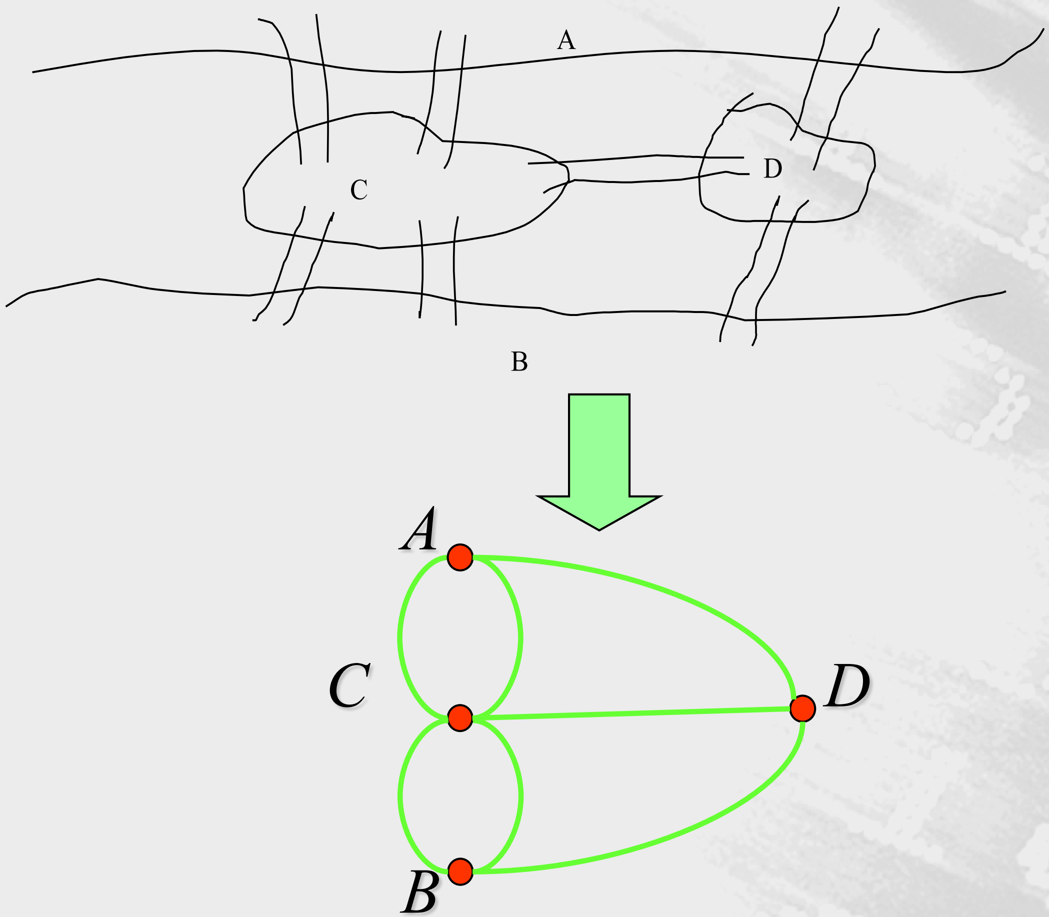
CHAPTER 9 Graphs

9.5.1 Euler Paths

Konigsberg Seven Bridge Problem 哥尼斯堡七桥问题



- 定义1　设G是一个图，G中包含所有边的迹(即每条边恰好出现一次的路径)称为Euler迹，闭的Euler迹称为Euler闭迹或Euler回路，具有Euler回路的图称为Euler图，开的Euler迹称为Euler开迹，具有Euler开迹的图称为半Euler图。

Terminologies:

- *Euler Circuit*

  图G里的欧拉回路是包含着G的每一条边的简单回路.

- *Euler Path*

  图G里的欧拉通路是包含着G的每一条边的简单通路

- *Euler Graph*

  A graph contains an Euler circuit.

Necessary and sufficient condition for Euler circuit and paths

欧拉回路和欧拉通路的充要条件

【 Theorem 1】连通多重图具有欧拉回路当且仅当它的每个顶点都有偶数度

Proof:

(1) Necessary condition必要条件

G has an Euler circuit $\Rightarrow$ Every vertex in V has even degree

Consider the Euler circuit.

- **the vertex a which the Euler circuit begins with**
- **the other vertex**

(2) sufficient condition

We will form a simple circuit that begins at an arbitrary vertex a of G.

