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Lecture Notes - MATH 455

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1 Abstract Metric and Topological Spaces

1.1 Metric Spaces Review

Throuhgout, assume X is a non empty set.

Definition 1.1.1: (Metric): $p : X \times X \to \mathbb{R}$ is called a *metric*, and thus (X, p) a metric space, if for all $x, y, z \in X$

- $p(x, y) \geq 0$,
- $p(x, y) = 0 \iff x = y$,
- $p(x, y) = p(y, x)$,
- $p(x, y) \leq p(x, z) + p(z, y)$ (Triangle Inequality).

Definition 1.1.2: (Norm): Let X be a vector space.[1] A function $\|\cdot\| : X \to [0, \infty)$ is called a *norm*, and thus $(X, \|.\|)$ a *normed vector space*, if for all $u, v \in X$ and $\alpha \in \mathbb{R}$

- $\|u\| = 0 \iff u = 0$,
- $\|u + v\| \leq \|u\| + \|v\|$,
- $\|\alpha u\| = |\alpha| \, \|u\|$.

[1]closed under linear combinations

Remark 1.1.3: A norm induces a metric by $p(x, y) := \|x - y\|$.

Example 1.1.4

Examples of normed vector spaces:

1. $(\mathbb{R}^n, |.|)$ where $|x| = ({x_1}^2 + \dots {x_n}^2)$
2. $L^p(E)$ for $E \subseteq \mathbb{R}^n, 1 \leq p \leq \infty$ where $\|f\|_{L^p(E)} = (\int_E |f(x)|^p \, dx)^{\frac{1}{p}}$
3. Discrete metric: if X is a non empty set, then $p(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
4. $C([a, b]) = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$ for $a, b \subseteq \mathbb{R}$. Then, $\|f\|_\infty := \sup_{x \in [a,b]} |f(x)| = \max_{x \in [a,b]} |f(x)|$, $p(f, g) = \|f - g\|_\infty$

Definition 1.1.5: Given two metrics p, σ on X, we say they are *equivalent* if $\exists$ a $C > 0$ such that $\frac{1}{C}\sigma(x, y) \leq p(x, y) \leq C\sigma(x, y)$ for every $x, y \in X$. A similar definition follows for equivalence of norms.

Given a metric space (X, P), then, we have the notion of

- open balls $B(x, r) = \{y \in X : p(x, y) \leq r\}$
- open sets (subsets of X with the property that for every $x \in X$, there is a constant $r > 0$ such that $B(x, r) \subseteq X$), closed sets, closures, and
- *convergence*

Definition 1.1.6 (Convergence): $\{x_n\}_{n=1}^\infty \subseteq X$ converges to x in (X, p) if $\lim_{n \to \infty} p(x_n, x) = 0$

We have several (equivalent) notions, then, of continuity; via sequences, $\epsilon - \delta$ definition, and by pullbacks (inverse images of open sets are open).

Definition 1.1.7 (Uniform Continuity): $f : (X, p) \to (\mathbb{R}, |.|)$ uniformly continuous if f has a "modulus of continuity", i.e. there is a continuous function $\omega : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to 0^+} \omega(t) = 0$, and

$$|f(x) - f(y)| \leq \omega(p(x, y))$$

for every $x, y \in X$

Remark 1.1.8: For instance, we say f Lipschitz continuous if there is a constant $C > 0$ such that $\omega(.) = C(.)$. Let $\alpha \in (0,1)$. We say f α-Holder continuous if $\omega(.) = C(.)^\alpha$ for some constant C.

Definition 1.1.9 (Completeness): We say (X, p) *complete* if every Cauchy sequence in (X, p) converges to a point in X.

Remark 1.1.10: let $E \subseteq X$ and (X, p) complete metric space. Then (E, p) is complete iff $E \subseteq X$ is closed (so limits belong to E)

1.2 Compactness, Separability

Definition 1.2.1 (Open Cover, Compactness): $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$ [1], where X_λ open in X and Λ an arbitrary index set, an *open cover* of X if for every $x \in X$, $\exists \lambda \in \Lambda$ such that $x \in X_\lambda$. [2] X is *compact* if every open cover of X admits a finite subcover. We say $E \subseteq X$ compact if (E, p) compact.

[1] 2^X denotes the power set of X, i.e. the set of all subsets of X.
[2] A cover is finite if $|\Lambda| < \infty$

Remark 1.2.2: for $E \subseteq X$, $X_\lambda \subseteq E$ is open in (E, p) iff X_λ is open in (X, p) Therefore, $E \subseteq X$ is compact iff every open cover of E (in X) has a finite subcover.

