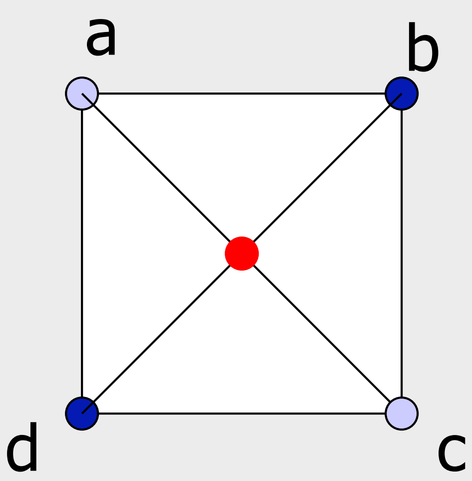
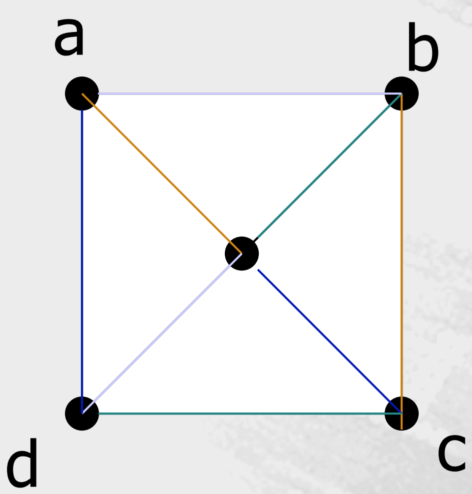
Graph Coloring

- **Coloring-** a coloring of a graph G assigns colors to the vertices of G so that adjacent vertices are given different colors.





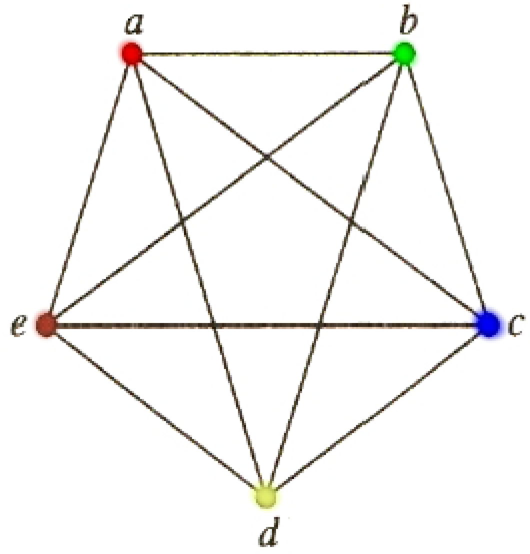
In the case of edge coloring, no edges that share a common vertex can be the same color.

Chromatic Number色数

- χ - least number of colors needed to color a graph
  - Chromatic number of a complete graph:

$$\chi(K_n) = n$$

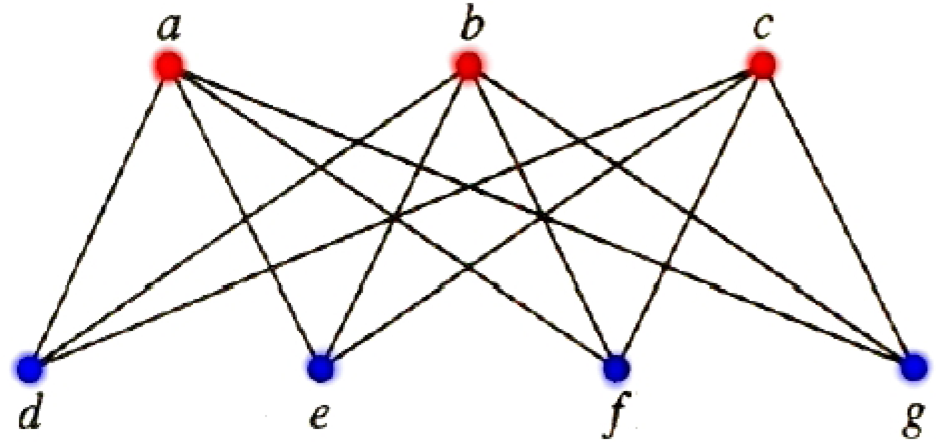
A coloring of K_5 using five colors is shown as follows



