The Realization of Symmetric Switching Functions with Linear-Input Logical Elements*

WILLIAM H. KAUTZ†, MEMBER, IRE

Summary—**The problem of synthesizing switching networks out of linear-input (threshold) elements is studied for the class of symmetric switching functions. Tight bounds are derived for the number of elements required in a minimal realization, and a method of synthesis is presented which yields economical networks. Minimal networks result for all symmetric functions of no more than about twelve variables, and for several other cases. In particular, it is shown how the parity function of any number n of variables can be realized with about $\log_2(n)$ elements.**

An outstanding problem in combinational switching theory concerns the realization of an arbitrary switching function f of n variables $x_1, x_2, \cdots, x_n$ in a network of logical elements, each of which is describable by a *linear-input* function,[1] also called setting function,[2] threshold function,[2] or linearly separable function.[3] A linear-input function is a switching function $h(y_1, y_2, \cdots, y_m)$ which takes on the value 1 or 0 in accordance with whether the linear equality

$$\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_m y_m \geq \alpha_0$$

is or is not satisfied. The *weights* α_i and *threshold* α_0 are real, positive or negative constants, which may be taken to be integers without loss of generality. A reasonable symbol for this element is shown in Fig. 1. Several simple linear-input devices have been conceived which show promise of leading to economical digital networks for the realization of arbitrary logical operations.[1]

At present, there exists no satisfactory analytical framework for the analysis of this class of networks, nor are there known any procedures for economical synthesis.

This paper concerns the realization of the class of *symmetric* switching functions with networks of linear-input elements. We present an approach to this synthesis problem which yields 1) tight bounds on the minimum number of such elements required to realize any symmetric function, 2) an analytical viewpoint which permits the synthesis of any symmetric function of up to about 12 variables, and several other cases, using the minimal number of elements, and 3) a minimal realization of the alternating symmetric or *parity* function

$$f_p = x_1 \oplus x_2 \oplus \cdots \oplus x_n$$

which requires only $1 + [\log_2(n)]$ elements. (The symbol $\oplus$ indicates exclusive-OR; the brackets denote the integer part of the quantity within.)

Previously published solutions for the case of the parity function have required from $1 + [n/2]$ to $1 + n$ elements.[1,4] The improved solution presented below makes possible simple parity-checking and error-correction circuits for applications to the recording and transmission of digital data. A 7-input parity gate, for example, requires only 3 linear-input elements (Fig. 2); a gate with up to 15 inputs requires 4 elements, etc.

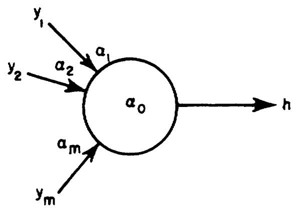


Fig. 1—Symbol for the linear-input element.

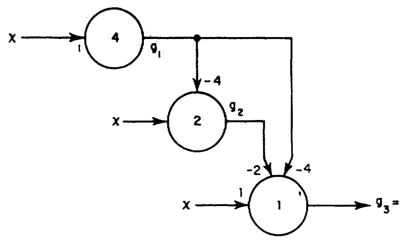


Fig. 2—Minimal linear-input network for the 7-variable parity function.

The Parity Function

Consider first the three-element network shown in Fig. 2. In this and subsequent figures, the generic input labeled x symbolically designates the entire set of inputs $x_1, x_2, \cdots, x_n$, each with unit weight. The variable x takes on the values $0, 1, 2, \cdots, n$, to indicate the number of x_i which have the value 1. It is well known that any symmetric function f_s of n (uncomplemented) variables may be described by a listing of these x values

* Received by the PGEC, January 13, 1961; revised manuscript received, May 9, 1961.

† Stanford Research Institute, Menlo Park, Calif.

[1] R. C. Minnick, "Linear-input logic," IRE TRANS. ON ELECTRONIC COMPUTERS, vol. EC-10, pp. 6–16; March, 1961.

[2] M. C. Paull and E. J. McCluskey, Jr., "Boolean functions realizable with single threshold devices," PROC. IRE, vol. 48, pp. 1334–1337; July, 1960.

[3] R. McNaughton, "Unate truth functions," Appl. Math, and Statistical Lab., Stanford University, Stanford, Calif., October, 1957; IRE TRANS. ON ELECTRONIC COMPUTERS, vol. EC-10, pp. 1–6, March, 1961.

[4] S. Muroga, "Logical elements on majority decision principle and complexity of their circuit," *Proc. Internatl. Conf. on Information Processing*, Paris, France, June, 1959, UNESCO House, Paris, pp. 400–407; 1960.

for which $f_s=1$; *e.g.*, in Shannon's notation,[5]

$$f_p(x_1, \cdots, x_7) = S_{1,3,5,7}(x_1, \cdots, x_7),$$

the x values subscripted to S indicate that the parity function of seven variables equals 1 when and only when an *odd* number of x_i equal 1.

