Basic Concepts of Graph Theory

Graph: A graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

Remark: The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set or an infinite no. of edges is called an infinite graph, and in comparison, a graph with a finite vertex set and a finite edge set is called a finite graph.

Simple graph: A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Consequently, when there is an edge of a simple graph associated to $\{u, v\}$, we can also say, without possible confusion, that $\{u, v\}$ is an edge of the graph.

Multigraphs: Graphs may have multiple edges connecting the same vertices are called multigraphs.

Note: when there are m different edges associated to the same unordered pair of vertices $\{u,v\}$, we also say that $\{u,v\}$ is an edge of multiplicity m. That is, we can think of this set of edges as m different copies of an ~~e~~ edge $\{u,v\}$

Loops: Edges that connect a vertex to itself are called loops.

Pseudographs: We may have more than one loop at a vertex. Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex itself, are sometimes called pseudographs.

So far ~~w~~ the graphs we have introduced are undirected graphs. Their edges are also said to be undirected. However, to construct a graph model, we may find it necessary to assign directions to the edges of a graph.

Directed Graph: A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

Note: We obtain a directed graph when we assign a direction to each edge in an undirected graph.

Simple directed graph: When a directed graph has no loops and has no multiple directed ~~g~~ edges, it is called a simple directed graph.

As a simple directed graph has at most one edge associated to each ordered pair of vertices (u, v), we call (u, v) an edge if there is an edge associated to it in the graph.

Directed Multigraphs: ~~Directed~~ graphs may have multiple directed edges from a vertex to a second (possibly the same) vertex. ~~are used to model such networks.~~ We called such graphs directed multigraphs.

Note: when there are m directed edges, each associated to an ordered pair of vertices (u,v), we say that (u,v) is an edge of multiplicity m

Mixed graph: A graph with both directed and undirected edges is called a mixed graph.

Graph terminology

Type	Edges	Multiple edges allowed?	Loops allowed
Simple graph	Undirected	No	No.
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No.
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes.

Basic Terminology and their properties

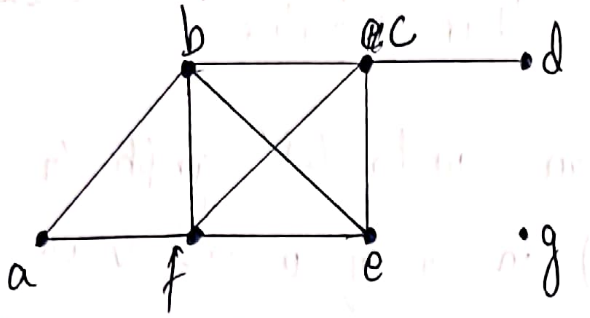
* Two Vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. such an edge e is called incident with the vertices u and v and e is said to connect u and v.

* The set of all neighbors of a vertex v of $G = (V, E)$ denoted by $N(v)$, is called the neighborhood of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A.

So $N(A) = \bigcup_{v \in A} N(v)$.

* The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $deg(v)$.

Example: what are the degrees and what are the neighborhoo of the vertices in the Graphs G?



Undirected graph G

Solution: In G, deg(a) = 2 , deg (b) = 4,

deg (c) = 4   deg (e) = 3   deg (f) = 4,   deg(d) = 1

deg(g) = 0.

The neighborhoods of these vertices are N(a) = {b,f}

N(b) = {a, c, e,f},   N(c) = {b, d, e, f}   , N(d) = {c}

N(e) = { b, c, f} , N(f) = { a, b, c, e}   and   N(g) = ϕ.

* A vertex of degree zero is called isolated. It means that an isolated vertex is not adjacent to any vertex.

* A vertex v is pendent if and only if it has degree one.

The Handshaking theorem:

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v)$$

(Note that this applies even if multiple edges and loops are present)

Example: How many edges are there in a graph with 10 vertices each of degree six?

Sol: Because the sum of the degrees of the vertices is $6 \times 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$.

* **Note:** The above theorem shows that the sum of the degrees of the vertices of an undirected graph is even.

Theorem: An undirected graph has even number of vertices of odd degree.

Proof: Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph $G = (V, E)$ with m edges. Then,

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

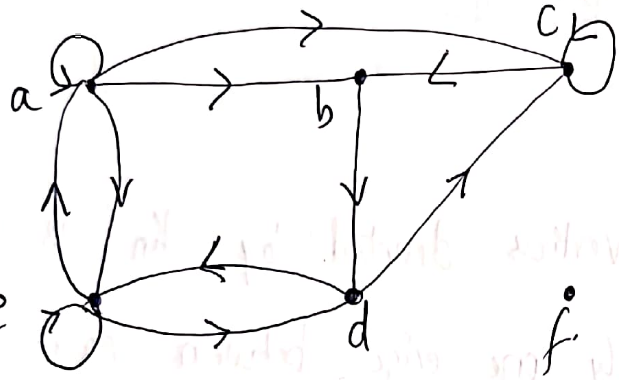
As $deg(v)$ is even for $v \in V_1$, the first term in the right-hand side of the last equality is even. Furthermore, the sum of the two terms on the right-hand side of the last equality is even, because this sum is $2m$. Hence, the second term in the sum is also even. As all the terms in this sum are odd, there must be an even number of such terms. Thus there are even number of vertices of odd degree.

* When ~~u and v~~ (u,v) is an edge of the graph, G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex of (u,v), and v is called the terminal or end vertex of (u,v). The initial vertex and terminal vertex of a loop are the same.

* In a graph with directed edges the in-degree of a vertex v, denoted by $deg^-(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $degree^+(v)$, is the number of edges with v as their initial

vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out degree of this vertex.)

