8.4 SCALAR AND VECTOR POINT FUNCTIONS

(1) *If to each point P*(**R**) *of a region E in space there corresponds a definite scalar denoted by f* (**R**), *then f* (**R**) *is called a* **scalar point function** *in E. The region E so defined is called a* **scalar field.**

The temperature at any instant, density of a body and potential due to gravitational matter are all examples of scalar point functions.

(2) *If to each point P*(**R**) *of a region E in space there corresponds a definite vector denoted by* **F**(**R**), *then it is called the* **vector point function** *in E. The region E so defined is called a* **vector field.**

The velocity of a moving fluid at any instant, the gravitational intensity of force are examples of vector point functions.

Differentiation of vector point functions follows the same rules as those of ordinary calculus. Thus if $\mathbf{F}(x, y, z)$ be a vector point function, then

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{dt}$$ [See (*iii*) p. 203]

and
$$d\mathbf{F} = \frac{\partial \mathbf{F}}{\partial x}dx + \frac{\partial \mathbf{F}}{\partial y}dy + \frac{\partial \mathbf{F}}{\partial z}dz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)\mathbf{F} \qquad ...(i)$$

(3) Vector operator del. The operator on the right side of the equation (*i*) is in the form of a scalar product of $\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}$ and $\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

If ∇ (*read as del*) *be defined by the equation* $\nabla = \mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}$ ...(*ii*)

then (*i*) may be written as $d\mathbf{F} = (\nabla \,.\, d\mathbf{R})\,\mathbf{F}$ for when $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, $d\mathbf{R} = \mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

8.5 DEL APPLIED TO SCALAR POINT FUNCTIONS—GRADIENT

(1) Def. *The vector function ∇f is defined as the gradient of the scalar point function f and is written as grad f.*

Thus
$$\text{grad } f = \nabla f = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}$$

(2) Geometrical interpretation. Consider the scalar point function $f(\mathbf{R})$, where $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.

If a surface $f(x, y, z) = c$ be drawn through any point $P(\mathbf{R})$ such that at each point on it, the function has the same value as at P, then such a surface is called a *level surface* of the function f through P, *e.g.*, equipotential or isothermal surface (Fig. 8.6).

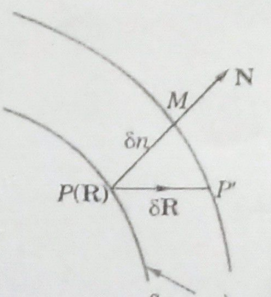

Fig. 8.6

Let $P'(\mathbf{R} + \delta\mathbf{R})$ be a point on a neighbouring level surface $f + \delta f$. Then

$$\nabla f.\, \delta\mathbf{R} = \left[\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right] . (\mathbf{I}\delta x + \mathbf{J}\delta y + \mathbf{K}\delta z)$$
$$= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = \delta f.$$

Now if P' lies on the same level surface as P, then $\delta f = 0$, *i.e.*, $\nabla f \,.\, \delta\mathbf{R} = 0$. This means that ∇f is perpendicular to every $\delta\mathbf{R}$ lying on this surface. Thus ∇f *is normal to the surface* $f(x, y, z) = c$.

$$\therefore \qquad \nabla f = |\nabla f|\,\mathbf{N}$$

where $\mathbf{N}$ is a unit vector normal to this surface. If the perpendicular distance PM between the surfaces through P and P' be δn, then the rate of change of f normal to the surface through P

$$= \frac{\partial f}{\partial n} = \underset{\delta n \to 0}{\text{Lt}} \frac{\delta f}{\delta n} = \underset{\delta n \to 0}{\text{Lt}} \nabla f \,.\, \frac{\delta\mathbf{R}}{\delta n}$$
$$= |\nabla f| \underset{\delta n \to 0}{\text{Lt}} \frac{\mathbf{N}\,.\,\delta\mathbf{R}}{\delta n} = |\nabla f|. \qquad [\because \ \mathbf{N}\,.\,\delta\mathbf{R} = |\delta\mathbf{R}| \cos\theta = \delta n]$$

Hence the magnitude of $\nabla f = \partial f / \partial n$.

