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THEORETICAL INVESTIGATION AND APPLICATION OF TRANSONIC SIMILARITY LAW FOR TWO DIMENSIONAL FLOW

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SUMMARY

The transonic similarity law for two-dimensional flow derived by von Kármán has been investigated by an iteration procedure similar to that of the Rayleigh-Janzen and Ackeret-Prandtl-Glauert methods. The results, which show that the potential can be expressed as a power series in a single parameter that depends on Mach number, thickness ratio, and ratio of specific heats, are in agreement with those of von Kármán. The iteration procedure has been applied to obtain the second approximation for the flow past a Kaplan section in similarity form. The exact solution by Kaplan for the second approximation has been examined and found expressible in the same similarity form. The exact numerical results to three approximations obtained by Kaplan for the Kaplan section and the circular arc have been reduced to transonic similarity form.

INTRODUCTION

The difficulty of calculating and measuring explicitly fluid flow patterns past given bodies has led to attempts at extracting information from the equations of motion without actually solving them, that is, in the form of similarity laws. A similarity law gives the solution for a whole class of related bodies at related conditions if the solution for only one of these bodies at one set of conditions is known. A well-known example of a similarity law is the Prandtl-Glauert rule, which relates potential flows past thin airfoils at low subsonic speeds.

Similarity rules for two-dimensional steady-state potential flow in the transonic speed range have been derived by von Kármán in reference 1. The treatment has been extended by Lin, Reissner, and Tsien in reference 2 to include unsteady two-dimensional transonic potential flows. These derivations utilized small-perturbation methods in the physical plane in which two basic simplifying assumptions were made: namely, (1) that the effect of stagnation points, if these exist in the flow field, was negligible in the region of interest in the flow field; and (2) that the boundary condition of zero velocity component normal to the body contour could be satisfied near the body rather than on the body itself. The first assumption is also made herein. The validity of the second assumption was investigated at the NACA Lewis laboratory and is reported herein. 1379

A rigorous justification of the assumption that the exact boundary condition may be replaced by another one would appear to require the study of the behavior under these various boundary conditions of an explicit solution of the equations of motion. An explicit solution, however, is just what the similarity analysis purports to do without. A dilemma is thus apparent. That this dilemma is nontrivial is illustrated by the fact that the preceding boundary-condition assumption has been known to lead to substantially incorrect results in the case of slender bodies of revolution in the subsonic speed range (see, for example, references 3 to 6), although, as is well known, this general assumption yields correct results in this speed range for two-dimensional airfoils.

The explicit class of solutions that may therefore be needed to establish a similarity law does not, however, render superfluous the similarity law contained implicitly therein. The expression of these solutions in a form that brings out the similarity features is both useful and significant, as is evident from the example of the Prandtl-Glauert rule. Accordingly, an explicit solution for the two-dimensional, continuous, potential flow past thin airfoils at high subsonic speeds is herein rigorously derived from the point of view of the possibility of expressing this solution in the form required by the transonic similarity law (reference 1). The limiting form of the solution is found to be expressible in this form as the airfoil thickness ratio approaches zero and the free-stream Mach number approaches unity. Hence, the boundary-condition assumption in question, underlying the derivations of references 1 and 2, is validated for the two-dimensional case treated herein. Such a validation, based on the particular solution for the Kaplan section given in reference 7, has already been indicated in references 8 and 9.

The method of analysis used herein is based on a conventional iteration procedure similar to that of the Rayleigh-Janzen method (reference 10) and the Ackeret-Prandtl-Glauert method (reference 7). The Rayleigh-Janzen method expresses the potential as a power series in the Mach number, whereas the Ackeret-Prandtl-Glauert method yields the potential as a power series in a parameter characterizing the body thickness. The present method expresses the potential as a power series in a parameter that depends on all the physical parameters of the problem, namely, thickness ratio, free-stream Mach number, and specific heat ratio. The form of this dependence is not specified initially. Instead, by going to the simultaneous limit of zero thickness ratio and free-stream Mach number unity in each step of the iteration procedure and retaining only highest-order terms, a power-series parameter (if it exists) is obtained as some combination of the previously mentioned physical parameters. The method thus constitutes a unification of the Rayleigh-Janzen and Ackeret-Prandtl-Glauert procedures in the small-perturbation transonic limit.

The preceding method of analysis turns out to be convenient for numerical application. Accordingly, as a by-product of the investigation, such application is made to the Kaplan section, and the results are compared with those of references 7 and 8. In particular, a value of the previously mentioned parameter of expansion of the power series for the potential is derived, which is estimated to represent a boundary value between convergence and divergence of the series. The significance of such a boundary value is discussed in reference 11 in connection with the so-called potential-limit phenomenon.

ANALYSIS

General formulation. - The partial differential equation for the potential of an irrotational compressible, two-dimensional flow with free-stream velocity U is, in Cartesian coordinates x,y (fig. 1).

$$\left[a^2 - (U+u)^2\right]\varphi_{xx} + (a^2-v^2)\varphi_{yy} - 2(U+u)v\varphi_{xy} = 0 \tag{1}$$

The following notation has been used: (A more complete list of symbols is given in the appendix.)

a local speed of sound

$U+u$ resultant velocity in x direction

v resultant velocity in y direction

φ perturbation potential, defined by $u = \varphi_x$, $v = \varphi_y$.

The subscripts denote differentiation.

The local speed of sound a is related to the free-stream speed of sound a_0, the ratio of specific heats γ, and the local velocity by the Bernoulli equation

$$a^2 = a_0^2 - \frac{\gamma-1}{2}\left(2Uu+u^2+v^2\right) \tag{2}$$

In accordance with the Ackeret-Prandtl-Glauert type of procedure, equation (1) will be written in a form in which the linear terms appear on the left-hand side of the equation and the nonlinear terms on the right-hand side. A solution will be sought for high subsonic free-stream velocity and small lateral-distance ratio such that the flow pattern obtained by inclusion of the nonlinear terms will differ by only a small amount from that obtained with only the linear terms. The coefficient of φ_{xx} in equation (1) is therefore expressed, with the aid of equation (2), in the form

$$a^2 - (U+u)^2 = \beta^2 a^2 - a_0^2\left\{\Gamma_M\left[\frac{2u}{U} + \left(\frac{u}{U}\right)^2\right] + \frac{\gamma-1}{2}M_0^4\left(\frac{v}{U}\right)^2\right\} \tag{3}$$

where

M_0 free-stream Mach number $(=U/a_0)$

$$\beta^2 \equiv 1 - M_0^2 \tag{4a}$$

$$\Gamma_M \equiv M_0^2\left(1 + \frac{\gamma-1}{2}M_0^2\right) \tag{4b}$$

For convenience the free-stream velocity is taken as the unit of velocity, so that u/U and v/U may be written as u and v, respectively. The differential equation (1) can now be expressed in the form

$$\left(\beta^2\varphi_{xx}+\varphi_{yy}\right)\left[1 - \frac{\gamma-1}{2}M_0^2\left(2\varphi_x+\varphi_x^2+\varphi_y^2\right)\right] = \left[\Gamma_M\left(2\varphi_x+\varphi_x^2\right) + \frac{\gamma-1}{2}M_0^4\varphi_y^2\right]\varphi_{xx} + M_0^2\varphi_y^2\varphi_{yy} + 2M_0^2\left(1+\varphi_x\right)\varphi_y\varphi_{xy} \tag{5}$$

The boundary conditions of the problem are that the perturbation velocities vanish at infinity and that the flow follow the contour of the body. Thus, at ∞,

$$\varphi_x = \varphi_y = 0 \tag{6}$$

The body is defined by

$$y_b = \tau g(x) \tag{7a}$$

On the body,

$$\varphi_y = \tau(1+\varphi_x)g_x \tag{7b}$$

where

τ lateral distance ratio of body

$g(x)$ function characterizing shape of body and together with its derivative g_x is of order of magnitude unity

and all lengths are expressed in terms of the chord c of the body as unit.

The lateral distance ratio τ, introduced in equation (7a) to yield the order of magnitude unity for $g(x)$, is a parameter to be regarded as including both thickness ratio and angle of attack. The parameter τ reduces to thickness ratio for a symmetrical airfoil at zero lift, to angle of attack for a flat plate with lift, and to camber ratio for a fore and aft symmetrical curved plate at zero angle of attack.

In order to obtain the Laplacian on the left-hand side of equation (5), the affine transformation

$$\eta = \beta y \tag{8a}$$

$$F(x,\eta) = \frac{\Gamma_M}{\beta^2}\varphi(x,y) \tag{8b}$$

is introduced. The transformation from φ to F in equation (8b) is made in order to incorporate the factor Γ_M/β^2 indicated by equations (5) and (8a) into the solution for the potential. The differential equation (5) thus becomes

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$$\Delta F = \frac{\gamma-1}{2}\frac{M_0^2}{\Gamma_M}\beta^2\left(2F_x + \frac{\beta^2}{\Gamma_M}F_x^2 + \frac{\beta^4}{\Gamma_M}F_\eta^2\right)\Delta F +$$

$$\left(2F_x + \frac{\beta^2}{\Gamma_M}F_x^2 + \frac{\gamma-1}{2}M_0^4\frac{\beta^4}{\Gamma_M^2}F_\eta^2\right)F_{xx} + M_0^2\frac{\beta^6}{\Gamma_M^2}F_\eta^2F_{\eta\eta} +$$

$$2M_0^2\frac{\beta^2}{\Gamma_M}\left(1 + \frac{\beta^2}{\Gamma_M}F_x\right)F_\eta F_{x\eta} \tag{9}$$

where ΔF is the Laplacian

$$\Delta F \equiv F_{xx} + F_{\eta\eta} \tag{10}$$

In terms of the new variables, the boundary conditions (6) and (7) become at ∞,

$$F_x = F_\eta = 0 \tag{11}$$

and on the body (subscript b)

$$\eta_b = \beta\tau g(x) \tag{12a}$$

$$F_\eta - \frac{\tau g_x}{\beta}F_x = \frac{\tau\Gamma_M}{\beta^3}g_x \tag{12b}$$

The formulation of the problem is thus far exact. A solution is now sought applicable in the limiting case $\tau \to 0$, $\beta \to 0$; this limit will be referred to as "the small-perturbation transonic limit." In the range of values of τ and β considered, the perturbation velocities will be assumed small compared to free-stream velocity, or $|u|,|v| \ll 1$. The right-hand side of equation (9) is thus considered as producing a small perturbation from the linear case so that a solution to the systems (9), (11), and (12) will be sought in the form

$$F = \overset{1}{F} + \overset{2}{F} + \overset{3}{F} + \ldots \tag{13}$$

in which each term is of a lesser order of magnitude than the preceding one. Accordingly, as the equivalent of boundary conditions (11) and (12) the following boundary conditions on $\overset{1}{F}, \overset{2}{F} \ldots$ will be taken at ∞:

$$\overset{n}{F}_x = \overset{n}{F}_\eta = 0, \quad n = 1, 2, 3 \ldots \tag{14}$$

on the body

$$\eta_b = \beta\tau g(x) \tag{15a}$$

$$\overset{1}{F}_\eta - \frac{\tau}{\beta} g_x \overset{1}{F}_x = \frac{\tau\Gamma_M}{\beta^3} g_x \tag{15b}$$

$$\overset{n}{F}_\eta - \frac{\tau}{\beta} g_x \overset{n}{F}_x = 0 \quad (n \neq 1) \tag{15c}$$

In order to obtain a solution of the systems (9), (14), and (15), equation (13) is inserted into the differential equation (9) and a typical Laplacian term on the left, such as $\Delta\overset{n}{F}$, is equated to the sum of those terms on the right that satisfy the following two conditions: (a) The superscript $n-1$ is present; and (b) any or none of the superscripts $n-2, n-3 \ldots 1$ are present. The resulting solution of the non-homogeneous Laplace equation for $\overset{n}{F}$ consists of a sum of terms of which, for a range of M_0 near unity and τ near zero, some will be of highest order of magnitude. These terms constitute the solution for $\overset{n}{F}$ in the small-perturbation transonic limit. It will be shown that if τ and β are allowed to approach zero in a particular manner then the remaining terms will vanish and the solution for $\overset{n}{F}$ will have a unique finite limit.