Example: Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in the figure



G.

Directed graph G.

Sol: The in-degree in G are $degree^-(a) = 2$, $degree^-(b) = 2$, $degree^-(c) = 3$, $degree^-(d) = 2$, $deg^-(e) = 3$, $deg^-(f) = 0$

The out-degrees are $deg^+(a) = 4$, $deg^+(b) = 1$, $deg^+(c) = 2$, $deg^+(d) = 2$, $deg^+(e) = 3$, and $degree^+(f) = 0$.

Theorem: Let $G = (V,E)$ be a graph with directed edges.

Then, $$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|.$$

Definition: The undirected graph that results from ignoring

directions of edges is called the underlying undirected graph.
A graph with directed edges and its underlying undirected graph have the same number of edges

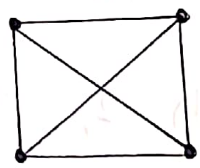
Some Special simple graphs

Example: Complete graphs
A complete graph on n vertices denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. The graph K_n, for $n = 1, 2, 3, 4, 5, 6$; are displayed below. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called noncomplete.

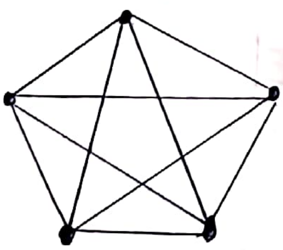
K_1

K_2

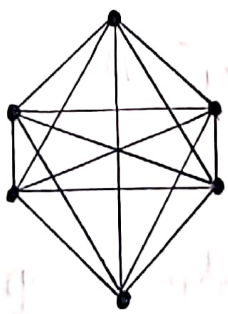
K_3



K_4



K_5



K_6.

Example: Cycle

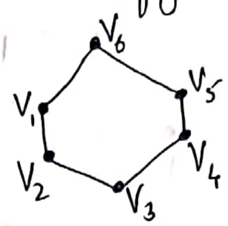
A cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\ldots$, $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.

The cycle C_3, C_4, C_5, C_6, are displayed in the figure:

C_3

C_4

C_5

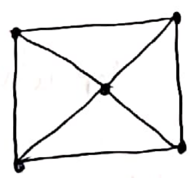


C_6.

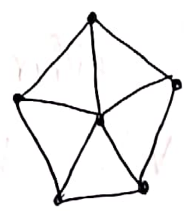
Example wheels

We obtain a wheel W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connects this new vertex to each of the n vertices in C_n, by the new edges. The wheels W_3, W_4, W_5 and W_6 are displayed in Fig:

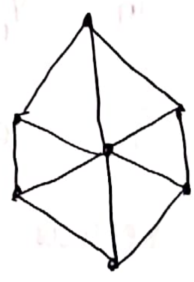
W_3



W_4



W_5

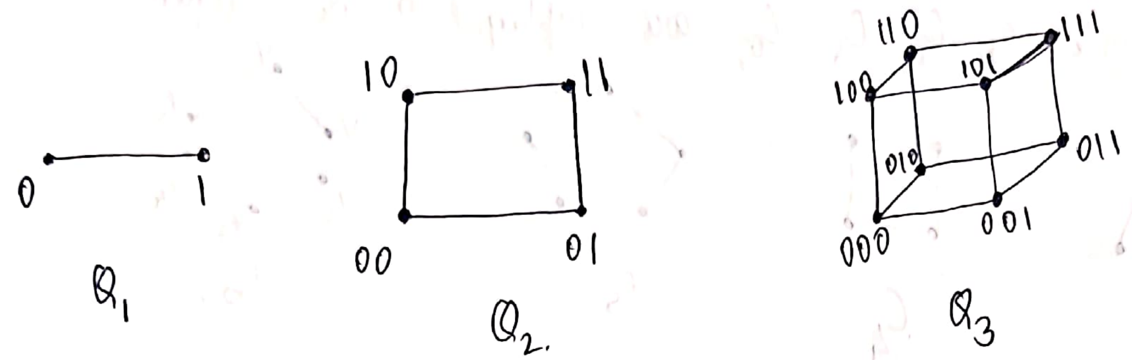


W_6.

Example n- Cubes

An n-dimensional hypercubes or n-cube denoted by Q_n, is a graph that has vertices representing the 2^n

bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in one ~~bu~~ bit position. Q_1, Q_2, Q_3 are displayed below:

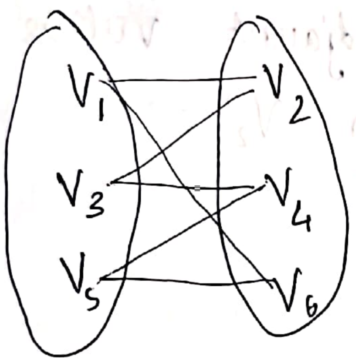


Bipartite Graphs

* A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Example: K_3 is not bipartite. Note that if we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain 2 vertices. If the graph were bipartite, these two vertices, could not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge.

Example: C_6 is bipartite. Because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.



Theorem: A simple graph is bipartite if and only if it is possible to assign one of two different colours to each vertex of the graph so that no two adjecent vertices are assigned the same order.

Proof: First, suppose that $G = (V, E)$ is a bipartite simple graph. Then $V = V_1 \cup V_2$, where V_1 and V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2. If we assign one color to each vertex in V_1 and a second colour to each vertex in V_2, then no two adjecent vertices are assigned the same ~~colour~~ color.

Now suppose that it is possible to assign colors to the vertices of the graph using just two colors so that no two adjacent vertices are assigned the same color.

Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other color.

Then V_1 and V_2 are disjoint and $V = V_1 \cup V_2$.

