

Stability of Diamond Turning Processes That Use Round Nosed Tools*

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Abstract

In this article, a multi-mode model is developed for a diamond-turning process which takes into account the cutting forces that result from the geometry of the chip area cut by a round nosed tool. These cutting forces are assumed proportional to the uncut chip area. The chip area generated during the cutting action is approximately modeled as a parabolic segment that is a function of the tool feed rate, the depth of cut, and the current and previous tool-displacement histories. This leads to a system of retarded differential equations that is studied to determine the stability of the cutting process with respect to parameters such as feed rate, depth of cut, and spindle speed. The results are presented in the form of stability charts. For round nosed tools, these charts are found to be perturbed forms of those obtained for orthogonal cutting tools. Furthermore, for a given fixed feed rate, the two most significant parameters that affect the stability regions are found to be the tool nose radius and the material damping ratio. The predicted stability results are found to be consistent with observations made during experiments. The model formulation, which is fairly general, should be applicable for determining stable parameter values for a wide variety of turning processes. This work also points to the importance of considering a multi-mode formulation for high-speed turning processes.

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1 Nomenclature

a_i	=	amplitude for mode i
d	=	depth of cut
f	=	feed per revolution
q_i	=	mode i amplitude about equilibrium
v	=	planar displacement of the boring bar
v_τ	=	planar displacement at previous tool pass
w_i	=	coefficients of perturbed frequency
z	=	axial coordinate along the boring bar
z_0	=	half chip width
A	=	cross-sectional area of the boring bar
A_c	=	approximate area of chip
A_p	=	parabolic chip area
A_0	=	circular chip area
E	=	modulus of elasticity of the boring bar
E_i	=	mode i equilibrium point
G_i	=	real part of mode i transfer function
H_i	=	imaginary part of mode i transfer function
I	=	moment-of-inertia of the boring bar
K	=	scaled workpiece specific energy coefficient
K_c	=	specific cutting energy of the workpiece
L	=	length of the boring bar
R	=	radius of diamond tool tip
W_i	=	spatial shape, mode i
α_i, β_i	=	approximate chip area coefficients
γ	=	damping coefficient of the boring bar
δ	=	Dirac delta function
ϵ	=	ratio of feed to chip width
λ	=	fraction of cutting energy transmitted to workpiece
ξ_i	=	damping factor, mode i
ρ	=	material density of the boring bar
τ	=	time per spindle revolution
τ_0	=	first term of perturbed delay
ψ	=	phase angle
Φ_i	=	mode i transfer function
ω_i	=	natural frequency, mode i
Ω	=	rotation rate of the spindle

2 Introduction

The scientific study of undesirable vibrations of the tool-workpiece system in machining processes dates back more than fifty years (Arnold,1946). This phenomenon can occur due to regenerative and non-regenerative effects. The chatter that arises due to regenerative effects (oscillations induced as the tool removes chips from the surface that was produced by the tool during its preceding pass) has been studied extensively in the literature (Arnold, 1946; Merritt, 1965; Tobias, 1965; Koenigsberger and Thusty, 1971; Hanna and Tobias, 1974; Thusty, 1985). In a recent study, Davies and Balachandran (1996) studied a milling process and

demonstrated that chatter can also occur due to impact dynamics resulting from intermittent engagement of the workpiece and the tool.

It is well known that modeling a turning process with a regenerative forcing term leads to a system of differential equations that involve a time delay in the state variable describing the tool displacement from its nominal position. The characteristic equation for the stability analysis of such a system is usually a complex transcendental equation that is difficult to solve for specific eigenvalues. A graphical tool, called a *stability chart*, has been introduced to show the rotational speeds at which the sytem is supposed to be stable, unstable, or at a threshold of stability (Tobias and Fishwick, 1958). Various authors have used this technique with one-degree-of-freedom models for analyzing stability under orthogonal cutting conditions (e.g., Merritt (1965) , Lemon and Ackermann (1965), Hanna and Tobias (1974), Tlusty (1985), Smith and Tlusty (1992). Recently Budak and Altintas (1995a,b) and Özdoğanlar and Endres (1997) have extended the method to the multi-mode case for orthogonal cutting. To the authors' knowledge, no general algorithm has been introduced in the literature for systematically producing multi-mode stability charts for the practical problem of a boring bar with a round nosed tool tip in a high precision diamond turning process.

The principal contributions of this work are the following: (a) development of a finite-dimensional model for a diamond-turning process taking into account the geometry of the chip area cut by a round nosed tool, (b) stability analysis for a multi-mode system characterized by a single time delay, (c) demonstration that the stability charts for a round nosed tool are modifications of those obtained for orthogonal cutting, and (d) sensitivity analysis of the critical stability region as a function of tool length, tool tip radius, and depth of cut. The model upon which these contributions are made is based on the following assumptions:

- (1) Motions are planar.
- (2) The cutting force is proportional to the uncut chip area of a rigid workpiece.
- (3) The tool, attached to a rigid machine, can be modeled as a uniform and homogeneous cantilever beam.
- (4) The uncut chip area is affected by tool position after only one previous pass.
- (5) The circular tool tip profile can be approximated by a parabolic tool tip profile with a depth of cut much smaller than the tool tip radius.
- (6) The feed rate is small relative to the chip width.

The rest of this paper is organized as follows. In the second section, the authors develop a model for the boring bar. To investigate the stability of the process, the characteristic equation and the associated stability boundaries are derived in Section 3 along with a discussion of an efficient numerical algorithm that can be used to graph multi-modal stability charts. In Section 4, a description of an experiment that influenced the model parameter selections is provided. A discussion of results is given in Section 5 with a closure given in Section 6. A determinant used in the paper is evaluated in APPENDIX A, and an estimation of the modeling errors involved in the perturbation analysis is given in APPENDIX B.

3 Model Formulation

The boring bar used in the diamond turning process is a solid round metal shaft with a single crystal diamond tool brazed to the end of the shaft, as illustrated in Figure 1 a. It is modeled as a uniform and homogeneous cantilever beam with the z -axis oriented along the axial direction as shown in Figure 1b. The location $z = 0$ is at the fixed end and the location $z = L$ is at the free end. The cutting portion of the tool point is assumed to be located at $z = L - R$, where R is the radius of the tool nose. The x -axis is oriented vertically with positive direction into the workpiece. The diamond turning machine and the workpiece are assumed not to displace in the x direction and the workpiece is assumed to rotate counterclockwise. The force that deflects the bar in the sensitive direction is assumed to be a point force proportional to the chip area (Kalpakjian, 1995) and is directed in the negative x direction. The chip area is modeled as a function of the tool feed rate, the depth of cut, the tool tip radius, the surface description during the previous revolution, and the surface description during the current revolution.

Assuming that the boring bar has uniform and homogeneous properties, the equation governing the planar flexural oscillations is of the form

$$\rho A \frac{\partial^2 v}{\partial t^2} + \gamma \frac{\partial v}{\partial t} + EI \frac{\partial^4 v}{\partial z^4} = -\delta(z - L + R) \lambda K_c A_c(f, d, R, v(t - \tau), v(t)) \quad (1)$$

for $0 < z < L$, with boundary conditions

$$v(0, t) = \frac{\partial v}{\partial z}(0, t) = \frac{\partial^2 v}{\partial z^2}(L, t) = \frac{\partial^3 v}{\partial z^3}(L, t) = 0 \quad (2)$$

and initial conditions $v(z, 0) = v_0(z)$, $\partial v / \partial t(z, 0) = v_1(z)$.

