二. Differential

① Partial Derivative

assume y is a constant; assume x is a constant

$z = f(x,y) \Leftrightarrow f_x(x,y) = \frac{\partial z}{\partial x}$, $f_y(x,y) = \frac{\partial z}{\partial y}$ partial derivatives of f with respect to x & y.

$f_x(a,b) = \frac{\partial z}{\partial x}\Big|_{(a,b)}$, $f_y(a,b) = \frac{\partial z}{\partial y}\Big|_{(a,b)}$ at (a,b).

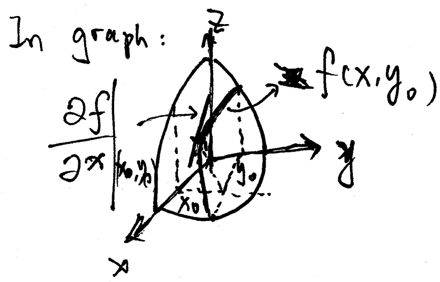
$\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$ & $\lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}$

△ higher order: $f_{xx} = \frac{\partial^2 z}{\partial x}$, $f_{xy} = \frac{\partial^2 z}{\partial x \partial y}$

eg. $z = x^2+3xy+y^2$. point $(1,2)$.

$\frac{\partial z}{\partial x} = 2x+3y$; $\frac{\partial z}{\partial y} = 3x+2y$, partial derivative at $(1,2)$: $\frac{\partial z}{\partial x}\Big|_{x=1, y=2} = 8$; $\frac{\partial z}{\partial y}\Big|_{x=1, y=2} = 7$.

In graph:



◦ For single variable: $dy = f'(x)dx \Rightarrow$ linear Approximation

◦ For two variables: plane Approximation: $z_0 = f(x_0, y_0)$

Function of plane: $A(x-x_0)+B(y-y_0)+C(z-z_0) = 0$

included two tangent line at (x_0, y_0, z_0)

$\therefore z = z_0 + \underbrace{f_x(x_0,y_0)}_{\text{slope}}\cdot(x-x_0)$; $z = z_0 + \underbrace{f_y(x_0,y_0)}_{\text{slope}}\cdot(y-y_0)$

Get: $z = f(0,0) + f_x(x) + f_y(y)$, f_x, f_y are the function of the partial derivative $f(x_0,y_0)$

$\therefore dz = f_x \cdot dx + f_y \cdot dy \Rightarrow$ ② total differential

$\left(dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)$

◦ For three variables: $f(x,y,z)$:

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$, ... for n variables

– parameterization (Chain rule)

$z = f(x,y)$, $x = x(u,v)$, $y = y(u,v) \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$;

$\Leftrightarrow$ Compound function: $z = f(x(u,v), y(u,v))$ → multi- in multi-variable

$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$ ——————

$f(\overset{u}{x(t)}, \overset{v}{y(t)}, \overset{w}{z(t)})$ → single- in multi-variable

$\Rightarrow$ ~~$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$~~

$\frac{df}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial w}\frac{dw}{dt}$ → total derivative $\left[df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw\right]$ Partial Derivative

$\boxed{\text{eg}}$: $z = e^u \sin v$, $u = xy$, $v = x+y$, find $\frac{\partial z}{\partial x}$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$

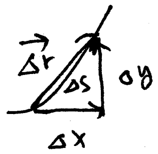
$= e^u \sin v \cdot y + e^u \cos v \cdot 1$

$= e^{xy}[y\sin(x+y) + \cos(x+y)]$

- Polar coordinates.

$x = r\cos\theta$, $y = r\sin\theta$, $f = f(x,y) \Rightarrow \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\sin\theta$.

- arc: $\vec{r}(t) = x(t)i + y(t)j = \langle x, y\rangle$



$\Delta s \approx |\Delta\vec{r}| = \sqrt{\Delta x^2 + \Delta y^2}$

when $\Delta t \to 0$, $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = |\vec{V}| = \text{speed}$.