Transformation to elliptic coordinates. - The Cartesian coordinates x,η are suitable for obtaining the flow past a two-dimensional profile that is periodic with respect to x (for example, a wavy wall) but are inconvenient in problems involving isolated airfoils. In order to obtain explicit solutions for the flow past an isolated two-dimensional body, it is convenient to introduce a system of elliptic coordinates (reference 12, p. 156). The transformation from Cartesian coordinates x,η to elliptic coordinates s,t is given by (fig. 1)

$$x + i\eta = \cosh\,(s+it) \tag{16}$$

The real and imaginary parts of equation (16) yield

$$x = \cosh s \cos t \tag{17a}$$

$$\eta = \sinh s \sin t \tag{17b}$$

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The curves s = constant, t = constant are confocal ellipses and hyperbolas, respectively, with the common foci at $x = \pm 1$, $\eta = 0$. The values of s range from 0 to ∞, whereas t varies between 0 and 2π. The Laplacian ΔF is, in elliptic coordinates,

$$\Delta F = \frac{F_{ss} + F_{tt}}{J(s,t)} \tag{18}$$

where $J(s,t)$, the Jacobian of the transformations (17), is given by

$$J(s,t) = x_s\eta_t - x_t\eta_s \tag{19a}$$

$$= \sinh^2 s + \sin^2 t \tag{19b}$$

$$= \cosh^2 s - \cos^2 t \tag{19c}$$

Transformations from derivatives with respect to x and η to derivatives with respect to s and t will be needed for the subsequent analysis. From the transformation equations (17) these relations are

$$F_x = \frac{1}{J(s,t)} \left(\sinh s \cos t \, F_s - \cosh s \sin t \, F_t\right) \tag{20}$$

$$F_\eta = \frac{1}{J(s,t)} \left(\cosh s \sin t \, F_s + \sinh s \cos t \, F_t\right) \tag{21}$$

<u>First approximation.</u> - The solution for the first approximation $\overset{1}{F}$ satisfies the homogeneous Laplace equation

$$\Delta\overset{1}{F} = \overset{1}{F}_{ss} + \overset{1}{F}_{tt} = 0 \tag{22}$$

and the boundary conditions (14), (15a), and (15b), which in elliptic coordinates are at ∞

$$\frac{1}{J(s,t)}\left(\sinh s \cos t \overset{1}{F}_s - \cosh s \sin t \overset{1}{F}_t\right) = 0 \qquad (23a)$$

$$\frac{1}{J(s,t)}\left(\cosh s \sin t \overset{1}{F}_s + \sinh s \cos t \overset{1}{F}_t\right) = 0 \qquad (23b)$$

on the body

$$\eta_b = \beta\tau g(x) \qquad (24a)$$

$$\frac{1}{J(s,t)}\left(\cosh s \sin t \overset{1}{F}_s + \sinh s \cos t \overset{1}{F}_t\right) -$$

$$\frac{\tau}{\beta} g_x \frac{1}{J(s,t)}\left(\sinh s \cos t \overset{1}{F}_s - \cosh s \sin t \overset{1}{F}_t\right) = \frac{\tau\Gamma_M}{\beta^3} g_x \qquad (24b)$$

A solution of equation (22) that satisfies the boundary conditions (23) is

$$\overset{1}{F} = \sum_{n=0}^{\infty}\left(\overset{1}{A}_n \sin nt + \overset{1}{B}_n \cos nt\right)e^{-ns} + \overset{1}{C}t \qquad (25)$$

It is necessary to exclude the positive exponentials e^{ns} from equation (25) because, from equations (23) and (19b), the order of $\overset{1}{F}_s$ and $\overset{1}{F}_t$ at infinity must be less than e^s. The constants $\overset{1}{A}_n$ and $\overset{1}{B}_n$ in equation (25) are determined from the boundary condition on the body (equations (24a) and (24b)). The constant $\overset{1}{C}$ is subsequently determined to yield a circulation around the body by a Kutta-type condition (such as rear stagnation point at trailing edge). Small-perturbation analysis usually neglects the term in $\overset{1}{F}_x$ occurring in equation (24b). This term will therefore be assumed negligible, an assumption to be justified subsequently on the basis of the resulting solution.

In obtaining the small-perturbation transonic limiting solution for $\overset{1}{F}$, equation (25) is inserted into the limiting form, for

$s_b \longrightarrow 0$, of equation (24b) with the $\overset{1}{F}_x$ term neglected. The limiting forms of equations (19), (20), and (21) as $s \longrightarrow 0$ are 1379

$$\lim_{s \longrightarrow 0} J(s,t) = \sin^2 t \tag{26}$$

$$\lim_{s \longrightarrow 0} F_x = -\frac{F_t}{\sin t} \tag{27}$$

$$\lim_{s \longrightarrow 0} F_\eta = \frac{F_s}{\sin t} \tag{28}$$

and the resulting equation for determining the constants $\overset{1}{A}_n$ and $\overset{1}{B}_n$ is

$$\sum_{n=0}^{\infty} n(\overset{1}{A}_n \sin nt + \overset{1}{B}_n \cos nt) = -\frac{\tau \Gamma_M}{\beta^3} g_x(x) \sin t \tag{29}$$

The more rigorous alternative process of inserting equation (25) into equation (24b) with the $\overset{1}{F}_x$ term dropped and only then going to the limit $s_b \longrightarrow 0$ yields the same equation (29). Evaluation of the boundary condition for $\overset{1}{F}$ on the axis, that is, by equation (29), will therefore yield the correct limiting solution for $\overset{1}{F}$. It may be noted that in order to obtain a solution for $\overset{1}{F}$ it is necessary that the quantity $\tau \Gamma_M / \beta^3$ in equation (29) be finite in the small-perturbation transonic limit $\tau \longrightarrow 0$, $\beta \longrightarrow 0$.

Equation (29) does not contain the variable s inasmuch as the limiting form of $g_x(x)$ as $s \longrightarrow 0$ is understood to be substituted into equation (29). The right-hand side of equation (29) may now be expanded into a Fourier series in t and the coefficients of corresponding functions on both sides equated. The resulting solution for $\overset{1}{A}_n$ and $\overset{1}{B}_n$ may be indicated as

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$$\overset{1}{A}_n = \overset{1}{a}_n(b)K \tag{30a}$$

$$\overset{1}{B}_n = \overset{1}{b}_n(b)K \tag{30b}$$

where

$$K = \frac{\tau \Gamma_M}{\beta^3} \tag{31}$$

and $\overset{1}{a}_n(b)$, $\overset{1}{b}_n(b)$ are functions of the body shape. (The symbol b within parentheses denotes the numerical coefficients characterizing the body shape, including angle of attack, that arise in expressing $g_x(x)$ as a trigonometric series in t.) Numerical evaluations such as for the Kaplan section to be made herein indicate that $\overset{1}{a}_n(b)$, $\overset{1}{b}_n(b)$ are of order of magnitude unity.

The solution for $\overset{1}{F}$ is thus by equations (25) and (30)

$$\overset{1}{F} = K\left[\sum_{n=0}^{\infty}(\overset{1}{a}_n \sin nt + \overset{1}{b}_n \cos nt)e^{-ns} + \overset{1}{c}t\right] \tag{32}$$

in which the substitution $\overset{1}{C} = K\overset{1}{c}(b)$ has been made, where $\overset{1}{c}(b)$ may be considered of order unity if $\overset{1}{a}_n(b)$, $\overset{1}{b}_n(b)$ are of order unity.

The functions $\overset{1}{a}_n$ and $\overset{1}{b}_n$ do not depend on the coordinates s_b because they were obtained from equations evaluated on the axis of the body $(s_b = 0)$. The reduced potential function $\overset{1}{F}$ therefore depends on s only through the exponential e^{-ns}, which is nearly constant for small values of s and approaches unity for $s \longrightarrow 0$. Hence, $\overset{1}{F}$ approaches a finite limiting solution as τ and β both approach zero in such a way that K is kept finite. This limiting solution for $\overset{1}{F}$ holds for a class of profiles with a common-shape factor $g(x)$ and for a fixed value of K, so that a similarity rule exists. If the limiting solutions for τ, $\beta \longrightarrow 0$ in the higher approximations can likewise be expressed in terms of the one parameter K, it will then be possible to say that a transonic similarity law exists for two-dimensional flows. 1379

The horizontal and vertical disturbance velocities are, from equations (8), (20), (21), and (32),

$$\overset{1}{\varphi}_x = \frac{\beta^2}{\Gamma_M}\overset{1}{F}_x = -\frac{\tau}{\beta}\frac{1}{J(s,t)}\left\{\sinh s \cos t \sum_{n=0}^{\infty} n(\overset{1}{a}_n \sin nt + \overset{1}{b}_n \cos nt)e^{-ns} + \cosh s \sin t\left[\sum_{n=0}^{\infty} n(\overset{1}{a}_n \cos nt - \overset{1}{b}_n \sin nt)e^{-ns} + \frac{1}{c}\right]\right\} \tag{33}$$

$$\overset{1}{\varphi}_y = \frac{\beta^3}{\Gamma_M}\overset{1}{F}_\eta = -\frac{\tau}{J(s,t)}\left\{\cosh s \sin t \sum_{n=0}^{\infty} n(\overset{1}{a}_n \sin nt + \overset{1}{b}_n \cos nt)e^{-ns} - \sinh s \cos t\left[\sum_{n=0}^{\infty} n(\overset{1}{a}_n \cos nt - \overset{1}{b}_n \sin nt)e^{-ns} + \frac{1}{c}\right]\right\} \tag{34}$$

The total disturbance velocity $\overset{1}{\Lambda}$ is defined by

$$\overset{1}{\Lambda} = \left[(1+\overset{1}{\varphi}_x)^2 + \overset{1}{\varphi}_y{}^2\right]^{\frac{1}{2}} - 1 \tag{35}$$

The trigonometric and exponential functions occurring within the summation signs in equations (33) and (34) are of order unity and it is reasonable to assume that the results of summation will be of the same order. The ratio of $\overset{1}{\varphi}_y$ to $\overset{1}{\varphi}_x$ is of order β so that the quantity $\overset{1}{\varphi}_y{}^2$ is negligible with respect to $\overset{1}{\varphi}_x$ in the small-perturbation case $\overset{1}{\varphi}_x \ll 1$. The disturbance velocity $\overset{1}{\Lambda}$ is therefore approximated by $\overset{1}{\varphi}_x$. In the first approximation given by equations (8), (31), (33), and (34), the similarity rule is therefore simply the well-known proportionality of the resultant perturbation velocity to thickness ratio and angle of attack (represented by the symbol τ) and to $1/\beta$, which is the Prandtl-Glauert rule.