Remark 1.2.3: This definition leads to another definition of compactness based on the finite intersection property.

Useful consequence: if (X, p) is compact metric space, and $\{E_k\}_{k=1}^\infty \subseteq X$ closed, and $E_{k+1} \subseteq E_k \forall k$, $\cap_{k=1}^\infty E_k \neq \emptyset$

Definition 1.2.4 (Totally Bounded, ϵ-nets): (X, p) is *totally bounded* if $\forall \epsilon > 0$, there is a finite cover of X of balls with radius $\epsilon > 0$. [1] If $E \subseteq X$, an ϵ-net of E is a collection $\{B(x_i, \epsilon)\}_{i=1}^N$ such that $E \subseteq \bigcup_{i=1}^N B(x_i, \epsilon)$ and $x_i \in X$ (note that x_i need not be in E).

[1] Totally bounded implies (X, p) is bounded

Definition 1.2.5 (Sequentially Compact): (X, p) *sequentially compact* if every sequence in X has a convergent subsequence whose limit is in X.

Definition 1.2.6 (Relatively/Pre-Compact): $E \subseteq X$ *precompact* if $\overline{E}$ compact.

Theorem 1.2.7

TFAE:

1. X complete and totally bounded;
2. X compact;
3. X sequentially compact.

Remark 1.2.8: TFAE:

1. E is totally bdd and Cauchy Seq. converge
2. E is precompact
3. $\forall \{x_k\}_{k=1}^{\infty} \subseteq E, \exists$ a convergent subsequence

Let $f : (X, p) \to (\mathbb{R}, |.|)$ continuous with (X, p) compact. Then,

- $f(X)$ compact in $(\mathbb{R}, |.|)$;
- The max and min of f over X are attained;
- f is uniformly continuous.

Lemma 1.2.9: Any cauchy sequence [1] converges iff it has a convergent subsequence.

[1] $\forall \epsilon > 0, \exists N > 0$ s.t. $\forall m, n > N$, $\|x_n - x_m\| < \epsilon$

Proof.
$\Rightarrow$ If $\{f_n\}_{n=1}^{\infty}$ converges, then $\exists f : X \to \mathbb{R}$ s.t. $\|f_n - f\|_\infty \to 0$, so all subsequences also converge to f.
$\Leftarrow$ Now assume $\exists$ a subsequence $\{f_{n_k}\}_{k=1}^{\infty} \subseteq C(X)$ s.t. $\lim_{k\to\infty} f_{n_k} = f$ in $C(X)$ $\iff$ $\|f_{n_k} - f\|_\infty \to 0$. Suppose for the purpose of contradiction that $f_n \not\to f$. Thus, $\exists \epsilon > 0$, and a subsequence $\{f_{n_j}\}_{j=1}^{\infty} \subseteq C(X)$ s.t. $\|f_{n_j} - f\|_\infty > \epsilon$ for every $j \geq 1$. Then,

$$\|f_{n_k} - f_{n_j}\|_\infty \geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

for k sufficiently large and for n_k, n_j large enough. But this violates $\{f_n\}_{n=1}^{\infty}$ being cauchy. (Contradiction), so we must have $f_n \to f$ in $C(X)$. $\square$

Let $C(X) := \{f : X \in \mathbb{R} \mid f \text{ continuous}\}$ and $\|f\|_\infty := \max_{x\in X} |f(x)|$ the sup norm. Then,

Proposition 1.2.10: Let (X, p) compact. Then $(C(X), \|.\|_\infty)$ is complete.

Proof. let $\{f_n\}_{n=1}^\infty \subseteq C(X)$ be Cauchy. Fix $k \in \mathbb{N}$. By Cauchy defn, let $\epsilon = 2^{-k}$, so $\exists N_k$ sufficiently large s.t. $\|f_{N_k} - f_{N_k+1}\|_\infty < 2^{-k}$. We can then choose $\{n_k\}_{k=1}^\infty$ s.t. $n_k \to \infty$ and $\|f_{n_k} - f_{n_{k+1}}\| < 2^{-k} \quad \forall k \in \mathbb{N}$. Let $j \in \mathbb{N}$. Then

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell=k}^{k+j-1} 2^{-\ell} \leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow[k\to\infty]{} 0$$

