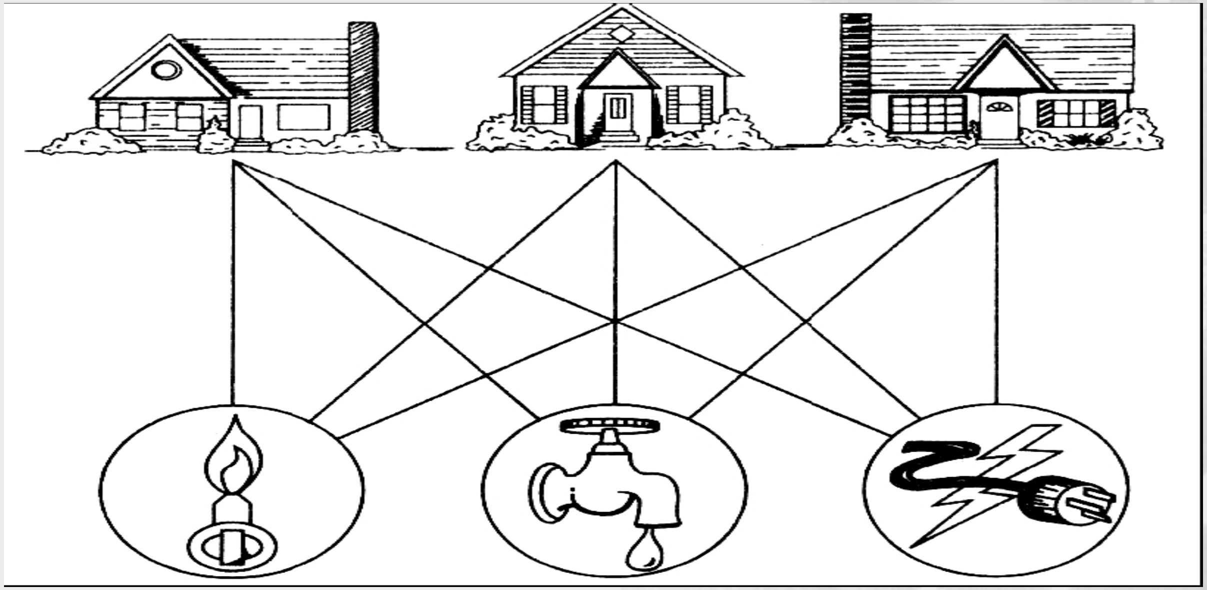
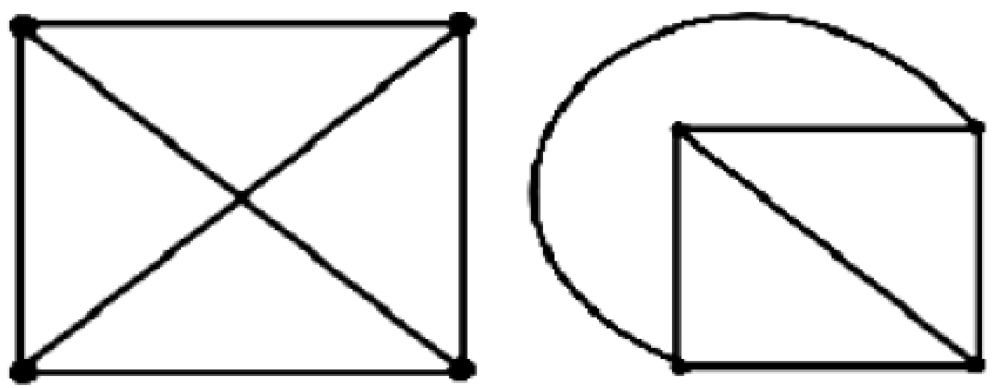
- **Planar Graphs and Graph Coloring**
  - Consider the problem of joining three houses to each of three separate utilities as shown below.



- A graph is *planar* if it has a drawing without crossings.
  - Such a drawing is a *planar embedding* of *G*.

A graph may be planar even if it is usually drawn with crossings, since it may be possible to draw it in a different way without crossings.

Example K_4 is a planar graph.

