大家好，这篇是有关台大机器学习课程作业六的详解。

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作业地址:
https://www.csie.ntu.edu.tw/~htlin/course/ml15fall/

参考资料:
https://blog.csdn.net/a1015553840/article/details/51085129
http://www.vynguyen.net/category/study/machine-learning/page/6/
http://book.caltech.edu/bookforum/index.php
http://beader.me/mlnotebook/
https://blog.csdn.net/qian1122221/article/details/50130093
https://acecoooool.github.io/blog/

Problem 1

首先计算p_n

$$
\begin{aligned}
p_n &= \theta(-y_n(Az_n+B)) \\
&= \frac{\exp(-y_n(Az_n+B))}{1+\exp(-y_n(Az_n+B))} \\
&= \frac{1}{1+\exp(y_n(Az_n+B))}
\end{aligned}
$$

现在对式子进行化简

$$
\begin{aligned}
F(A,B) &= \frac{1}{N}\sum_{n=1}^{N}\ln\Big(1+\exp\Big(-y_n\Big(Az_n+B\Big)\Big)\Big) \\
&= \frac{1}{N}\sum_{n=1}^{N}\ln\Big(\frac{1+\exp\Big(y_n\Big(Az_n+B\Big)\Big)}{\exp\Big(y_n\Big(Az_n+B\Big)\Big)}\Big) \\
&= -\frac{1}{N}\sum_{n=1}^{N}\ln\Big(\frac{\exp\Big(y_n\Big(Az_n+B\Big)\Big)}{1+\exp\Big(y_n\Big(Az_n+B\Big)\Big)}\Big) \\
&= -\frac{1}{N}\sum_{n=1}^{N}\ln\Big(1-p_n\Big)
\end{aligned}
$$

现在计算梯度

$$
\begin{aligned}
\nabla F(A,B) &= -\frac{1}{N}\sum_{n=1}^{N}\frac{1}{1-p_n}(-1)p_n(1-p_n)(-y_n)\begin{pmatrix} z_n \\ 1 \end{pmatrix} \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n p_n \begin{pmatrix} z_n \\ 1 \end{pmatrix}
\end{aligned}
$$

Problem 2

现在要计算Hessian矩阵，由上一题可知

$$
\begin{aligned}
\frac{\partial F(A,B)}{\partial A} &= -\frac{1}{N}\sum_{n=1}^{N} y_n z_n p_n \\
\frac{\partial F(A,B)}{\partial B} &= -\frac{1}{N}\sum_{n=1}^{N} y_n p_n
\end{aligned}
$$

在计算$\frac{\partial^2 F(A,B)}{\partial A^2}$, $\frac{\partial^2 F(A,B)}{\partial B^2}$, $\frac{\partial^2 F(A,B)}{\partial A \partial B}$之前，先计算$\frac{\partial p_n}{\partial A}$, $\frac{\partial p_n}{\partial B}$

$$
\begin{aligned}
\frac{\partial p_n}{\partial A} &= p_n(1-p_n)(-y_n)z_n \\
\frac{\partial p_n}{\partial B} &= p_n(1-p_n)(-y_n)
\end{aligned}
$$

接下来分别计算上述三个式子，注意$y_n^2 = 1$

$$
\begin{aligned}
\frac{\partial^2 F(A,B)}{\partial A^2} &= -\frac{1}{N}\sum_{n=1}^{N} y_n z_n \frac{\partial p_n}{\partial A} \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n z_n p_n(1-p_n)(-y_n)z_n \\
&= \frac{1}{N}\sum_{n=1}^{N} y_n^2 z_n^2 p_n(1-p_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} z_n^2 p_n(1-p_n)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 F(A,B)}{\partial B^2} &= -\frac{1}{N}\sum_{n=1}^{N} y_n \frac{\partial p_n}{\partial A} \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n p_n(1-p_n)(-y_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} y_n^2 p_n(1-p_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} p_n(1-p_n)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 F(A,B)}{\partial A \partial B} &= -\frac{1}{N}\sum_{n=1}^{N} y_n z_n \frac{\partial p_n}{\partial B} \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n z_n p_n(1-p_n)(-y_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} y_n^2 z_n p_n(1-p_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} z_n p_n(1-p_n)
\end{aligned}
$$

结合这几个式子，我们可知

$$
H(F) = \begin{pmatrix} \frac{1}{N}\sum_{n=1}^{N} z_n^2 p_n(1-p_n) & \frac{1}{N}\sum_{n=1}^{N} z_n p_n(1-p_n) \\ \frac{1}{N}\sum_{n=1}^{N} z_n p_n(1-p_n) & \frac{1}{N}\sum_{n=1}^{N} p_n(1-p_n) \end{pmatrix}
$$

Problem 3

首先回顾下Gaussian kernel的形式

$$
K(x,x^{'}) = \exp(-\gamma ||x-x^{'}||^2)
$$

所以如果$\gamma \to \infty$，那么$K(x,x^{'}) \to 0$，从而kernel matrix $K \to 0$ ，注意最后的0是零矩阵的意思。现在回顾讲义上β的式子

$$
\beta = (\lambda I + K)^{-1} y
$$

现在$K \to \infty$，那么

$$
\beta \to \lambda^{-1} y
$$

Problem 4

本题的目的是将条件极值改写为无条件极值，先看下本题的条件。

$$
-\epsilon - \xi_n^{\vee} \le y_n - w^T\phi(x_n) - b \le \epsilon + \xi_n^{\wedge}
$$

由几何意义可知，

$$
\begin{aligned}
&\text{当}\, y_n - w^T\phi(x_n) - b \ge 0\text{时，}\ \xi_n^{\vee} = 0,\ \xi_n^{\wedge} = \max\Big(0, |w^T z_n + b - y_n| - \epsilon\Big) \\
&\text{当}\, y_n - w^T\phi(x_n) - b < 0\text{时，}\ \xi_n^{\wedge} = 0,\ \xi_n^{\vee} = \max\Big(0, |w^T z_n + b - y_n| - \epsilon\Big)
\end{aligned}
$$

