What is discrete Mathematics?

Discrete Mathematics is the part of Mathematics devoted to the study of discrete objects. (Here discrete means consisting of distinct or unconnected elements.)

Partition of a set

A partition of a set divides the set into nonoverlapping non empty subsets whose union equals the original sets.

Logic and Propositional Calculus

Proposition

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Example: All the following declarative sentences are propositions

① Washington, D.C, is the capital of the United States of America

② Toronto is the capital of Canada

③ $1+1=2$

④ $2+2=3$

Propositions 1 and 3 are true, where as 2 and 4 are false.

Some sentences that are not propositions are given in Example 2.

Eg. 2 Consider the following sentences

1. What time is it?
2. Read this carefully
3. $x+1 = 2$
4. $x+y = z$

Sentence 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

* We use letters to denote propositional variables (or sentential variables), that is, variables that represent propositions just as letters are used to denote numerical variables. The conventional letters used for propositional variables are $p, q, r, s \ldots$. The truth value of a proposition is true, is denoted by T, if it is a true proposition, and the truth value of the proposition is false,

denoted by F, if it is a false proposition.

* The propositions that cannot be expressed in terms of simpler propositions are called atomic propositions.

* The Area of logic that ~~that~~ deals with propositions is called the propositional calculus or propositional logic.

* New propositions, called compound propositions, are formed from existing propositions using logical operators.

Definition 1 Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement

"It is not the case that p".

The proposition $\neg p$ is read as "not p". The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Example: Find the negation of the proposition

"Suresh's PC run Linux"

Solution: The negation is "It is not the case that Suresh's PC runs Linux.

In simpler words "Suresh's PC does not run Linux

Truth table.

The truth table for negation operator

p	$\neg p$
T	F
F	T

* The negation of a proposition can also be considered the result of the operation of the negation operator on a proposition.

* The negation operator constructs a new proposition from a single existing proposition.

* The logic operators that are used to form new propositions from two or more existing propositions are called connectives.

Definition: Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q"

The conjuction $p \wedge q$ is true when both p and q are true and is false otherwise.

Truth table for the conjunction of two propositions

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: Find the conjunction of the propositions p and q where p is the proposition "Rebecca's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Rebecca's PC runs faster than 1 GHz.

Solution: The conjunction of these propositions $p \wedge q$, is the proposition "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz.

This can be expressed more simply as "Rebecca's PC has more than 16 GB free hard disk space and its processor runs faster than 1 GHz." For this conjunction to be true, both conditions given must be true. It is false when one or both of these conditions are false.

Def: Let p and q be propositions. The disjunction of p and q denoted by $p \vee q$, is the proposition p or q. The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Truth table for disjunction of two propositions

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The use of the connective 'or' in a disjunction corresponds to one of the two ways the word 'or' is used in English, namely, as an 'inclusive or'.

A disjunction is true when atleast one of the two proposition is true. That is, $p \vee q$ is true when both p and q are true or when exactly one of p and q are true.

Example Translate the statement "Students who have taken calculus or introductory computer science can take this class." in a statement in propositional logic using the proposition p : "A student who has taken calculus can take this class" and q: A student who has taken introductory computer science can take this class.

Sol: We assume that this statement means that students who have taken both calculus and introductory computer science can take the class, as well as the students who have taken only one of the two subjects. Hence this statement can be expressed as $p \vee q$, the inclusive or, or disjunction of p and q.

Def: Let p and q be propositions. The 'exclusive or' of p and q, denoted by $p \oplus q$ (or p xor q), is the proposition that is true when exactly one of p and q is true and is false otherwise.

Truth table for 'Exclusive or'

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Eg: Let p and q be the propositions that state "A student can have a salad with dinner." and "A student can have soup with dinner", respectively. What is $p \oplus q$, the exclusive or of p and q?

Sol: The 'exclusive or' of p and q is the statement that is true when exactly one of the p and q is true. That is, $p \oplus q$ is the statement "A student can have soup or salad, but not both, with dinner.

Conditional Statements

Def Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence)

* The statement $p \to q$ is called a conditional statement because $p \to q$ asserts that q is true on the condition that p holds. A conditional statement is also called an implication.

