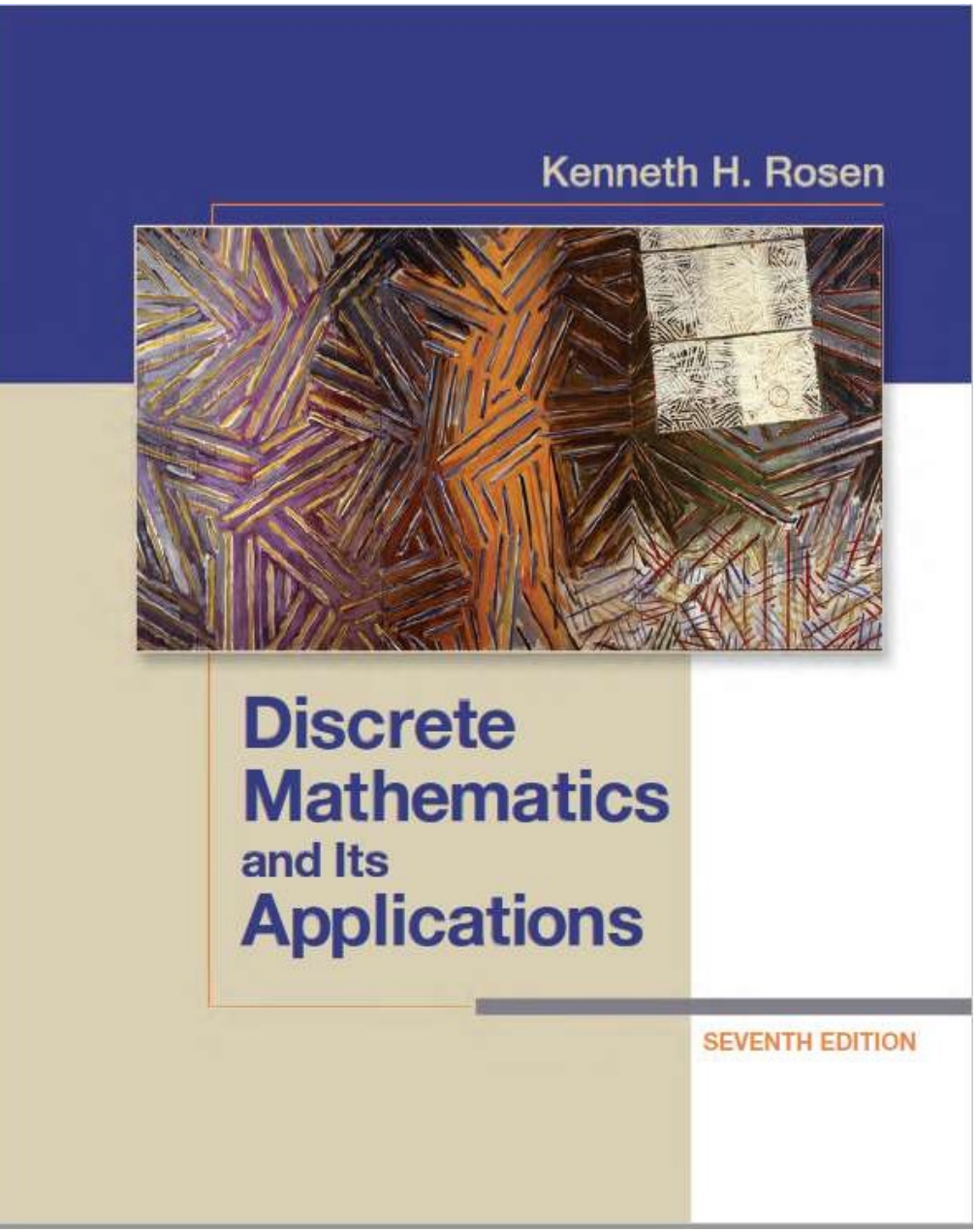
离散数学

Discrete Mathematics

2020年新冠疫情网络教学版

Introduction
概述

- 离散数学(**Discrete mathematics**) 研究离散量的结构和相互间的关系为主要目标，充分描述了计算机科学离散性的特点。
- 内容包含：数理逻辑、集合论/关系、代数结构、图论、组合数学、数论、自动机等。
- 离散数学是计算机学科的数学基础



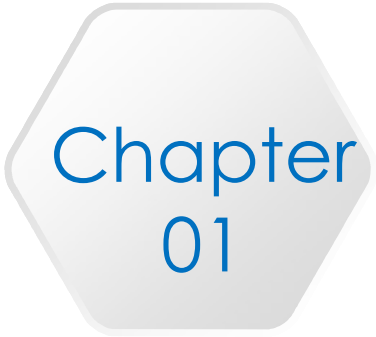
教材

- 1. Kenneth H. Rosen著， Discrete Mathematics and Its Applications （Seventh Edition）
- 软件学院内部编写教材

Syllabus

教学大纲

Contents



逻辑

The Foundations: Logic and Proofs

CONTENT

Propositional Logic
命题逻辑

- Logic -- The study of reasoning
- The applications of logic 逻辑学的应用
  - In mathematics: to prove theorems（在数学上：定理的证明）
  - In computer science（在计算机科学上）

    to prove the correctness of programs ……（程序正确性的证明）

Proposition（命题）

- 【Definition】: a declarative sentence that is either true or false,
  but not both.（命题是一个或真或假的陈述句，但不能既真又假）

  Note: 陈述句;或者真,或者假.

  The *truth value* of a proposition： T(1), F(0)（命题的真值： T(1), F(0)）

- 〖Example〗

  (1) The sun is bigger than earth.（太阳比地球大）

  ---- Proposition (TRUE)

  (2) The integer 9 is prime.（整数9是素数）

  ---- Proposition (FALSE)

  (3) This statement is false.（这个陈述是错误的）

  ---- Not a proposition

〖Example〗

(4)Please open the book.（请打开课本）

---- Not a proposition

(5)What time is it？（现在几点了？）

---- Not a proposition

(6) $x + 1 = 2$.

----Not a proposition

这些不能再分的命题称为原子命题或简单命题

Logical operators (Connectives)

- 命题可以通过逻辑联结词(逻辑运算)构成新的命题----复合命题.

  复合命题的真值依赖于其中简单命题的真值

- **We formalize this by denoting propositions variable with letters such as *p*, *q*, *r*, *s*, and introducing several *logical operators*.**（我们通过诸如p,q,r,s这样的字母来表示命题变量并引入几个连接词进行组合形成复合命题）

- **We will examine the following logical operators:**（我们将要检验下面的连接词）
  - **Negation (NOT)** 否定词
  - **Conjunction (AND)** 合取词
  - **Disjunction (OR)** 析取词
  - **Exclusive or (XOR)** 异或词
  - **Implication (if – then)** 蕴涵词
  - **Biconditional (if and only if)** 等价词

1. Negation (NOT)

联结词 “ ¬ ” 称为否定词

p: Today is Tuesday.（今天是星期二）

¬ *p*: Today is not Tuesday（今天不是星期二）

¬*p*: not *p* （*It is not the case that P*）

True when *p* is false, false when *p* is true. （当p为假时，非p为真；当p为真时，非p为假）

The Truth Table for the Negation of a Proposition（命题否定的真值表）

p	¬*p*
T	F
F	T

2. Conjunction (AND)

联结词 “∧”称为合取词

- **p and q :** 称为**p** 和**q** 的合取, 记做 **p ∧ q**

  True when both *p* and *q* are true. （当p和q都为真时， **p ∧ q** 才为真）

The Truth Table

p	*q*	*p*∧*q*
T	T	T
T	F	F
F	T	F
F	F	F

3. Disjunction (OR)

联结词 “∨”称为析取词

- p or q ：称为p和q的析取,记做p ∨ q

  False when both p and q are false.（当p和q都为假时， p ∨ q才为假）

The Truth Table

p	*q*	*p* ∨ *q*
T	T	T
T	F	T
F	T	T
F	F	F

〖Example 〗 Consider the following propositions:（思考下列命题）

***p*: Today is Tuesday.**（今天是星期二）

***q*: It is raining today.**（今天在下雨）

Then,

***p* ∧ *q*:** **Today is Tuesday and It is raining today.**（今天是星期二并且还下雨）

***p* ∨ *q*:** **Today is Tuesday or It is raining today.**（今天是星期二或者今天下雨）

¬*p* ∧ *q*: **Today is not Tuesday and It is raining today.**（今天不是星期二并且今天下雨）

¬*p* ∧ ¬ *q*: **Today is not Tuesday and It is not raining today.**（今天不是星期二并且今天没下雨）

4. Exclusive Or (XOR)

p: Today is Tuesday.（今天是星期二）

q: It is raining today.（今天在下雨）

$p \oplus q$: Either today is Tuesday or It is raining today.