In equations (1) and (2), $v(z, t)$ is the vertical displacement of the boring bar center line relative to its nominal position in the plane, ρ is the boring bar's material density, A is the boring bar's cross sectional area, E is the boring bar's modulus of elasticity, I is the bar's moment-of-inertia, γ is the damping coefficient

of the boring bar material, K_c is the specific cutting energy of the workpiece material, and λ is the fraction of the cutting energy transmitted vertically to the tool tip. The δ function is used to simulate a point force at the tool tip. The chip area A_c is a function of the feed rate f , the nominal depth of cut, d , the tool tip radius, R , the tool displacements at the previous cut, $v(t - \tau)$, and the present cut, $v(t)$, where τ is the delay time for one spindle rotation. An underlying assumption of this model is that during one feed pass of the round nosed tool it produces a helical groove in the workpiece that overlaps with itself depending on the feed rate and spindle speed. The current model assumes that the feed rate is slow relative to the spindle speed so that the grooves overlap and can lead to regenerative effects.

Under an orthogonal cutting assumption the area of the chip that can be modeled by A_c is proportional to $v(t - \tau) - v(t)$. However, in diamond turning with a round nosed tool this proportionality does not apply and one must compute the chip area scalloped from the surface. In Figure 2a, the area of a chip removed in one pass of a round nosed tool of tip radius R with a nominal depth of cut d is shown. The half width for the cut is given by

$$z_0 = \sqrt{2Rd - d^2} \quad (3)$$

and the chip area is given by

$$A_0 = R^2 \sin^{-1} \left(\frac{z_0}{R} \right) - (R - d)z_0 \quad (4)$$

In a recent study by Marsh et. al (1997), it is shown that the chip area cut by a round nosed tool tip in diamond turning can be approximated by the area cut by a tool with a parabolic tip having an appropriately set focal length. In particular, if the focal length is set as $R/2 - d/4$ then the half width of the parabolic tool tip cut is the same as the half width given in equation (3) and the depth of cut is d . The resulting area is given by

$$A_p = \frac{4}{3}dz_0 \quad (5)$$

If we use the first two terms of the Taylor series expansion for $\sin^{-1}(x)$ around $x = 0$ the relative error between A_0 and A_p can be shown to be

$$\left| \frac{A_0 - A_p}{A_0} \right| \approx \frac{d}{8R - d} \quad (6)$$

A typical depth of cut in high precision diamond turning is $2 \mu m$. For a tool tip with a radius of $500 \mu m$, the relative error computed from equation (6) would then be 0.05%. Thus, for diamond turning, the parabolic model of a tool tip is adequate.

Given a single feed after one revolution of f , the profiles in Figure 2b for the parabolic tool tip model are

given by

$$x = -\frac{z^2}{2R-d} + \frac{R}{2} - \frac{d}{4} + v_\tau \quad (7)$$

$$x_f = -\frac{(z-f)^2}{2R-d} + \frac{R}{2} - \frac{d}{4} + v \quad (8)$$

where $v_\tau = v(t - \tau)$. The points defining the chip area are

$$z_1 = \frac{f}{2} + (v_\tau - v)\frac{2R-d}{2f} \quad (9)$$

$$z_2 = \sqrt{(2R-d)(d+v_\tau)} \quad (10)$$

$$z_3 = f + \sqrt{(2R-d)(d+v)} \quad (11)$$

The area A_c is given by

$$A_c = \int_{z_1}^{z_2} \left[\frac{z^2 - (z-f)^2}{2R-d} + (v - v_\tau) \right] dz + \int_{z_2}^{z_3} \left[-\frac{(z-f)^2}{2R-d} + d + v \right] dz \quad (12)$$

In order to study the stability of the system a symbolic manipulator was used to compute the linear approximation to A_c as

$$A_c = \alpha_0 + \alpha_1 v + \beta_1 v_\tau \quad (13)$$

where

$$\alpha_0 = \frac{24dfR - f^3 - 12d^2f}{24R - 12d} \quad (14)$$

$$\alpha_1 = \frac{f}{2} + z_0 \quad (15)$$

$$\beta_1 = \frac{f}{2} - z_0 \quad (16)$$

Note that as f decreases relative to the other variables in the problem the chip area can be approximated by $A_c \approx df + z_0(v(t) - v(t - \tau))$ which is the same form as that obtained for orthogonal cutting. Geometrically this makes sense since, as $\frac{f}{R}$ decreases, the tool can be approximated by a triangular wedge and the chip area is approximately the sum of two parallelograms, one with area df and the other with area $z_0(v(t) - v(t - \tau))$.

When equation (13) is used in equation (1), that system is reduced to a set of ordinary differential equations that are more amenable to analysis by using a Galerkin projection, similar to that used by Lee, Liu, and Chang (1991) and Davies and Balachandran (1996). The solution is assumed to be of the form

$$v(z, t) = \sum_{i=1}^N a_i(t) W_i(z) \quad (17)$$

where the spatial mode shapes $W_i(z)$ are approximated by the spatial modes associated with the free vibrations of an undamped, isotropic cantilever beam (Meirovitch, 1986) and $a_i(t)$ represents the time dependent

which is true since c_2 is positive for a positive feed rate. Thus the equations (22) can be inverted to solve for a nontrivial equilibrium position E_i .

Assume that the motions described by the system (19) are considered about the equilibrium point, one will need to study the stability of the following system of N equations

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q_i = -W_i K\alpha_1 \sum_{n=1}^N q_n W_n - W_i K\beta_1 \sum_{n=1}^N q_{n\tau} W_n \quad (24)$$

System (24) can also be put into vector form by defining $\mathbf{q} = (q_1 \dots q_N)^T$ and rewriting the system as

$$\begin{aligned} \ddot{\mathbf{q}} + \begin{pmatrix} 2\xi_1\omega_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 2\xi_N\omega_N \end{pmatrix} \dot{\mathbf{q}} + \begin{pmatrix} \omega_1^2 + K\alpha_1 W_1 W_1 & \cdots & K\alpha_1 W_1 W_N \\ \cdots & \cdots & \cdots \\ K\alpha_1 W_N W_1 & \cdots & \omega_N^2 + K\alpha_1 W_N W_N \end{pmatrix} \mathbf{q} \\ = -K\beta_1 \begin{pmatrix} W_1 W_1 & \cdots & W_1 W_N \\ \cdots & \cdots & \cdots \\ W_N W_1 & \cdots & W_N W_N \end{pmatrix} \mathbf{q}_\tau \end{aligned} \quad (25)$$

with initial conditions $\mathbf{q}(0) = \mathbf{0}$, $\dot{\mathbf{q}}(0) \neq \mathbf{0}$, and $\mathbf{q}(t) = \mathbf{0}$ for $t \in [-\tau, 0]$. The existence and uniqueness of solutions to (25) are guaranteed by the general existence and uniqueness conditions given in Hale and Lunel (1993).

4 Stability of the Process

The stability of the process corresponds to the stability of the equilibrium position $(q_1 \dots q_N) = (0 \dots 0)$ of the system (25). If this equilibrium position is stable (unstable), the diamond turning process is said to be stable (unstable). The instability of this equilibrium position will mean undesirable oscillations of the tool during the manufacturing process. The characteristic equation of system (25) permits us to determine the stability boundaries between stable and unstable motions.

4.1 Characteristic Equation

In order to find the characteristic equation, the Laplace transform of system (25) is taken. The system $\mathbf{M}\mathcal{L}(\mathbf{q}) = \dot{\mathbf{q}}(0)$ is obtained, noting $\mathbf{q}(0) = \mathbf{0}$, where $\mathcal{L}(\mathbf{q})$ is the Laplace transform of \mathbf{q} and

$$\mathbf{M} = \begin{pmatrix} s^2 + 2\xi_1\omega_1 s + \omega_1^2 + K(\alpha_1 + \beta_1 e^{-s\tau})W_1 W_1 & \cdots & K(\alpha_1 + \beta_1 e^{-s\tau})W_1 W_N \\ \cdots & \cdots & \cdots \\ K(\alpha_1 + \beta_1 e^{-s\tau})W_N W_1 & \cdots & s^2 + 2\xi_N\omega_N s + \omega_N^2 + K(\alpha_1 + \beta_1 e^{-s\tau})W_N W_N \end{pmatrix} \quad (26)$$

The characteristic equation is computed by setting the determinant of \mathbf{M} to zero. The determinant is evaluated, by using the result in APPENDIX A, as

$$\det(\mathbf{M}) = \prod_{j=1}^N (s^2 + 2\xi_j\omega_j s + \omega_j^2) + K(\alpha_1 + \beta_1 e^{-s\tau}) \sum_{m=1}^N \left(\prod_{\substack{k=1 \\ k \neq m}}^N (s^2 + 2\xi_k\omega_k s + \omega_k^2) \right) W_m^2 \quad (27)$$

For $i = 1, \dots, N$, with $\xi_i \neq 0$, the polynomials $s^2 + 2\xi_i\omega_i s + \omega_i^2$ cannot have a zero on the imaginary axis.