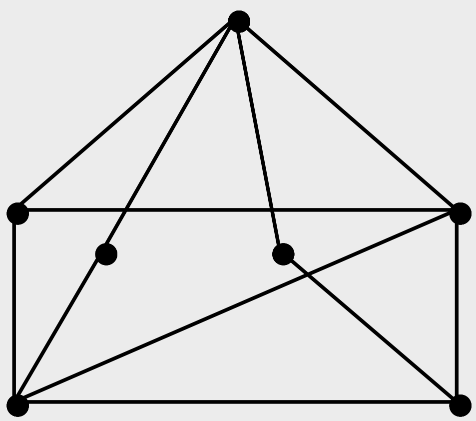


平面图

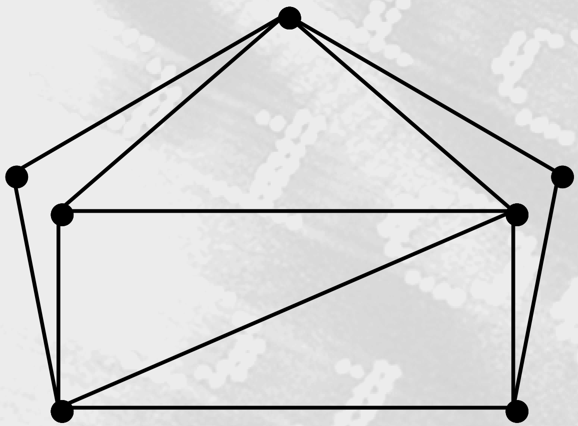
- 定义1　如果一个图能画在平面上，使得它的边仅在端点相交，则称这个图为平面图，或说它是可平面嵌入的，平面图*G*的这样一种画法，称为*G*的一个平面嵌入。

- 平面图*G*的平面嵌入称为平图。

◈ $K_{3,}$， K_4， K_5

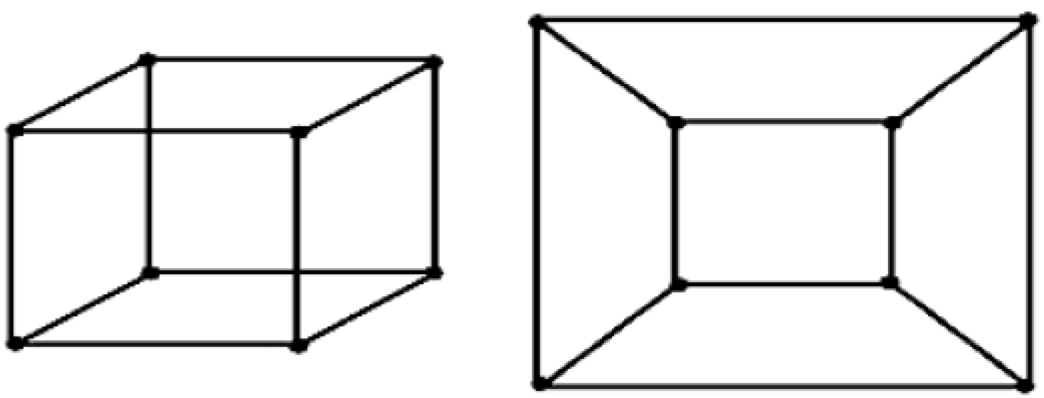


(a)

