Vector Differential Calculus

Definitions

① **Scalar point function:**

If to each point P(R) of a region E in space there corresponds to a definite scalar denoted by f(R), then f(R) is called a scalar point function in E. The region E is called a scalar field.

② **Vector point function:**

If to each point P(R) of a region E in space there corresponds a definite vector denoted by F(R) then it is called the vector point function in E. The region E is called a vector field.

③ **Vector operator** ∇

∇ (read as del) is defined as $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

④ **Gradient of f**

The vector function ∇f is defined as the gradient of the scalar point function f and is written as grad f.

$$\text{grad} f = \nabla f = \bar{i}\frac{\partial}{\partial x} f + \bar{j}\frac{\partial}{\partial y} f + \bar{k}\frac{\partial}{\partial z} f$$

⑤ Directional derivative:

For a scalar point function f, and a unit vector $\bar{u}$, the directional derivative is given by the dot product of ∇f and the unit vector $\bar{u}$ at the point (x_0, y_0, z_0).

Directional derivative at the point $(x_0, y_0, z_0) =$

$$\bar{u} \cdot \nabla f_{(x_0, y_0, z_0)}$$

⑥ Divergence:

The divergence of a continuously differentiable vector point function F is denoted by $\text{div } F$ and is defined by the equation

$$\text{div } F = \nabla \cdot F = \bar{i}\frac{\partial}{\partial x} F + \bar{j}\frac{\partial}{\partial y} F + \bar{k}\frac{\partial}{\partial z} F$$

If $F = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$

$$\text{div } F = \nabla \cdot F = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k})$$

$$= \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3$$

⑦ <u>Curl</u> : The curl of a continuously differentiable vector point function F is defined by the equation

$$\text{curl } F = \nabla \times F = \bar{i} \times \frac{\partial F}{\partial x} + \bar{j} \times \frac{\partial F}{\partial y} + \bar{k} \times \frac{\partial F}{\partial z}$$

If $F = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then $\text{curl } F = \nabla \times F$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

⑧ <u>Solenoidal Vector</u>:

If $\text{div } F = 0$ every where, then such a vector is called a solenoidal vector function

⑨ <u>Irrotational Vector</u>:

If the curl of the vector is zero then it is said to be irrotational, otherwise rotational.

Laplace Transforms

Definitions and solutions for Homework problems

Laplace Transforms:

Let $f(t)$ be a function of t defined for all positive values of t. Then the Laplace transforms of $f(t)$, denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

provided that the integral exists. s is a parameter which may be a real or complex number.

Inverse Laplace Transforms

$L\{f(t)\}$ being clearly a function of s is briefly written as $\bar{f}(s)$ i.e, $L\{f(t)\} = \bar{f}(s)$, which can be written as $f(t) = L^{-1}\{\bar{f}(s)\}$

Transforms of elementary functions

① $L\{1\} = \frac{1}{s}$ $\quad s>0$

$$\int_0^\infty e^{-st}.1.dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty = 0+\frac{1}{s} = \frac{1}{s}$$

② $L(t^n) = \int_0^\infty t^n e^{-st}\,dt$ = put $st = p$

$t = \frac{p}{s} \Rightarrow t^n = \left(\frac{p}{s}\right)^n$

$dt = \frac{dp}{s}$

$= \int_0^\infty \frac{p^n}{s^n} e^{-p} \frac{dp}{s} = \frac{1}{s^{n+1}} \int_0^\infty p^n e^{-p}\,dp$

$= \frac{\Gamma(n+1)}{s^{n+1}} = \underline{\underline{\frac{n!}{s^{n+1}}}}$ if $n > -1$, and $s > 0$.

$\left(\int_0^\infty p^n e^{-p}\,dp = \Gamma(n+1) = n!\right.$

$\Gamma\tfrac{1}{2} = \sqrt{\pi}$

③ $L(e^{at}) = \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$

$= \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = 0 + \frac{1}{(s-a)} = \underline{\underline{\frac{1}{s-a}}}$

④ $L(\sin at) = \int_0^\infty \cancel{e^{at}\sin} \int_0^\infty e^{-st} \sin at\,dt$

1st Method: By integration by parts

$\int e^{-st} \sin at\,dt = \sin at \frac{e^{-st}}{-s} - \int a\cos at \frac{e^{-st}}{-s}\,dt$

$$= -\frac{\sin at\ e^{-st}}{s} + \frac{a}{s}\int \cos at\ e^{-st}\,dt$$

$$= -\frac{\sin at\ e^{-st}}{s} + \frac{a}{s}\left[\cos at\ \frac{e^{-st}}{-s} + \int a\sin at\ \frac{e^{-st}}{-s}\,dt\right]$$

$$= -\frac{\sin at\ e^{-st}}{s} + \frac{a}{s}\left[-\frac{\cos at\ e^{-st}}{s} - \frac{a}{s}\int \sin at\ e^{-st}\,dt\right]$$

Let $\int e^{-st}\sin at\,dt = I$

$$\Rightarrow I = -\frac{\sin at\ e^{-st}}{s} - \frac{a}{s^2}\cos at\ e^{-st} - \frac{a^2}{s^2} I$$

$$\Rightarrow I + \frac{a^2}{s^2} I = -\frac{\sin at\ e^{-st}}{s} - \frac{a}{s^2}\cos at\ e^{-st}$$

$$I = -\left(\frac{s^2}{s^2+a^2}\right)\left(\frac{\sin at\ e^{-st}}{s} + \frac{a}{s^2}\cos at\ e^{-st}\right)$$

$$= \frac{-1}{s^2+a^2}\left(s\sin at\ e^{-st} + a\cos at\ e^{-st}\right)$$

$$I = -\frac{e^{-st}}{s^2+a^2}\left(s\sin at + a\cos at\right)$$

$$\int_0^\infty e^{-st}\sin at\,dt = \left.\frac{-e^{-st}}{s^2+a^2}\left(s\sin at + a\cos at\right)\right]_0^\infty$$

$$= \frac{1}{s^2+a^2}\left[s\sin 0 + a\cos 0\right] = \frac{a}{s^2+a^2}$$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