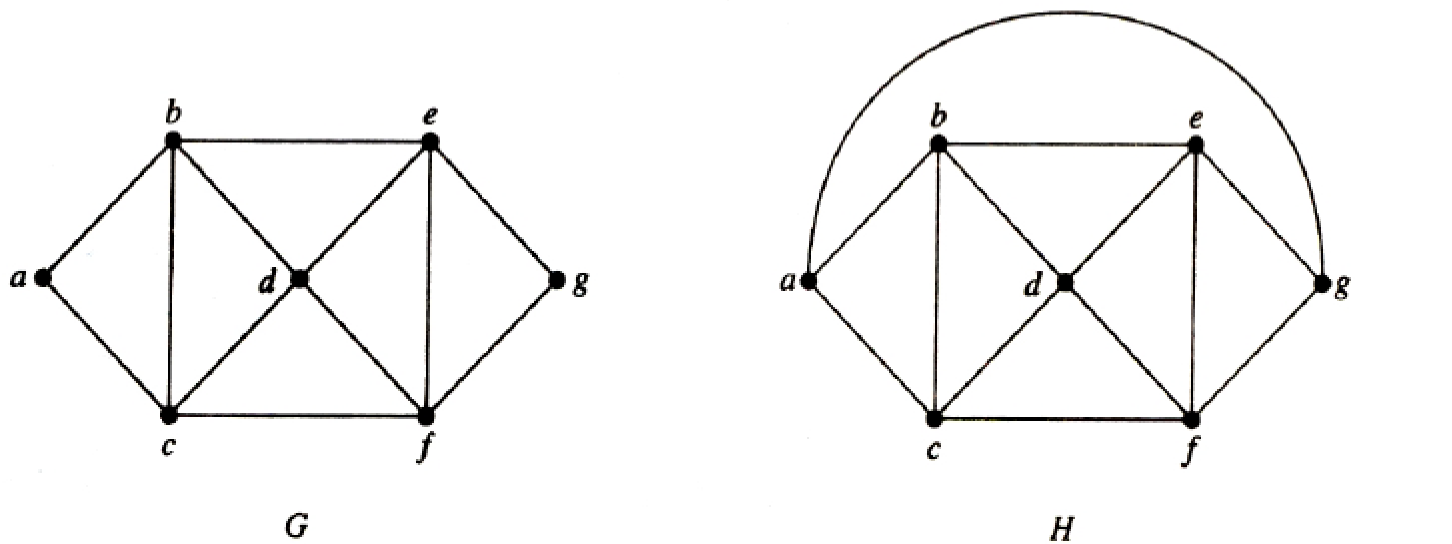
Example What is the chromatic number of the complete bipartite graph $K_{m,n}$, where m and n are positive integers?

Solution The number of colors needed may seem to depend on m and n. However, only two colors are needed. Color the set of m vertices with one color and the set of n vertices with a second color. Since edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same color. ◄

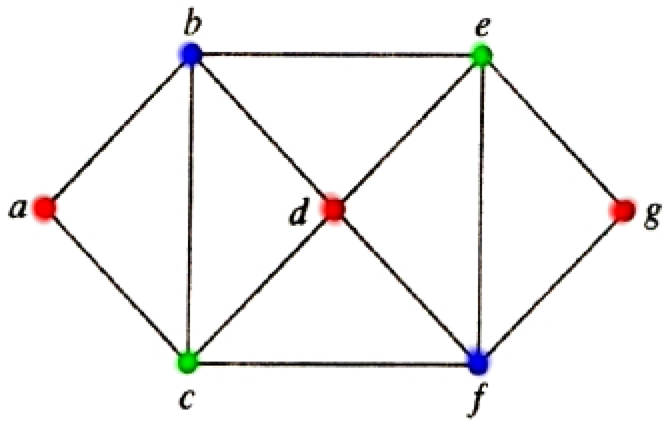
A coloring of $K_{3,4}$ with two colors is displayed below.