所以

$$
(\xi_n^{\vee})^2 + (\xi_n^{\wedge})^2 = \Big(\max\Big(0, |w^T z_n + b - y_n| - \epsilon\Big)\Big)^2
$$

所以原问题可以转化为以下问题

$$\min_{b,w} \frac{1}{2}w^Tw + C\sum_{n=1}^{N}\left(\max\Big(0, |w^Tz_n + b - y_n| - \epsilon\Big)\right)^2$$

Problem 5

对Problem 4最后的结果进行改写

$$\min_{b}\min_{w} \frac{1}{2}w^Tw + C\sum_{n=1}^{N}\left(\max\Big(0, |w^Tz_n + b - y_n| - \epsilon\Big)\right)^2$$

对于第一个最小化问题$\min\limits_{w} \frac{1}{2}w^Tw + C\sum_{n=1}^{N}\left(\max\Big(0, |w^Tz_n + b - y_n| - \epsilon\Big)\right)^2$，由Representer Theorem可知，该问题的最优解为

$$w_* = \sum_{m=1}^{N}\beta_m z_m$$

带入上式可得，现在问题转化为

$$\min_{b} \frac{1}{2}w_*^Tw_* + C\sum_{n=1}^{N}\left(\max\Big(0, |w_*^Tz_n + b - y_n| - \epsilon\Big)\right)^2$$

将$\beta_1,\ldots,\beta_N$视为参数，结合$K(x_n,x_m) = (\varphi(x_n))^T(\varphi(x_m))$，该问题转化为

$$\min_{b,\beta} F(b,\beta) = \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N}\beta_n\beta_m K(x_n,x_m) + C\sum_{n=1}^{N}\left(\max\Big(0, |\sum_{m=1}^{N}\beta_m K(x_n,x_m) + b - y_n| - \epsilon\Big)\right)^2$$

题目中记$s_n = \sum_{m=1}^{N}\beta_m K(x_n,x_m) + b$，所以上式可以变形为

$$\min_{b,\beta} F(b,\beta) = \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N}\beta_n\beta_m K(x_n,x_m) + C\sum_{n=1}^{N}\left(\max\Big(0, |s_n - y_n| - \epsilon\Big)\right)^2$$

现在计算$\frac{\partial F(b,\beta)}{\partial \beta_m}$，分两种情形讨论

当$|s_n - y_n| - \epsilon \le 0$时，$C\sum_{n=1}^{N}\left(\max\Big(0, |s_n - y_n| - \epsilon\Big)\right)^2 = 0$

$$\begin{aligned}\frac{\partial F(b,\beta)}{\partial \beta_i} &= \sum_{n=1}^{N}\beta_n K(x_n,x_i)\\ &= \sum_{n=1}^{N}\beta_n K(x_n,x_i)\end{aligned}$$

当$|s_n - y_n| - \epsilon > 0$时，$C\sum_{n=1}^{N}\left(\max\Big(0, |s_n - y_n| - \epsilon\Big)\right)^2 = C\sum_{n=1}^{N}\Big(|s_n - y_n| - \epsilon\Big)^2$

$$
\begin{aligned}
\frac{\partial F(b,\beta)}{\partial \beta_i} &= \sum_{n=1}^{N}\beta_n K(x_n,x_i) + 2C\sum_{n=1}^{N}(|s_n-y_n|-\epsilon)\frac{\partial s_n}{\partial \beta_i}\\
&= \sum_{n=1}^{N}\beta_n K(x_n,x_i) + 2C\sum_{n=1}^{N}(|s_n-y_n|-\epsilon)K(x_n,x_i)\\
&= \sum_{n=1}^{N}(\beta_n + 2C(|s_n-y_n|-\epsilon))K(x_n,x_i)
\end{aligned}
$$

如果统一起来，可以写成

$$
\frac{\partial F(b,\beta)}{\partial \beta_i} = \sum_{n=1}^{N}\Big(\beta_n + 2C[\![|s_n-y_n|-\epsilon]\!]\Big)K(x_n,x_i)
$$

Problem 6

我们把$E_{\text{test}}(g_t) = \frac{1}{M}\sum_{m=1}^{M}(g_t(\tilde{x}_m)-\tilde{y}_m)^2 = e_t(t=0,1,2,\ldots,T)$这个式子打开，记$z_t = \frac{2}{M}\sum_{m=1}^{M} g_t(\tilde{x}_m)\tilde{y}_m$，注意$\frac{1}{M}\sum_{m=1}^{M}(g_t(\tilde{x}_m))^2 = s_t$

$$
\begin{gathered}
\frac{1}{M}\sum_{m=1}^{M}(g_t(\tilde{x}_m)-\tilde{y}_m)^2 = e_t\\
\frac{1}{M}\sum_{m=1}^{M}(g_t(\tilde{x}_m))^2 - \frac{2}{M}\sum_{m=1}^{M}g_t(\tilde{x}_m)\tilde{y}_m + \sum_{m=1}^{M}y_m^2 = e_t\\
s_t - z_t + \sum_{m=1}^{M}y_m^2 = e_t(t=0,1,2,\ldots,T)
\end{gathered}
$$

我们要求的量是z_t，已知的量是s_t,e_t，还有两个条件为$g_0(x)=0, s_0 = \frac{1}{M}\sum_{m=1}^{M}(g_0(\tilde{x}_m))^2 = 0$，所以

$$
\begin{gathered}
z_0 = 0\\
s_0 - z_0 + \sum_{m=1}^{M}y_m^2 = e_0\\
\sum_{m=1}^{M}y_m^2 = e_0 - s_0 = e_0
\end{gathered}
$$

