大家好，这篇是有关台大机器学习课程作业七的详解。

我的github地址：
https://github.com/Doraemonzzz

个人主页：
http://doraemonzzz.com/

作业地址:
https://www.csie.ntu.edu.tw/~htlin/course/ml15fall/

参考资料:
https://blog.csdn.net/a1015553840/article/details/51085129
http://www.vynguyen.net/category/study/machine-learning/page/6/
http://book.caltech.edu/bookforum/index.php
http://beader.me/mlnotebook/
https://blog.csdn.net/qian1122221/article/details/50130093
https://acecoooool.github.io/blog/

Problem 1

$$
\begin{aligned}
1-\mu_{+}^{2}-\mu_{-}^{2} &= 1-\mu_{+}^{2}-(1-\mu_{+})^{2} \\
&= 1-\mu_{+}^{2}-\mu_{+}^{2}+2\mu_{+}-1 \\
&= -2\mu_{+}^{2}+2\mu_{+} \\
&= -2(\mu_{+}-\frac{1}{2})^{2}+\frac{1}{2}
\end{aligned}
$$

因为$\mu_{+}\in[0,1]$，所以$1-\mu_{+}^{2}-\mu_{-}^{2}\in[0,\frac{1}{2}]$，最大值为$\frac{1}{2}$

Problem 2

$$
\begin{aligned}
\mu_{+}(1-(\mu_{+}-\mu_{-}))^{2}+\mu_{-}(-1-(\mu_{+}-\mu_{-}))^{2} &= \mu_{+}[1-2(\mu_{+}-\mu_{-})+(\mu_{+}-\mu_{-})^{2}]+\mu_{-}[1+2(\mu_{+}-\mu_{-})+(\mu_{+}-\mu_{-})^{2}] \\
&= \mu_{+}+\mu_{-}+(\mu_{+}+\mu_{-})(\mu_{+}-\mu_{-})^{2}-2(\mu_{+}-\mu_{-})(\mu_{+}-\mu_{-}) \\
&= 1+(\mu_{+}-\mu_{-})^{2}-2(\mu_{+}-\mu_{-})^{2} \\
&= 1-(\mu_{+}-\mu_{-})^{2} \\
&= 1-(2\mu_{+}-1)^{2} \\
&= 4\mu_{+}-4\mu_{+}^{2}
\end{aligned}
$$

根据Problem 1以及正规化错误的定义可知

$$
\text{normalized Gini index} = (-2\mu_{+}^{2}+2\mu_{+})/(\frac{1}{2}) = 4\mu_{+}-4\mu_{+}^{2}
$$

所以

$$
\text{normalized Gini index} = \text{normalized squared regression error}
$$

Problem 3

回顾课件可知，一个数据不被选择的概率为

$$(1-\frac{1}{N})^{N'}=(1-\frac{1}{N})^{pN}=[(1-\frac{1}{N})^{N}]^{p}\approx e^{-p}$$

因为一共有N组数据，所以没有被选择的数据数量大概为

$$e^{-p}N$$

Problem 4

根据下一题可知

$$E_{out}(G)\le\frac{2}{3+1}(0.15+0.25+0.35)=0.375$$

显然

$$E_{out}(G)\ge 0$$

所以

$$E_{out}(G)\in[0,0.375)$$

Problem 5

根据Random Forest的算法，我们知道如果要把一个点误分，那么K个binary classification trees中必然至少要$\frac{K+1}{2}$个分类器犯错，我们知道一共有$\sum_{k=1}^{K}e_k$个错误，所以$E_{out}(G)$最多为

$$\sum_{k=1}^{K}e_k/\frac{K+1}{2}=\frac{2}{K+1}\sum_{k=1}^{K}e_k$$

Problem 6

计算公式为

$$U_{t+1}=2U_t\sqrt{\epsilon_t(1-\epsilon_t)}$$

具体的推导过程可以看作业6的22题，这里直接带入

$$U_3=2U_2\sqrt{\epsilon_2(1-\epsilon_2)}=4U_1\sqrt{\epsilon_2(1-\epsilon_2)}\sqrt{\epsilon_1(1-\epsilon_1)}$$