The three element outputs of Fig. 2 are shown plotted against x in Fig. 3. The circuit operates as follows. The first element generates g_1, which equals 0 until x is increased to the threshold 4, and equals 1 for $x \geq 4$. The second element behaves similarly with threshold 2 so long as $x<4$, since $g_1=0$; when $x\geq 4$, however, the presence of $g_1=1$ increases the apparent threshold of the second element to $2+4=6$. Thus g_2 has two positive transitions in value instead of one. The number of positive transitions is redoubled in g_3: the third element operates with threshold 1 when $g_1=g_2=0$, but the apparent threshold increases to 3 when $g_1=0$, $g_2=1$; to 5 when $g_1=1$, $g_2=0$; and finally to 7 when $g_1=g_2=1$. Thus, the circuit of Fig. 2 realizes the parity function $f_p=g_3$ of seven variables (or fewer, since $x\leq n$).

It is clear that this process may be continued to achieve 8 positive transitions at the output of a fourth element, etc., and in general 2^{r-1} positive transitions at the output of the rth element, provided only that the weights and thresholds are scaled up properly as each additional element is added. Using the notation indicated in Fig. 4(a) for a general network of this "feed-forward" type, we may apply the above analysis directly to verify that the set of values

$$\beta_{jk} = 2^{r-j}, \qquad j \leq k, j, k = 1, 2, \cdots, r,$$

gives the output function

$$f_p = S_{1,3,5,\cdots 2^r-1}(x_1, x_2, \cdots, x_{2^r-1})$$

—that is, the parity function of $n=2^r-1$ (or fewer) variables. The next lower value of r cannot be used until n is reduced to $2^{r-1}-1$, so that a number of elements equal to

$$r = 1 + [\log_2(n)]$$

is always adequate for the parity function of n variables.

For the *even* parity function $\bar{f}_p=S_{0,2,4}, \cdots$, a network transformation proposed by Minnick allows the output of any network to be complemented by systematically changing only the weights and thresholds of the elements, and does not increase the number of elements.[1] Thus this lower bound is valid for both the odd and even alternating symmetric functions.

It is clear from the construction that in such a "feed-forward" type of network, the number of positive transitions in g_r cannot exceed 2^{r-1}, the maximum number of different ways in which the $r-1$ previous element outputs can be combined together to yield different apparent thresholds for the rth element. Thus no fewer number of elements can produce this parity function, which has n transitions, and the proposed realization is minimal in the class of "feed-forward" networks.

The class of "feed-forward" networks is that in which the elements can be numbered 1, 2, $\cdots$, r such that

[5] C. Shannon, "A symbolic analysis of relay and switching circuits," *Trans. AIEE*, vol. 57, pp. 713–723; 1938.

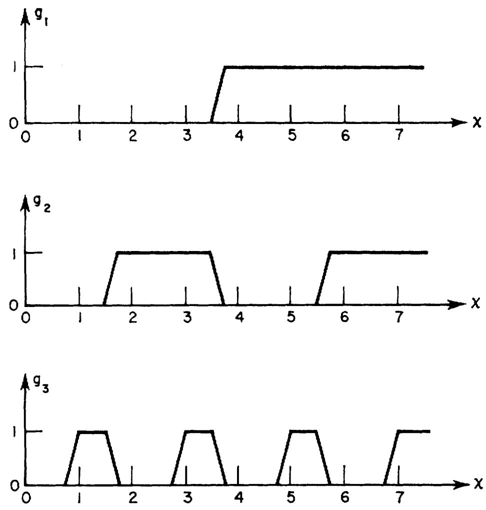


Fig. 3—Element-output functions for the network of Fig. 2

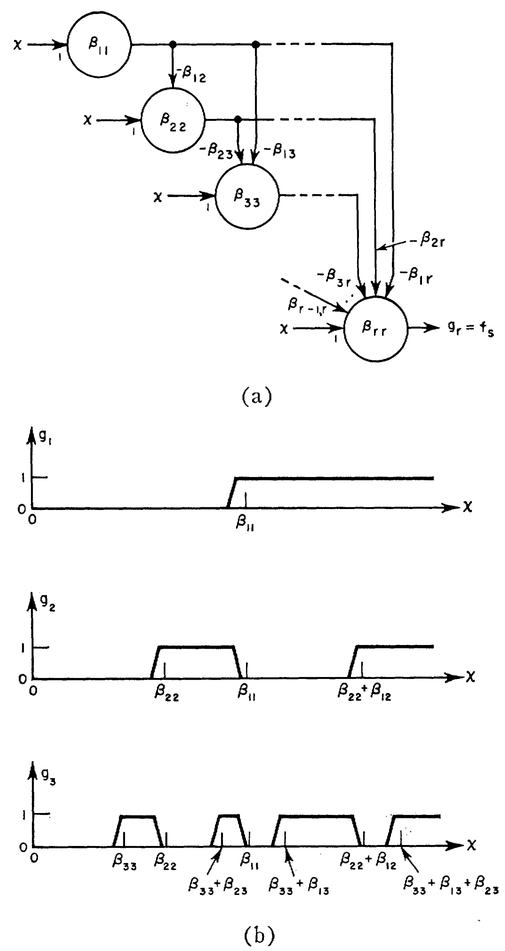


Fig. 4—General feed-forward linear-input network, and the corresponding element-output functions.

each element receives inputs only from x and from the outputs of lower-numbered elements. However, a simple argument shows that a feed-forward network is the most general form of a *loop-free* network. For, in the absence of any closed loops, a path traced backwards from the output f through non-x input arrows must eventually terminate on some element; call this element #1, momentarily delete it, and repeat to locate element #2, etc. By this construction, each element output can be connected only to higher-numbered element inputs, and the feed-forward feature follows. Thus the realization presented is minimal over the class of loop-free networks.