Thus *grad f is a vector normal to the surface f = constant and has a magnitude equal to the rate of change of f along this normal.*

(3) Directional derivative. If δr denotes the length PP' and $\mathbf{N}'$ is a unit vector in the direction PP', then the limiting value of $\delta f/\delta r$ as $\delta r \to 0$ (*i.e.*, $\partial f/\partial r$) is known as the *directional derivative of f at P along the direction PP'*.

Since $$\delta r = \delta n/\cos\alpha = \delta n/\mathbf{N}\,.\,\mathbf{N}'$$

$$\therefore \quad \frac{\partial f}{\partial r} = \underset{\delta r \to 0}{\text{Lt}} \left[\mathbf{N}\,.\,\mathbf{N}'\,\frac{\delta f}{\delta n}\right] = \mathbf{N}'\,.\,\frac{\partial f}{\partial n}\,\mathbf{N} = \mathbf{N}'\,.\,\nabla f$$

Thus the directional derivation of f in the direction of $\mathbf{N}'$ is the resolved part of ∇f in the direction $\mathbf{N}'$.

Since $$\nabla f\,.\,\mathbf{N}' = |\,\nabla f\,|\cos\alpha \le |\,\nabla f\,|$$

It follows that ∇f *gives the maximum rate of change of f.*

Example 8.9. *Prove that* $\nabla r^n = nr^{n-2}\,\mathbf{R}$, *where* $\mathbf{R} + x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.

(*Bhopal, 2007 ; Anna, 2003 S ; V.T.U., 2000*)

Solution. We have $f(x, y, z) = r^n = (x^2 + y^2 + z^2)^{n/2}$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}.\ 2x = nxr^{n-2}. \text{ Similarly, } \frac{\partial f}{\partial y} = nyr^{n-2} \text{ and } \frac{\partial f}{\partial z} = nzr^{n-2}$$

Thus $$\nabla r^n = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z} = nr^{n-2}(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) = nr^{n-2}\,\mathbf{R}.$$

Otherwise : The level surfaces for f = constant, *i.e.*, r^n = constant are concentric spheres with centre O and hence unit normal $\mathbf{N}$ to the level surface through P is along the radius $\mathbf{R}$

i.e., $$\mathbf{N} = \hat{\mathbf{R}}.$$

$$\therefore \quad \nabla f = \frac{\partial f}{\partial n}\,.\,\mathbf{N} = \frac{df}{dr}\hat{\mathbf{R}} = nr^{n-1}\,\hat{\mathbf{R}} \qquad [\because\ f = r^n]$$

$$= nr^{n-1}(\mathbf{R}/r) = nr^{n-2}\mathbf{R}.$$

Example 8.10. *If* $\nabla u = 2r^4\,\mathbf{R}$, *find u.* (*Mumbai, 2008*)

Solution. We have $\nabla u = 2(x^2 + y^2 + z^2)^2\,\mathbf{R}$ $\qquad [\because\ r = \sqrt{(x^2 + y^2 + z^2)}\,]$

$$= 2(x^2 + y^2 + z^2)^2\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) \qquad ...(i)$$

But $$\nabla u = \frac{\partial u}{\partial x}\mathbf{I} + \frac{\partial u}{\partial y}\mathbf{J} + \frac{\partial u}{\partial z}\mathbf{K} \qquad ...(ii)$$

Comparing (*i*) and (*ii*), we get

$$\frac{\partial u}{\partial x} = 2x(x^2 + y^2 + z^2)^2,\ \frac{\partial u}{\partial y} = 2y(x^2 + y^2 + z^2)^2\ ;\ \frac{\partial u}{\partial z} = 2z(x^2 + y^2 + z^2)$$

Also $$du(x, y, z) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 2(x^2 + y^2 + z^2)^2\,(xdx + ydy + zdz)$$

$$= 2t^2\,.\,\frac{dt}{2}, \text{ taking } x^2 + y^2 + z^2 = t \text{ and } 2(xdx + ydy + zdz) = dt$$

