CHAPTER 21

Laplace Transforms

21.1 INTRODUCTION

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850–1925), to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by Bromwich and Carson during 1916–17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.*

The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready tables of Laplace transforms reduce the problem of solving differential equations to mere algebraic manipulation.

21.2 (1) DEFINITION

Let $f(t)$ be a function of t defined for all positive values of t. Then the **Laplace transforms** *of $f(t)$, denoted by $L\{f(t)\}$ is defined by*

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad \text{...(1)}$$

provided that the integral exists. s is a parameter which may be a real or complex number.

$L\{f(t)\}$ being clearly a function of s is briefly written as $\bar{f}(s)$ *i.e.*, $L\{f(t)\} = \bar{f}(s)$,

which can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$.

Then $f(t)$ is called the **inverse Laplace transform** of $\bar{f}(s)$. The symbol L, which transforms $f(t)$ into $\bar{f}(s)$, is called the *Laplace transformation operator.*

**Pierre de Laplace* (1749–1827) (See footnote p. 18) used such transforms, much earlier in 1799, while developing the theory of probability.

(2) Conditions for the existence

The Laplace transform of $f(t)$ *i.e.,* $\int_0^\infty e^{-st} f(t)\,dt$ *exists for* $s > a$, *if*

(i) $f(t)$ *is continuous* *(iii)* $\underset{t\to\infty}{\text{Lt}} \{e^{-at} f(t)\}$ *is finite.*

It should however, be noted that the above conditions are sufficient and not necessary.

For example, $L(1/\sqrt{t}\,)$ exists, though $1/\sqrt{t}$ is infinite at $t = 0$.

21.3 TRANSFORMS OF ELEMENTARY FUNCTIONS

The direct application of the definition gives the following formulae :

(1) $\mathbf{L\,(1) = \dfrac{1}{s}}$ $(s > 0)$

(2) $\mathbf{L\,(t^n) = \dfrac{n!}{s^{n+1}}}$, when $n = 0, 1, 2, 3, \ldots$ $\left[\text{Otherwise } \dfrac{\Gamma(n+1)}{s^{n+1}}\right]$

(3) $\mathbf{L\,(e^{at}) = \dfrac{1}{s-a}}$ $(s > a)$

(4) $\mathbf{L\,(\sin at) = \dfrac{a}{s^2+a^2}}$ $(s > 0)$

(5) $\mathbf{L\,(\cos at) = \dfrac{s}{s^2+a^2}}$ $(s > 0)$

(6) $\mathbf{L\,(\sinh at) = \dfrac{a}{s^2-a^2}}$ $(s > |\,a\,|)$

(7) $\mathbf{L\,(\cosh at) = \dfrac{s}{s^2-a^2}}$ $(s > |\,a\,|)$

Proofs. **(1)**
$$L\,(1) = \int_0^\infty e^{-st}\,.\,1\,dt = \left| -\frac{e^{-st}}{s} \right|_0^\infty = \frac{1}{s} \text{ if } s > 0.$$

(2)
$$L\,(t^n) = \int_0^\infty e^{-st}\,.\,t^n\,dt = \int_0^\infty e^{-p}\,.\left(\frac{p}{s}\right)^n \frac{dp}{s}, \text{ on putting } st = p$$
$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p}\,.\,p^n\,dp = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } n > -1 \text{ and } s > 0. \text{ [Page 302]}$$

In particular $L\,(t^{-1/2}) = \dfrac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\dfrac{\pi}{s}}$; $L\,(t^{1/2}) = \dfrac{\Gamma(3/2)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2\,s^{3/2}}$

In n be a positive integer, $\Gamma(n+1) = n\,!$ [(*v*) p. 302],

therefore, $L\,(t^n) = n\,!/s^{n+1}$.

(3)
$$L\,(e^{at}) = \int_0^\infty e^{-st}\,.\,e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left| \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty = \frac{1}{s-a}, \text{ if } s > a.$$

(4)
$$L\,(\sin at) = \int_0^\infty e^{-st} \sin at\,dt = \left| \frac{e^{-st}}{s^2+a^2}(-\,s \sin at - a \cos at) \right|_0^\infty = \frac{a}{s^2+a^2}$$

Similarly, the reader should prove **(5)** himself.

(6)
$$L\,(\sinh at) = \int_0^\infty e^{-st} \sinh at\,dt = \int_0^\infty e^{-st} \left(\frac{e^{at}-e^{-at}}{2}\right) dt$$
$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2} \text{ for } s > |\,a\,|.$$

Similarly, the reader should prove **(7)** himself.