Second approximation. - The second approximation $\overset{2}{F}$ is a solution of the non-homogeneous Laplace equation

$$\overset{2}{F}_{ss} + \overset{2}{F}_{tt} = \frac{\gamma-1}{2}\frac{M_0{}^2\beta^2}{\Gamma_M}\left(2\overset{1}{F}_x + \frac{\beta^2}{\Gamma_M}\overset{1}{F}_x{}^2 + \frac{\beta^4}{\Gamma_M}\overset{1}{F}_\eta{}^2\right)\left(\overset{1}{F}_{ss} + \overset{1}{F}_{tt}\right) +$$

$$J(s,t)\left[\left(2\overset{1}{F}_x + \frac{\beta^2}{\Gamma_M}\overset{1}{F}_x{}^2 + \frac{\gamma-1}{2}M_0{}^4\frac{\beta^4}{\Gamma_M{}^2}\overset{1}{F}_\eta{}^2\right)\overset{1}{F}_{xx} +\right.$$

$$\left. M_0{}^2\frac{\beta^6}{\Gamma_M{}^2}\overset{1}{F}_\eta{}^2 F_{\eta\eta} + 2M_0{}^2\frac{\beta^2}{\Gamma_M}\left(1 + \frac{\beta^2}{\Gamma_M}\overset{1}{F}_x\right)\overset{1}{F}_\eta\overset{1}{F}_{x\eta}\right] \tag{36}$$

which satisfies the boundary conditions (14), (15a), and (15c). As previously mentioned, in solving equation (36) only those terms on the right-hand side will be retained that are of highest order of magnitude in the small-perturbation transonic limit $\tau,\beta \longrightarrow 0$, K finite. It is thus first noted that the term involving the Laplacian factor $\left(\overset{1}{F}_{ss}+\overset{1}{F}_{tt}\right)$ vanishes by virtue of the solution for the first approximation. Secondly, equations (20), (21), and (32) show that $\overset{1}{F}$ and its derivatives are of order of magnitude K. In the numerical applications hereinafter given, the permissible range of K, is approximately $0 < K < 0.5$. It is therefore convenient to assign the order of magnitude unity to K. The order of magnitude, as $\beta \longrightarrow 0$, of any term on the right-hand side of equation (36) therefore depends on the power of β occurring in that term, and only one term, involving $\overset{1}{F}_x\overset{1}{F}_{xx}$, does not have a factor of β to some positive power. In the small-perturbation transonic limit, all terms on the right-hand side of equation (36) except the term in $\overset{1}{F}_x\overset{1}{F}_{xx}$ are therefore negligible. Equation (36) for the second approximation $\overset{2}{F}$ thus simplifies to

$$\overset{2}{F}_{ss} + \overset{2}{F}_{tt} = 2J(s,t)\overset{1}{F}_x\overset{1}{F}_{xx} \tag{37}$$

For the solution of equation (37), $\frac{1}{F}$ is first expressed as 1379

$$\frac{1}{F} = KR(s,t)$$

where, by equation (32),

$$R(s,t) = \sum_{n=0}^{\infty} \left(\overset{1}{a}_n \sin nt + \overset{1}{b}_n \cos nt\right)e^{-ns} + \overset{1}{c}t \tag{38}$$

The x derivatives of R are given by, from equation (20),

$$R_x = \frac{1}{J}\left(\sinh s \cos t\, R_s - \cosh s \sin t\, R_t\right) \tag{39}$$

$$\begin{aligned} R_{xx} = \frac{1}{J^2}\Bigg\{&\sinh^2 s \cos^2 tR_{ss} + \cosh^2 s \sin^2 tR_{tt} - \\ &2 \cosh s \sinh s \cos t \sin tR_{st} + \\ &\sinh s \cosh s\left[1 - \frac{2\cos^2 t}{J}\left(\sinh^2 s - \sin^2 t\right)\right]R_s + \\ &\cos t \sin t\left[1 + \frac{2\cosh^2 s}{J}\left(\sinh^2 s - \sin^2 t\right)\right]R_t\Bigg\} \end{aligned} \tag{40}$$

In terms of R, equation (37) becomes

$$\overset{2}{F}_{ss} + \overset{2}{F}_{tt} = 2K^2JR_xR_{xx} \tag{41}$$

where R_x and R_{xx} are obtained from equations (38) to (40). The right-hand side of equation (41) is now to be expanded in a Fourier series in t. A typical term of equation (41) has a singularity at the end points $t = 0,\pi$ for $s = 0$, because of the factor $1/J$, which becomes infinite at these points. Because the limiting solution for $\overset{2}{F}$ for $s \longrightarrow 0$ is of principal interest, it may be necessary to exclude a small and finite region around the end points from the domain of definition of $\overset{2}{F}$. This objection is not serious, however, because the end points must in any event be excluded from consideration if stagnation points, which violate 1379 the assumption of small perturbations, occur there. When the right-hand side is expanded in a Fourier series in t, equation (41) becomes

$$\overset{2}{F}_{ss} + \overset{2}{F}_{tt} = K^2 \sum_{n=0}^{\infty} \left[h_n(s) \sin nt + \bar{h}_n(s) \cos nt\right] \tag{42}$$

where $h_n(s)$ and $\bar{h}_n(s)$ are functions of s.

A solution for $\overset{2}{F}$ is now sought in the form

$$\overset{2}{F} = K^2 \sum_{n=0}^{\infty} \left[q_n(s) \sin nt + \bar{q}_n(s) \cos nt\right] \tag{43}$$

where $q_n(s)$ and $\bar{q}_n(s)$ are functions of s to be determined. Insertion of equation (43) in equation (42) yields

$$\sum_{n=0}^{\infty} \left[(q_n''-n^2q_n) \sin nt + (\bar{q}_n''-n^2\bar{q}_n) \cos nt\right] = \sum_{n=0}^{\infty} (h_n \sin nt + \bar{h}_n \cos nt) \tag{44}$$

where the primes denote differentiation with respect to s. Because equation (44) must hold for each value of n and for all values of t, the coefficients of corresponding trigonometric functions on both sides may be equated, which yields the ordinary differential equations

$$q_n'' - n^2q_n = h_n \tag{45a}$$

$$\bar{q}_n'' - n^2\bar{q}_n = \bar{h}_n \tag{45b}$$

Equations (45a) and (45b) are of the same type so that attention may be confined, say, to equation (45a). The solution for $q_n(s)$ consists of a complementary function, taken as $\overset{2}{a}_n e^{-ns}$ to satisfy the boundary conditions at infinity, and of a particular integral that may be expressed in terms of two indefinite integrals by

$$r_n(s) = e^{-ns}\int e^{2ns}\,ds\int e^{-ns}\,h_n(s)ds \tag{46}$$

In order to determine the behavior of the particular integral $r_n(s)$ as $s \longrightarrow 0$, the form of $h_n(s)$ is first examined. Because a region around the end points is excluded if stagnation points occur, the quantity $\frac{\cosh^2 s}{J}$ occurring in equation (40) may be expanded in an absolutely convergent power series in $\frac{\cos^2 t}{\cosh^2 s}$. The right-hand side of equation (41) may now be expressed as a Fourier series in t. The function $h_n(s)$ may therefore be expressed as a sum of terms each of which has the form $\cosh^k se^{-ms}$ or $\sinh s \cosh^k se^{-ms}$ where k and m are integers, m positive, and each term is multiplied by a coefficient that depends on a body shape and is of order unity. The particular integrals corresponding to these forms are, from equation (46), expressible as functions also having these forms. It then follows that the function $r_n(s)$, corresponding to $h_n(s)$, and its derivatives $r_n'(s)$, $r_n''(s)$ are of order of magnitude unity and have a finite limit as $s \longrightarrow 0$. Writing $\bar{r}_n(s)$ as the particular integral corresponding to $\bar{h}_n(s)$ gives the following solution for $\overset{2}{F}$:

$$\overset{2}{F} = K^2\left[\sum_{n=0}^{\infty}(\overset{2}{a}_n e^{-ns}+r_n)\sin nt + \sum_{n=0}^{\infty}(\overset{2}{b}_n e^{-ns}+\bar{r}_n)\cos nt + \overset{2}{c}t\right] \tag{47}$$

A circulation term $\overset{2}{c}t$ is included in equation (47) in order to satisfy the Kutta condition as in the first approximation. The coefficients $\overset{2}{a}_n$ and $\overset{2}{b}_n$ are determined from the boundary condition on the body (15c) which, with neglect of the F_x term and by equation (21), is in elliptic coordinates

$$\cosh s \sin t\,\overset{2}{F}_s + \sinh s \cos t\,\overset{2}{F}_t = 0 \tag{48}$$

The result of combining equations (47) and (48) and going to the limit $s_b \longrightarrow 0$ is the same equation as that obtained from combining equation (47) with the limiting form, as $s_b \longrightarrow 0$, of equation (48). It is therefore permissible, as in the first approximation, to evaluate the boundary condition on the axis of the body for which equation (48) becomes

$$\overset{2}{F}_s \doteq 0 \tag{49}$$

Substitution of equation (47) in equation (48) yields

$$\sum_{n=0}^{\infty} (-n\overset{2}{a}_n + r_n') \sin nt + \sum_{n=0}^{\infty} (-n\overset{2}{b}_n + \overline{r}_n') \cos nt = 0 \tag{50}$$

For equation (50) to hold for each value of n and all values of t, it is necessary that

$$-n\overset{2}{a}_n + r_n' = 0 \tag{51a}$$

$$-n\overset{2}{b}_n + \overline{r}_n' = 0 \tag{51b}$$

Solving equations (51a) and (51b) for the coefficients $\overset{2}{a}_n$ and $\overset{2}{b}_n$ yields

$$\overset{2}{a}_n = r_n'/n \tag{52a}$$

$$\overset{2}{b}_n = \overline{r}_n'/n \tag{52b}$$

Equations (52) show that the coefficients $\overset{2}{a}_n$ and $\overset{2}{b}_n$ are of order of magnitude unity.

The small-perturbation transonic limiting solution for the second-approximation potential $\overset{2}{F}$ is thus proportional to K^2 and to a function of body shape and position of order of magnitude unity, which approaches a finite limit as $s \rightarrow 0$. The derivatives of $\overset{2}{F}$ and, in particular $\overset{2}{F}_x$, also have these properties. From the results obtained thus far it is clear that similar solutions will be obtained in succeeding approximations because the right-hand side of equation (9), in the small-perturbation transonic limit, will always, in any approximation, be made up of power of K and of functions of order of magnitude unity that approach a finite limit for $s \longrightarrow 0$. The various approximations may be conveniently arranged in the form of a power series in K and the final results indicated as follows: 1379

On the body $s = s_b \sim 0$

$$F_x = \frac{\Gamma_M}{\beta^2} \Lambda = a_1 K + a_2 K^2 + a_3 K^3 + \ldots \tag{53}$$

In the flow field $s \sim 1$

$$F_x = \frac{\Gamma_M}{\beta^2} \nu = b_1 K + b_2 K^2 + b_3 K^3 + \ldots \tag{54}$$

where

Λ disturbance velocity on body

ν disturbance velocity in flow field

$a_1, a_2 \ldots, b_1, b_2 \ldots$ functions of body shape and position of order of magnitude unity

The velocity on the body and in the flow field in two-dimensional transonic flow past a thin airfoil therefore depends on Mach number M_0, specific-heat ratio γ, and lateral distance ratio τ only through the combination $K = \tau\Gamma_M/\beta^3$. The small-perturbation transonic limiting form of the parameter K is $\tau\Gamma/\beta^3$, where $\Gamma = \lim_{M_0 \to 1} \Gamma_M = \frac{\gamma+1}{2}$. This limiting form of the parameter K is a simple function of the similarity parameter of reference 1. Thus the transonic similarity law derived here agrees, in the small-perturbation transonic limit, with that of reference 1. It is apparent that the validity of the derivation of this law depends principally upon the properties of the potential in the neighborhood of the axis.