Integrating both sides, $u = \int t^2\,dt + c = \frac{1}{3}t^3 + c = \frac{1}{3}(x^2 + y^2 + z^2)^3 + c$

Hence $$u = \frac{1}{3}r^{3/2} + c.$$

Example 8.11. *If* $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, *prove that grad u, grad v and grad w are coplanar.* (*U.T.U., 2010 ; U.P.T.U., 2002*)

Solution. $$\text{grad } u = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)(x + y + z) = \mathbf{I} + \mathbf{J} + \mathbf{K}$$

$$\text{grad } v = 2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K},\ \text{grad } w = (y + z)\,\mathbf{I} + (z + x)\,\mathbf{J} + (z + x)\,\mathbf{K}$$

We know that three vectors are coplanar if their scalar triple product is zero.

Here [grad u, grad v, grad w]

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & y+z+x & z+x+y \\ y+z & z+x & x+y \end{vmatrix} \quad \text{[Operate } R_2 + R_3\text{]}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0.$$

Hence grad u, grad v and grad w are coplanar.

• **Example 8.12.** *Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.* (*Mumbai, 2008*)

Solution. A vector normal to the given surface is $\nabla(xy^3z^2)$

$$= \mathbf{I}\frac{\partial}{\partial x}(xy^3z^2) + \mathbf{J}\frac{\partial}{\partial y}(xy^3z^2) + \mathbf{K}\frac{\partial}{\partial z}(xy^3z^2) = \mathbf{I}(y^3z^2) + \mathbf{J}(3xy^2z^2) + \mathbf{K}(2xy^3z)$$

$$= -4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K} \text{ at the point } (-1, -1, 2).$$

Hence the desired unit normal to the surface

$$= \frac{-4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K}}{\sqrt{[(-4)^2 + (-12)^2 + 4^2]}} = -\frac{1}{\sqrt{11}}(\mathbf{I} + 3\mathbf{J} - \mathbf{K}).$$

• **Example 8.13.** *Find the directional derivative of $f(x, y, z) = xy^3 + yz^3$ at the point $(2, -1, 1)$ in the direction of vector $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$.* (*Bhopal, 2008 ; Kurukshetra, 2006 ; Rohtak, 2003*)

Solution. Here $\nabla f = \mathbf{I}(y^2) + \mathbf{J}(2xy + z^3) + \mathbf{K}(3yz^2) = \mathbf{I} - 3\mathbf{J} - 3\mathbf{K}$ at the point $(2, -1, 1)$.

$\therefore$ directional derivative of f in the direction $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$

$$= (\mathbf{I} - 3\mathbf{J} - 3\mathbf{K}) \cdot \frac{\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}}{\sqrt{(1^2 + 2^2 + 2^2)}} = (1 \cdot 1 - 3 \cdot 2 - 3 \cdot 2)/3 = -3\frac{2}{3}.$$

Example 8.14. *Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point $(5, 0, 4)$. Also calculate the magnitude of the maximum directional derivative.*

Solution. We have $\nabla f = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = 2x\mathbf{I} - 2y\mathbf{J} + 4z\mathbf{K}$

$$= 2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K} \text{ at } P(1, 2, 3)$$

Also $\qquad \vec{PQ} = \vec{OQ} - \vec{OP} = (5\mathbf{I} + 0\mathbf{J} + 4\mathbf{K}) - (\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}) = 4\mathbf{I} - 2\mathbf{J} + \mathbf{K} = \mathbf{A}$ (say)

$\therefore \qquad$ unit vector of $\mathbf{A} = \hat{A} = \dfrac{\mathbf{A}}{a} = \dfrac{4\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{\sqrt{(16 + 4 + 1)}} = \dfrac{4\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{21}}$

Thus the directional derivative of f in the direction of $\vec{PQ}$

$$\nabla f \cdot \hat{A} = (2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K}) \cdot (4\mathbf{I} - 2\mathbf{J} + \mathbf{K})/\sqrt{21}$$

$$= (8 + 8 + 12)/\sqrt{21} = 28/\sqrt{21}$$

The directional derivative of its maximum in the direction of the normal to the surface *i.e.*, in the direction of ∇f.