$\vec{V}(t) = \frac{d\vec{r}}{dt} = \langle x'(t), y'(t)\rangle \Rightarrow$ tangent vector, always tangent to the trace.

unit tangent vector $\vec{T} = \frac{\vec{V}}{|\vec{V}|} = \frac{\text{velocity}}{\text{speed}} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{v}}{s}$

③ Gradient $\Rightarrow \nabla f = \langle f_x, f_y, f_z\rangle$, $df = \langle f_x, f_y, f_z\rangle \cdot \langle dx, dy, dz\rangle$. $f = f(x,y,z)$

(∇f; $d\vec{r} \Rightarrow$ differential of direction)

△ gradients are orthogonal to level curves & level surfaces
(level curves → contour line for 2 variable; level surfaces → contour surface.)

$df = \nabla f \cdot d\vec{r}$

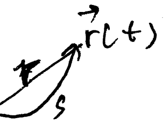
$\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt}$.

$\nabla f(x_0, y_0, z_0)$ at point (x_0, y_0, z_0)

$= f_x(x_0,y_0,z_0)\,i + f_y(x_0,y_0,z_0)\,j + f_z(x_0,y_0,z_0)\,k$.

④ directional derivative

$\vec{r}$ is a part curve of $f = f(x,y,z)$



$\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = |\nabla f|\left|\frac{d\vec{r}}{dt}\right|\cdot\cos\theta = |\nabla f|\cdot|v|\cos\theta \to \theta = 0 \Rightarrow$ maximum.
($|v| = \frac{ds}{dt}$; $\theta = 0 \Leftarrow$ dir(∇f))

$\therefore \frac{df}{ds} = \nabla f \cdot \hat{u} < |\nabla f|$, $\hat{u}$ is the directional vector of curve r,
$\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$

$\left.\frac{df}{ds}\right|_{(x,y,\dots)} = \nabla f \cdot \vec{u}$ - Function ~~declines~~ changes the fastest at the ∇ direction, increase for $|\nabla f|$, decrease for $-|\nabla f|$
when dir$\left(\left.\frac{df}{ds}\right|_{P,\vec{u}}\right) = $ dir(∇f), $\left.\frac{df}{ds}\right|_P = |\nabla f|$.

e.g. $z = x\cdot e^{2y}$, find the directional derivative of $P(1,0)$ ~~from~~ of the direction $P(1,0) \to Q(2,-1)$

$\Rightarrow \vec{S} = \vec{PQ} = \langle 1, -1\rangle$, $\vec{u} = \langle\cos\alpha, \cos\beta\rangle = \langle\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle$

dir derivative

$= \left.\frac{\partial z}{\partial \vec{s}}\right|_{(1,0)} = \nabla f \cdot \langle\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle = \langle \left.\frac{\partial z}{\partial x}\right|_{(1,0)}, \left.\frac{\partial z}{\partial y}\right|_{(1,0)}\rangle \cdot \langle\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle = \langle \left.e^{2y}\right|_{(1,0)}, \left.2xe^{2y}\right|_{(1,0)}\rangle \cdot \langle\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle$

$= -\frac{\sqrt{2}}{2}$

Δ – using the ∇ to find the tangent plane.

eg. $f(x,y,z) = x^2+y^2+z = f$. find the T. plane at $P_0\ (1,2,4)$

$$\nabla f|_{P_0} = \langle 2x, 2y, 1\rangle|_{(1,2,4)} = \langle 2,4,1\rangle (= 2i+4j+k)$$

the direction of ∇f is the direction of the normal plane of the countour plane at $f(x,y,z)=f$.

$P_0(1,2,4)$, Tangent Plane vector: $\langle (x-1), (y-2), (z-4)\rangle$

$\therefore$ Tangent Plane $\perp \nabla f$, $\therefore \langle (x-1), (y-2), (z-4)\rangle \cdot \langle 2,4,1\rangle = 0$

$\therefore 2(x-1) + 4(y-2) + z-3 = 0$

Δ Change rate of f

eg2. if $f(x,y) = \frac{1}{2}(x^2+y^2)$, $P_0(1,1)$, analyze the change direction.

