离散数学二 期末复习

2020-2021学年第一学期

期末试卷组成

- 八道大题，共100分
  - 计数高级计数：4道大题，共50分，其中15分英文题目。答题可以全部用中文
  - 图论(2019级)： 4道大题，共50分，其中15分英文题目。答题可以全部用中文
  - 代数系统(重修)： 4道大题，共50分，无英文题目

计数与高级计数

THE BASICS OF COUNTING

- THE PRODUCT RULE
- THE SUM RULE
- THE SUBTRACTION RULE
- THE DIVISION RULE

BASIC COUNTING PRINCIPLES: THE PRODUCT RULE

THE PRODUCT RULE: A PROCEDURE CAN BE BROKEN DOWN INTO A SEQUENCE OF TWO TASKS. THERE ARE N_1 WAYS TO DO THE FIRST TASK AND N_2 WAYS TO DO THE SECOND TASK. THEN THERE ARE $N_1 \cdot N_2$ WAYS TO DO THE PROCEDURE.

BASIC COUNTING PRINCIPLES: THE SUM RULE

THE SUM RULE: IF A TASK CAN BE DONE EITHER IN ONE OF N_1 WAYS OR IN ONE OF N_2 WAYS TO DO THE SECOND TASK, WHERE NONE OF THE SET OF N_1 WAYS IS THE SAME AS ANY OF THE N_2 WAYS, THEN THERE ARE $N_1 + N_2$ WAYS TO DO THE TASK.

BASIC COUNTING PRINCIPLES: SUBTRACTION RULE

SUBTRACTION RULE: IF A TASK CAN BE DONE EITHER IN ONE OF n_1 WAYS OR IN ONE OF n_2 WAYS, THEN THE TOTAL NUMBER OF WAYS TO DO THE TASK IS $n_1 + n_2$ MINUS THE NUMBER OF WAYS TO DO THE TASK THAT ARE COMMON TO THE TWO DIFFERENT WAYS.

- ALSO KNOWN AS, THE *PRINCIPLE OF INCLUSION-EXCLUSION*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

BASIC COUNTING PRINCIPLES: DIVISION RULE

DIVISION RULE: THERE ARE N/D WAYS TO DO A TASK IF IT CAN BE DONE USING A PROCEDURE THAT CAN BE CARRIED OUT IN N WAYS, AND FOR EVERY WAY W, EXACTLY D OF THE N WAYS CORRESPOND TO WAY W.

- RESTATED IN TERMS OF SETS: IF THE FINITE SET A IS THE UNION OF N PAIRWISE DISJOINT SUBSETS EACH WITH D ELEMENTS, THEN $N = |A|/D$.
- IN TERMS OF FUNCTIONS: IF F IS A FUNCTION FROM A TO B, WHERE BOTH ARE FINITE SETS, AND FOR EVERY VALUE $Y \in B$ THERE ARE EXACTLY D VALUES $X \in A$ SUCH THAT $F(X) = Y$, THEN $|B| = |A|/D$.

THE PIGEONHOLE PRINCIPLE

- THE PIGEONHOLE PRINCIPLE
- THE GENERALIZED PIGEONHOLE PRINCIPLE

THE PIGEONHOLE PRINCIPLE

- **PIGEONHOLE PRINCIPLE**: IF K IS A POSITIVE INTEGER AND $K + 1$ OBJECTS ARE PLACED INTO K BOXES, THEN AT LEAST ONE BOX CONTAINS TWO OR MORE OBJECTS.

  PROOF: WE USE A PROOF BY CONTRAPOSITION. SUPPOSE NONE OF THE K BOXES HAS MORE THAN ONE OBJECT. THEN THE TOTAL NUMBER OF OBJECTS WOULD BE AT MOST K. THIS CONTRADICTS THE STATEMENT THAT WE HAVE $K + 1$ OBJECTS.

THE GENERALIZED PIGEONHOLE PRINCIPLE

THE GENERALIZED PIGEONHOLE PRINCIPLE: IF N OBJECTS ARE PLACED INTO K BOXES, THEN THERE IS AT LEAST ONE BOX CONTAINING AT LEAST $[N/K]$ OBJECTS.

PROOF: WE USE A PROOF BY CONTRAPOSITION. SUPPOSE THAT NONE OF THE BOXES CONTAINS MORE THAN $[N/K] - 1$ OBJECTS. THEN THE TOTAL NUMBER OF OBJECTS IS AT MOST

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

WHERE THE INEQUALITY $[N/K] < [N/K] + 1$ HAS BEEN USED. THIS IS A CONTRADICTION BECAUSE THERE ARE A TOTAL OF N OBJECTS.

