Modal analysis of a dynamical system using Proper Orthogonal Decomposition (POD) and Discrete Fourier Transform (DFT)

Project description

The aim of this project is to apply the POD and the DFT to analyse a dynamical system. POD [1] (also known as PCA or KLT) is a decomposition technique which aims at finding the structure in the dataset associated with the maximum energy. It is often used in the construction of Reduced Order Models (ROMs), because it allows the reduction of the system's dimensionality.

The POD is generally performed by applying the singular value decomposition to the dataset:

$$\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \tag{1}$$

where $\mathbf{X}$ is the data matrix, $\mathbf{U}$ and $\mathbf{V}$ contain the spatial and temporal modes, and $\mathbf{\Sigma}$ is a diagonal matrix containing the singular values.

The goal of DFT [2] is to decompose the original dataset into a series of harmonics, using the Fourier tansform:

$$\mathbf{X} = \mathbf{\Phi}_F\mathbf{\Sigma}_F\mathbf{\Psi}_F \tag{2}$$

The DFT is not a data-driven decomposition technique, because the new basis is independent from the dataset. However, it remains very important in the study of quasi-periodic signals, because it can highlight the dominant frequencies in the dataset.

The objective of this work is to apply POD and DFT to two different dynamical systems: the first one is a simple damped harmonic oscillator, while the second one is represented by a pair of pulsating flame. In both cases, the solution should include a critical assessment of the POD and DFT modes retrieved, highlighting the pros and cons of one approach over the other.

Optionally, if you are feeling brave, you can try to apply a Convolutional AutoEncoder (CAE) [3] to the study of the pulsating flames. The CAE is a type of convolutional neural network that contains an encoder, which compresses the dataset into the latent space, and a decoder that projects the dataset from the latent space to the original space. By exploiting the non-linearity of the CAE, you should be able to achieve a better compression, and a better reconstruction accuracy, than both POD and DFT.

Tasks

In this project, you will complete the following tasks:

Task 1: The damped harmonic oscillator is a simple dynamical system in which the forces experienced by the mass are the restoring force of the spring and the dampening force of friction:

$$m\frac{d^2s}{dt^2} + c\frac{ds}{dt} + ks = 0 \tag{3}$$

where m is the mass, c is the dampening coefficient, k is the spring constant and s is the axis of movement. The system has an analytic solution:

$$s(t) = A_0e^{-c/2m\ t}\cos(\omega t + \phi) \tag{4}$$

where A_0 and ϕ depend on the initial conditions and $\omega = \sqrt{k/m - (c/2m)^2}$.

The steps to solve the first exercise are:

- Construct an underdamped ($c < \sqrt{4mk}$) harmonic oscillator with your choice of characteristic frequency and starting conditions.
- Build a synthetic dataset by pretending that the position of the mass is tracked in time by two sensors in the x, y directions. Each sensor has a 20% uncertainty (random white noise) and the angle between the s and x axis is equal to $\theta = 35°$, as reported in Fig. 1. Collect the position for $6\ \tau$, where $\tau = 2m/c$ is the characteristic time length.

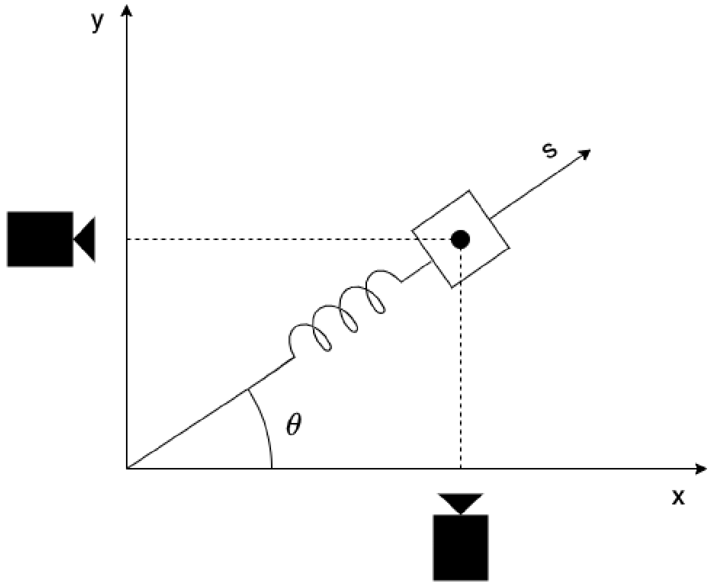

Figure 1: Sketch of the virtual experiment.

- Apply the POD and DFT algorithms to the data matrix $\mathbf{X}$ containing the x and y positions. To perform the POD and DFT, you can use the standard python libraries (`numpy.linalg.svd()` for the SVD algorithm and `numpy.fft.fft()` to perform the DFT). Or, you can use the `modulo-vki` package that you can find here, along with the instruction for the installation and the documentation.
- Compare the POD and DFT results.

