ISyE 6420 Homework 3

1.

$$y \mid \theta_1, \theta_2 \sim N(\theta_1 + \theta_2, 1)$$

$$\theta_i \overset{iid}{\sim} N(0, 1) \qquad i = 1, 2.$$

i.e.

$$y \mid \beta \overset{iid}{\sim} N(X'\beta, 1) \quad \text{where } \beta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \quad X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{2\times1}$$

$$P(\beta \mid y) \propto P(y \mid \beta) P(\beta)$$

$$\propto e^{-\frac{1}{2\times1}(y-(\theta_1+\theta_2))^2} \times e^{-\frac{1}{2\times1}(\beta' \Sigma_0^{-1} \beta)} \quad \text{where } \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\propto e^{-\frac{1}{2}\left[(y - X'\beta)'(y - X'\beta) + \beta'\Sigma_0^{-1}\beta\right]}$$

$$\propto e^{-\frac{1}{2}\left[y'y - 2y'_{1}X'\beta + \beta'XX'\beta + \beta'\Sigma_0^{-1}\beta\right]}$$

$$\propto e^{-\frac{1}{2}\left[\beta'(XX' + \Sigma_0^{-1})\beta - 2y'X'\beta + y'y\right]}$$

So

$$\beta \mid y \sim N_2(\beta_1, \Sigma_1)$$

$$(\beta - \beta_1)'\Sigma_1^{-1}(\beta - \beta_1) = \beta'\Sigma_1^{-1}\beta - 2\beta_1'\Sigma_1^{-1}\beta + \beta_1'\Sigma_1^{-1}\beta_1$$

$$\Sigma_1^{-1} = (XX' + \Sigma_0^{-1}), \qquad \beta_1'\Sigma_1^{-1} = \underset{1\times1}{y'}\underset{1\times2}{X'}$$

Thus, $\Sigma_1 = (XX' + \Sigma_0^{-1})^{-1}$, $\quad \beta_1 = (y'X'(XX'+\Sigma_0^{-1})^{-1})' = \left[(X'X+\Sigma_0^{-1})^{-1}\right]' Xy$

$$\Rightarrow \beta \mid y \sim N_2\left(\left[(X'X + \Sigma_0^{-1})^{-1}\right]' Xy, \ (XX' + \Sigma_0^{-1})^{-1}\right)$$

(1) Continue.

By hint (ii), I_n: $n\times n$ identity matrix, J_n: $n\times n$ matrix of 1's

Then, $(I_n + bJ_n)^{-1} = I_n - \frac{b}{1+nb} J_n$

So

$$XX' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \Sigma_0^{-1} = \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Rightarrow (\Sigma_0 + XX')^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-1} = \Sigma_0 - \frac{1}{1+2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Since $\beta | y \sim N_2\left(\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\right) = N_2\left(\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\right)$

then

$$P(\beta|y) = \frac{1}{2\pi \times \frac{2}{3} \times \frac{2}{3} \times \sqrt{1-(-\frac{1}{3})^2}} \exp\left(-\frac{1}{2(1-(-\frac{1}{3})^2)}\left(\frac{(\theta_1 - \frac{1}{3})^2}{\frac{2}{3}} + \frac{(\theta_2 - \frac{1}{3})^2}{\frac{2}{3}} - \frac{2(-\frac{1}{3}) \times (\theta_1 - \frac{1}{3})(\theta_2 - \frac{1}{3})}{\frac{2}{3} \times \frac{2}{3}}\right)\right)$$

So

$$P(\theta_1|y) \sim N\left(\frac{1}{3}, \frac{2}{3}\right)$$

$$P(\theta_2|y) \sim N\left(\frac{1}{3}, \frac{2}{3}\right)$$

3.

$y_i | \theta_i \overset{iid}{\sim} \text{Poisson}(\theta_i)$

$\theta_i | \eta \overset{iid}{\sim} \text{Exp}(\lambda)$ where η is a parameter of λ.

Sol)

$$m(\underset{\sim}{y} | \eta) = \int_0^\infty P(\underset{\sim}{y} | \underset{\sim}{\theta}) P(\underset{\sim}{\theta} | \eta) \, d\underset{\sim}{\theta}$$

$$= \int_0^\infty \prod_{i=1}^n \frac{\theta_i^{y_i} e^{-\theta_i}}{y_i!} \times \prod_{i=1}^n \lambda e^{-\lambda \theta_i} \, d\underset{\sim}{\theta}$$

$$= \prod_{i=1}^n \int_0^\infty \frac{\lambda \theta_i^{y_i}}{y_i!} e^{-\theta_i(\lambda+1)} \, d\theta_i$$

$$= \prod_{i=1}^n \lambda \int_0^\infty \frac{1}{y_i!} \theta_i^{y_i} e^{-\theta_i(\lambda+1)} \, d\theta_i$$

$$= \prod_{i=1}^n \lambda \times \frac{1}{y_i!} \times \frac{1}{\lambda+1} \times \int_0^\infty \theta_i^{y_i} \left[(\lambda+1) e^{-(\lambda+1)\theta_i} \right] d\theta_i$$

$$= \prod_{i=1}^n \lambda \times \frac{1}{y_i!} \left(\frac{1}{\lambda+1} \right)^{y_i+1}$$

$$\log m(\underset{\sim}{y} | \eta) = n \log \lambda + \left(-\log \prod_{i=1}^n y_i!\right) - \sum_{i=1}^n y_i + n \log(\lambda+1)$$

$$\frac{\partial}{\partial \lambda} \log m(\underset{\sim}{y} | \eta) = \frac{n}{\lambda} - \left(\sum_{i=1}^n y_i + n\right) \times \frac{1}{\lambda+1} = 0$$

$$\Rightarrow \frac{n\lambda + n - \lambda \sum_{i=1}^n y_i - n\lambda}{\lambda(\lambda+1)} = 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^n y_i}$$