The right hand side of equation (27) is referred to in the literature as a quasipolynomial (El'sgol'ts and Norkin, 1973). The solutions of the transcendental characteristic equation (27) are also called characteristic roots. Since there are an infinite number of them, only the stability boundaries between regions having all characteristic roots with negative real parts (stable regions) and those with at least one characteristic root with a positive real part (unstable regions) are usually computed. The graphs of these boundaries are the *stability charts*. For a given depth of cut, d and feed rate f these stability charts will be drawn in two dimensional parameter space (Ω, K) where $\Omega = \frac{1}{\tau}$ is the spindle rotation rate and K is a material parameter proportional to the specific cutting energy, K_c , given by equation (20). In this paper we will be concerned with the effect that different materials will have on the stability of the process so that K becomes a natural parameter. The width of cut parameter that is conventionally used in the stability analysis of orthogonal cutting is appears in the round nosed tool problem in a nonlinear manner and is not directly available as an independent parameter.

4.2 Stability Boundaries

Because this problem is somewhat different than that of simple orthogonal cutting, calculation of stability boundaries is somewhat more involved. However, for feeds, f , and depths of cut, d , much smaller than the tool tip radius, R , the equations are amenable to analytic solution. Following the orthogonal cutting analysis, let s be a complex variable. Then $\det(\mathbf{M}) = 0$ translates to

$$1 + K(\alpha_1 + \beta_1 e^{-s\tau}) \sum_{m=1}^N \Phi_m(s) W_m^2 = 0 \quad (28)$$

where

$$\Phi_m(s) = \frac{1}{s^2 + 2\xi_m\omega_m s + \omega_m^2} \quad (29)$$

For the stability boundary analysis, let $s = i\omega$. Then

$$\Phi_m(i\omega) = G_m(\omega) + iH_m(\omega) \quad (30)$$

where

$$\begin{aligned}
G_m(\omega) &= \frac{1}{\omega_m^2} \frac{\left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right]}{\left[\left(1 - \left(\frac{\omega}{\omega_m}\right)^2\right)^2 + \left(2\xi_m \left(\frac{\omega}{\omega_m}\right)\right)^2\right]} \\
H_m(\omega) &= \frac{1}{\omega_m^2} \frac{\left[-2\xi_m \left(\frac{\omega}{\omega_m}\right)\right]}{\left[\left(1 - \left(\frac{\omega}{\omega_m}\right)^2\right)^2 + \left(2\xi_m \left(\frac{\omega}{\omega_m}\right)\right)^2\right]}
\end{aligned} \tag{31}$$

After substituting equations (30) and (31) into equation (28) and equating real and imaginary parts to zero one obtains:

$$1 + K \left[\sum_{m=1}^N W_m^2 G_m(\omega) \right] (\alpha_1 + \beta_1 \cos \omega\tau) + K \left[\sum_{m=1}^N W_m^2 H_m(\omega) \right] \beta_1 \sin \omega\tau = 0 \tag{32}$$

$$- \left[\sum_{m=1}^N W_m^2 H_m(\omega) \right] (\alpha_1 + \beta_1 \cos \omega\tau) + \left[\sum_{m=1}^N W_m^2 G_m(\omega) \right] \beta_1 \sin \omega\tau = 0 \tag{33}$$

In order to simplify the notation let

$$G(\omega) = \sum_{m=1}^N W_m^2 G_m(\omega) \tag{34}$$

$$H(\omega) = \sum_{m=1}^N W_m^2 H_m(\omega) \tag{35}$$

From equation (33), one can obtain

$$\frac{H(\omega)}{G(\omega)} = \frac{\frac{\beta_1}{\alpha_1} \sin \omega\tau}{1 + \frac{\beta_1}{\alpha_1} \cos \omega\tau} \tag{36}$$

The method of perturbations will now be used in two steps to show that the right hand side of equation (36) can be put into a form similar to one that arises in the stability analysis of orthogonal cutting. From equation (15) and (16), one obtains

$$\frac{\beta_1}{\alpha_1} = \frac{-1 + \frac{f}{2z_0}}{1 + \frac{f}{2z_0}} \tag{37}$$

For the sake of notation let

$$\epsilon = \frac{f}{2z_0} \tag{38}$$

To be meaningful the feed must be less than the denominator, which is the chip width; the essential condition we require is that $0 < \epsilon < 1$. In the case of diamond turning ϵ is often very small. For example, using a typical feed rate of $10 \mu\text{m/s}$ at 300 revolutions per minute, one has $f = 2 \mu\text{m}$ per revolution. Then for a tool tip of radius $R = 500 \mu\text{m}$ and a depth of cut of $d = 2 \mu\text{m}$, one can compute $\epsilon = 0.0224$ from equation (38).

By the geometric series, for $0 < \epsilon < 1$,

$$\frac{1}{1 + \epsilon} = 1 - \epsilon + \epsilon^2 - \dots \tag{39}$$

By using equations (37), (38) and (39) one can write

$$\frac{\beta_1}{\alpha_1} = (-1 + \epsilon)(1 - \epsilon + \epsilon^2 - \dots) \approx -1 + 2\epsilon \quad (40)$$

This implies that the right hand side of (36) can be estimated by

$$\frac{\frac{\beta_1}{\alpha_1} \sin \omega \tau}{1 + \frac{\beta_1}{\alpha_1} \cos \omega \tau} \approx \frac{(-1 + 2\epsilon) \sin \omega \tau}{1 + (-1 + 2\epsilon) \cos \omega \tau} \quad (41)$$

Because the right hand side of (41) is near the ratio $\frac{-\sin \omega \tau}{1 - \cos \omega \tau}$ (a ratio obtained under orthogonal cutting conditions) we conjecture that, as a first step, if we consider a perturbed frequency

$$\omega(\epsilon) = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \dots \quad (42)$$

then we can solve for the coefficients w_i for $i = 0, 1, 2, \dots$ so that to a first order ϵ approximation

$$\frac{H(w_0 + \epsilon w_1)}{G(w_0 + \epsilon w_1)} \approx \frac{(-1 + 2\epsilon) \sin(w_0 + \epsilon w_1) \tau_0}{1 + (-1 + 2\epsilon) \cos(w_0 + \epsilon w_1) \tau_0} \quad (43)$$

where w_0 and τ_0 are selected to satisfy the orthogonal cutting conditions obtained when $\epsilon = 0$ (see APPENDIX B). Using the Taylor series, sum of angles formulas and the assumption that $\epsilon w_1 \tau$ is small we can rewrite (43) as

$$\frac{H(w_0) + \epsilon w_1 H'(w_0)}{G(w_0) + \epsilon w_1 G'(w_0)} \approx \frac{-\sin w_0 \tau_0 + \epsilon(2 \sin w_0 \tau_0 - w_1 \tau_0 \cos w_0 \tau_0)}{(1 - \cos w_0 \tau_0) + \epsilon(2 \cos w_0 \tau_0 - w_1 \tau_0 \sin w_0 \tau_0)} \quad (44)$$