21.4 PROPERTIES OF LAPLACE TRANSFORMS

I. Linearity property. *If a, b, c be any constants and f, g, h any functions of t, then*

$$\mathbf{L\,[af(t) + bg(t) - ch(t)] = aL\,\{f(t)\} + bL\,\{g(t)\} - cL\,\{h(t)\}}$$

For by definition,

$$\text{L.H.S.} = \int_0^\infty e^{-st}\,[af(t) + bg(t) - ch(t)]\,dt$$

$$= a\int_0^\infty e^{-st}\,f(t)\,dt + b\int_0^\infty e^{-st}\,g(t)\,dt - c\int_0^\infty e^{-st}\,h(t)\,dt = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

This result can easily be generalised.

Because of the above property of L, it is called a *linear operator*.

II. First shifting property. *If* $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\,\{e^{at} f(t)\} = \bar{f}(s - a)}.$$

By definition, $L\{e^{at} f(t)\} = \int_0^\infty e^{-st}\,e^{at}\,f(t)\,dt = \int_0^\infty e^{-(s-a)t}\,f(t)\,dt$

$$= \int_0^\infty e^{-rt}\,f(t)\,dt, \text{ where } r = s - a = \bar{f}(r) = \bar{f}(s - a).$$

Thus, if we know the transform $\bar{f}(s)$ *of* $f(t)$, *we can write the transform of* $e^{at} f(t)$ *simply replacing s by* $s - a$ *to get* $\bar{f}(s - a)$.

Application of this property leads us to the following useful results :

(1) $\mathbf{L\,(e^{at}) = \dfrac{1}{s - a}}$ $\qquad\left[\because\ L(1) = \dfrac{1}{s}\right.$

(2) $\mathbf{L\,(e^{at}t^n) = \dfrac{n\,!}{(s - a)^{n+1}}}$ $\qquad\left[\because\ L(t^n) = \dfrac{n\,!}{s^{n+1}}\right.$

(3) $\mathbf{L\,(e^{at}\sin bt) = \dfrac{b}{(s - a)^2 + b^2}}$ $\qquad\left[\because\ L(\sin bt) = \dfrac{b}{s^2 + b^2}\right.$

(4) $\mathbf{L\,(e^{at}\cos bt) = \dfrac{s - a}{(s - a)^2 + b^2}}$ $\qquad\left[\because\ L(\cos bt) = \dfrac{s}{s^2 + b^2}\right.$

(5) $\mathbf{L\,(e^{at}\sinh bt) = \dfrac{b}{(s - a)^2 - b^2}}$ $\qquad\left[\because\ L(\sinh bt) = \dfrac{b}{s^2 - b^2}\right.$

(6) $\mathbf{L\,(e^{at}\cosh bt) = \dfrac{s - a}{(s - a)^2 - b^2}}$ $\qquad\left[\because\ L(\cosh bt) = \dfrac{s}{s^2 - b^2}\right.$

where in each case $s > a$.

Example 21.1. *Find the Laplace transforms of*

(i) $\sin 2t \sin 3t$ *(ii)* $\cos^2 2t$ *(iii)* $\sin^3 2t$.

Solution. (*i*) Since $\sin 2t \sin 3t = \frac{1}{2}[\cos t - \cos 5t]$

$$\therefore \quad L(\sin 2t \sin 3t) = \tfrac{1}{2}[L(\cos t) - L(\cos 5t)] = \frac{1}{2}\left[\frac{s}{s^2 + 1^2} - \frac{s}{s^2 + 5^2}\right] = \frac{12s}{(s^2 + 1)(s^2 + 25)}$$

(*ii*) Since $\qquad \cos^2 2t = \dfrac{1}{2}(1 + \cos 4t)$

$$\therefore \quad L(\cos^2 2t) = \frac{1}{2}[L(1) + L\cos 4t] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2 + 16}\right)$$

(*iii*) Since $\sin 6t = 3 \sin 2t - 4 \sin^3 2t$

or $\sin^3 2t = \frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t$

$$\therefore \quad L(\sin^3 2t) = \tfrac{3}{4} L(\sin 2t) - \tfrac{1}{4} L(\sin 6t)$$

$$= \frac{3}{4} \cdot \frac{2}{s^2 + 2^2} - \frac{1}{4} \cdot \frac{2}{s^2 + 6^2} = \frac{48}{(s^2 + 4)(s^2 + 36)}.$$

Example 21.2. *Find the Laplace transform of*

(i) $e^{-3t}(2 \cos 5t - 3 \sin 5t)$. *(ii)* $e^{2t} \cos^2 t$ *(V.T.U., 2006)* *(iii)* $\sqrt{t}e^{3t}$. *(P.T.U., 2009)*

Solution. (*i*) $L\{e^{-3t}(2 \cos 5t - 3 \sin 5t)\} = 2L(e^{-3t} \cos 5t) - 3L(e^{-3t} \sin 5t)$

$$= 2 \cdot \frac{s+3}{(s+3)^2 + 5^2} - 3 \cdot \frac{5}{(s+3)^2 + 5^2} = \frac{2s - 9}{s^2 + 6s + 34}.$$

(*ii*) Since $L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2 + 4}\right\}$

$\therefore$ by shifting property, we get

$$L(e^{2t} \cos^2 t) = \frac{1}{2}\left\{\frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4}\right\}.$$

(*iii*) Since $L(\sqrt{t}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{(1/2) . \Gamma\pi}{s^{3/2}}$

$\therefore$ by shifting property, we obtain $L(e^{3t}\sqrt{t}) = \frac{\sqrt{\pi}}{2} \frac{1}{(s-3)^{3/2}}$.