Truth table for Conditional Statement

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Following ways are used to express conditional statement:

i) If p, then q

ii) if p, q

p only if q

p implies q

a sufficient condition for q is p

q whenever p

q ~~is~~ ~~act~~ follows from p

q provided that p.

q when p.

Example: Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job."
Express the statement $p \to q$ as a statement in English

Sol: From the definition of conditional statements, we see that when p is the statement "Maria learns discrete mathematics" and q is the statement "Maria will find a good job" $p \to q$ represents the statement
"If Maria learns discrete mathematics, then she will find a good job".

Converse, contrapositive and inverse

We can form some new conditional statements with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \to p$ is called the converse of $p \to q$. The contrapositive of $p \to q$ is the proposition $\neg q \to \neg p$. The proposition $\neg p \to \neg q$ is called the inverse of $p \to q$. We will see that of these conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$

Truth table for converse $q \to p$

p	q	$q \to p$
T	T	T
T	F	T
F	T	F
F	F	T

Truth table for Inverse $\neg p \to \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Truth Value for Contrapositive $\neg q \to \neg p$

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	~~F~~T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

When two compound propositions always have the same truth values, regardless of the truth values of its propositional variables, we call them equivalent.

Hence, a conditional statement and its contrapositive are equivalent.

The converse and inverse of a conditional statement are also equivalent, but neither is equivalent to the original conditional statement.

Example Find the contrapositive, the converse, and the inverse of the conditional statement.

"The home team wins whenever its raining"

Sol: As 'q whenever p' is one of the ways to express the conditional statements $p \to q$, the original statements can be rewritten as

"If it is raining, then the home team wins"

Contrapositive of the statement is "If the home team does not win, then its not raining".

The converse is "If the home team wins, then its raining"

The inverse is "If it is not raining, then the home team does not win".

only contrapositive is equivalent to the original statement

Biconditionals:

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

Truth Table for biconditional $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
~~T~~F	~~T~~F	~~F~~T

Different ways of expressing biconditional statement $p \leftrightarrow q$

~~uses~~ ~~the~~ "p is necessary and sufficient for q"

"if p then q, and conversely"

"p iff q" "p exactly when q"

* Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$

Truth table for $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p\to q)\wedge(q\to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example: Let p be the statement "You can take the flight" and q be the statement "You buy a ticket".

Then $p \leftrightarrow q$ is the statement "You can take the flight if and only if you buy a ticket".

Truth table of compound propositions

Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \to (p \wedge q)$$

Sol: Truth table

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Propositional Equivalences.

Def: A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, ~~it~~ is called a tautology. A compound proposition that is always false is called a contradiction. A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Example: We can construct example of tautology and contradiction using just one propositional variable. Consider the truth table of $p \vee \neg p$ and $p \wedge \neg p$. Because $p \vee \neg p$ is always true, it is a tautology. And $p \wedge \neg p$ is always false, a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logic equivalences

Compound proposition that have the same truth values in all possible cases are called logically equivalent

Def: Compound proposition p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

The notation $p \equiv q$ denotes that p and q are logically equivalent.

De Morgan's law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Eg: Prove that $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ are logically equivalent.

Truth table for $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$

Sol:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The truth values of the compound propositions $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \vee q) \leftrightarrow (\neg p) \wedge (\neg q)$ is a tautology and that these compound propositions are logically equivalent.

Eg: Show that $\neg(p \wedge q)$ and $(\neg p) \vee (\neg q)$ are logically equivalent

Sol: Truth table for $\neg(p \wedge q)$ and $(\neg p) \vee (\neg q)$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The truth value of the compound propositions $\neg(p \wedge q)$ and $(\neg p) \vee (\neg q)$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$ is a tautology and that these compound propositions are logically equivalent

Eg: Show that $p \to q$ and $\neg p \vee q$ are logically equivalent (This is known as the conditional disjunction equivalence)

Sol: Truth table for $\neg p \vee q$ and $p \to q$

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The truth values of $\neg p \vee q$ and $p \to q$ agree, they are logically equivalent.

Example Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is a distributive law of disjunction over conjunction.

Truth table $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

The truth value of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ agree these compound propositions are logically equivalent.