The appropriate choice for w_1 is shown in APPENDIX B as

$$w_1 = \frac{2w_0 \sin w_0 \tau_0 G(w_0) - 2w_0 \cos w_0 \tau_0 H(w_0)}{w_0 \tau_0 H(w_0) \sin w_0 \tau_0 + w_0 \tau_0 G(w_0) \cos w_0 \tau_0 + w_0(1 - \cos w_0 \tau_0) H'(w_0) + w_0 \sin w_0 \tau_0 G'(w_0)} \quad (45)$$

In order to use the methods from orthogonal cutting to develop stability charts for round nosed tools we must, in the second step, find a function $\tau(\epsilon)$, defined on an interval about $\epsilon = 0$ such that $\tau(0) = \tau_0$ and

$$\frac{(-1 + \epsilon) \sin \omega(\epsilon) \tau_0}{(1 + \epsilon) + (-1 + \epsilon) \cos \omega(\epsilon) \tau_0} = \frac{-\sin w_0 \tau(\epsilon)}{1 - \cos w_0 \tau(\epsilon)} \quad (46)$$

The details of this argument can also be found in APPENDIX B. Note that in the limit as $\epsilon \rightarrow 0$ both sides of (46) tend to $\frac{-\sin w_0 \tau_0}{1 - \cos w_0 \tau_0}$. This fraction plays a prominent part in the stability analysis of orthogonal cutting. Although the left hand side of (46) is non-singular, it approaches the right hand singular function in the limit as $\epsilon \rightarrow 0$. The two sides are equal up to a neighborhood of the singularity which occurs for $w_0 \tau(\epsilon)$ equal to a multiple of 2π . This is so since the equality only holds for some interval about $\epsilon = 0$ (see APPENDIX B). The closeness of these two functions is shown in Figure 3. The figures differ in a neighborhood of 2π .

Equation (36) can now be rewritten in the form

$$\frac{H(\omega(\epsilon))}{G(\omega(\epsilon))} = \frac{-\sin w_0 \tau(\epsilon)}{1 - \cos w_0 \tau(\epsilon)} = \tan \psi \quad (47)$$

where ψ is the phase of the transfer function that is given by

$$\psi = \arctan \left(\frac{H(\omega(\epsilon))}{G(\omega(\epsilon))} \right) \quad (48)$$

To solve for $w_0\tau(\epsilon)$, half angle formulas are used in equation (47) to show

$$\tan \psi = -\frac{\cos \left(\frac{w_0\tau(\epsilon)}{2} \right)}{\sin \left(\frac{w_0\tau(\epsilon)}{2} \right)} = -\cot \left(\frac{w_0\tau(\epsilon)}{2} \right) = \tan \left(\frac{\pi}{2} + \frac{w_0\tau(\epsilon)}{2} \pm n\pi \right) \quad (49)$$

Then $\psi = \frac{\pi}{2} + \frac{w_0\tau(\epsilon)}{2} \pm n\pi$. Selecting the negative sign and $n = 2 + p$ for $p = 0, 1, 2, \dots$, one obtains

$$w_0\tau(\epsilon) = 2(\psi + p\pi) + 3\pi \quad (50)$$

for $p = 0, 1, 2, \dots$. Using equation (50), the spindle rotation rate is computed as

$$\Omega(\epsilon) = \frac{1}{\tau(\epsilon)} = \frac{w_0}{2(\psi(\epsilon) + p\pi) + 3\pi} \quad (51)$$

Next substitute equation (40) and the first equation in (47) into (32) and keep only first order terms in ϵ to get

$$K(\epsilon) \approx \frac{-1}{2\alpha_1 G(w_0) + \epsilon\alpha_1 P(w_0)} \quad (52)$$

where

$$\begin{aligned} P(w_0) = & G(w_0)[2 \cos w_0\tau_0 + \left(\frac{w_1}{w_0} \right) (w_0\tau_0) \sin w_0\tau_0] \\ & + H(w_0)[2 \sin w_0\tau_0 - \left(\frac{w_1}{w_0} \right) (w_0\tau_0) \cos w_0\tau_0] \\ & + w_1 G'(w_0)(1 - \cos w_0\tau_0) \\ & - w_1 H'(w_0) \sin w_0\tau_0 \end{aligned} \quad (53)$$

As $\epsilon \rightarrow 0$, the cutting energy parameter K approaches the form for orthogonal cutting.

Equations (48) and (52) are related to the recent results of Özdoganlar and Endres (1997). However, they only develop equations for orthogonal cutting for systems up to three degrees-of-freedom. Their results do not account for the round nosed tool correction.

A stability chart is computed by the following algorithm, which has been implemented in software optimized to carry out vector calculations. First, the zeroes of $G(\omega)$ are computed. Subsequently, the frequency intervals in which $G(\omega)$ is negative are determined in order to select the frequency intervals associated with each mode. The denominator of (52) must be selected as negative in order to make $K(\epsilon)$ positive and physically meaningful. If we consider two modes of oscillation only, then $N = 2$ in equation (52) and there

will be two intervals corresponding to these two modes. For example, in the case of a 101.6 mm (4 in) bar with a tool tip radius of 500 μm and a depth of cut of 2.0 μm the graph of the denominator, within the parentheses, of $K(\epsilon)$ is shown in Figure 4. The two intervals, for this case, are given by $(2.73 \times 10^3 \text{s}^{-1}, 1.23 \times 10^4 \text{s}^{-1})$ and $(1.71 \times 10^4 \text{s}^{-1}, +\infty)$. One next discretizes these intervals into frequency vectors. For each of these frequency vectors one computes a K vector using (52). Then, for each $p = 1, 2, \dots$, one uses equation (51) to compute a vector Ω associated with each vector $K(\epsilon)$. Finally, the K -vectors are plotted versus the Ω -vectors for several values of p . This plot, called a *stability chart*, will show a sequence of lobes for each mode.

5 Selecting Model Parameters

5.1 Experimental Arrangement

The experiment described in this section was conducted by Evans and McGlauffin (1988) at the National Institute of Standards and Technology (NIST) and influenced the modeling efforts undertaken in this work. The workpiece, a copper tube, was kinematically mounted to a vacuum chuck of a diamond turning machine. A diamond tipped tool of nose radius of 500 μm was brazed to a steel boring bar of diameter 6.35 mm and length of 101.6 mm. This boring bar was subsequently mounted to a tool post. For a schematic diagram of the boring bar, diamond tipped tool and the workpiece, refer to Figure 1a. Many tool passes were made into and out of the copper tube. The initial feed rate was 20 mm per minute at 300 revolutions per minute. The initial depth of cut on the inward passes was 3 μm and the depth of cut on the outward finishing passes was 2 μm . The lowest feed value used was 0.67 μm per revolution and the smallest depth of cut used was 0.5 μm . The revolutions per minute, feed rates, and depths of cut were all varied but the boring bar was found to encounter severe vibrations in all cases. The best finish was achieved with the lowest feed and smallest depth of cut.

5.2 Model Parameters

The boring bar parameters selected for the stability analysis were based on the properties of the boring bar used in the experiment. The diameter of the boring bar was 6.35 mm (1/4 in). The density of the steel was taken to be $\rho = 7850 \text{ kg/m}^3$ and the modulus of elasticity was taken to be $E = 200 \times 10^9 \text{ Pa}$. The specific cutting energy of copper was taken as $K_c = 2.35 \times 10^9 \text{ N/m}^2$ (Kalpakjian, 1995). The damping factor for the first mode, ξ_1 , was estimated as 0.001 (Evans, 1996), and the damping factor for the second, ξ_2 , was determined from

$$\xi_2 = \xi_1 \frac{\omega_1}{\omega_2} \quad (54)$$

Several parameter ranges in the stability analysis were selected based on the diamond turning machine performance ranges. For the machine used in this work, a diamond turning operator (McGlauffin,1997) suggested that a good rotation rate range to study for stability analysis be restricted between 200 and 1000 revolutions per minute although up to 3000 revolutions per minute were feasible. The machine used in the experiment had a depth of cut resolution of 10 nm and a similar resolution for feed per revolution. For the stability analysis a range of feed rates from 10 nm per revolution to 1.5 μm per revolution and depths of cut from 10 nm to 2 μm were selected.