Example 21.3. *If* $L f(t) = \bar{f}(s)$, *show that*

$$L[(\sinh at) f(t)] = \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)]$$

$$L[(\cosh at) f(t)] = \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$$

Hence evaluate (i) $\sinh 2t \sin 3t$ *(ii)* $\cosh 3t \cos 2t$.

Solution. We have $L[(\sinh at) f(t)] = L\left[\frac{1}{2}(e^{at} - e^{-at}) f(t)\right] = \frac{1}{2}[L\{e^{at} f(t)\} - L\{e^{-at} f(t)\}]$

$= \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)]$, by shifting property.

Similarly, $L[(\cosh at) f(t)] = \frac{1}{2}[L\{e^{at} f(t)\} + \{e^{-at} f(t)\}]$

$= \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$, by shifting property.

(*i*) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, the first result gives

$$L(\sinh 2t \sin 3t) = \frac{1}{2}\left\{\frac{3}{(s-2)^2 + 3^2} - \frac{3}{(s+2)^2 + 3^2}\right\} = \frac{12s}{s^4 + 10s^2 + 169}$$

(*ii*) Since $L(\cos 2t) = \frac{s}{s^2 + 2^2}$, the second result gives

$$L(\cosh 3t \cos 2t) = \frac{1}{2}\left\{\frac{s-3}{(s-3)^2 + 2^2} + \frac{s+3}{(s+3)^2 + 2^2}\right\} = \frac{2s(s^2 - 5)}{s^4 - 10s^2 + 169}.$$

Example 21.4. *Show that*

(*i*) $L(t \sin at) = \dfrac{2as}{(s^2+a^2)^2}$ (*Bhopal, 2001*) (*ii*) $L(t \cos at) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$.

Solution. Since $L(t) = 1/s^2$. $\therefore$ $L(te^{iat}) = \dfrac{1}{(s-ia)^2} = \dfrac{(s+ia)^2}{[(s-ia)(s+ia)]^2}$

or $$L[t(\cos at + i \sin at)] = \frac{(s^2-a)^2 + i(2as)}{(s^2+a^2)^2}$$

Equating the real and imaginary parts from both sides, we get the desired results.

Example 21.5. *Find the Laplace transform of $f(t)$ defined as*

(*i*) $f(t) = t/\tau$, *when* $0 < t < \tau$

$= 1$, *when* $t > \tau$. (*Kerala, 2005*)

(*ii*) $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$ (*J.N.T.U., 2006 ; W.B.T.U., 2005*)

Solution. (*i*) $$Lf(t) = \int_0^\tau e^{-st} \cdot \frac{t}{\tau} dt + \int_\tau^\infty e^{-st} \cdot 1\, dt = \frac{1}{\tau}\left[\left| t \cdot \frac{e^{-st}}{-s}\right|_0^\tau - \int_0^\tau 1 \cdot \frac{e^{-st}}{-s} dt\right] + \left|\frac{e^{-st}}{-s}\right|_\tau^\infty$$

$$= \frac{1}{\tau}\left[\frac{\tau e^{-s\tau} - 0}{-s} - \left|\frac{e^{-st}}{s^2}\right|_0^\tau\right] + \frac{0 - e^{-s\tau}}{-s} = \frac{-e^{-s\tau}}{s} - \frac{e^{-s\tau} - 1}{\tau s^2} + \frac{e^{-s\tau}}{s} = \frac{1 - e^{-s\tau}}{\tau s^2}.$$

(*ii*) $$L\{f(t)\} = \int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} \cdot t\, dt + \int_2^\infty e^{-st} \cdot (0)\, dt$$

$$= \left|\frac{e^{-st}}{-s}\right|_0^1 + \left| t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_1^2 = \frac{1-e^{-s}}{s} + \left\{\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\}$$

$$= \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}.$$

Example 21.6. *Find the Laplace transform of* (*i*) $\left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)^3$. (*Kuruksherta, 2005*)

(*ii*) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$ (*Mumbai, 2009*)

Solution. (*i*) Since $(\sqrt{t} - 1/\sqrt{t})^3 = t^{3/2} - 3t^{1/2} + 3t^{-1/2} - t^{-3/2}$