注意$(u_1,\ldots,u_N)=(\frac{1}{N},\ldots,\frac{1}{N})$，所以$U_1=1$

$$U_3 = 2U_2\sqrt{\epsilon_2(1-\epsilon_2)} = 4U_1\sqrt{\epsilon_2(1-\epsilon_2)}\sqrt{\epsilon_1(1-\epsilon_1)} = 4\sqrt{\epsilon_2(1-\epsilon_2)}\sqrt{\epsilon_1(1-\epsilon_1)}$$

Problem 7

结合题目以及课件17页可知

$$\eta\text{为使得}\frac{1}{N}\sum_{n=1}^{N}\Big((y_n - s_n) - \eta g_1(x_n)\Big)^2\text{最小的值}$$

对上式关于η求偏导可得

$$-\frac{2}{N}\sum_{n=1}^{N} g_1(x_n)\Big((y_n - s_n) - \eta g_1(x_n)\Big) = 0$$

此处$g_1(x_n) = 2, s_n = 0$带入可得

$$\sum_{n=1}^{N} 2\Big((y_n - 0) - 2\eta\Big) = 0$$

$$\eta = \frac{1}{2N}\sum_{n=1}^{N} y_n$$

由于更新规则为$\alpha_1 = \eta$，$s_n = \alpha_1 g_1(x_n)$，所以

$$s_n = \alpha_1 g_1(x_n) = \frac{1}{2N}\sum_{n=1}^{N} y_n \times 2 = \frac{1}{N}\sum_{n=1}^{N} y_n$$

Problem 8

回顾课件19页可知

$$\alpha_t = \eta = \frac{\sum_{n=1}^{N} g_t(x_n)(y_n - s_n)}{\sum_{n=1}^{N} g_t^2(x_n)}$$

$$\sum_{n=1}^{N} g_t(x_n)(y_n - s_n) = \alpha_t \sum_{n=1}^{N} g_t^2(x_n)$$

$$\sum_{n=1}^{N} g_t(x_n) s_n = \sum_{n=1}^{N} g_t(x_n) y_n - \alpha_t \sum_{n=1}^{N} g_t^2(x_n)$$

Problem 9

OR运算的特点是只有当每个值都为False，结果才为False，结合这个特点可以取

$$w_0 = d - \frac{1}{2}, w_i = 1 (i = 1, \ldots, d)$$

Problem 10

由Problem 21，D的最小值为5，具体过程见Problem 21

Problem 11

初始的$w_{ij}^{(l)}$都为0，所以前向传播之后$s_j^{(l)}=0,(l=1,\ldots,L)$

回顾反向传播的更新规则

$$\frac{\partial e_n}{\partial w_{ij}^{(l)}}=\delta_j^{(l)}\left(x_i^{(l-1)}\right)$$
$$\delta_j^{(L)}=-2\left(y_n-s_j^{(L)}\right)$$
$$\delta_j^{(l)}=\sum_k\left(\delta_k^{(l+1)}\right)\left(w_{jk}^{(l+1)}\right)\left(\tanh'(s_j^{(l)})(l=0,\ldots,L-1)\right.$$

所以根据上式，

$$\delta_j^{(l)}=0,(l=0,\ldots,L-1)$$

因为原始的$w_{ij}^{(l)}$都为0，所以根据更新规则

$$w_{ij}^{(l)}=w_{ij}^{(l)}-\eta\delta_j^{(l)}x_i^{(l-1)}$$

可知

$$w_{ij}^{(l)}=0(l=0,\ldots,L-1)$$

Problem 12

初始的$w_{ij}^{(l)}$都为1，假设输入为$x_1,\ldots,x_d$，偏移项为$x_0=1$，那么

$$s_i^{(1)}=\sum_{i=0}^d w_{ij}^{(l)}x_i=\sum_{i=0}^d x_i$$

说明第一个隐藏层的$s_i^{(1)}$都相等，根据递推公式

$$x_i^{(l-1)}=\tanh(s_i^{(l)})$$
$$s_i^{(l)}=\sum_{i=0}^{d^{(l-1)}} w_{ij}^{(l)}x_i^{(l-1)}$$