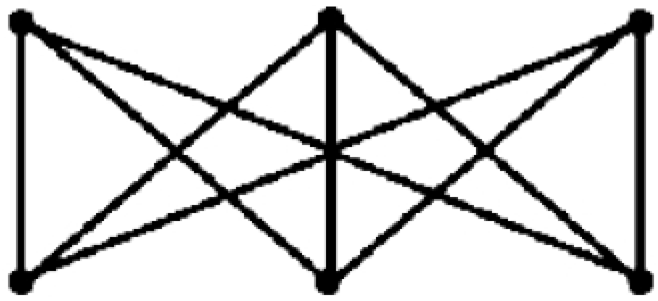


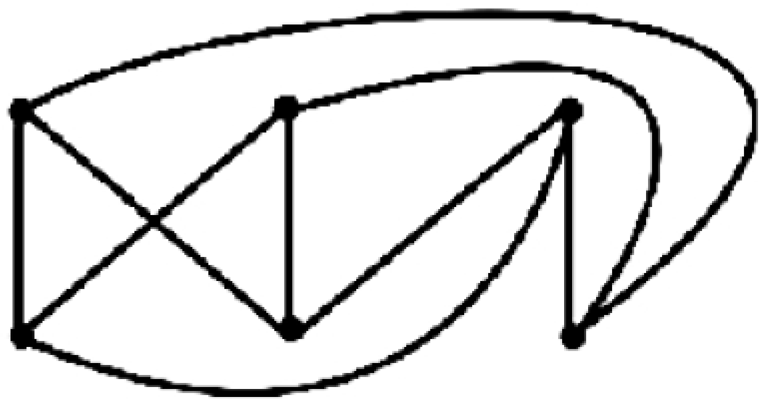
(b)

Example Q_3 is a planar graph.



Example $K_{3,3}$ is not planar.

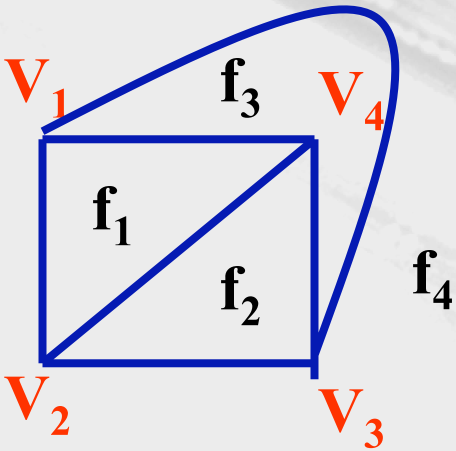


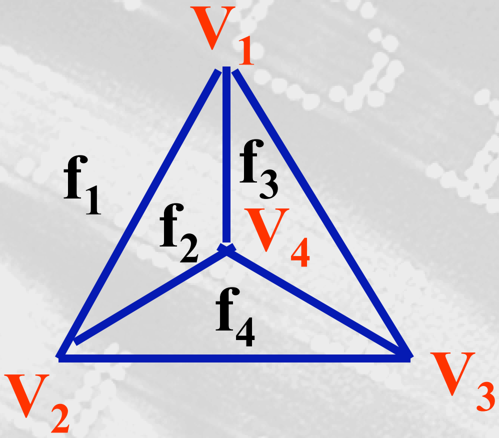


◈ 平面图的域(面)　内部域、无限域

一个域的边界

例





- 定义 2　一条连续的、自身不相交的封闭曲线称为Jordon曲线。
- J的外部，ext$J$，外点，ext$J$与J之并称为ext$J$的闭包，记为Ext$J$；另一部分(不含曲线J)称为J的内部，记为int$J$，int$J$的点称为J的内点，int$J$与J之并称为int$J$的闭包，记为Int$J$。
- 引理　设J是一条Jordon曲线，任何连接J的内点与外点的曲线必与J相交。