$\therefore$ $$L(\sqrt{t} - 1/\sqrt{t}) = L(t^{3/2}) - 3L(t^{1/2}) + 3L(t^{-1/2}) - L(t^{-3/2})$$

$$= \frac{\Gamma(3/2+1)}{s^{3/2+1}} - 3\frac{\Gamma(1/2+1)}{s^{1/2+1}} + 3\frac{\Gamma(-1/2+1)}{s^{-1/2+1}} - \frac{\Gamma(-3/2+1)}{s^{-3/2+1}}$$

$$= \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} - 3\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} + 3\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}}$$

$$= \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}} - \frac{3}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{2\sqrt{\pi}}{s^{-1/2}} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{4}\left(\frac{3}{s^{5/2}} - \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} + \frac{8}{s^{-1/2}}\right).$$

(*ii*) We know that $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \infty$

$$\therefore \qquad \cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \ldots$$

and
$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \ldots$$

and
$$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} + \ldots$$

$$= \frac{\Gamma 1/2}{\sqrt{s}} - \frac{1}{2}\cdot\frac{1/2\,\Gamma 1/2}{s^{3/2}} + \frac{1}{4!}\frac{3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{5/2}} - \frac{1}{6!}\frac{5/2\,.\,3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{7/2}} + \ldots$$

$$= \sqrt{\left(\frac{\pi}{2}\right)}\left[1 - \frac{1}{(4s)} + \frac{1}{2!}\frac{1}{(4s)^2} - \frac{1}{3!}\frac{1}{(4s)^3}\ldots\right] = \sqrt{\left(\frac{\pi}{s}\right)}\, e^{-1/4s}.$$

Example 21.7. *Find the Laplace transform of the function*
(i) $f(t)\ |\ t-1\ |\ +\ |\ t+1\ |, t \geq 0$ *(S.V.T.U., 2009)*
(ii) $f(t) = [t]$, *where* [ ] *stands for the greatest integer function.* *(P.T.U., 2010)*

Solution. (*i*) Given function is equivalent to

$$f(t) = \begin{cases} 2, & 0 \leq t < 1 \\ 2t, & t \geq 1 \end{cases}$$

$$\therefore \qquad L f(t) = \int_0^1 e^{-st}(2)\,dt + \int_1^\infty e^{-st}(2t)dt = 2\left[\left|\frac{e^{-st}}{-s}\right|_0^1 + 2\left|\frac{t.e^{-st}}{-s}\right|_1^\infty - \left|\frac{e^{-st}}{(-s)^2}\right|_1^\infty\right]$$

$$= 2\left(\frac{e^{-s}}{-s} + \frac{1}{s}\right) + 2\left(\frac{0 - e^{-s}}{-s} - \frac{0 - e^{-s}}{s^2}\right) = \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right)$$

(*ii*) Given function is equivalent to

$$[t] = 0 \text{ in } (0, 1) + 1 \text{ in } (1, 2) + 2 \text{ in } (2, 3) + 3 \text{ in } (3, 4) + \ldots$$

$$\therefore \qquad L[f(t)] = \int_0^\infty e^{-st}[f(t)]dt = \int_0^\infty e^{-st}\,[t]\,dt$$

$$= \int_0^1 e^{-st}(0)dt + \int_1^2 e^{-st}(1)dt + \int_2^3 e^{-st}(2)dt + \int_3^4 e^{-st}(3)dt + \ldots \infty$$

$$= 0 + \left|\frac{e^{-st}}{-s}\right|_1^2 + 2\left|\frac{e^{-st}}{-s}\right|_2^3 + 3\left|\frac{e^{-st}}{-s}\right|_3^4 + \ldots \infty$$

$$= -\frac{1}{s}[(e^{-2s} - e^{-s}) + 2(e^{-3s} - e^{-2s}) + 3(e^{-4s} - e^{-3s}) + \ldots \infty]$$

$$= \frac{1}{s}(e^{-s} + e^{-2s} + e^{-3s} + \ldots \infty) = \frac{1}{s}\left(\frac{e^{-s}}{1 - e^{-s}}\right) = \frac{1}{s(e^s - 1)}.$$

III. Change of scale property. *If* $L\{f(t)\} = \bar{f}(s)$, *then* $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt = \int_0^\infty e^{-su/a} f(u)\,.\,du/a \qquad \left|\begin{array}{l}\text{Put } at = u \\ \quad dt = du/a\end{array}\right.$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)du = \frac{1}{a}\bar{f}(s/a).$$

Example 21.8. *Find* $L\left\{\frac{\sin at}{t}\right\}$, *given that* $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$.