Logic equivalences

Equivalences	Name
$p \land T \equiv p$ $p \lor F \equiv p$	Identity Laws
$p \lor T \equiv T$ $p \land F \equiv F$	Domination Laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation laws
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's Law
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv T$ $p \land \neg p \equiv F$	Negation laws

Constructing New logical equivalences

Eg: Show that $\neg(p \rightarrow q)$ and $(p \wedge \neg q)$ are logically equivalent

Sol: We can use a truth table to show that these compound propositions are equivalent.

However, we will illustrate it using logical identities

$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ by the conditional disjunction equivalence)

$\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan's law

$\equiv p \wedge \neg q$ by double negation law

Eg: Show that $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ are logically equivalent by developing a series of logical equivalences.

Sol. $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ by the second DeMorgan's law

$\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$ by the first De Morgan's law

$\equiv \neg p \wedge (p \vee \neg q)$ by double negation law

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ by the second distributive law

$\equiv F \vee (\neg p \wedge \neg q)$ $\quad \neg p \wedge p \equiv F$

$\equiv (\neg p \wedge \neg q) \vee F$ by commulative law for disjunction

$\equiv (\neg p \wedge \neg q)$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent

Eg: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Sol: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$; $(p \rightarrow q \equiv \neg p \vee q)$

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ by first De Morgan's law.

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ by associative and commutative law for disjunction

$\equiv T \vee T$ by Negation law

$\equiv T$ by domination law.

Hence, $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Satisfiability

A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that makes it true. (That is, when it is a tautology or a ~~conting~~ contingency).

When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is unsatisfiable.

* Note that a compound proposition is unsatisfiable if and only if it is negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.

Solution: when we find a particular assignment of truth values that makes a compound proposition true, we have shown that it is satisfiable; such an assignment is called an solution of this particular satisfiability problem.

Eg: Determine whether each of the compound propositions $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ and $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

is ~~satisfiability~~ satisfiable.

Sol: Note that $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when the three $(p \vee \neg q)$, $(q \vee \neg r)$ and $(r \vee \neg p)$ are true.

$(p \vee \neg q)$ is false when p is ~~true~~ false and q is true

$(q \vee \neg r)$ is false when q is false and r is true

$(r \vee \neg p)$ is false when r is false and p is true.

If either p is true or q is false, we get $(p \vee \neg q)$ is true. or if either q is true or r is false, we get $(q \vee \neg r)$ is true.

If r is true or p is false, we get $(r \vee \neg p)$ is true

Hence if p, q, r have the same truth value $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is ~~true~~. satisfiable.

Next, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true, when both $(p \vee q \vee r)$ ~~A~~ and $(\neg p \vee \neg q \vee \neg r)$ is true.

Therefore, atleast one of p, q, r must be true and atleast one of the p, q, r must be false.

Hence, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable.

Finally, for $(p \vee \neg q) \wedge (q \wedge \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

to be true, $(p \vee \neg q) \wedge (q \wedge \neg r) \wedge (r \vee \neg p)$ and $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ must both be true. For the first to be true, the three variables must have the same truth values, and for the second to be true atleast one of the three must be true and at least one must be false. However, these conditions are contradictory. From these observations we conclude that no assignment of truth values to p, q, and r ~~must~~ ~~to~~ makes $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ true.

Hence it is unsatisfiable.

Predicates

Statements involving variables, such as "$x > 3$", "$x = y + 3$", "$x + y = z$" and "computer x is functioning properly" are often found in mathematical assertions. These statements are neither true nor false when the values of variables are not specified.

We will see how these statements can be converted to propositions.

The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. ~~can~~ have. The second part - the predicate "is greater than 3" - refer to a property that the subject of the statement can have. We can denote the statement "x is greater than 3" by $P(x)$ where P denotes the predicate "is greater than 3" and x is the variable.

The statement $P(x)$ is also said to be the value of the <u>propositional function</u> P at x.

Once a value has been assigned to the variables x, the statement $P(x)$ ~~is~~ ~~also~~ becomes a proposition and has a truth value.

<u>Eg:</u> Let $P(x)$ denote the statement "$x>3$". What are the truth values of $P(4)$ and $P(2)$?