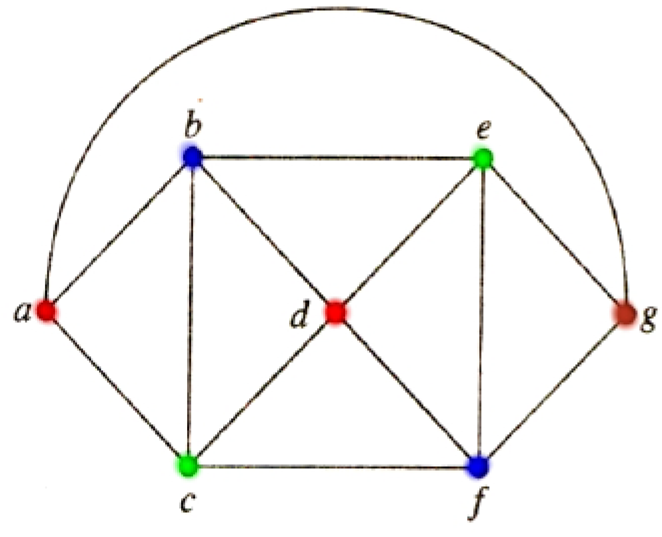


Example What are the chromatic numbers of the graphs G and H shown below.



Solution





- The best algorithm known for finding the chromatic number of a graph have exponential worst-case complexity (in the number of vertices of the graph).
- Even the problem of finding an approximation to the chromatic number of a graph is difficult!

Properties of $\chi(G)$

- $\chi(G) = 1$ if and only if G is totally disconnected
