_1) = \frac{1}{1 + e^{-\theta^T x}}$$

$$\text{where } \theta = \begin{bmatrix} \log \frac{1-\phi}{\phi} - \frac{1}{2}(\mu_0^T \Sigma^{-1} \mu_0 - \mu_1^T \Sigma^{-1} \mu_1) \\ \Sigma^{-1}(\mu_0 - \mu_1) \end{bmatrix}$$

**Hint:** the canonical link function of logistic regression  
 $y|x \sim \text{Bernoulli}(\psi)$  is  $\log \frac{\psi}{1-\psi}$ ,

$$\theta^T x = \log \frac{\psi}{1-\psi} = \log \frac{p(y=1|x)}{p(y=0|x)} = \log \frac{p(x|y=1)p(y=1)}{p(x|y=0)p(y=0)}$$

If  $p(x|y) \sim \mathcal{N}(\mu, \Sigma)$ ,  $p(y|x)$  is a logistic function.

# GDA and Logistic Regression

## Logistic Regression

- ▶ Maximizes the **conditional likelihood**  $\prod_{i=1}^m p(y^{(i)}|x^{(i)})$
- ▶ Modeling assumptions:  $p(y|x)$  is a logistic function; no restriction on  $p(x)$
- ▶ More robust and less sensitive to incorrect modeling assumptions.

## GDA

- ▶ Maximizes the **joint likelihood**  $\prod_{i=1}^m p(x^{(i)}, y^{(i)})$
- ▶ Modeling assumptions:  $x|y=b \sim \mathcal{N}(\mu_b, \Sigma)$ ,  $y \sim \text{Bernoulli}(\phi)$
- ▶ When modeling assumptions are correct, GDA is more data efficient (i.e., requires less training data to learn “well”)

# Naïve Bayes

A simple generative learning algorithm for discrete input variables

## Example: Spam filter

Classify email messages  $x$  to spam ( $y = 1$ ) and non-spam ( $y = 0$ ) classes.

## Binary text features

Given a dictionary of size  $n$ , represent a message composed of dictionary words as  $x \in \{0, 1\}^n$ :

$$x_i = \begin{cases} 1 & i\text{-th dictionary word is in message} \\ 0 & \text{otherwise} \end{cases}$$

$$x = \begin{bmatrix} 1 & a \\ 0 & aardvark \\ \vdots & \vdots \\ 1 & buy \\ \vdots & \vdots \end{bmatrix}$$

# Naïve Bayes Model

Probability of observing email  $x_1, \dots, x_n$  given spam class  $y$  :

$$p(x_1, \dots, x_n | y) = p(x_1 | y)p(x_2 | y, x_1), \dots, p(x_n | y, x_1, \dots, x_{n-1})$$

Naïve Bayes (NB) assumption

$x_i$ 's are conditionally independent given  $y$ :

$$p(x_i | y, x_1, \dots, x_{i-1}) = p(x_i | y)$$

$$p(x_1, \dots, x_n | y) = p(x_1 | y)p(x_2 | y) \dots p(x_n | y) = \prod_{i=1}^n p(x_i | y)$$

# Naïve Bayes Parameters

Multi-variate Bernoulli event model

$$p(x, y) = p(y)p(x|y) = p(y) \prod_{i=1}^n p(x_i|y)$$

- ▶ Assume email class (spam vs no-spam) is randomly generated with prior  $p(y = 1) = \phi_y$
- ▶ Given  $y$ , each word  $x_i$  is included in the message independently with  $p(x_i = 1|y) = \phi_{i|y}$

Model parameters:

- ▶  $\phi_y = p(y = 1)$
- ▶  $\phi_{i|y=1} = p(x_i = 1|y = 1)$
- ▶  $\phi_{i|y=0} = p(x_i = 1|y = 0)$

## Naïve Bayes Parameter Learning

Likelihood of training data  $(x^{(1)}, y^{(1)}), \dots, (x^{(m)}, y^{(m)})$ :

$$L(\phi_y, \phi_{j|y=0}, \phi_{j|y=1}) = \prod_{i=1}^m p(x^{(i)}, y^{(i)})$$

Maximum likelihood estimation of parameters:

$$\phi_y = \frac{1}{m} \sum_{i=1}^m \mathbf{1}\{y^{(i)} = 1\}$$

$$\phi_{j|y=b} = \frac{\sum_{i=1}^m \mathbf{1}\{x_j^{(i)} = 1, y^{(i)} = b\}}{\sum_{i=1}^m \mathbf{1}\{y^{(i)} = b\}} \text{ for } b = 1, 0$$