Solution. By the above property,

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\tan^{-1}\left\{\frac{1}{(s/a)}\right\} = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right) \text{ i.e., } L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left\{\frac{a}{s}\right\}.$$

PROBLEMS 21.1

Find the Laplace transforms of

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$. (*J.N.T.U., 2003*)
2. $1 + 2\sqrt{t} + 3/\sqrt{t}$.
3. $3\cosh 5t - 4\sinh 5t$. (*Nagarjuna, 2006*)
4. $\cos(at + b)$.
5. $(\sin t - \cos t)^2$.
6. $\sin 2t \cos 3t$. (*Kottayam, 2005*)
7. $\sin\sqrt{t}$.
8. $\sin^5 t$. (*Mumbai, 2007*)
9. $\cos^3 2t$.
10. $e^{-at}\sinh bt$
11. $e^{2t}(3t^5 - \cos 4t)$. (*P.T.U., 2007*)
12. $e^{-3t}\sin 5t \sin 3t$. (*V.T.U., 2006*)
13. $e^{-t}\sin^2 t$. (*Mumbai, 2009*)
14. $e^{2t}\sin^4 t$ (*Mumbai, 2007*)
15. $\cosh at \sin at$. (*Delhi, 2002*)
16. $\sinh 3t \cos^2 t$ (*Madras, 2000*)
17. $t^2 e^{2t}$. (*V.T.U., 2008 S*)
18. $(1 + te^{-t})^3$
19. $t\sqrt{(1 + \sin t)}$ (*Mumbai, 2007*)
20. $f(t) = \begin{cases} 4, & 0 \le t <= 1 \\ 3, & t > 1 \end{cases}$ (*U.P.T.U., 2009*)
21. $f(t) = \begin{cases} \sin t, 0 < t < \pi \\ 0, t > \pi. \end{cases}$ (*Madras, 2000 S*)
22. $f(x) = \begin{cases} \sin(x - \pi/3), & x > \pi/3 \\ 0, & x < \pi/3 \end{cases}$ (*Rajasthan, 2006*)
23. $f(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$
24. $f(t) = \begin{cases} t^2, 0 < t < 2 \\ t - 1, 2 < t < 3 \\ 7, t > 3. \end{cases}$ (*Mumbai, 2007*)
25. If $L[f(t)] = \frac{1}{s(s^2+1)}$, find $L[e^{-t}f(2t)]$.

21.5 TRANSFORMS OF PERIODIC FUNCTIONS

If $f(t)$ is a periodic function with period T, i.e., $f(t + T) = f(t)$, then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

We have $\quad L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \ldots$

In the second integral put $t = u + T$, in the third integral put $t = u + 2T$, and so on. Then

$$L\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(u+T)} f(u+T)du + \int_0^T e^{-s(u+2T)} f(u+2T)du + \ldots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-su} f(u)du + e^{-2sT}\int_0^T e^{-su} f(u)du + \ldots$$

$$[\because \quad f(u) = f(u+T) = f(u+2T) \text{ etc.}]$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-st} f(t)dt + e^{-2sT}\int_0^T e^{-st} f(t)dt + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots \infty)\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1 - e^{-sT}}\int_0^T e^{-st} f(t)dt. \qquad \text{(V.T.U., 2008 ; Mumbai, 2006)}$$

Example 21.9. *Find the Laplace transform of the function*

$$f(t) = \sin \omega t, \quad 0 < t < \pi/\omega$$
$$= 0, \quad \pi/\omega < t < 2\pi/\omega.$$

(Kurukshetra, 2005 ; Madras, 2003)

Solution. Since $f(t)$ is a periodic function with period $2\pi/\omega$.

$$\therefore \quad L\{f(t)\} = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)dt$$

$$= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \, . \, 0 \, dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s/\omega}} \left| \frac{e^{-st}(-s \sin \omega t - \omega \cos \omega t)}{s^2 + \omega^2} \right|_0^{\pi/\omega} = \frac{\omega e^{-\pi s/\omega} + \omega}{(1 - e^{-2\pi s/\omega})(s^2 + \omega^2)} = \frac{\omega}{(1 - e^{-\pi s/\omega})(s^2 + \omega^2)}.$$

Example 21.10. *Draw the graph of the periodic function*

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi. \end{cases}$$

and find its Laplace transform. *(U.P.T.U., 2003)*

Solution. Here the period of $f(t) = 2\pi$ and its graph is as in Fig. 21.1.

$$\therefore \quad L f(t) = \frac{1}{1 - e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st} t \, dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t) dt \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \left| t \left(\frac{e^{-st}}{-s} \right) - 1 . \left(\frac{e^{-st}}{s^2} \right) \right|_0^{\pi} + \left| (\pi - t) \left(\frac{e^{-st}}{-s} \right) - (-1) \left(\frac{e^{-st}}{s^2} \right) \right|_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2} \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{\pi}{s} \left(e^{-2\pi s} - e^{-\pi s} \right) + \frac{1}{s^2} \left(1 + e^{-2\pi s} - 2e^{-\pi s} \right) \right\}.$$

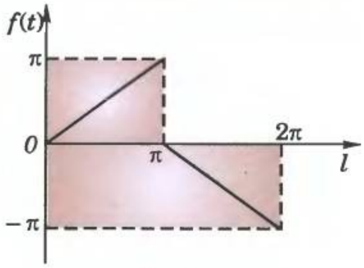

Fig. 21.1

21.6 TRANSFORMS OF SPECIAL FUNCTIONS

(1) Transform of Bessel functions $J_0(x)$ and $J_1(x)$.