In particular, $\forall x \in X$ fixed, let $c_k := f_{n_k}(x)$. Then $\left|c_{k+j} - c_k\right| \leq \|f_{n_{k+j}} - f_{n_k}\|_\infty \to 0 \quad \forall j \in \mathbb{N}$. Thus $\{c_k\}_{k=1}^\infty \subseteq \mathbb{R}$ is cauchy, so by completeness of $\mathbb{R}$, $\exists \bar{c} \in \mathbb{R}$ s.t. $\lim_{k\to\infty} c_k = \bar{c} =: f(x)$ Doing this $\forall x \in X$, we have

$$\begin{aligned}
|f_{n_k}(x) - f(x)| &= \lim_{j\to\infty} \left|f_{n_k}(x) - f_{n_{k+j}}(x)\right| \\
&\leq \lim_{j\to\infty} \|f_{n_k} - f_{n_{k+j}}\|_\infty \\
&\leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow[k\to\infty]{} 0
\end{aligned}$$

$\Rightarrow \|f_{n_k} - f\|_\infty = \sup_{x\in X} |f_{n_k}(x) - f(x)| \xrightarrow[k\to\infty]{} 0$, so $f_{n_k} \to f$ in $C(X)$. Finally, by the lemma this implies $f_n \to f$ in $C(X)$, so $(C(X), \|.\|_\infty)$ is complete. □

Definition 1.2.11 (Density/Separability): A set $D \subseteq X$ is called *dense* in (X, p) if for every [1] nonempty open subset $A \subseteq X$, $D \cap A \neq \emptyset$. We say that X is *separable* if there is a countable dense subset $D \subseteq X$.

[1] If A dense in X, then $\overline{A}$ dense in X

Proposition 1.2.12: If X compact, then X is separable

Proof. Since X is compact, it is totally bounded. Therefore, for $n \in \mathbb{N}$, there is some K_n and $\{x_i^n\} \subseteq X$ such that $X \subseteq \cup_{i=1}^{K_n} B(x_i^n, \frac{1}{n})$. Then, $D = \cup_{n=1}^\infty \cup_{i=1}^{K_n} \{x_i^n\}$ countable and dense in X □

1.3 Arzelà-Ascoli

Goal: Find suitable conditions for a sequence to have a convergent subsequence in $(C(X), \|.\|_\infty)$.

Definition 1.3.1 (Equicontinuous): A family $\mathcal{F} \subseteq C(X)$ is called *equicontinuous* at $x \in X$ if $\forall \epsilon > 0$ there exists a $\delta_x > 0$ such that if $p(x, x') < \delta_x$ then $|f(x) - f(x')| < \epsilon$ for every $f \in \mathcal{F}$. $\mathcal{F}$ is pointwise equicontinuous on X if $\mathcal{F}$ is equicontinuous at every point $x \in X$. [1]

[1] if $|\mathcal{F}| < \infty$, then $\mathcal{F}$ is pointwise equicontinuous on X.

Example 1.3.2

Fix $M > 0$, $[a,b] \subseteq \mathbb{R}$. $\mathcal{F} := \{f \in C([a,b]) \cap C'((a,b)) \mid |f'| \leq M\}$. By Mean Value Theorem, $|f(x) - f(y)| \leq |f'(x^*)|\,|x-y| \leq M\,|x-y|$ for some $x^* \in [x,y]$, so $\forall x \in [a,b]$ if $|x-y| < \frac{\epsilon}{M}$ then $|f(x) - f(y)| < \epsilon$, $\forall f \in \mathcal{F}$, therefore $\mathcal{F}$ is pointwise equicontinuous on $[a,b]$.

Example 1.3.3

Consider $f_n(x) := x^n$ on $[0,1]$. Then $\{f_n\}_{n=1}^{\infty}$ is non equicontinuous at $x = 1$. $f_n(1) = 1\ \forall n$, but the threshold to be close to $f_n(1)$ is not uniform on n.

Definition 1.3.4 (Pointwise, Uniform Boundedness): $\{f_n\}$ pointwise bounded if $\forall x \in X, \exists M(x) > 0$ such that $|f_n(x)| \leq M(x)\ \forall n$, and uniformly bounded if such an M exists independent of X.

Definition 1.3.5 (Uniform Equicontinuous): $\mathcal{F} \subseteq C(X)$ is uniformly equicontinuous on X if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\forall x, y \in X$ if $p(x,y) < \delta$, then $|f(x) - f(y)| < \epsilon$, $\forall f \in \mathcal{F}$.

Remark 1.3.6: $\mathcal{F}$ equicontinuous at x $\iff$ all $f \in \mathcal{F}$ share the same modulus of continuity at x, i.e. $\exists \omega_x$ s.t. $|f(x) - f(y)| \leq \omega_x\,|x-y|$, $\forall f \in \mathcal{F}$.