Hence maximum value of this directional derivative

$$= |\nabla f| = |2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K}| = (4 + 16 + 144) = \sqrt{164}.$$

Example 8.15. *Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1, 1, 1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$.* (Bhopal, 2008 ; U.P.T.U., 2004)

Solution. We have
$$\nabla\phi = \mathbf{I}\frac{\partial \phi}{\partial x} + \mathbf{J}\frac{\partial \phi}{\partial y} + \mathbf{K}\frac{\partial \phi}{\partial z}$$
$$= (10xy + 2.5z^2)\,\mathbf{I} + (5x^2 - 10yz)\,\mathbf{J} + (-5y^2 + 5zx)\,\mathbf{K}$$
$$= 12.5\mathbf{I} - 5\mathbf{J} \text{ at } P(1, 1, 1)$$

Also direction of the given line is $\hat{A} = \dfrac{2\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{3}$

Hence the required directional derivative
$$= \nabla\phi \,.\, \hat{A} = (12.5\mathbf{I} - 5\mathbf{J})\,.\,(2\mathbf{I} - 2\mathbf{J} + \mathbf{K})/3 = (25 + 10)/3 = 11\tfrac{2}{3}.$$

Example 8.16. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.* (V.T.U., 2010 ; Kottayam, 2005 ; U.P.T.U., 2003)

Solution. Let $f_1 = x^2 + y^2 + z^2 - 9 = 0$ and $f_2 = x^2 + y^2 - z - 3 = 0$

Then $N_1 = \nabla f_1$ at $(2, -1, 2) = (2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K})$ at $(2, -1, 2) = 4\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$

and $N_2 = \nabla f_2$ at $(2, -1, 2) = (2x\mathbf{I} + 2y\mathbf{J} - \mathbf{K})$ at $(2, -1, 2) = 4\mathbf{I} - 2\mathbf{J} - \mathbf{K}$

Since the angle θ between the two surfaces at a point is the angle between their normals at that point and $\mathbf{N}_1, \mathbf{N}_2$ are the normals at $(2, -1, 2)$ to the given surfaces, therefore
$$\cos\theta = \frac{\mathbf{N}_1 \,.\, \mathbf{N}_2}{n_1 n_2} = \frac{(4I - 2J + 4K)\,.\,(4I - 2J - K)}{\sqrt{(16+4+16)}\,\sqrt{(16+4+1)}}$$
$$= \frac{4(4) + (-2)(-2) + 4(-1)}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

Hence the required angle $\theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$.

Example 8.17. *Find the values of a and b such that the surface $ax^2 - byz = (a + 2)\,x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1, -1, 2)$.* (Madras, 2004)

Solution. Let $f_1 = ax^2 - byz - (a + 2)\,x = 0$ ...(*i*)

and $f_2 = 4x^2y + z^3 - 4 = 0$ ...(*ii*)

Then $\nabla f_1 = (2ax - a - 2)\mathbf{I} - 4z\mathbf{J} - by\mathbf{K} = (a - 2)\,\mathbf{I} - 2b\mathbf{J} + b\mathbf{K}$ at $(1, -1, 2)$.

$\nabla f_2 = 8xy\mathbf{I} + 4x^2\mathbf{J} + 3z^2\mathbf{K} = -8\mathbf{I} + 4\mathbf{J} + 12\mathbf{K}$ at $(1, -1, 2)$.

The surfaces (*i*) and (*ii*) will cut orthogonally if $\nabla f_1 \,.\, \nabla f_2 = 0$, *i.e.*, $-8(a - 2) - 8b + 12b = 0$

or $-2a + b + 4 = 0$ ...(*iii*)

Also since the point $(1, -1, 2)$ lies on (*i*) and (*ii*),

$\therefore \quad a + 2b - (a + 2) = 0$ or $b = 1$

From (*iii*), $-2a + 5 = 0$ or $a = 5/2$.

Hence $a = 5/2$ and $b = 1$.