If loops are allowed in a network of linear-input elements, a form of memory or storage can result. This storage can be used constructively for the design of sequential networks, but can produce nonunique or oscillating outputs in a network intended for purely combinational use. This does not imply that loops are never desired, for it is conceivable that in networks of sufficient complexity, nonstorage loops must be allowed if minimality is to be achieved. Examples of this phenomenon are already known in other types of combinational switching networks.[6] However, it is not known whether or not loops are required in minimal linear-input networks.

General Analysis

Consider now the form of the element outputs $g_1, g_2, \cdots, g_r$ when the weights and thresholds are not so restricted as above. The graphs of Fig. 4(b) display the transition x values in terms of the weights β_{kj} of Fig. 4 for $r=3$. We assume for the moment an ordering of β values which allows the maximum number (namely, four) of positive transitions in g_3. It is apparent from the parity example and from this figure that, in general, 1) the x values at which positive transitions occur in g_k define negative transitions in $g_{k+1}, g_{k+2}, \cdots, g_r$, and 2) whereas these positive-transition x values are completely unconstrained for g_1 and g_2, falling at the values

$$g_1\colon \beta_{11} \qquad g_2\colon \begin{array}{l}\beta_{22}\\ \beta_{22}+\beta_{12},\end{array}$$

they are somewhat restricted for g_3, g_4, etc.; *e.g.*,

$$g_3\colon \begin{array}{l}\beta_{33}\\ \beta_{33}+\beta_{23}\\ \beta_{33} \qquad +\beta_{13}\\ \beta_{33}+\beta_{23}+\beta_{13}.\end{array}$$

Thus, if three of these values are specified, the fourth is determined. This constraint may be conveniently expressed by noting that if these x values are written as the exponents of a polynomial in a variable z, then this polynomial may be factored, *e.g.*,

$$g_1\colon P_1(z) = z^{\beta_{11}} \qquad g_2\colon P_2(z) = z^{\beta_{22}} + z^{\beta_{22}+\beta_{12}} = z^{\beta_{22}}(1+z^{\beta_{12}})$$

and

$$g_3\colon P_3(z) = z^{\beta_{33}} + z^{\beta_{33}+\beta_{23}} + z^{\beta_{33}+\beta_{13}} + z^{\beta_{33}+\beta_{23}+\beta_{13}} = z^{\beta_{33}}(1+z^{\beta_{23}})(1+z^{\beta_{13}}).$$

In general, the r constants $\beta_{kr} (k=1, 2, \cdots, r)$ associated with element r are made manifest in the factored form of $P_r(z)$, namely

$$P_r(z) = z^{\beta_{rr}}(1+z^{\beta_{r-1,r}}) \cdots (1+z^{\beta_{1r}}),$$

the expansion of which yields 2^{r-1} terms whose exponents define the 2^{r-1} positive transitions of g_r. In this way, an r-element loop-free network may be completely characterized in terms of the sequence of polynomials $P_1(z)$, $P_2(z), \cdots, P_r(z)$, provided only that the sequence of ordered exponents in the factored form of each $P_k(z)$ corresponds to the sequence of input weights on element k in accordance with the numbering of the elements in the network. This condition will be met automatically if

$$\beta_{jk} \leq \beta_{j-1,k}$$

for $j=1, 2, \cdots, k$, $k=1, 2, \cdots, r$. Further, the maximum number of transitions will be achieved provided only that the sequence of positive transition points alternate with the sequence of negative transition points for each $g_k(z)$. That is, the ordered exponents of each $P_k(z)$ in expanded form must alternate with the ordered exponents of the set of *all* previous polynomials, $P_1(z), P_2(z), \cdots, P_{k-1}(z)$.

An example in which this alternation condition is satisfied is provided by the network of three elements in which

$$\begin{array}{lll}\beta_{11}=5 & \beta_{12}=4 & \beta_{13}=5.\\ & \beta_{22}=3 & \beta_{23}=3\\ & & \beta_{33}=1.\end{array}$$

For this case

$$P_1(z) = z^5$$
$$P_2(z) = z^3 + z^7$$
$$P_3(z) = z + z^4 + z^6 + z^9.$$

Thus the exponents (3, 7) of P_2 alternate with the exponent (5) of P_1, and the exponents (1, 4, 6, 9) of P_3 alternate with the exponents (3, 5, 7) of P_1 and P_2. Hence these three polynomials define a g_3 function with

[6] R. A. Short, "A Theory of Relations Between Sequential and Combinational Realizations of Switching Functions," Stanford Electronics Lab., Stanford University, Stanford, Calif., Tech. Rept. No. 098–1; December, 1960.