Finally, the previous assumption that in the body boundary condition (12) the F_x term could be neglected with respect to the F_η term will be justified. Because F_x and F_η are of the same order, the ratio of the F_x term to the F_η term in equation (12b) is of order τ/β. For a transonic flow with the parameter K of order one, τ is of order β^3. The ratio τ/β is therefore of order β^2, which is small compared to unity in the transonic limit $\beta \longrightarrow 0$. Hence, the neglect of the F_x term was initially permissible.

ILLUSTRATIVE EXAMPLE

Although the prime purpose of the method of analysis was to verify the transonic similarity law, the method of analysis may now be used to calculate flows past specific airfoil sections. As an illustration, the first two approximations for the flow past a Kaplan section will be obtained in the limiting form given by equation (53). The parametric equations of the Kaplan section are (reference 7)

$$x_b = \cos\theta - \frac{\tau}{4}(\cos\theta - \cos 3\theta)$$

$$y_b = \frac{\tau}{4}(3\sin\theta - \sin 3\theta)$$

where the parameter θ varies between 0 and 2π and the coordinates x_b, y_b, in conformity with the convention of reference 7, refer to the semichord as the unit of length.

<u>First approximation.</u> - The slope dy_b/dx_b is given to the first order in τ by

$$\frac{dy_b}{dx_b} = -\frac{3}{4}\tau\left(\frac{\cos\theta - \cos 3\theta}{\sin\theta}\right)$$

$$= -3\tau \sin\theta \cos\theta$$

In terms of x_b the slope is then given, to the first order in τ, by

$$\frac{dy_b}{dx_b} = -3\tau x_b \sqrt{1-{x_b}^2} \tag{55}$$

In general, the slope is, from equation (7a),

$$\frac{dy_b}{dx_b} = \tau\, g_x(x)$$

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so that here

$$g_x(x) = -3x_b \sqrt{1-{x_b}^2} \tag{56}$$

The coordinates x_b,η_b are given in terms of elliptic coordinates by equation (17)

$$x_b = \cosh s \cos t \tag{17a}$$

$$\eta_b = \sinh s \sin t \tag{17b}$$

Because the boundary condition is to be evaluated at $s = 0$, equation (17) becomes

$$\left.\begin{aligned} x_b &= \cos t \\ \eta_b &= 0 \end{aligned}\right\} \tag{57}$$

The function $g_x(x)$ is therefore given by, at $s = 0$,

$$g_x(x) = -3 \cos t \sin t \tag{58}$$

Equation (58) could have been obtained directly from the expression for $g_x(x)$ in terms of θ by noting that, in the limit $s \longrightarrow 0$, the parameter θ becomes identical to the coordinate t.

The first approximation $\overset{1}{F}$ is, from equation (25),

$$\overset{1}{F} = \sum_{n=0}^{\infty} (\overset{1}{A}_n \sin nt + \overset{1}{B}_n \cos nt)e^{-ns} \tag{25}$$

The boundary condition for $\overset{1}{F}$ is, from equation (29),

$$\sum_{n=0}^{\infty} n(\overset{1}{A}_n \sin nt + \overset{1}{B}_n \cos nt) = -Kg_x(x) \sin t \tag{29}$$

Inserting the value of $g_x(x)$ from equation (58) in equation (29) yields

$$\sum_{n=0}^{\infty} n\left(\overset{1}{A}_n \sin nt + \overset{1}{B}_n \cos nt\right) = \frac{3K}{4} (\cos t - \cos 3t)$$

The coefficients $\overset{1}{A}_n$ and $\overset{1}{B}_n$ are therefore given by

$$\left.\begin{aligned} \overset{1}{A}_n &= 0 \\ \overset{1}{B}_1 &= \frac{3}{4} K \\ \overset{1}{B}_3 &= -\frac{1}{4} K \\ \overset{1}{B}_n &= 0, \quad n \neq 1, 3 \end{aligned}\right\} \tag{59}$$

The first approximation $\overset{1}{F}$ is thus

$$\overset{1}{F} = \frac{K}{4} \left(3e^{-s} \cos t - e^{-3s} \cos 3t\right) \tag{60}$$

The limiting form of $\overset{1}{F}$ for $s \longrightarrow 0$ is

$$\overset{1}{F} = \frac{K}{4} (3 \cos t - \cos 3t) \tag{61}$$

The limiting form of the velocity $\overset{1}{\Lambda}$ for $s \longrightarrow 0$ is, by equations (27), (33), and (35),

$$\Gamma_M \frac{\overset{1}{\Lambda}}{\beta^2} = -\frac{3}{2} K \cos 2t \tag{62}$$

The maximum value of $\overset{1}{\Lambda}$ occurs at the midchord $(t = \pi/2)$ and is given by

$$\Gamma_M \overset{1}{\Lambda}_{max}/\beta^2 = \frac{3}{2} K \tag{63}$$

<u>Second approximation.</u> - The second approximation $\overset{2}{F}$ is the solution of the non-homogeneous Laplace equation (37) 1379

$$\overset{2}{F}_{ss} + \overset{2}{F}_{tt} = 2J(s,t)\overset{1}{F}_x\overset{1}{F}_{xx} \tag{37}$$

The x-derivatives of $\overset{1}{F}$ may be obtained by successive applications of equation (20). It is more convenient, however, to first introduce the conjugate complex variables

$$\left.\begin{aligned} \zeta &= s + it \\ \bar{\zeta} &= s - it \end{aligned}\right\} \tag{64}$$

Equation (60) becomes, in terms of ζ and $\bar{\zeta}$,

$$\overset{1}{F} = \frac{K}{8}\left(3e^{-\zeta} - e^{-3\zeta} + 3e^{-\bar{\zeta}} - e^{-3\bar{\zeta}}\right) \tag{65}$$

The derivatives with respect to x are given in terms of derivatives with respect to ζ and $\bar{\zeta}$ by

$$F_x = \frac{F_\zeta}{\sinh\,\zeta} + \frac{F_{\bar{\zeta}}}{\sinh\,\bar{\zeta}} \tag{66}$$

$$\begin{aligned} F_{xx} = {} & \frac{1}{\sinh^2\zeta}F_{\zeta\zeta} - \frac{\cosh\,\zeta}{\sinh^3\zeta}F_\zeta + \frac{1}{\sinh^2\bar{\zeta}}F_{\bar{\zeta}\bar{\zeta}} - \frac{\cosh\,\bar{\zeta}}{\sinh^3\bar{\zeta}}F_{\bar{\zeta}} + {} \\ & \frac{2}{\sinh\,\zeta\,\sinh\,\bar{\zeta}}F_{\zeta\bar{\zeta}} \end{aligned} \tag{67}$$

The x-derivatives of $\overset{1}{F}$ are, from equations (65) to (67)

$$\overset{1}{F}_x = -\frac{3}{4}K\left(e^{-2\zeta} + e^{-2\bar{\zeta}}\right) \tag{68}$$

$$\overset{1}{F}_{xx} = \frac{3}{2}K\left(\frac{e^{-2\zeta}}{\sinh\,\zeta} + \frac{e^{-2\bar{\zeta}}}{\sinh\,\bar{\zeta}}\right) \tag{69}$$

The Jacobian $J(\zeta,\overline{\zeta})$ and the Laplacian $\Delta F(\zeta,\overline{\zeta})$ are given in
1379 terms of ζ and $\overline{\zeta}$ by

$$J(\zeta,\overline{\zeta}) = \sinh \zeta \sinh \overline{\zeta} \tag{70}$$

$$\Delta F(\zeta,\overline{\zeta}) = \frac{4}{\sinh \zeta \sinh \overline{\zeta}} F_{\zeta\overline{\zeta}} \tag{71}$$

The differential equation (37) for the second approximation $\overset{2}{F}$ becomes

$$4\overset{2}{F}_{\zeta\overline{\zeta}} = -\frac{9}{4} K^2 (e^{-2\zeta} + e^{-2\overline{\zeta}})(-2\zeta \sinh \overline{\zeta} e + -2\overline{\zeta} \sinh \zeta e) \tag{72}$$

If equation (72) is transformed back to s,t coordinates, there is obtained

$$\overset{2}{F}_{ss} + \overset{2}{F}_{tt} = \frac{9}{4} K^2 \left[(e^{-5s} - e^{-3s}) \cos t + e^{-5s} \cos 3t - e^{-3s} \cos 5t\right] \tag{73}$$

In order to solve equation (73), a solution is assumed of the form

$$\overset{2}{F} = \frac{9}{4} K^2 \left[\overset{2}{q}_1(s) \cos t + \overset{2}{q}_3(s) \cos 3t + \overset{2}{q}_5(s) \cos 5t\right] \tag{74}$$

Insertion of equation (74) into equation (73) yields the following ordinary differential equations for the q functions:

$$\left.\begin{aligned} \overset{2}{q}_1{}'' - \overset{2}{q}_1 &= e^{-5s} - e^{-3s} \\ \overset{2}{q}_3{}'' - 9\overset{2}{q}_3 &= e^{-5s} \\ \overset{2}{q}_5{}'' - 25\overset{2}{q}_5 &= -e^{-3s} \end{aligned}\right\} \tag{75}$$

The solutions for the differential equations (75) are

$$\left.\begin{aligned}
\overset{2}{q}_1(s) &= \overset{2}{A}_1 e^{-s} + \overset{2}{r}_1(s) \\
\overset{2}{q}_3(s) &= \overset{2}{A}_3 e^{-3s} + \overset{2}{r}_3(s) \\
\overset{2}{q}_5(s) &= \overset{2}{A}_5 e^{-5s} + \overset{2}{r}_5(s)
\end{aligned}\right\} \tag{76}$$

The particular integrals $\overset{2}{r}_1(s)$, $\overset{2}{r}_3(s)$ and $\overset{2}{r}_5(s)$ may be evaluated by means of equation (46) 1379

$$r_n(s) = e^{-ns}\int e^{2ns}\,ds\int e^{-ns}\,h_n(s)\,ds \tag{46}$$

The result is

$$\left.\begin{aligned}
\overset{2}{r}_1(s) &= \frac{1}{24}\,e^{-5s} - \frac{1}{8}\,e^{-3s} \\
\overset{2}{r}_3(s) &= \frac{1}{16}\,e^{-5s} \\
\overset{2}{r}_5(s) &= \frac{1}{16}\,e^{-3s}
\end{aligned}\right\} \tag{77}$$

The constants $\overset{2}{A}_1$, $\overset{2}{A}_3$, and $\overset{2}{A}_5$ are determined from the boundary condition (49) which, when applied to equation (76), yields

$$\left.\begin{aligned}
\overset{2}{A}_1 &= 1/6 \\
\overset{2}{A}_3 &= -5/48 \\
\overset{2}{A}_5 &= -3/80
\end{aligned}\right\} \tag{78}$$

The solution for $\overset{2}{F}$ is therefore

$$\overset{2}{F} = \frac{9}{4}K^2\left[\left(\frac{1}{6}\,e^{-s} - \frac{1}{8}\,e^{-3s} + \frac{1}{24}\,e^{-5s}\right)\cos t + \left(\frac{-5}{48}\,e^{-3s} + \frac{1}{16}\,e^{-5s}\right)\cos 3t + \right.$$
$$\left.\left(\frac{-3}{80}\,e^{-5s} + \frac{1}{16}\,e^{-3s}\right)\cos 5t\right] \tag{79}$$