PROBLEMS 8.3

1. (*a*) Find $\nabla\phi$, if $\phi = \log(x^2 + y^2 + z^2)$. (*b*) Show that grad $(1/r) = -\mathbf{R}/r^3$.
2. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$. (P.T.U., 1999)
3. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2\mathbf{I} - \mathbf{J} - 2\mathbf{K}$. (V.T.U., 2007 ; Rohtak 2006 S ; J.N.T.U., 2006 ; U.P.T.U., 2006)
4. What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$? (S.V.T.U., 2009)

5. Find the values of constants a, b, c so that the directional derivative of $p = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in the direction parallel to the z-axis. *(Rajasthan, 2006)*
6. Find the directional derivative of $\phi = x^4 + y^4 + z^4$ at the point $A\,(1, -2, 1)$ in the direction AB where B is $(2, 6, -1)$. Also find the maximum directional derivative of ϕ at $(1, -2, 1)$. *(Mumbai, 2009)*
7. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$, find the values of a, b and c. *(U.P.T.U., 2002)*
8. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum ? Find also the magnitude of this maximum. *(Rohtak, 2003)*
9. What is the greatest rate of increase of $u = xyz^2$ at the point $(1, 0, 3)$? *(Bhopal, 2008)*
10. The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move ?
11. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.
12. Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1$, $x^2y = 2 - z$ at the point $(1, 1, 1)$. *(Hissar, 2005 S ; J.N.T.U., 2003)*
13. Find the values of a and b so that the surface $5x^2 - 2yz - 9z = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1, -1, 2)$. *(Nagpur, 2009)*
14. If f and $\mathbf{G}$ are point functions, prove that the components of the latter normal and tangential to the surface $f = 0$ are

$$\frac{(\mathbf{G}\,.\,\nabla f)\,\nabla f}{(\nabla f)^2} \text{ and } \frac{\nabla f \times (\mathbf{G} \times \nabla f)}{(\nabla f)^2}$$ [Cf. Ex. 3.24]

8.6 DEL APPLIED TO VECTOR POINT FUNCTIONS

(1) Divergence. *The divergence of a continuously differentiable vector point function* $\mathbf{F}$ *is denoted by* div *and is defined by the equation*

$$\text{div } \mathbf{F} = \nabla\,.\,\mathbf{F} = \mathbf{I}\,.\,\frac{\partial \mathbf{F}}{\partial x} + \mathbf{J}\,.\,\frac{\partial \mathbf{F}}{\partial y} + \mathbf{K}\,.\,\frac{\partial \mathbf{F}}{\partial z}$$

If $\quad \mathbf{F} = f\,\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$

then
$$\text{div } \mathbf{F} = \nabla\,.\,\mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right).\,(f\,\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}) = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$$

(2) Curl. *The curl of a continuously differentiable vector point function* $\mathbf{F}$ *is defined by the equation*

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}$$

If $\mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$ then $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right) \times (f\,\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K})$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = \mathbf{I}\left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial z}\right) + \mathbf{J}\left(\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\right) + \mathbf{K}\left(\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial y}\right).$$

Example 8.18. *If* $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, *show that*
(i) $\nabla\,.\,\mathbf{R} = 3$ *(ii)* $\nabla \times \mathbf{R} = 0$. *(V.T.U. 2008 ; P.T.U., 2006 ; U.P.T.U., 2006)*

Solution. *(i)* $\nabla\,.\,\mathbf{R} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3.$

(ii)
$$\nabla \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{I}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \mathbf{J}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \mathbf{K}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \mathbf{I}(0 - 0) - \mathbf{J}(0 - 0) + \mathbf{K}(0 - 0) = \mathbf{0}.$$

[Remember : *div* $\mathbf{R} = 3$; *curl* $\mathbf{R} = 0$**]**

Example 8.19. *Find div* **F** *and curl* **F**, *where* $\mathbf{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$.