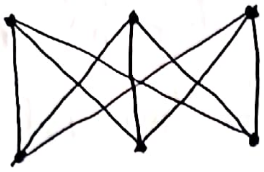
Furthermore, every edge connects a vertex in V_1 and a vertex in V_2 because no two adjacent vertices are either both in V_1 or both in V_2.

Consequently, G is bipartite.

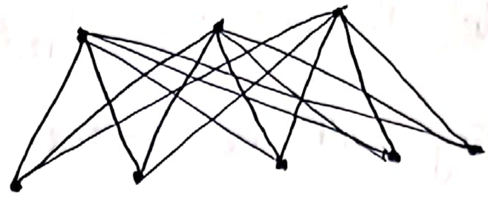
Complete bipartite Graphs

A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset. The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$ are displayed below:

$K_{2,3}$



$K_{3,3}$



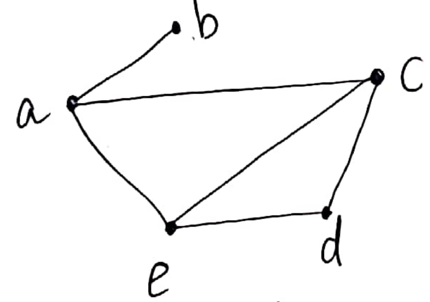
$K_{3,5}$

Representing Matrices

One way to represent a graph without multiple edges is to list all the edges of this graph.

Another way to represent a graph without no multiple edges is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph.

Example: Use adjacency list to describe the simple graph given in the figure below;

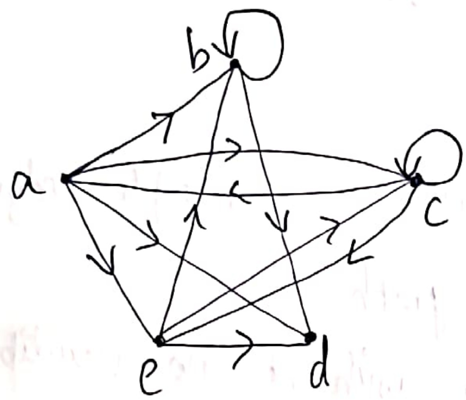


A simple graph

Sol: Adjacency list

Vertex	Adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Example: Represent the directed graph (or digraph) in the figure by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph



A directed graph

Solution: An adjacency list for a directed graph

Initial Vertex	Terminal Vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d.

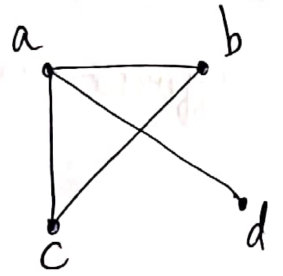
Adjacency Matrices

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots v_n$. The adjacency matrix A (or A_G) of G with respect to this listing of the vertices, is the $n \times n$

zero-one matrix with 1 as its (i,j)th entry when v_i and v_j are adjacent, and 0 as its (i,j)th entry when they are not adjacent. In other words, if its adjacency matrix is $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 & \text{otherwise.} \end{cases}$$

<u>Example</u>: Use an adjacency matrix to represent the graph:



<u>Sol</u>: We order the vertices as a, b, c, d. The matrix representing this graph is

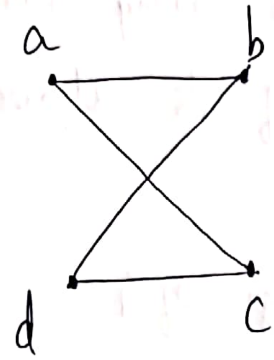
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

<u>Example</u>: Draw a graph with the adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \cancel{0}\,1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

with respect to the ordering of vertices a, b, c, d.

Sol: The graph is;



Note: ① An adjacency matrix of a graph is based on the ordering chosen for the ~~mat~~ vertices. Hence, there may be as many as $n!$ different adjacency matrices, for a graph with n vertices, because there are $n!$ different ordering of n vertices.

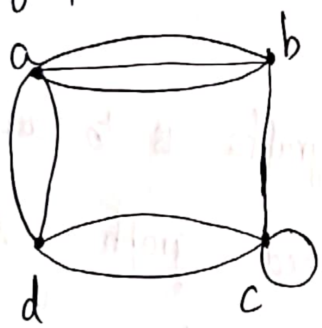
* The adjacency matrices of a simple graph is symmetric, that is, $a_{ij} = a_{ji}$, because both of these entries are 1 when V_i and V_j are adjacent, and both are zero otherwise. Furthermore, because a simple graph has no loops, each entry a_{ii}, $i = 1, 2, 3, \ldots n$ is 0.

* An adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges. A loop at the vertex V_i is represented by a 1 at the (i, i) th ~~posd~~ position of the adjacency matrix.

When multiple edges connecting the same pair of vertices v_i and v_j or multiple loops at the same vertex, are present, the adjacency matrix is no longer a zero-one matrix, because the (i, j)th entry of this matrix equals the number of edges that are associated to $\{v_i, v_j\}$.

All ~~the~~ undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

Example: Use an adjacency matrix to represent the pseudograph:



Sol: The adjacency matrix using the ordering of vertices a, b, c, d is:

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

Note: The matrix for directed graph $G = (V, E)$ has a 1 in its (i, j)th position if there is an edge from v_i to v_j where $v_1, v_2, \ldots v_n$ is an arbitrary listing of the vertices of the directed graph. In other words, if $A = [a_{ij}]$ is the adjacency matrix for the directed graph with respect to this listing of the vertices, then $a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$

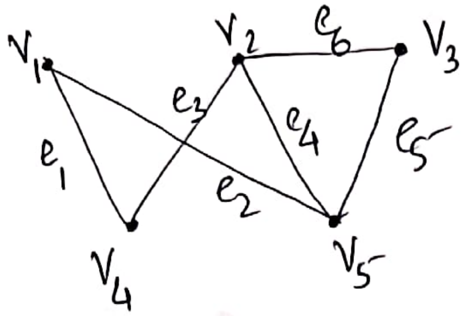
The adjacency matrix for a directed graph does not have to be symmetric, because there may not be an edge from v_j to v_i when there is an edge from v_i to v_j.