transition at $x=1$, 3, 4, 5, 6, 7, and 9. If $n=10$, for example, then

$$g_3 = S_{1,2,4,6,9,10}(x_1, \cdots, x_{10}).$$

The parity function $f_p(x_1, \cdots, x_n)$ introduced above provides another example in which the alternation condition is satisfied. From the values $\beta_{jk}=2^{r-j}$ ($j \leq k$, j, $k=1, 2, \cdots, r$), we have

$$P_k(z) = z^{2^{r-k}}(1 + z^{2^{r-k+1}}) \cdots (1 + z^{2^{r-2}})(1 + z^{2^{r-1}}), \quad k = 1, 2, \cdots, r,$$

so that the exponents of

$$P_r(z) = z + z^3 + z^5 + \cdots + z^{2^r - 1}$$

locate the positive transitions in g_r, and the exponents of all lower-order $P_k(z)$, namely,

$$\sum_{k=1}^{r-1} P_k(z) = z^2 + z^4 + z^6 + \cdots + z^{2^r - 2}$$

locate the negative transitions in g_r.

Consider now the consequences of a violation of the alternation condition on the exponents of successive $P_k(z)$. With reference to Fig. 4(b), if threshold β_{22} were increased so that

$$\beta_{11} \geq \beta_{22} \geq \beta_{33} + \beta_{23},$$

the second and third transition points would pass and cancel, merging the first two "pulses" of g_3 into a single pulse. On the other hand, if β_{22} were reduced so that

$$\beta_{22} \leq \beta_{33},$$

the first and second transition points would pass and cancel, eliminating the first pulse entirely. In general, if a pair of positive and negative transition points in g_k ever occur out of order, they will cancel out, eliminating either a pulse or an interpulse gap, and modifying the pattern of transitions in successive element outputs $g_{k+1}, \cdots, g_r$.

The succession of g functions corresponding to a sequence of polynomials $P_1(z), P_2(z), \cdots, P_r(z)$ for which the alternation condition is not necessarily satisfied may be derived by the following procedure. Let us place in an r-row array the exponents c_{kj} of the expanded polynomials,

$$P_k(z) = z^{c_{k1}} + z^{c_{k2}} + \cdots + z^{c_{k2^{k-1}}}$$

keeping the terms in the same order in which they naturally occur as a result of the expansion of the product (not necessarily in order of increasing value):

P_1:								c_{11}								
P_2:				c_{21}								c_{22}				
P_3:		c_{31}				c_{32}				c_{33}				c_{34}		
P_4:	c_{41}		c_{42}		c_{43}		c_{44}		c_{45}		c_{46}		c_{47}		c_{48}	

etc.

Note that row k of this array may be thought of as being divided into 2^k-intervals by the set of exponents in row k and the rows above it. We now enter in these intervals the values assumed by the g functions, successively in rows 1, 2, $\cdots$, r for $g_1, g_2, \cdots, g_r$. Enter a *0* to the left and a *1* to right of c_{11}. In row k, we note for each interval whether or not the exponents defining the ends of the interval increase from left to right. (We imagine a *0* and an ∞ at the extreme left and right ends of the array.)

1. For odd-numbered intervals, enter a *0* if they increase, and a *1* otherwise.
2) For even-numbered intervals, enter a *1* if they increase, and a *0* otherwise.
3) In every interval in which an increase did not occur, delete all exponents c_{ij} in this same interval which fall in rows *below* row k.
4) Proceed to row $(k+1)$.

The values of the exponents in the bottom row $[P_r(z)]$ which fall between a *0* and a *1* are now the positive-transition x values in g_r, and those (leftmost) exponents in other rows which fall between a *1* and a *0* in the bottom row are the negative-transition x values in g_r.

The validity of this procedure follows from the recognition of c_{ki} as the apparent threshold of element k when the previous g outputs $(g_1, g_2, \cdots, g_{k-1})$ are regarded as the binary form of the number i ($i=1, 2, \cdots, 2^{k-1}$). Consequently, a positive transition can occur at $x=c_{ki}$ if and only if its value falls between the limits of the interval of the previous row in which it is positioned in the array. The deleted terms arise when and only when a particular sequence of g outputs does not occur.

The network of Fig. 5, for example, can be completely described in terms of the polynomials

$$P_1(z) = z^{13}$$

$$P_2(z) = z^6(1 + z^{20}) = z^6 + z^{26}$$

$$P_3(z) = z^4(1 + z)(1 + z^{12}) = z^4 + z^5 + z^{16} + z^{17}$$

$$P_4(z) = z^2(1 + z^3)(1 + z^8)(1 + z^{16})$$

$$= z^2 + z^5 + z^{10} + z^{13} + z^{18} + z^{21} + z^{26} + z^{29}.$$

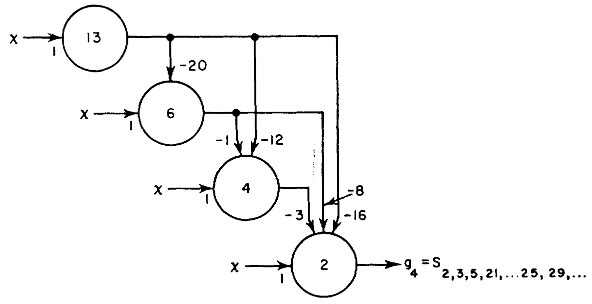


Fig. 5—An example of a network in which the alternation condition is not satisfied.