<u>Sol:</u> We obtain the statement $P(4)$ by setting $x=4$ in the statement "$x>3$". Hence, $P(4)$, which is the statement "$4>3$" is true.

However, $P(2)$, which is the statement "$2>3$", is false.

Eg: Let $Q(x,y)$ denote the statement "$x = y+3$". what are the truth values of the propositions $Q(1,2)$ and $Q(3,0)$?

Sol: To obtain $Q(1,2)$, set $x=1$ and $y=2$ in the statement $Q(x,y)$. Hence, $Q(1,2)$ is the statement "$1 = 2+3$" which is false. The statement $Q(3,0)$ is the proposition "$3 = 0+3$", which is true.

Eg: what are the truth values of the propositions $R(1,2,3)$ and $R(0,0,1)$ if $R(x,y,z) =$ "$x+y = z$"

Sol: The proposition $R(1,2,3)$ is obtained by setting $x=1$, $y=2$, and $z=3$ in the statement $R(x,y,3)$

We see that $R(1,2,3)$ is the statement "$1+2=3$", which is true.

$R(0,0,1)$ which is the statement "$0+0=1$" is false.

* A statement of the form $P(x_1, x_2, \ldots x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots x_n)$ and P is also called an n-place predicate or an n-ary predicate.

Preconditions and Post Conditions

Predicates are used to establish the correctness of computer programs, that is, to show that computer programs always produce the desired output when given valid input.

The statements that describe valid inputs are known as preconditions and the conditions that the output should satisfy when the program has run are known as post conditions.

Quantifiers

When the variable in a propositional functions are assigned values, the resulting statement becomes a proposition with a certain truth value.

There is another way, quantification, to create a proposition from a propositional function.

Quantification expresses the extent to which a predicate is true over a range of elements.

In English, the words all, some, many, none and few are used in quantifications.

The area of logic that deals with predicates and quantifiers is called the predicate calculus.

Universal Quantifiers

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, Ho called the domain of the discourse (or the universe of discourse), often just referred to as the domain. Such statements are expressed using universal quantification.

Def The universal quantification of $P(x)$ is the statement
"$P(x)$ for all values of x in the domain"
The notation $\forall x\, P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifier. We read $\forall x\, P(x)$ as "for all x $P(x)$" or "for every x $P(x)$".

An element for which $P(x)$ is false is called a counter example to $\forall x\, P(x)$.

Eg: Let $P(x)$ be the statement "$x+1>x$" what is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Sol: Because $P(x)$ is true for all real numbers x, the quantification $\forall x\, P(x)$ is true.

Remark: An implicit assumption is made that all domain of discourse for quantifiers are nonempty. If domain is empty, then $\forall x P(x)$ is true for any propositional function $P(x)$ because there is no elements x in the domain for which $P(x)$ is false.

Besides "for all" and "for every", universal quantification can be expressed in many other ways, including "all of", "for each", "given any," for arbitrary", "for each".

* The statement $\forall x P(x)$ is true for every x. and is false if there is an x for which $P(x)$ is false.

Eg: Let $Q(x)$ be the statement "$x<2$". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Sol: $Q(x)$ is not true for every real number x; because for instance $Q(3)$ is false. That is $x=3$ is a counter example for the statement $\forall x Q(x)$. Thus, $\forall x Q(x)$ is false.

Eg: Suppose that $P(x)$ is "$x^2 > 0$" To show that the statement is false, when the universe of discourse consists of all integers, we give a counter example.

$x = 0$ is a counter example, because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$

Eg: what does the statement $\forall x\, N(x)$ mean if $N(x)$ is "computer x is connected to the network" and the domain consists of all computers on campus?

Sol: The statement $\forall x\, N(x)$ means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed as "Every computer on campus is connected to the network.

Existential Quantifier

Def: The existential quantifier of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$"

We use the notation $\exists x\, P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

* Besides the phrase "there exists" we can also express existential quantification in many other ways, such as by using the words

"for some", "for atleast one", or "there is"

The existential quantification $\exists x P(x)$ is read as

"There is an x such that $P(x)$"

"there is atleast one x such that $P(x)$"

Remark: Generally, an implicit assumptions is made that all domains of discourse for quantifiers are nonempty. If the domain is empty, then $\exists x Q(x)$ is false whenever $Q(x)$ is a propositional function because when the domain is empty, there can be no element x in the domain for which $Q(x)$ is true.