## Naïve Bayes Prediction

Given new example with feature  $x$ , compute the posterior probability

$$\begin{aligned} p(y = 1|x) &= \frac{p(x|y = 1)p(y = 1)}{p(x)} \\ &= \frac{p(x|y = 1)p(y = 1)}{p(x|y = 1)p(y = 1) + p(x|y = 0)p(y = 0)} \\ &= \frac{\prod_{i=1}^n p(x_i|y = 1)p(y = 1)}{\prod_{i=1}^n p(x_i|y = 1)p(y = 1) + \prod_{i=1}^n p(x_i|y = 0)p(y = 0)} \end{aligned}$$

Choose label  $y = 1$  if  $p(y = 1|x) > 0.5$

## Laplace smoothing

Issue with Naïve Bayes prediction:

- ▶ Suppose word  $x_j$  hasn't been seen in the training data,  
 $\phi_{j|y=1} = \phi_{j|y=0} = 0$
- ▶ Can not compute class posterior  $p(y = 1|x) = \frac{0}{0}$ .

Let  $z \in \{1, \dots, k\}$  be a multinomial random variable. Given  $m$  independent observations  $z^{(1)} \dots z^{(m)}$ , maximum likelihood estimation of  $\phi_j = p(z = j)$  with **Laplace smoothing** is

$$\phi_j = \frac{\sum_{i=1}^m 1\{z^{(i)} = j\} + 1}{m + k}$$

- ▶  $\phi_j \neq 0$  for all  $j$
- ▶  $\sum_{j=1}^k \phi_j = 1$

## Naïve Bayes with Laplace smoothing

Apply Laplace smoothing to  $\phi_{j|y=b}$  for  $b \in \{0, 1\}$

$$\phi_{j|y=b} = \frac{\sum_{i=1}^m \mathbf{1}\{x_j^{(i)} = 1, y^{(i)} = b\} + 1}{\sum_{i=1}^m \mathbf{1}\{y^{(i)} = b\} + 2}$$

In practice we don't apply Laplace smoothing to  $\phi_y = p(y = 1)$ , which is greater than 0.

# Naïve Bayes and Multinomial Event Model

## Alternative text representation

- ▶  $x_i \in \{1, \dots, |V|\}$  where  $|V|$  is the dictionary size
- ▶ Represent email of  $n$  words as  $x = \{x_1, \dots, x_n\}$

“a free gift ...”  $\rightarrow \{x_1 = 1, x_2 = 1300, x_3 = 2433, \dots\}$

dictionary id	1	2	...	1300	...	2433	...
word	a	aa	...	free	...	gift	...

## Multinomial event model

- ▶ assume spam/non-spam is determined randomly by  $p(y)$
- ▶ Select  $x_1, x_2, \dots, x_n$  independently from the same Multinomial distribution
- ▶ Overall probability:  $p(x_1, \dots, x_n, y) = p(y) \prod_{i=1}^n p(x_i|y)$

## Multinomial event model parameters

Assume  $p(x_j = k|y)$  is the same for all  $j$

- ▶  $\phi_y = p(y = 1)$
- ▶  $\phi_{k|y=1} = p(x_j = k|y = 1)$  for any  $j \in \{1, \dots, n\}$
- ▶  $\phi_{k|y=0} = p(x_j = k|y = 0)$  for any  $j \in \{1, \dots, n\}$

Likelihood of training set  $(x^{(1)}, y^{(1)}), \dots, (x^{(m)}, y^{(m)})$ :

$$\begin{aligned} L(\phi_y, \phi_{k|y=0}, \phi_{k|y=1}) &= \prod_{i=1}^m p(x^{(i)}, y^{(i)}) \\ &= \prod_{i=1}^m p(y^{(i)}; \phi_y) \prod_{j=1}^{n_i} p(x_j^{(i)} | y^{(i)}; \phi_{k|y=0}, \phi_{k|y=1}) \end{aligned}$$

where  $n_i$  is the # words in the  $i$ -th email.

## Maximum likelihood estimation with Laplace smoothing

- ▶  $\phi_y = \frac{1}{m} \sum_{i=1}^m \mathbf{1}\{y^{(i)} = 1\}$
- ▶  $\phi_{k|y=1} = \frac{\sum_{i=1}^m \sum_{j=1}^{n_i} \mathbf{1}\{x_j^{(i)} = k, y^{(i)} = 1\} + 1}{\sum_{i=1}^m \mathbf{1}\{y^{(i)} = 1\} n_i + |V|}$
- ▶  $\phi_{k|y=0} = \frac{\sum_{i=1}^m \sum_{j=1}^{n_i} \mathbf{1}\{x_j^{(i)} = k, y^{(i)} = 0\} + 1}{\sum_{i=1}^m \mathbf{1}\{y^{(i)} = 0\} n_i + |V|}$