The array of exponents, with g values entered in accordance with the steps above, then appears as follows:

P_1: 0 13 1

P_2: 0 6 1 0 26 1

P_3: 0 4 1 1 5 1 0 16 1 1 17 1

P_4: 0 2 1 0 5 1 ~~10~~ 0 13 0 0 18 0 0 21 1 ~~26~~ 0 29 1.

Thus, g_4 has positive transitions at $x=2$, 5, 21, and 29, and negative transitions at $x=4$, 6, and 26. Note the two deletions in row 4 which occurred because of the decreases over the 3rd and 7th intervals in row 3.

In this way, the output $f=g_r$ of any feed-forward network of linear-input elements may be determined with a minimum of calculation. It is apparent that, while a succession of misorderings of the exponent values will always reduce the number of pulses in g_r, it will allow at the same time fewer constraints between the remaining transition x values because of the flexibility in the amount of overlap of pulses and interpulse gaps, and freedom of choice of which pulses are eliminated. We will now see how this flexibility may be used to advantage.

Some Remarks on Synthesis

The synthesis problem is one of deriving from a given symmetric function at least one network (that is, a set of β values) which realizes this function over the range $0 \leq x \leq n$ with a minimum number $r_{\min}$ of elements. While the synthesis process for arbitrary n appears to be very intricate, all symmetric functions of up to about twelve variables can be handled fairly easily by inversion of the above analysis procedure. In fact, symmetric functions of *any* number of variables, and having a total number τ of transitions up to about nine, can be synthesized in this way.

We have already seen that an r-element network can give rise to no more than 2^{r-1} positive transitions, or 2^r-1 transitions in all, in the output function g_r. Therefore, for all symmetric functions

$$r_{\min}(\tau) \geq 1 + [\log_2(\tau)].$$

This lower bound is achieved for the parity function, for which $\tau=n$, so the bound cannot be improved.

Minnick[1] describes a procedure for the synthesis of an arbitrary symmetric function which provides the upper bound

$$r_{\min}(\tau) \leq 1 + [\tau/2].$$

A simple argument shows that this bound cannot be improved. In an r-element network, only r of the 2^{r-1} possible positive transitions in g_r are arbitrary, since the entire set is determined by the r constants $\beta_{1r}, \beta_{2r}, \cdots, \beta_{rr}$. Thus by choosing successive positive-transition values large enough, we can certainly find a symmetric function having just r positive transitions, in which the other $2^{r-1}-r$ potential positive transitions are not desired and must be cancelled by negative transitions produced by the other $r-1$ elements. If the largest transition is positive (τ odd), then $\tau \leq 2r-1$. If the largest transition is negative (τ even), there will be a hidden positive transition at some value of $x>n$ (since $g_r \to 1$ as $x \to \infty$), so $\tau \leq 2r-2$. In either case, $r \geq 1+[\tau/2]$ for this function, and the tightness of the bound is established.

Thus, the minimum number $r_{\min}$ of elements required for the realization of any symmetric function having τ transitions in value as x increases from 0 to n satisfies the inequality

$$1 + [\log_2(\tau)] \leq r_{\min}(\tau) \leq 1 + [\tau/2],$$

and this range cannot be further narrowed. Since $\tau \leq n$, and $\tau=n$ only for the parity function, the same inequality is valid if τ is replaced by n, except that the upper bound is no longer tight, and indeed may be rather poor:

$$1 - [\log_2(n)] \leq r_{\min}(n) < 1 + [n/2].$$

Values of these two bounds are listed in the upper rows of Table I. Note that the bounds are equal up to $\tau=5$.

Analysis of numerous special cases leads to the results summarized in Table I, which lists for each pair of values of n and τ the number $r_{\min}$ of elements which is required in the minimal realization. The number in parentheses following an entry $r_{\min}$ indicates the number of different symmetric functions (symmetry types) whose minimal realization requires just $r_{\min}$ elements. When no such number is given, all $\binom{n}{\tau}$ symmetric functions of n variables having τ transitions require the same number $r_{\min}$ of elements.[7]

Let the transition x values of a given function to be synthesized be (in increasing order) $a_1, b_1, a_2, b_2, a_3, \cdots$. At best, no cancellations are necessary, and the polynomial

$$z^{a_1} + z^{a_2} + z^{a_3} + z^{a_4} + z^{a_5} + \cdots$$

[7] An argument which we do not present here reveals that the number of 6-transition, 3-element symmetry types for any value of n greater than 6 exceeds the corresponding number for $n-1$ by the sum $\Sigma\alpha_1\alpha_2\alpha_3$ over all additive partitions of $n-3$ into just three positive integers, the first of which is at least as big as the second: $n-3=\alpha_1+\alpha_2+\alpha_3$, $\alpha_1 \geq \alpha_2 > 0$, $\alpha_3 > 0$. A related expression can be derived when $\tau=7$.