$p \oplus q$: the exclusive or of *p* and *q*（*p*和*q*的异或）

True when exactly one of *p* and *q* is true.（当p和q中恰有一个成真时它成真，否则它为假）

The Truth Table

p	*q*	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Note **:**

- $\vee$ ---- **Inclusive or(同或、兼或)**
- $\oplus$ ---- **Exclusive or (异或)**

〖Example 〗How can the following sentence be translated into a logical expression?（怎样把下面的句子翻译成逻辑表达式）

- **Students who have taken calculus or computer science can take this class.（学过微积分学或者学过计算机的学生可以学这门课程）**
- **George Boole was born in 1815 or 1816.（George Boole 不是出生在1815年就是出生在1816年）**

5. Implication (if - then)

“ →”称为 蕴涵词

p → q : if p then q .

p is called the hypothesis (假设),and q is called the conclusion(结论)

False when *p* is true and *q* is false. （只在p为真且q为假时， p → q为假

The Truth Table

p	*q*	***p → q***
T	T	T
T	F	F
F	T	T
F	F	T

〖Example 〗 Consider the following implication. （考虑下列蕴含关系）

- If today is Tuesday, then 1+1=2.（如果今天是星期二，那么1+1=2）
- If today is Tuesday, then 1+1=3.（如果今天是星期二，那么1+1=3）

Some of the more common ways of expressing the implication $p \rightarrow q$ are: （一些常用的表示蕴含$p \rightarrow q$ *的表示方式*)

- **if *p*, then *q*** *如果p，那么q*
- ***p* implies *q*** *(p蕴含q)*
- **if *p*, *q*** *(如果p，则q)*
- ***p* only if *q*** （*p仅当q*）
- ***p* is sufficient for *q*** （*q的充分条件是p*）
- ***q*, whenever *p*** *(q每当p)*
- ***q* is necessary for *p*** *(q是p的必要条件)*
- ***q unless*** $\neg p$
- ***q follows from p***

Converse of an Implication（蕴含的逆）

Implication: $p \rightarrow q$

Converse(逆): $q \rightarrow p$

- For example,
  - Implication:（蕴含）

    If it is raining now, **then** there are clouds outside.（如果现在在下雨，那么外面有很多乌云）
  - Converse:（蕴含的逆）

    If there are clouds outside, **then** it is raining now.（如果外面有很多乌云，那么现在在下雨）

    注：真值的比较

Contrapositive of an Implication （蕴含的倒置）

Implication: $p \rightarrow q$

Contrapositive（倒置、逆否）: $\neg q \rightarrow \neg p$

- For example,
  - Implication:

    If it is raining now, then there are clouds outside. （如果现在在下雨，那么外面有乌云）

  - Contrapositive: （逆否）

    If there are not clouds outside, then it is not raining now. （如果外面没有乌云，那么现在没下雨）

Note : p → q 等价于 ¬q → ¬p

Inverse of an Implicationv(反蕴含)

Implication: $p \rightarrow q$

Inverse: $\neg p \rightarrow \neg q$ *（反蕴含）*

- For example,
  - Implication:

    If it is raining now, **then** there are clouds outside. （如果现在在下雨，那么外面有乌云）

  - **Inverse** :

    If it is not raining now, **then** there are not clouds outside. （如果现在不下雨，那么外面没有乌云）

6. Biconditional (if and only if)

“↔” :等价词

- **p ↔ q : p if and only if q**（**p**当且仅当**q**）

  True when p and q have the same truth values. 当**p**和**q**有着相同的真值时， **p ↔ q** 为真）

p	*q*	***p ↔ q***
T	T	T
T	F	F
F	T	F
F	F	T

Some of the more common ways of expressing the implication $p \leftrightarrow q$ are:（一些常用的表示蕴含$p \leftrightarrow q$的表示方式）

p* if and only if *q（*p*当且仅当*q*）

- ***p* is necessary and sufficient for *q***（*p*是*q*的必要充分条件）
- **if *p* then *q* and conversely**（如果p，那么q且如果q，那么p）
- **p iff q**
- **(p→q) ∧ (q→p)**

Precedence of Logical Operators(逻辑运算符的优先级)

- Parentheses gets the highest precedence

  Then $\neg$ $\wedge$ $\vee$ $\rightarrow$ $\leftrightarrow$

- 圆括号优先级最高，其次是$\neg$ $\wedge$ $\vee$ $\rightarrow$ $\leftrightarrow$

For example:

- $p \wedge q \vee r$ means $(p \wedge q) \vee r$, not $p \wedge (q \vee r)$
- $p \vee q \rightarrow r$ means $(p \vee q) \rightarrow r$

〖Example 〗

将以下的句子翻译成逻辑表达式

- “*You are a computer science major or you are not a freshman only if you can access the Internet from campus.*”（只要你主修计算机科学或不是新生，就可以从校内网访问因特网）
- ***Solution:***

Let *a*, *c* and *f* represent “you can access the Internet from campus”, “you are a computer science major” and “you are a freshman”.（令a、c和f分别表示“你可以从校内网访问因特网”、“你主修计算机科学”和“你是个新生”。)

This sentence can be represented as : $(c \vee \neg f) \rightarrow a$

Constructing a Truth Table 构造真值表

〖Example 6〗Construct the truth value table for the following formula.
（根据下列表达式构造真值表）

$$(p \wedge q) \rightarrow r$$

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Homework:(ONE)

- P.13 14 (d, e, f),
- P.14 22, 23

CONTENT

Some Terminologies

- Tautology永真式: **A compound proposition that is always true, no matter what the truth values of the propositions that occur in it.** （如果无论复合命题中出现的命题的真值是什么，它的真值总是真。
- Contradiction矛盾式: **A compound proposition that is always false.** （真值永远为假的复合命题）
- Contingency可能式: **A proposition that is neither a tautology nor contradiction.** （既不是永真式又不是矛盾的命题称为可能式）
- **Examples:**
  - $p \wedge \neg p$ -- Contradiction
  - $p \vee \neg p$ -- Tautology
  - $p \vee \neg q$ -- Contingency
  - $(p \vee \neg q) \wedge (q \vee \neg p)$ -- ??

Logically Equivalences（逻辑等价）

【Definition】 The propositions *p* and *q* are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.（如果是永真式，命题**p**和**q**称为是逻辑等价的）

Notation: $p \Leftrightarrow q$ **or** $p \equiv q$

〖Example 〗Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution:

(1) Construct the truth value table for $\neg(p \vee q)$ and $\neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

(2) Show that $\neg (p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology

Note:

- **$\neg (p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ ---- De Morgan's law（德摩根律)**
- **Show that $A \Leftrightarrow B$ using the truth value table**

  A truth table of a compound proposition involving three different propositions requires eight rows, one for each possible combination of truth values of the three propositions.

  In general, 2^n rows are required if a compound proposition involves *n* propositions.

Some well known logical equivalences

Logical Equivalences	
Name	***Equivalence***
Identity laws 恒等律	$p \wedge T \Leftrightarrow p$ $p \vee F \Leftrightarrow p$
Domination laws 支配律	$p \vee T \Leftrightarrow T$ $p \wedge F \Leftrightarrow F$
Idempotent laws 幂等律	$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$
Double negation law 双非律	$\neg\neg p \Leftrightarrow p$
Commutative laws 交换律	$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$
Associative laws 结合律	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

Some well known logical equivalences

Logical Equivalences	
Name	***Equivalence***
Distributive laws 分配律	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
De Morgan's laws 德摩根律	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$
Absorption laws 吸收律	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$
Negation laws 否定律	$p \vee \neg p \Leftrightarrow T$ Tautology law 永真律 $p \wedge \neg p \Leftrightarrow F$ Contradiction law 矛盾律

Some well known logical equivalences

Logical Equivalences Involving Conditional Statements
$\boldsymbol{p \rightarrow q \Leftrightarrow \neg p \vee q}$
$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$
$p \vee q \Leftrightarrow \neg p \rightarrow q$
$p \wedge q \Leftrightarrow \neg (p \rightarrow \neg q)$
$\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \Leftrightarrow p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \Leftrightarrow (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \Leftrightarrow p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

Some well known logical equivalences

Logical Equivalences Involving Biconditional Statements
$\boldsymbol{p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)}$
$p \leftrightarrow q \Leftrightarrow \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \neg q$

连接词的完备性

- {¬， ∧}是完备的，足以表达{¬，∨， ∧， →， ↔}
  - $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$ {↔}可由{→， ∧}表达
  - $p \rightarrow q \Leftrightarrow \neg p \vee q$ {→}可由{¬，∨}表达
  - $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ (德摩根律)