The tool tip radius had a potentially wide range. Although the tool radius in the experiment was 500 μm the tool tip radius could have been set as low as 1 μm . For the stability analysis, several tool radii were considered from 500 μm to 1 μm , with properly selected feeds and depths of cut so that $0 < \epsilon < 1$.

Although the Galerkin projection allows for an arbitrary mode expansion, for convenience, only a two mode expansion in equation (17) was assumed. The higher mode was added in order to examine the stability effects of the second mode at higher rotation rates. Third and higher modes do not affect the stability of the process in the chosen parameter space.

6 Results and Discussion

As was noted earlier in this paper, one would conventionally present stability chart information graphically as a plot of a chip width parameter versus spindle rotation rate. From equation (3) the chip width, $2z_0$, for the round nosed tool case, is seen to be a function of the tool nose radius and depth of cut. Thus, the chip width for a round nosed tool is not an independent parameter and cannot be used in as direct a manner as it can be in computing stability charts for orthogonal cutting. The parameter, from equation (24), that can be computed directly is a material parameter proportional to the specific cutting energy of the workpiece, given by equation (20). We have therefore chosen to concentrate on the effect that material has on the stability of diamond turning with round nosed tools. We present in this paper, for convenience, the stability charts as a plot of a specific cutting energy parameter versus spindle rotation rate.

6.1 Critical Parameter Values and Stability Lobes

In Table 1, the specific cutting energies (in N/m^2) of workpiece materials bored by eight tool lengths ranging from 0.0127 m (1/2 in) to 0.102 m (4 in) are given. The critical cutting energy is the highest value that K_c can take on before instability occurs at a given feed rate, depth of cut and tool nose radius. The stability chart given in Figure 5 shows that at rotation rates within the usual operating range the lobes overlap each other, allowing relatively small gaps between them for stable operation. This figure shows the stability lobes for a bar, such as the 101.6 mm (4 in) boring bar, with a round nosed tool of $500\ \mu\text{m}$ radius at a feed of $1\ \mu\text{m}$ per revolution and $0.1\ \mu\text{m}$ depth of cut. For example, at a rotation rate of 250 revolutions per minute, the critical cutting energy is approximately $4 \times 10^7\ \text{N/m}^2$. For 240 revolutions per minute the critical cutting energy is approximately $2 \times 10^7\ \text{N/m}^2$ and falls at the bottom of a lobe. This particular critical cutting energy provides an idea of the value below which the boring bar is uniformly stable, but as is clear from the figure there are regions above this minimum critical cutting energy, especially at high rotation rates, where stable operations can be realized. These regions fall in between the lobes. Similar stable regions between lobes have been exploited by Smith and Thusty (1992) who have developed an algorithm to automatically adjust the rotation rate of a milling machine to find the nearest stable region between lobes when their system detects chatter conditions.

All of the parameter combinations selected in Table 1 satisfy the convergence criteria, $0 < \epsilon < 1$. The first three rows of the table show the length, depth of cut and tool nose radius used for each column. Below these rows the first column shows the feeds selected for examination. The other eight columns show the critical workpiece material cutting energy for a given length, depth of cut, tool tip radius and feed. The entries that are blank are critical specific cutting energies that are below that of copper, the principal workpiece for this study. Thus, copper would be an unstable material for those parameter combinations. The cutting energy of copper is taken as $2.35 \times 10^9\ \text{N/m}^2$. This is the average of the values for copper given in Kalpakjian (1995) and is the value used here for comparison. We note that as the length of the bar increases the critical cutting energy decreases. Above the critical cutting energy value the bar motions are unstable, leading to chatter, except in some regions of the parameter space. For example, for a 15.9 mm (5/8 in) bar at a $2\ \mu\text{m}$ depth of cut, with a $500\ \mu\text{m}$ tool tip radius and a feed of $1.5\ \mu\text{m}$ per revolution the critical cutting energy is $5.92 \times 10^9\ \text{N/m}^2$ for the first mode. Copper's critical cutting energy is below this value, so, for this combination of parameters, copper would be stable.

Since the critical cutting energy of copper is greater by two orders of magnitude than the lobe minima of $4.17 \times 10^7\ \text{N/m}^2$ for the case in Figure 5, there is no possibility of stability, which helps explain the observed

Table 1: Critical Specific Cutting Energies at 600 r/min

Critical Cutting Energy $K_c(\text{N/m}^2) \times 10^{-9}$								
L(mm)	12.7	15.9	19.1	22.2	25.4	50.8	76.2	101.6
d(μm)	2	2	2	1	0.5	0.1	0.1	0.05
R (μm)	500	500	500	500	500	10	1	1
f ($\mu\text{m}/r$)								
0.10	12.1	6.01	3.41	3.04	2.85	5.24	4.87	3.63
0.20	12.1	6.00	3.41	3.04	2.84	5.09	4.11	2.49
0.30	12.0	6.00	3.41	3.03	2.83	4.91	3.51	
0.40	12.0	5.99	3.40	3.03	2.83	4.69	3.00	
0.50	12.0	5.98	3.40	3.02	2.82	4.48	2.52	
1.00	12.0	5.95	3.38	3.00	2.79	3.64		
1.50	12.0	5.92	3.36	2.98	2.76	2.86		

instabilities of the experiments at NIST. The initial feed rate selected for the NIST experiment was $66.6 \mu\text{m}$ per revolution. For a $500 \mu\text{m}$ radius tool and a $2 \mu\text{m}$ depth of cut the critical cutting energy for this feed is $7.92 \times 10^6 \text{ N/m}^2$, which is approximately three orders of magnitude lower than copper. The boring bar would clearly be unstable, as the experiment indicated. However, for a bar of length 19.1 mm, the results shown in Table 1 indicate that it is possible to machine in a stable manner at feeds commonly used in diamond turning operations.

6.2 Sensitivity Analysis Results

Figure 6 shows the critical cutting energy values at the low feed of $0.10 \mu\text{m}$ per revolution as a function of boring bar length and tool tip radius for a depth of cut of $1 \mu\text{m}$. As shown in Table 1 this feed value can allow stable diamond turning of copper for all of the lengths selected depending on depth of cut and tool nose radius. The figure shows the critical values of K_c for the range of bar lengths from about 30 mm to 101.6 mm and tool tip radii from $1 \mu\text{m}$ to $500 \mu\text{m}$. For smaller bar lengths the critical cutting energies were above those for most practical materials including hard steel. The most significant observation here is that for smaller tool tip radii and smaller bar lengths the critical cutting energies rise dramatically. The figures for shallower depths of cut are similar and form a family of surfaces one above the other with the higher surfaces associated with shallower depths of cut. Although decreasing the depth of cut makes gains in increasing the stability region, its gains are relatively small compared to reducing the boring bar tool tip radius. Combining a reduction both in the bar length and the tool tip radius brings about a significant increase in the stability region as would be expected.

Figure 7 shows the quadratic behavior of the critical cutting energy as a function of the damping ratio. This

figure represents the results for a 101.6 mm (4 in) bar with a $500\text{ }\mu\text{m}$ tool tip radius at a $2\text{ }\mu\text{m}$ depth of cut and a feed of $0.1\text{ }\mu\text{m}$ per revolution. A fit of a quadratic equation to this data indicates that the function is essentially composed of a linear plus quadratic term. This indicates that implicit in the model is a result similar to the result that, in orthogonal cutting, the chip width limit can be analytically modeled as a linear plus quadratic function of the damping ratio. A fit was used because the exact calculation of the quadratic relation is much too cumbersome to display and requires a symbolic manipulator to generate in the present case.