The limiting form of $\overset{2}{F}$ as $s \longrightarrow 0$ is

$$\overset{2}{F} = \frac{9}{4} K^2 \left(\frac{1}{12} \cos t - \frac{1}{24} \cos 3t + \frac{1}{40} \cos 5t \right) \tag{80}$$

The limiting form of the velocity $\overset{2}{\Lambda}$ as $s \longrightarrow 0$ is, by equations (27), (33), and (35),

$$\overset{2}{F}_x = \frac{\Gamma_M \overset{2}{\Lambda}}{\beta^2} = \frac{9}{4} \frac{K^2}{\sin t} \left(\frac{1}{12} \sin t - \frac{1}{8} \sin 3t + \frac{1}{8} \sin 5t \right) \tag{81}$$

The maximum value of $\overset{2}{\Lambda}$ occurs at the midchord $(t = \pi/2)$ and is given by

$$\frac{\Gamma_M \overset{2}{\Lambda}_{max}}{\beta^2} = \frac{3}{4} K^2 \tag{82}$$

COMPARISON OF RESULTS WITH KAPLAN

The first and second approximations for the velocity potential for the Kaplan section as obtained by the Ackeret-Prandtl-Glauert iteration method are given by equations (B-17) and (C-5), respectively, of reference 7. The limiting forms of these expressions as $\tau \longrightarrow 0$ and $\beta \longrightarrow 0$ are in agreement with the present results, (equations (60) and (79), respectively). Hence the legitimacy in the present report of obtaining a limiting solution by going to the limit in the differential equation rather than in the final solution is verified. The first procedure requires, of course, much less work than the second. It may now also be inferred that the limiting forms of Kaplan's results in the third approximation are the same as would be obtained by the present method in the third approximation. In particular, the third approximation for the maximum velocity increment (at midchord) $\overset{3}{\Lambda}_{max}$, given by equation (17) of reference 7, has as limiting form

$$\frac{\Gamma_M \overset{3}{\Lambda}_{max}}{\beta^2} = \frac{87}{80} K^3 \tag{83}$$

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The small-perturbation transonic limiting form for the maximum velocity on the surface of the Kaplan section is therefore given, to three approximations, by

$$\frac{\beta\Lambda_{max}}{\tau} = \frac{3}{2} + \frac{3}{4} K + \frac{87}{80} K^2 \tag{84}$$

The maximum velocity increments given in similarity form by equation (84) and the corresponding values obtained from reference 7 for several thickness ratios and Mach numbers are plotted in figure 2 as a function of K. It is seen that the results for the finite thickness ratios do not differ negligibly from those for the limiting solution, so that some error would be incurred in extrapolating the limiting solution for zero thickness ratio to a finite thickness ratio. For airfoil sections more closely approximating the ellipse in shape, however, the effect of thickness ratio on velocity may be expected to be smaller than for the Kaplan section. This conclusion is indicated in unpublished calculations by the method of reference 11 and mentioned in presenting reference 9.

The local Mach number M corresponding to a given velocity increment may be obtained from Bernoulli's equation which, in small-perturbation form, is given by (reference 11)

$$M^2-1 = -\beta^2 + 2\Gamma_M \Lambda \tag{85}$$

Equation (85) can be expressed in the similarity form

$$\frac{\beta\Lambda}{\tau} = \frac{1}{2K}\left(1 + \frac{M^2-1}{\beta^2}\right) \tag{86}$$

Curves of constant value of the parameter $(M^2-1)/\beta^2$ in equation (86) have been plotted in figure 2 to indicate the local Mach number corresponding to a given velocity increment.

It may be noted that the small-perturbation transonic limiting value of K derived in reference 8, corresponding to the first appearance of locally sonic velocity on the Kaplan section, is given by the intersection in figure 2 of the $\tau = 0$ curve with the $(M^2-1)/\beta^2 = 0$ local Mach number contour. In order to obtain local Mach numbers corresponding to the finite-thickness-ratio velocities in figure 2, the exact form of Bernoulli's equation should be used, rather than the small-perturbation form that leads to equation (86). Accordingly, contours of local Mach number unity, corresponding to Λ_{max}, based on the exact form of Bernoulli's equation are shown in figure 2(b) for several thickness ratios. In order to aid in obtaining sonic values of K for other thickness ratios, the interpolation curve AB has been drawn through the intersection of these contours with the corresponding velocity curves. Local Mach numbers other than unity for finite thickness ratio may be estimated by comparison with the various contours of local Mach number unity.

Each velocity curve of figure 2(b) has been terminated at a particular upper value of K, indicated by a circle. These values of K, called K_l, were obtained from the values of $M_{0,l}$ for given τ derived in reference 7. The value of $M_{0,l}$ is the limiting Mach number above which, for the given τ, the series solution for the velocity may be expected to diverge. The value of K_l for the limiting solution $\tau \longrightarrow 0$ was estimated from equation (84), as in reference 7, by comparison with the corresponding terms of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. The ratio of the terms of the series solution to the corresponding terms of the harmonic series indicated a value of K below which this ratio formed a decreasing sequence. For a discussion of the significance of these so-called potential limit quantities, see reference 11.

For convenience in converting the similarity parameters in figure 2 to more directly usable quantities, such as velocity increment Λ, a plot of the similarity parameter K against free-stream Mach number M_0 for several values of the thickness ratio τ is presented in figure 3.

The actual values of the velocity increment Λ, free-stream Mach number M_0, and local Mach number M at the potential limit and the first two of these quantities at the critical Mach number $M_{0,cr}$ (local Mach number unity) obtained from the data of reference 7 and shown in figure 2 are presented as functions of thickness ratio τ in figure 4. For comparison are included the corresponding values derived from the similarity parameters for $\tau = \beta = 0$ given also in figure 2. The local Mach number for the limiting solution should, for consistency, be calculated from the small-perturbation form of Bernoulli's equation (85). In order to indicate the approximation introduced by equation (85), the local value of M for both the critical and potential limit conditions has been also obtained from the exact form of Bernoulli's equation.

In order to compare the exact and limiting solutions with regard to flow-field conditions, calculations have been made of the maximum lateral extent of the local supersonic region at the potential limit Mach number for several thickness ratios by means of both solutions. The results for the exact solution were obtained by the method of reference 7 (equation (18)), whereas those for the limiting solution were obtained in a similar manner by equations (8a), (60), and (79). As in reference 7, only the first two approximations were used in making the calculations, because, as noted in reference 7, the disturbance velocity decreases rapidly with distance from the body. The results are shown in figure 5, in which the maximum height in chord lengths of the supersonic region has been plotted as a function of thickness ratio. The results for the limiting solution are in surprisingly good agreement with those for the exact solution, especially in view of the rapid change of lateral extent of the local supersonic region with Mach number in the neighborhood of the potential-limit Mach number shown in references 7 and 13. It is noted that the height increases rapidly with decrease in thickness ratio at the small thickness ratios. This behavior is in conformity with the similarity law, which gives a fixed value of the nondimensional parameter η_l corresponding to the height of the supersonic region. The parameter β_l approaches zero as the thickness ratio goes to zero so that the height $y_l = \eta_l/\beta_l$ (equation (8a)) correspondingly becomes infinite.

The results given by Kaplan for the circular arc (reference 14) may be analyzed in a similar manner. The maximum velocity increment obtained from equation (18) of reference 14 yields, in the small-perturbation transonic limit,

$$\frac{\beta\Lambda_{max}}{h} = 4 + 6K + \frac{94}{3}K^2 \qquad (87)$$

where h is the camber ratio (ratio of maximum height to chord length), and $K = h\Gamma_M/\beta^3$. The maximum velocity increments given by equation (87) and reference 14 are plotted in similarity form in figure 6. The velocity increment Λ and free-stream Mach number M_0 at the critical Mach number $M_{0,cr}$ for both the exact and limiting solutions are presented as functions of h in figure 7. In the same manner as for the Kaplan section, the local Mach number obtained from the exact form of Bernoulli's equation is included. The values of Λ_{cr} for the exact and limiting solutions are in better agreement for the circular arc than for the Kaplan section.

CONCLUDING REMARKS

A transonic similarity law for two-dimensional flow has been derived by von Kármán; in the derivation of this law, the assumption has been made that the boundary condition at the body may be satisfied on the axis. This assumption has been investigated herein by an iteration procedure similar to the Rayleigh-Janzen and Ackeret-Prandtl-Glauert procedures, in each step of which the boundary condition is satisfied exactly on the body. The resulting solution is of the form required by the existing transonic similarity theory, thereby validating the boundary-condition assumption. The question of the effect of stagnation points upon the similarity law, however, has not been investigated.

In using the method, the assumption was made that the small-perturbation transonic limiting form of a solution of the governing differential equation is the same as the solution of the limiting form of the differential equation. This assumption was verified by comparison with the exact solution for the Kaplan section.

The present method of analysis offers some advantages in simplicity and amount of computational work as compared to the exact method in applications to specific airfoils.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, May 24, 1950.

APPENDIX - SYMBOLS

The following symbols are used in this report:

A_n, B_n, C	arbitrary constants (equation (25))
a	local speed of sound
a_n, b_n, c	function of body shape (equations (30) and (32))
a_0	speed of sound in free air
c	chord of body
F	transformed velocity potential $\left(= \frac{\Gamma_M}{\beta^2} \varphi\right)$
$g(x)$	function characterizing shape of body
h	camber ratio
$h_n, \bar{h}_n$	functions of s (equation (42))
J	Jacobian
K	similarity parameter $\left(= \frac{\tau \Gamma_M}{\beta^3}\right)$
M_0	free-stream Mach number $(= U/a_0)$
$q_n, \bar{q}_n$	functions of s (equation (43))
R	function defined by equation (38)
r_n	particular integral (equation (46))
s,t	elliptic coordinates
U	free-stream velocity
u	disturbance velocity in x direction
v	disturbance velocity in y direction
x,y	Cartesian coordinates
β	$\sqrt{1 - M_0^2}$

Γ $\frac{\gamma+1}{2}$

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Γ_M $M_0^2 \left(1 + \frac{\gamma-1}{2} M_0^2\right)$

γ ratio of specific heats

Δ Laplacian

ζ $(= s + it)$

$\bar{\zeta}$ $(= s - it)$

η transformed y-coordinate

Λ disturbance velocity on body

ν disturbance velocity in flow field

τ lateral distance or thickness ratio

φ disturbance velocity potential

Subscripts:

b on the body

cr critical

l potential limit

max maximum

The symbols x, y, η, s, and t used as subscripts indicate differentiation with respect to that variable.