- 定义 3　设G是一个平图，则G把平面划分成若干个连通区域，每个连通区域的闭包称为G的一个面，其中恰有一个无界的面，称为外部面。

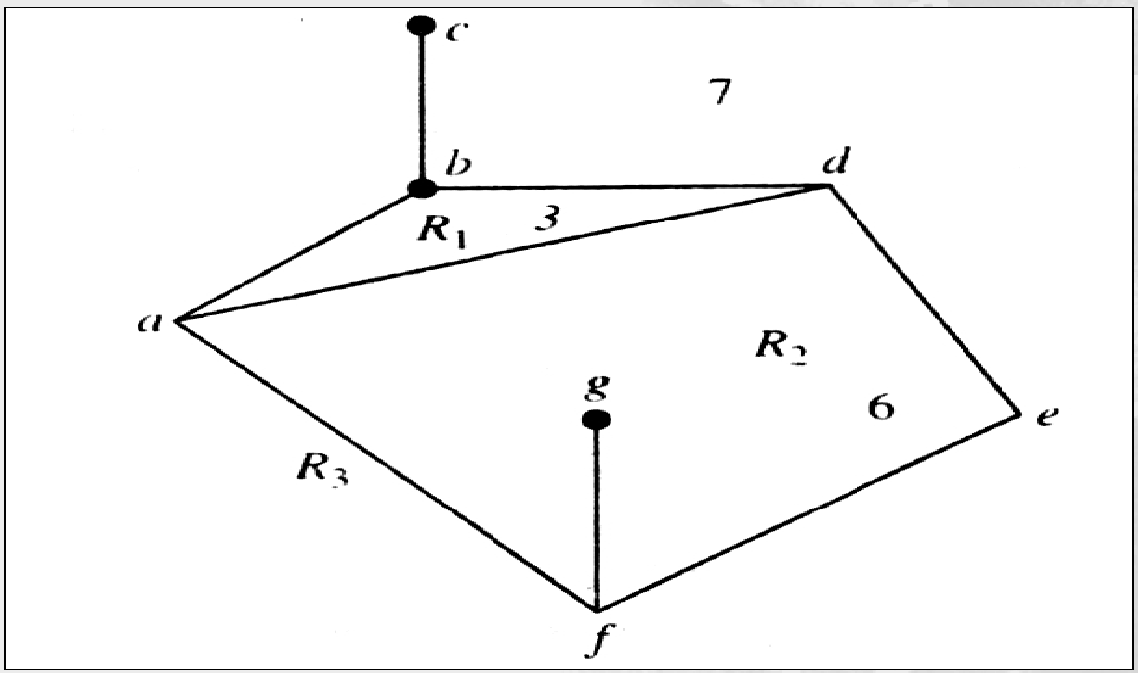
Euler's Formula: If a connected plane graph G has exactly n vertices, e edges, and f faces, then n-e+f=2.

Example Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Solution We have $2e = 3v = 3 \cdot 20 = 60$, or $e = 30$. From Euler's formula, the number of regions is $r = e - v + 2 = 30 - 20 + 2 = 12$. ◄

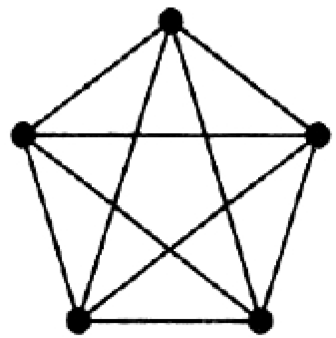
- 推论1　给定平面连通图G，则G的所有平面嵌入有相同的面数。

Corollary *If G is a connected planar simple graph with e edges and v vertices where v ≥ 3, then e ≤ 3v – 6.*



Example Show that K_5 is non-planar.

Solution The graph K_5 has 5 vertices and 10 edges. However, the inequality $e \leq 3v - 6$ is not satisfied for this graph. Therefore, K_5 is not planar. ◄



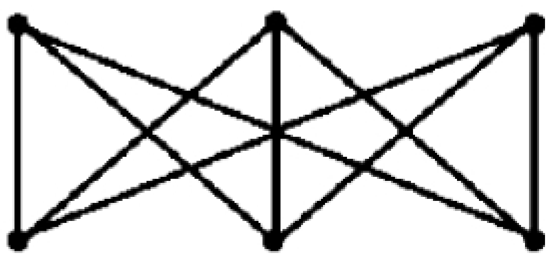
K_5

Corollary *If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3, then $e \leq 2v - 4$.*

Proof The proof is similar to that of last corollary, except that in this case the fact that there are no circuits of length 3 implies that the degree of a region must be at least 4. Thus $2e \geq 4r$. But $r = e - v + 2$, so we have $e - v + 2 \leq e/2$, which implies that $e \leq 2v - 4$. ◄

Example Show that $K_{3,3}$ is non-planar.