EXAMPLE: AMONG 100 PEOPLE THERE ARE AT LEAST $[100/12] = 9$ WHO WERE BORN IN THE SAME MONTH.

PERMUTATIONS AND COMBINATIONS

- PERMUTATIONS
- COMBINATIONS
- COMBINATORIAL PROOFS

A FORMULA FOR THE NUMBER OF PERMUTATIONS

THEOREM 1: IF N IS A POSITIVE INTEGER AND R IS AN INTEGER WITH $1 \leq R \leq N$, THEN THERE ARE

$$P(N, R) = N(N - 1)(N - 2) \cdots (N - R + 1)$$

R-PERMUTATIONS OF A SET WITH N DISTINCT ELEMENTS.

PROOF: USE THE PRODUCT RULE. THE FIRST ELEMENT CAN BE CHOSEN IN N WAYS. THE SECOND IN $N - 1$ WAYS, AND SO ON UNTIL THERE ARE $(N - (R - 1))$ WAYS TO CHOOSE THE LAST ELEMENT.

- NOTE THAT $P(N,0) = 1$, SINCE THERE IS ONLY ONE WAY TO ORDER ZERO ELEMENTS.

COROLLARY 1: IF N AND R ARE INTEGERS WITH $1 \leq R \leq N$, THEN

$$P(n, r) = \frac{n!}{(n-r)!}$$

COMBINATIONS

DEFINITION: AN *R-COMBINATION* OF ELEMENTS OF A SET IS AN UNORDERED SELECTION OF *R* ELEMENTS FROM THE SET. THUS, AN *R*-COMBINATION IS SIMPLY A SUBSET OF THE SET WITH *R* ELEMENTS.

- THE NUMBER OF *R*-COMBINATIONS OF A SET WITH N DISTINCT ELEMENTS IS DENOTED BY C(*N*, *R*). THE NOTATION $\binom{n}{r}$ IS ALSO USED AND IS CALLED A *BINOMIAL COEFFICIENT*. (*WE WILL SEE THE NOTATION AGAIN IN THE BINOMIAL THEOREM IN SECTION* 6.4.)

COMBINATIONS

THEOREM 2: THE NUMBER OF *R*-COMBINATIONS OF A SET WITH *N* ELEMENTS, WHERE $N \geq R \geq 0$, EQUALS

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

PROOF: BY THE PRODUCT RULE $P(N, R) = C(N,R) \cdot P(R,R)$. THEREFORE,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

COMBINATIONS

COROLLARY 2: LET n AND r BE NONNEGATIVE INTEGERS WITH $r \leq n$. THEN $C(n, r) = C(n, n - r)$.

PROOF: FROM THEOREM 2, IT FOLLOWS THAT

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

AND

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}\ .$$

HENCE, $C(n, r) = C(n, n - r)$. ◀

COMBINATORIAL PROOFS

- **DEFINITION 1**: A *COMBINATORIAL PROOF* OF AN IDENTITY恒等式的组合证明 IS A PROOF THAT USES ONE OF THE FOLLOWING METHODS.
  - A *DOUBLE COUNTING PROOF* USES COUNTING ARGUMENTS TO PROVE THAT BOTH SIDES OF AN IDENTITY COUNT THE SAME OBJECTS, BUT IN DIFFERENT WAYS.
  - A *BIJECTIVE PROOF* SHOWS THAT THERE IS A BIJECTION BETWEEN THE SETS OF OBJECTS COUNTED BY THE TWO SIDES OF THE IDENTITY.

BINOMIAL COEFFICIENTS AND IDENTITIES

- THE BINOMIAL THEOREM
- PASCAL'S IDENTITY AND TRIANGLE
- OTHER IDENTITIES INVOLVING BINOMIAL COEFFICIENTS

BINOMIAL THEOREM

BINOMIAL THEOREM: LET *X* AND *Y* BE VARIABLES, AND *N* A NONNEGATIVE INTEGER. THEN:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

A USEFUL IDENTITY

COROLLARY 1: WITH $N \geq 0$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}.$$

A USEFUL IDENTITY

COROLLARY 2 Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof: When we use the binomial theorem with $x = -1$ and $y = 1$, we see that

$$0 = 0^n = ((-1) + 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

This proves the corollary. ◁

Remark: Corollary 2 implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots .$$