    $\therefore (p \vee q) \Leftrightarrow \neg(\neg p \wedge \neg q)$ {∨}可由{¬， ∧}表达
- {¬， ∧} 能表达⊕吗？
  - $p \oplus q \Leftrightarrow \neg(p \leftrightarrow q)$
- {¬， ∨}是完备的吗？

Examples:

1) $p \wedge \neg p$ -- **Contradiction**
2) $p \vee \neg p$ -- **Tautology**
3) $p \vee \neg q$ -- **Contingency**
4) $(p \vee \neg q) \wedge (q \vee \neg p)$ -- **??**

- $(p \vee \neg q) \wedge (q \vee \neg p)$
- $\Leftrightarrow (q \rightarrow p) \wedge (p \rightarrow q)$
- $\Leftrightarrow p \leftrightarrow q$

〖Example 2〗(1)Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution:

$$\neg(p \vee (\neg p \wedge q)) \Leftrightarrow \neg p \wedge \neg(\neg p \wedge q)$$ **De Morgan' law**德摩根律

$$\Leftrightarrow \neg p \wedge (\neg\neg p \vee \neg q)$$ **De Morgan' law**德摩根律

$$\Leftrightarrow \neg p \wedge (p \vee \neg q)$$ **The double negation law** 双非律

$$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$$ **The distributive law**分配律

$$\Leftrightarrow F \vee (\neg p \wedge \neg q)$$ **The contradiction law**矛盾律

$$\Leftrightarrow \neg p \wedge \neg q$$ **The identity law**恒等律

〖**Example 2**〗 **(2)Show that** $((P \to Q) \to R) \to ((R \to P) \to (S \to P))$ **is a tautology.**

Solution:

$$((p \to q) \to r) \to ((r \to p) \to (s \to p))$$

$$\Leftrightarrow \neg(\neg(\neg p \vee q) \vee r) \vee (\neg(\neg r \vee p) \vee (\neg s \vee p))$$

$$\Leftrightarrow ((\neg p \vee q) \wedge \neg r) \vee (r \wedge \neg p) \vee (\neg s \vee p)$$

$$\Leftrightarrow (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee (r \wedge \neg p) \vee (\neg s \vee p)$$

$$\Leftrightarrow (\neg p \wedge (\neg r \vee r)) \vee (q \wedge \neg r) \vee (\neg s \vee p)$$

$$\Leftrightarrow \neg p \vee (q \wedge \neg r) \vee (\neg s \vee p)$$

$$\Leftrightarrow T$$

证明 $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) = R$

证：

左端 $= (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$

$= ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$

$= (\neg P \wedge \neg Q) \vee (Q \vee P)) \wedge R$

$= (\neg (P \vee Q) \vee (P \vee Q)) \wedge R$

$= 1 \wedge R$

$= R$

证明$((P \vee Q) \wedge \neg (\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) = 1$

证：

左端＝

$((P \vee Q) \wedge (P \vee (Q \wedge R))) \vee \neg ((P \vee Q) \wedge (P \vee R))$

$= ((P \vee Q) \wedge (P \vee Q) \wedge (P \vee R)) \vee \neg ((P \vee Q) \wedge (P \vee R))$

$= ((P \vee Q) \wedge (P \vee R)) \vee \neg ((P \vee Q) \wedge (P \vee R))$

$= 1$

范式（补充内容）

定义 1　命题变元及其否定统称为文字.

一些文字的合取称为基本合取式或短语，

一些文字的析取称为基本析取式或子句.

特别地，一个文字既是短语又是子句.

范式（补充内容）

- 定义 2　有限个短语的析取，即形如$A_1 \vee A_2 \vee \ldots \vee A_n$的公式，其中$A_i$(i= 1，2，…，n) 为短语，称为析取范式；有限个子句的合取，即形如 $B_1 \wedge B_2 \wedge \ldots \wedge B_n$的公式，其中$B_i$(i= 1，2，…，n)为子句，称为合取范式.

- 短语既是析取范式又是合取范式，子句同样既是析取范式又是合取范式.

范式（补充内容）

- 定理 1　对于任意命题公式，都存在等价于它的析取范式和合取范式.

- 证明　对于任意公式A，通过如下算法即可得出等价于它的范式.
  - 使用基本等价式，将A中逻辑联结词$\rightarrow$、$\leftrightarrow$去除.
  - 使用Morgan律和双重否定律，将A中所有的否定词$\neg$都放在命题变元之前，形成文字.
  - 反复使用分配律，即可得到等价的范式.

范式（补充内容）

□例 1　$A = (P \wedge (Q \rightarrow R)) \rightarrow S$

$$
\begin{aligned}
A &= \neg(P \wedge (\neg Q \vee R)) \vee S \\
&= \neg P \vee \neg(\neg Q \vee R) \vee S \\
&= \neg P \vee (Q \wedge \neg R) \vee S \text{ (析取范式)} \\
&= ((\neg P \vee Q) \wedge (\neg P \vee \neg R)) \vee S \\
&= (\neg P \vee Q \vee S) \wedge (\neg P \vee \neg R \vee S) \quad \text{(合取范式)}
\end{aligned}
$$

范式（补充内容）

- 利用析取范式，可以判断一个公式是否永真或永假

- 方法：一个公式*A*永假，当且仅当其析取范式中每个短语永假，而一个短语永假当且仅当该短语中同时含有某命题变元及其否定.

范式（补充内容）

□例 2　判断$A=(P\to Q)\wedge(Q\to R)\wedge(R\to P)$是否永假.

$A=(P\to Q)\wedge(Q\to R)\wedge(R\to P)$

$=(\neg P\vee Q)\wedge(\neg Q\vee R)\wedge(\neg R\vee P)$

$=((\neg P\wedge\neg Q)\vee(\neg P\wedge R)\vee(Q\wedge\neg Q)\vee(Q\wedge R))\wedge(\neg R\vee P)$

$=(\neg P\wedge\neg Q\wedge\neg R)\vee\ldots$

故公式A不是永假的.

范式（补充内容）

□例 3　$A=(P\rightarrow Q)\wedge P\wedge \neg Q$是否永假.

$$A = (P\rightarrow Q)\wedge P\wedge \neg Q$$

$$=(\neg P\vee Q)\wedge P\wedge \neg Q$$

$$=(\neg P\wedge P\wedge \neg Q)\vee(Q\wedge P\wedge \neg Q)$$

$$= 0$$

故A是永假的.

范式（补充内容）

- 范式并非是唯一的. 例如， 析取范式

$$(P \wedge Q) \vee (Q \wedge \neg R)$$

可等价地写成

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (Q \wedge \neg R)$$

或

$$(P \wedge Q \wedge R) \vee (Q \wedge \neg R),$$

范式（补充内容）

定义 3　设P_1，P_2，…，P_n是n个命题变元，形如

$$\tilde{p}_1 \wedge \tilde{p}_2 \wedge \cdots \wedge \tilde{p}_n$$

的公式，称为由P_1，P_2，…，P_n生成的极小项，其中 (i＝ 1，2，…，n).

$$\tilde{p}_i = \begin{cases} P_i \\ \neg P_i \end{cases}$$

范式（补充内容）

- $P \wedge Q \wedge R$，$\neg P \wedge Q \wedge \neg R$是由$P$，$Q$，$R$生成的极小项，但不是$Q$，$P$，$R$生成的极小项

范式（补充内容）

- 设P_1，P_2，…，P_n是n个命题变元，若将P_i对应 1 ，$\neg P_i$对应 0 ，则极小项

$$\tilde{p}_1 \wedge \tilde{p}_2 \wedge \cdots \tilde{p}_n$$

对应n位二进制数$\delta_1\delta_2...\delta_n$，

- 极小项可记为 $m_{\delta_1\delta_2\cdots\delta_n}$

- 例如m_{10}，m_{1100}，也可写为m_2，m_{12}

范式（补充内容）

□ 极小项具有如下性质：

1）n个命题变元生成的极小项共有 2^n个.

2）对于每个极小项，存在唯一一个指派使该极小项为 1 .

3）极小项两两不等价，且

$$m_i \wedge m_j = 0 \quad (i \neq j),$$

$$\sum_{i=0}^{2^n-1} m_i = 1$$

范式（补充内容）

定义 4　设公式A中出现的所有命题变元为P_1，P_2，…，P_n，如果在A的析取范式A'中，每个短语均为关于P_1，P_2，…，P_n的极小项，则称A'为A的主析取范式.

范式（补充内容）

- 定理 2　对任意公式A，都存在唯一一个与之等价的主析取范式.