The effect on stability of adjusting the damping ratio was exploited in a later experiment at NIST. The experiment involved the diamond turning of a long cantilever rod. A damping mechanism was attached to the free end of the rod. It provided sufficient damping to allow stable turning of the rod (Clary, 1997).

6.3 Multi-Mode Case: Stability Chart

In Figure 8, one can observe the effect of the second mode on the stability regions of the first mode. This figure shows that stability lobes for the second mode can overlay the stable regions for the first mode. Although, for diamond turning, the rotation rates shown in the range of 20,000 revolutions per minute are not used, the fact that second mode stability lobes can be encountered within first mode stability regions is a result that could affect high-speed turning or milling when high spindle speeds are used with long tools.

7 Closure

Based on the findings of the current work, the following points are made:

- (a) The stability charts for round nosed tools used in diamond turning can be computed as a perturbation of the stability charts obtained for orthogonal cutting. The perturbation parameter relates the feed to the chip width. As the feed approaches zero the stability analysis approaches that for orthogonal cutting.
- (b) A general method of generating stability charts for boring bars and an associated vector based algorithm for computing them has been presented. In this algorithm, one makes use of general transfer function techniques for multiple-mode systems with a single time delay.
- (c) The stability charts confirm rules developed from diamond turning operator's experience at NIST. Machinists (McGlauffin, 1997) have a rule-of-thumb that the length of a boring bar should ordinarily be no longer than 2.5 times the width of the boring bar. Thus, for the 6.35 mm (1/4 in) width bar with a $500\text{ }\mu\text{m}$

tool tip radius, at $2\text{ }\mu\text{m}$ depth of cut, the rule-of-thumb would indicate that the proper boring bar should be approximately 15.9 mm (5/8 in) in length. From Table 1, one can discern that for a $500\text{ }\mu\text{m}$ tool tip radius, at $2\text{ }\mu\text{m}$ depth-of-cut, a machinist could select a 19.1 mm (6/8 in) bar, which confirms the rule-of-thumb and also indicates that this rule is somewhat conservative in this case. Beyond these lengths, for the specified tool tip radius and depth of cut, the critical cutting energies drop below the critical cutting energy of copper unless, as shown, the depth of cut or other parameters, including the material damping ratio, are adjusted. The rule-of-thumb does not assume adjustments to the material damping ratio. However, adjustments to the material damping ratio clearly affect the stability regions as found in the NIST experiments.

(d) Higher vibration modes can affect the stability of the machining process in high speed machining. High speed machining has been of interest for some time (King,1985). Spindles with speeds greater than 20,000 revolutions per minute are being produced and utilized in industry for milling. At high rotation rates single-mode models are inadequate to predict stability regions at cutting energies above the minimal critical cutting energy. The reason for this is that although at higher speeds the stability lobes of the first mode spread apart and leave gaps between them the lobes of higher modes begin to fill in portions of these gaps with fine structure lobes thus preventing the gaps from being entirely stable. To optimize regions of stability at higher rotation rates, one would need to examine gaps in between the higher mode lobes.

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9 References

- Altintas, Y. and Budak, E., 1995, "Analytical Prediction of Chatter Stability in Milling", *Annals of the CIRP*, Vol. 44, 1, pp. 357-362.
- Apostol, T.M., 1974, *Mathematical Analysis*, Addison-Wesley Publishing Company, Reading.
- Arnold, R.N., 1946, "The Mechanism of Tool Vibration in the Cutting of Steel", *Proceedings Institute of Mechanical Engineers*, p. 154.

- Budak, E. and Altintas, Y., 1995a, "Analytical Prediction of Chatter Stability in Milling, Part I: General Formulation", *Proc. ASME Dyn. Sys. and Ctrl. Div.*, DSC-Vol. 57-1, pp. 545-556.
- Budak, E. and Altintas, Y., 1995b, "Analytical Prediction of Chatter Stability in Milling, Part II: Application of the General Formulation to Common Milling Systems", *Proc. ASME Dyn. Sys. and Ctrl. Div.*, DSC-Vol. 57-1, pp. 557-565.
- Clary, R., 1997, Private Communication, National Institute of Standards and Technology, Gaithersburg, MD.
- Davies, M. and Balachandran, B., 1996, "Nonlinear Oscillations in Milling of Thin Walled Structures", *Journal of Sound and Vibration*, submitted for publication.
- El'sgol'ts, L.E. and Norkin, S.B., 1973, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, New York.
- Evans, C., 1996, Private Communication, National Institute of Standards and Technology, Gaithersburg, MD.
- Evans, C. and McGlauffin, M., 1988, Private Communication, National Institute of Standards and Technology, Gaithersburg, MD.
- Hale, J.K. and Lunel, S.M.V., 1993, *Introduction to Functional Differential Equations*, Springer-Verlag, New York.
- Hanna, N.H. and Tobias, S.A., 1974, "A Theory of Nonlinear Regenerative Chatter," *Journal of Engineering for Industry*, Vol. 96, pp. 247-255.
- Kalpakjian, S., 1995, *Manufacturing Engineering and Technology*, Addison Wesley, Reading, Massachusetts.
- King, R.I. (Ed.), 1985, *Handbook of High-speed Machining Technology*, Chapman and Hall, New York.
- Koenigsberger, I. and Thusty, J. (1971), *Structures of Machine Tools*, Pergamon Press.
- Lee, A.C., Liu, C.S., and Chiang, S.T., 1991, "Analysis of Chatter Vibration in a Cutter-Workpiece System", *International Journal of Machine Tools and Manufacture*, Vol. 31 (2), pp. 221-234.
- Lemon, J.R. and Ackermann, P.C., 1965, "Application of Self-Excited Machine-Tool Chatter Theory, Contribution to Machine-Tool Chatter Research-4", *Journal of Engineering for Industry*, Vol. 87, pp. 471-479.

Marsh, E.R., Yantek, D.S., Davies, M.A. and Gilsinn, D.E., 1997, "Simulation and Measurement of Chatter in Diamond Turning", submitted for publication, *Journal of Manufacturing Science and Engineering*.

McGlauffin, M., 1997, Private Communication, National Institute of Standards and Technology, Gaithersburg, MD.

Meirovitch, L., 1986, *Elements of Vibration Analysis*, McGraw-Hill, New York.

Merritt, H.E., 1965, "Theory of Self-Excited Machine-Tool Chatter, Contribution to Machine-Tool Chatter Research-1," *Journal of Engineering for Industry*, Vol. 87, pp. 447-454.

Özdoğanlar, O.B. and Endres, W.J., 1997, "A Structure Approach to Analytical Multi-Degree-of-Freedom Time-Invariant Stability Analysis for Machining", submitted to ASME IMECE 1997.

Parker, E. W., 1970, "Dynamic Stability of a Cantilever Boring Bar with Machined Flats Under Regenerative Cutting Conditions", *Journ. Mech. Eng. Sci.*, Vol. 12, 2, pp. 104-115.

Smith, S. and Tlustý, J., 1992, "Stabilizing Chatter by Automatic Spindle Speed Regulation", *Annals of the CIRP*, Vol. 41, 1, pp.433-436.

Tlustý, J., 1985, "Machine Dynamics," *Handbook of High-Speed Machining Technology*, R. I. King, ed., Chapman and Hall, New York.

Tobias, S.A., 1965, *Machine Tool Vibration*, Wiley, New York.

Tobias, S.A. and Fishwick, W., 1958, "The Chatter of Lathe Tools Under Orthogonal Cutting Conditions," *Transactions of the ASME*, Vol. 80, pp. 1079-1088.

