Problem 1

Answer to the problem goes here.

(a) marginal distribution $f_X(x)$ is exponential $Exponential(\lambda)$.

$$
\begin{aligned}
f_X(x) &= \int_x^\infty \lambda^2 e^{-\lambda y} dy \\
&= \lambda^2 \left[-\frac{1}{\lambda} e^{-\lambda y} \right]_x^\infty \\
&= -\lambda \left[\frac{1}{e^{\lambda y}} \right]_x^\infty \\
&= -\lambda[0 - e^{-\lambda x}] = \lambda e^{-\lambda x}, x \geq 0 \\
&\sim Exponential(\lambda)
\end{aligned}
$$

(b) marginal distribution $f_Y(y)$ is $Gamma(2, \lambda)$

$$
\begin{aligned}
f_Y(y) &= \int_0^y \lambda^2 e^{-\lambda y} dx \\
&= \lambda^2 e^{-\lambda y} \left[x \right]_0^y \\
&= \lambda^2 e^{-\lambda y} y \\
&= \frac{\lambda^2}{\Gamma(2-1)} y^{2-1} e^{-\lambda y},\ y \geq 0 \\
&\sim Gamma(2, \lambda)
\end{aligned}
$$

(c) conditional distribution $f(y|x)$ is shifted exponential, $f(y|x) = \lambda e^{-\lambda(y-x)}, y \geq x$.

$$
\begin{aligned}
f(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} \\
&= \lambda e^{-\lambda(y-x)},\ y \geq x
\end{aligned}
$$

(d) conditional distribution $f(x|y)$ is uniform $U(0, y)$.

$$\begin{aligned} f(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{\lambda^2 e^{-\lambda y}}{\frac{\lambda^2}{\Gamma(2-1)} y^{2-1} e^{-\lambda y}} \\ &= \frac{1}{y} \\ &\sim U(0, y) \end{aligned}$$

Problem 2

(a) For the prior suggested by the expert, find the posterior distribution of θ.

$$\begin{aligned} P(\theta|x) &\propto P(x|\theta)P(\theta) \\ &= 27\theta^3(x_1x_2x_3)^2 e^{\theta\sum_{i=1}^3 x_i^3} \times 2e^{-2\theta} \\ &= 54\theta^3(x_1x_2x_3)^2 e^{-\theta(2+\sum_{i=1}^3 x_i^3)} \\ &\propto \theta^3 e^{-\theta(2+\sum_{i=1}^3 x_i^3)} \\ &= \frac{(2+\sum_{i=1}^3 x_i^3)^4}{\Gamma(4)} x^{4-1} e^{-(2+\sum_{i=1}^3 x_i^3)\theta} \\ &\sim Gamma(4, 2+\sum_{i=1}^3 x_i^3) \\ &= Gamma(4, 45) \end{aligned}$$

(b) What are the posterior mean and variance? No need to integrate if you recognize to which family of distributions the posterior belongs.

$$\begin{aligned} E_{\theta|x}(\theta) &= \frac{4}{45} \\ Var_{\theta|x}(\theta) &= \frac{4}{45^2} \end{aligned}$$

Problem 3

(a) Suppose $\lambda = 1/5$, find the probabilities that

(i) a run continues for at least 5 hours.

$$\int_5^\infty 5e^{-5x}dx$$

$$\begin{aligned}&= 5\int_5^\infty e^{-5x}dx\\ &= 5\Big[-\frac{1}{5}e^{-5x}\Big]_5^\infty\\ &= -\Big[\lim_{x\to\infty}\frac{1}{e^{5x}} - e^{-25}\Big]\\ &= e^{-25}\end{aligned}$$

(ii) a run lasts less than 10 hours.

$$\begin{aligned}\int_0^{10} 5e^{-5x}dx &= 5\int_0^{10} e^{-5x}dx\\ &= 5\Big[-\frac{1}{5}e^{-5x}\Big]_0^{10}\\ &= -e^{-50}+1\end{aligned}$$

(iii) a run continues for at least 10 hours, given that it has lasted 5 hours.

$$P(x \geq 10|x \leq 5) = \frac{e^{50}}{1-e^{-25}}$$

(b) Now suppose that the rate parameter λ is unknown, but there are three measurements of interblockage times, $T_1 = 2,\ T_2 = 4,\ and\ T_3 = 8$.

(i) How would classical statistician estimate λ?

$$L(\lambda|x) = \prod_{i=1}^{3}\lambda e^{-\lambda(\sum_{i=1}^{3}x_i)}$$

$$l(\lambda) = 3\log\lambda - \lambda(\sum_{i=1}^{3}x_i)$$

$$\frac{\partial}{\partial\lambda}l(\lambda) = \frac{3}{\lambda} - \sum_{i=1}^{3}x_i = 0$$

$$3 = \lambda\sum_{i=1}^{3}x_i$$

$$\lambda = \frac{3}{\sum_{i=1}^{3}x_i} = \frac{3}{14}$$

(ii) What is the Bayes estimator of λ if the prior is $\pi(\lambda) = \frac{1}{\sqrt{\lambda}},\ \lambda \geq 0$.

$$\pi(\lambda) = \frac{1}{\sqrt{\lambda}},\ \lambda > 0$$

$$T \sim Exponential(\lambda)$$

$$\begin{aligned}
P(\lambda|T) &\propto P(T|\lambda)P(\lambda) \\
&= \left[\prod_{i=1}^{3} \lambda e^{-Ti\lambda}\right]\frac{1}{\sqrt{\lambda}} \\
&= \lambda^3 e^{-\lambda \sum_{i=1}^{3} T_i}\frac{1}{\sqrt{\lambda}} \\
&= (\lambda)^{\frac{7}{2}-1} e^{-\lambda \sum_{i=1}^{3} T_i} \\
&= \frac{(\sum_{i=1}^{3} T_i)^{\frac{7}{2}}}{\Gamma(\frac{7}{2})}(\lambda)^{\frac{7}{2}-1} e^{-\lambda \sum_{i=1}^{3} T_i} \\
&\sim Gamma(\frac{7}{2}, \sum_{i=1}^{3} T_i) \\
E(\lambda|T) &= \frac{\frac{7}{2}}{\sum_{i=1}^{3} x_i} \\
&= \frac{7}{28}
\end{aligned}$$