- $\chi(G) = 3$ if G is an odd cycle
- $\chi(G) \leq \Delta(G)+1$ (maximum degree)

顶点着色

- 定义　设G是一个图，对G的每个顶点着色，使得没有两个相邻的顶点着上相同的颜色，这种着色称为图的正常着色，若图
  G的顶点可用k种颜色正常着色，称G为k可着色的，使G是k可着色的数k的最小值称为G的色数，记为$\chi(G)$，如果$\chi(G)=k$，则称G是k色的。

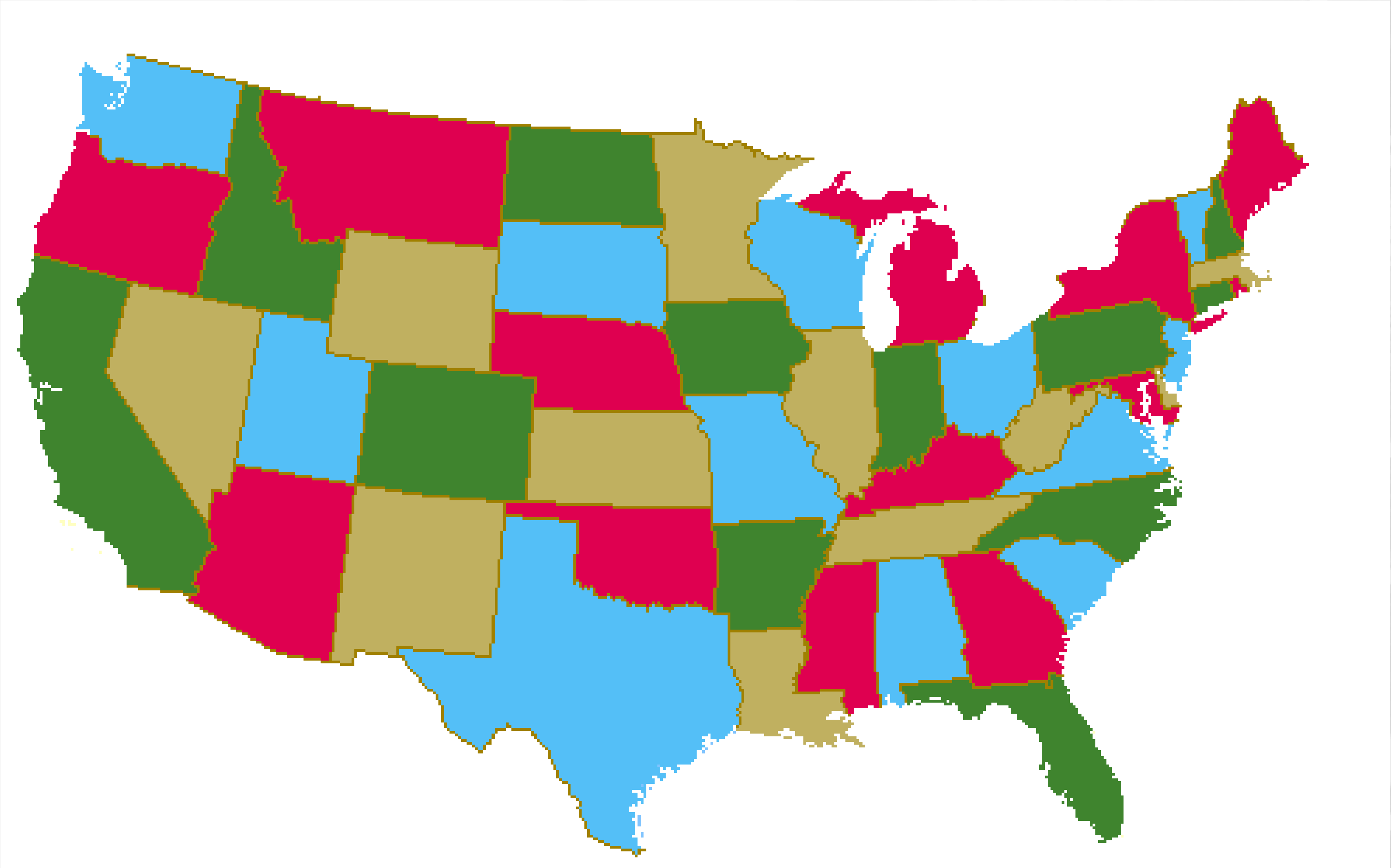
- 假设G是简单连通图。
- 定理1
- (1)对于完全图Kn，有$\chi(Kn)=n$，$\chi(\sim Kn)=1$。
- (2)对于n个顶点构成的圈Cn，当n是偶数时，$\chi(Cn)=2$，当n是奇数时，$\chi(Cn)=3$。
- (3)对于非平凡树T，有$\chi(T)=2$。
- (4)G是二分图，当且仅当$\chi(G)=2$。