我们知道每个隐藏层的$s_i^{(l)},x_i^{(l)}$都相等，从而$\delta_j^{(L)}=-2\left(y_n-s_j^{(L)}\right)$都相等，根据反向传播的更新公式

$$\delta_j^{(l)}=\sum_k\left(\delta_k^{(l+1)}\right)\left(w_{jk}^{(l+1)}\right)\left(\tanh'(s_j^{(l)})\right)$$

可得，对于固定的l，$\delta_j^{(l)}$都相等，特别的，第一层的$\delta_j^{(1)}$都相等，根据更新规则

$$w_{ij}^{(1)}=w_{ij}^{(1)}-\eta\delta_j^{(1)}x_i^{(0)}$$

以及初始的$w_{ij}^{(l)}$都为1可得

$$w_{ij}^{(1)}=w_{i(j+1)}^{(1)}$$

Problem 13

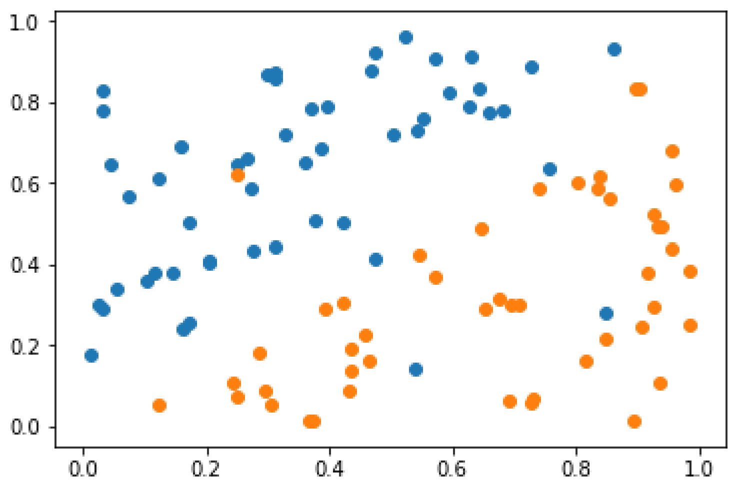
略过

Problem 14

```
import numpy as np
import matplotlib.pyplot as plt

#构造类
class DTree:
    def __init__(self, node, theta, d, left, right):
        self.node = node
        #阈值
        self.theta = theta
        #维度
        self.d = d
        self.left = left
        self.right = right

    #判断是否都为一类
    def ispure(self):
        num = np.sum(self.node[:, 2] == 1)
        return num == 0 or num == len(self.node)

#读取数据
def readdata(file):
    Data = []
    with open(file) as data:
        for i in data.readlines():
            i.strip()
            Data.append(list(map(float,i.split())))
    return np.array(Data)
```

读取数据并作图。

```
train = readdata('hw7_train.dat')
test = readdata('hw7_test.dat')

#作图
plt.scatter(train[:, 0][train[:, 2] == -1], train[:, 1][train[:, 2] == -1])
plt.scatter(train[:, 0][train[:, 2] == 1], train[:, 1][train[:, 2] == 1])
plt.show()
```



定义Gini index

```
def Gini(y):
    N = len(y)
    if(N == 0):
        return 1
    t = np.sum(y == -1)/ N
    return 1 - t**2 - (1 - t)**2
```

定义impurty

```
def lossfunc(theta, data, d):
    '''
    d为数据的维度，theta为decision stump的阈值
    '''
    index1 = (data[:, d] < theta)
    index2 = (data[:, d] >= theta)
    Gini1 = Gini(data[index1][:, 2])
    Gini2 = Gini(data[index2][:, 2])
    return len(index1) * Gini1 + len(index2) * Gini2
```

