Differential

Basic: $\frac{dy}{dx} = -\frac{F_x}{F_y}$, $F(x,y)=0$, $F_y(x_0,y_0) \neq 0$,

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

for $F(x,y,z)=0$, $F_z(x_0,y_0,z_0)\neq 0$.

$f_{xx} = \frac{\partial^2 z}{\partial x}$, $f_{xy} = \frac{\partial^2 z}{\partial x \partial y}$, $(uv)_x = u_x v + u v_x$

total differential: $dz = f_x dx + f_y dy$

$df = f_x dx + f_y dy + f_z dz \cdots$

if $g(x,y) = f(x,y,z(x,y))$, $df = f_x dx + f_y dy + \nabla z(x,y) d\langle x,y\rangle = dg$.

Chain rule: $\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \Leftrightarrow z_u = f_x x_u + f_y y_u \cdots$

$\frac{\partial z}{\partial v} = z_v = f_x x_v + f_y y_v$

for $z = f(x,y) = f(x(u,v), y(u,v))$, $x = x(u,v)$, $y = y(u,v)$

$f(x(t), y(t), z(t))$: $df = f_x dx + f_y dy + f_z dz = \nabla f \cdot d\vec{r}$

$\therefore \frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt}$

- unit Tangent vector: $\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{v}}{s}$

$\vec{r} = \langle x, y, z\rangle$ → $\langle x(t), y(t), z(t)\rangle$

$s = |\vec{r}|$

- directional derivative: $\frac{df}{ds}$

$f(x,y,z)$

$\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = |\nabla f| \cdot \left|\frac{d\vec{r}}{dt}\right| \cos\theta$

$= |\nabla f| \cdot |u| \cdot \cos\theta$.

$\frac{\partial f}{\partial s} = \nabla f \cdot \langle \cos\alpha, \cos\beta, \cos\gamma\rangle$

$= \nabla f \cdot \hat{n} \leq \nabla f$, fastest rate as $\hat{n} = \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$

$\hat{n} = \frac{\nabla f(x_0, y_0 \dots)}{|\nabla f(x_0, y_0 \dots)|}$

$\frac{\partial f}{\partial n} = |\nabla f(x_0,y_0,z_0)|$, if $f(x,y,z) = c$

→ directional dirivative of f

- Tangent Plane of $f(x,y,z)$ at $P_0(x_0,y_0,z_0)$:

- Plane: $\langle x-x_0, y-y_0, z-z_0\rangle \cdot \nabla f(x_0,y_0,z_0) = 0$,

$\because \nabla f \perp$ counter plane, same direction as normal vector

$\therefore \nabla f \perp$ tangent plane at P_0

- $\boxed{\text{Optimization}}$

1° extrema: $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ ⇒ critical point (x_0,y_0) $\Leftrightarrow$ $\nabla f(x_0,y_0) = 0$ at (x_0,y_0)

$A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

- $AC - B^2 = 0$, can't determine
- $AC - B^2 < 0$, saddle point → has the opposite trend of x & y,
- $AC - B^2 > 0$ & $A > 0$, (x_0,y_0) is min
- $AC - B^2 > 0$ & $A < 0$, (x_0,y_0) is max.

2° Customized optimization → constrained, test $\begin{cases}\text{max}\\ \text{min}\\ \text{saddle}\end{cases}$

$z = f(x,y)$, & $\varphi(x,y) = 0$, (x_0,y_0) is extrema,

$L(x,y) = f(x,y) + \lambda \varphi(x,y)$ where $L_x(x_0,y_0) = L_y(x_0,y_0) = 0$,

Lagrange ~~multiplier~~ function; Lagrange multiplier

In general,

- multi-variables

$L = F + \lambda \Psi$.

$\nabla L = 0$

$\therefore \nabla F = \lambda \nabla \Psi$.

⇒ get (x,y), λ

extrema.

- multi-constraints

$L = F + \sum \lambda_i \Psi_i$, $\nabla L = 0$ ⇒ get extrema

- inequility Constrains

use 2° & find critical points (as "=")

compare all extrema & with domain

⇒ get max/min