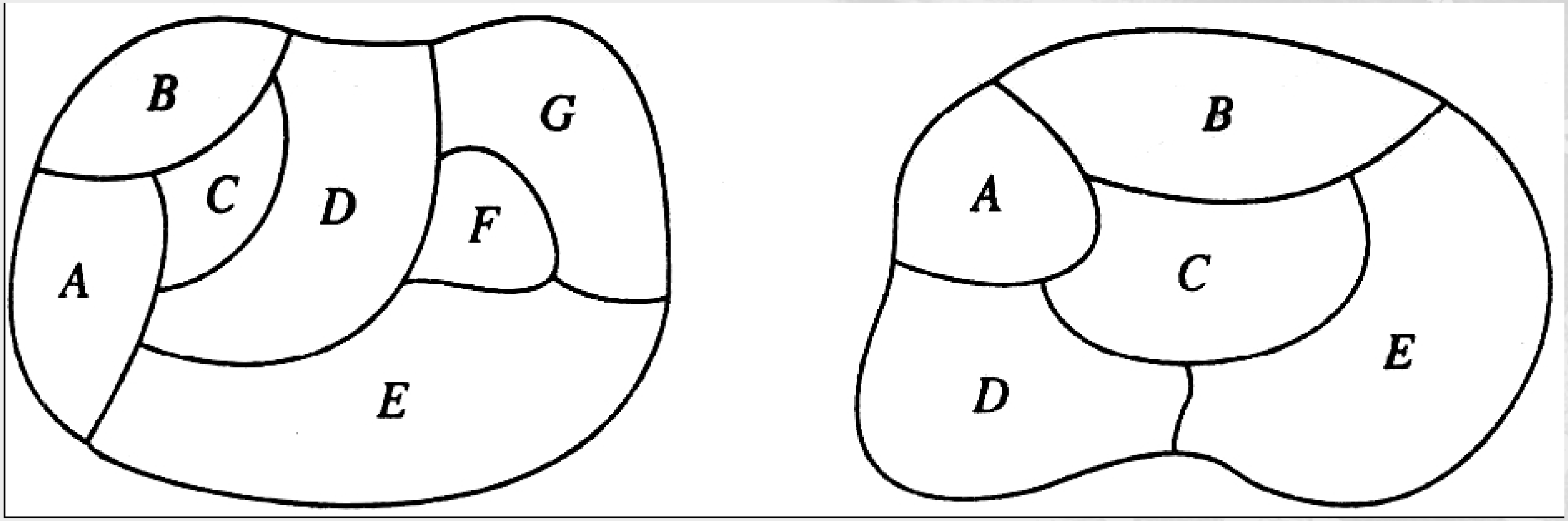
◈ 定理 2　对于任意连通简单图G，有

$$\chi(G)\leq 1+\Delta(G)。$$

证明　往证 G是$1+\Delta(G)$可着色的。对G的顶点数施行归纳法，……

Face Coloring





On the left, four colors suffice, but three colors are not enough. On the right, three colors are sufficient but two are not.

- 定义 2　一个没有割边的连通平图，称为地图。

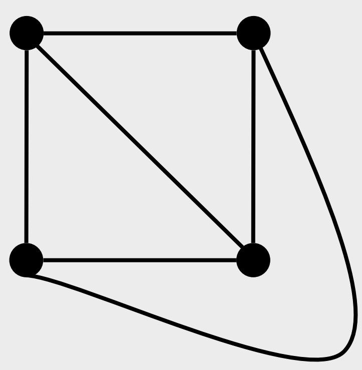
- 定义 3　设G是一个地图，对G的每个面着色，使得没有两个相邻的面着上相同的颜色，这种着色称为地图的正常面着色，地图G可用k种颜色正常面着色，称G是k面可着色的，使得G是k面可着色的数k的最小值称为G的面色数，记为$\chi^*(G)$，若$\chi^*(G)=k$，则称G是k面色的。

- 定理1*　(五色定理)任何无自环的平面图G是5可着色的。

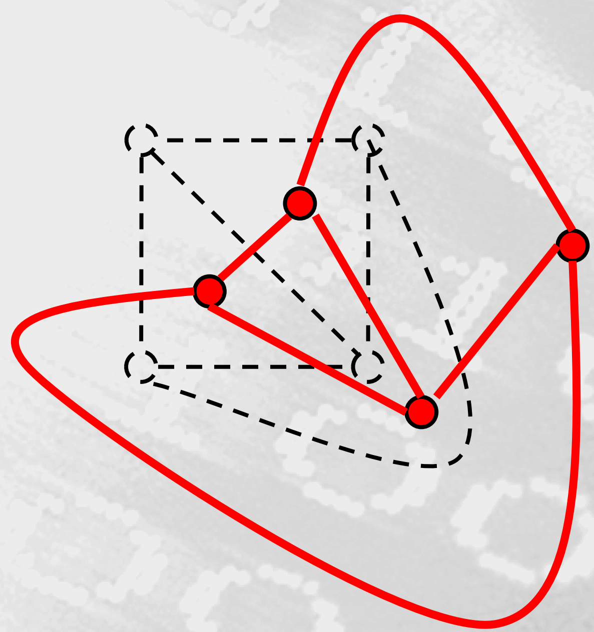
Dual Graph G* of a Plane Graph:

(1) A plane graph whose vertices corresponding to the faces of G.

(2) The edges of G* corresponds to the edges of G as follows: if e is an edge of G with face X on one side and face Y on the other side, then the endpoints of the dual edge e* in E(G*) are the vertices x and y of G* that represents the faces X and Y of G.