We know that $\quad J_0(x) = 1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 . 4^2} - \dfrac{x^6}{2^2 . 4^2 . 6^2} + \ldots$ [§ 16.7 (1), p. 553]

$$\therefore \quad L\{J_0(x)\} = \frac{1}{s} - \frac{1}{2^2} \frac{2!}{s^3} + \frac{1}{2^2 . 4^2} \cdot \frac{4!}{s^5} - \frac{1}{2^2 . 4^2 . 6^2} \cdot \frac{6!}{s^7} + \ldots$$

$$= \frac{1}{s} \left\{ 1 - \frac{1}{2} \left(\frac{1}{s^2} \right) + \frac{1 . 3}{2 . 4} \left(\frac{1}{s^4} \right) - \frac{1 . 3 . 5}{2 . 4 . 6} \left(\frac{1}{s^6} \right) + \ldots \right\}$$

$$= \frac{1}{s} \left(1 + \frac{1}{s^2} \right)^{-1/2} = \frac{1}{\sqrt{(s^2 + 1)}} \quad \ldots(1)$$

Also since $\quad J_0'(x) = -J_1(x).$ [Problem 4(*i*), p. 557]

$$\therefore \quad L\{J_1(x)\} = -L\{J_0'(x)\} = -[sL\{J_0(x)\} - 1] = 1 - \frac{s}{\sqrt{(s^2 + 1)}} \quad \ldots(2)$$

(2) Transform of Error function

We know that $\quad erf(\sqrt{x}) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{\sqrt{x}} e^{-t^2} dt$ (§ 7.18, p. 312)

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \ldots\right) dt = \frac{2}{\sqrt{\pi}} \left(x^{1/2} - \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5.2!} - \frac{x^{7/2}}{7.3!} + \ldots\right)$$

$$\therefore \quad L\{erf(\sqrt{x})\} = \frac{2}{\sqrt{\pi}} \left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5.2!\,s^{7/2}} - \frac{\Gamma(9/2)}{7.3!\,s^{9/2}} + \ldots\right\}$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2}\frac{1}{s^{5/2}} + \frac{1.3}{2.4}\cdot\frac{1}{s^{7/2}} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^{9/2}} + \ldots$$

$$= \frac{1}{s^{3/2}} \left\{1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1.3}{2.4}\cdot\frac{1}{s^2} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^3} + \ldots\right\}$$

$$= \frac{1}{s^{3/2}} \left[1 + \frac{1}{s}\right]^{-1/2} = \frac{1}{s\sqrt{(s+1)}}. \qquad (Mumbai, 2009) \quad ...(3)$$

(3) Transform of Laguerre's polynomials $L_n(x)$

We know that $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$ (§ 16.18, p. 571)

$$L[L_n(t)] = \int_0^\infty e^{-st} e^t \frac{d^n}{dt^n}(t^n e^{-t}) dt = \int_0^\infty e^{-(s-1)t} \frac{d^n}{dt^n}(e^{-t} t^n) dt$$

$$= \left| e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n)\right|_0^\infty + \int_0^\infty e^{-(s-1)t\,(s-1)}\,(s-1) \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt$$

$$= (s-1) \int_0^\infty e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt. \qquad \text{(Integrating by parts)}$$

$$= (s-1)^n \int_0^\infty e^{-(s-1)t} . e^{-t} . t^n \, dt = (s-1)^n \int_0^\infty e^{-st} . t^n \, dt$$

$$= (s-1)^n L(t^n) = (s-1)^n . \frac{n!}{s^{n+1}}$$

Hence $\quad L[L_n(x)] = \frac{n!(s-1)^n}{s^n + 1} \; (s > 1).$

Example 21.11. *Evaluate (i) $L\{e^{-at} J_0(at)\}$ (ii) $L(erf\, 2\sqrt{t})$.* *(Mumbai, 2006)*

Solution. (*i*) We know that $L\{J_0(at)\} = \frac{1}{\sqrt{(s^2 + a^2)}}$

By shifting property, we get

$$L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{[(s+a)^2 + a^2]}} = \frac{1}{\sqrt{(s^2 + 2sa + 2a^2)}}$$

(*ii*) We know that $\quad L(erf\sqrt{t}\,) = \frac{1}{s(s+1)}$

$$\therefore \quad L(erf\, 2\sqrt{t}\,) = L[erf\,\sqrt{(4t)}] = \frac{1}{4}\cdot\frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}.$$

PROBLEMS 21.2

1. Find the Laplace transform of the *saw-toothed wave* of period T, given $f(t) = t/T$ for $0 < t < T$. *(V.T.U., 2007)*
2. Find the Laplace transform of the *full-wave rectifier*

   $f(t) = E \sin wt$, $0 < t < \pi/w$, having period π/w.