所以

$$
\begin{gathered}
z_t = s_t + \sum_{m=1}^{M}y_m^2 - e_t = s_t + e_0 + e_t\\
\sum_{m=1}^{M}g_t(\tilde{x}_m)\tilde{y}_m = \frac{M}{2}z_t = \frac{M}{2}(s_t+e_0+e_t)
\end{gathered}
$$

Problem 7

设两个点的坐标为$(x_1,y_1),(x_2,y_2),y_1=x_1^2,y_2=x_2^2$，由公式可知，最小二乘解为

$$w=\frac{x_1y_1+x_2y_2-2\frac{x_1+x_2}{2}\frac{y_1+y_2}{2}}{(x_1-\frac{x_1+x_2}{2})^2+(x_2-\frac{x_1+x_2}{2})^2}=\frac{(x_1-x_2)(y_1-y_2)}{(x_1-x_2)^2}=\frac{y_1-y_2}{x_1-x_2}=\frac{x_1^2-x_2^2}{x_1-x_2}=x_1+x_2,$$
$$b=\frac{y_1+y_2}{2}-w\frac{x_1+x_2}{2}=\frac{x_1^2+x_2^2}{2}-(x_1+x_2)\frac{x_1+x_2}{2}=-x_1x_2$$

因为x_1,x_2服从$[0,1]$上的均匀分布，所以

$$\mathbb{E}w=\mathbb{E}(x_1+x_2)=\mathbb{E}(x_1)+\mathbb{E}(x_2)=1$$
$$\mathbb{E}b=\mathbb{E}(-x_1x_2)=-\mathbb{E}(x_1)\mathbb{E}(x_2)=-\frac{1}{2}\times\frac{1}{2}=-\frac{1}{4}$$
$$\bar{g}(x)=x-\frac{1}{4}$$

Problem 8

$$\min_w E_{in}^u(w)=\frac{1}{N}\sum_{n=1}^{N}u_n(y_n-w^Tx_n)^2$$

由于$u_n\ge 0$，所以可以对$E_{in}^u(w)$进行如下处理

$$E_{in}^u(w)=\frac{1}{N}\sum_{n=1}^{N}u_n(y_n-w^Tx_n)^2=\frac{1}{N}\sum_{n=1}^{N}(\sqrt{u_n}y_n-w^T\sqrt{u_n}x_n)^2$$

现在记$(\tilde{x}_n,\tilde{y}_n)=\sqrt{u_n}(x_n,y_n)$，那么$E_{in}^u(w)$可以转化为

$$E_{in}^u(w)=\frac{1}{N}\sum_{n=1}^{N}(\tilde{y}_n-w^T\tilde{x}_n)^2$$

这样就转化为常规形式。

Problem 9

我们知道$g_1(x)$的正确率为99%，只在negative example上预测错误，根据讲义8第11到13页可知

$$\frac{u_+^{(2)}}{u_-^{(2)}}=\frac{1}{99}$$

Problem 10

首先回顾假设的形式

$$g_{s,i,\theta}(x)=s\cdot\operatorname{sign}(x_i-\theta)(i\in\{1,2,\ldots,d\})$$

首先考虑两种最极端的情况，$\theta<L,\theta\ge R$，在这两种情形下，$\operatorname{sign}(x_i-\theta)$或者都为$1$，或者全为$-1$，所以在这两种条件下一共有两个$g(x)$，注意这种情形是和$i$无关，最后计算的时候要注意这点。

现在考虑$L\le\theta<R$，根据题目中的定义，决定$\operatorname{sign}(x_i-\theta)$只是$\theta$相对于$x_i$的位置，所以对于

$$\theta \in [k, k+1), k \in \{L, L+1, \ldots, R-1\}$$

$\mathrm{sign}(x_i - \theta)$表示的都是同一个函数，因此一共有$R-L$种$\mathrm{sign}(x_i - \theta)$，

由于$s \in \{+1, -1\}$，所以$g_{s,i,\theta}(x) = s \cdot \mathrm{sign}(x_i - \theta)$一共有$2(R-L)$种。我们现在考虑的是一个维度上的，因为一共有$d$个维度，每个维度代表一种分类器，最后加上最开始讨论的全1或者全-1的情况，所以一共有

$$2d(R-L)+2$$

此题将$d=2, L=1, R=6$带入可得

$$2 \times 2 \times 5 + 2 = 22$$

Problem 11

先计算$g_t(x)g_t(x^{'})$

$$\begin{aligned} g_t(x)g_t(x^{'}) &= (s_t.\mathrm{sign}(x_i - \theta_t))(s_t.\mathrm{sign}(x_i^{'} - \theta_t)) \\ &= \mathrm{sign}(x_{t_i} - \theta_t)\mathrm{sign}(x_{t_i}^{'} - \theta_t) \\ & t_i\text{的含义为}g_t(x)\text{对应的}i \end{aligned}$$

所以

$$\begin{aligned} K_{ds}(x, x^{'}) &= (\phi_{ds}(x))^T \phi_{ds}(x^{'}) \\ &= \sum_{t=1}^{|\mathcal{G}|} g_t(x)g_t(x^{'}) \\ &= \sum_{t=1}^{|\mathcal{G}|} \mathrm{sign}(x_{t_i} - \theta_t)\mathrm{sign}(x_{t_i}^{'} - \theta_t) \\ & t_i\text{的含义为}g_t(x)\text{对应的}i \end{aligned}$$