$\rightarrow$

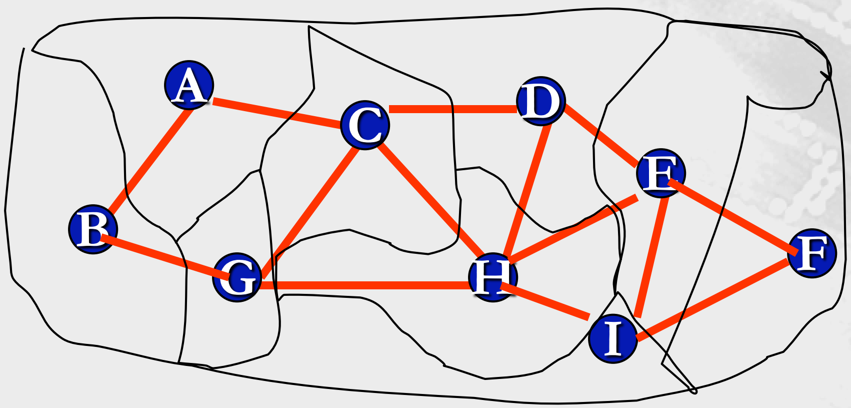


K_4

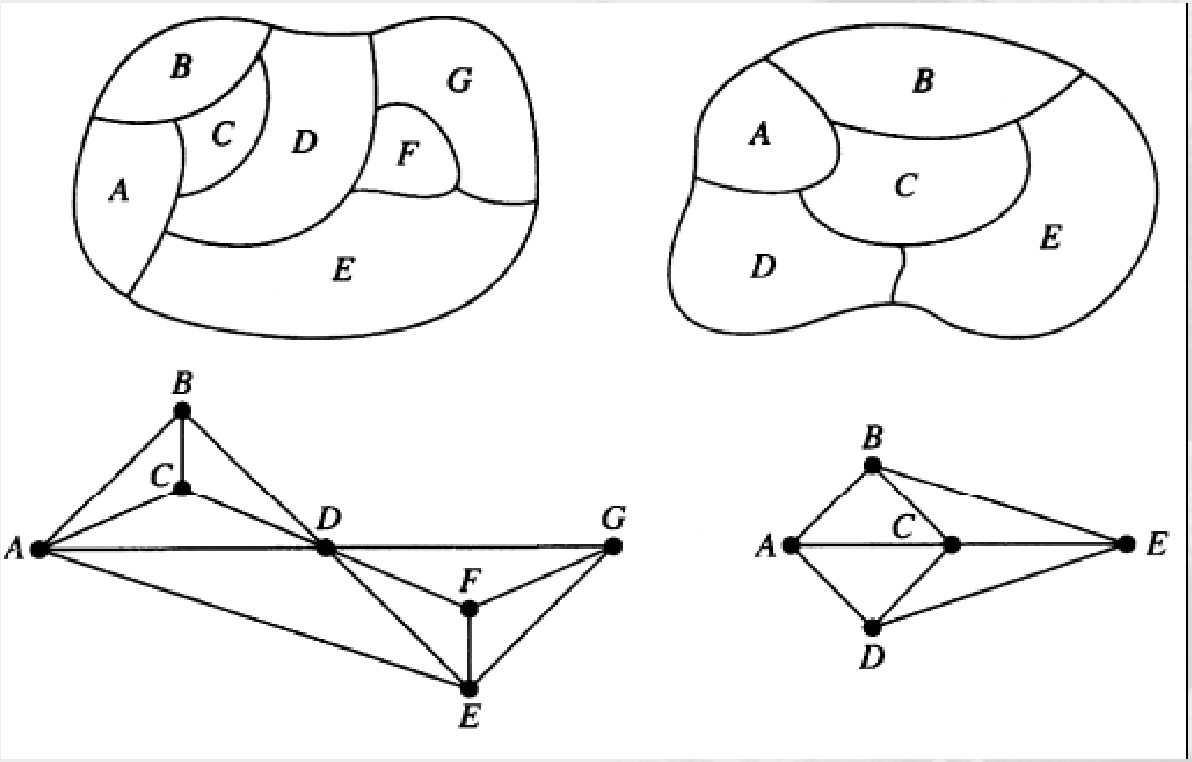
Dual Map

Region → vertex
Common border → edge

G



Dual graphs



Theorem :Every planar graph is 5-colorable.

Proof. 1. We use induction on $n(G)$, the number of nodes in G.