Incidence Matrices

Another common way to represent graphs is to use incidence matrices. Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

Example: Represent the graph with an incidence matrix.



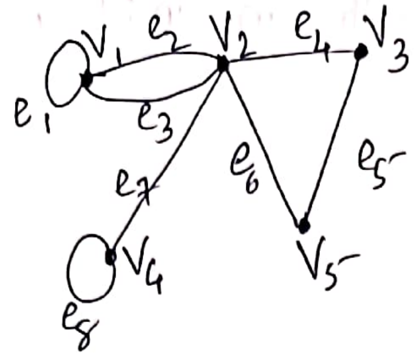
Sol: The incidence matrix is;

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Note: Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries, because these edges are incidence with the same pair of vertices. Loops are represented using a columns with exactly one entry equals to 1, corresponding to the vertex that is incident with this loop.

Example: Represent the pseudograph shown in the fig. using an incidence matrices.

~~Sol~~:



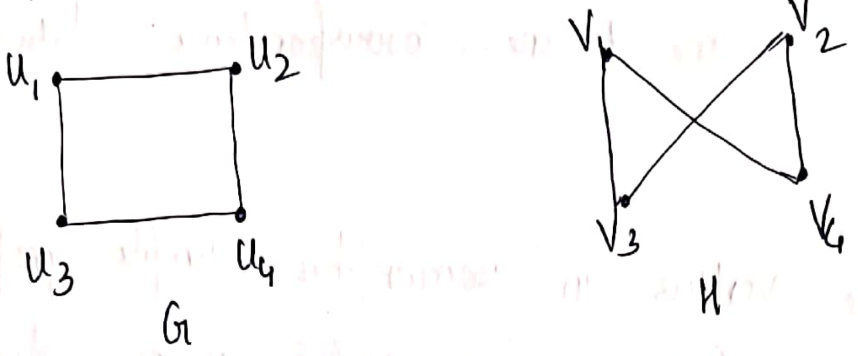
Sol:

$$
\begin{array}{c c}
 & \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

Isomorphism of Graphs

Definition: A simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.
Such a function f is called an isomorphism. Two simple graphs that are not isomorphic are called non isomorphic.

Example: show that the graphs $G = (V, E)$ and $H = (W, F)$ displayed below are isomorphic.



Sol: The function f with $f(u_1) = V_1$, $f(u_2) = V_4$, $f(u_3) = V_3$, and $f(u_4) = V_2$ is a one one correspondence between V and W. To see this correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_3 and u_4, u_2 and u_4 and each of the pairs $f(u_1) = V_1$ and $f(u_2) = V_4$, $f(u_1) = V_1$ and $f(u_3) = V_3$, ~~and~~ $f(u_2) = V_4$ and $f(u_4) = V_2$, $f(u_3) = V_3$ and $f(u_4) = V_2$ consists of two adjacent vertices in H.

Determining whether two simple Graphs are isomorphic

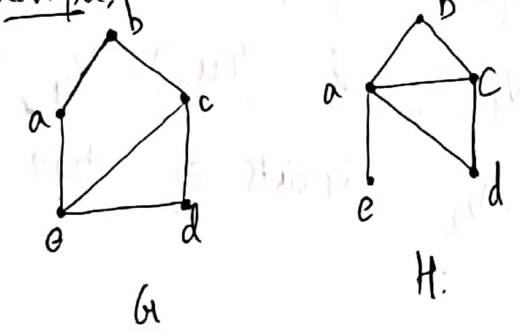
* A property preserved by isomorphism of graphs is called a graph invariant

For example, Isomorphic simple graphs must have the same number of vertices, because there is a one to one correspondence between the sets of vertices of the graphs.

* Isomorphic simple graphs also must have the same number of edges, because the one to one correspondence between vertices establishes a one to one correspondence between edges.

* The degrees of the vertices in isomorphic simple graphs must be the same. That is, a vertex v of degree d in G must corresponds to a vertex f(v) of degree d in H, because a vertex w in G is adjacent to v if and only if f(v) and f(w) are adjacent in H.

* <u>Example</u>: Show that the graphs G and H are not isomorphic.



<u>Sol:</u> Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic.

* Another way to see that G and H are not isomorphic is to note that the subgraphs of G and H made up of vertices of degree three and the edge connecting them must be isomorphic if ~~these~~ the two graphs are isomorphic.

Connectivity

Many problems can be modelled with paths formed by traveling along the edges of graphs. For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model.

Path A path is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

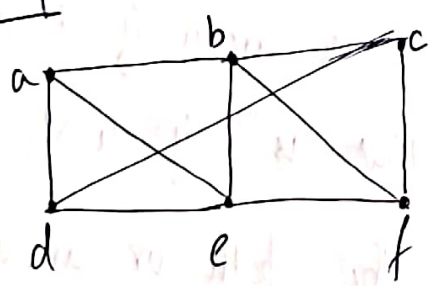
Def: Let n be a nonnegative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, \ldots n$ the endpoints x_{i-1} and x_i. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots x_n$ (because listing these vertices uniquely determines the path). The path is a circuit if it begins and ends at the same vertex; that is, if $u = v$; and has a length greater than zero. The path or circuit is said to pass through the vertices $x_1, x_2, \ldots x_{n-1}$ or transverse the edges $e_1, e_2 \ldots e_n$. A path or circuit is simple if it does not contain the same edge more than once.