现在考虑$\mathrm{sign}(x_{t_i} - \theta_t)\mathrm{sign}(x_{t_i}^{'} - \theta_t)$，分两种情况考虑，如果$\theta_t \in [\min(x_i, x_i^{'}), \max(x_i, x_i^{'}))$，那么$\mathrm{sign}(x_{t_i} - \theta_t)\mathrm{sign}(x_{t_i}^{'} - \theta_t)$异号，其余情况$\mathrm{sign}(x_{t_i} - \theta_t)\mathrm{sign}(x_{t_i}^{'} - \theta_t)$同号，总结如下

$$\mathrm{sign}(x_{t_i} - \theta_t)\mathrm{sign}(x_{t_i}^{'} - \theta_t) = \begin{cases} -1, & \theta_t \in [\min(x_{t_i}, x_{t_i}^{'}), \max(x_{t_i}, x_{t_i}^{'})) \\ 1, & \text{其他} \end{cases}$$

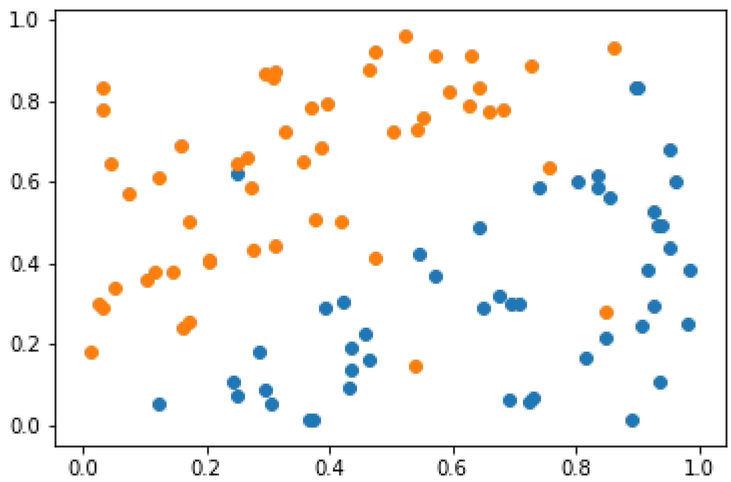
所以上述求和式中$\sum_{t=1}^{|\mathcal{G}|} \mathrm{sign}(x_{t_i} - \theta_i)\mathrm{sign}(x_{t_i}^{'} - \theta_t)$中$+1, -1$的数量取决于$x_{t_i}, x_{t_i}^{'}$，在$[\min(x_{t_i}, x_{t_i}^{'}), \max(x_{t_i}, x_{t_i}^{'}))$中，一共有$|x_{t_i} - x_{t_i}^{'}|$个整数，所以使得$\mathrm{sign}(x_{t_i} - \theta_t)\mathrm{sign}(x_{t_i}^{'} - \theta_t) = -1$的分类器一共有$2\sum_{j=1}^{d} |x_j - x_j^{'}| = 2||x - x^{'}||_1$，这里乘以2是因为还要考虑$s$有两种可能，从而使得$\mathrm{sign}(x_{t_i} - \theta_t)\mathrm{sign}(x_{t_i}^{'} - \theta_t) = 1$的数量一共有$|\mathcal{G}| - 2||x - x^{'}||_1$，所以

$$\begin{aligned} K_{ds}(x, x^{'}) &= \sum_{t=1}^{|\mathcal{G}|} \mathrm{sign}(x_{t_i} - \theta_t)\mathrm{sign}(x_{t_i}^{'} - \theta_t) \\ &= |\mathcal{G}| - 2||x - x^{'}||_1 - 2||x - x^{'}||_1 \\ &= |\mathcal{G}| - 4||x - x^{'}||_1 \\ &= 2d(R-L) - 4||x - x^{'}||_1 + 2 \end{aligned}$$

Problem 12

题目的思路是这样的，利用decision stump来产生原始模型，然后用Adaptive Boosting算法得到最终结果，先作图看下。

```
import numpy as np
import matplotlib.pyplot as plt

train = np.genfromtxt('hw2_adaboost_train.dat')
test = np.genfromtxt('hw2_adaboost_test.dat')
plt.scatter(train[:, 0][train[:, 2] == 1], train[:, 1][train[:, 2] == 1])
plt.scatter(train[:, 0][train[:, 2] == -1], train[:, 1][train[:, 2] == -1])
plt.show()
```




```
#按第一个下标排序
train1 = np.array(sorted(train, key=lambda x:x[0]))

#按第二个下标排序
train2 = np.array(sorted(train, key=lambda x:x[1]))

#获得临界点
x1 = train1[:, 0]
threshold1 = np.append(np.array(x1[0]-0.1), (x1[:-1] + x1[1:])/2)
threshold1 = np.append(threshold1, x1[-1]+0.1)

x2 = train1[:, 1]
threshold2 = np.append(np.array(x2[0]-0.1), (x2[:-1] + x2[1:])/2)
threshold2 = np.append(threshold2, x2[-1]+0.1)

threshold = [threshold1, threshold2]

y = train1[:, 2 ]

n = len(train)

def decision_stump(X, U, threshold):
    #获得数据
```