(*V.T.U., 2008 ; Kurukshetra, 2006 ; Burdwan, 2003*)

Solution. If $u = x^3 + y^3 + z^3 - 3xyz$, then

$$\mathbf{F} = \nabla u = \mathbf{I}\frac{\partial u}{\partial x} + \mathbf{J}\frac{\partial u}{\partial y} + \mathbf{K}\frac{\partial u}{\partial z} = \mathbf{I}(3x^2 - 3yz) + \mathbf{J}(3y^2 - 3zx) + \mathbf{K}(3z^2 - 3xy)$$

$$\therefore \quad \text{div } \mathbf{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial z}(3z^2 - 3xy) = 6(x + y + z)$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3(x^2 - yz) & 3(y^2 - zx) & 3(z^2 - xy) \end{vmatrix}$$

$$= \mathbf{I}(-3x + 3x) - \mathbf{J}(-3y + 3y) + \mathbf{K}(-3z + 3z) = \mathbf{0}.$$

8.7 (1) PHYSICAL INTERPRETATION OF DIVERGENCE

Consider the motion of the fluid having velocity $\mathbf{V} = v_x\mathbf{I} + v_y\mathbf{J} + v_z\mathbf{K}$ at a point $P(x, y, z)$. Consider a small parallelopiped with edges δx, δy, δz parallel to the axes in the mass of fluid, with one of its corners at P (Fig. 8.7).

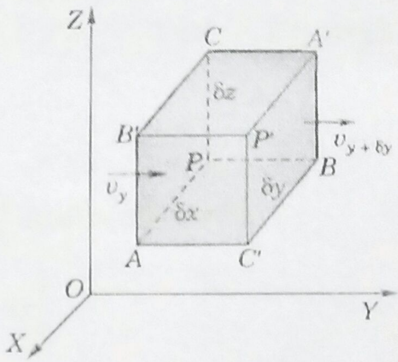

Fig. 8.7

$\therefore$ the amount of fluid entering the face PB' in unit time $= v_y\,\delta z\,\delta x$ and the amount of fluid leaving the face $P'B$ in unit time

$$= v_{y+\delta y}\,\delta z\,\delta x = \left(v_y + \frac{\partial v_y}{\partial y}\delta y\right)\delta z\,\delta x \quad \text{nearly}$$

$\therefore$ the net decrease of the amount of fluid due to flow across these two faces $= \dfrac{\partial v_y}{\partial y}\,\delta x\,\delta y\,\delta z$.

Finding similarly the contributions of other two pairs of faces, we have the total decrease of amount of fluid inside the parallelopiped per unit time $= \left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}\right)\delta x\,\delta y\,\delta z$.

Thus the rate of loss of fluid per unit volume

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \text{div } \mathbf{V}.$$

Hence *div* **V** *gives the rate at which fluid is originating at a point per unit volume.*

Similarly, if V represents an electric flux, div **V** is the amount of flux which diverges per unit volume in unit time. If **V** represents heat flux, div **V** is the rate at which heat is issuing from a point per unit volume. In general, the divergence of a vector point function representing any physical quantity gives at each point, the rate per unit volume at which the physical quantity is issuing from that point. This explains the justification for the name *divergence of a vector point function.*

If the fluid is incompressible, there can be no gain or loss in the volume element. Hence div $\mathbf{V} = 0$, which is known in Hydrodynamics as the **equation of continuity** for incompressible fluids.

Def. *If the flux entering any element of space is the same as that leaving it, i.e., div* $\mathbf{V} = 0$ *everywhere then such a point function is called a* **solenoidal vector function.**

(2) Physical interpretation of curl. Consider the motion of a rigid body rotating about a fixed axis through O. If Ω be its angular velocity, then the velocity **V** of any particle $P(\mathbf{R})$ of the body is given by $\mathbf{V} = \Omega \times \mathbf{R}$

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If $\quad \Omega = \omega_1\mathbf{I} + \omega_2\mathbf{J} + \omega_3\mathbf{K}$ and $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$

then
$$\mathbf{V} = \Omega \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \mathbf{I}(\omega_2 z - \omega_3 y) + \mathbf{J}(\omega_3 x - \omega_1 z) + \mathbf{C}(\omega_1 y - \omega_2 x)$$

$$\therefore \qquad \text{curl } \mathbf{V} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y, & \omega_3 x - \omega_1 z, & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \mathbf{I}(\omega_1 + \omega_1) + \mathbf{J}(\omega_2 + \omega_2) + \mathbf{K}(\omega_3 + \omega_3) \qquad [\because \ \omega_1, \omega_2, \omega_3 \text{ are constants.}]$$

$$= 2(\omega_1 \mathbf{I} + \omega_2 \mathbf{J} + \omega_3 \mathbf{K}) = 2\Omega. \quad \text{Hence} \quad \Omega = \frac{1}{2} \text{ curl } \mathbf{V}$$