- **Build a simple path $x_0=a, x_1, x_2,\ldots,x_n=a$.**
- **An Euler circuit has been constructed if all the edges have been used. otherwise,**
- **Consider the subgraph H obtained from G.**

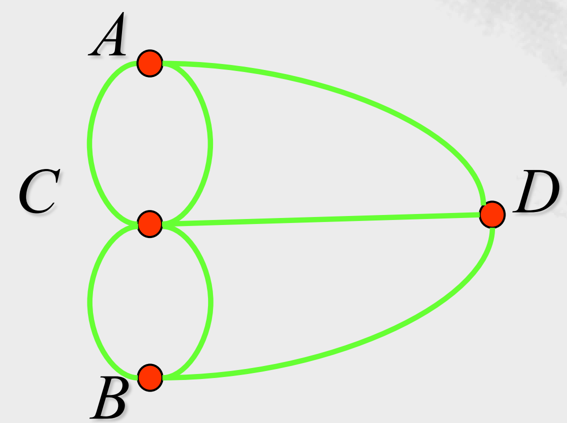
  Let w be a vertex which is the common vertex of the circuit and H.

  Beginning at w, construct a simple path in H.

【 Theorem 2】 连通多重图具有欧拉通路而无欧拉回路，当且仅当它恰有两个奇数度顶点

〖Example 1〗 Konigsberg Seven Bridge Problem

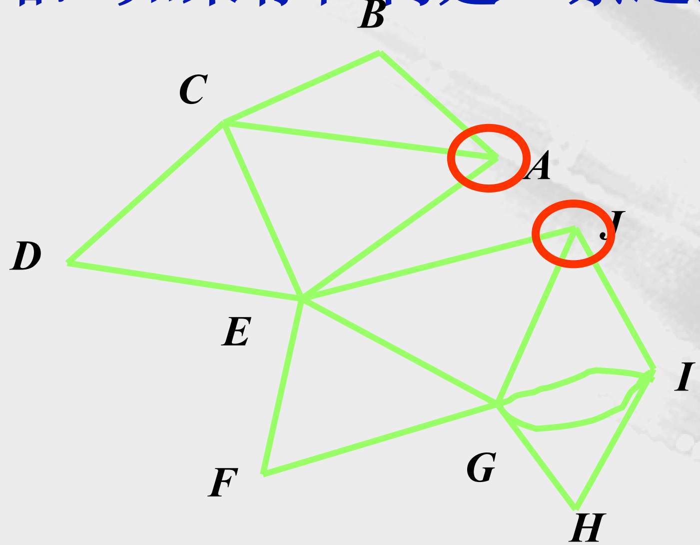
哥尼斯堡七桥问题



Solution:

The graph has four vertices of odd degree. Therefore, it does not have an Euler circuit.欧拉回路

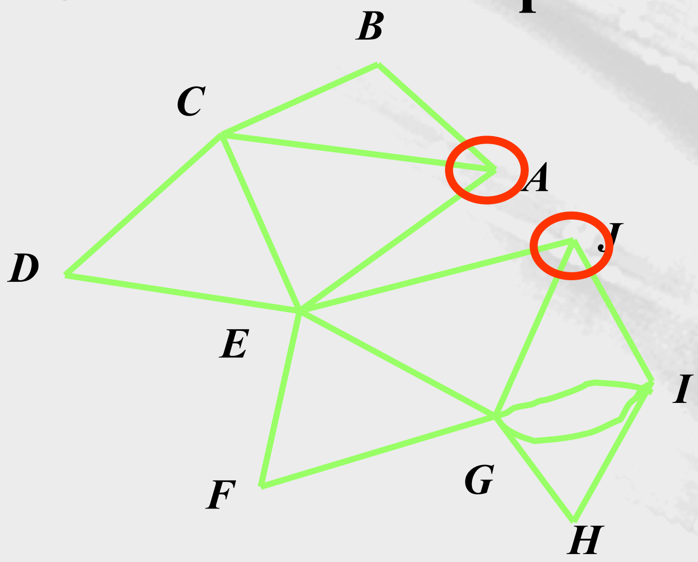
〖Example 2〗 Determine whether the following graph has an Euler path. Construct such a path if it exists.判断下图是否具有欧拉通路，如果存在构建一条通路



Solution:

The graph has 2 vertices of odd degree, and all of other vertices have even degree . Therefore, this graph has an Euler path.