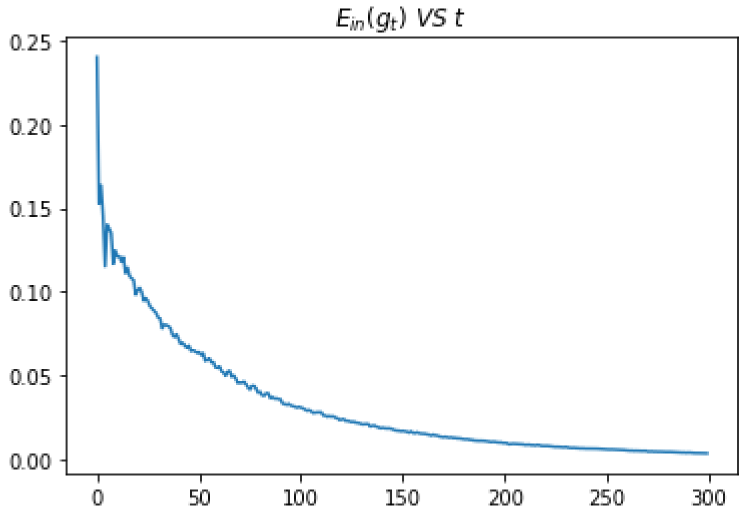
```python
    x1 = X[:, 0]
    x2 = X[:, 1]
    y = X[:, 2]

    #获得数据数量
    n = len(x1)

    #记录维度
    d = 0
    #记录索引
    index = 0
    #记录Ein
    Ein = 1
    #记录s
    s = 1

    for i in range(n+1):
        t1 = threshold[0][i]
        #计算第一个维度的Ein
        E11 = (np.sign(x1 - t1) != y).dot(U)
        E12 = (np.sign(t1 - x1) != y).dot(U)
        if(E11 < Ein):
            d = 0
            index = i
            Ein = E11
            s = 1
        if(E12 < Ein):
            d = 0
            index = i
            Ein = E12
            s = -1
        #计算第二个维度的Ein
        t2 = threshold[1][i]
        E21 = (np.sign(x2 - t2) != y).dot(U)
        E22 = (np.sign(t2 - x2) != y).dot(U)
        if(E21 < Ein):
            d = 1
            index = i
            Ein = E21
            s = 1
        if(E22 < Ein):
            d = 1
            index = i
            Ein = E22
            s = -1
    return Ein,s,d,index

def Adaptive_Boosting(X,  threshold, T = 300):
    n = len(X)
    u = np.ones(n)/n

    #记录需要的数据
    Alpha = np.array([])
```

```python
    U = np.array([])
    Epsilon = np.array([])
    Ein = np.array([])
    G = np.array([])

    #准备数据
    x1 = X[:, 0]
    x2 = X[:, 1]
    x = [x1, x2]
    y = X[:, 2]

    for t in range(T):
        ein,s,d,index = decision_stump(X, u, threshold)
        epsilon = u.dot((s*np.sign(x[d] - threshold[d][index])) != y)/np.sum(u)
        k = np.sqrt((1 - epsilon)/epsilon)
        #找到错误的点
        i1 = s*np.sign(x[d] - threshold[d][index]) != y
        #更新权重
        u[i1] = u[i1]*k
        #找到正确的点
        i2 = s*np.sign(x[d] - threshold[d][index]) == y
        #更新权重
        u[i2] = u[i2]/k
        alpha = np.log(k)

        #存储数据
        Ein = np.append(Ein, ein)
        if(t == 0):
            U = np.array([u])
        else:
            U = np.concatenate((U, np.array([u])),axis = 0)
        Epsilon = np.append(Epsilon, epsilon)
        Alpha = np.append(Alpha, alpha)
        g = [[s,d,index]]
        if(t == 0):
            G = np.array(g)
        else:
            G = np.concatenate((G,np.array(g)),axis = 0)
    return Ein, U, Epsilon, Alpha, G
```

训练数据

```python
Ein, U, Epsilon, Alpha, G = Adaptive_Boosting(train, threshold, T = 300)
```

```python
T = 300
t = np.arange(T)

plt.plot(t, Ein)
plt.title("$E_{in}(g_t)\ VS\ t$")
plt.show()
print("Ein(g1) =", Ein[0], ",alpha1 =", Alpha[0])
```




```
Ein(g1) = 0.24 ,alpha1 = 0.576339754969
```

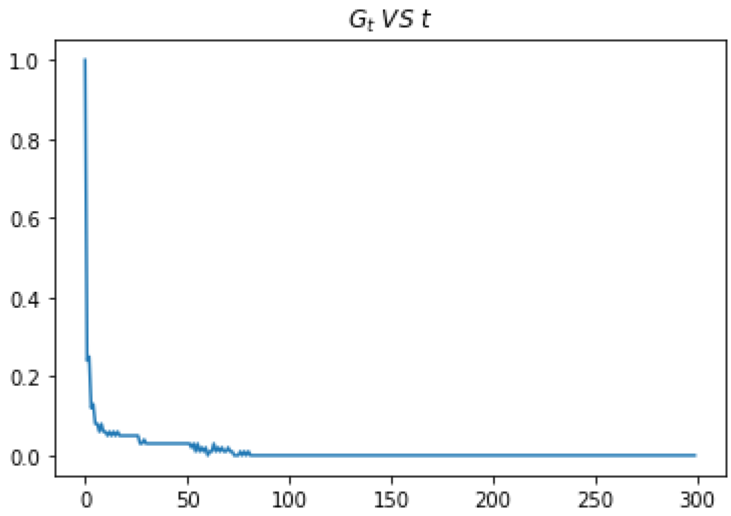
Problem 13

$E_{in}(g_t)$在逐渐变小，因为Adaptive Boosting算法每次对错误的点增加权重，正确的点减小权重，所以每一次比前一次的分类效果都会逐渐变好。

problem 14

```
def predeict(X, G, Alpha, t, threshold):
    "预测Ein(Gt)"
    x1 = X[:, 0]
    x2 = X[:, 1]
    x = [x1, x2]
    y = X[:, 2]
    N = len(X)

    s = G[:t, 0]
    d = G[:t, 1]
    thresh = G[:t, 2]
    alpha = Alpha[:t]

    result = []
    for i in range(t):
        s1 = s[i]
        d1 = d[i]
        t1 = thresh[i]
        #print(s1,d1,t1)
        result.append(s1*np.sign(x[d1] - threshold[d1][t1]))
    result = alpha.dot(np.array(result))
```