A USEFUL IDENTITY

COROLLARY 3 Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

Proof: We recognize that the left-hand side of this formula is the expansion of $(1+2)^n$ provided by the binomial theorem. Therefore, by the binomial theorem, we see that

$$(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Hence

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

◁

PASCAL'S IDENTITY

PASCAL'S IDENTITY: IF *N* AND *K* ARE INTEGERS WITH $N \geq K \geq 0$, THEN

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

VANDERMONDE'S IDENTITY

THEOREM 3 **VANDERMONDE'S IDENTITY** Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Links

Remark: This identity was discovered by mathematician Alexandre-Théophile Vandermonde in the eighteenth century.

Proof: Suppose that there are m items in one set and n items in a second set. Then the total number of ways to pick r elements from the union of these sets is $\binom{m+n}{r}$.

Another way to pick r elements from the union is to pick k elements from the second set and then $r-k$ elements from the first set, where k is an integer with $0 \le k \le r$. Because there are $\binom{n}{k}$ ways to choose k elements from the second set and $\binom{m}{r-k}$ ways to choose $r-k$ elements from the first set, the product rule tells us that this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways. Hence, the total number of ways to pick r elements from the union also equals $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$.

We have found two expressions for the number of ways to pick r elements from the union of a set with m items and a set with n items. Equating them gives us Vandermonde's identity. ◁

VANDERMONDE'S IDENTITY

COROLLARY 4 If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Proof: We use Vandermonde's identity with $m = r = n$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

The last equality was obtained using the identity $\binom{n}{k} = \binom{n}{n-k}$. ◁

THEOREM 4 Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: We use a combinatorial proof. By Example 14 in Section 6.3, the left-hand side, $\binom{n+1}{r+1}$, counts the bit strings of length $n+1$ containing $r+1$ ones.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones. This final one must occur at position $r+1, r+2, \ldots,$ or $n+1$. Furthermore, if the last one is the kth bit there must be r ones among the first $k-1$ positions. Consequently, by Example 14 in Section 6.3, there are $\binom{k-1}{r}$ such bit strings. Summing over k with $r+1 \leq k \leq n+1$, we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

bit strings of length n containing exactly $r+1$ ones. (Note that the last step follows from the change of variables $j = k-1$.) Because the left-hand side and the right-hand side count the same objects, they are equal. This completes the proof. ◁

GENERALIZED PERMUTATIONS AND COMBINATIONS

- PERMUTATIONS WITH REPETITION
- COMBINATIONS WITH REPETITION

SUMMARIZING THE FORMULAS FOR COUNTING PERMUTATIONS AND COMBINATIONS WITH AND WITHOUT REPETITION

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

APPLICATIONS OF RECURRENCE(递推) RELATIONS

- 构造递推关系及初始条件

SOLVING LINEAR RECURRENCE RELATIONS

- LINEAR HOMOGENEOUS RECURRENCE RELATIONS
- SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS.
- SOLVING LINEAR NONHOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS.

1. 首先考虑2阶线性齐次递推关系，存在两个不等特征根的情况

【 Theorem 1 】 Let c_1, c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1, r_2 . Then the Sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ *for* $n = 0, 1, 2, \cdots$ where α_1, α_2 are constants.

当存在二重特征根时定理**1**不再适用，推出定理**2**可处理这种情况

【 Theorem 2】 Let c_1, c_2 be real numbers with $c_2 \neq 0$.
Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0.
A sequence $\{a_n\}$ is a solution of the recurrence relation
$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if
$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ *for* $n = 0,1,2,\cdots$, where α_1, α_2 are constants.

定理2：设 c_1, c_2 **是实数，**c_2**≠0.**假设 $r^2 - c_1 r - c_2 = 0$只有一个根r_0

序列 $\{a_n\}$ 是递推关系 $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ 的解，当且仅当

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n, n = 0,1,2,\cdots,$$

其中 α_1, α_2 是常数。

SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS OF ARBITRARY DEGREE

THEOREM 3: LET C_1, C_2 ,…, C_K BE REAL NUMBERS. SUPPOSE THAT THE CHARACTERISTIC EQUATION

$$R^K - C_1R^{K-1} - \cdots - C_K = 0$$

HAS K DISTINCT ROOTS R_1, R_2, …, R_K. THEN A SEQUENCE $\{A_N\}$ IS A SOLUTION OF THE RECURRENCE RELATION

$$A_N = C_1A_{N-1} + C_2A_{N-2} + \ldots.. + C_K\, A_{N-K}$$

IF AND ONLY IF

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

FOR $N = 0, 1, 2, \ldots,$ WHERE A_1, A_2,…, A_K ARE CONSTANTS.