* ~~Obser~~ observe that the statement $\exists x P(x)$ is false if and only if there is no element x in the domain for which $P(x)$ is true. That is, $\exists x P(x)$ is false if and only if $P(x)$ is false for every element of the domain.

Eg: Let $P(x)$ denote the statement "$x > 3$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all the real numbers.?

Sol: Because $x > 3$ is sometimes true, for instance, when $x = 4$, the existential quantification of $P(x)$, which is $\exists x P(x)$ is true

Uniqueness Quantifier

Uniqueness quantifier denoted by $\exists!$ or $\exists_1$.

The notation $\exists! x\, P(x)$ states "there exists a unique x such that $P(x)$ is true

Other phrases for uniqueness quantification include "there is exactly one" and "there is one and only one")

Eg: $\exists! x\, (x-1=0)$, where the domain is the set of real numbers, states that there is a unique real number x such that $x-1=0$

This is a true statement, as $x=1$ is the unique real number such that $x-1=0$.

Negating Quantified Expressions

The negation of the universal quantification is the existential quantification of the negation of the original propositional function.

$$\boxed{\neg \forall x P(x) \equiv \exists x\ \neg P(x)}$$

To show that the $\neg \forall x P(x)$ and $\exists x \neg P(x)$ are logically equivalent no matter what the propositional function $P(x)$ is and what the domain is, first note that $\neg \forall x P(x)$ is true or if and only if, $\forall x P(x)$ is false. Next note that $\forall x P(x)$ is false if and only if there is an element x

in the domain for which $P(x)$ is false.

This holds if there is an element x in ~~to~~ the domain for which $\neg P(x)$ is true. Finally note that there is an element x in the domain for which $\neg P(x)$ is true if and only if $\exists x \neg P(x)$ is true.

$\therefore$ we conclude that $\neg \forall x P(x)$ is true if and only if $\exists x \neg P(x)$ is true. It follows that $\neg \forall x P(x)$ and $\exists x \neg P(x)$ are logically equivalent.

* The negation of the existential quantification is the universal quantification of negation of the original propositional function.

$$\boxed{\neg \exists x Q(x) \equiv \forall x \neg Q(x)}$$

To show that $\neg \exists x Q(x)$ and $\forall x \neg Q(x)$ are logically equivalent no matter what $Q(x)$ is and what the domain is, first note that $\neg \exists x Q(x)$ is true if and only if $\exists x Q(x)$ is false. This is true if and only if no x exists in the domain for which $Q(x)$ is true.

Next, note that no x exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for ~~which~~ every x in the domain. Finally note that $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for all

x in the domain, which holds true if and only if $\forall x \neg Q(x)$ is true. Putting these steps together, we see that $\neg \exists x Q(x)$ is true if and only if $\forall x \neg Q(x)$ is true.

We conclude that $\neg \exists x Q(x)$ and $\forall x \neg Q(x)$ are logically equivalent.

De Morgan's Law for Quantifiers

Negation	Equivalent Statement
$\neg \exists x P(x)$	$\forall x \neg P(x)$
$\neg \forall x P(x)$	$\exists x \neg P(x)$

Example: what are the negation of the statements $\forall x (x^2 > x)$ and $\exists x (x^2 = 2)$?

Sol: The negation of $\forall x (x^2 > x)$ is the statement $\neg \forall x (x^2 > x)$ which is equivalent to $\exists x \neg (x^2 > x)$, This can be rewritten as $\exists x (x^2 \leq x)$.

The negation of $\exists x (x^2 = 2)$ is the statement $\neg \exists x (x^2 = 2)$ which is equivalent to $\forall x \neg (x^2 = 2)$. This can be written as $\forall x (x^2 \neq 2)$. The Truth values of these statements depends on the domain.

Rules of Inference

Proof in mathematics are valid arguments that establish the truth of mathematical statements.

By an argument, ~~we~~ is a sequence of statements that end with a conclusion.