〖Example 2〗 Determine whether the following graph has an Euler path. Construct such a path if it exists.



The Euler path:

A,C,E,F,G,I,J,E,A,B, C,D,E,G, H,I,G,J

Solution:

The graph has 2 vertices of odd degree, and all of other vertices have even degree . Therefore, this graph has an Euler path.

Euler circuit and paths in directed graphs

有向图中的欧拉回路与欧拉通路

A directed multigraph having no isolated vertices has an Euler circuit if and only if 一个没有孤立顶点的有向多重图含有欧拉回路的充要条件是：

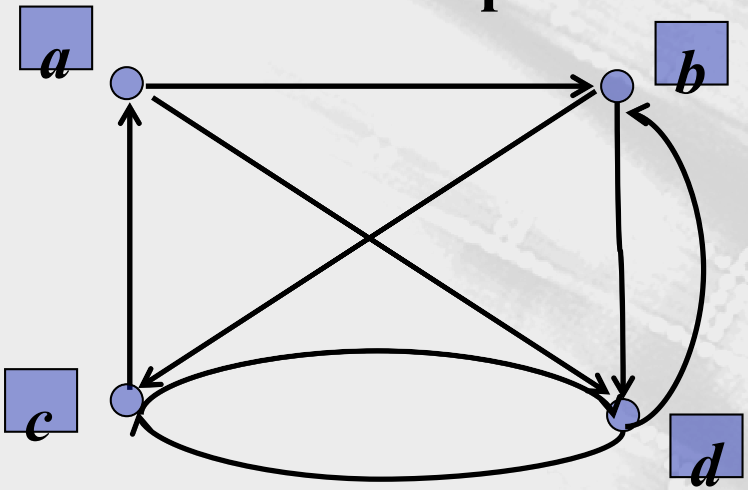
- **the graph is weakly connected** 弱连通的
- **the in-degree and out-degree of each vertex are equal** 每个顶点的出度和入度相等

A directed multigraph having no isolated vertices has an Euler path but not an Euler circuit if and only if一个没有孤立顶点的有向多重图含有欧拉通路但不含欧拉回路的充要条件是：

-- the graph is weakly connected 弱连通的

-- the in-degree and out-degree of each vertex are equal for all but two vertices, one that has in-degree 1 larger than its out-degree and the other that has out-degree 1 larger than its in-degree.除去两个顶点外每个顶点的出度和入度相等，其中一个顶点的出度比入度大1，另一个顶点的入度比出度大1.

〖Example 3〗 Determine whether the directed graph has an Euler path. Construct an Euler path if it exists.



Solution:

	$deg^-(v)$	$deg^+(v)$
a	1	2
b	2	2
c	2	2
d	3	2

Hence, the directed graph has an Euler path.

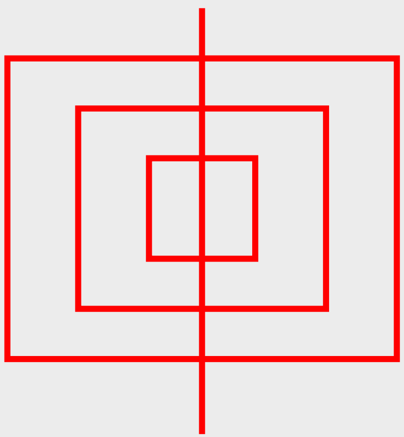
Application

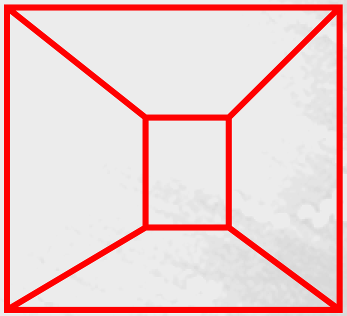
A type of puzzle

Draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced.

The equivalent problem: Whether the graph exist an Euler path or circuit.

