[Methods for Integration]

c in plane c in space

$\vec{r}(t)=\langle x(t),y(t)\rangle$ or $\langle x(t),y(t),z(t)\rangle$ ($\Rightarrow$ definite integral)

$$=\int_a^b \vec{F}[\vec{r}(t)]\cdot\vec{r}'(t)\,dt=\int_c \vec{F}\hat{n}\,dS$$

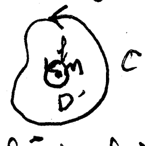
curve: $\int_c \vec{F}d\vec{r} \rightarrow$

use Green Theorem: $\left[\begin{array}{l}(Q_x-P_y)\text{ is easy to calculate \& simple.}\end{array}\right]$ or $\vec{F}$ is complex.

1° c closed in plane, $\oint_c Pdx+Qdy=\iint_R (Q_x-P_y)\,dxdy$ ($\Rightarrow$ double integral)

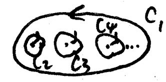
2° $M(x_0,y_0)$ is a singularity (undefined in P, Q, Q_x or P_y)

$\cancel{\oint_c Pdx+Qdy}=\iint_{D'}(Q_x-P_y)\,dxdy+\oint_l Pdx+Qdy$



In general:

$\sum\oint_{C_i}\vec{F}d\vec{r}=\iint_R \text{curl}\,\vec{F}\,dA$



particularly if $Q_x=P_y$, $\oint_c \vec{F}d\vec{r}=\oint_l \vec{F}d\vec{r}$

l needs to be selected properly

△ curve in a plane (xoy)

3° if ~~∄~~ c isn't closed but path independent:

$\Leftrightarrow \vec{F}=\nabla f$, $C: P_0\rightarrow P_1$

then $\int_c \vec{F}dr=f(P_1)-f(P_0)$

$\left\{\begin{array}{l} P_y=Q_x,\ P_z=R_x,\ Q_z=R_y \\ \text{get } f(x,y,z) \text{ potential function:} \\ f=\int f_x dx=\int P dx=g(x,y)+h_0(y,z) \\ f_y=Q=g_y+h_{0y}\Rightarrow h_0(y,z)=\int(Q-g_y)dy \\ \qquad =h(y,z)+l_0(z) \\ f_z=R=h_z+l_0'(z)\Rightarrow l_0(z)=\int(R-h_z)dz+C \\ \therefore f=g(x,y)+h(y,z)+l(z)+C. \end{array}\right.$

- close & path independent

if $\vec{F}=\nabla f$, $\oint_c \vec{F}d\vec{r}=0$ ← 4° if c isn't close & not path independent

conservative field.

$\int_c \vec{F}d\vec{r}=\iint_D (Q_x-P_y)\,dxdy-\int_l Pdx+Qdy$

hard to get directly. easy to get.

use Stoke's Theorem: 1° closed curve which is intersection line of a plane & surface,

c is a boundary curve of ~~[illegible]~~ surface S, then

△ (closed) curve in a space

$\oint_c \vec{F}d\vec{r}=\iint_S \text{curl}(\vec{F})\,d\vec{A}$ (has right-hand rule)

($\Rightarrow$ surface integral) $\nabla\times\vec{F}$ is easy to get.

Flux

Surface: $\iint_S \vec{F}d\vec{A} \rightarrow =\iint_{R_{xoy}}\vec{F}[x,y,z(x,y)]\cdot\langle -z_x,-z_y,1\rangle\,dxdy=\iint_R \vec{F}(\vec{r})\,|\vec{r}_u\times\vec{r}_v|\,dA$, In general

$S: z=z(x,y)$ ($\Rightarrow$ double integral) $S: \vec{r}=\langle x(u,v),y(u,v),z(u,v)\rangle$

$R: z=0,\ z(x,y)=0$

$=\iint_S \vec{F}\hat{n}\,dS$ if $S\perp xoy$, $\iint_S R\,dxdy=0$; $S\perp yoz$, $\iint_S P\,dydz=0$; $S\perp zox$, $\iint_S Q\,dzdx=0$

use Divergence Theorem: 1° surface's closed, $\nabla\vec{F}$ exist

($\Rightarrow$ triple integral) $\oiint_S \vec{F}d\vec{A}=\iiint_R \nabla\cdot\vec{F}\,dv=\iiint_R \text{div}(\vec{F})\,dv$

2° if singularity $M\in R$, if $\nabla\cdot\vec{F}=0$, $\iint_S \vec{F}d\vec{A}=\iint_{S'}\vec{F}d\vec{A}$

S' is the small enough sphere included $M\in R$

3° S isn't closed, but $\nabla\cdot\vec{F}$ is simple: get a closed surface by adding a surface S' ~~to~~ which has the same boundary as S and function as S (always another side of function of S)

$\Rightarrow \iint_S \vec{F}d\vec{A}=\oiint_{S+S'}\vec{F}d\vec{A}-\iint_{S'}\vec{F}d\vec{A}$

$=\iiint_R \text{div}\vec{F}\,dv-\iint_{S'}\vec{F}\,d\vec{A}$

easy to get ← → sometimes $S'\perp xoy/zoz/yoz$, so has "0".

use Stoke's Theorem: when evaluate $\iint_S \text{curl}(\vec{F})\,d\vec{A}$:

$=\oint_c \vec{F}d\vec{r}$