Proposition 1.3.7 (Sufficient Conditions for Uniform Equicontinuity):

1. $\mathcal{F} \subseteq C(X)$ is uniformly Lipschitz continuous, i.e. $\exists M > 0$ s.t. $|f(x) - f(y)| \leq M p(x,y)\ \forall f \in \mathcal{F}$;
2. $\mathcal{F} \subseteq C(X) \cap C^1(X)$ has a uniform L^∞ bound on the 1st derivative (same as earlier example, by MVT);
3. If (X, p) is compact and $\mathcal{F} \subseteq C(X)$ is pointwise equicontinuous on $X \Rightarrow \mathcal{F}$ is uniformly equicontinuous (Homework).

Lemma 1.3.8 (Arzelà-Ascoli Lemma): Let X be separable and let $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ be pointwise bounded and equicontinuous. Then, there is a function $f \subseteq C(X)$ and a subsequence

> $\{f_{n_k}\}_{k=1}^{\infty}$ which converges pointwise to f on all of X.

Proof. Let $D = \{x_j\}_{j=1}^{\infty} \subseteq X$ be a countable dense subset of X. Since $\{f_n\}$ is pointwise bounded, $\{f_n(x_1)\}$ as a sequence of real numbers is bounded and so by Bolzano-Weierstrass, there is a convergent subsequence $\left\{f_{n(1,k)}(x_1)\right\}_k$ that converges to some $a_1 \in \mathbb{R}$. Consider now $\left\{f_{n(1,k)}(x_2)\right\}_k$, which is again a bounded sequence of $\mathbb{R}$ and so has a convergent subsequence, call it $\left\{f_{n(2,k)}(x_2)\right\}_k$, which converges to some $a_2 \in \mathbb{R}$. Note that $\left\{f_{n(2,k)}\right\} \subseteq \left\{f_{n(1,k)}\right\}$, so also $f_{n(2,k)}(x_1) \to a_1$ as $k \to \infty$. We can repeat this procedure, producing a sequence of real numbers $\{a_\ell\}$, and for each $j \in \mathbb{N}$ a subsequence $\left\{f_{n(j,k)}\right\}_k \subseteq \{f_n\}$ such that $f_{n(j,k)}(x_\ell) \to a_\ell$ for each $1 \leq \ell \leq j$. Define then

$$f : D \to \mathbb{R}, \quad f(x_j) := a_j$$

Consider now

$$f_{n_k} := f_{n(k,k)}, \quad k \geq 1$$

the "diagonal sequence", and remark that $f_{n_k}(x_j) \to a_j = f(x_j)$ as $k \to \infty$ for every $j \geq 1$. Hence, $\{f_{n_k}\}_k$ converges to f on D, pointwise.

We claim now that $\{f_{n_k}\}_k$ converges on all of X to some function $f : X \to \mathbb{R}$, pointwise. Put $g_k := f_{n_k}$ for notational convenience. Fix $x_0 \in X, \epsilon > 0$, and let $\delta_{x_0} > 0$ be such that if $x \in X$ such that $p(x, x_0) < \delta_{x_0}$, $|g_k(x) - g_k(x_0)| < \frac{\epsilon}{3}$. Since D is dense in X, $\exists x_j \in D$ s.t. $p(x_j, x_0) < \delta_{x_0}$. Since $\{g_k(x_j)\}_k$ converges, it is thus Cauchy, and hence for every $k, \ell \geq K$, $|g_k(x_j) - g_\ell(x_j)| < \frac{\epsilon}{3}$. Therefore,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \epsilon$$

And thus $\{g_k(x_0)\}_k$ Cauchy as a sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, then $\{g_k(x_0)\}_k$ also converges, to, say, $f(x_0) \in \mathbb{R}$. Since x_0 was arbitrary, this means there is some function $f : X \to \mathbb{R}$ such that $g_k \to f$ pointwise on X as we aimed to show. $\square$

Theorem 1.3.9: Arzelà-Ascoli Theorem

> Let X be compact and let $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ be uniformly bounded and uniformly equiontinuous. Then, $\exists$ subseq $\{f_{n_k}\}_{k=1}^{\infty}$ and $f \in C(X)$ s.t. $f_{n_k} \xrightarrow[k \to \infty]{} f$ in $C(X)$ (i.e. uniformly)

Proof. Since (X, p) is compact, it is thus separable. Also, uniform bounded/equicontinuous implies pointwise bounded/equicontinuous. Therefore, by Arzelà-Ascoli lemma, $\exists f : X \to \mathbb{R}$ and $\{f_{n_k}\}_{k=1}^{\infty}$ s.t. $f_{n_k} \to f$ pointwise in X.
Now let $g_k := f_{n_k}$.