A.B. Valid, means the conclusion, or final statement of the argument, must follow from the truth of the preceding statement, or premises, of the arguments.

That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be ~~true~~ false.

Some common forms of incorrect reasoning is called fallacy.

Definition: An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

An argument is valid if the truth of all its premises implies that the conclusion is true.

An argument form in propositional logic is a sequence of compound propositions involving propositional variables.

An argument form is valid if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Remark: From the definition of a valid argument form we see that the argument form ~~which~~ with premises $p_1, p_2, \dots p_n$ and conclusion q is valid exactly when $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \to q$ is a tautology.

Rules of Inference for Propositional logic

We can show that ~~to~~ an ~~of~~ argument is valid, using some relatively simpler argument forms, called rules of inference.

* The tautology $(p \wedge (p \to q)) \to q$ is the basis of the rule of inference called modus ponens or (latin word for mode that affirms) law of detachment.

p

$p \to q$

$\therefore q$

Example: "If you have a current password, then you can log on to the network."

"You have a current password"

Therefore "You can log onto the network."

Example: Suppose that the conditional statement "If it snows today then we will go skiing," and its hypothesis, "It is ~~now~~ snowing today," are true. Then, by modus ponens, it follows that the conclusion of the conditional statement "we will go skiing" is true.

Introduction to Proofs

Theorem: Theorem is a statement that can be shown to be true. In mathematical writing, the terms theorem is usually reserved for a statement that is considered atleast somewhat important. Less important theorems sometimes are called propositions.

Theorems can also be referred as facts or results.

We demonstrate that a theorem is true with a proof.

A proof is a valid argument that establishes the truth of a theorem.

The statements used in a proof can include axioms (or postulates), which are statements we assume to be true.

A less important theorem that is helpful in the proof of other results is called a lemma.

A corollary is a theorem that can be established directly from a theorem that has been proved.

A conjecture is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert.

When a proof of a conjecture is found, the conjecture becomes a theorem.

Understanding How Theorems are Stated.

We need to understand how mathematical theorems are stated. Many theorems assert that a property holds for all elements in a domain such that the integers or the real numbers.

Although the precise statement of such theorems needs to include a universal quantifier, the standard convention in mathematics is to omit it. For example, the statement

"If $x > y$, where x and y are positive real numbers then $x^2 > y^2$"

really means

"For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$".

Further more, when theorems of this type are proved, the first step of the proof usually involves selecting a general element of the domain. Subsequent steps shows that this elements has the property in question. Finally, universal generalization implies that the theorem holds for all members of the domain.

Method of proving theorem

① Direct Proof

A direct proof of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference; with the final step showing that q must also be true.

A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs. In direct proof we assume that p is true and use axioms, definitions and previously proven theorems, together with rules of inference to show that q must also be true.

You may find that direct proofs of many results are quite straight-forward. Starting with the hypothesis and leading to the conclusion, the way forward is essentially dictated by the premises available at that step.

Definition: An integer n is even if there exists an integer k such that $n = 2k$ and n is odd if there exists an integer k such that $n = 2k+1$ (Note that every integer is either odd or even, no integer is both even and odd) Two integers have the same parity when both are even or both are odd; they have opposite parity when one is even and the other is odd.

<u>Example</u>: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

<u>Sol</u>: Note that this theorem states $\forall n \; P(n) \rightarrow Q(n)$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd". We will show $P(n)$ implies $Q(n)$.

To begin with the direct proof, we assume that the hypothesis of this conditional statement is true, i.e., we assume that n is odd. By the definition of an odd integer, it follows that $n = 2k+1$, where k is some integer. We want to show that n^2 is also odd. We can square both the sides of the equation $n = 2k+1$ to obtain a new equation that expresses n^2. When we do this, we find that $n^2 = (2k+1)^2 = 4k^2+1+4k$
$= 2(2k^2+2k)+1$. By the definition of an odd integer we can conclude that n^2 is an odd integer. Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

Proof by contrapositions

The ~~propos~~ proofs of theorems which are not direct proofs, that is, that do not start with the premises and end with the conclusion, are called indirect proofs

Proofs by contraposition make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive conditional statements $\neg q \to \neg p$. This means that the $p \to q$ can be proved by showing that its contrapositive $\neg q \to \neg p$ is true.