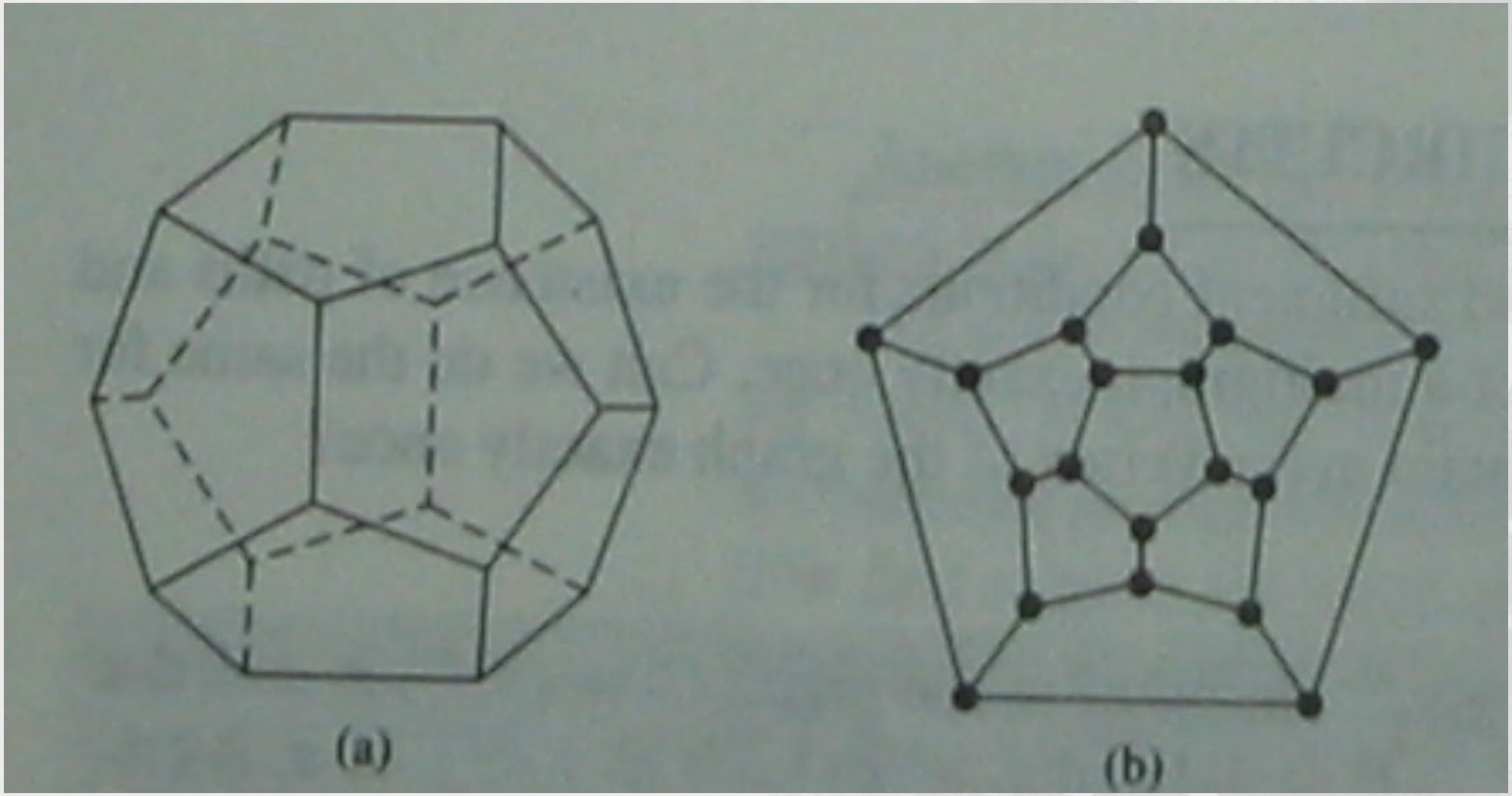
For example,





9.5.2 Hamilton paths and circuit 哈密顿通路和回路

Hamilton's puzzle



(a) (b)

A *Hamilton path* in a graph *G* is a path which visits ever vertex in *G* exactly once.

哈密顿通路是一个访问图G中每个顶点次数有且仅有一次的通路

A *Hamilton circuit* (or *Hamilton cycle*) is a cycle which visits every vertex exactly once, *except for the first vertex*, which is also visited at the end of the cycle.

哈密顿回路，仅访问每个顶点一次，但除去始点，这个始点同样也是终点。

If a connected graph *G* has a Hamilton circuit, then *G* is called a *Hamilton graph*.

如果一个连通图G含有哈密顿回路，那么G是哈密顿图

Note: 定义适用与所有类型的有向图和无向图.