$\Rightarrow$ 1° $f(x,y)$ increases at the fastest rate at the direction along $\nabla f(1,1) = \langle 1,1\rangle$

direction vector $\vec{n} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$ $\quad (= \frac{\nabla f(1,1)}{|\nabla f(1,1)|})$

directional derivative $= \frac{\partial f}{\partial s}\Big|_{(1,1),\vec{n}} = |\nabla f(1,1)| = \sqrt{2}$

2° $f(x,y)$ decreases at the fastest rate at the direction of $-\nabla f(1,1) =$

$\vec{u}' = \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\rangle$, directional derivative $= \frac{\partial f}{\partial s}\Big|_{(1,1),-\vec{n}} = -|\nabla f(1,1)| = -\sqrt{2}$

3° when the change rate is 0,

$\vec{u} = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$ or $\vec{u} = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\rangle$

B. Optimization

– max/min: $\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases}$ to get critical point (x_0, y_0)

$\Leftrightarrow \nabla f(x_0, y_0) = \vec{0}$ at (x_0, y_0)

let: $A = f_{xx}$, $B = f_{xy} = f_{yx}$, $C = f_{yy}$

if $AC-B^2 = 0$, can't determine

$AC-B^2 < 0$, critical point is saddle point — has the opposite direction trend in x & y dir.

$AC-B^2 > 0$ & $A>0$, critical point is min

$AC-B^2 > 0$ & $A<0$, critical point is max

optimization
- Practice: Max Profit & Min Cost: Profit (P) = Revenue − Cost = $R - C$

$$R = P \cdot Q$$

two products: q_1, q_2, P_1, P_2, has: $P_1 = 600 - 0.3q_1$, $P_2 = 500 - 0.2q_2$

$$C = 16 + 1.2q_1 + 1.5q_2 + 0.2q_1q_2$$

To maximize the profit, produce how much each?

Solution: $R = P_1q_1 + P_2q_2 = 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2$

$P = R - C = -16 + 598.8q_1 - 0.3q_1^2 + 498.5q_2 - 0.2q_2^2 - 0.2q_1q_2$

$\therefore P = f(q_1, q_2)$, $q_1, q_2 \geq 0$.

$$\begin{cases} \dfrac{\partial P}{\partial q_1} = 598.8 - 0.6q_1 - 0.2q_2 = 0 \\ \dfrac{\partial P}{\partial q_2} = 498.5 - 0.4q_2 - 0.2q_1 = 0 \end{cases} \Rightarrow \begin{cases} q_1 = 699.1, \ P_1 = 390.27 \\ q_2 = 896.7, \ P_2 = 320.66 \end{cases}$$

$$\frac{\partial^2 P}{\partial q_2^2} = -0.4, \quad \frac{\partial^2 P}{\partial q_1^2} = -0.6, \quad \frac{\partial^2 P}{\partial q_1 \partial q_2} = -0.2$$

it's local max,

$\because AC - B^2 = 0.2 > 0$, $\therefore P$ is an upside-down paraboloid,

$\therefore (699.1, 896.7)$ is a global max.

- Constrained optimization: Lagrange Multipliers

$z = f(x,y)$, $\varphi(x,y) = 0 \to$ Constrianed condition

~~if (x_0,y_0) is max/min, $\because \varphi(x,y)$, $\exists \psi(x) = y$~~

~~$\therefore z = f[x, \psi(x)]$~~ (x_0, y_0) max/min.

$$\underbrace{L(x,y)}_{\text{Lagrange function}} = f(x,y) + \underbrace{\lambda}_{\text{Lagrange multiplier}}\varphi(x,y)$$

$\Rightarrow L_x(x_0,y_0) = L_y(x_0,y_0) = 0$

We have $\begin{cases} f_x(x,y) + \lambda\varphi_x(x,y) = 0 \\ f_y(x,y) + \lambda\varphi_y(x,y) = 0 \\ \varphi(x,y) = 0 \end{cases}$

get $(x, y), \lambda$.
$\downarrow$
max/min

Multi-variables functions:

$\begin{cases} L_x \\ L_y \\ L_z \\ \vdots \end{cases} = 0 \Rightarrow \begin{cases} f_x(x,y,\ldots) + \lambda\varphi_x(x,y,\ldots) = 0 \\ f_y(\ldots) + \lambda\varphi_y(\ldots) = 0 \\ \ldots \end{cases}$

$\therefore \nabla f = \lambda \nabla \varphi$

another proof: $\left.\dfrac{df}{ds}\right|_{\vec{u}} = \nabla f \cdot \vec{u} = 0$, $\vec{u}$ the direction of $\varphi = C$

$\vec{u}$ tangent to $\varphi = C$

velocity of

$\therefore \nabla f \perp \varphi$, $\therefore \nabla f \parallel \nabla\varphi$, $\therefore \nabla f = \lambda \nabla \varphi$

- e.g. for many constrained optimization

① $x = Me^{-\alpha p}$ $(M>0, \alpha>0)$; ② $C = C_0 - k\ln x$ $(k>0, x>1)$

x: sale quantity, p: price, C: costs

Get the max profit.

Solution: Profit $u = (p-C)x$, constrained: ①. ②.

$\nabla u = u_x\vec{i} + u_p\vec{j} + u_c\vec{k}$,

$L(x,p,c) = (p-C)x + \lambda(x - Me^{-\alpha p}) + \mu(C - C_0 + k\ln x)$

$\{ L_x = L_p = L_c = 0$

$\Rightarrow p = \dfrac{C_0 - k\ln M + \frac{1}{\alpha} - k}{1-\alpha k}$

Methods for Q.

$\sum (f_x + \lambda \varphi_x + \mu g_x \dots) = 0$.

$\varphi_x = g_x = \cdots = 0$

- eg of optimization with inequality Constraints

$f(x,y) = (x-1)^2 + (y-2)^2$, $x^2+y^2 \le 45$.

step ① $\Rightarrow f_x(x,y) = 2x-2 = f_y(x,y) = 2y-4 = 0$

$1^2+2^2 < 45$, $\therefore (x,y) = (1,2) \in$ region.

critical point

step ② let $g(x) = x^2+y^2-45 = 0$.

$\nabla f = \lambda \nabla g$

$\Rightarrow \begin{cases} 2x-2 = \lambda \cdot 2x \\ 2(y-2) = \lambda \cdot 2y \\ x^2+y^2-45 = 0 \end{cases}$ $\Rightarrow y = 2x$, $x = \pm 3$

$\begin{cases} x=3, & y=6 \\ x=-3, & y=-6 \end{cases}$

Compare three points

step ③ $f(1,2) = 0$, $f(3,6) = 20$, $f(-3,-6) = 80$

$\therefore \min = 0$, $\max = 80$.

(Constrained Differentials)

⑤ Non-independent variables

$w = f(x,y,z)$, $z = g(x,y)$

Find $\left(\dfrac{\partial f}{\partial z}\right)_y$ — consider y constant

$df = f_x dx + f_y dy + f_z dz$

$= f_x dx + f_z dz$, since $y = C$

$dg = f_x dx + f_z dz = 0$,

$\therefore dx = -\dfrac{g_z}{g_x} dz$

$df = (-f_x \frac{g_z}{g_x} + f_z) dz$

$\Rightarrow \left.\dfrac{\partial f}{\partial z}\right|_y = -f_x \dfrac{g_z}{g_x} + f_z$

ef. $\nabla f(x,y,z) = \langle 1,-2,3\rangle$, $z(x,y) = -x^3+y^2$, $P=(1,2,3)$, $g(x,y) = f(x,y,z(x,y))$, Find ∇g at $(1,2)$.

Solution: Total differential: $\begin{cases} df = dx - 2dy + 3dz \\ dz = -3x^2dx + 2y\,dy = -3dx + 6dy \text{ at } (1,2) \end{cases}$

$\therefore df = -8dx + 10dy = dg$

$\therefore \nabla g = \langle -8, 10\rangle$ at $(1,2)$.