Solution Since $K_{3,3}$ has no circuits of length 3 (this is easy to see since it is bipartite). $K_{3,3}$ has 6 vertices and 9 edges. Since $e = 9$ and $2v - 4 = 8$, the corollary shows that $K_{3,3}$ is non-planar. ◀



- 定理 2　在平面简单图*G*中，至少存在一个顶点*v*0，使*d*(*v*0)≤5。
- 证明　假设一个平面简单图的所有顶点度数均大于 5，则，

$$6\upsilon \le \sum_{v\in V} d(v) = 2\varepsilon \le 6\upsilon - 12$$

- 矛盾，因此，平面简单图中至少有一个顶点*v*0，使*d*(*v*0)≤5。

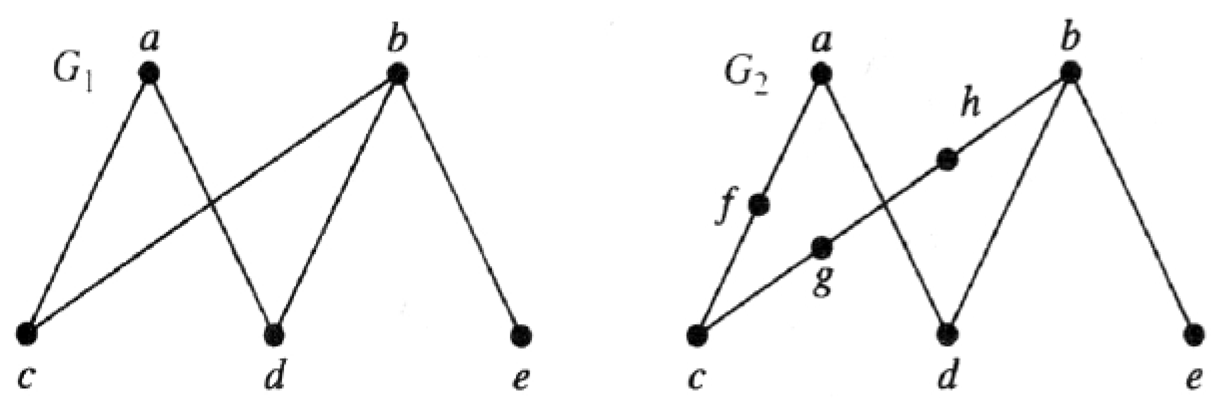
KURATOWSKI'S THEOREM

We have seen that $K_{3,3}$ and K_5 are not planar. Clearly, a graph is not planar if it contains either of these two graphs as a subgraph. Furthermore, all non-planar graphs must contain a subgraph that can be obtained from $K_{3,3}$ or K_5 using certain permitted operations.

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision. 初等细分

同胚

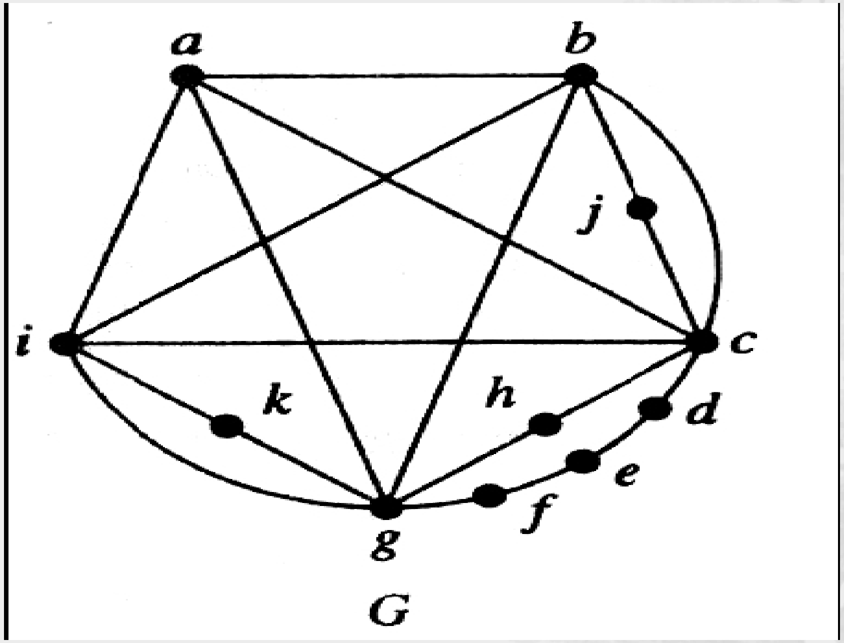
The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.



Theorem 5 *A graph is non-planar if and only if it contains a subgraph homeomorphic to* $K_{3,3}$ *or* K_5.