Note: A path of length zero consists of a single vertex.

Remark: The term walk is also used instead of path, where a walk is defined to be an alternating sequence of vertices and edges of a graph, $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$, where v_{i-1} and v_i are the endpoints of e_i for $i = 1, 2, \ldots n$. When this terminology is used, closed walk is used instead of circuit to indicate a walk that begins and ends at the same vertex.

A trial is used to denote a walk that has no repeated edge (replacing the term simple path.)

Example:



G.

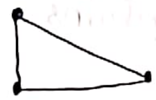
In the simple graph G, a, d, c, f, e is a simple path of length 4., because {a,d}, {d,c}, {c,f} and {f,e} are all edges. However, d, e, c, a is not a path, because {e, c} is not an edge. Note that b, c, f, e, b is a circuit of length 4 because {b,c}, {c,f}, {f, e}, and {e,b} are edges, and this path begins and ends at b. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains an edge {a,b} twice.

Def: Let n be a nonnegative integer and G a directed graph. A path of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2) and so on, with e_n associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, \ldots x_n$. A path of length greater than zero, that begins and ends at the same vertex is called a circuit or cycle. A path or circuit is called simple if it does not contain the same edge more than once.

Def: An undirected graph is connected if there is a path between every pair of distinct vertices of the graph.
An undirected graph that is not connected is called disconnected. We say that we disconnect a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

Example: The graph: G_1 and G_2 are connected.

G_1



G_2

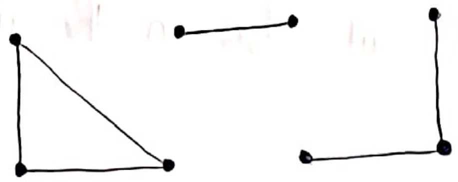
As every pair of distinct vertices there is a path between them.

Theorem: There is a simple path between every ~~path~~ pair of distinct vertices of a connected undirected graph.

Connected component: A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.
That is, a connected component of a graph G is a maximal connected subgraph of G. A graph that is not connected has two or more connected components that are

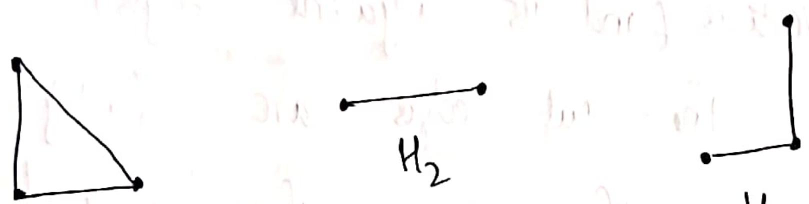
disjoint and have G as their union.

<u>Example:</u> What are the connected components of the Graph H?



H

<u>Sol:</u> The Graph H is the union of three disjoint connected subgraphs H_1, H_2, H_3.
These 3 subgraphs are the connected components of H.



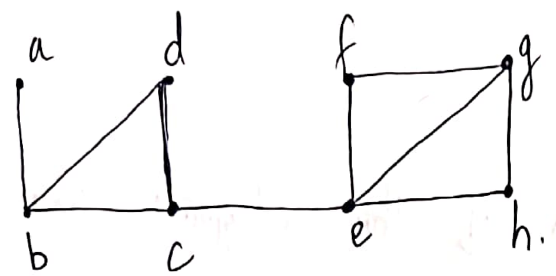
<u>Cut Vertices:</u> Removal of a vertex and all incident edges produces a subgraph with more connected components. Such vertices are called <u>cut vertices</u>. (or articulation points)

<u>Note:</u> The removal of a cut vertex from a connected graph produces a subgraph that is not connected.

Cut edge or bridge:

An ~~ed.~~ edge whose removal produces a graph with more connected components than in the original graph is called a cut edge or bridge.

Example: Find the cut vertices and cut edges in the graph G_1



G_1

Sol: The cut vertices of G_1 are b, c, and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are $\{a, b\}$ and $\{c, e\}$. Removing either one of these edges disconnects G_1.

Vertex Connectivity

Not all graphs have cut vertices. For example, the complete graph K_n, where $n \geq 3$, has no cut vertices. When you remove a vertex from K_n and all edges incident to it, the resulting subgraph is the complete graph K_{n-1}, a connected graph.

Connected graphs without cut vertices are called <u>non separable graphs</u>

* A subset V' of the vertex set V of $G = (V, E)$ is a <u>vertex cut</u>, or separating set, if $G - V'$ is disconnected.

* Every connected graph, except a completed graph, has a vertex cut.

* <u>Vertex connectivity</u> of a noncomplete graph G, denoted by $k(G)$, is the minimum number of vertices in a vertex cut

* When G is a complete graph, it has no vertex cuts, because removing any subset of its vertices and all incident edges still leaves a complete graph. Consequently, we cannot define $k(G)$ as the minimum number of vertices in a vertex cut when G is complete. Instead we set $k(G) = n-1$, the number of vertices needed to be removed to produce a graph with a single vertex.

→ We have, $0 \leq k(G) \leq n-1$, if G has n vertices.

→ $k(G) = 0$ if and only if G is disconnected. or $G = K_1$.

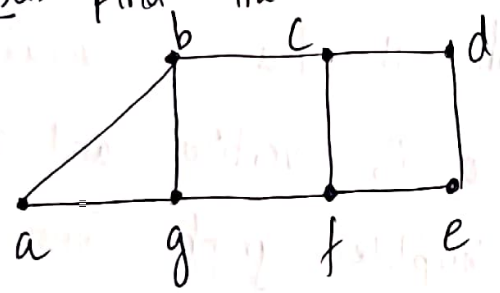
→ $k(G) = n-1$ if and only if ~~K~~ G is complete.