Task 2: The second dataset [4] contains a series of swirling flames' images, that are forced using a 100 Hz and a 290 Hz signal. The forcing is achieved by modulating the air flow using a periodic signal.

In the study of turbulent flames, an external forcing is often employed to study the response of the flame to external excitements. The use of modal analysis techniques such as POD and DFT can be very helpful to isolate the coherent flow structures and to study the flame's behavior.

The steps to follow to solve the exercise are:

- Download the pulsating flames' datasets from the UV. Each dataset contains 1024 64×64 images, with a sampling frequency of 2500 Hz. You can load the datasets using the `numpy.load()` command.
- Apply the POD and DFT algorithms to the datasets as in the previous task.
- (*Optional*) Build and apply the CAE to the dataset. To build the CAE, you can use `PyTorch` (more complex but more flexible), or `TensorFlow` (easier to use as a black box). If you use `Colab`, you can also request GPU resources to train the CAE.
- Compare the results.

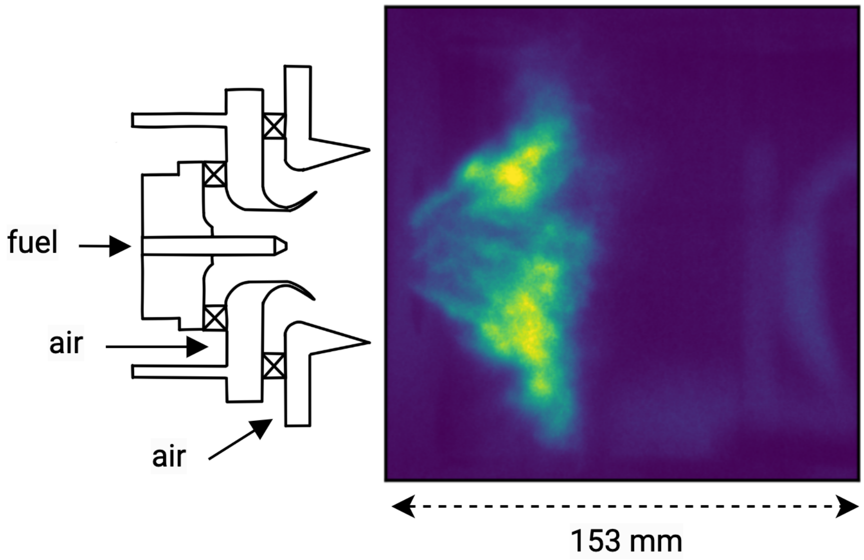

Figure 2: Schematic representation of the experimental apparatus used for the swirling forced flames.

Written report

Your written report should contain the following:

Section 1: A brief introduction of the mathematical formulation of POD and DFT (also the CAE if you chose to build one). In particular, highlight the computational advantages and drawbacks of the two (three) techniques.

Section 2: Results corresponding to **Tasks 1**, compared to the analytical solution.

Section 3: Results corresponding to **Tasks 2**, including relevant figures and a discussion of the differences between POD and DFT (and CAE).

Section 4: Final discussion and conclusions, highlighting the role of these decomposition techniques in the study of dynamical systems. Don't forget to cite the relevant literature to strengthen your conclusions.

Resources

You can find a series of introductory videos on modal analysis from Pr. Mendez here.

Regarding the CAE, a simple introduction on autoeconders can be found here, while a more in-depth discussion on autoencoders is contained in this book [5].

The papers and books introducing POD and DFT are cited in the bibliography. If you encounter problems in the download of the data or installation of the libraries you can contact Alberto Procacci on TEAMS or at alberto.procacci@ulb.be.

References

[1] G Berkooz, P Holmes, and J L Lumley. The proper orthogonal decomposition in the analysis of turbulent flows. *Annual Review of Fluid Mechanics*, 25(1):539–575, 1993.

[2] Julius O. Smith. *Mathematics of the Discrete Fourier Transform (DFT)*. W3K Publishing, `http://www.w3k.org/books/`, 2007.

[3] Noriyasu Omata and Susumu Shirayama. A novel method of low-dimensional representation for temporal behavior of flow fields using deep autoencoder. *AIP Advances*, 9(1), 2019.

[4] Antoine Renaud, Sébastien Ducruix, and Laurent Zimmer. Experimental Study of the Precessing Vortex Core Impact on the Liquid Fuel Spray in a Gas Turbine Model Combustor. *Journal of Engineering for Gas Turbines and Power*, 141(11), 10 2019. 111022.

[5] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep Learning*. MIT Press, 2016. `http://www.deeplearningbook.org`.