A A Useful Determinant

In this Appendix a determinant, used in this paper, will be established by an inductive argument. It will be shown that

$$\begin{aligned}
 D &= \begin{vmatrix} a_1 + cb_1b_1 & cb_1b_2 & \cdots & cb_1b_N \\ cb_2b_1 & a_2 + cb_2b_2 & \cdots & cb_2b_N \\ \cdots & \cdots & \cdots & \cdots \\ cb_Nb_1 & cb_Nb_2 & \cdots & a_N + cb_Nb_N \end{vmatrix} \\
 &= \prod_{n=1}^N a_n + c \sum_{n=1}^N \left(\prod_{\substack{k=1 \\ k \neq n}}^N a_k \right) b_n^2
 \end{aligned} \tag{55}$$

For $N = 1$ the result holds since in that case $D = a_1 + cb_1^2$. Now assume the result is true for all determinants

of the above form of order $\leq N - 1$. One can then show that it also holds for order N . To do this the determinant D is expanded by the first column to get

$$\begin{aligned}
D = & (a_1 + cb_1^2) \begin{vmatrix} a_2 + cb_2b_2 & cb_2b_3 & \cdots & cb_2b_N \\ \cdots & \cdots & \cdots & \cdots \\ cb_Nb_2 & cb_Nb_3 & \cdots & a_N + cb_Nb_N \end{vmatrix} \\
& + (-1)^3 cb_2b_1 \begin{vmatrix} cb_1b_2 & cb_1b_3 & \cdots & cb_1b_N \\ cb_3b_2 & a_3 + cb_3b_3 & \cdots & cb_3b_N \\ \cdots & \cdots & \cdots & \cdots \\ cb_Nb_2 & cb_Nb_3 & \cdots & a_N + cb_Nb_N \end{vmatrix} \\
& \vdots \\
& + (-1)^{N+1} cb_Nb_1 \begin{vmatrix} cb_1b_2 & \cdots & cb_1b_{N-1} & cb_1b_N \\ a_2 + cb_2b_2 & \cdots & cb_2b_{N-1} & cb_2b_N \\ \cdots & \cdots & \cdots & \cdots \\ cb_{N-1}b_2 & \cdots & a_{N-1} + cb_{N-1}b_{N-1} & cb_{N-1}b_N \end{vmatrix}
\end{aligned} \tag{56}$$

The inductive hypothesis is used on the first term of (56). On all of the other terms cb_1 is factored from the first row and set to multiply the determinant in that term. Then, the first row of the determinant in each term is multiplied by the appropriate factor $-cb_i$ and is then added to each of the lower rows. This operation does not affect the value of the determinant. The result of these operations is given by

$$\begin{aligned}
D = & (a_1 + cb_1^2) \left(\prod_{n=2}^N a_n + c \sum_{n=2}^N \left(\prod_{\substack{k=2 \\ k \neq n}}^N a_k \right) b_n^2 \right) \\
& - c^2 b_2 b_1^2 \begin{vmatrix} b_2 & b_3 & \cdots & b_N \\ 0 & a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_N \end{vmatrix} \\
& + \sum_{n=3}^{N-1} (-1)^{n+1} c^2 b_n b_1^2 \begin{vmatrix} b_2 & b_3 & \cdots & b_{n-1} & b_n & b_{n+1} & \cdots & b_N \\ a_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{n-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_{n+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_N \end{vmatrix} \\
& + (-1)^{N+1} c^2 b_N b_1^2 \begin{vmatrix} b_2 & b_3 & \cdots & b_{N-1} & b_N \\ a_2 & 0 & \cdots & 0 & 0 \\ 0 & a_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{N-1} & 0 \end{vmatrix}
\end{aligned} \tag{57}$$

The determinant in the second term can be computed by multiplying the diagonal elements. Since the permutation of two columns of a determinant multiplies the determinant by -1 the determinants in the remaining terms can be computed by permuting the n -th column $n - 2$ columns to the left. Then

$$\begin{aligned}
D = & (a_1 + cb_1^2) \left(\prod_{n=2}^N a_n + c \sum_{n=2}^N \left(\prod_{\substack{k=2 \\ k \neq n}}^N a_k \right) b_n^2 \right) - c^2 b_1^2 b_2 (b_2 a_3 \cdots a_N) \\
& + \sum_{n=3}^{N-1} (-1)^{n+1} c^2 b_1^2 b_n (-1)^{n-2} (b_n a_2 \cdots a_{n-1} a_{n+1} \cdots a_N) \\
& + (-1)^{N+1} c^2 b_1^2 b_N (-1)^{N-2} (b_N a_2 \cdots a_{N-1})
\end{aligned} \tag{58}$$

$$\begin{aligned}
&= (a_1 + cb_1^2) \left(\prod_{n=2}^N a_n + c \sum_{n=2}^N \left(\prod_{\substack{k=2 \\ k \neq n}}^N a_k \right) b_n^2 \right) - c^2 b_1^2 \left(\sum_{n=2}^N \left(\prod_{\substack{k=2 \\ k \neq n}}^N a_k \right) b_n^2 \right) \\
&= \prod_{n=1}^N a_n + c \sum_{n=1}^N \left(\prod_{\substack{k=1 \\ k \neq n}}^N a_k \right) b_n^2
\end{aligned}$$

which was to be shown.

B Perturbation Details

In order to show that there is a perturbed frequency series (42) that satisfies equation (43) we will use the implicit function theorem (Apostol, 1974) in the following form. Let $F(\omega, \epsilon)$ be an infinitely differentiable function of ω and ϵ . Suppose there is an $(w_0, 0)$ such that $F(w_0, 0) = 0$ and $\frac{\partial F}{\partial \omega}(w_0, 0) \neq 0$. Then there is an infinitely differentiable function $\omega(\epsilon)$, defined on an interval about $\epsilon = 0$, such that $\omega(0) = w_0$ and $F(\omega(\epsilon), \epsilon) = 0$.

For the current problem we can define, by cross multiplying equation (43), a function

$$F(\omega, \epsilon) = H(\omega)[1 + (-1 + 2\epsilon) \cos \omega \tau] - G(\omega)(-1 + 2\epsilon) \sin \omega \tau \quad (59)$$

In order to find an w_0 such that $F(w_0, 0) = 0$ note that

$$F(\omega, 0) = H(\omega)(1 - \cos \omega \tau) - G(\omega)(-\sin \omega \tau) \quad (60)$$

implies

$$\frac{H(\omega)}{G(\omega)} = \frac{-\sin \omega \tau}{1 - \cos \omega \tau} \quad (61)$$

The phase, ψ , of the left hand side of equation (61) can be computed as

$$\psi = \arctan \left(\frac{H(\omega)}{G(\omega)} \right) \quad (62)$$

On the right hand side of equation (61) apply the half angle formulas to show that

$$\frac{-\sin \omega \tau}{1 - \cos \omega \tau} = -\frac{\cos \left(\frac{\omega \tau}{2} \right)}{\sin \left(\frac{\omega \tau}{2} \right)} = -\cot \left(\frac{\omega \tau}{2} \right) = \tan \left(\frac{\pi}{2} + \frac{\omega \tau}{2} \pm n\pi \right) \quad (63)$$

Then $\psi = \frac{\pi}{2} + \frac{\omega \tau}{2} \pm n\pi$. Selecting the negative sign and $n = 2 + p$ for $p = 0, 1, 2, \dots$, one obtains

$$\omega \tau = 2(\psi + p\pi) + 3\pi \quad (64)$$

for $p = 0, 1, 2, \dots$. This shows that the τ 's are related to the ω 's by

$$\tau = \frac{2(\psi + p\pi) + 3\pi}{\omega} \quad (65)$$

The set of feasible ω 's is selected as follows. Note that, using equation (15) and equation (38) for the case of $\epsilon = 0$ equation (61) can be inserted into equation (32) to show that

$$K = -\frac{1}{2z_0 G(\omega)} \quad (66)$$

In order for K to be physically meaningful, however, it must be positive. Thus, for each w_0 such that $G(w_0) < 0$, compute τ_0 from equation (65) so that the first condition of the implicit function theorem is satisfied as

$$F(w_0, 0) = H(w_0)(1 - \cos w_0 \tau_0) - G(w_0)(-\sin w_0 \tau_0) = 0 \quad (67)$$

Using equation (59) it is not hard to show that

$$\frac{\partial F}{\partial \omega}(w_0, 0) = \frac{1 - \cos w_0 \tau_0}{G(w_0)} [H'(w_0)G(w_0) - G'(w_0)H(w_0)] - \tau_0 G(w_0) \quad (68)$$

which can numerically be shown to be nonzero on the intervals for which $G(w_0) < 0$. Therefore, by the implicit function theorem there is a function $\omega(\epsilon)$, such that $\omega(0) = w_0$ and $F(\omega(\epsilon), \epsilon) = 0$ which implies equation (43).