Claim: $\{g_k\}_{k=1}^{\infty}$ is uniformly Cauchy.[1]

[1] Cauchy sequence in $(C(X), \|\cdot\|_\infty)$

Fix $\epsilon > 0$. By uniform equicontinuity, $\exists \delta > 0$ s.t.

$$p(x,y) < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3} \quad \forall n \in \mathbb{N}.$$

Letting $n = n_k$,

$$p(x,y) < \delta \implies |g_k(x) - g_k(y)| < \epsilon \quad \forall k \in \mathbb{N}.$$

Since X is compact, it is totally bounded, so $\exists \{x_i\}_{i=1}^N$ s.t. $X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$.

Moreover, $\forall 1 \leq i \leq N$ fixed, we know $\{g_k(x_i)\}_{k=1}^\infty \subseteq \mathbb{R}$ converges because $\{g_k\}_{k=1}^\infty$ converges pointwise, so $\{g_k(x_i)\}_{k=1}^\infty$ is a Cauchy sequence. So $\exists K_i > 0$ s.t. $\forall k, \ell \geq K_i$,

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3}.$$

Let $K := \max_{1 \leq i \leq N} K_i$. Then, $\forall k, \ell \geq K$, we have

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3} \quad \forall 1 \leq i \leq N.$$

So $\forall x \in X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$, $\exists x_i$ s.t. $p(x, x_i) < \delta$, and $\forall k, \ell > K$,

$$|g_k(x) - g_\ell(x)| \leq |g_k(x) - g_k(x_i)| + |g_k(x_i) - g_\ell(x_i)| + |g_\ell(x_i) - g_\ell(x)| < \epsilon.$$

This implies $\forall \epsilon > 0, \exists K > 0$ s.t. $\forall k, \ell > K$,

$$\|g_k - g_\ell\|_\infty = \sup_{x \in X} |g_k(x) - g_\ell(x)| < \epsilon,$$

so $\{g_k\}_{k=1}^\infty$ is uniformly Cauchy. Since (X, p) is compact, $C(X)$ is complete, so $\{g_k\}_{k=1}^\infty = \{f_{n_k}\}_{k=1}^\infty$ converges uniformly. Since $f_{n_k} \to f$ pointwise in X, it must be that $f_{n_k} \to f$ uniformly, and thus $f \in C(X)$. $\square$

> **Remark 1.3.10:** How do we use the AA theorem? To extract convergent subsequence, which may give us convergence of the original sequence.

Fact: Let $\{f_n\}_{n=1}^\infty \subseteq C(X)$. If $\exists!\, f$ s.t. for every subsequence, $\exists$ a further subsequence $\{f_{n_{k_j}}\}_{j=1}^\infty$ s.t. $f_{n_{k_j}} \xrightarrow[j \to \infty]{} f$ uniformly, then $f_n \xrightarrow[n \to \infty]{} f$ uniformly.

Typical Application

- Verify $\{f_n\}$ satisfies hypothesis of AA;
- For every subseq $\{f_{n_k}\}$ also satisfies hypothesis of AA;
- Use AA to extract $\{f_{n_{k_j}}\}_{j=1}^\infty$ s.t. $f_{n_{k_j}} \to f$ uniformly on X.
- If you can show f is unique, then $f_n \to f$ in $C(X)$.

Corollary 1.3.11: Let (X, p) be a compact metric space. Let $\mathcal{F} \subseteq C(X)$ be uniformly bounded and uniformly equicontinuous. Then, $\mathcal{F}$ is precompact in $(C(X), \|.\|_\infty)$.

Proof. If f is uniformly bounded and uniformly equicontinuous, then by the AA theorem, $\forall$ sequence $\{f_n\}_{n=1}^\infty \subseteq \mathcal{F}$, there is a subseq. $\{f_{n_j}\}_{j=1}^\infty$ and $f \in C(X)$ s.t. $f_{n_j} \to f$ in $C(X)$. Note, f may not be in $\mathcal{F}$. So $\mathcal{F}$ is precompact. $\square$

Example 1.3.12

Let $M > 0$, $\mathcal{F} = \{f \in C([a,b]) \cap C^1([a,b]) : \|f\|_\infty + \|f'\|_\infty < M\}$. $\mathcal{F}$ is uniformly bounded and uniformly equicontinuous. So by AA then, for $\{f_n\} \subseteq \mathcal{F}, \exists \{f_{n_k}\}_{k=1}^\infty$ s.t. $f_{n_k} \to f$ uniformly. But, f may not be in $C^1([a,b])$. (So, f may not be in $\mathcal{F}$)

Extra stuff left in the assignment, go back and look at it.