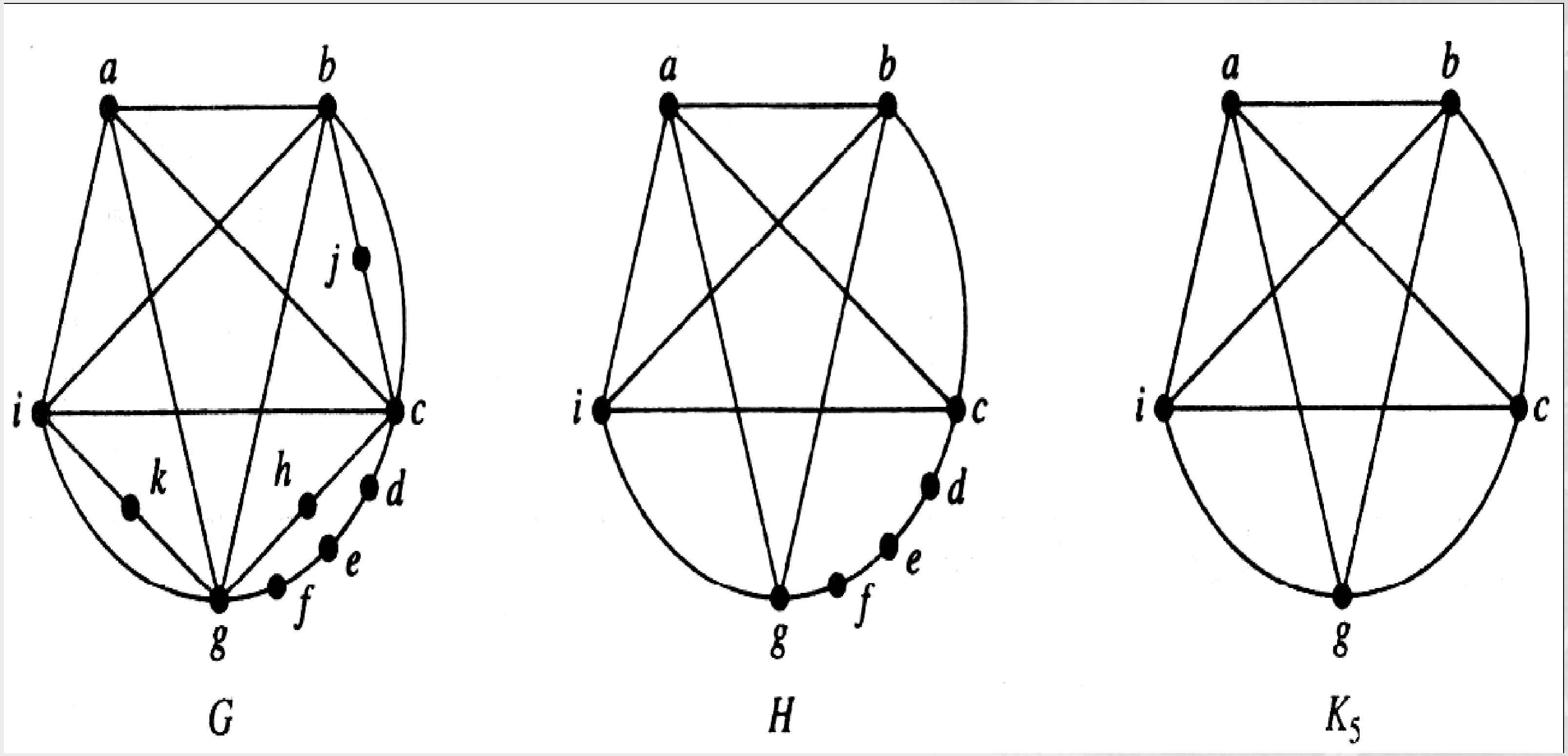
It is clear that a graph containing a subgraph homeomorphic to $K_{3,3}$ or K_5 is non-planar. However, the proof of the converse is complicated and will not be given here.

◈ Example Determine whether the graph G shown below is planar?