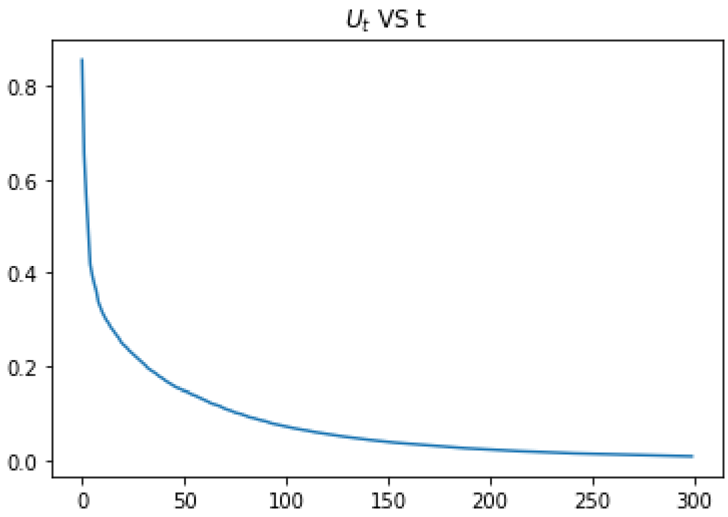
```
    return np.sum(np.sign(result) != y)/len(y)

T = 300
t = np.arange(T)
G1 = [predeict(train, G, Alpha, i, threshold) for i in t]

plt.plot(t, G1)
plt.title("$G_t\ VS\ t$")
plt.show()

print("Ein(G) =",G1[-1])
```




```
Ein(G) = 0.0
```

problem 15

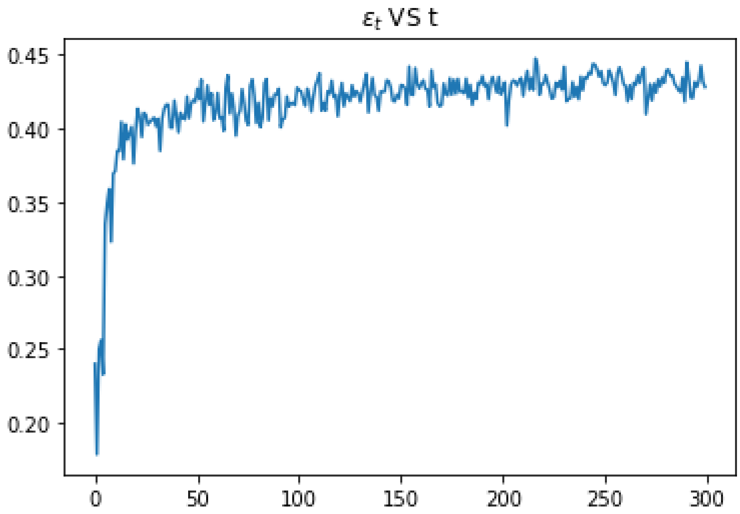
```
U1 = U.sum(axis = 1)

plt.plot(t,U1)
plt.title('$U_t$ VS t')
plt.show()

print("U2 =",U1[1],"UT =",U1[-1])
```




```
U2 = 0.654503963774 UT = 0.00859677507496
```

problem 16

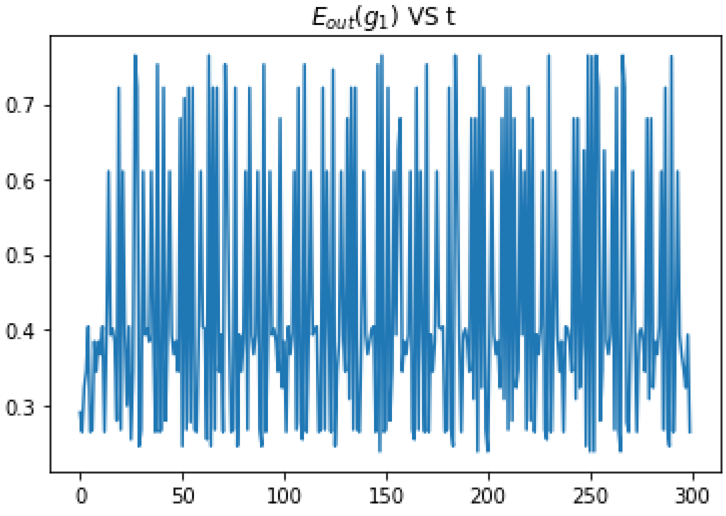
```
plt.plot(t,Epsilon)
plt.title('$\epsilon_t$ VS t')
plt.show()

print("minimun epsilon =",np.min(Epsilon))
```




```
minimun epsilon = 0.178728070175
```

problem 17

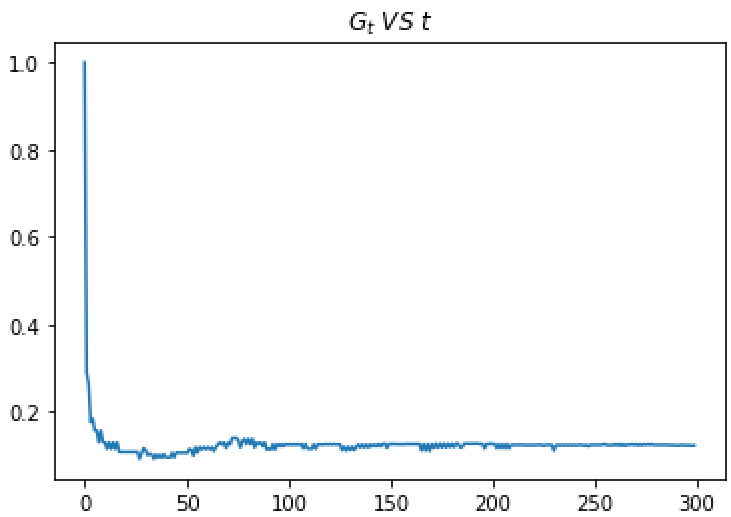
```
x1 = test[:, 0]
x2 = test[:, 1]
xtest = [x1, x2]
ytest = test[:, 2]
N = len(x1)

s = G[:, 0]
d = G[:, 1]
thresh = G[:, 2]

g = []
for i in range(300):
    s1 = s[i]
    d1 = d[i]
    t1 = thresh[i]
    #print(s1,d1,t1)
    g.append(np.sum(s1*np.sign(xtest[d1] - threshold[d1][t1]) != ytest)/N)

plt.plot(t, g)
plt.title('$E_{out}(g_1)$ VS t')
plt.show()

print("Eout(g1) =",g[0])
```




```
Eout(g1) = 0.29
```

problem 18

```
T = 300
t = np.arange(T)
G2 = [predeict(test, G, Alpha, i, threshold) for i in t]

plt.plot(t, G2)
plt.title("$G_t\ VS\ t$")
plt.show()

print("Ein(G) =",G2[-1])
```




```
Ein(G) = 0.123
```