The sufficient condition for the existence of Hamilton path and Hamilton circuit哈密顿通路和哈密顿回路存在的充分条件

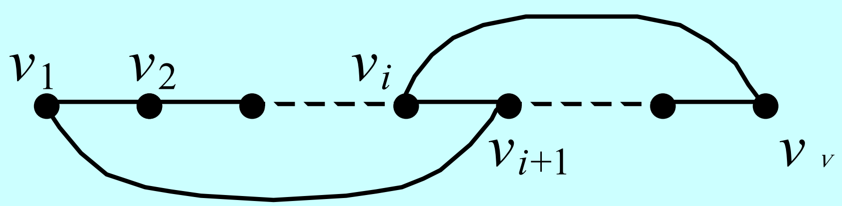
【 Theorem 3】 DIRAC'THEOREM 狄拉克定理

如果G是带n个顶点的连通简单图，其中 n>=3，则G有哈密顿回路的充分条件是每个顶点的度都至少为 $n/2$

证明 假设G不是Hamilton图，设G为极大非Hamilton图；

$G+uv$是Hamilton图;

$d(u)+d(v)=|S|+|T|=|S\cup T|<u$ 。



引理 设G是简单图，顶点u和v一对不相邻的顶点，且满足 $deg(u)+deg(v) >= n$ 则G是哈密顿图当且仅当G+uv是哈密顿图

The sufficient condition for the existence of Hamilton path and Hamilton circuit哈密顿通路和哈密顿回路存在的充分条件

引理 设G是简单图，顶点u和v一对不相邻的顶点，且满足deg(u)+deg(v) >= n 则G是哈密顿图当且仅当G+uv是哈密顿图

【 Theorem 4】ORE'THEOREM 奥尔定理

If G is a simple graph with n vertices with n>=3 such that deg(u)+deg(v) >= n for every pair of nonadjacent vertices u and v in G , then G has a Hamilton circuit.

如果G是带n个顶点的连通简单图，其中n>=3 ，并且对于G中每一对不相邻的顶点u和v来说，都有deg(u)+deg(v) >= n ，则G有哈密顿回路。

- 设G是一个图，反复连接满足$d(u)+d(v)\geq u$的不相邻顶点u，v，直到没有这样的顶点对为止，这样得到的图称作图G的闭包，记为$C(G)$。
- 定理 3　简单图G是Hamilton图当且仅当$C(G)$是Hamilton图。

- 推论1　若$C(G)$是完全图，则G是Hamilton图。
- 推论 2　若G中任意不相邻顶点u，v均满足$d(u)+d(v)\geq u$，则G是Hamilton图。

The necessary condition for Hamilton path and Hamilton circuit

For undirected graph:

The necessary condition for the existence of Hamilton path:

- ***G* is connected;**
- **There are at most two vertices which degree are less than 2.**

The necessary condition for the existence of Hamilton circuit:

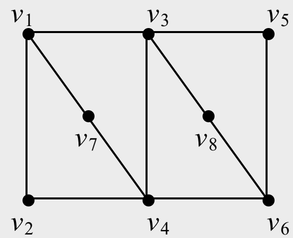
- **The degree of each vertex is larger than 1.**

Remark Certain properties can be used to show that a graph has no Hamilton circuit.

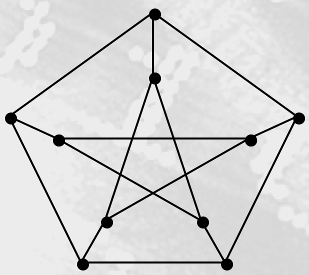
- **A graph with a vertex of degree 1 cannot have a Hamilton circuit.**
- **If a vertex in the graph has degree 2, then both edges that are incident with this vertex must be part of any Hamilton circuit.**
- **A Hamilton circuit cannot contain a smaller circuit within it.**

- 哈密顿图的必要条件
- 定理1　设G是Hamilton图，则对于顶点集V的任一非空真子集S，均有
- $\omega(G-S)\leq|S|$。
- 这里G-S表示从图G中删去S中的所有顶点以及所关联的边

◈ 例1　在下图(a)中，取S＝{$v1$，$v4$}，则G－S有3个连通分支，故该图不是Hamilton图。



(a)


(b)

〖Example 5〗 There are seven people denoted by A, B, C, D, E, F, G. Suppose that the following facts are known.