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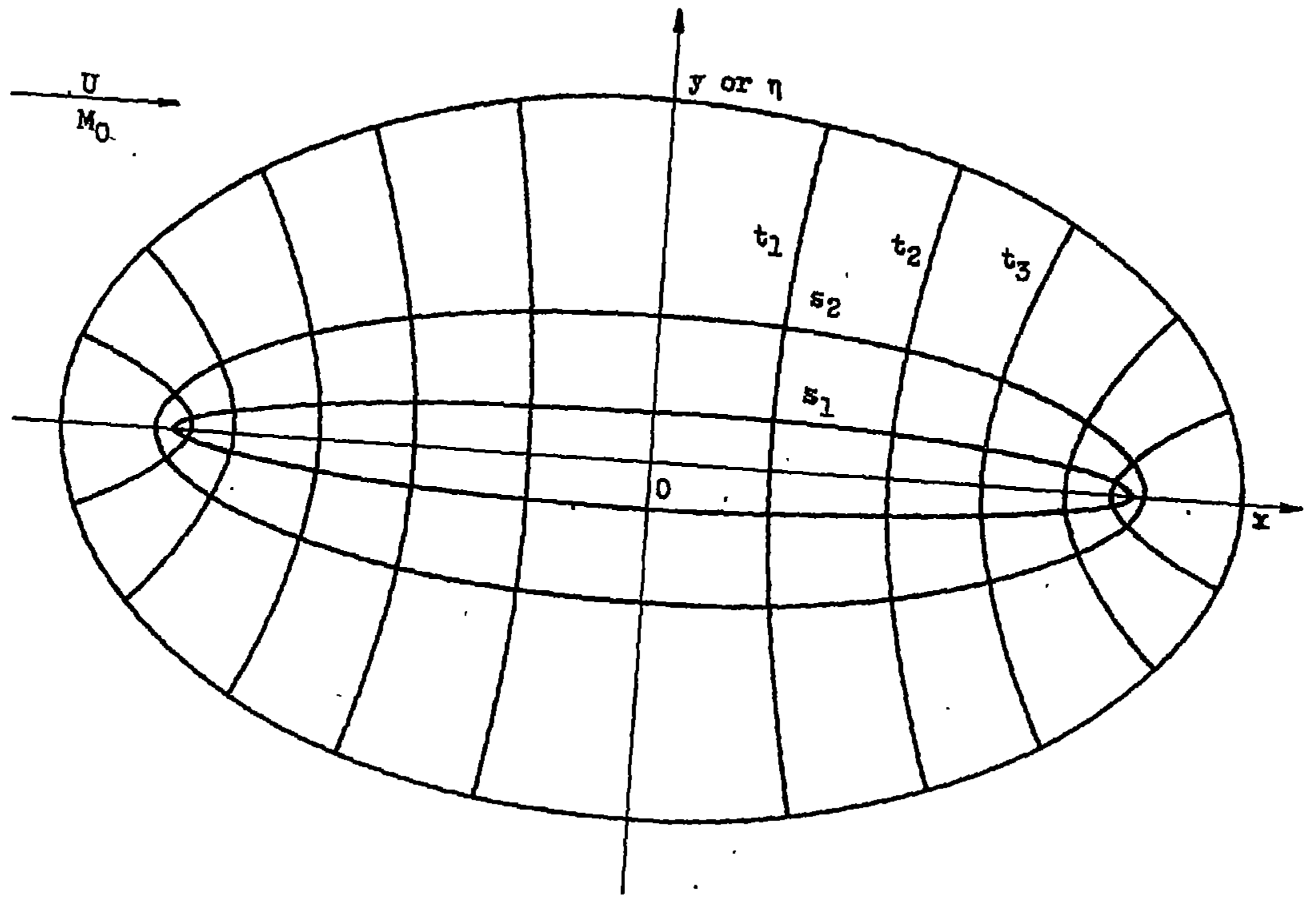
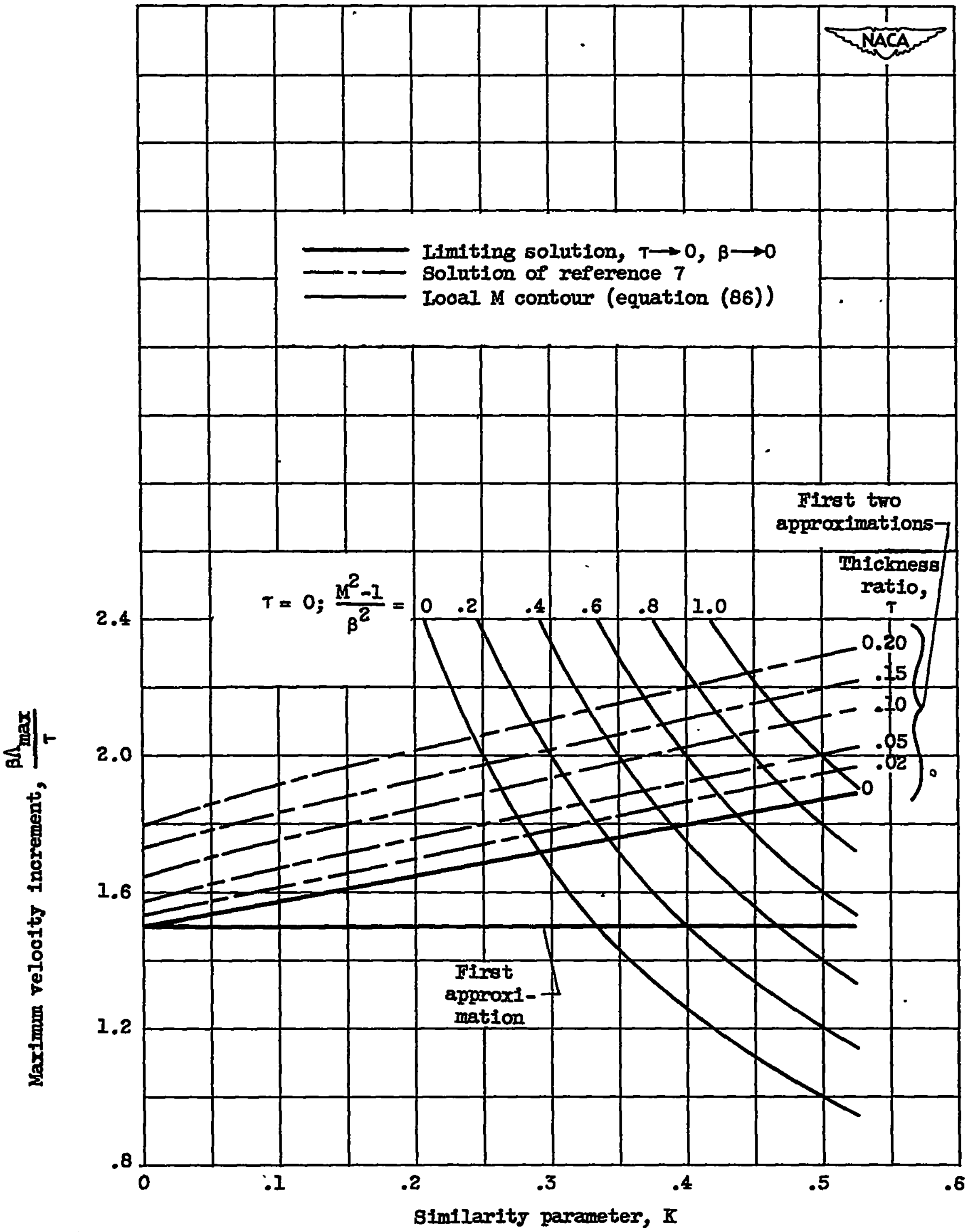
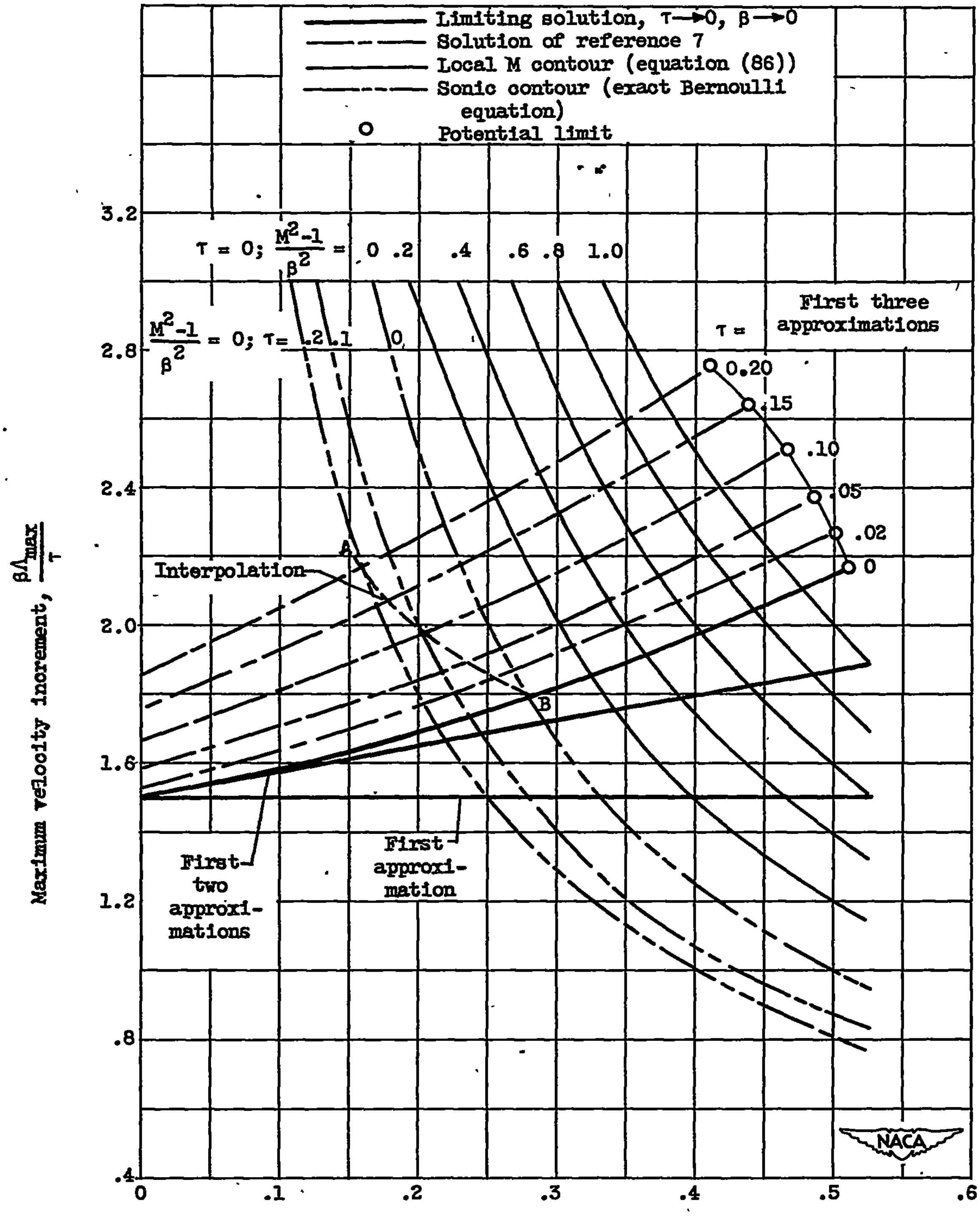


Figure 1. - Cartesian- and elliptic-coordinate systems.



(a) First and second approximations.

Figure 2. - Variation of maximum velocity increment with similarity parameter for Kaplan section.



(b) First three approximations.

Figure 2. - Concluded. Variation of maximum velocity increment with similarity parameter for Kaplan section.