范式（补充内容）

- 例 5　求$P \leftrightarrow Q$的主析取范式.
- $P \leftrightarrow Q$的真值表如下：

P	Q	$P \leftrightarrow Q$
0	0	1
0	1	0
1	0	0
1	1	1

- 主析取范式：$P \leftrightarrow Q = m_{00} \vee m_{11}$

范式（补充内容）

例 6　$A=(P\wedge Q)\vee(\neg P\wedge R)\vee(Q\wedge R)$

P	Q	R	A
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$A=m_1\vee m_3\vee m_6\vee m_7$

Homework:(TWO)

- P35. 8, 22, 23, 29

Rules of Inference (推理规则)

- **Addition** 附加
- **Simplification** 简化
- **Conjunction** 合取
- **Modus ponens** 假言推理
- **Modus Tollens** 取拒式
- **Hypothetical syllogism** 假言三段论
- **Disjunctive syllogism** 析取三段论
- **Constructive dilemma** 构造性二难
- **Resolution** 消解推理

- **Addition** $p \rightarrow (p \vee q)$

$$\frac{p}{\therefore p \vee q}$$

附加

- **Simplification** $(p \wedge q) \rightarrow p$

$$\frac{p \wedge q}{\therefore p}$$

化简

- **Conjunction** $((p) \wedge (q)) \rightarrow p \wedge q$

$$\frac{\begin{array}{l} p \\ q \end{array}}{\therefore p \wedge q}$$

合取

- **Modus ponens** $(p \wedge (p \to q)) \to q$

  p

  $p \to q$

  $\therefore q$ *假言推理*

- **Modus Tollens** $(\neg q \wedge (p \to q)) \to \neg p$

  $\neg q$

  $p \to q$

  $\therefore \neg p$ *取拒式*

- **Hypothetical syllogism** $((p \to q) \wedge (q \to r)) \to (p \to r)$

  $p \to q$

  $q \to r$

  $\therefore p \to r$ *假言三段论*

- **Disjunctive syllogism** $((p \vee q) \wedge \neg p) \rightarrow q$

  $p \vee q$
  $\neg p$
  $\therefore q$

  析取 三段论

- **Constructive dilemma** $((p \rightarrow r) \wedge (q \rightarrow r) \wedge (p \vee q)) \rightarrow r$

  $(p \rightarrow r) \wedge (q \rightarrow r)$
  $p \vee q$
  $\therefore r$

  构造性两难

- **Resolution** $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

  $p \vee q$
  $\neg p \vee r$
  $\therefore q \vee r$

  消解

Valid Arguments （有效的论证）

***Deductive reasoning*: the process of reaching a conclusion *q* from a sequence of propositions** $p_1, p_2, \cdots, p_n$

Where p_i are called the *premises* 前提or *hypotheses*假设

and*q* is the *conclusion*.

则称q是$p_1, \cdots, p_n$的逻辑结果，或者称$p_1, \cdots, p_n$共同蕴涵q

An argument form is called *valid* if whenever all the hypotheses are true, the conclusion is also true, namely若每当所有的前提都为真时，结论也为真，则这样的论证称为有效的。也就是等价于下列蕴含式为真。

The implication $p_1 \wedge p_2 \wedge \cdots \wedge p_n \rightarrow q$ **is tautology.**

As a rule of inference they take the symbolic form:

P_1 他们可以采用下列符号形式作为一种推理规则：

p_2

.

.

p_n

$\therefore q$

where $\therefore$ means 'therefore' or 'it follows that.'

or $p_1 \wedge p_2 \wedge \cdots \wedge p_n \Rightarrow q$

Formal Proofs

To prove an argument is valid or the conclusion follows *logically* from the hypotheses:

- **Assume the hypotheses are true.**

假定假设是正确的

- **Use the rules of inference and logical equivalences to determine that the conclusion is true.**

用一些推论和逻辑等价式来确定结论的正确

〖Example 1〗Consider the following logical argument, is it valid?

If horses fly or cows eat artichokes(笋), then the mosquito （蚊子）is the national bird（国鸟）. If the mosquito is the national bird then peanut butter（花生酱） tastes good on hot dogs. But peanut butter tastes terrible on hot dogs.

Therefore, cows don't eat artichokes.

solution:

■ **Assign propositional variabl**

argument:

***h*: Horses fly, *a*: Cows eat artic**
the national bird, *p*: Peanut bu

1. $(h \vee a) \rightarrow m$
2. $m \rightarrow p$
3. $\neg p$
$\therefore \neg a$

■ **Use the hypotheses 1, 2, and 3. and the above rules of inference and any logical equivalences to construct the proof.**

Step	Reasons
1. $(h \vee a) \rightarrow m$	前提
2. $m \rightarrow p$	前提
3. $(h \vee a) \rightarrow p$	假言三段论 using steps 1 and 2
4. $\neg p$	前提
5. $\neg(h \vee a)$	取拒式using steps 3 and 4
6. $\neg h \wedge \neg a$	DeMorgan
7. $\neg a$	化简using step 6

〖Example 2〗Show that $\neg w$ logically follows from the hypotheses

$$(w \vee r) \rightarrow v, v \rightarrow (c \vee s), s \rightarrow u, \neg c \wedge \neg u$$

solution:

	Step	Reason
1.	$(w \vee r) \rightarrow v$	Hypothesis（前提）
2.	$v \rightarrow (c \vee s)$	Hypothesis （前提）
3.	$(w \vee r) \rightarrow (c \vee s)$	假言三段论 using steps 1 and 2
4.	$\neg c \wedge \neg u$	Hypothesis （前提）
5.	$\neg u$	Simplification using step 4（化简）
6.	$s \rightarrow u$	Hypothesis （前提）
7.	$\neg s$	Modus tollens using steps 5 and 6（取拒式）
8.	$\neg c$	Simplification using step 4 （化简）
9.	$\neg s \wedge \neg c$	Conjunction using step 7 and 8 （合取）
10.	$\neg (c \vee s)$	Step 9 and De morgan（德摩根定律）
11.	$\neg (w \vee r)$	Modus tollens using steps 3 and 10 （取拒式）
12.	$\neg w \wedge \neg r$	Step 11 and De morgan（德摩根定律）
13.	$\neg w$	Simplification using step 12（化简）

Note:

If the conclusion is given in form $p \to q$, we can convert the original problem to （如果有这样**p→q**的结论形式，我们就能把原问题转换成为下面这个式子：）

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge p \Rightarrow q$$

This method is based on the tautology 这方法是根据下面这个永真式的：

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge p) \to q \Leftrightarrow (p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to (p \to q)$$

〖Example 3〗Show that $r \to s$ **logically follows from the hypotheses**

$$p \to (q \to s), \neg r \vee p, q$$

solution:

Step	Reason
1. $\neg r \vee p$	Hypothesis
2. r	Hypothesis
3. p	Disjunctive syllogism using step 1 and 2析取三段论
4. $p \to (q \to s)$	Hypothesis
5 $q \to s$	Modus ponens using steps 3 and 4假言推理
6. q	Hypothesis
7. s	Modus ponens using steps 5 and 6假言推理

Note:

另一个重要的证明方法是归缪法

The proof of $p \rightarrow q$ by contradiction consists of the following steps:

1) **Assume p is true and q is false**假设p为真并且q为假
2) **Show that $\neg p$ is also true.**证明 $\neg p$为真。

   Since the statement $p \wedge (\neg p)$ is always false.

   —Contradiction!

〖Example 4〗Show that $s \vee r$ logically follows from the hypotheses $(p \vee q), (p \to r), (q \to s)$

solution:

	Step	Reason
1.	$\neg (s \vee r)$	Additional hypothesis附加前提
2.	$\neg s \wedge \neg r$	Step 1 and De morgan 德摩根定律
3.	$\neg s$	Simplification using step 2 化简
4.	$\neg r$	Simplification using step 2 化简
5.	$p \to r$	Hypothesis
6.	$\neg p$	Modus tollens using steps 4 and 5 取拒式
7.	$q \to s$	Hypothesis
8.	$\neg q$	Modus tollens using steps 3 and 7取拒式
9.	$\neg p \wedge \neg q$	Conjunction using step 6 and 8 合取
10.	$\neg (p \vee q)$	Step 9 and De morgan德摩根定律
11.	$p \vee q$	Hypothesis

Resolution 消解

Resolution can be used to build automatic theorem proving system.消解能够被用在生成自动的定理证明系统上。

To construct proofs in propositional logic

-- using resolution as the only rule of inference,

-- the hypotheses and the conclusion must be expressed as clauses.