Example: Prove that if n is an integer and $3n+2$ is odd then n is odd.

Sol: We first attempt a direct proof. To construct a direct proof, we first assume that $3n+2$ is an odd integer. From the definition of the odd integer, we know that $3n+2 = 2k+1$ for some integer k.

which gives $3n+1 = 2k$, but there does not seem to be any direct way to conclude that n is odd. As our attempt at a direct proof failed, we try a proof by contraposition.

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n+2$

is odd then n is odd." is false. Let us assume that n is even. Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$.

This tells that ~~3t~~ $3n+2$ is even, (because it is multiple of 2) therefore not odd. This is the negation of the premise of the theorem.

Our proof by contraposition is complete. Hence, ~~3t~~ If $3n+2$ is odd, then n is odd.

Example: Prove that if $n = ab$, where a and b are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

Sol: The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $n = ab$ where a and b are positive integers, then '$a \le \sqrt{n}$ or $b \le \sqrt{n}$'" is false. That is, we assume that the statement $(a \le \sqrt{n}) \vee (b \le \sqrt{n})$ is false.

Using the De Morgan's law, we see that both $a \le \sqrt{n}$ and $b \le \sqrt{n}$ are false. This implies $a > \sqrt{n}$ and ~~a~~ $b > \sqrt{n}$

We can multiply these inequalities together to obtain

$ab > \sqrt{n} \cdot \sqrt{n} = n$ (Using the fact that if $0 < s < t$ and $0 < u < v$, then $su < tv$

This shows that $ab \neq n$, which contradicts the statement $n = ab$.

Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Hence, if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Vacuous and Trivial Proofs

We can quickly prove that a conditional statement $p \to q$ is true when we know that p is false, because $p \to q$ must be true when p is false. This method is called Vacuous proof, of the conditional statement $p \to q$.

Example: Prove that if n is an integer with $10 \leq n \leq 15$ which is a perfect square, then n is also a perfect ~~square~~ cube.

Sol: Note that there ~~is~~ are no perfect squares n with $10 \leq n \leq 15$, because $3^2 = 9$ and $4^2 = 16$. Hence, the statement that n is an integer with $10 \leq n \leq 15$ which is a perfect square is false for all integers n. Consequently, the statement to be proved is true for all integers n.

Trivial Proof

We can quickly prove a conditional statement $p \to q$ if we know that the conclusion q is true. By showing that q is true, it follows that $p \to q$ must also be true. A proof of $p \to q$ that uses the fact that q is true is called a "trivial proof".

Example: Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$", where the domain consists of all nonnegative integers. Show that $P(0)$ is true.

Sol: The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$".

Because $a^0 = b^0 = 1$, the ~~conditional~~ conclusion of the conditional statement "If $a \geq b$, then $a^0 \geq b^0$" is true. Hence, this conditional statement, which is $P(0)$, is true.

This is an example of a trivial proof. Note that the hypothesis, which is the statement "$a \geq b$" was not needed in this proof.

Proof by Contradiction

* Show that atleast four of any 22 days must fall on the same day of the week.

Sol: Let p be the proposition " At least four of 22 chosen days fall on the same day of the week." suppose that $\neg p$ is true. This means that atmost three of the 22 days falls on the same day of the week. Because there are seven days of the week, this implies that atmost 21 days could have been ~~chosen~~ chosen, as for each of the days of the week, at most three of the chosen days could fall on that day. This contradicts that premises that we have 22 days under consideration. That is, if r is the statement that 22 days are chosen, ~~we~~ then we have shown that $\neg p \rightarrow (r \wedge \neg r)$. Consequently, we know that p is true. We have proved that atleast four of 22 chosen days fall on the same day of the week.

Proofs of equivalence.

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we ~~shows~~ show that $p \to q$ and $q \to p$ are both true. The validity of this approach is based on the tautology

$$(p \leftrightarrow q) \leftrightarrow (p \to q) \wedge (q \to p)$$

Example: Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd.

Sol: This theorem has the form "p if and only if q," where p is "n is odd" and q is "n^2 is odd".

To prove this theorem, we need to show that $p \to q$ and $q \to p$ are true.