在两个维度上分别利用decision stump计算，找到损失函数的最小值，返回维度以及阈值

```
def branch(data):
    '''
    在两个维度上分别利用decision stump计算，找到损失函数的最小值，返回维度以及阈值
    '''
```

```
    train = data
    #记录最优阈值以及损失函数的最小值以及维度
    theta = 0
    error = 10000
    d = 0

    #根据第一个维度
    train = np.array(sorted(train, key = lambda x: x[0]))
    #计算decision stump的阈值
    segmentx = train[:, 0]
    #
    for i in segmentx:
        error1 = lossfunc(i, train, 0)
        if error1 < error:
            error = error1
            theta = i

    #根据第二个维度排序
    train = np.array(sorted(train, key = lambda x: x[1]))
    #计算decision stump的阈值
    segmenty = train[:, 1]
    for i in segmenty:
        error2 = lossfunc(i, train, 1)
        if error2 < error:
            error = error2
            theta = i
            d = 1
    return theta, d
```

构造学习函数

```
def isstop(data):
    '''
    判断是否停止，有两种情形，一种是没有数据，另一种是所有数据都为一类
    '''
    n = len(data)
    num = np.sum(data[:, 2] == -1)
    return num == n or num == 0

def learntree(data):
    if isstop(data):
        return DTree(data[0][2], 0, 0, None, None)
    else:
        theta, d = branch(data)
        tree = DTree(None, theta, d, None, None)
        #划分数据
        leftdata = data[data[:, d] < theta]
        rightdata = data[data[:, d] >= theta]
        #学习左树
        leftTree = learntree(leftdata)
        #学习右树

        rightTree = learntree(rightdata)
```

```
        #返回
        tree.left = leftTree
        tree.right = rightTree
        return tree
```

预测函数

```
def pred(tree, data):
    if tree.left == None and tree.right == None:
        return tree.node
    if data[tree.d] < tree.theta:
        return pred(tree.left, data)
    else:
        return pred(tree.right, data)
```

计算误差

```
def error(Dtree, data):
    ypred = [pred(Dtree, i) for i in data]
    return 1 - np.sum(ypred == data[:, 2]) / len(data)
```

训练数据

```
dtree = learntree(train)
```

```
error(dtree, train)
```

```

```

Problem 15

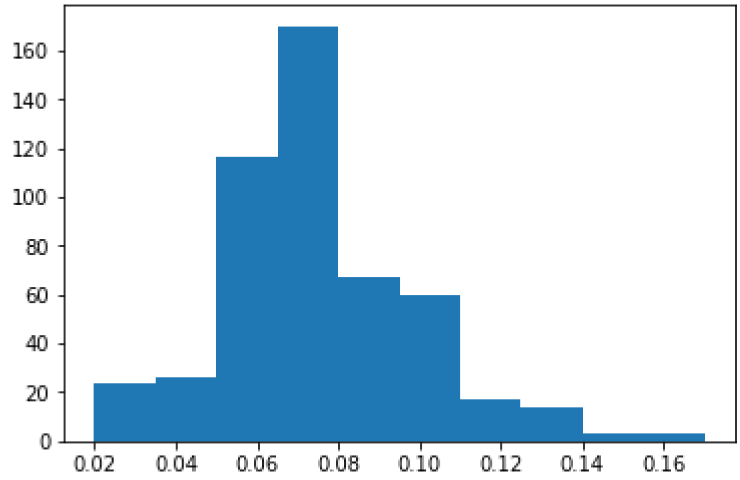
```
error(dtree, test)
```

```
0.14800000000000002
```

Problem 16

求出30000棵树对应的$E_{in}(g_t)$，为了减少运算量，这里取500棵。

```
N = 500
Ein = np.array([])
tree = []
m, n = train.shape
for i in range(N):
    index = np.random.randint(0, m, (m))
    traindata = train[index, :]
    dtree = learntree(traindata)
    tree.append(dtree)
    Ein = np.append(Ein, error(dtree, train))

plt.hist(Ein)
plt.show()
```



