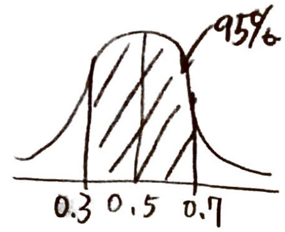
ISyE 6416 Homework 02



1. Rejection: $0.5 \pm 0.2 = [0.3, 0.7]$

$$X \sim N(0.5, 0.01)$$

$$P\left(\frac{0.3-0.5}{\sqrt{Var(X)}} < \frac{X-0.5}{\sqrt{Var(X)}} < \frac{0.7-0.5}{\sqrt{Var(X)}}\right) = 0.95$$

$$Z_{0.025} = \frac{0.7-0.5}{\sqrt{Var(X)}}$$

$$\Rightarrow 1.96 \times \sqrt{Var(X)} = 0.2$$

$$\Rightarrow \sqrt{Var(X)} = \frac{0.2}{1.96}$$

$$\Rightarrow Var(X) = \frac{0.04}{3.8416} = 0.01$$

$$\begin{cases} \frac{\alpha}{\alpha+\beta} = 0.5 & -① \\ \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.01 & -② \end{cases}$$

$$\Rightarrow \alpha = 0.5\alpha + 0.5\beta \Rightarrow 0.5\alpha = 0.5\beta \Rightarrow \alpha = \beta$$

$$\frac{\cancel{\alpha^2}}{4\cancel{\alpha^2}(2\alpha+1)} = 0.01 \Rightarrow \frac{1}{2\alpha+1} = 0.04 \Rightarrow 0.08\alpha + 0.04 = 1$$

$$\Rightarrow 0.08\alpha = 1 - 0.04$$

$$\Rightarrow \alpha = 12$$

$$\Rightarrow \beta = 12$$

$$\theta \sim Beta(12, 12) \#$$

2.

Maxwell distribution

$$P(y\mid\alpha) = \sqrt{\frac{2}{\pi}}\,\alpha^{\frac{3}{2}} y^2 e^{-\frac{1}{2}\alpha y^2}, \quad \alpha > 0, \quad y \in (0,\infty)$$

Sol)

$$\log P(y\mid\alpha) = \text{constant} + \frac{3}{2}\log\alpha + 2\log y - \frac{1}{2}\alpha y^2$$

$$\frac{\partial}{\partial\alpha}\log P(y\mid\alpha) = \frac{3}{2}\times\frac{1}{\alpha} - \frac{1}{2}y^2$$

$$\frac{\partial^2}{\partial\alpha^2}\log P(y\mid\alpha) = \frac{3}{2}\times\frac{-1}{\alpha^2} = -\frac{3}{2\alpha^2}$$

$$I(\alpha) = -E\left[\frac{\partial^2}{\partial\alpha^2}\log P(y\mid\alpha)\,\middle|\,\alpha\right] = \frac{3}{2}E\left[\frac{1}{\alpha^2}\,\middle|\,\alpha\right] = \frac{3}{2\alpha^2}$$

$$P(\alpha) \propto \sqrt{I(\alpha)} \propto \frac{1}{\alpha}, \quad \alpha > 0$$

Transformation of this parameter:

$$P(\psi) = \underset{\frac{1}{\alpha}}{P(\alpha)}\left|\frac{\partial\alpha}{\partial\psi}\right| \propto \sqrt{I(\alpha)}\left|\frac{\partial\alpha}{\partial\psi}\right| = \sqrt{I(\psi)} =$$

$$\sqrt{\cancel{I(\alpha)}} = \sqrt{\cancel{I(\psi)}} \times \frac{1}{\left|\frac{\partial\alpha}{\partial\psi}\right|}$$

$$\Rightarrow \sqrt{I(\psi)} = \sqrt{I(\alpha)} \times \left|\frac{\partial\alpha}{\partial\psi}\right| = c = P(\psi)$$

$$P(\psi) = \begin{cases} c, & \psi \in [0, \frac{1}{c}] \\ 0 & \psi \notin [0, \frac{1}{c}] \end{cases} \sim \text{Uniform}(0, \tfrac{1}{c})$$

3.

$$p(\theta) \propto \sqrt{\det(I(\theta))}$$

$$\pi_1(\theta_1) = \sqrt{I(\theta_1)} = \sqrt{-E\left(\frac{\partial^2}{\partial\theta_1^2} P_1(y_1|\theta_1)\right)}$$

$\vdots$

$$\pi_n(\theta_n) = \sqrt{I(\theta_n)} = \sqrt{-E\left[\frac{\partial^2}{\partial\theta_n^2} P_n(y_n|\theta_n)\right]}$$

Since y_i's are independent.

$$-E\left[\frac{\partial^2}{\partial_i\partial_j} \log P(y_i, y_j|\theta_i, \theta_j)\right] = -E\left[\frac{\partial}{\partial_i} \log P(y_i|\theta_i) \frac{\partial}{\partial_j} \log P(y_j|\theta_j)\right]$$

$$= -E\left[\frac{\partial}{\partial_i} \log P(y_i|\theta_i)\right] E\left[\frac{\partial}{\partial_j} \log P(y_j|\theta_j)\right] = 0$$

By Lemma.

$$E\left[\frac{\partial L}{\partial\theta}\right] = 0 \qquad L: \text{log likelihood}$$

pf/

$$E\left[\frac{\partial L}{\partial\theta}\right] = \int \left[\frac{\partial \log l}{\partial\theta}\right] \overset{P}{\overset{\shortparallel}{p(x|\theta)}}\, dx = \int \frac{\partial \log P}{\partial\theta}\, P\, dx$$

$$= \int \frac{\partial P}{\partial\theta} \times \frac{1}{P} \times P\, dx = \int \frac{\partial P}{\partial\theta} dx = \frac{d}{d\theta} \int P dx = \frac{d}{d\theta} 1 = 0$$

Thus.

$$P(\theta) \propto \sqrt{\begin{vmatrix} I(\theta_1) & 0 & \cdots & 0 \\ 0 & I(\theta_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I(\theta_n) \end{vmatrix}} = \sqrt{\det(I(\theta))} = \sqrt{I(\theta_1) \times \cdots \times I(\theta_n)} = \pi_1(\theta_1) \times \cdots \pi_n(\theta_n)$$

$$= \prod_{i=1}^{n} \pi_i(\theta_i)$$

4.