TABLE I

MINIMAL NUMBER $r_{\min}$ OF LINEAR-INPUT ELEMENTS REQUIRED FOR THE REALIZATION OF SYMMETRIC SWITCHING FUNCTIONS OF n VARIABLES HAVING τ TRANSITIONS IN VALUE

	τ:	1	2	3	4	5	6	7	8	9	10	11
Lower bound		1	2	2	3	3	3	3	4	4	4	4
Upper bound		1	2	2	3	3	4	4	5	5	6	6
n	1	1										
	2	1	2									
	3	1	2	2								
	4	1	2	2	3							
	5	1	2	2	3	3						
	6	1	2	2	3	3	3					
	7	1	2	2	3	3	3 (5) 4 (2)	3				
	8	1	2	2	3	3	3 (19) 4 (9)	3 (4) 4 (4)	4			
	9	1	2	2	3	3	3 (53) 4 (31)	3 (14) 4 (22)	4	4		
	10	1	2	2	3	3	3 (129) 4 (81)	3 (36) 4 (84)	4	4	4	
	11	1	2	2	3	3	3 (275) 4 (187)	3 (84) 4 (246)	4	4 (47) 5 (8)	4	4

can be put into factorable form as $P_r(z)$ merely by adding as necessary terms with exponents x outside of the range $0 \leq x \leq n$. The degrees of the factors may then be identified with the weights β_{jr}, as defined previously. For example, the symmetric function

$$f_s = S_{1,2,4,6,7,10,12}(x_1, \cdots, x_{13})$$

has transition values

$$a_1 = 1 \quad a_2 = 4 \quad a_3 = 6 \quad a_4 = 10 \quad a_5 = 12$$
$$b_1 = 3 \quad b_2 = 5 \quad b_3 = 8 \quad b_4 = 11 \quad b_5 = 13,$$

and $P_4(z)$ may be formed as follows, with the original terms in parentheses:

$$P_4(z) = z^{-1} + (z + z^4 + z^6 + z^{10} + z^{12}) + z^{15} + z^{17}$$
$$= z^{-1}(1 + z^2)(1 + z^5)(1 + z^{11}),$$

so

$$\beta_{44} = -1, \quad \beta_{34} = 2, \quad \beta_{24} = 5, \quad \text{and} \quad \beta_{14} = 11.$$

$P_3(z)$ may then be formed from alternate terms of the sequence of negative transitions: 0, 3, 5, 8, 11, 13, $\cdots$. Thus

$$P_3(z) = z^0 + (z^5 + z^{11}) + z^{16} = z^0(1 + z^5)(1 + z^{11}),$$

so

$$\beta_{33} = 0, \quad \beta_{23} = 5, \quad \text{and} \quad \beta_{13} = 11.$$

Similarly,

$$P_2(z) = z^3 + z^{13} = z^3(1 + z^{10})$$
$$P_1(z) = z^8,$$

so $\beta_{22} = 3$, $\beta_{12} = 10$, $\beta_{11} = 8$. The network with these β values then realizes the given symmetric function.

In general, the possibility of cancellation of terms must be taken into account, in which case the requirement of alternating terms in successive $P_k(z)$ must be replaced by a more detailed analysis of the network under consideration.

It can be seen from Fig. 4(b) and the form of $P_3(z)$ that if $r \leq 3$, all of the transition x values of g_1, g_2, and g_3 are completely arbitrary except one of the positive transitions (*e.g.*, the last one) of g_3. Thus, all symmetric functions of five or fewer transitions can be realized with the minimum number of elements. From the form of the polynomial $P_3(z)$, the four positive transition x values must satisfy the equality $a_4 - a_3 = a_2 - a_1$. For $\tau = 6$, therefore, only those functions for which n falls between the third and fourth pulses are realizable with three elements:

$$b_3 \leq n < a_3 + a_2 - a_1,$$

and all others require four elements. For $\tau = 7$, the equality

$$a_4 = a_3 + a_2 - a_1$$

must hold if three elements are to be adequate, otherwise, four elements are needed.

For $\tau > 7$, at least four elements are necessary. The selection of a set of β values to produce the desired sequence of transitions, with cancellations as necessary, and simultaneously to satisfy the multiplicative condition implicit in the polynomial expansion, may be executed with a trial-and-error process. As an example, consider the function

$$f_5 = S_{1,3,5,8}(x_1, \cdots, x_{11}),$$

which has the transition values

$$\begin{array}{llll} a_1 = 1 & a_2 = 3 & a_3 = 5 & a_4 = 8 \\ b_1 = 2 & b_2 = 4 & b_3 = 6 & b_4 = 9. \end{array}$$

Since $\tau = 8$, the bounds indicate that either 4 or 5 elements are required. With $r = 4$, the four a_k might be identified with the transition points β_{44}, $\beta_{44} + \beta_{34}$, $\beta_{44} + \beta_{24}$, and $\beta_{44} + \beta_{14}$, giving

$$\begin{aligned} P_4(z) &= z(1 + z^2)(1 + z^4)(1 + z^7) \\ &= z + z^3 + z^5 + z^7 + z^8 + z^{10} + z^{12} + z^{14}, \end{aligned}$$

but then unwanted positive transitions occur at $x = 7$, 10, 12, and 14. For $n = 11$, the first two must be cancelled with negative transitions, but this can be done with the set of β values expressed in the polynomials

$$\begin{aligned} P_3(z) &= z^2 + z^6 + z^9 + z^{13} = z^2(1 + z^4)(1 + z^7) \\ P_2(z) &= z^4 + z^{10} = z^4(1 + z^6) \\ P_1(z) &= z^7. \end{aligned}$$

The final network is shown in Fig. 6(a), and has four elements.