Problem 17

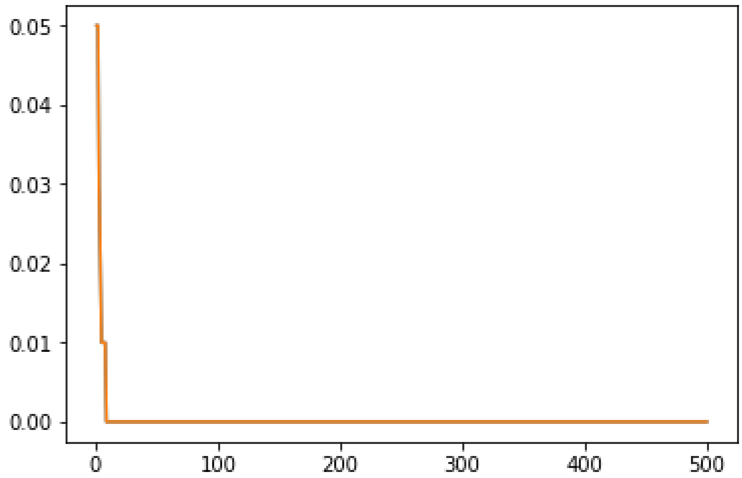
每次取前t棵数构成随机森林，计算结果并作图。

```
def random_forest_error(tree, data):
    '''
    利用前k个树计算结果
    '''
    Error = np.array([])
    N = len(tree)
    for i in range(N):
        E = []
        for j in range(1+i):
            #E = np.append(E, error(tree[j], train))
            E.append([pred(tree[j], k) for k in data])
        E = np.array(E)
        #0视为1
        ypred = np.sign(E.sum(axis = 0) + 0.5)
        error = 1 - np.sum(ypred == data[:, 2]) / len(data)
        Error = np.append(Error, error)

    return Error
```

```
Ein_G = random_forest_error(tree, train)
```