So

$$P(\theta_i | y_i) \propto P(y_i | \theta_i) P(\theta_i)$$

$$= \frac{\theta_i^{y_i} e^{-\theta_i}}{y_i!} \times \hat{\lambda} e^{-\hat{\lambda}\theta_i} = \frac{1}{y_i! \times \frac{1}{\hat{\lambda}}} \theta_i^{y_i} e^{-\frac{\theta_i}{\left(\frac{1}{1+\hat{\lambda}}\right)}}$$

$$\propto \theta_i^{(y_i+1)-1} e^{-\frac{\theta_i}{\left(\frac{1}{1+\hat{\lambda}}\right)}} \sim \text{Gamma}\left(y_i+1, \frac{1}{1+\hat{\lambda}}\right)$$

$$\hat{\theta}^{EB} = \frac{y_i+1}{1+\hat{\lambda}} = (y_i+1)\bar{y} / \bar{y}+1 \quad \text{where } \bar{y} = \frac{\sum y_i}{n}$$

4.

$X_i \mid \phi_i \overset{i.i.d}{\sim} N(0, \phi_i)$, $\frac{1}{\phi_i} \overset{i.i.d}{\sim} Exp(\lambda) = Gamma(1, \lambda)$ $i = 1, 2, \dots, n$

$$P(\phi_i) = P\left(\frac{1}{\phi_i}\right) \left| \frac{d}{d\phi_i}\left(\frac{1}{\phi_i}\right) \right|$$

$$= \lambda e^{-\lambda/\phi_i} \times \frac{1}{\phi_i^2} = \lambda \phi_i^{-2} e^{-\lambda/\phi_i}$$

$$m(\underset{\sim}{X} \mid \underset{\sim}{\phi}) = \int_0^\infty \prod_{i=1}^n P(X_i \mid \phi_i) P(\phi_i \mid \lambda) d\phi_i$$

$$= \int_0^\infty \prod_{i=1}^n \frac{1}{\sqrt{2\pi \phi_i}} e^{-\frac{X_i^2}{2\phi_i}} \lambda \phi_i^{-2} e^{-\lambda/\phi_i} d\phi_i$$

$$= \prod_{i=1}^n \frac{\lambda}{\sqrt{2\pi}} \int_0^\infty \phi_i^{-\frac{5}{2}} e^{-\frac{\frac{X_i^2+2\lambda}{2}}{\phi_i}} d\phi_i$$

$$= \prod_{i=1}^n \frac{\lambda}{\sqrt{2\pi}} \times \frac{\Gamma(\frac{3}{2})}{\left(\frac{X_i^2+2\lambda}{2}\right)^{3/2}} \underbrace{\int_0^\infty \frac{\left(\frac{X_i^2+2\lambda}{2}\right)^{3/2}}{\Gamma(\frac{3}{2})} X^{-\frac{3}{2}-1} e^{-\frac{X_i^2+2\lambda/2}{\phi_i}} d\phi_i}_{1}$$

since it's inverse-gamma$\left(\frac{3}{2}, \frac{X_i^2}{2}+\lambda\right)$

$$\log m(\underset{\sim}{X} \mid \underset{\sim}{\phi}) = \text{constant} + n \log \lambda - \frac{3}{2} \sum_{i=1}^n \log\left(\frac{X_i^2}{2} + \lambda\right)$$

$$\frac{\partial}{\partial \lambda} \log m(\underset{\sim}{X} \mid \underset{\sim}{\phi}) = \frac{n}{\lambda} - \frac{3}{2} \sum_{i=1}^n \frac{1}{\frac{X_i^2+2\lambda}{2}} = \frac{n}{\lambda} - 3 \sum_{i=1}^n \frac{1}{X_i^2+2\lambda} = 0$$

$$\frac{n}{\lambda} - 3\left[\frac{1}{X_1^2+2\lambda} + \frac{1}{X_2^2+2\lambda} + \dots + \frac{1}{X_n^2+2\lambda}\right] = 0$$

Since $\log m(\underset{\sim}{X} \mid \underset{\sim}{\phi})$ has no closed form solution,

$$\frac{n}{\hat{\lambda}} - 3 \sum_{i=1}^n \frac{1}{X_i^2 + 2\hat{\lambda}} = 0$$

$$E(\hat{\phi}) = \frac{X_i^2 + 2\lambda/2}{\frac{1}{2}} = X_i^2 + 2\lambda \Rightarrow \hat{\phi}_i = X_i^2 + 2\hat{\lambda}$$