Problem Set 1

Issued: Monday 14th September, 2020 **Due:** Monday 21st September, 2020

Notations: We use $\mathsf{x}, \mathsf{y}, \mathsf{w}$ and $\underline{\mathsf{x}}, \underline{\mathsf{y}}, \underline{\mathsf{w}}$ to denote random variables and random vectors.

1.1. *Mathematical Expectation, Variance, and Covariance Matrix* Prove the following properties, where $\underline{\mathsf{x}}$ is a random vector in $\mathbb{R}^k$.

(a) i. $\mathbb{E}[\mathsf{x}|\mathsf{y}] = \mathbb{E}[\mathbb{E}[\mathsf{x}|\mathsf{yz}]|\mathsf{y}]$.
  ii. $\mathbb{E}[\mathsf{x}g(\mathsf{y})|\mathsf{y}] = g(\mathsf{y})\,\mathbb{E}[\mathsf{x}|\mathsf{y}]$, for all functions $g\colon \mathcal{Y} \to \mathbb{R}$.
  iii. $\mathbb{E}[\mathsf{x}\,\mathbb{E}[\mathsf{x}|\mathsf{y}]] = \mathbb{E}[(\mathbb{E}[\mathsf{x}|\mathsf{y}])^2]$.
  iv. $\mathrm{var}(\mathsf{x}) = \mathbb{E}[\mathrm{var}(\mathsf{x}|\mathsf{y})] + \mathrm{var}(\mathbb{E}[\mathsf{x}|\mathsf{y}])$.

(b) i. $\mathrm{cov}(\underline{\mathsf{x}}) = \mathbb{E}[\mathrm{cov}(\underline{\mathsf{x}}|\mathsf{y})] + \mathrm{cov}(\mathbb{E}[\underline{\mathsf{x}}|\mathsf{y}])$.
  ii. $\det(\mathrm{cov}(\underline{\mathsf{x}})) = 0 \iff \exists\, \underline{c} \in \mathbb{R}^k \setminus \{\underline{0}\}$, such that $\underline{c}^{\mathrm{T}}\underline{\mathsf{x}}$ is a constant.

Solution: The former 3 can be verified by definition. On variance or covariance, let's take variance as an example: $\mathbb{E}[\mathrm{var}(\mathsf{x}|\mathsf{y})] + \mathrm{var}(\mathbb{E}[\mathsf{x}|\mathsf{y}]) = \mathbb{E}[\mathbb{E}[\mathsf{x}^2|\mathsf{y}] - \mathbb{E}^2[\mathsf{x}|\mathsf{y}]] + \mathbb{E}[\mathbb{E}^2[\mathsf{x}|\mathsf{y}]] - \mathbb{E}^2[\mathbb{E}[\mathsf{x}|\mathsf{y}]] = \mathbb{E}[\mathbb{E}[\mathsf{x}^2|\mathsf{y}]] - \mathbb{E}^2[\mathbb{E}[\mathsf{x}|\mathsf{y}]] = \mathbb{E}[\mathsf{x}^2] - \mathbb{E}^2[\mathsf{x}] = \mathrm{var}(\mathsf{x})$.

To obtain the last property, note the following facts:

- $\det(\mathrm{cov}(\underline{\mathsf{x}})) = 0 \iff \exists\, \underline{c} \in \mathbb{R}^k \setminus \{\underline{0}\}$, such that $\mathrm{cov}(\underline{\mathsf{x}})\underline{c} = \underline{0}$.
- $\underline{c}^{\mathrm{T}} \mathrm{cov}(\underline{\mathsf{x}})\underline{c} = 0 \iff \mathrm{cov}(\underline{\mathsf{x}})\underline{c} = \underline{0}$. To obtain "$\Rightarrow$", note that since $\mathrm{cov}(\underline{\mathsf{x}})$ is PSD, we can find a PSD matrix $\mathbf{A}$, such that $\mathbf{A}^2 = \mathrm{cov}(\underline{\mathsf{x}})$. Therefore, we have
$$\underline{c}^{\mathrm{T}} \mathrm{cov}(\underline{\mathsf{x}})\underline{c} = 0 \implies \|\mathbf{A}\underline{c}\|^2 = 0 \implies \mathbf{A}\underline{c} = \underline{0} \implies \mathrm{cov}(\underline{\mathsf{x}})\underline{c} = \mathbf{A}^2\underline{c} = \underline{0}.$$
- $\underline{c}^{\mathrm{T}} \mathrm{cov}(\underline{\mathsf{x}})\underline{c} = 0 \iff \underline{c}^{\mathrm{T}}\underline{\mathsf{x}}$ is a constant (with probability one). This can be obtained by noting that
$$\mathrm{var}(\underline{c}^{\mathrm{T}}\underline{\mathsf{x}}) = \underline{c}^{\mathrm{T}} \mathrm{cov}(\underline{\mathsf{x}})\underline{c}.$$

1.2. The Pearson correlation coefficient $\rho(\mathsf{x}, \mathsf{y})$ of two random variables x and y is defined as
$$\rho(\mathsf{x}, \mathsf{y}) \triangleq \frac{\mathbb{E}[(\mathsf{x} - \mathbb{E}[\mathsf{x}])(\mathsf{y} - \mathbb{E}[\mathsf{y}])]}{\sqrt{\mathrm{var}(\mathsf{x})\,\mathrm{var}(\mathsf{y})}}. \tag{1}$$

(a) When $\mathrm{var}(\mathsf{x}) = \mathrm{var}(\mathsf{y})$, prove that $\rho = a^*$ where a^* is the coefficient in the linear regression problem:
$$(a^*, b^*) \triangleq \operatorname*{arg\,min}_{(a,b)\in\mathbb{R}^2} \mathbb{E}[(\mathsf{y} - a\mathsf{x} - b)^2].$$

(b) Prove that

$$\mathsf{x} \perp\!\!\!\perp \mathsf{y} \iff \forall f, g,\ \rho(f(\mathsf{x}), g(\mathsf{y})) = 0.$$

Solution:

(a) Compute (a^*, b^*) via

$$\frac{\partial}{\partial a}\mathbb{E}[(\mathsf{y} - a\mathsf{x} - b)^2] = \frac{\partial}{\partial b}\mathbb{E}[(\mathsf{y} - a\mathsf{x} - b)^2] = 0.$$

(b) "$\Rightarrow$" is obvious. To obtain "$\Leftarrow$", consider $f(x) = \mathbb{1}_{\{x=x_0\}}$ and $g(y) = \mathbb{1}_{\{y=y_0\}}$ for all possible choices of (x_0, y_0).

1.3. Suppose x has a normal distribution $\mathsf{x} \sim \mathcal{N}(0,1)$. Please compute the density of $\exp(\mathsf{x})$. (The answer is called the **lognormal distribution**.)

Solution: Let $\mathsf{y} = \exp(\mathsf{x})$. Since the exponential function is monotonic

$$f_{\mathsf{y}}(y) = \frac{\mathrm{d}}{\mathrm{d}y}(1 - Q(\ln y)) = -Q'(\ln y)\frac{1}{y} = \frac{1}{\sqrt{2\pi}y}\exp(-\frac{\ln^2 y}{2}), y > 0$$