THE GENERAL CASE WITH REPEATED ROOTS ALLOWED

THEOREM 4: LET C_1, C_2 ,…, C_K BE REAL NUMBERS. SUPPOSE THAT THE CHARACTERISTIC EQUATION

$$R^K - C_1 R^{K-1} - \cdots - C_K = 0$$

HAS T DISTINCT ROOTS R_1, R_2, …, R_T WITH MULTIPLICITIES M_1, M_2, …, M_T, RESPECTIVELY SO THAT $M_I \geq 1$ FOR $I = 0, 1, 2, \ldots, T$ AND $M_1 + M_2 + \ldots + M_T = K$. THEN A SEQUENCE $\{A_N\}$ IS A SOLUTION OF THE RECURRENCE RELATION

$$A_N = C_1 A_{N-1} + C_2 A_{N-2} + \ldots + C_K A_{N-K}$$

IF AND ONLY IF

$$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+(\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

FOR $N = 0, 1, 2, \ldots$, WHERE $A_{I,J}$ ARE CONSTANTS FOR $1 \leq I \leq T$ AND $0 \leq J \leq M_{I-1}$.

SOLVING LINEAR NONHOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

THEOREM 5: IF $\{a_n^{(p)}\}$ IS A PARTICULAR SOLUTION OF THE NONHOMOGENEOUS LINEAR RECURRENCE RELATION WITH CONSTANT COEFFICIENTS

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k\, a_{n-k} + F(n)\ ,$$

THEN EVERY SOLUTION IS OF THE FORM $\{a_n^{(p)} + a_n^{(h)}\}$, WHERE $\{a_n^{(h)}\}$ IS A SOLUTION OF THE ASSOCIATED HOMOGENEOUS RECURRENCE RELATION

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k\, a_{n-k}\ .$$

【 Theorem 6】 Assume a linear nonhomogeneous recurrence equation with constant coefficients with the nonlinear part *F(n)* of the form

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$$

If *s* is not a root of the characteristic equation of the associated homogeneous recurrence equation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

If *s* is a root of multiplicity *m* , a particular solutions is of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

GENERATING FUNCTIONS

- GENERATING FUNCTIONS
- COUNTING PROBLEMS AND GENERATING FUNCTIONS
- USEFUL GENERATING FUNCTIONS
- SOLVING RECURRENCE RELATIONS USING GENERATING FUNCTIONS (*NOT YET COVERED IN THE SLIDES*)

图论

- 图的相关基本概念(图、顶点、边、顶点的度、路径、连通、连通分支、权、等等等等)
- 图的表示(邻接矩阵，关联矩阵)
- 特殊的图(完全图、二分图、等等基本概念)

【 Theorem 1】 The Handshaking Theorem握手理论

Let $G = (V, E)$ be an undirected graph G with e edges.

Then设G=(V,E)是e条边的无向图，则

$$\sum_{v \in V} \deg(v) = 2e$$

【 Theorem 2】 An undirected graph has an even number of vertices of odd degree. 无向图有偶数个奇数度顶点

【 Theorem 3】

在带有向边的图里，所有顶点的入度之和等于出度之和。这两个和都等于图的边数。

Necessary and sufficient condition for Euler circuit and paths

欧拉回路和欧拉通路的充要条件

【Theorem 1】连通多重图具有欧拉回路当且仅当它的每个顶点都有偶数度

Proof:

(1) Necessary condition必要条件

G has an Euler circuit $\Rightarrow$ Every vertex in V has even degree

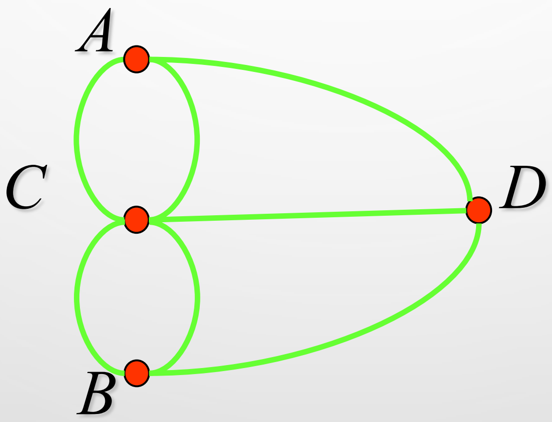
Consider the Euler circuit.