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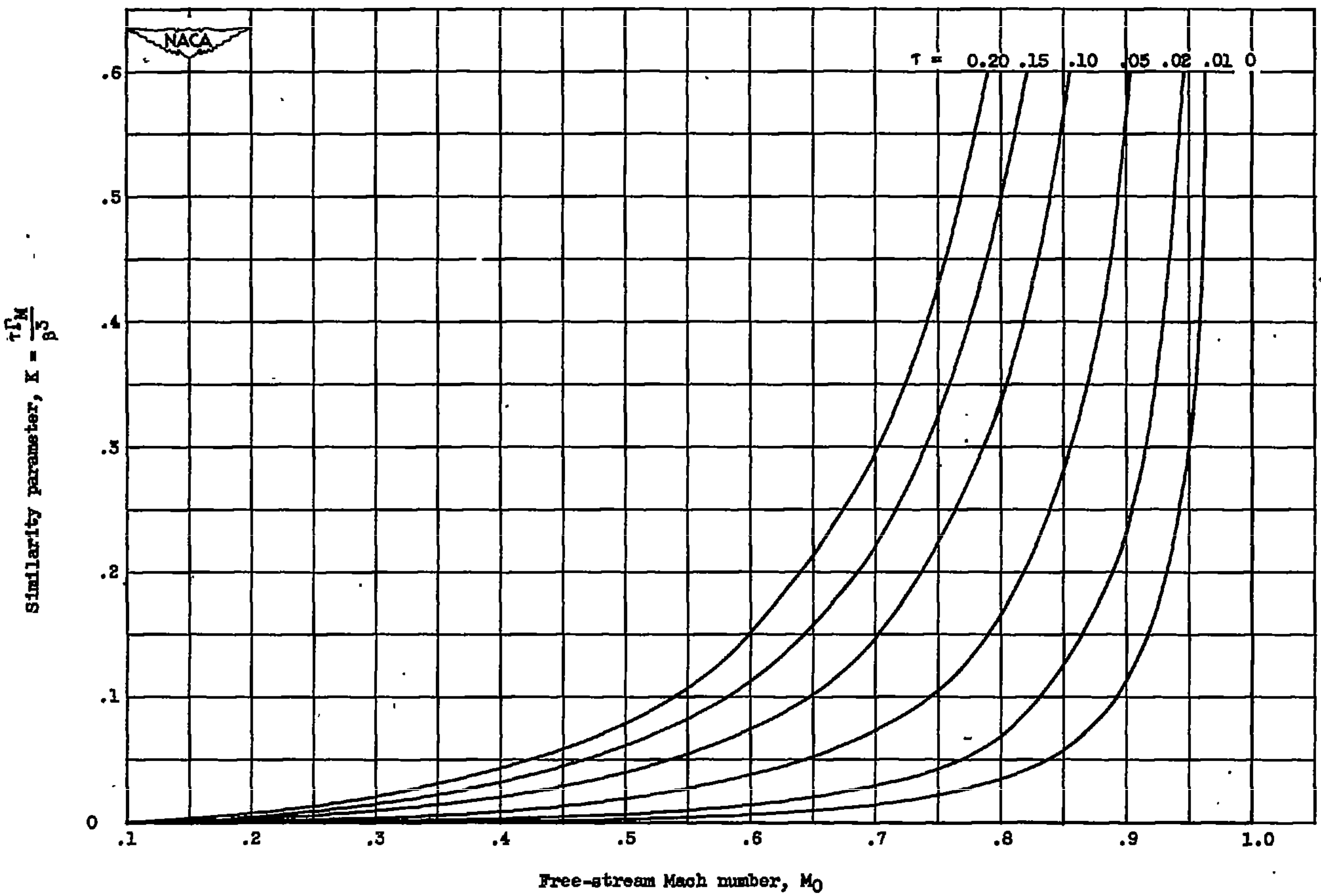

Figure 3. - Curves of similarity parameter against free-stream Mach number for several values of thickness ratio.

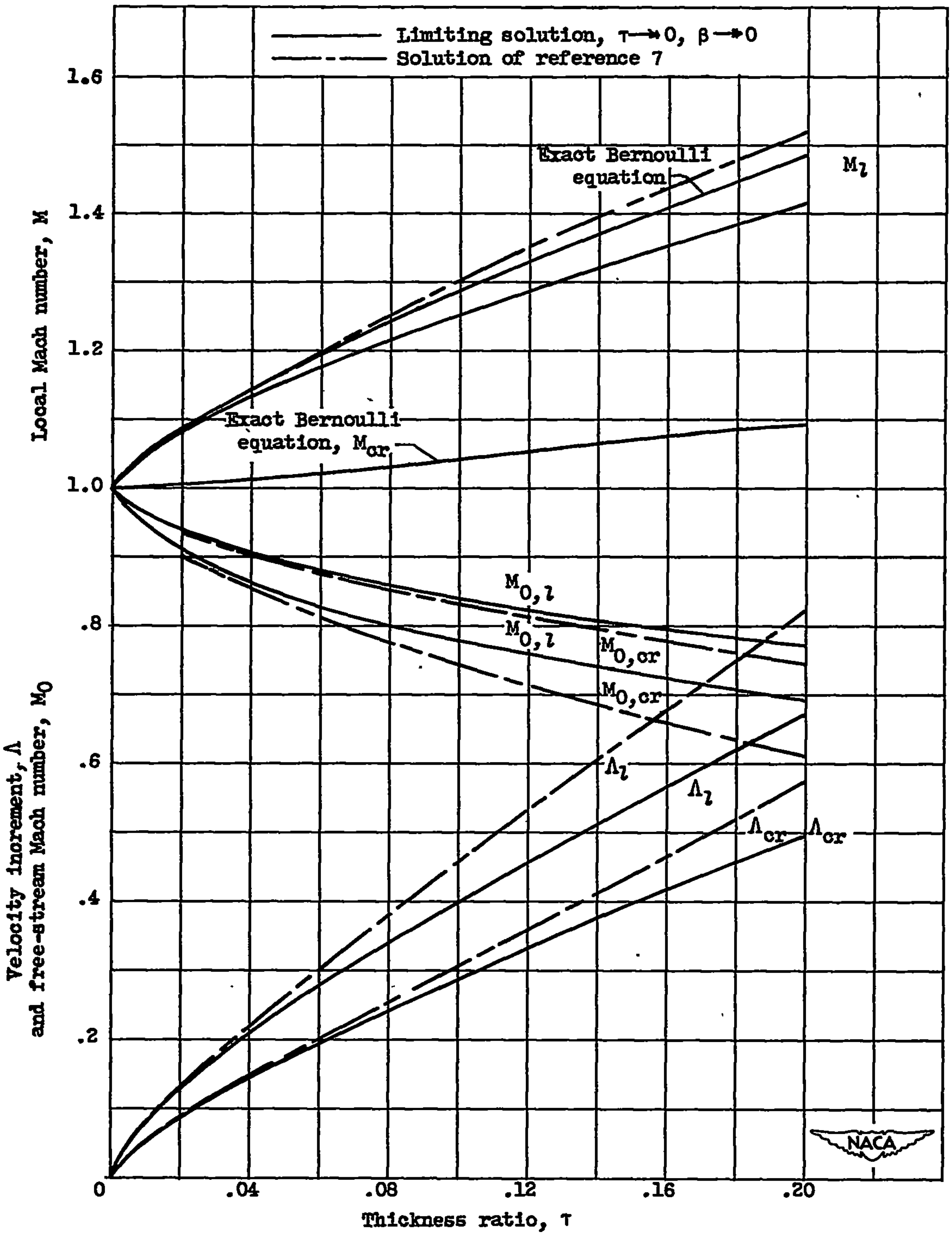

Figure 4. - Variation of maximum velocity increment, free-stream Mach number, and local Mach number with thickness ratio at critical Mach number and potential limit for Kaplan section.