* We say that a graph is k- connected (or k- vertex connected), if $K(G) \geq k$.

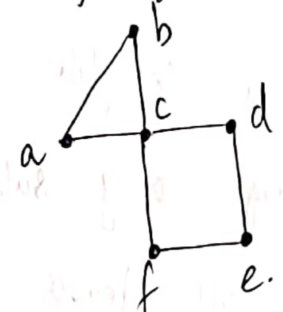
→ A graph G is 1-connected if it is connected and not a graph containing a single vertex, a graph is 2- connected or biconnected, if it is non separable and has at least three vertices.

→ Note that if G is a k- connected graph, then G is a j- connected graph for all j with $0 \leq j \leq k$.

Example: Find the vertex connectivity for the graph: G_1 and G_2



G_1



G_2

Sol: G_1 has no cut vertices. $\{b, g\}$ is a vertex cut.

Hence, $K(G_1) = 2$

Note that c is a cut vertex of G_2.

Hence $K(G_2) = 1$.

Edge Connectivity

We can also measure the connectivity of G in terms of the minimum number of edges that we can remove to disconnect it. If a graph has a cut edge, then we need to remove it to disconnect G. If G does not have a cut edge, we need to look for the smallest set of edges that can be removed to disconnected it.

A set of edges E' is called an edge cut of G if the subgraph $G-E'$ is disconnected. The edge connectivity of a graph G, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of G. This defines $\lambda(G)$ for all connected graphs with more than one vertex because it is always possible to disconnect such a graph by removing all edges incident to one of its vertices.

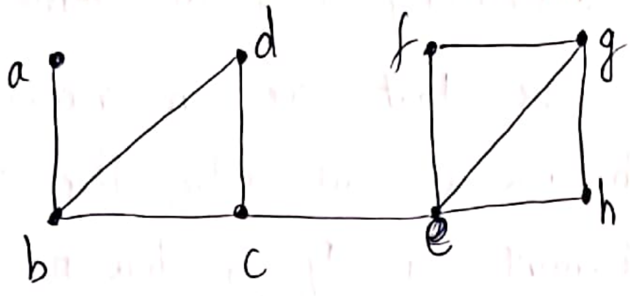
→ $\lambda(G) = 0$ if G is not connected.

→ $\lambda(G) = 0$ if G is a graph with single vertex

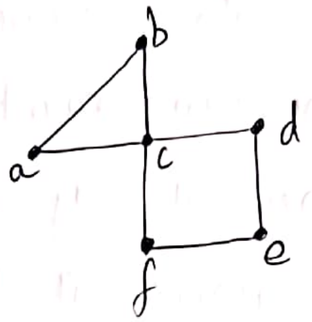
→ If G is a graph with n vertices, then $0 \le \lambda(G) \le n-1$.

→ $\lambda(G) = n-1$ where G is a graph with n vertices if and only if $G = K_n$, therefore, $\lambda(G) \le n-2$, when G is not a complete graph.

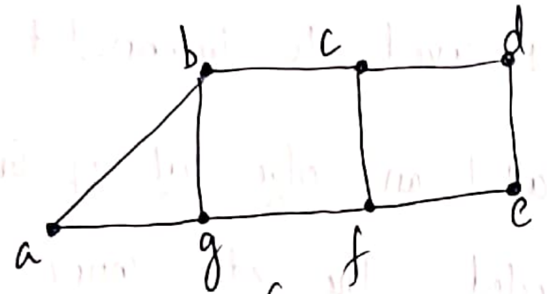
Example: Find the edge connectivity of the following graph :



G_1



G_2



G_3

Sol: Each of the three graphs are connected and has more than one vertex, so we ~~can~~ know that all of them have ~~poss~~ positive edge connectivity.

G_1 has a cut edge $\{c, e\}$ so $\lambda(G_1) = 1$

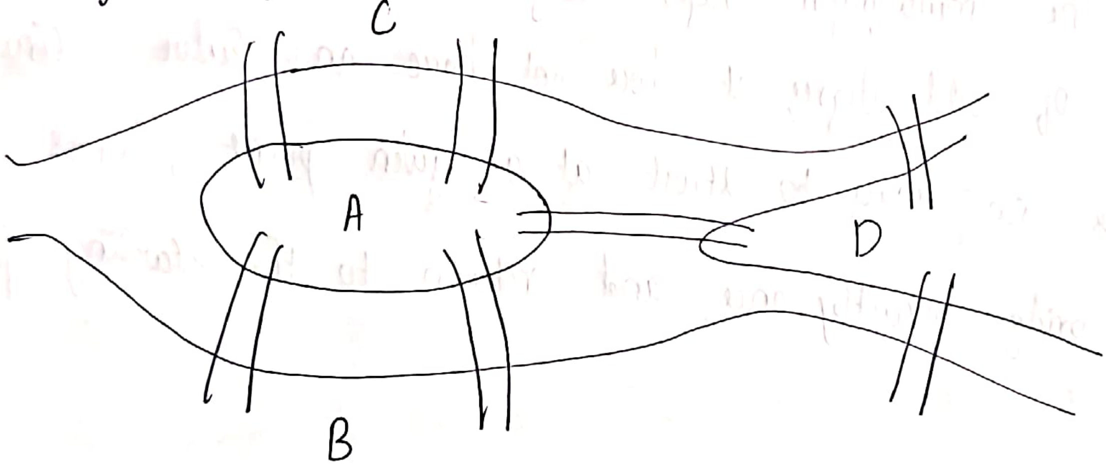
The graph G_2 has no cut edges, but the removal of the two edges $\{a, b\}$ and $\{a, c\}$ disconnects it.