If n is odd, $n = 2k+1$, then $n^2 = (2k+1)^2 = 4k^2+4k+1$

$= 2(2k^2+2k)+1$ which is odd.

Hence $p \to q$ is true.

We prove $q \to p$ by the method of contraposition.

We take as our hypothesis the statement that n is not odd.

This means that n is even. Hence, $n = 2k$

$\Rightarrow n^2 = (2k)^2 = 4k^2 = \cancel{2k^2}\ 2(2k^2)$ which implies that n^2 is also even. Hence if n is an integer and n^2 is odd, then n is odd.

Because we have shown that both $p \to q$ and $q \to p$ are true, we have shown that the theorem is true.

Example: Show that these statements about the integers n are equivalent

P_1: n is even $\quad$ P_2: $n-1$ is odd $\quad$ P_3: n^2 is even.

Sol: We will show that these three statements are equivalent by showing that the conditional statements $P_1 \to P_2$, $P_2 \to P_3$ and $P_3 \to P_1$ are true.

We use a direct proof to show that $P_1 \to P_2$. Suppose that n is even. Then $n = 2k$, for some integer k. Consequently, $n-1 = 2k-1 = 2(k-1)+1$. This means that $n-1$ is odd because it is of the form $2m+1$ where m is the integer $k-1$.

We also use a direct proof to show $P_2 \to P_3$.

Suppose if $(n-1)$ is odd, then $n-1 = 2k+1$ for some integer k. Hence $n = 2k+2$ so that $n^2 = (2k+2)^2$ $= 4k^2+8k+4 = 2(2k^2+4k+2)$. Hence n^2 is even.

Next to prove that $P_3 \to P_1$, we use proof by ~~contradiction~~ contraposition. That is, we prove that if n is not even, then n^2 is not even. This is same as proving if n is odd, n^2 is odd. This completes the proof.

Counter Examples

To show that a statement of the form $\forall x\, P(x)$ is false we need to only find a counter example, that is, an example x for which $P(x)$ is false.

Example: Show that the statement "Every positive integer is the sum of the squares of two integers." is false.

Sol:- To show that this statement is false, we look for a counter example, which is a particular integer that is not the sum of the squares of two integers.

3 cannot be written as the sum of the squares of two integers. Note that the only perfect squares not exceeding 3 are $0^2=0$ and $1^2=1$. Furthermore, there is no way to get 3 as the sum of two terms each of which is 0 and 1.

Hence, "Every positive integer is sum of the squares of two integers" is false.

Extra Problems

* Show that the conditional statement is a tautology.

① $(p \wedge q) \rightarrow p$ (Apply conditional disjunction equivalence)

Sol: $\neg(p \wedge q) \vee p \equiv (\neg p \vee \neg q) \vee p$

(By De Morgan's law)

$\equiv \neg p \vee (\neg q \vee p)$ by associativity

$\equiv \neg p \vee (p \vee \neg q)$ Commutativity

$\equiv (\neg p \vee p) \vee \neg q$ associativity

$\equiv T \vee \neg q$ negation law

$\equiv T$ by Domination law

② $\neg p \rightarrow (p \rightarrow q)$ (prove that it is tautology)

$\neg p \rightarrow (p \rightarrow q) \equiv \neg p \rightarrow (\neg p \vee q)$ by conditional disjunction equivalence.

$\equiv \neg(\neg p) \vee (\neg p \vee q)$ by conditional disjunction equivalence

$\equiv p \vee (\neg p \vee q)$ Double negation law

$\equiv (p \vee \neg p) \vee q$ associativity

$\equiv T \vee q$ negation law

$\equiv T$ Domination law

$\neg(p \to q) \to p$ (Prove that it is a tautology)

$\equiv \neg(\neg p \vee q) \to p$ conditional disjunction equivalence

$\equiv \neg(\neg(\neg p \vee q)) \vee p$ conditional disjunction equivalence

$\equiv (\neg p \vee q) \vee p$ Double negation law

$\equiv (q \vee \neg p) \vee p$ Commutative law

$\equiv q \vee (\neg p \vee p)$ associative law

$\equiv q \vee T$ negation law

$\equiv T$ Domination law