- the vertex a which the Euler circuit begins with
- the other vertex

【 Theorem 2】 连通多重图具有欧拉通路而无欧拉回路，当且仅当它恰有两个奇数度顶点

〖Example 1〗 Konigsberg Seven Bridge Problem

哥尼斯堡七桥问题



Solution:

The graph has four vertices of odd degree. Therefore, it does not have an Euler circuit.欧拉回路

The sufficient condition for the existence of Hamilton path and Hamilton circuit哈密顿通路和哈密顿回路存在的充分条件

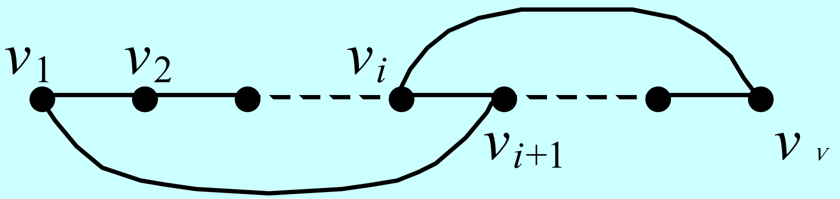
【 Theorem 3】 DIRAC‘THEOREM 狄拉克定理

如果*G*是带n个顶点的连通简单图，其中 $n>=3$，则G有哈密顿回路的充分条件是每个顶点的度都至少为 $n/2$

证明　假设*G*不是Hamilton图，设G为极大非Hamilton图；

G＋*uv*是Hamilton图;

$d(u)+d(v)=|S|+|T|=|S\cup T|<u$ 。



The sufficient condition for the existence of Hamilton path and Hamilton circuit哈密顿通路和哈密顿回路存在的充分条件

引理 设G是简单图，顶点u和v一对不相邻的顶点，且满足deg(u)+deg(v) >= n 则G是哈密顿图当且仅当G+uv是哈密顿图

【 Theorem 4】ORE‘THEOREM 奥尔定理

If G is a simple graph with n vertices with n>=3 such that deg(u)+deg(v) >= n for every pair of nonadjacent vertices u and v in G , then G has a Hamilton circuit.

如果G是带n个顶点的连通简单图，其中n>=3 ，并且对于G中每一对不相邻的顶点u和v来说，都有deg(u)+deg(v) >= n ，则G有哈密顿回路。

EULER'S FORMULA: IF A CONNECTED PLANE GRAPH *G* HAS EXACTLY *N* VERTICES, *E* EDGES, AND *F* FACES, THEN N-E+F=2.

Corollary *If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$.*

Corollary *If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3, then $e \leq 2v - 4$.*

Proof The proof is similar to that of last corollary, except that in this case the fact that there are no circuits of length 3 implies that the degree of a region must be at least 4. Thus $2e \geq 4r$. But $r = e - v + 2$, so we have $e - v + 2 \leq e/2$, which implies that $e \leq 2v - 4$. ◀

Theorem 5 *A graph is non-planar if and only if it contains a subgraph homeomorphic to* $K_{3,3}$ *or* K_5.

It is clear that a graph containing a subgraph homeomorphic to $K_{3,3}$ or K_5 is non-planar. However, the proof of the converse is complicated and will not be given here.

树

- 定义1　连通无回路的图称为树，树中度为1的点称为树叶，度大于1的点称为分枝点或内点，每个连通分支均为树的图称为森林。

- **THEOREM**: LET G BE A GRAPH WITH N NODES AND E EDGES
  - 1.G IS A TREE (CONNECTED, ACYCLIC)
  - 2. G IS ACYCLIC AND E = N - 1
  - 3.G IS CONNECTED AND E = N - 1
  - 4.G IS ACYCLIC AND IF ANY TWO NON-ADJACENT POINTS ARE JOINED BY AN EDGE, THE RESULTING GRAPH HAS EXACTLY ONE CYCLE
  - 5. G IS CONNECTED, BUT IF ANY EDGE IS DELETED, IT WILL BE NON-CONNECTED
  - 6. EVERY TWO NODES OF G ARE JOINED BY A UNIQUE PATH
- **THEOREM**: THERE EXISTS AT LEAST TWO NODES OF DEGREE ONE FOR EVERY TREE.