2. Basis Step: All graphs with $n(G) \leq 5$ are 5-colorable.

3. Induction Step: $n(G) > 5$.

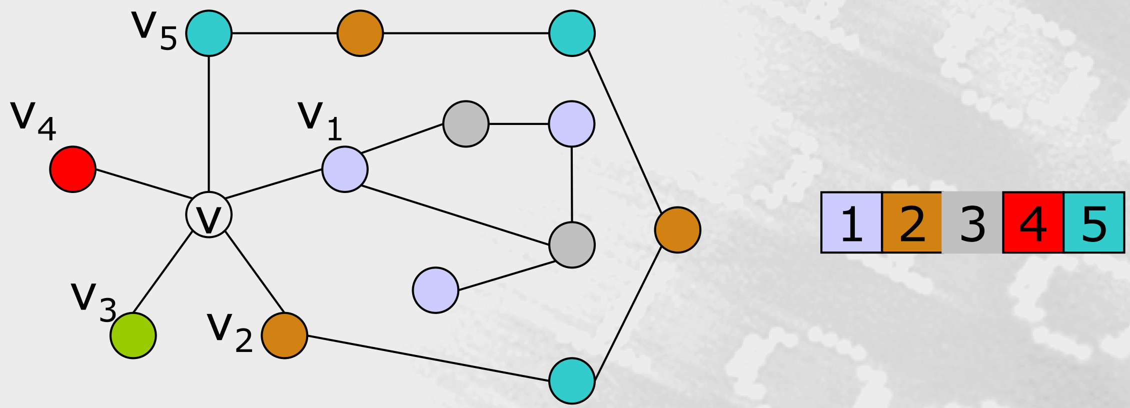
4. G has a vertex, v, of degree at most 5 because $e(G) \leq 3n(G)-6$

5. G-v is 5-colorable by Induction Hypothesis.

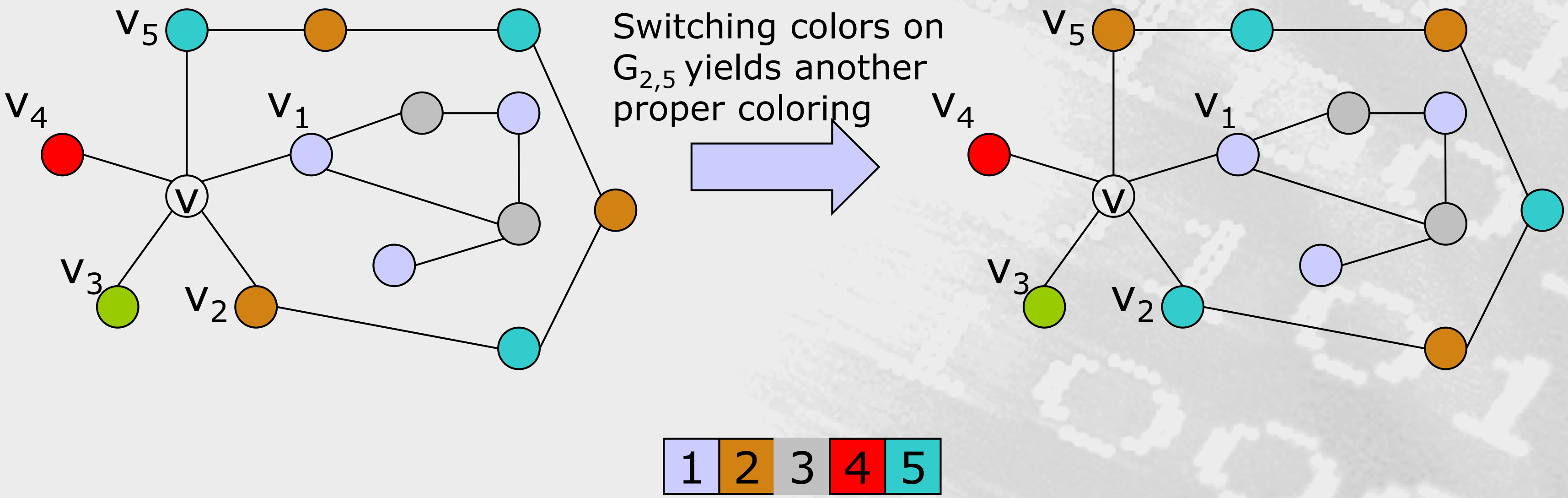
6. Let f be a proper 5-coloring of G-v.

7. If G is not 5-colorable, f assigns each color to some neighbor of v, and hence d(v)=5.

8. Let v_1, v_2, v_3, v_4, and v_5 be the neighbors of v in clockwise order around v, and name the colors so that $f(v_i)=i$.



9. 10. Switching the two colors on any component of $G_{i,j}$ yields another proper coloring of G-v. Let $G_{i,j}$ denote the subgraph of G-v induced by the vertices of colors i and j.



Theorem Appel-Haken-Koch[1977]

- Every planar graph is 4-colorable.
  - Using 1200hours of computer time in 1976, they found an unavoidable set of 1936 reducible configurations, all with ring size at most 14