〖Example 5〗 Show that the hypotheses $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

solution:

$$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$$

$$r \rightarrow s \equiv \neg r \vee s$$

$$(p \vee r) \wedge (\neg r \vee s) \Rightarrow p \vee s$$

符号化下列命题, 并证明其结论: 一公安人员审查一件盗窃案, 已知事实如下:

(1) 张平或王磊盗窃了机房的计算机一台;

(2) 若张平盗窃了计算机, 则作案时间不可能发生在午夜之前;

(3) 若王磊的证词正确, 则午夜时机房里的灯未灭;

(4) 若王磊的证词不正确, 则作案时间发生在午夜之前;

(5) 午夜时机房灯光灭了。

问: 盗窃计算机的是张平?王磊?

设

P : 张平盗窃了计算机;

Q: 王磊盗窃了计算机;

R: 作案时间发生在午夜前;

S: 王磊的证词正确;

U : 午夜时灯光灭了。

则前提可符号化为:

$P \vee Q , P \rightarrow \neg R , S \rightarrow \neg U , \neg S \rightarrow R , U$

证明的结论为P 或Q。

Fallacies （谬误）

1. The Fallacy of affirming the conclusion断定结论的谬误

Method:

Reasoning based on $((p \to q) \wedge q) \to p$ *把此式当作重言式的不正确的论证*

〖Example 6〗Let p: $n \equiv 1 \pmod 3$; and q: $n^2 \equiv 1 \pmod 3$.

The implication $p \to q$ is true. If q is true, so that $n^2 \equiv 1 \pmod 3$, does it follow that p is true, namely, that $n \equiv 1 \pmod 3$?

solution:

It would be incorrect to conclude that p is true.

2. The Fallacy of denying the hypothesis 否定假设的谬误

Method:

Reasoning based on $((p \to q) \wedge \neg p) \to \neg q$

〖Example 7〗Is it correct to conclude that

$$n^2 \not\equiv 1 (\mathrm{mod}\, 3) \quad if \quad n \not\equiv 1 (\mathrm{mod}\, 3) \ ,$$

using the implication: if $n \equiv 1 (\mathrm{mod}\ 3)$, then $n^2 \equiv 1 \ (\mathrm{mod}\ 3)$?

CHAPTER 1

The Foundations: Logic and Proof

Predicates 谓词

Consider the statement "$x > 3$" or "x is greater than 3".

This can be represented by $P(x)$ where P is the *predicate* or the *propositional function* representing the property "is greater than 3" and x is the variable.(考虑下列陈述句 "$x > 3$" 或者 "x是个比3大的数"。它可以用$P(x)$ 表示，这里P是谓词或者是能代表 "大于3"这个性质的命题函数，而x是变量）

$P(x)$: x is greater than 3.

〖Example 1〗

(1) x is greater than y. $P(x, y)$

(2) x is between y and z. $B(x, y, z)$

Note:

- **Propositional function has not a definite truth value.**

  命题函数没有明确的真值

- **Once a value has been assigned to the variable x, $P(x)$ becomes a proposition and has a truth value.**

  当变量x被赋予一个值时，$P(x)$ 变为一个有真假值的命题

- **The *truth value of* $P(x)$ can be determined when x is assigned a value. (The variable x is bound.)（当x被指派一个值时，P（x）的真值就能确定了）**

〖Example 2〗Let $P(x)$ denote the statement "$x > 0$." What are the truth values of $P(-3)$, $P(0)$ and $P(3)$?

〖Example 3〗Let $Q(x, y)$ denote the statement "$x < y$". $Q(4, 3)$ means "$4 < 3$" which is false, $Q(2, 7)$ means "$2 < 7$" which is true.

〖Example 4〗Let *P*(*x*) denote the statement "*x* > 0."

(1) Is *P*(*y*) ∨ ¬*P*(0) a proposition?

(2) Is *P*(3) ∨ ¬*P*(0) a proposition?

In general, a statement of the form $P(x_1, x_2, \cdots, x_n)$ **is the value of the propositional function *P* at the *n*-tuple** $(x_1, x_2, \cdots, x_n)$ **, and *P* is also called a *predicate*（一般来说，形为** $P(x_1, x_2, \cdots, x_n)$ **的语句是命题函数P在n元组** $(x_1, x_2, \cdots, x_n)$ **的值，P也称为谓词。）**

Quantifiers 量词

There are two ways to create a proposition from a propositional function:（可以从命题函数中通过两种方式 产生命题）

- **assigning a value to every variable**对每个变量赋予值
- **quantifying it** 对它进行定量化

The universe of discourse, or domain **(论域)**

---- A particular domain for all values of a variable. (对一个变量取尽该域中所有值的一个特殊的域)

Universal quantification 全称量化

$\forall x\, P(x)$ ---- $P(x)$ is true for all values of x in the universe of discourse.
（对论域中任意一个x而言， $P(x)$ 的真值都为真。）

For all $x\, P(x)$ For every $x\, P(x)$

$\forall$ **----** ***Universal quantifier*** （全称量词）

〖Example 5〗Express the following statement as a universal quantification.

All lions are fierce.

Solution:

Let $Q(x)$ denote the statement "x is fierce".

(1)Assuming that the universe of discourse is the set of all lions.

$$\forall x Q(x)$$

(2) Assuming that the universe of discourse is the set of all creatures.

Let $P(x)$ denote the statement "x is a lion".

$$\forall x(P(x) \rightarrow Q(x))$$

NOTE:从上面的例子我们可以看出，将命题符号化的时候，必须明确所涉及到的个体集合，即论域。论域的不同，可能得到的符号化的表达式完全不同

〖Example 6〗Assume that the universe of discourse is $\{1,2,3\}$ **.**

$\forall x\, P(x)$**=?**

Solution:

$$\forall x P(x) \Leftrightarrow P(1) \wedge P(2) \wedge P(3)$$

In general, the universe of discourse is $\{x_1, x_2, \cdots, x_n\}$ **.**（当域中的所有元素可以列成 $\{x_1, x_2, \cdots, x_n\}$ 时，量化语句 $\forall x P(x)$ 与合取 $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$ 是等价的。）

$$\forall x P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

Existential quantification （存在量化）

∃ *x P*(*x*) ---- **There exists an element *x* in the universe of discourse such that *P(x)* is true.** （在论域中存在一个x使*P(x)* 的真值为真）

For some *x* *P*(*x*);（对某个x, *P(x)* ）

There is an *x* such that *P*(*x*);（有一个x使得*P(x)* ）

There is at least one *x* such that *P*(*x*);（至少有一个x使得*P(x)* ）

I can find an *x* such that *P(x)*. （我可以找出一个x使得*P(x)* ）

∃ ---- *existential quantifier* （存在量词）

〖Example 7〗Express the following statement as a existential quantification.

Some real numbers are rational numbers. （一些实数是有理数）

Solution:

Let $Q(y)$: y is a rational numbers

(1)Assuming that the universe of discourse is the set of all real numbers. （论域为实数集合）

$$\exists y Q(y)$$

(2) Assuming that the universe of discourse is the set of all numbers. Let $R(y)$: y is a real number（论域为全体数）

$$\exists y(R(y) \wedge Q(y))$$

〖Example 8〗Assume that the universe of discourse is $\{1,2,3\}$.

$\exists x\, P(x)$=?

Solution:

$$\exists x P(x) \Leftrightarrow P(1) \vee P(2) \vee P(3)$$

In general, the universe of discourse is $\{x_1, x_2, \cdots, x_n\}$.（当域中的所有元素可以列成{ $x_1, x_2, \cdots, x_n$时，存在量化 $\exists x P(x)$;析取 $\exists x P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$ 是等价的）$\exists x P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$

Statement	When true?	When false?
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Binding Variables

在一个谓词公式中，变量的出现是绑定的，当且仅当有量词作用于它或者给它赋值时；变量的出现说是自由的，当且仅当它的出现不是绑定的。

***Bound* variable【绑定变量（约束变量）】: Quantified or assigned a specific value**

***Free* variable（自由变量）: Neither quantified nor assigned a specific value**

Example: $\forall x\, P(x)$: x是绑定变量

$\exists x\, Q(x, y)$: x是绑定变量，y是自由变量

***Scope* of quantifiers（量词的范围）: Part of a logical expression to which a quantifier is applied**

Example: $\exists x\,(P(x) \wedge Q(x)) \vee \forall x\, R(x)$

Negations of Quantifiers量词的否定

Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg\forall x A(x) \quad \Leftrightarrow \quad \exists x \neg A(x)$$
$$\neg\exists x A(x) \quad \Leftrightarrow \quad \forall x \neg A(x)$$

Consider:

***P*(*x*) : *x* has taken a course in calculus**

（ ***x***学过一门微积分课）

Summary of Negations of Quantifiers量词的否定

Negation	Equivalent Statement	When is Negation True?	When False?
$\neg\exists x\, P(x)$	$\forall x\, \neg P(x)$	$P(x)$ is false for every x	There is an x for which $P(x)$ is true
$\neg\forall x\, P(x)$	$\exists x\, \neg P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for every x

Translating from English into Logical Expression

Goal:

To produce a logical expression that is simple and can be easily used in subsequent reasoning.

