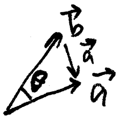
一. Vectors & Matrix.

1. Vectors.

- Dot Product: eg. $\vec{a}=\langle a_1,a_2,a_3\rangle$, $\vec{b}=\langle b_1,b_2,b_3\rangle$, then

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3 \qquad \vec{a}\cdot\vec{a}=\|\vec{a}\|^2$$

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cdot\cos\theta$.

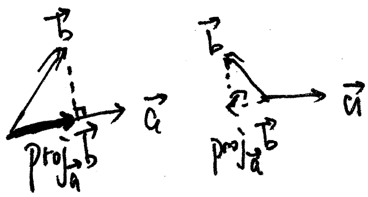


proof: $|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\cos\theta$.

$\Leftrightarrow (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})=|\vec{a}|^2-2\vec{a}\cdot\vec{b}+|\vec{b}|^2$

$\Leftrightarrow \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cdot\cos\theta$.

$\vec{a}\cdot\vec{b}=0 \Leftrightarrow$ two vectors are orthogonal.



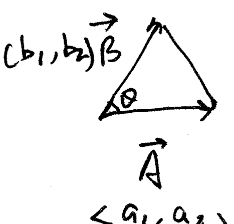
$$\text{proj}_{\vec{a}}\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$$

or: $\text{proj}_{\vec{a}}\vec{b}=|\vec{b}|\cdot\cos\theta$

$=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\vec{u}$ → unit vector

$\vec{u}=\frac{\vec{a}}{|\vec{a}|}$

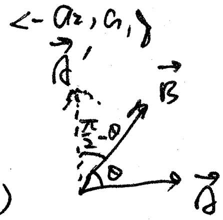
- Area

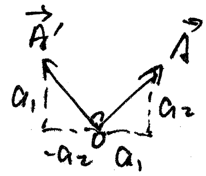


$S_\triangle=\frac{1}{2}|\vec{A}|\cdot|\vec{B}|\cdot\sin\theta$

$=\frac{1}{2}|\vec{A}'|\cdot|\vec{B}|\cdot\cos(\frac{\pi}{2}-\theta)$

$=\frac{1}{2}\vec{A}'\cdot\vec{B}$





$=\frac{1}{2}\langle -a_2,a_1\rangle\cdot\langle b_1,b_2\rangle=\frac{1}{2}(a_1b_2-a_2b_1)=\frac{1}{2}\det(\vec{A},\vec{B})=\frac{1}{2}\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}$

∴ $S_\triangle=\det(\vec{A},\vec{B})$

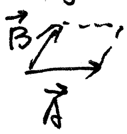
2-D R^2: $|\vec{A}\times\vec{B}|=\det(\vec{A},\vec{B})=\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}=\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}$

- Cross Product

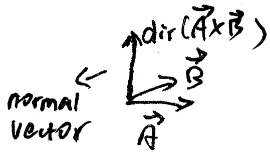
$i=(1,0,0)$ $j=(0,1,0)$ $k=\langle 0,0,1\rangle$

3-D R^3: $\vec{A}\times\vec{B}=\begin{vmatrix}i&j&k\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}=\begin{vmatrix}a_y&a_z\\b_y&b_z\end{vmatrix}i-\begin{vmatrix}a_x&a_z\\b_x&b_z\end{vmatrix}j+\begin{vmatrix}a_x&a_y\\b_x&b_y\end{vmatrix}k$ ⇒ is a vector

☆ $=\langle a_yb_z-b_ya_z,\ -(a_xb_z-b_xa_z),\ a_xb_y-b_xa_y\rangle$

~~2-D~~: $|\vec{A}\times\vec{B}|$ = ~~Aea~~ Area of 

$\text{dir}(\vec{A}\times\vec{B})=\perp$ to plane of the ▱ with right hand rule. (from $\vec{A}$ to $\vec{B}$)



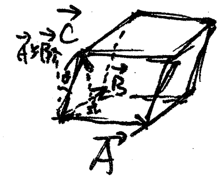
∴ $S_\triangle=\frac{1}{2}|\vec{A}\times\vec{B}|=\frac{1}{2}\det(\vec{A},\vec{B})$

$\vec{A}\times\vec{B}=-(\vec{B}\times\vec{A})$

△ ∴ $\vec{A}\times\vec{A}=0$.