作图

```
plt.plot(np.arange(1, N+1), Ein_G)
plt.show()
```

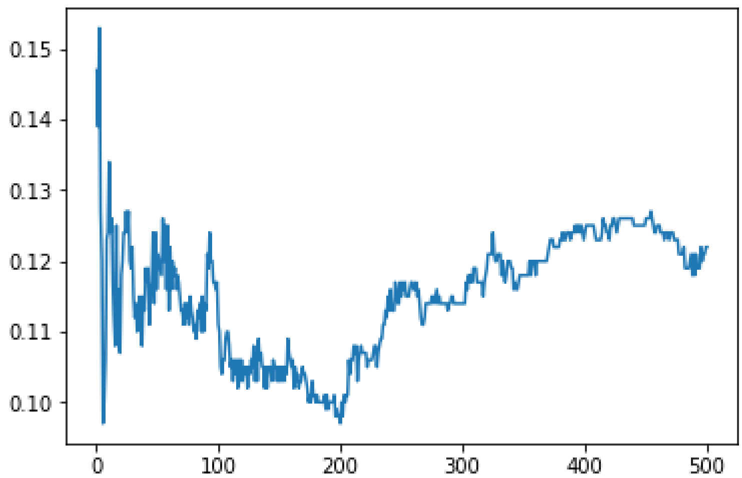


Problem 18

```
Eout_G = random_forest_error(tree, test)
```

作图

```
plt.plot(np.arange(1, N+1), Eout_G)
plt.show()
```



Problem 19

依旧取前t棵数构成随机森林，但是没棵树只有一个branch，即每棵树对应了二元分类。

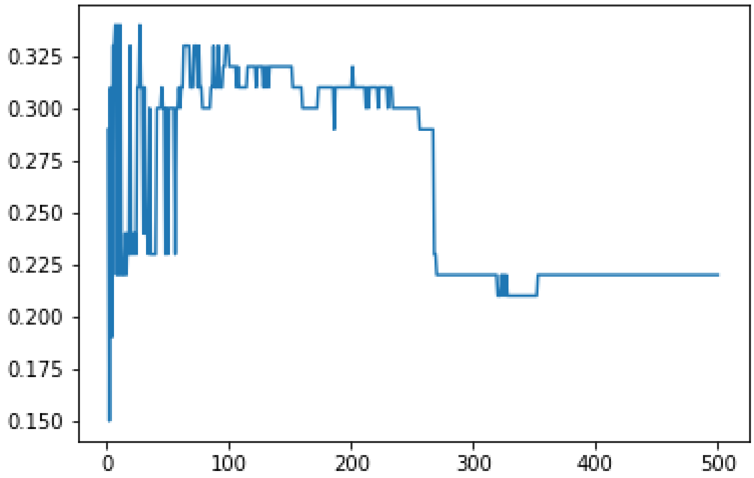
```
def learntree_new(data):
    theta, d = branch(data)
    tree = DTree(None, theta, d, None, None)
    #划分数据
    leftdata = data[data[:, d] < theta]
    rightdata = data[data[:, d] >= theta]
    #左树
    k1 = np.sign(np.sum(leftdata[:, 2]) + 0.5)#+0.5是为了防止出现0
    leftTree = DTree(k1, None, None, None, None)
    #右树
    k2 = np.sign(np.sum(rightdata[:, 2]) + 0.5)
    rightTree = DTree(k2, None, None, None, None)
    #返回
    tree.left = leftTree
    tree.right = rightTree
    return tree
```

```
N = 500
newtree = []
m, n = train.shape
for i in range(N):
    index = np.random.randint(0, m, (m))
    traindata = train[index, :]
    dtree = learntree_new(traindata)
    newtree.append(dtree)
```

```
newEin_G = random_forest_error(newtree, train)
```

作图

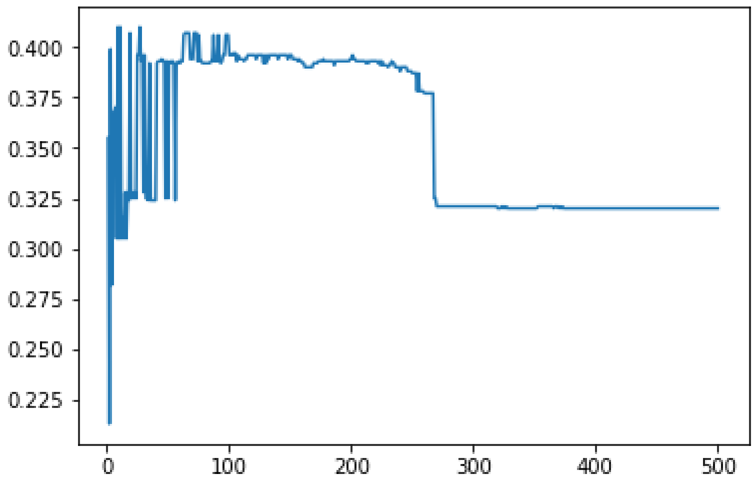
```
plt.plot(np.arange(1, N+1), newEin_G)
plt.show()
```



Problem 20

```
newEout_G = random_forest_error(newtree, test)
```

```
plt.plot(np.arange(1, N+1), newEout_G)
plt.show()
```



Problem 21

由之前讨论可以知道，我们可以利用$\text{sign}(s)$表示NOT,AND,OR逻辑，从而第一层可以表示如下逻辑关系

$$\prod_{i=d_1}^{d_n} x_i^t, x_i^t \in \{x_i, \overline{x}_i\}(1 \le d_1 \le \ldots \le d_n \le d)$$

第二层我们利用这种逻辑关系来表达$\text{XOR}\left(x_1, x_2, \ldots, x_d\right)$，给出以下命题

命题：

$$\text{记}\, x_1, \ldots, x_d \text{为} d \text{个逻辑单元,}$$
$$z_j = \prod_{i=d_1}^{d_{a_j}} x_i^t, x_i^t \in \{x_i, \overline{x}_i\}(1 \le d_1 \le \ldots \le d_{a_j} \le d)$$
$$f_m = \sum_{j=1}^{m} z_j^t, \text{其中} z_j^t \in \{z_j, \overline{z}_j\}$$
$$\text{那么存在}\, f_d = \mathrm{XOR}\Big(x_1, x_2, \ldots, x_d\Big)\text{，且} d \text{为表达异或逻辑的神经元数量的最小值}$$

证明:

关于d利用数学归纳法。

这里的基础情况为$d = 2$，因为1个逻辑单元无法表示异或逻辑，回顾课件可知

$$f_2 = \overline{x}_1 x_2 + x_1 \overline{x}_2$$

可以表示异或逻辑，所以$d = 2$时结论成立。

假设$d = k$时结论，现在证$d = k + 1$时结论也成立。假设逻辑单元为$x_1, \ldots, x_k, x_{k+1}$，根据归纳假设，存在

$$f_k = \sum_{j=1}^{k} z_j^t, \text{其中} z_j^t \in \{z_j, \overline{z}_j\}$$
$$z_j = \prod_{i=d_1}^{d_{a_j}} x_i^t, x_i^t \in \{x_i, \overline{x}_i\}(1 \le d_1 \le \ldots \le d_{a_j} \le d)$$
$$f_k = \mathrm{XOR}\Big(x_1, \ldots, x_k\Big)$$
$$k \text{表达异或逻辑的神经元数量的最小值}$$

根据异或的定义，有如下关系

$$\mathrm{XOR}\Big(x_1, x_2, \ldots, x_k, x_{k+1}\Big) = \mathrm{XOR}\Big(\mathrm{XOR}\Big(x_1, \ldots, x_d\Big), x_{k+1}\Big) = \mathrm{XOR}\Big(f_k, x_{k+1}\Big)$$

因为f_k也为逻辑单元，所以表示$\mathrm{XOR}\Big(f_k, x_{k+1}\Big)$至少需要关于$f_k, x_{k+1}$的2个逻辑单元，可以表示如下

$$\mathrm{XOR}\Big(f_k, x_{k+1}\Big) = \overline{f}_k x_{k+1} + f_k \overline{x}_{k+1}$$

根据逻辑运算规则，

$$\overline{f_k} = \prod_{j=1}^{k} \overline{z}_j^t$$
$$z_j = \prod_{i=d_1}^{d_{a_j}} x_i^t, x_i^t \in \{x_i, \overline{x}_i\}(1 \le d_1 \le \ldots \le d_{a_j} \le d)$$

从而$\overline{f_k}$为一个逻辑单元，将$f_k = \sum_{j=1}^{k} z_j^t$一起带入可得

$$
\begin{aligned}
\mathrm{XOR}\Big(x_1,x_2,\ldots,x_k,x_{k+1}\Big) &= \mathrm{XOR}\Big(f_k,x_{k+1}\Big)\\
&= \overline{f}_k x_{k+1} + f_k\overline{x}_{k+1}\\
&= x_{k+1}\prod_{j=1}^{k}\overline{z}_j^t + (\sum_{j=1}^{k} z_j^t)\overline{x}_{k+1}
\end{aligned}
$$

由逻辑学知识可知，NOT,AND逻辑可以表达所有的逻辑，从而$x_{k+1}\prod_{j=1}^{k}\overline{z}_j^t, z_j^t\overline{x}_{k+1}$可以表达为

$$
\prod_{i=1}^{d_{a_{j+1}}}\overline{x}_i^t\text{或}\overline{\prod_{i=1}^{d_{a_{j+1}}}\overline{x}_i^t},\text{其中}x_i^t\in\{x_i,\overline{x}_i\}
$$

记$z_j^{'}=\prod_{i=d_1}^{d_{a_{j+1}}}x_i^t, x_i^t\in\{x_i,\overline{x}_i\}(1\le d_1\le\ldots\le d_{a_{j+1}}\le k+1)$，那么

$$
\mathrm{XOR}\Big(x_1,x_2,\ldots,x_k,x_{k+1}\Big)=\sum_{j=1}^{k+1}z_j^{'t}
$$

$$
\text{其中}z_j^{'t}\in\{z_j^{'},\overline{z}_j^{'}\}
$$

所以结论对于$n=k+1$也成立，从而结论得证。

Problem 22

直接给出结论，最小值为

$$
1+[\log_2 n]
$$

方法很巧妙，可以参考以下两篇文献，主要是文献2，文献已经下载在文件夹中。

Neural network computation with DNA strand displacement cascades

The Realization of Symmetric Switching Functions with Linear-Input Logical Elements