Steps:

- **Clearly identify the appropriate quantifier(s)**
  确定恰当的量词
- **Introduce variable(s) and predicate(s)**
  引入变量和谓词
- **Translate using quantifiers, predicates, and logical operators**
  用量词，谓词，和逻辑操作来转化

〖Example 9〗 $C(x)$: x is a CS student, $E(x)$: x is an CM student
$S(x)$: x is a smart student, U = {all students in our class}

1) **Everyone is a CS student.**
   $\forall x\, C(x)$
2) **Nobody is an CM student.**
   $\forall x \neg E(x)$ **or** $\neg \exists x\, E(x)$
3) **All CS students are smart students.**
   $\forall x\, (C(x) \rightarrow S(x))$
4) **Some CS students are smart students.**
   $\exists x\, (C(x) \wedge S(x))$

〖Example 9〗 $C(x)$: x is a CS student, $E(x)$: x is an CM student
$S(x)$: x is a smart student, U = {all students in our class}

5) No CS student is an CM student.

- **If x is a CS student, then that student is not an CM student.**

$$\forall x\,(C(x) \rightarrow \neg\, E(x))$$

- **There does not exist a CS student who is also an CM student.**

$$\neg\, \exists x\,[C(x) \wedge E(x)]$$

6) If any CM student is a smart student then he is also a CS student.

$$\forall x\,((E(x) \wedge S(x)) \rightarrow C(x))$$

- 例10. 通过两个不同点至多有一条直线.
- 令：$T(x)$：x是点，　　$L(x)$：x是直线.

  $P(x, y, z)$：x通过y，z.

  $E(x, y)$：$x=y$.
- 则有

$$\forall x \forall y((T(x) \wedge T(y) \wedge \neg E(x, y)) \rightarrow \forall z_1 \forall z_2((L(z_1) \wedge L(z_2) \wedge P(z_1, x, y) \wedge P(z_2, x, y)) \rightarrow E(z_1, z_2))).$$

换名

- $\forall x P(x) \vee Q(x)$
- 与
- $\forall y P(y) \vee Q(x)$

换名

- 任一公式均可用如下换名规则给其中的变元换名.
- (1) 将某个个体变元在量词中作为指导变元的出现和它在该量词辖域中的所有出现都用同一个新个体变元去替换.
- (2) 所用新个体变元在原式中不出现.

换名

- $\forall x(P(x，y)\rightarrow\exists yQ(x，y，z))\wedge R(f(x，z))$
- 经过换名可化为
- $\forall v(P(v，y)\rightarrow\exists wQ(v，w，z))\wedge R(f(x，z))$
- 经过换名以后，总可以使任意约束变元不是自由变元.

Some Logical Equivalences 一些逻辑等价

1.
$$\forall x(A(x) \wedge B(x)) \quad \Leftrightarrow \quad \forall x A(x) \wedge \forall x B(x)$$
$$\exists x(A(x) \vee B(x)) \quad \Leftrightarrow \quad \exists x A(x) \vee \exists x B(x)$$

Note:

$$\forall x(A(x) \vee B(x)) \quad \not\Leftrightarrow \quad \forall x A(x) \vee \forall x B(x)$$
$$\exists x(A(x) \wedge B(x)) \quad \not\Leftrightarrow \quad \exists x A(x) \wedge \exists x B(x)$$

$$\exists x(A(x) \wedge B(x)) \quad \Rightarrow \quad \exists x A(x) \wedge \exists x B(x)$$
$$\forall x A(x) \vee \forall x B(x) \quad \Rightarrow \quad \forall x(A(x) \vee B(x))$$

2. x is not occurring in P and B.

(1) $\forall x A(x) \vee P \Leftrightarrow \forall x(A(x) \vee P)$

(2) $\forall x A(x) \wedge P \Leftrightarrow \forall x(A(x) \wedge P)$

(3) $\exists x A(x) \vee P \Leftrightarrow \exists x(A(x) \vee P)$

(4) $\exists x A(x) \wedge P \Leftrightarrow \exists x(A(x) \wedge P)$

2. x is not occurring in P and B.

(1) $\forall x A(x) \vee P \Leftrightarrow \forall x(A(x) \vee P)$

(2) $\forall x A(x) \wedge P \Leftrightarrow \forall x(A(x) \wedge P)$

(3) $\exists x A(x) \vee P \Leftrightarrow \exists x(A(x) \vee P)$

(4) $\exists x A(x) \wedge P \Leftrightarrow \exists x(A(x) \wedge P)$

(5) $\forall x(B \to A(x)) \Leftrightarrow B \to \forall x A(x)$

Proof:

$$\forall x(B \to A(x)) \Leftrightarrow \forall x(\neg B \vee A(x))$$

$$\Leftrightarrow \neg B \vee \forall x A(x)$$

$$\Leftrightarrow B \to \forall x A(x)$$

Homework:

P.53 13, 19（b,c,d), 22

CHAPTER 1

The Foundations: Logic and Proof, Sets, and Functions

Nested quantifiers嵌套的量词

Quantifiers that occur within the scope of other quantifiers.
它是出现在其他量词的作用域内的量词

For example,

$\forall x\, \exists y(x+y=0)$（我们可以把辖域内的 部分看成命题函数Q(x)）

Translate Statements Involving Nested Quantifiers

〖Example 1〗Assume that the universe of discourse for the variables *x* ,*y* ,and *z* consists of all real number .

$$\forall x\, \forall\, y\, \forall z(x+(y+z)=(x+y)+z)$$

答案：对于任意三个实数x,y,z而言，他们之间满足加法的结合律。

〖**Example 2**〗

(1) Translate the statement $\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$

into English, where *C*(*x*) is "*x* has a computer," *F*(*x*,*y*) is "*x* and *y* are friends," and the universe of discourse for both *x* and *y* is the set of all students in NBU.

答案：对学校的每一个学生x，或者x有台计算机，或者另有学生y，他有台计算机，并且x和y是朋友。换言之，学校的每个学生或有计算机，或有个有计算机的朋友。

(2) Translate the statement

$$\exists x \forall y \forall z(((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z)))$$

into English, where *F*(*a*,*b*) means *a* and *b* are friends and the universe of discourse for *x*, y, and *z* is the set of all students in NBU.

答案：有一个学生x，对所有学生y及不同于y的所有学生z，只要x和y是朋友，x和z也是朋友，那么y和z就不是朋友。换句话说，有个学生，他的朋友之间都不是朋友。

Translate Sentences Into Logical Expressions

〖Example 3〗 Express the following statements as logical expressions.

(1) ***Everyone has exactly one best friend.***

Solution:

Let the universe of discourse for the variables be the set of all people in the world.

Let *B*(*x*, *y*) be the statement "*y* is the best friend of *x*".

Consequently, we can translate the sentence as

$$\forall x \exists y \forall z (B(x, y) \wedge ((z \neq y) \rightarrow \neg B(x, z)))$$

〖Example 3〗 Express the following statements as logical expressions.

(2) *If somebody is female and is a parent, then this person is someone's mother.*

Solution:

Let *F*(*x*) be the statement "*x* is female" ,

let "*P*(*x*) be the statement "*x* is a parent" ,

and let *M*(*x*, *y*) be the statement "*x* is the mother of *y*".

Since the statement in the example pertains to all people,

we can write it symbolically as

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y))$$

〖Example 3〗 Express the following statements as logical expressions.

(3) ***Not all of the real numbers are rational numbers.***

Solution:

$R(x)$: x is a real number,

$Q(x)$: x is a rational number

we can write it symbolically as

$$\neg\forall x(R(x)\rightarrow Q(x))$$

$$\Leftrightarrow \exists x(R(x)\wedge\neg Q(x))$$

〖Example 3〗 Express the following statements as logical expressions.