2nd Method: $e^{iat} = \cancel{\sin at}\ \cos at + i \sin at$

$$e^{-iat} = \cos at - i \sin at$$

$$\frac{e^{ait} + e^{-ait}}{2} = \cos at$$

$$\frac{e^{ait} - e^{-ait}}{2i} = \sin at$$

$$L(\sin at) = \frac{L\left(e^{ait} - e^{-ait}\right)}{2i}$$

$$= \frac{L(e^{ait}) - L(e^{-ait})}{2i} = \frac{\frac{1}{s-ai} - \frac{1}{s+ai}}{2i}$$

$$= \frac{\frac{s+ai - (s-ai)}{s^2-(ai)^2}}{2i} = \frac{s+ai - s + ai}{(s^2+a^2)\, 2i}$$

$$= \underline{\underline{\frac{a}{s^2+a^2}}}$$

(5) $L(\cos at)$ = Method 1: similar to (4)

Method 2: $L(\cos at) = \dfrac{L(e^{ait} + e^{ait})}{2}$

$$= \frac{L(\cancel{ae}\, e^{ait}) + L(e^{ait})}{2}$$

$$= \frac{\frac{1}{s-ai} + \frac{1}{s+ai}}{2}$$

$$= \frac{(s+ai) + (s-ai)}{(s^2+a^2)\,2} = \frac{2s}{(s^2+a^2)\,2} = \underline{\underline{\frac{s}{s^2+a^2}}}$$

⑥ $L(\sinh at) = \int_0^\infty e^{-st} \sinh at \, dt$

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$\cosh at = \frac{e^{at} + e^{-at}}{2}$$

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{L(e^{at}) - L(e^{-at})}{2}$$

$$= \frac{\frac{1}{s-a} - \frac{1}{s+a}}{2} = \frac{s+a-(s-a)}{(s^2-a^2)\,2}$$

$$= \frac{a}{s^2-a^2}$$

⑦ $L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right)$

$$= \frac{L(e^{at}) + L(e^{-at})}{2} = \frac{\frac{1}{s-a} + \frac{1}{s+a}}{2}$$

$$= \frac{s+a+s-a}{(s^2-a^2)\,2} = \underline{\underline{\frac{s}{s^2-a^2}}}$$

Linear Property

If a, b, c be any constants and f, g, h any functions of t, then $L[af(t) + bg(t)] = aL(f(t)) + bL(g(t))$

First Shifting Property

If $L(f(t)) = \bar{f}(s)$, then

$$L(e^{at} f(t)) = \bar{f}(s-a)$$

Eg ① $L(e^{at}) = \frac{1}{s-a}$

② $L(e^{at} e^{bt}) = \frac{1}{(s-b)-a}$

③ $L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}$.

④ $L(e^{at} \sin bt) = \frac{b}{(s-a)^2+b^2}$

⑤ $L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2+b^2}$

⑥ $L(e^{at} \cosh at) = \frac{s-a}{(s-a)^2-b^2}$

⑦ $L(e^{at} \sinh at) = \frac{b}{(s-a)^2-b^2}$

Transform of Periodic functions

If $f(t)$ is a periodic function with period T, i.e.

$f(t+T) = f(t)$, then

$$L(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Transforms of derivatives

If $f'(t)$ be continuous and $L(f(t)) = \bar{f}(s)$, then $L(f'(t)) =$

$s\bar{f}(s) - f(0)$

$$L(f^n(t)) = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$$

Transform of Integrals

If $L(f(t)) = \bar{f}(s)$, then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$

Multiplication by t^n

If $L(f(t)) = \bar{f}(s)$, then

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\bar{f}(s))$, where $n = 1, 2, 3, \dots$

Convolution theorem

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$

$$L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du = F*G$$

$F*G$ is the convolution of F and G.

Inverse Laplace Transforms

① $L^{-1}\left(\frac{1}{s}\right) = 1$

② $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

③ $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$, $n = 1, 2, 3, \ldots$

④ $L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{e^{at}\,t^{n-1}}{(n-1)!}$

⑤ $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

⑥ $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

⑦ $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

(8) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

(9) $L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{1}{b}\sin bt\, e^{at}$

(10) $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$

(11) $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a} t \sin at$

(12) $L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^3}\left(\sin at - at\cos at\right)$

Proof: $L(t\sin at) = \frac{2as}{(s^2+a^2)^2}$ $\qquad L(t\cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}$

$$t\sin at = 2a L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$$

$$t\cos at = L^{-1}\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{(s^2+a^2)-2a^2}{(s^2+a^2)^2}\right)$$

$$t\cos at = L^{-1}\left(\frac{1}{(s^2+a^2)}\right) - 2a^2 L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)$$

$$t\cos at = \frac{1}{a}\sin at - 2a^2 L^{-1}\left(\frac{1}{\cancel{\sin}(s^2+a^2)^2}\right)$$

$$\Rightarrow L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^3}\left(\sin at - at\,\cancel{\sin}\cos at\right)$$