$X \sim \text{Binomial}(n, \pi)$

$\pi \sim \text{Beta}(1,3)$

$Y \sim \text{Binomial}(n, \rho)$

$\rho \sim \text{Beta}(3,1)$

Loss Function: $(\pi - \rho - a)^2$

$$P(\rho | Y) = P(Y|\rho)P(\rho)$$
$$= \binom{n}{y}\rho^y(1-\rho)^{n-y} \times \frac{\rho^2(1-\rho)^0}{B(3,1)} \propto \rho^{y+2}(1-\rho)^{n-y}$$

$$P(\pi|X) = P(X|\pi)P(\pi)$$
$$= \binom{n}{x}\pi^x(1-\pi)^{n-x}\frac{\pi^0(1-\pi)^2}{B(1,3)}$$
$$\propto \pi^x(1-\pi)^{n-x+2}$$

$$\rho_0(a,(X,Y)) = E[((\pi-\rho)-a)^2 | X,Y]$$
$$= [E(\pi-\rho|X,Y)-a]^2 + Var((\pi-\rho)|X,Y)$$

To minimize $\rho(a,(X,Y)) \Rightarrow a = E(\pi-\rho|X,Y)$

$$= E(\pi|X) - E(\rho|Y)$$
$$= \frac{x+1}{n+4} - \frac{y+3}{n+4} = \frac{x-y-2}{n+4} \quad *$$

Note:

Posterior distribution of π, ρ.

$$P(\pi,\rho|X,Y) = \binom{n}{x}\pi^x(1-\pi)^{n-x}\binom{n}{y}\rho^y(1-\rho)^{n-y}$$
$$\propto \pi^x(1-\pi)^{n-x}\rho^y(1-\rho)^{n-y}$$

Since $X \perp Y$,

$$P(\pi,\rho|X,Y) = P(\pi|X)P(\rho|Y) \text{ where } P(\pi|X) = \pi^x(1-\pi)^{n-x+2}$$
$$P(\rho|Y) = \rho^{y+2}(1-\rho)^{n-y}$$

$\pi | X \sim \text{Beta}(X+1, n-X+3)$

$\rho | Y \sim \text{Beta}(y+3, n-y+1)$

5.

$$\begin{cases} L(\theta, a) = c_1(a-\theta) & \text{if } a \geq \theta \Rightarrow a-\theta \geq 0 \\ L(\theta, a) = c_2(\theta - a) & \text{if } a \leq \theta \Rightarrow a-\theta \leq 0 \end{cases}$$

$|a-\theta| \geq 0$

$\Rightarrow a-\theta \geq 0$

$\theta - a \leq 0$

$$P(a) = E[L(\theta, a)]$$

$$= \int_{\theta \geq a} c_2(\theta - a) P(\theta|X)\, d\theta + \int_{a \geq \theta} c_1(a-\theta) P(\theta|X)\, d\theta$$

$$\frac{\partial}{\partial a} P(a) = \frac{\partial}{\partial a}\left[\int_a c_2(\theta - a) P(\theta|X)\, d\theta + \int_{-\infty}^{a} c_1(a-\theta) P(\theta|X)\, d\theta\right] = 0$$

$$= \int_a^{\infty} -c_2 P(\theta|X)\, d\theta + \underbrace{f(x, \overset{\infty}{\overset{\shortparallel}{b(x)}}) \frac{\partial b(x)}{\partial x}}_{0} - \underbrace{c_2(a-a) P(\theta|X) \frac{da}{da}}_{0}$$

$$+ \int_{-\infty}^{a} c_1 P(\theta|X)\, d\theta + \underbrace{f(x.a) \frac{da}{da}}_{0} - \underbrace{f(x, \overset{-\infty}{\overset{\shortparallel}{a(x)}}) P(\theta|X) \frac{d\, a(x)}{da}}_{0}$$

$$= 0$$

$$\Rightarrow \int_a^{\infty} c_2 P(\theta|X)\, d\theta = \int_{-\infty}^{a} c_1 P(\theta|X)\, d\theta$$

$$\Rightarrow \int_a^{\infty} c_2 P(\theta|X)\, d\theta + \int_{-\infty}^{a} c_2 P(\theta|X)\, d\theta = \int_{-\infty}^{a} c_1 P(\theta|X)\, d\theta + \int_{-\infty}^{a} c_2 P(\theta|X)\, d\theta$$

$$\Rightarrow c_2\left[\int_a^{\infty} P(\theta|X) + \int_{-\infty}^{a} P(\theta|X)\, d\theta\right] = c_1 + c_2 \left[\int_{-\infty}^{a} P(\theta|X)\, d\theta\right]$$

$$\Rightarrow \frac{c_2}{c_1 + c_2} = \int_{-\infty}^{a} P(\theta|X)\, d\theta = P(\theta \leq a)$$