3. Find the Laplace transform of the *square-wave* (or *meander*) function of period a defined as

$$f(t) = k, \quad \text{when} \quad 0 < t < \alpha$$
$$= -k, \quad \text{when} \quad \alpha < t < 2\alpha.$$

(V.T.U., 2011)

4. Find the Laplace transform of the *triangular wave* of period $2a$ given by

$$f(t) = t, \qquad 0 < t < a$$
$$= 2a - t, \qquad a < t < 2a.$$

(Nagarjuna, 2008 ; V.T.U., 2008 S ; U.P.T.U., 2002)

Find the Laplace transform of the following functions :

5. $J_0(ax)$. 6. $e^{-at} J_0(bt)$. 7. $e^{2t}\, erf(\sqrt{t})$.

21.7 TRANSFORMS OF DERIVATIVES

(1) *If $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$, then $L\{f'(t)\} = s\bar{f}(s) - f(0)$.*

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt \qquad \textbf{[Integrate by parts]}$$

$$= \left| e^{-st} f(t) \right|_0^\infty - \int_0^\infty (-s) e^{-st} \cdot f(t)\,dt.$$

Now assuming $f(t)$ to be such that $\underset{t\to\infty}{\text{Lt}}\; e^{-st} f(t) = 0$. When this condition is satisfied, $f(t)$ is said to be *exponential order s*.

Thus, $$L\{f'(t)\} = f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

whence follows the desired result.

(2) *If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then*

$$\mathbf{L\{f^n(t)\} = s^n\, \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{n-1}(0).}$$

Using the general rule of integration by parts (Footnote p. 398).

$$L\{f^n(t)\} = \int_0^\infty e^{-st} f^n(t)\,dt$$

$$= \left| e^{-st} f^{n-1}(t) - (-s) e^{-st} f^{n-2}(t) + (-s)^2 e^{-st} f^{n-3}(t) - \ldots \right.$$

$$\left. + (-1)^{n-1} (-s)^{n-1} e^{-st} \cdot f(t) \right|_0^\infty + (-1)^n (-s)^n \int_0^\infty e^{-st} f(t)\,dt$$

$$= -f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) - \ldots - s^{n-1} f(0) + s^n \int_0^\infty e^{-st} f(t)\,dt$$

Assuming that $\underset{t\to\infty}{\text{Lt}}\; e^{-st} f^m(t) = 0$ for $m = 0, 1, 2, \ldots, n-1$.

This proves the required result.

21.8 TRANSFORMS OF INTEGRALS

If $L\{f(t)\} = \bar{f}(s)$, then $\mathbf{L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s).}$

Let $\quad \phi(t) = \int_0^t f(u)\,du$, then $\phi'(t) = f(t)$ and $\phi(0) = 0$

$\therefore \quad L\{\phi'(t)\} = s\bar{\phi}(s) - \phi(0)$ **[By § 21.7 (1)]**

or $\quad \bar{\phi}(s) = \frac{1}{s} L\{\phi'(t)\}$ *i.e.,* $L\left(\int_0^t f(u)\,du\right) = \frac{1}{s}\bar{f}(s)$.

21.9 MULTIPLICATION BY t^n

If $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]}, \textit{ where } n = 1, 2, 3 \ldots$$

We have $\int_0^\infty e^{-st} f(t)\, dt = \bar{f}(s)$.

Differentiating both sides with respect to s, $\frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)\, dt\right\} = \frac{d}{ds}\{\bar{f}(s)\}$

or By Leibnitz's rule for differentiation under the integral sign (p. 233).

$$\int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\, dt = \frac{d}{ds}\{\bar{f}(s)\}$$

or $$\int_0^\infty \{-te^{-st} f(t)\}\, dt = \frac{d}{ds}[\bar{f}(s)] \quad \text{or} \quad \int_0^\infty e^{-st} [tf(t)\}\, dt = -\frac{d}{ds}[\bar{f}(s)]$$

which proves the theorem for $n = 1$.

Now assume the theorem to be true for $n = m$ (say), so that

$$\int_0^\infty e^{-st} [t^m f(t)]\, dt = (-1)^m \frac{d^m}{ds^m}[\bar{f}(s)]$$

Then $$\frac{d}{ds}\left[\int_0^\infty e^{-st}\, t^m f(t)\, dt\right] = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$$

or By Leibnitz's rule, $\int_0^\infty (-te^{-st}) . t^m f(t)\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$

or $$\int_0^\infty e^{-st} [t^{m+1} f(t)]\, dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)].$$

This shows that, if the theorem, is true for $n = m$, it is also true for $n = m + 1$. But it is true for $n = 1$. Hence it is true for $n = 1 + 1 = 2$, and $n = 2 + 1 = 3$ and so on.