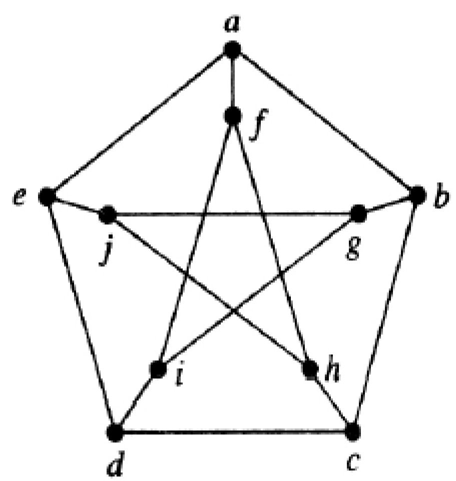


Solution G has a subgraph H homeomorphic to K_5. H is obtained by deleting h, j, and k and all edges incident with these vertices. H is homeomorphic to K_5 since it can be obtained from K_5 (with vertices a, b, c, g and i) by a sequence of elementary subdivisions, adding the vertices d, e, and f.

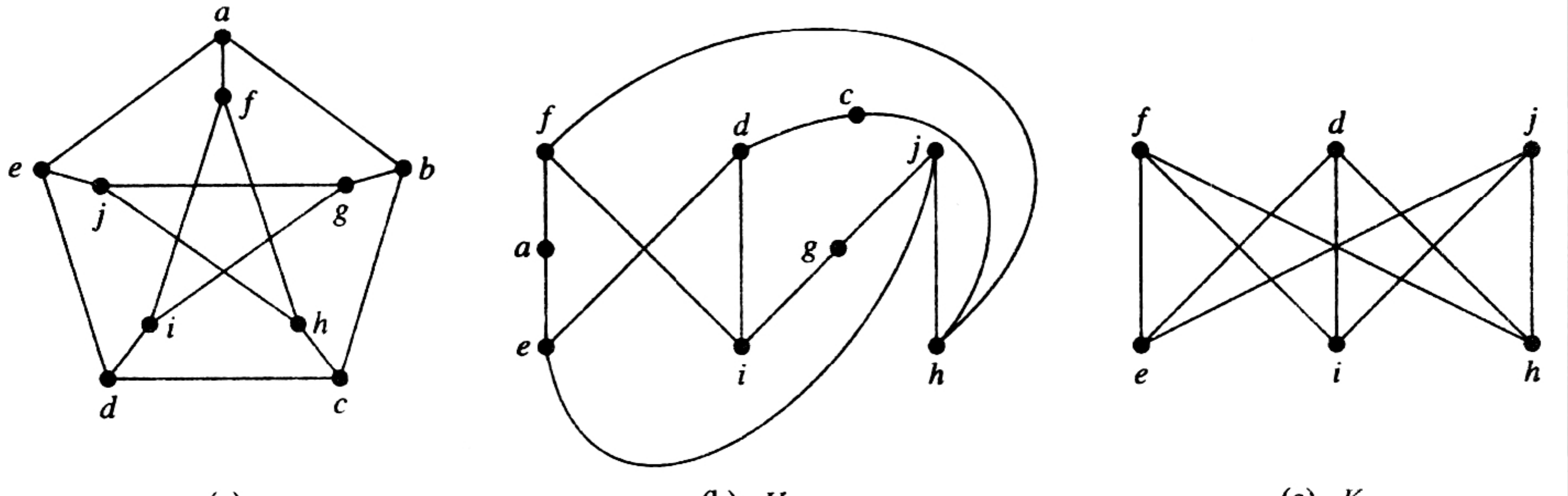
Hence, G is nonplanar.


a
b
j
i
c
k
h
d
e
f
g
G
a
b
i
c
d
e
f
g
H
a
b
i
c
g
K_5

Example Is the Petersen graph, shown below, planar?



- Solution The subgraph H of the Petersen graph obtained by deleting b and the three edges that have b as an endpoint, is homeomorphic to $K_{3,3}$, with vertex sets {f, d, j} and {e, i, h}, since it can be obtained by a sequence of elementary subdivisions, deleting {d, h} and adding {c, h} and {c, d}, deleting {e, f } and adding {a, e} and {a, f }, and deleting {i, j} and adding {g, i} and {g, j}.Hence, the Petersen graph is not planar.



(a) (b) H (c) $K_{3,3}$