(4) ***All men have some shortcoming.***

Solution:

***M*(*x*): *x* is a man**

***G*(*y*): *y* is shortcoming,**

***P*(*x*, *y*): *x* has *y*.**

we can write it symbolically as

$$\forall x(M(x) \to \exists y(G(y) \wedge P(x,y))$$

$$\forall x \exists y(M(x) \to (G(y) \wedge P(x,y))$$

(5) ***Express the definition of a limit using quantifiers.*** ***用量词表示极限的定义***

Solution:

回顾定义： **Recall the definition of the statement** $\lim_{x \to x_0} f(x) = A$ **is: For every real number ε>0 there exists a real number δ>0 such that** $|f(x) - A| < \varepsilon$ *wherever* $|x - x_0| < \delta$

***R*(*x*): *x* is a real number**

P*(*x*,*y*): *x*<*y

This definition of limit can be phrased in terms of quantifiers by

$$\forall \varepsilon (R(\varepsilon) \wedge P(0,\varepsilon) \rightarrow \exists \delta (R(\delta) \wedge P(0,\delta) \wedge \forall x (R(x) \wedge P(|x - x_0|, \delta) \rightarrow P(|f(x) - A|, \varepsilon))))$$

Negating Nested Quantifiers否定嵌套量词

Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier. 带嵌套量词的语句可以通过连续地应用否定带单个量词的语句的规则成为否定的。

〖**Example 4**〗 **Express the negation of the statement**

$\forall x \exists y\ (xy=1)$ **so that no negation precedes a quantifiers.**表达上面语句的否定，使得量词前面没有否定词。

Solution:

$$\neg\forall x \exists y\ (xy=1)$$

$$\exists x \neg \exists y\ (xy=1)$$

$$\exists x \forall y \neg\ (xy=1)$$

$$\exists x \forall y\ (xy\neq 1)$$

〖Example 5〗 Use quantifiers and predicates express the fact that $\lim_{x \to x_0} f(x)$ does not exist.

Solution:

$$\neg \forall \varepsilon (R(\varepsilon) \wedge P(0,\varepsilon) \to \exists \delta (R(\delta) \wedge P(0,\delta) \wedge \forall x (R(x) \wedge P(|x - x_0|, \delta) \to P(|f(x) - A|, \varepsilon))))$$

……

$$\exists \varepsilon (R(\varepsilon) \wedge P(0,\varepsilon) \wedge \forall \delta (R(\delta) \wedge P(0,\delta) \to \exists x (R(x) \wedge P(|x - x_0|, \delta) \wedge P(\varepsilon, |f(x) - A|))))$$

The Order of Quantifiers（量词的顺序）

Predicates with multiple variables may involve multiple quantifications. The *order of the multiple quantifiers*多个量词的顺序 is significant. 许多数学语句需要对多变量命题函数作多重量化。除非所有量词均为全称量词或均为存在量词，否则量词的顺序是重要的。

〖Example 6〗 Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications

$$\exists y \forall x Q(x, y) \text{ and } \forall x \exists y Q(x, y)$$

Solution: *第一个语句：有个实数y能使Q（x，y）对每一个实数x成立。*
第二语句：对每个实数x都有一个实数y使Q(x,y)成立。

Read left to right

$\exists y \forall x Q(x, y)$ *is flase.*

$\forall x \exists y Q(x, y)$ *is true.*

Note: $\exists y \forall x Q(x, y) and \forall x \exists y Q(x, y)$ **are not logically equivalent.**

How to determine the truth values of

$$\exists y \forall x Q(x, y) \; and \; \forall x \exists y Q(x, y) \quad ?$$

Statement	*When True?*	*When False?*
$\forall x \, \forall y \, P(x, y)$ $\forall y \, \forall x \, P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \, \exists y \, P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x for which $P(x, y)$ is false for every y.
$\exists x \, \forall y \, P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \, \exists y \, P(x, y)$ $\exists y \, \exists x \, P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

1.For example P52

2、假设有如下一个解释：

论域**D={1,2}, a = 2, f(x) = x^2(mod 2) + 1, P(x,y)：x > y**

求在上述解释下公式 **∀x∃y(P(x,a) →P(f(x),f(y)))** 的真值。

解: ∀x∃y(P(x,a) →P(f(x),f(y)))= ∀x((P(x,a) →P(f(x),f(1)))
∨(P(x,a) →P(f(x),f(2))))

= ((P(1,a)→P(f(1),f(1)))∨(P(1,a)→P(f(1),f(2)))∧
((P(2,a)→P(f(2),f(1)))∨(P(2,a) →P(f(2),f(2))))

= ((P(1,2) →P(2,2)) ∨(P(1,2) →P(2,1))) ∧ ((P(2,2) →P(1,2))
∨(P(2,2) →P(1,1)))

= ((F →F) ∨(F →T)) ∧((F →F) ∨(F →F))=T

□定义 2　形如

$$Q_1 x_1 Q_2 x_2 \ldots Q_n x_n M$$

的公式称为前束范式，其中$Q_i x_i$ ($1 \le i \le n$) 是$\forall x_i$ 或 $\exists x_i$，称为首标，M中不含任何量词，称为母式．如果将M中的原子公式及其否定视为文字，当M是析取范式时，上述公式称为前束析取范式；当M是合取范式时，上述公式称为前束合取范式.

定理 1　对任意谓词公式A，都存在与其等价的前束范式 .

证明　通过如下步骤，可将A化为前束范式.

(1)使用等价式

$$A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A),\ A \rightarrow B = \neg A \vee B$$

去除$\rightarrow$与$\leftrightarrow$，

(2)使用Morgan律和双重否定律及量词否定型等价式将 $\neg$ 放在原子公式之前.

(3)利用量词分配等价式，将所有量词提到公式前面(必要时换名).

第二类型Skolem范式

- 生成方法：去掉存在量词
- 例： $\exists x\ \forall y\ \forall z\ \exists u\ \forall v\ \exists w\ P(x,y,z,u,v,w)$

- 定理 2：公式A永假 当且仅当 其Skolem范式永假。

Homework:

P.64 5,7,21

CHAPTER 1

The Foundations: Logic and Proof, Sets, and Functions

Formal Reasoning in Predicate Logic

Rules of Inference for Quantifiers

- **Universal Instantiation (全称量词消去)**
- **Universal Generalization （全称量词引入）**
- **Existential Instantiation （存在量词消去）**
- **Existential Generalization （存在量词引入）**

(1) Universal Instantiation

$$\frac{\forall x P(x)}{\therefore P(c) \text{ if } c \in U}$$

(2) Universal Generalization

$P(c)$ **for an arbitrary** $c \in U$

$\therefore \forall x P(x)$

(3) Existential Instantiation

$\exists x P(x)$

$\therefore P(c)$ **for some element** $c \in U$

(4) Existential Generalization

$P(c)$ **for some element** $c \in U$

$\therefore \exists x P(x)$

〖Example 8〗Show that the premises "*All men are mortal*（凡人）" and "*Socrates is a man*" imply the conclusion" *Socrates is mortal*".

solution:

Let *H*(*x*) denote "*x* is mortal" and let *M*(*x*) denote "*x* is a man" , and let *s* denote Socrates .

Then the premises are $\forall x(M(x) \to H(x))$ **and *M*(*s*), the conclusion is *H*(*s*).**

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(M(x) \to H(x))$	**Premises**
2. $M(s) \to H(s)$	**UI using step 1**全称量词消去
3. $M(s)$	**Premises**
4. $H(s)$	**Modus ponens from step 2 and 3** 假言推理

〖Example 9〗 Consider the following logical argument, is it valid?
Each member of the committee is teacher and expert. Some members of the committee are younger. Therefore, some members of the committee are young experts.

solution:

Let *F*(*x*) be "*x* is a member of the committee", let *G*(*x*) be "*x* is an expert", let *H*(*x*) be "*x* is a teacher", and let *R*(*x*) be " *x* is a younger."

Then the premises is $\forall x(F(x) \to G(x) \wedge H(x)) \wedge \exists x(F(x) \wedge R(x))$

The conclusion is $\exists x(F(x) \wedge G(x) \wedge R(x))$

Step		Reason
(1)	$\exists x(F(x) \wedge R(x))$	**Premises**
(2)	$F(c) \wedge R(c)$	**Existential instantiation using Step 1**
(3)	$\forall x(F(x) \to G(x) \wedge H(x))$	**Premise**
(4)	$F(c) \to G(c) \wedge H(c)$	**Universal instantiation using Step 3**
(5)	$F(c)$	**Simplification using Step 2**
(6)	$G(c) \wedge H(c)$	**Modus ponens using Steps 4 and 5**
(7)	$G(c)$	**Simplification using Step 6**
(8)	$R(c)$	**Simplification using Step 2**
(9)	$F(c) \wedge G(c) \wedge R(c)$	Conjunction using Step 5 and 7 and 8
(10)	$\exists x(F(x) \wedge G(x) \wedge R(x))$	**EG using Step 9**

〖Example 10〗Determine whether the following argument is valid.