Thus the angular velocity of rotation at any point is equal to half the curl of the velocity vector which justifies the name *rotation* used for curl.

In general, the curl of any vector point function gives the measure of the angular velocity at any point of the vector field.

Def. *Any motion in which the curl of the velocity vector is zero is said to be* **irrotational**, *otherwise* **rotational.**

Example 8.20. *Prove that div $(r^n \mathbf{R}) = (n + 3) r^n$. Hence show that $\mathbf{R}/r^3$ is solenoidal.*
(V.T.U., 2006 ; U.P.T.U., 2006 ; P.T.U., 2005)

Solution. We have $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and $r = \sqrt{(x^2 + y^2 + z^2)}$

$$\therefore \qquad div\,(r^n \mathbf{R}) = \nabla \,.\, (x^2 + y^2 + z^2)^{n/2}\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$$

$$= \frac{\partial}{\partial x}[x(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial y}[y(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial z}[z(x^2 + y^2 + z^2)^{n/2}]$$

$$= \Sigma \left\{ 1\,.\,(x^2 + y^2 + z^2)^{n/2} + x\,.\,\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2} - 1}\,.\,2x \right\}$$

$$= \Sigma r^n + n\,\Sigma x^2 (x^2 + y^2 + z^2)^{\frac{n}{2} - 1} = 3r^n + nr^2\,.\,r^{n-2}$$

Thus $div\,(r^n \mathbf{R}) = (n + 3)\, r^n$

When $n = -3$, $div\,(\mathbf{R}/r^3) = 0$ *i.e.*, $\mathbf{R}/r^3$ is solenoidal.

Example 8.21. *Show that $r^\alpha \mathbf{R}$ is any irrotational vector for any value of α but is solenoidal if $\alpha + 3 = 0$ where $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and r is the magnitude of* $\mathbf{R}$. *(V.T.U., 2006 ; Kottayam, 2005)*

Solution. Let $\mathbf{A} = r^\alpha \mathbf{R} = (x^2 + y^2 + z^2)^{\alpha/2}\,(x\mathbf{I} + y\mathbf{I} + z\mathbf{K}) = \Sigma x\,(x^2 + y^2 + z^2)^{\alpha/2}\,\mathbf{I}$

$$\therefore \qquad \text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{\alpha/2} & y(x^2 + y^2 + z^2)^{\alpha/2} & z(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \Sigma\,\mathbf{I} \left\{ \frac{\alpha z}{2}(x^2 + y^2 + z^2)^{\alpha/2 - 1}\,(2y) - \frac{\alpha y}{2}(x^2 + y^2 + z^2)^{\alpha/2 - 1}\,.\,2z \right\} = 0$$

Hence $\mathbf{A}$ is irrotational for any value of α.

But $\qquad div\,\mathbf{A} = \nabla\,.\,(r^\alpha\,\mathbf{R}) = (\alpha + 3)\,r^\alpha$

which is zero for $\alpha + 3 = 0$, *i.e.*, $\mathbf{A}$ is solenoidal if $\alpha + 3 = 0$.

8.8 DEL APPLIED TWICE TO POINT FUNCTIONS

∇f and $\nabla \times \mathbf{F}$ being vector point functions, we can form their divergence and curl whereas $\nabla.\,\mathbf{F}$ being a scalar point function, we can have its gradients only. Thus we have the following five formulae :

(1) *div grad* $f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

(2) *curl grad* $f = \nabla \times \nabla f = 0$

(3) *div curl* $\mathbf{F} = \nabla\,.\,\nabla \times \mathbf{F} = 0$