We will next solve for $\omega(\epsilon)$ up to the first order ϵ term, w_1 . To do this we assume that ϵw_1 and $\epsilon w_1 \tau_0$ are small. Then to the first order ϵ

$$\frac{H(\omega(\epsilon))}{G(\omega(\epsilon))} \approx \frac{H(w_0) + \epsilon w_1 H'(w_0)}{G(w_0) + \epsilon w_1 G'(w_0)} \quad (69)$$

and, using sums of angles formulas,

$$\frac{(-1 + \epsilon) \sin \omega(\epsilon) \tau_0}{(1 + \epsilon) + (-1 + \epsilon) \cos \omega(\epsilon) \tau_0} \approx \frac{-\sin w_0 \tau_0 + \epsilon(2 \sin w_0 \tau_0 - w_1 \tau_0 \cos w_0 \tau_0)}{(1 - \cos w_0 \tau_0) + \epsilon(2 \cos w_0 \tau_0 - w_1 \tau_0 \sin w_0 \tau_0)} \quad (70)$$

Equating the right hand sides of equations (69) and (70) we can solve for w_1 as

$$w_1 = \frac{2w_0 \sin w_0 \tau_0 G(w_0) - 2w_0 \cos w_0 \tau_0 H(w_0)}{w_0 \tau_0 H(w_0) \sin w_0 \tau_0 + w_0 \tau_0 G(w_0) \cos w_0 \tau_0 + w_0(1 - \cos w_0 \tau_0)H'(w_0) + w_0 \sin w_0 \tau_0 G'(w_0)} \quad (71)$$

where again we set $w_0 \tau_0 = 2\psi_0 + 3\pi$.

The previous argument established that there is a frequency function of ϵ that satisfies equation (42). In order to extend the arguments used in orthogonal cutting to the case of generating stability charts for round nosed tools we next need to show that there also exists a function $\tau(\epsilon)$ such that $\tau(0) = \tau_0$ and

$$\frac{(-1 + \epsilon) \sin \omega(\epsilon) \tau_0}{(1 + \epsilon) + (-1 + \epsilon) \cos \omega(\epsilon) \tau_0} = \frac{-\sin w_0 \tau(\epsilon)}{1 - \cos w_0 \tau(\epsilon)} \quad (72)$$

Again, we apply the implicit function theorem to the following function of τ .

$$T(\tau, \epsilon) = (1 - \cos w_0 \tau)((-1 + \epsilon) \sin \omega(\epsilon) \tau_0) + \sin w_0 \tau[(1 + \epsilon) + (-1 + \epsilon) \cos \omega(\epsilon) \tau_0] \quad (73)$$

By selecting w_0 so that $G(w_0) < 0$ and τ_0 satisfying equation (65) we have $T(\tau_0, 0) = 0$. The derivative is easily seen to be

$$\frac{\partial T}{\partial \tau}(\tau_0, 0) = w_0(\cos w_0 \tau_0 - 1) \quad (74)$$

The right hand side is nonzero since we select $w_0 \tau_0 \neq 2n\pi$, for $n = 0, \pm 1, \pm 2, \dots$. Therefore from equations (43) and (72) we can write

$$\frac{H(\omega(\epsilon))}{G(\omega(\epsilon))} = -\frac{\sin w_0 \tau(\epsilon)}{1 - \cos w_0 \tau(\epsilon)} \quad (75)$$

Proceeding as in orthogonal cutting we define the phase angle by

$$\psi(\epsilon) = \arctan \left(\frac{H(\omega(\epsilon))}{G(\omega(\epsilon))} \right) \quad (76)$$

The rotation rate is then given by

$$\Omega(\epsilon) = \frac{1}{\tau(\epsilon)} = \frac{w_0}{2(\psi(\epsilon) + p\pi) + 3\pi} \quad (77)$$

for $p = 0, 1, 2, \dots$. Notice that to a first order ϵ

$$\tan \psi(\epsilon) \approx \frac{H(w_0) + \epsilon w_1 H'(w_0)}{G(w_0) + \epsilon w_1 G'(w_0)} \quad (78)$$

From equation (32) we can easily solve for $K(\epsilon)$. Using equations (40) and (69) and trigonometric formulas for the sums of angles we can approximate $K(\epsilon)$ up to the first order ϵ by

$$K(\epsilon) \approx \frac{-1}{2\alpha_1 G(w_0) + \epsilon \alpha_1 P(w_0)} \quad (79)$$

where

$$\begin{aligned} P(w_0) = & G(w_0)[2 \cos w_0 \tau_0 + \left(\frac{w_1}{w_0} \right) (w_0 \tau_0) \sin w_0 \tau_0] \\ & + H(w_0)[2 \sin w_0 \tau_0 - \left(\frac{w_1}{w_0} \right) (w_0 \tau_0) \cos w_0 \tau_0] \\ & + w_1 G'(w_0)(1 - \cos w_0 \tau_0) \\ & - w_1 H'(w_0) \sin w_0 \tau_0 \end{aligned} \quad (80)$$

where again $w_0 \tau_0 = 2\psi_0 + 3\pi$.

C CAPTIONS

Figure 1a: Boring bar and Workpiece Fixtured for Diamond Turning.

Figure 1b: Model of the Boring bar.

Figure 2a: Chip Area Parameters for One Pass of the Tool over the Workpiece with a Depth of Cut, d .

Figure 2b: Chip Area Parameters for Two Consecutive Passes of the Tool over the Workpiece. The Profile of the Current Pass is $x_f(t)$ and that of the Previous Pass is $x(t)$.

Figure 3: The Nonsingular Left Hand Ratio (solid) and the Singular Right Hand Ratio (dashes) of Equation (44) for the Case $\epsilon = 0.001$.

Figure 4: Plot of the Dimensionless Denominator in Equation (52) for the Case of a 101.6 mm (4 in) Bar with a $500 \mu\text{m}$ Tool Tip Radius at $2 \mu\text{m}$ Depth of Cut and Feed of $0.1 \mu\text{m}$ per Revolution.

Figure 5: Stability Chart for a 101.6 mm (4 in) Bar during a $2 \mu\text{m}$ Depth of Cut Showing the Overlapping Lobes at Normal Rotation Rates.

Figure 6: Critical Cutting Energy as a Function of Boring Bar Length and Tool tip Radius for a $1 \mu\text{m}$ Depth of Cut and Feed of $0.1 \mu\text{m}$ per Revolution.

Figure 7: Critical Cutting Energy as a Function of the Damping Ratio for a 101.6 mm (4 in) Bar with a $500 \mu\text{m}$ Tip Radius at $2 \mu\text{m}$ Depth of Cut and a Feed of $0.1 \mu\text{m}$ per Revolution.

Figure 8: Stability Chart for a 101.6 mm ($5/8 \text{ in}$) Bar during a $2 \mu\text{m}$ Depth of Cut Showing the Second Mode Lobes Overlapping Portions of the Stability Region Between First Mode Lobes at High Rotation Rates. Solid and Dashed Lines Correspond to First and Second Modes, Respectively.