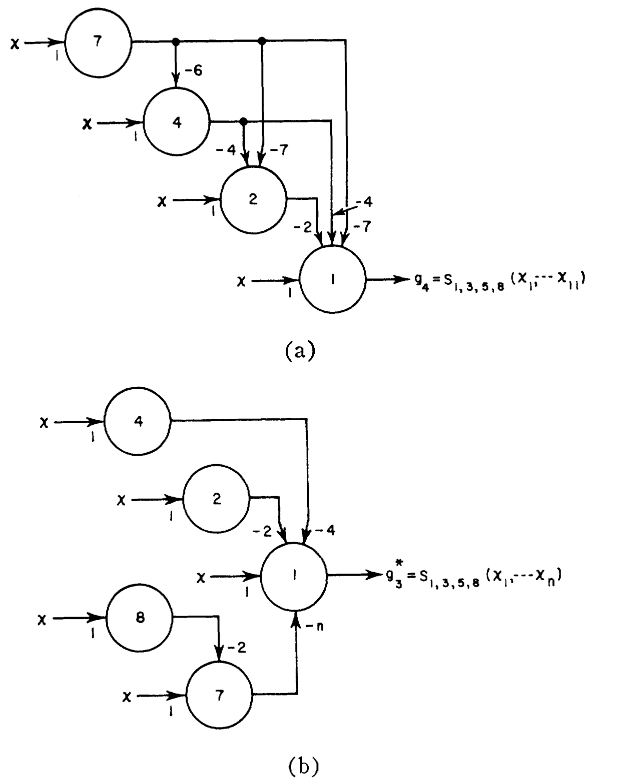
If the number n of variables had been greater than 11, other β values might be selected to cancel the positive transitions at 12 and 14, or to yield the desired transitions in another way. Trial of all of these possibilities, which are not very numerous, reveals that no such selection is possible with only four elements, however, so that five elements are required for this case. One possible network, valid for $n \geq 9$ and minimal for $n \geq 12$, is shown in Fig. 6(b).

Any symmetric function for which $\tau \leq 9$, and many others, can be handled easily by this approach. The cases $\tau = n$, $\tau = n - 1$, and $\tau = n - 2$ have either been considered already or are not difficult. Thus all cases for which $n \leq 12$ are amenable to the procedure described with a very modest amount of trial and error effort.

The previously mentioned procedure due to Minnick[1] yields a network whose $r = 1 + [\tau/2]$ elements are arranged as shown in Fig. 7(a), with the following weights and thresholds:

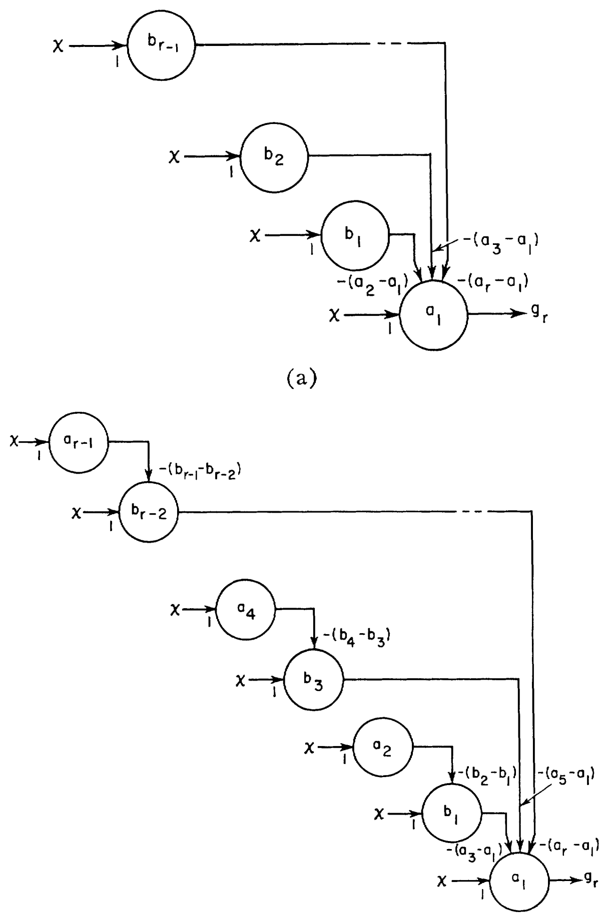
$$\begin{array}{ll} \beta_{jk} = 0 & j < k < r \\ \beta_{jr} = a_{j+1} - a_1 & j < r \\ \beta_{kk} = b_k & k < r. \\ \beta_{rr} = a_1 & \end{array}$$

Thus all elements except the rth are driven only by x inputs and drive only the rth element, which provides the output. An alternative configuration is shown in Fig. 7(b). Several circuit arrangements which are hybrid combinations of the general feed-forward structure and one of these circuits of Fig. 7 are possible, and can



(a) (b)

Fig. 6—Minimal linear-input networks for realization of the example $S_{1,3,5,8}(x_1, \cdots, x_n)$ for (a) $n < 12$, and (b) $n \geq 12$.