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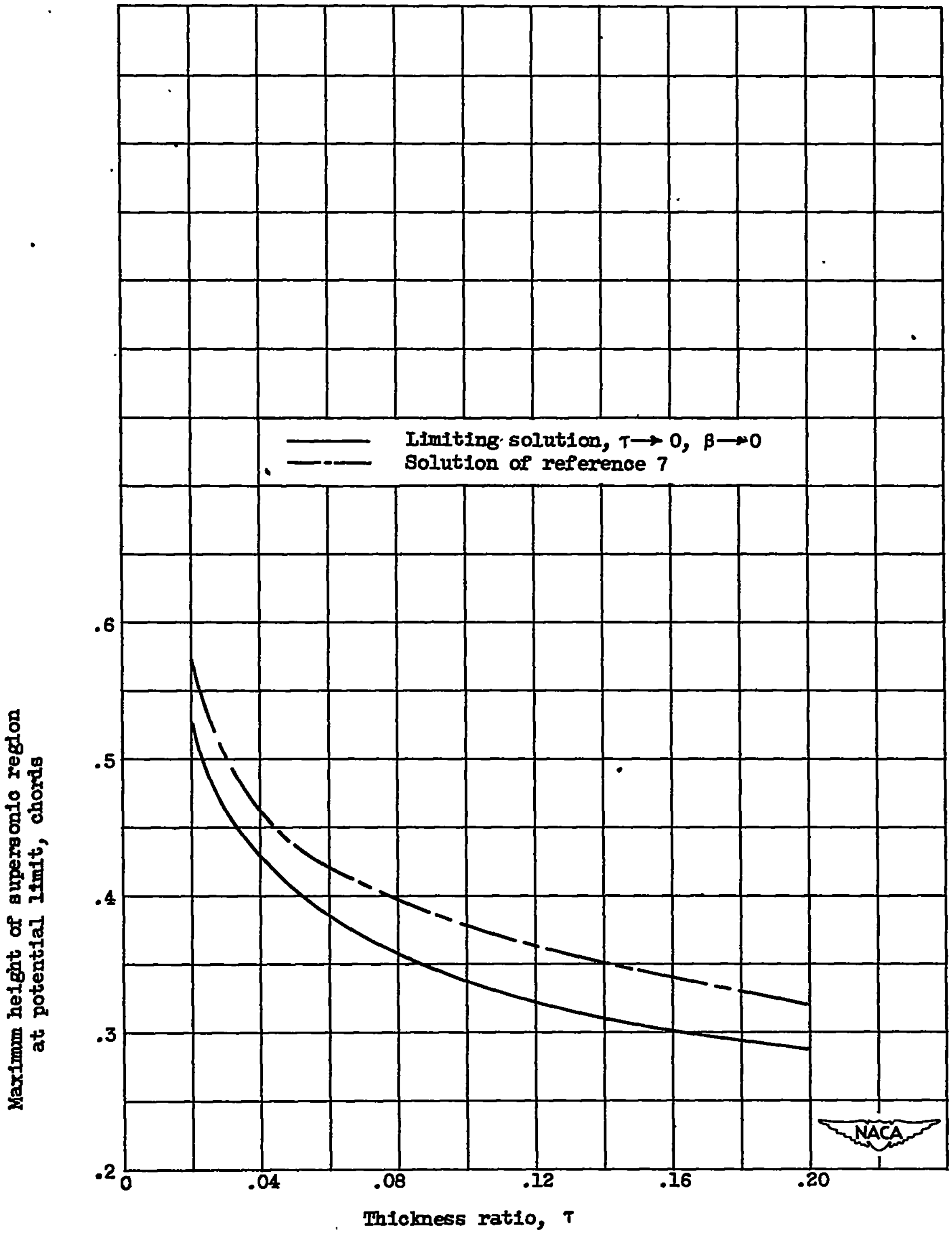
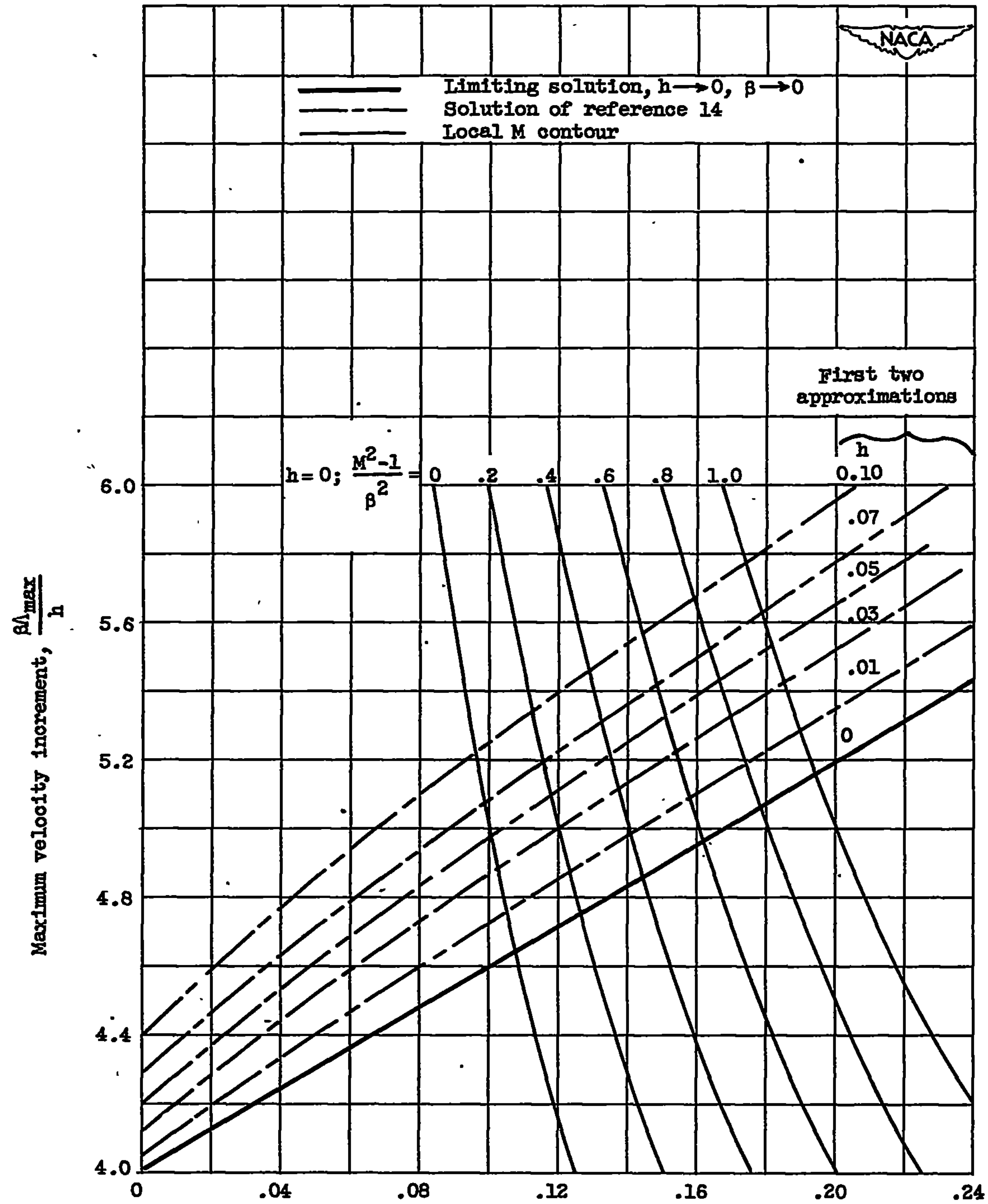
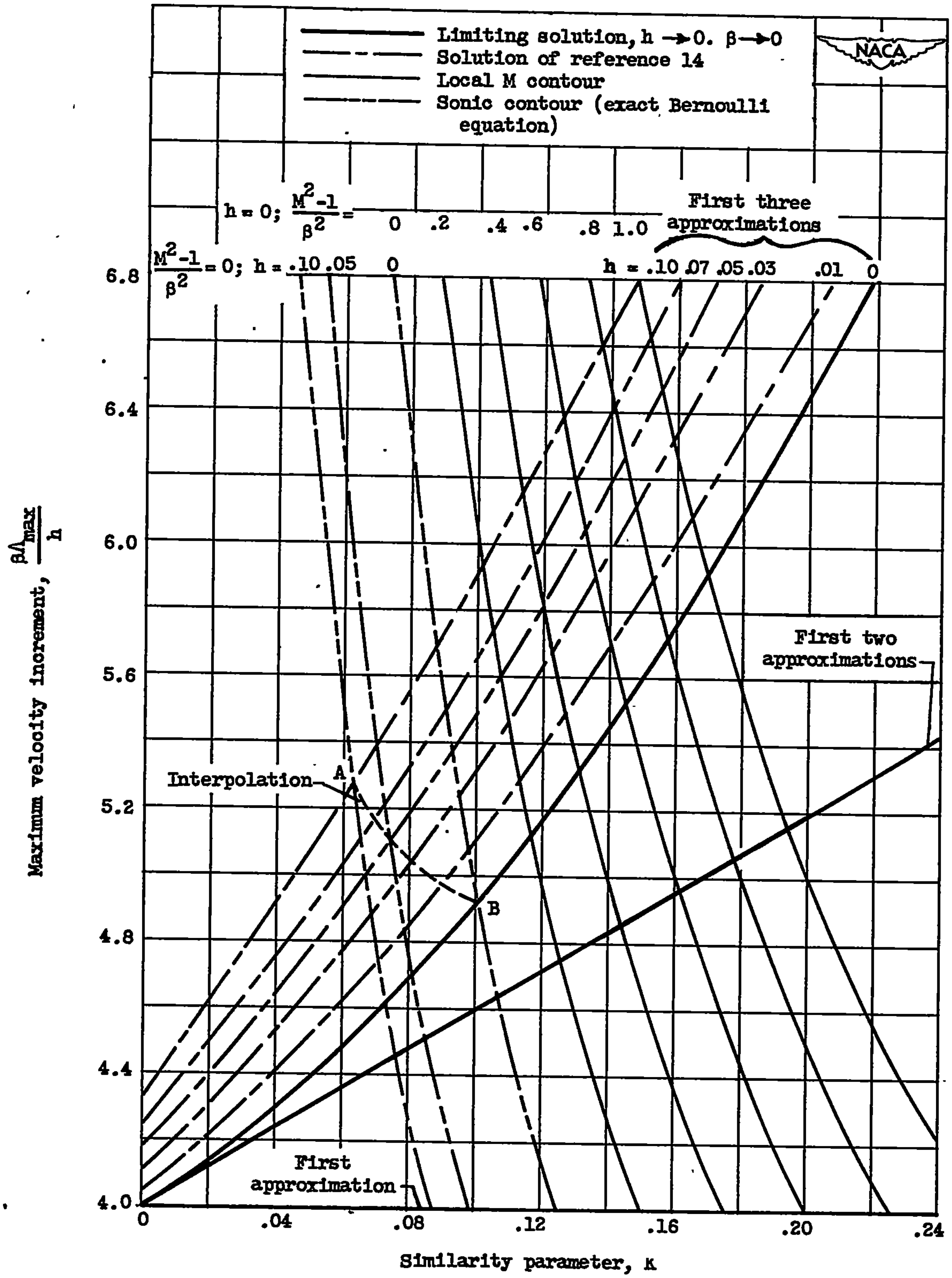


Figure 5. - Variation of maximum height of supersonic region at potential limit with thickness ratio for Kaplan section.



(a) First and second approximations.

Figure 6. - Variation of maximum velocity increment with similarity parameter for circular arc section.



(b) First three approximations.

Figure 6. - Concluded. Variation of maximum velocity increment with similarity parameter for circular arc section.

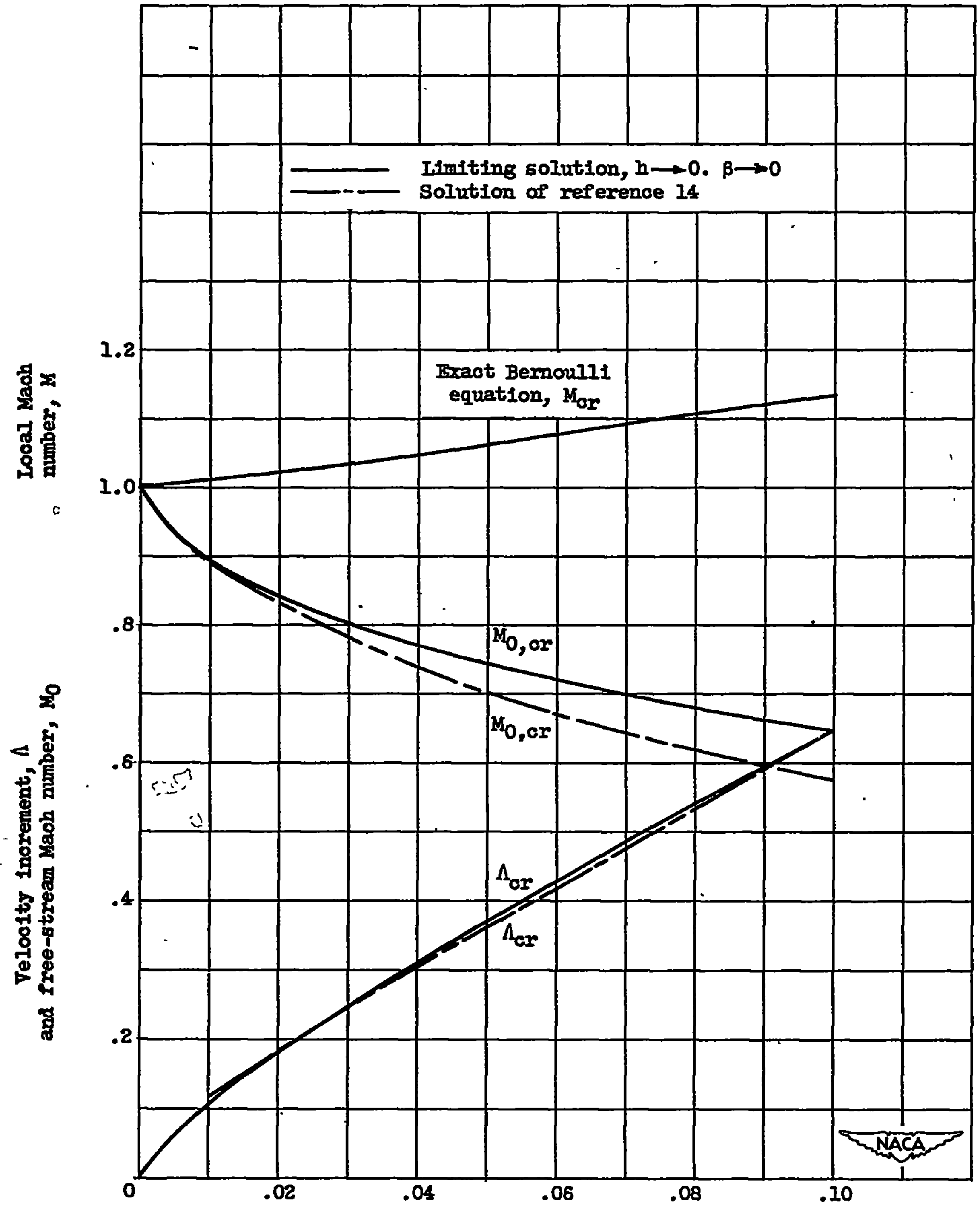


Figure 7. - Variation of maximum velocity increment, free-stream Mach number, and local Mach number with camber ratio at critical Mach number for circular arc section.

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