1.4 Baire Category Theorem

Definition 1.4.1 (Hollow/Nowhere Dense): We say a set E is *hollow* if $\mathrm{Int}(E) = \emptyset$. [1] We say $E \subseteq X$ *nowhere dense* if its closure is hollow, i.e. $\mathrm{Int}(\overline{E}) = \emptyset$.

[1] i.e. E contains no nontrivial open sets

Remark 1.4.2: E hollow $\iff$ E^c dense in X, since $\mathrm{Int}(E) = \emptyset \iff (\mathrm{Int}(E))^c = \overline{E^c} = X$.

Goal: When can we guarantee that

- a union of hollow sets is hollow?
- an intersection of dense sets is dense?

Theorem 1.4.3: Baire Category Theorem

Let (X, p) be a complete metric space.

1. Let $\{F_n\}_{n=1}^\infty \subseteq X$ be a collection of closed hollow sets. Then $\bigcup_{n=1}^\infty F_n$ is hollow.
2. Let $\{\mathcal{O}_n\}_{n=1}^\infty \subseteq X$ be a collection of open dense sets. Then $\bigcap_{n=1}^\infty \mathcal{O}_n$ is dense.

Proof. (2) $\Rightarrow$ (1) by taking complements and using the previous remark, so we prove only (2).

Claim: Let $G = \bigcap_{n=1}^\infty \mathcal{O}_n$. Then G is dense in X.

Fix $x \in X, r > 0$. $\forall n \in \mathbb{N}, \mathcal{O}_n$ is open and dense, so $\exists y \in \mathcal{O}_n$ and $s > 0$ s.t.

$$B(x,r) \cap \mathcal{O}_n \supseteq B(y,2s) \supseteq \overline{B(y,s)}.$$

Now we use this fact inductively in n. Let $x_1 \in X, r_1 < \frac{1}{2}$ s.t. $\overline{B(x_1,r_1)} \subseteq B(x,r) \cap \mathcal{O}_1$. Let $x_2 \in X, r_2 < 2^{-2}$ s.t. $\overline{B(x_2,r_2)} \subseteq B(x_1,r_1) \cap \mathcal{O}_2$. Repeating this process, take $x_n \in X$, $r_n < 2^{-n}$ s.t. $\overline{B(x_n,r_n)} \subseteq B(x_{n-1},r_{n-1}) \cap \mathcal{O}_n$.

$$\Rightarrow \overline{B(x_1,r_1)} \supseteq \overline{B(x_2,r_2)} \supseteq \cdots \supseteq \overline{B(x_n,r_n)} \supseteq \ldots,$$

and $r_n \to 0$. Therefore $\{x_n\}_{n=1}^{\infty}$ is Cauchy, and (X,p) is complete, so $\exists x_0 \in X$ s.t. $x_n \to x_0$. Thus,

$$x_0 = \bigcap_{n=1}^{\infty} \overline{B(x_n,r_n)}.$$

Since $x_0 \in \overline{B(x_n,r_n)} \subseteq \mathcal{O}_n \; \forall n$, and $x_0 \in \overline{B(x_1,r_1)} \subseteq B(x,r) \Rightarrow x_0 \in G \cap B(x,r)$.

$$\Rightarrow G \cap B(x,r) \neq \emptyset \; \forall x \in X, \forall r > 0.$$

$\Rightarrow G$ is dense in X. □

Another restatement of the Baire Category Theorem is as follows: If (X,p) is complete, the countable union of nowhere dense sets is hollow.

Proof. Let $\{E_n\}_{n=1}^{\infty}$ be nowhere dense sets. Then by BCT, $\cup_{n=1}^{\infty} \overline{E_n}$ is hollow. It follows that $\cup_{n=1}^{\infty} E_n \subseteq \cup_{n=1}^{\infty} \overline{E_n}$ so $\cup_{n=1}^{\infty} E_n$ is also hollow. □

The main way we will use the Baire Category Theorem is the following:

Corollary 1.4.4: Let (X,p) be complete. Suppose $\{F_n\}_{n=1}^{\infty}$ is a collection of closed sets. If $X = \cup_{n=1}^{\infty} F_n$, then $\exists n_0$ s.t. $\text{Int}(F_{n_0}) \neq \emptyset$.