(a) (b)

Fig. 7—Two alternative realizations which use a number of elements equal to the upper bound.

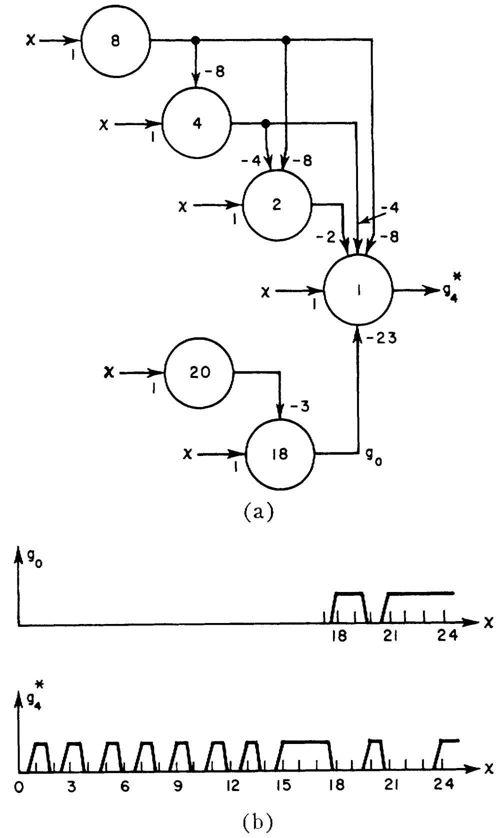


Fig. 8—An example of a hybrid realization and the element-output functions of its two main parts.

be applied in cases when τ is large to yield a network which is not necessarily minimal, but may represent a saving over the number of elements required for the upper bound. One example of a hybrid structure is provided by the example of Fig. 6(b), introduced earlier. The network of Fig. 8 provides another example: here $\tau=19$, so that $r_{\min}\leq 10$. The fact that the polynomial condition is not satisfied for the entire set of positive transitions, but is satisfied for the first eight, suggests that the last few pulses in g_r be realized separately. The hybrid network requires $r=6$ elements.

Discussion

While the approach to symmetric-function synthesis presented in the previous section provides solutions to all cases of any conceivable practical importance, there nevertheless remains the challenging theoretical problem of developing a general procedure for the minimal realization of arbitrary symmetric functions. As is frequently typical of synthesis algorithms, the main motivation for their development probably lies more in the understanding to be gained of the structure and properties of linear-input networks than in the possible direct utility of the method itself. In the present case, the knowledge of just how economical networks of linear-input elements can give rise to a desired type of symmetric-terminal behavior could very well provide much of the basis and insight needed for a solution of the synthesis problem for arbitrary switching functions.

The reader may have already observed the close relation between the method described and Markov's procedure for the synthesis of minimum-NOT networks.[8] A direct element-for-element conversion of Markov networks to linear-input networks, even for the symmetric-function case, generally does not appear to be possible, and, when it is possible, it does not necessarily lead to minimal linear-input networks. Nevertheless, the method of decomposition of the symmetric function into a succession of unate subfunctions which can be optimally combined is common to both procedures.[3] The possible extension of the above procedure to arbitrary switching functions through a development analogous to Markov's procedure for arbitrary functions suggests that the given function be decomposed into a succession of linear-input, rather than unate, subfunctions. (The two classes are identical when the given function is symmetric.) In this way, examples of non-symmetric switching functions of arbitrary complexity can be created which have a minimal realization for virtually any number of elements. These possibilities are presently under investigation.

It was pointed out to the author by D. A. Huffman of M.I.T. that an arbitrary switching function $f=\sum(a_1, a_2, \cdots, a_s)$ of m variables $x_1, x_2, \cdots, x_m$ (expressed here in "decimal" form, following Caldwell[9]) can be written as a symmetric function in 2^m-1 variables, $f=S_{a_1,a_2,\cdots,a_s}(x_1, x_2, x_2, x_3, x_3, x_3, x_3, x_4, \cdots, x_m.)$ Here variable x_i occurs with multiplicity 2^{i-1}. Thus, this equivalence enables the procedure of this paper to be applied to the synthesis of arbitrary switching functions, although, of course, the final result will in general not be very economical in the number of elements required.

Finally, it should be noted that for the networks under consideration, most or all of the element non-x input weights are *negative*. Certain circuit realizations of the linear-input element—*e.g.*, those using resitive adders with transistors—allow negative weights to be realized much more easily than positive weights, and therefore might be particularly desirable from an applied point of view. Since the x inputs all have positive weights and are common to all elements, these inputs can be collected and summed separately, then applied through an inverting amplifier to the entire set of elements.

Acknowledgment

The author is indebted to R. Minnick of Stanford Research Institute, whose work stimulated his serious interest in the subject area of this paper, and with whom many profitable technical discussions have taken place.

[8] A. A. Markov, "On the inversion complexity of a system of functions," *J. Assoc. Computing Mach.*, vol. 5, pp. 331–334, October, 1958; translated from *Doklady Akad. Nauk S.S.S.R.*, vol. 116, pp. 917–919, 1957.

[9] S. H. Caldwell, "Switching Circuits and Logical Design," John Wiley and Sons, Inc., New York, N. Y.; 1958. See especially p. 124.