A--English (A can speak English.)

B--English, Chinese

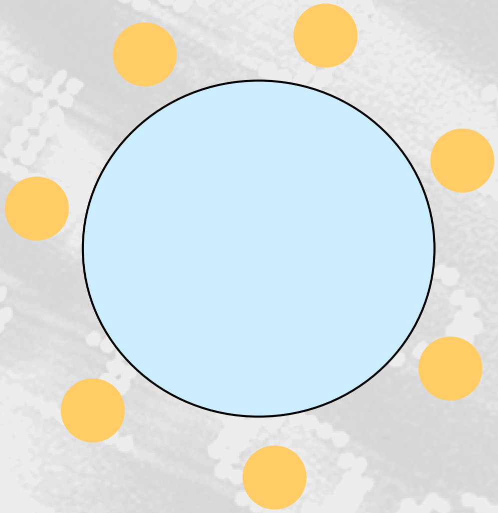
C--English, Italian, Russian

D--Japanese, Chinese

E--German, Italia

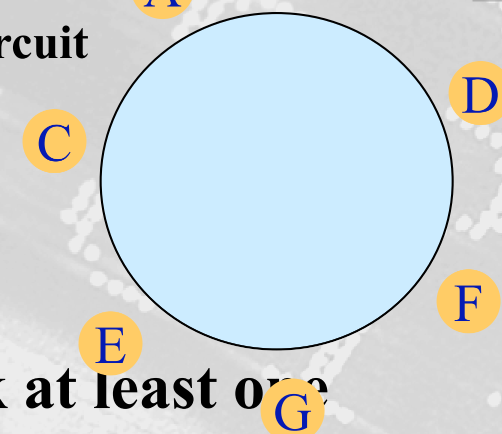
F--French, Japanese, Russian

G--French, German



How to arrange seat for the round desk such that the seven people can talk each other?

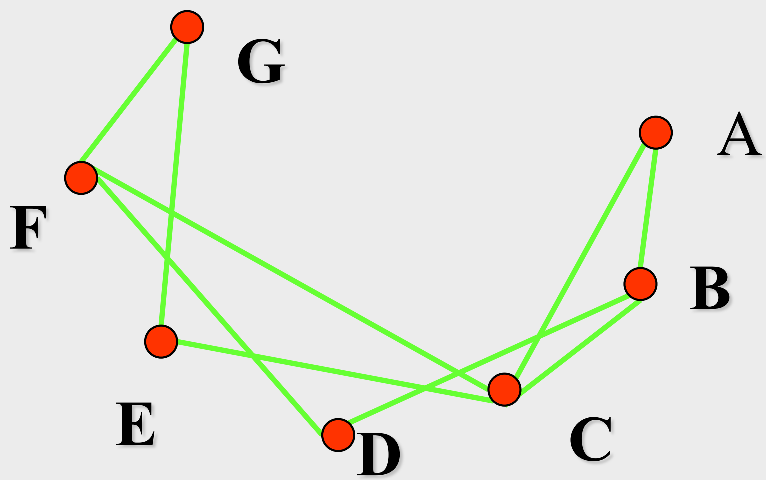
9.5.2 Hamilton paths and circuit



Solution:

(1) Construct graph

V={A,B,C,D,E,F,G}, E={(u,v)|u,v can speak at least one common language.}



A--English (A can speak English.)

B--English, Chinese

C--English, Italian, Russian

D--Japanese, Chinese

E--German, Italia

F--French, Japanese, Russian

G--French, German

(2) If there is a H circuit, then we can arrange seat for the round desk such that the seven people can talk each other.

H circuit: A,B,D,F,G,E,C,A

- 定义 2　设D是有向图，D中包含所有顶点的有向圈称为Hamilton有向圈，含有Hamilton有向圈的有向图称为Hamilton有向图，D中包含所有顶点的有向路，称为Hamilton有向路，含有Hamilton有向路的有向图称为半Hamilton有向图。
- Hamilton有向图必定是强连通的。

- 若有向图D中每两个顶点之间恰有一条弧，则称D为竞赛图

- D是竞赛图$\Leftrightarrow$$D$是完全图的定向图
- 下面的两个定理，指出了竞赛图与Hamilton图的关系。
- 定理　竞赛图必是半Hamilton有向图。
- 定理　强连通的竞赛图必是Hamilton有向图。