Proof. If $\nexists n_0$, then F_n is hollow $\forall n$, so by BCT $X = \cup_{n=1}^{\infty} F_n$ is hollow, but this is a contradiction because $X \subseteq X$ is open and nontrivial. □

Baire Category Theorem Application:

Theorem 1.4.5

Let $X \subseteq C(X)$ where (X,p) is complete. Suppose $\mathcal{F}$ is pointwise bounded. Then, $\exists$ non-empty open set $\mathcal{O} \subseteq X$ s.t. $\mathcal{F}$ is uniformly bounded on $\mathcal{O}$, i.e. $\exists M > 0$ s.t.

$$\sup_{f \in \mathcal{F}} \sup_{x \in \mathcal{O}} |f(x)| \leq M$$

Proof. Let $E_n = \{x \in X : |f(x)| \leq n \forall f \in \mathcal{F}\} = \bigcap_{f \in \mathcal{F}} \{x \in X : |f(x)| \leq n\} \Rightarrow E_n$ is closed $\forall n$. Since $\mathcal{F}$ is pointwise bounded, $\forall x \in X, \exists M_x > 0$ s.t. $\sup_{f \in \mathcal{F}} |f(x)| \leq M_x$ Thus, $\forall n$ s.t. $M_x \leq n$, then $x \in E_n$ $(|f(x)| \leq M_x \leq n)$. So, $X = \bigcup_{n=1}^{\infty} E_n$ and E_n is closed. By corollary, $\exists n_0$ s.t. $\text{Int}(E_{n_0}) \neq \emptyset$. So $\exists x_0 \in X, r$ s.t. $B^p(x_0, r) \subseteq E_{n_0}$. Letting $\mathcal{O} = B^p(x_0, r)$, we have $\sup_{x \in \mathcal{O}} |f(x)| \leq n_0 \quad \forall f \in \mathcal{F}$. □

Corollary 1.4.6: Let (X, p) be a complete metric space. Suppose $\{F_n\}_{n=1}^{\infty}$ is a collection of closed sets. Then $\bigcup_{n=1}^{\infty} \partial F_n$ is hollow.

Proof. Claim: ∂F_n is hollow $\forall n$. Suppose for contradiction that $\exists n$ s.t. $\text{Int}(\partial F_n) \neq \emptyset$. Then $\exists x_0 \in \partial F_n, r > 0$ s.t. $B^p(x_0, r) \subseteq \partial F_n$. But then,

$$B^p(x_0, r) \cap F_n^c = B^p(x_0, r) \cap \overline{F_n^c} = B^p(x_0, r) \cap (F_n \cup \partial F_n)^c = B^p(x_0, r) \cap \partial F_n^c \cap F_n^c = \emptyset$$

and this contradicts $x_0 \in \partial F_n$ by defn. $\Rightarrow \partial F_n$ is hollow $\forall n$. Furthermore, ∂F_n is closed, since it contains all of its limit points by definition. Thus, by BCT, $\bigcup_{n=1}^{\infty} \partial F_n$ is hollow. □

Now recall that in general, $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ and $f_n \to f$ pointwise, then f is not necessarily continuous.

Theorem 1.4.7

Let (X, p) be complete. Let $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ s.t. $f_n \to f$ pointwise in X. Then there is a dense subset $D \subseteq X$ where $\{f_n\}_{n=1}^{\infty}$ is pointwise equicontinuous on D and $\forall x_0 \in D$, f is continuous at x_0.

Proof. Let $m, n \in \mathbb{N}$. Define

$$E(m, n) = \left\{x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \forall j, k \geq n\right\} = \bigcap_{j,k \geq n} \underbrace{\left\{x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m}\right\}}_{\text{closed, since } f_k, f_j \in C(X)}.$$

So $E(m, n)$ is closed $\forall m, n$. Thus, by the corollary, $\bigcup_{m,n \in \mathbb{N}} \partial E(m, n)$ is hollow. This implies that

$$D := \left(\bigcup_{m,n \in \mathbb{N}} \partial E(m, n)\right)^c = \bigcap_{m,n \in \mathbb{N}} \partial E(m, n)^c \quad \text{is dense.}$$

Claim 1: If $\exists x \in X, m, n \in \mathbb{N}$ s.t. $x \in D \cap E(m, n)$, then $x \in \text{Int}(E(m, n))$.
If $x \in D$, then

$$x \in \underbrace{\partial E(m, n)^c}_{\text{open}} = \text{Int}(E(m, n)) \cup \text{Ext}(E(m, n)).$$