Hence, $\lambda(G_2) = 2$

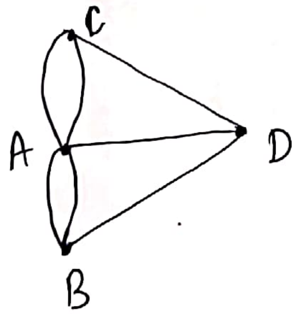
Similarly $\lambda(G_3) = 2$, because G_3 has no cut edges, but the removal of the two edges $\{b, c\}$ and $\{f, g\}$ disconnects it.

Königsberg Bridge Problem

In the town of ~~Go~~ Königsberg, Prussia was divided into four sections by the branches of the Pregel river. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions. Figure below depicts these regions and bridges



Seven bridges of Königsberg



Multigraph model of the town of Königsberg

The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice and return to the same starting point.

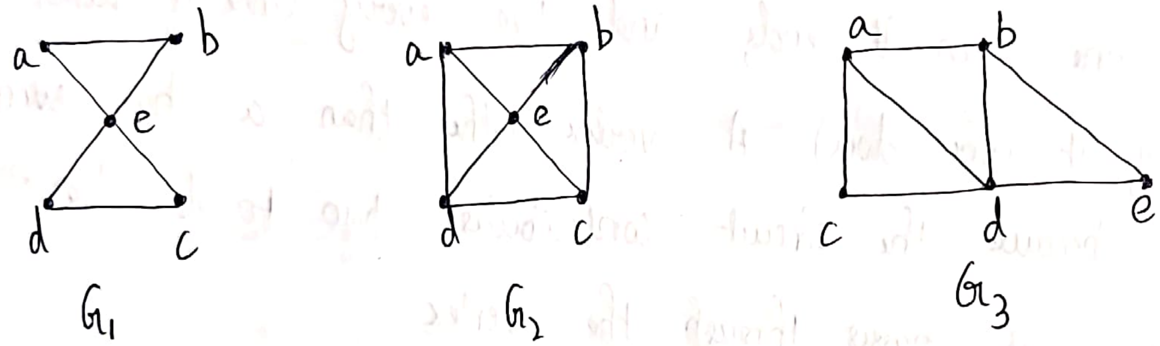
The Swiss Mathematician Leonhard Euler solved this problem. Euler studied this problem using the multigraph obtained when the four regions are represented by vertices and the bridges by edges.

As the multigraph representing these bridges, has four vertices of odd degree, it does not have an Euler circuit. There is no way to start at a given point, cross each bridge exactly once, and return to the starting point.

Euler Circuit and Euler path

An Euler Circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

Example Which of the undirected graphs have Euler Circuit? Of those that do not, which have an Euler path?



Sol: The graph G_1 has an Euler Circuit, for example, a, e, c, d, e, b, a. Neither of the graphs G_2 or G_3 has an Euler Circuit. However, G_3 has an Euler path, namely, a, c, d, e, b, d, a, b. G_3 does not have an Euler path.

Necessary and Sufficient conditions for Euler Circuits and paths

If a Connected multigraph has an Euler Circuit then every vertex must have an even degree. An Euler Circuit begins with a vertex a and continues with an edge incident with a,

say $\{a,b\}$. The edge $\{a,b\}$ contributes one to deg(a).
Each time the circuit passes through a vertex it contributes two to the vertex's degree, because the circuit enters via an edge incident with this vertex and leaves via another such edge. Finally, the circuit terminates where it started, contributing one to deg(a). Therefore, deg(a) must be even, because the circuit contributes one when it begins, one when it ends, and two every time it passes through a (if it ever does). A vertex other than a has even degree because the circuit contributes two to its degree each time it passes through the vertex.
We conclude that if a connected graph has an Euler circuit, then every vertex must have even degree.
This necessary condition is also sufficient to have an Euler circuit.

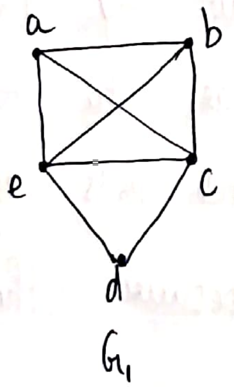
Theorem 1: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem 2 A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree

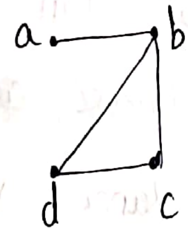
Hamiltoni Path and Circuits

Def: A simple path in a Graph G that passes through every vertex exactly once is called a Hamilton path, and a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton Circuit. That is, the simple path $x_0, x_1, \ldots x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots x_{n-1}, x_n$ is a Hamilton path.

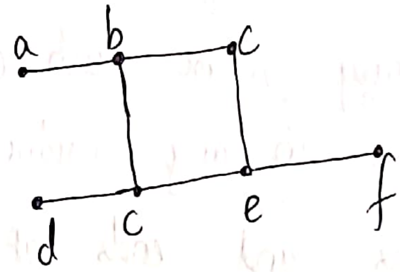
Example: which of the simple graphs in the figure have a Hamilton circuit or, if not, a Hamilton path?



G_1



G_2



G_3

Sol: G_1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G_2. (this can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice), but G_2 does have a Hamilton path, namely, a, b, c, d. G_3 has neither Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a,b\}$, $\{e,f\}$, and $\{c,d\}$ more than once.

<u>Result</u>: A graph with a vertex of degree one cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex is incident with two edges in a circuit.