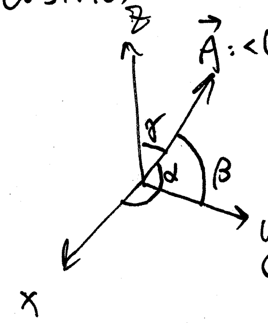
$\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

R^3 3-D: $\det(\vec{A},\vec{B},\vec{C})=\pm$ Volume of parallelepiped $|\vec{A}\times\vec{B}||\vec{C}|\cdot\cos\theta$



☆ $=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}=\vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{A}\times\vec{B})\cdot\vec{C}$ //

- Cosines:



$\vec{A}: \langle a_x, a_y, a_z \rangle$

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}|} = \frac{a_x}{|\vec{a}|}$$

$$\cos\beta = \frac{a_y}{|\vec{a}|}, \quad \cos\gamma = \frac{a_z}{|\vec{a}|}$$

$\sum \cos^2 \iota = 1$, unit vector: $\frac{\vec{a}}{|\vec{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

eg. Question: if $\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta = 1$, get $\tan\theta$.

Solve: assume unit vector: $\hat{e}_1 = \langle \sin\theta, \cos\theta \rangle$, $\vec{e}_2 = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$\vec{e}_1 \cdot \vec{e}_2 = 1 = |\vec{e}_1| \cdot |\vec{e}_2| \cdot \cos\alpha \Rightarrow \vec{e}_1 \parallel \vec{e}_2$

$\therefore \sin\theta = \lambda \cdot \frac{3}{5}$, $\cos\theta = \lambda \cdot \frac{4}{5} \Rightarrow \tan\theta = \frac{3}{4}$.

- derivative

vector function

$r(t) = \langle x(t), y(t), z(t) \rangle$, $\frac{d}{dt}(r\cdot s) = r\cdot\frac{ds}{dt} + s\cdot\frac{dr}{dt}$; $\frac{d}{dt}(r\times s) = r\times\frac{ds}{dt} + s\times\frac{dr}{dt}$

2. Matrix.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases} \qquad \vec{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \vec{A}x = \vec{b}$$

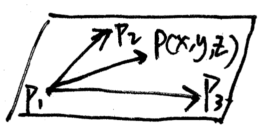
transposition

$A^T = (a_{ji})$

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} = \underset{m\times p}{C}, \quad c_{ij} = \sum_{k=1}^{n} a_{ik}\cdot b_{kj}$$

$$\text{eg. } \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \cdot \underset{3\times 1}{\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 1\times 7 + 2\times 8 + 3\times 9 \\ 4\times 7 + 5\times 8 + 6\times 9 \end{pmatrix}} = \begin{pmatrix} 50 \\ 122 \end{pmatrix}$$

3. Equations of Planes



Plane $P_1P_2P_3$, if $P \in P_1P_2P_3$, $\det(\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}, \overrightarrow{P_1P}) = 0$

or $\overrightarrow{P_1P} \cdot (\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) = 0$ △

- plane $ax+by+cz=d$, if we know normal vector, and another point, we could find a plane.

eg. normal vector $\vec{N} = \langle 1,5,10 \rangle$, pass $P_0 = (2,1,-1)$, the plane's ~~eqn~~ function:

assume $P \in$ Plane, $P = (x,y,z)$

$\Leftrightarrow \overrightarrow{P_0P} \cdot \vec{N} = 0 \Leftrightarrow \langle x-2, y-1, z+1 \rangle \cdot \langle 1,5,10 \rangle \Leftrightarrow \underbrace{1x + 5y + 10z}_{\text{coefficient is } \vec{N}\text{'s } x,y,z.} = -3$ (Find a plane)

- linear sys.

eg. 3x3: line $P_1 \cap P_2$ intersects P_3 in a point = solution

3 planes P_1, P_2, P_3

$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$, $|A| \neq 0$ $(A\cdot A^{-1} = \underset{\text{identity matrix}}{I_{(n\times n)}})$

A is $n\times n$ matrix, then $Ax=0 \Rightarrow$ homogeneous, $Ax=b$, $b\neq 0 \Rightarrow$ inhomogeneous.

$\boxed{\text{Theorem 1}}$. $|A| \neq 0 \Rightarrow Ax=b$ has the unique solution $x = A^{-1}b$, if $b=0$, $x=0$ trivial solutions

$\boxed{\text{Theorem 2}}$ $|A| = 0 \Rightarrow Ax=0$ has non-zero solutions (non-trivial solutions)

$\Rightarrow Ax=b$ usually has no solutions, but has ones for some b.