Homework 4

ISyE 6420
Fall 2020

1. Simple Metropolis: Normal Precision – Gamma. Suppose $X = -2$ was observed from the population distributed as $\mathcal{N}\left(0, \frac{1}{\theta}\right)$ and one wishes to estimate the parameter θ. (Here θ is the reciprocal of the variance σ^2 and is called the *precision parameter*). Suppose the analyst believes that the prior on θ is $\mathcal{G}a(1/2, 1)$.

Using Metropolis algorithm, approximate the posterior distribution and the Bayes' estimator of θ. As the proposal distribution, use gamma $\mathcal{G}a(\alpha, \beta)$ with parameters α, β selected to ensure efficacy of the sampling (this may require some experimenting).

2. Normal-Cauchy by Gibbs. Assume that $y_1, y_2, \ldots, y_n$ is a sample from $\mathcal{N}(\theta, \sigma^2)$ distribution, and that the prior on θ is Cauchy $\mathcal{C}a(\mu, \tau)$,

$$f(\theta|\mu, \tau) = \frac{1}{\pi} \cdot \frac{\tau}{\tau^2 + (\theta - \mu)^2}.$$

Even though the likelihood for $y_1, \ldots, y_n$ simplifies by sufficiency arguments to a likelihood of $\bar{y} \sim \mathcal{N}(\theta, \sigma^2/n)$, a closed form for the posterior is impossible and numerical integration is required.

The approximation of the posterior is possible by Gibbs sampler as well. Cauchy $\mathcal{C}a(\mu, \tau)$ distribution can be represented as a scale-mixture of normals:

$$[\theta] \sim \mathcal{C}a(\mu, \tau) \equiv [\theta|\lambda] \sim \mathcal{N}\left(\mu, \frac{\tau^2}{\lambda}\right), [\lambda] \sim \mathcal{G}a\left(\frac{1}{2}, \frac{1}{2}\right),$$

that is,

$$\frac{\tau}{\pi(\tau^2 + (\theta - \tau)^2)} \propto \int_0^\infty \sqrt{\frac{\lambda}{2\pi\tau^2}} \exp\left\{-\frac{\lambda}{2\tau^2}(\theta - \mu)^2\right\} \cdot \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\lambda}{2}\right\} d\lambda.$$

The full conditionals can be derived from the product of the densities for the likelihood and priors,

$$[\bar{y}|\theta, \sigma^2] \sim \mathcal{N}\left(\theta, \frac{\sigma^2}{n}\right),$$

$$[\theta|\lambda] \sim \mathcal{N}\left(\mu, \frac{\tau^2}{\lambda}\right),$$

$$[\lambda] \sim \mathcal{G}a\left(\frac{1}{2}, \frac{1}{2}\right).$$

(a) Show that full conditionals are normal and exponential,

$$[\theta|\bar{y}, \lambda] \sim \mathcal{N}\left(\frac{\tau^2}{\tau^2 + \lambda\sigma^2/n}\bar{y} + \frac{\lambda\sigma^2/n}{\tau^2 + \lambda\sigma^2/n}\mu, \frac{\tau^2 \cdot \sigma^2/n}{\tau^2 + \lambda\sigma^2/n}\right),$$

$$[\lambda|\bar{y}, \theta] \sim \mathcal{E}\left(\frac{\tau^2 + (\theta - \mu)^2}{2\tau^2}.\right)$$

(b) Jeremy models the score on his IQ tests as $\mathcal{N}(\theta, \sigma^2)$ with $\sigma^2 = 90$. He places Cauchy $\mathcal{C}a(110, \sqrt{120})$ prior on θ.

In 10 random IQ tests Jeremy scores $y = [100, 106, 110, 97, 90, 112, 120, 95, 96, 109]$. The average score is 103.5, which is the frequentist estimator of θ. Using Gibbs sampler described in (a) approximate the posterior mean and variance. Approximate 95% equi-tailed credible set by sample quantiles.