Problem 19

这两题主要计算出矩阵K即可，偷懒的话可以直接用sklearn的包。

```
import numpy as np
from scipy.linalg import inv

data = np.genfromtxt('hw2_lssvm_all.dat')

#获得K
def generateK(X, X1, gamma):
    n = X.shape[0]
    m = X1.shape[0]
    K = np.zeros((n,m))
    for i in range(n):
        K[i, :] = - np.sum((X1 - X[i])**2, axis = 1)
    return np.exp(gamma*K)

n = int(data.shape[0] * 0.8)
m = data.shape[0] - n
```

```
trainx = data[:n,:][:, :-1]
trainy = data[:n,:][:, -1]
testx = data[n:,:][:, :-1]
testy = data[n:,:][:, -1]

Gamma = [32, 2, 0.125]
Lambda = [0.001, 1, 1000]

gammatrain = Gamma[0]
lambdatrain = Lambda[0]
gammatest = Gamma[0]
lambdatest = Lambda[0]
Ein = 1
Eout = 1

for i in Gamma:
    K = generateK(trainx, trainx, i)
    K1 = generateK(trainx, testx, i)
    for j in Lambda:
        beta = inv(np.eye(n)*j + K).dot(trainy)
        r1 = beta.T.dot(K)
        r2 = beta.T.dot(K1).T
        ein = np.sum(np.sign(r1) != trainy)/n
        eout = np.sum(np.sign(r2) != testy)/m
        if(ein < Ein):
            Ein = ein
            gammatrain = i
            lambdatrain = j
        if(eout < Eout):
            Eout = eout
            gammatest = i
            lambdatest = j

print("minimum Ein =", Ein)
print("minimum Eout =", Eout)
```

```
minimum Ein = 0.0
minimum Eout = 0.39
```

Problem 19

这两题主要计算出矩阵K即可，偷懒的话可以直接用sklearn的包。

```
import numpy as np
from scipy.linalg import inv

data = np.genfromtxt('hw2_lssvm_all.dat')

#获得K
```

```python
def generateK(X, X1, gamma):
    n = X.shape[0]
    m = X1.shape[0]
    K = np.zeros((n,m))
    for i in range(n):
        K[i, :] = - np.sum((X1 - X[i])**2, axis = 1)
    return np.exp(gamma*K)

n = int(data.shape[0] * 0.8)
m = data.shape[0] - n

trainx = data[:n,:][:, :-1]
trainy = data[:n,:][:, -1]
testx = data[n:,:][:, :-1]
testy = data[n:,:][:, -1]

Gamma = [32, 2, 0.125]
Lambda = [0.001, 1, 1000]

gammatrain = Gamma[0]
lambdatrain = Lambda[0]
gammatest = Gamma[0]
lambdatest = Lambda[0]
Ein = 1
Eout = 1

for i in Gamma:
    K = generateK(trainx, trainx, i)
    K1 = generateK(trainx, testx, i)
    for j in Lambda:
        beta = inv(np.eye(n)*j + K).dot(trainy)
        r1 = beta.T.dot(K)
        r2 = beta.T.dot(K1).T
        ein = np.sum(np.sign(r1) != trainy)/n
        eout = np.sum(np.sign(r2) != testy)/m
        if(ein < Ein):
            Ein = ein
            gammatrain = i
            lambdatrain = j
        if(eout < Eout):
            Eout = eout
            gammatest = i
            lambdatest = j

print("minimum Ein =", Ein)
print("gamma =", gammatrain)
print("lambda =", lambdatrain)
```

```
minimum Ein = 0.0
gamma = 32
lambda = 0.001
```

Problem 20

```
print("minimum Eout =", Eout)
print("gamma =", gammatest)
print("lambda =", lambdatest)
```

```
minimum Eout = 0.39
gamma = 0.125
lambda = 1000
```

以下两题是证明Adaptive Boosting最终会导致$E_{out} \to 0$

Problem 21

首先看下题目中的条件，我们知道u_n^t的更新规则为

$$u_n^{t+1} = \begin{cases} u_n^t\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}, & y_n g_t(x_n) = -1 \\ u_n^t/\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}, & y_n g_t(x_n) = 1 \end{cases}$$

这个分段的式子可以合起来写为

$$u_n^{t+1} = u_n^t\left(\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}\right)^{-y_n g_t(x_n)}$$

回顾课件我们知道

$$\alpha_t = \ln\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}$$
$$\sqrt{\frac{1-\epsilon_t}{\epsilon_t}} = e^{\alpha_t}$$

这样可以把上式改写为

$$u_n^{t+1} = u_n^t\left(\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}\right)^{-y_n g_t(x_n)} = u_n^t e^{-y_n \alpha_t g_t(x_n)}$$

把这个式子递推下去可得

$$\begin{aligned} u_n^{t+1} &= u_n^t e^{-y_n \alpha_t g_t(x_n)} \\ &= u_n^{t-1} e^{-y_n (\sum_{i=t-1}^{t} \alpha_i g_i(x_n))} \\ &= \ldots \\ &= u_n^1 e^{-y_n (\sum_{i=1}^{t} \alpha_i g_i(x_n))} \\ &= \frac{1}{N} e^{-y_n (\sum_{i=1}^{t} \alpha_i g_i(x_n))} \end{aligned}$$