* A Hamilton circuit cannot contain a smaller circuit within it.

<u>Example</u>: Show that K_n has a Hamilton circuit whenever $n \geq 3$.

<u>Sol</u>: We can form a Hamilton circuit in K_n beginning ~~at~~ at any vertex. Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once. This is possible because there are edges in K_n between any two vertices.

Dirac's theorem:

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is atleast $n/2$, Then G has a Hamilton circuit.

Ore's Theorem:

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Weighted graph:

Graphs that have a number assigned to each edge are called weighted graphs

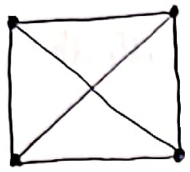
Planar graph

A graph is called planar if it can be drawn in the plane without any edge crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoints).

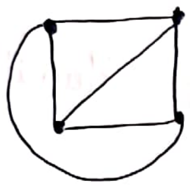
Such a drawing is called a planar representation of the graph.

Example 1 Is K_4 planar?

Sol: K_4 is :



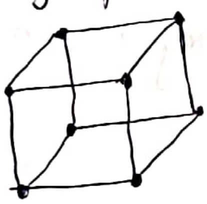
K_4 is planar because it can be drawn without crossings as shown below



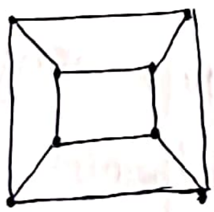
K_4 drawn with no crossing

Example: Is Q_3 planar?

Q_3

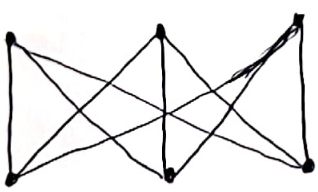


Sol: Q_3 is planar, because it can be drawn without any edges crossings,



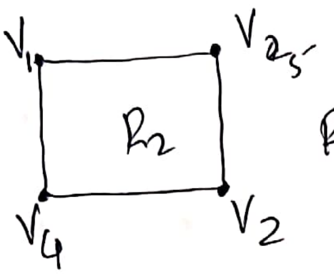
A planar representation of Q_3.

Example: Is $K_{3,3}$ planar?

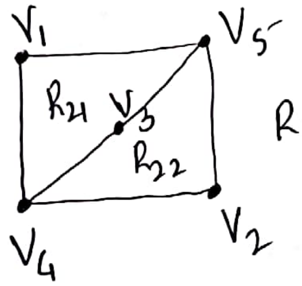


The graph $K_{3,3}$

Sol: In any planar representation of $K_{3,3}$ the vertices V_1 and V_2 must be connected to both V_4 and V_5. These ~~two~~ four edges form a closed curve that splits the plane into two regions R_1 and R_2 as shown below;



The vertex V_3 is in either R_1 or R_2. When V_3 is in R_2, the inside of the closed curve, the edges between V_3 and V_4 and between V_3 and V_5 separate R_2 into two subregions, R_{21} and R_{22}; as shown below



Next note that there is no way to place the final vertex V_6 without forming a crossing. For if V_6 is in R_1, then the edge between V_3 and V_6 cannot be drawn without a crossing. If V_6 is in R_{22}, then the edge between V_1 and V_6 cannot be drawn without crossing. If V_6 is in R_{21} then the edge between V_2 and V_6 cannot be drawn without a Crossing.

A similar argument can be used when V_3 is in R_1

It follows that $K_{3,3}$ is not planar.

Graph Colouring

Def: A colouring of a simple graph is the assignment of a ~~colour~~ color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Def: The chromatic number of a graph is the least number of colors needed for a coloring of this graph. Chromatic number of a graph G is denoted by $\chi(G)$. (χ - chi)

Four color theorem

The chromatic number of a planar graph is no greater than four.

Example: what is the chromatic number of K_n?

Sol: A coloring of K_n can be constructed using n colors by assigning a different color to each vertex. There is no way to color using fewer colors. Because every two vertices of a complete graph is adjacent. Hence the chromatic number is of K_n is n. That is $\chi(K_n) = n$.

(K_n is not planar when $n \geq 5$, so this result does not contradict the four color theorem).

Example: What is the chromatic number of the complete bipartite graph $K_{m,n}$ where m and n are possible positive integers?

Sol: The number of colors does not depend on m and n. Only two colors are needed, as $K_{m,n}$ is a bipartite graph. Hence $\chi(K_{m,n}) = 2$. This means that we can color the set of m vertices with one color and the set of n vertices with a second color. As the edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same color. ∎

Example: What is the chromatic number of C_n? where $n \geq 3$ (C_n is a cycle with n vertices)

Sol: Two colors are needed to color C_n when n is even. To construct such a coloring, simply pick a vertex and color it red. Proceed around the graph in a clockwise direction coloring the second vertex blue, the third vertex red, and so on. The n^{th} vertex can be colored blue, because the two vertices adjacent to it, namely the $(n-1)$ th and the first vertices, are both colored red.

When n is odd, and $n>1$, the chromatic number of C_n is 3. To see this, fix an initial vertex. To use only two colors, it is necessary to alternate colours as the graph is traversed in a clockwise direction. However, the n^{th} vertex reached is adjacent to two vertices of different colors, namely the first and $(n-1)^{th}$ vertex. Hence, the third color must be used.

$\chi(C_n) = 2$ if n is an even positive integer with $n \geq 4$

$\chi(C_n) = 3$ if n is an odd positive integer with $n \geq 3$

Applications of Graph coloring

① Scheduling final exams

To schedule final exams at a university so that no student has two exams at the same time.

② Frequency assignments

Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. The assignment of channels can be modelled by graph coloring.