(1)	$\forall x \exists y G(x, y)$	Premise
(2)	$\exists y G(a, y)$	UI using Step 1
(3)	$G(a, c)$	EI using Step 2
(4)	$\forall x G(x, c)$	UG using Step 3
(5)	$\exists y \forall x G(x, y)$	EG using Step 3

Homework:

P.78 3 , 9(b,d), 10 (b),13

CHAPTER 1

The Foundations: Logic and Proof

Methods Of Proving Theorems 证明定理的方法

A *theorem* is a *valid* logical assertion which can be proved using

- **other theorems**
- ***axioms*** **（*公理*）(statements which are given to be true) and**
- ***rules of inference*** **(logical rules which allow the deduction of conclusions from premises).**

We will discuss the different types of statement and the methods to prove them.

1. **Direct proof** 直接证明
2. **Indirect proof** 间接证明
3. **Vacuous proof** 空证明
4. **Trivial proof** 平凡证明
5. **Proof by contradiction** 反证法
6. **Proof by Cases** 分情形证明
7. **Existence Proof** 存在性证明
8. **Disproof by Counterexample** 通过反例反证
9. **Nonexistence Proof** 不存在性证明
10. **Universally Quantified Assertions** 普遍量化的断言

1. *Direct proof*

Goal: **To establish that $p \rightarrow q$ is true.**
p may be a conjunction of other hypotheses.

To establish that $p \rightarrow q$ is true.

- **assumes the hypotheses are true**
- **uses the rules of inference, axioms and any logical equivalences to establish the truth of the conclusion.**

〖Example 11〗 Give a direct proof of the theorem "If n is odd, then n^2 is odd."

Proof:

Assume that the hypothesis of this implication is true, namely, suppose that n is odd.

Then $n = 2k + 1$, where k is an integer.

It follows that

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Therefore,

n^2 is odd (it is 1 more than twice an integer).

2. *Indirect proof*

Goal: **To establish that $p \rightarrow q$ is true.**

Recall: $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

Using the method of *indirect proof* to establish that $p \rightarrow q$ is true.

- **assumes the conclusion of $p \rightarrow q$ is false ($\neg q$ is true)**
- **uses the rules of inference, axioms and any logical equivalences to establish the premise p is false.**

Note:

In order to show that a conjunction of hypotheses is false is suffices to show just one of the hypotheses is false.

〖Example 12〗Theorem: *A perfect number is not a prime.*

A *perfect* number is one which is the sum of all its divisors except itself. （如果一个数的所有约数之和正好等于这个数，那么这个数就是完美数）

For example, 6 is perfect since 1 + 2 + 3 = 6.

Proof:

We assume the number *s* is a prime and show it is not perfect.

But the only divisors of a prime are 1 and itself.

Hence the sum of the divisors less than *s* is 1 which is not equal to *s*.

Hence *s* cannot be perfect.

3. *Vacuous proof*

Goal: **To establish that $p \rightarrow q$ is true.**

If one of the hypotheses in p is false then $p \rightarrow q$ is *vacuously* true.

> **Using the method of *vacuous proof* to establish that $p \rightarrow q$ is true.**
>
> - **Show that p is false**

〖Example 13〗If Tom is both handsome and ugly then he feels unhappy.

Solution:

This is of the form $(p \wedge \neg p) \rightarrow q$.

The hypotheses form a contradiction.

Hence q follows from the hypotheses vacuously.

4. *Trivial proof*

Goal: **To establish that $p \rightarrow q$ is true.**

If we know q is true then $p \rightarrow q$ is true.

> **Using the method of *trivial proof* to establish that $p \rightarrow q$ is true.**
>
> **■ Show that q is true.**

〖Example 14〗If the earth is smaller than moon then the void set is a subset of every set .

Solution:

The assertion is *trivially* true independent of the truth of p.

5. *Proof by contradiction*

Goal: **To establish the truth of the 'theorem' p.**

Using the method of *proof by contradiction* to establish the truth of the 'theorem' p

- **assumes the conclusion p is false**
- **derives a contradiction, usually of the form $q \wedge \neg q$ which establishes $\neg p \rightarrow F$.**

Note: An indirect proof of an implication can be rewritten as a proof by contradiction.

〖Example 15〗Theorem: There is no largest prime number.

Proof:

Let p by the proposition 'there is no largest prime number' .

Suppose that $\neg p$ is true, namely, there is a largest prime number, denoted by s.

Hence, the set of all primes lie between 1 and s.

Form the product of these primes:

$r = 2 \bullet 3 \bullet 5 \bullet 7 \bullet 11 \bullet .... \bullet s$.

But $r + 1$ is a prime larger than s. Why?

This is a contradiction since we have shown that $\neg p$ implies both q and $\neg q$ where q is the statement that s is the largest prime number.

Hence, $\neg p$ is false, so that p: 'there is no largest prime number' is true.

6. *Proof by Cases*

Goal: **To prove an implication of the form**

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q \ .$$

Recall:

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q \Leftrightarrow (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)$$

Each of the implications $p_i \rightarrow q$ is a case.

To show that $(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$

- **establish all implications $p_i \rightarrow q$**

〖Example 16〗 Prove that if n is an integer not divisible by 3, then $n^2 \equiv 1 (\text{mod } 3)$.

Proof:

p: n an integer is not divisible by 3

q: $n^2 \equiv 1 (\text{mod } 3)$

Then p is equivalent to $p_1 \vee p_2$, where p_1 is "$n \equiv 1 (\text{mod } 3)$" and p_2 is"$n \equiv 2 (\text{mod } 3)$".

Hence, to show that $p \rightarrow q$ it can be shown that $p_1 \rightarrow q$ and $p_2 \rightarrow q$.

It is easy to give direct proves of those two implications.

Question:

(1) How to prove the proposition "*p* if and only if *q*"?

(2) How to prove that several propositions $p_1, p_2, \ldots, p_n$ are equivalent ?

To show that $p_1, p_2, \ldots, p_n$ are equivalent

- **establish the implications $p_1 \rightarrow p_2$, ..., $p_{n-1} \rightarrow p_n$, $p_n \rightarrow p_1$**

7. *Existence Proofs*

Goal: **To establish the truth of $\exists x P(x)$.**

(1) *Constructive* existence proof

Using *constructive* existence proof to establish the truth of $\exists x P(x)$.

- **Establish *P(c)* is true for some *c* in the universe.**
- **Then $\exists x P(x)$ is true by Existential Generalization (EG).**

〖Example 17〗 Show that there are n consecutive composite positive integers for every positive integer n.

Proof: $\forall n \exists x (x + i \text{ is composite for } i = 1, 2, \ldots, n)$.

Let $x = (n + 1)! + 1$.

Consider the integers $x + 1, x + 2, \ldots, x + n$.

Note that $i + 1$ divides $x + i = (n + 1)! + (i + 1)$ for $i = 1, 2, \ldots, n$.

Hence, n consecutive composite positive integers have been given.

Note that in the solution a number x such that $x + i$ is composite for $i = 1, 2, \ldots, n$ has been produced.

Hence, this is an example of constructive existence proof.

(2) *Nonconstructive* existence proof

Using *nonconstructive* existence proof to establish the truth of $\exists x P(x)$.

- **Assume no c exists which makes $P(c)$ true and derive a contradiction**

〖Example 18〗*Theorem: There exists an irrational number.*

Proof:

Assume there doesn't exist an irrational number. Then all numbers must be rational.

Then the set of all numbers must be countable.

Then the real numbers in the interval [0, 1] is a countable set.

But we have already shown this set is not countable.

Hence, we have a contradiction (The set [0,1] is countable and not countable).

Therefore, there must exist an irrational number.

8. *Disproof by Counterexample*

Goal: **To establish that $\neg\forall x P(x)$ is true (or $\forall x P(x)$ is false).**

Recall: $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$

Using the method of *disproof by counterexample* to establish that $\neg\forall x P(x)$ is true.

- **To construct a c such that $\neg P(c)$ is true or $P(c)$ is false.**

9. *Nonexistence Proofs*

Goal: **To establish that $\neg\exists x P(x)$ is true .**

Recall: $\neg\,\exists x\, P(x) \Leftrightarrow \forall\, x \neg\, P(x)$

To establish that $\neg\exists x P(x)$ is true .

- **Use a proof by contradiction by assuming there is a c which makes $P(c)$ true .**

10. *Universally Quantified Assertions*

Goal: **To establish the truth of $\forall x P(x)$.**

To establish the truth of $\forall x P(x)$.

- **We assume that x is an arbitrary member of the universe and show $P(x)$ must be true.**
- **Using UG it follows that $\forall x P(x)$.**

〖Example 19〗*Theorem: For the universe of integers, x is even iff x^2 is even.*

Proof:

$\forall x$[x is even $\leftrightarrow$ x^2 is even] .

Recall that $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.

Case 1. *sufficiency*

Show that if x is even then x^2 is even using a direct proof .

Case 2. *necessity*

We use an indirect proof.

Assume x is not even and show x^2 is not even.