Thus the theorem is true for all positive integral values of n. *(U.P.T.U., 2005)*

Example 21.12. *Find the Laplace transforms of*

(i) $t \cos at$ *(Raipur, 2005)* *(ii) $t^2 \sin at$* *(S.V.T.U., 2007)*

(iii) $t^3 e^{-3t}$ *(Kottayam, 2005)* *(iv) $te^{-t} \sin 3t$.* *(Kurukshetra, 2005)*

Solution. (*i*) Since $L(\cos at) = s/(s^2 + a^2)$

$$\therefore \quad L(t \cos at) = -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right) = -\frac{s^2 + a^2 - s\,.\,2s}{(s^2 + a^2)^2}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$ [*cf.* Example 21.4]

(*ii*) Since $\sin at = \frac{a}{s^2 + a^2}$,

$$\therefore \quad L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2 + a^2}\right) = \frac{d}{ds}\left\{\frac{-2as}{(s^2 + a^2)^2}\right\} = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}.$$

(*iii*) Since $L(e^{-3t}) = 1/(s + 3)$,

$$\therefore \quad L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s + 3}\right) = -\frac{(-1)^3 . 3!}{(s + 3)^{3+1}} = 6/(s + 3)^4.$$

(*iv*) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, therefore $L(t \sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2 + 3^2}\right) = \frac{6s}{(s^2 + 9)^2}$

Now using the shifting property (§ 21.4 II), we get

$$L(e^{-t}\, t \sin 3t) = \frac{6(s + 1)}{[(s + 1)^2 + 9]^2} = \frac{6(s + 1)}{(s^2 + 2s + 10)^2}.$$

Example 21.13. *Evaluate (i) $L\{t\,J_0(at)\}$ (ii) $L\{t\,J_1(t)\}$ (iii) $L\{t\,erf\,2\sqrt{t}\}$.*

Solution. (*i*) Since $L\{J_0(at)\} = \dfrac{1}{\sqrt{(s^2+a^2)}}$

$$\therefore \quad L\{tJ_0(at)\} = -\frac{d}{ds}\{L\,J_0(at)\} = -\frac{d}{ds}\frac{1}{\sqrt{(s^2+a^2)}} = \frac{s}{(s^2+a^2)^{3/2}}$$

(*ii*) Since $\quad L\{J_1(t)\} = 1 - \dfrac{s}{\sqrt{(s^2+1)}}$

$$\therefore \quad L\{t\,J_1(t)\} = -\frac{d}{ds}\{L\,J_0(t)\} = -\frac{d}{ds}\left\{1 - \frac{s}{(\sqrt{s^2+1})}\right\} = \frac{1}{(s^2+1)^{3/2}}$$

(*iii*) Since $\quad L(erf\,\sqrt{t}) = \dfrac{1}{s\sqrt{(s+1)}}$

$$\therefore \quad L(erf\,2\sqrt{t}) = L\,erf\,\sqrt{4(t)} = \frac{1}{4}\cdot\frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}$$

Thus $$L(t\,erf\,2\sqrt{t}) = -\frac{d}{ds}\left\{\frac{2}{s\sqrt{(s+4)}}\right\} = -\frac{d}{ds}\left\{\frac{2}{\sqrt{(s^3+4s^2)}}\right\} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

21.10 DIVISION BY t

If $L\{f(t)\} = \bar{f}(s)$, *then* $\mathbf{L}\left\{\dfrac{\mathbf{1}}{\mathbf{t}}\,\mathbf{f(t)}\right\} = \int_s^\infty \bar{\mathbf{f}}\,\mathbf{(s)\,ds}$ *provided the integral exists.*

We have $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

Integrating both sides with respect to s from s to ∞.

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds = \int_0^\infty \int_s^\infty f(t)\,e^{-st}\,ds\,dt$$

[Changing the order of integration]

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt \qquad [\because \ t \text{ is independent of } s]$$

$$= \int_0^\infty f(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty e^{-st}\cdot\frac{f(t)}{t}\,dt = L\left\{\frac{1}{t}f(t)\right\}.$$

Example 21.14. *Find the Laplace transform of (i) $(1-e^t)/t$* *(Madras, 2000)*

(ii) $\dfrac{\cos at - \cos bt}{t} + t \sin at.$ *(V.T.U., 2010)*

Solution. (*i*) Since $\quad L(1-e^t) = L(1) - L(e^t) = \dfrac{1}{s} - \dfrac{1}{s-1}$

$$\therefore \quad L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = \left|\log s - \log(s-1)\right|_0^\infty$$

$$= \left|\log\left(\frac{s}{s-1}\right)\right|_s^\infty = -\log\left[\frac{1}{1-1/s}\right] = \log\left(\frac{s-1}{s}\right)$$

(*ii*) Since $\quad L(\cos at - \cos bt) = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$