比较题目的的式子

$$U_{t+1}=\frac{1}{N}\sum_{n=1}^{N}\exp\Big(-y_n\sum_{\tau=1}^{t}\alpha_\tau g_\tau(x_n)\Big)$$

可得

$$U_{t+1}=\sum_{n=1}^{N}u_n^{t+1}$$

现在来证明题目中的结论，利用$u_n^{t+1}=u_n^t e^{-y_n\alpha_t g_t(x_n)},\epsilon_t=\frac{\sum_{y_n\neq g_t(x_n)}u_n^t}{\sum_{n=1}^N u_n^t},\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}=e^{\alpha_t}$

$$\begin{aligned}
U_{t+1}&=\sum_{n=1}^{N}u_n^{t+1}\\
&=\sum_{n=1}^{N}u_n^t e^{-y_n\alpha_t g_t(x_n)}\\
&=\sum_{y_n=g_t(x_n)}u_n^t e^{-\alpha_t}+\sum_{y_n\neq g_t(x_n)}u_n^t e^{\alpha_t}\\
&=\Big(\sum_{n=1}^{N}u_n^t\Big)\Big(e^{-\alpha_t}\frac{\sum_{y_n=g_t(x_n)}u_n^t}{\sum_{n=1}^N u_n^t}+e^{\alpha_t}\frac{\sum_{y_n\neq g_t(x_n)}u_n^t}{\sum_{n=1}^N u_n^t}\Big)\\
&=\Big(\sum_{n=1}^{N}u_n^t\Big)\Big(e^{-\alpha_t}(1-\epsilon_t)+e^{\alpha_t}\epsilon_t\Big)\\
&=U_t\Big(\sqrt{\frac{\epsilon_t}{1-\epsilon_t}}(1-\epsilon_t)+\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}\epsilon_t\Big)\\
&=2U_t\sqrt{\epsilon_t(1-\epsilon_t)}
\end{aligned}$$

因为$\epsilon_t\le\epsilon<\frac{1}{2}$，所以由二次函数的性质可得

$$U_{t+1}=2U_t\sqrt{\epsilon_t(1-\epsilon_t)}\le 2U_t\sqrt{\epsilon(1-\epsilon)}$$

最后补充证明下$E_{\text{in}}(G_T)\le U_{T+1}$，这里需要利用$G_T(x_n)=\text{sign}\Big(\sum_{\tau=1}^T\alpha_\tau g_\tau(x_n)\Big)$以及$[\![\text{sign}(x)\neq 1]\!]\le e^{-x}$

$$\begin{aligned}
E_{\text{in}}(G_T)&=\frac{1}{N}\sum_{n=1}^{N}[\![y_n\neq G_T(x_n)]\!]\\
&=\frac{1}{N}\sum_{n=1}^{N}[\![y_nG_T(x_n)\neq 1]\!]\\
&=\frac{1}{N}\sum_{n=1}^{N}[\![y_n\text{sign}\Big(\sum_{\tau=1}^{T}\alpha_\tau g_\tau(x_n)\Big)\neq 1]\!]\\
&=\frac{1}{N}\sum_{n=1}^{N}[\![\text{sign}\Big(\sum_{\tau=1}^{T}y_n\alpha_\tau g_\tau(x_n)\Big)\neq 1]\!]\\
&=\frac{1}{N}\sum_{n=1}^{N}e^{-y_n\Big(\sum_{\tau=1}^T\alpha_\tau g_\tau(x_n)\Big)}
\end{aligned}$$

注意

$$U_{T+1}=\frac{1}{N}\sum_{n=1}^{N}e^{-y_n\sum_{\tau=1}^{T}\alpha_\tau g_\tau(x_n)}$$

所以

$$E_{\text{in}}(G_T)\le U_{T+1}$$

Problem 22

首先把题目给出的条件简单证明下，利用的结论是$1-x\le e^{-x}$

$$\sqrt{\epsilon(1-\epsilon)}=\sqrt{\frac{1}{4}-(\epsilon-\frac{1}{2})^2}=\frac{1}{2}\sqrt{1-4(\epsilon-\frac{1}{2})^2}\le\frac{1}{2}\sqrt{e^{-4(\epsilon-\frac{1}{2})^2}}=\frac{1}{2}e^{-2(\epsilon-\frac{1}{2})^2}$$

所以该结论成立。

利用上题$U_{t+1}\le U_t.2\sqrt{\epsilon(1-\epsilon)},U_1=1$可得

$$U_{t+1}\le U_t.2\sqrt{\epsilon(1-\epsilon)}\le U_te^{-2(\epsilon-\frac{1}{2})^2}$$
$$U_{t+1}\le U_te^{-2(\epsilon-\frac{1}{2})^2}\le U_{t-1}e^{-2\times2(\epsilon-\frac{1}{2})^2}\le\ldots\le U_1e^{-2\times t(\epsilon-\frac{1}{2})^2}=e^{-2t(\epsilon-\frac{1}{2})^2}$$
$$U_{T+1}\le e^{-2T(\epsilon-\frac{1}{2})^2}$$

如果$e^{-2T(\epsilon-\frac{1}{2})^2}<\frac{1}{N}$，那么$E_{\text{in}}(G_T)\le U_{T+1}<\frac{1}{N}$，因为误差函数为$0,1$误差，所以此时$E_{\text{in}}(G_T)=0$，现在解$e^{-2T(\epsilon-\frac{1}{2})^2}<\frac{1}{N}$这个不等式

$$e^{-2T(\epsilon-\frac{1}{2})^2}<\frac{1}{N}$$
$$N<e^{2T(\epsilon-\frac{1}{2})^2}$$
$$\ln N<2T(\epsilon-\frac{1}{2})^2$$
$$T=O(\log N)$$

所以结论成立。