For the exterior term:

$$\text{Ext}(E(m, n)) = X \setminus (\text{Int}(E(m, n)) \cup \partial E(m, n))$$

$$= X \setminus E(m,n) = E(m,n)^c.$$

Since we also have $x \in E(m,n)$, this means $x \in \mathrm{Int}(E(m,n))$.
Claim 2: $\{f_n\}_{n=1}^{\infty}$ is equicontinuous on D.
Let $x_0 \in D$ and $\epsilon > 0$. Choose m s.t. $\frac{1}{m} < \frac{\epsilon}{4}$. Since $\{f_n\}_{n=1}^{\infty}$ converges, $\{f_n(x_0)\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is a Cauchy sequence. So $\exists N$ s.t. $\forall j,k \geq N$,

$$|f_j(x_0) - f_k(x_0)| \leq \frac{1}{m}.$$

This means $x_0 \in E(m,n) \cap D$, so by Claim 1, $x_0 \in \mathrm{Int}(E(m,n))$. Let $B^p(x_0,r) \subseteq E(m,N)$, so $\forall j,k \geq N, \forall x \in B(x_0,r)$,

$$|f_j(x) - f_k(x)| \leq \frac{1}{m}.$$

Since f_N is continuous at x_0, $\exists \delta_{x_0} > 0$ (which WLOG we can choose $< r$), s.t. $\forall x \in B^p(x_0,\delta_{x_0})$,

$$|f_N(x) - f_N(x_0)| \leq \frac{1}{m}.$$

So $\forall j \geq N, \forall x \in B^p(x_0,\delta_{x_0})$,

$$|f_j(x) - f_j(x_0)| \leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_j(x_0)| \leq \frac{3}{m} \leq \frac{3\epsilon}{4}.$$

Since this holds $\forall j \geq N$, this implies that $\{f_n\}_{n=1}^{\infty}$ is equicontinuous at x_0. Furthermore, $\forall x \in B^p(x_0,\delta_{x_0})$, sending $j \to \infty$, we obtain that $\forall x \in B^p(x_0,\delta_{x_0})$,
$|f(x) - f(x_0)| \leq \frac{3\epsilon}{4}$, so f is continuous at $x_0 \in D$. □

1.5 Topological Spaces

We'll consider topological spaces, where we will define all concepts using open sets, and we will generalize what we have learned from Metric Spaces.

Definition 1.5.1: Let X be a non empty set. A *topology* $\mathcal{T}$ on X is a collection of subsets of X, such that

- $X, \emptyset \in \mathcal{T}$;
- If $\{E_n\} \subseteq \mathcal{T}, \bigcap_{n=1}^{N} E_n \in \mathcal{T}$ (closed under finite intersections);
- If $\{E_n\}_{\lambda \in \Lambda} \subseteq \mathcal{T}, \bigcup_{\lambda \in \Lambda} E_\lambda \in \mathcal{T}$ (closed under arbitrary unions).

We say $(X, \mathcal{T})$ is a *topological space*.
If $E \in \mathcal{T}$, then we call E an open set (with respect to $\mathcal{T}$).
If $x \in X$, a set $E \in \mathcal{T}$ containing x is called a *neighborhood* of x.

Remark 1.5.2: By definition of $\mathcal{T}$, $E \in \mathcal{T}$ iff $\forall x \in E, \exists$ a neighbourhood of x, contained in E. (consistent with metric space definition of open set)

Example 1.5.3: Metric topology

Let (X, p) be a metric space. Define

$$\mathcal{T} := \{\text{open sets w.r.t. } p\}.$$

Then, $\mathcal{T}$ is a topology on X, called the metric topology induced by p.
Given a topology $\mathcal{T}$, if $\exists$ a metric p s.t. $\mathcal{T}$ is the metric topology induced by p, then we say $\mathcal{T}$ is *metrizable*.

Example 1.5.4: Trivial Topology

Let X be a non empty set. Define

$$\mathcal{T} = \{\varnothing, X\}.$$

Then, $\mathcal{T}$ is a topology on X, called the trivial topology.

Example 1.5.5: Discrete Topology

Let X be a non empty set. Let $p(x, y)$ be the discrete metric on X. Define Then $B^p(x_0, r) = \begin{cases} \{x_0\}, & 0 < r \leq 1 \\ X, & r > 1 \end{cases}$ So $\forall E \subseteq X, \forall x \in E, B^p(x, \frac{1}{2}) = \{x\} \subseteq E \Rightarrow E$ is open. Then, $\mathcal{T} = P(X) = \{\text{All possible subsets of } X\}$ is a topology on X, called the discrete topology, and it contains all subsets of X.

Example 1.5.6: Relative Topology

Let $(X, \mathcal{T})$ be a topological space. Let $Y \subseteq X$. Then,

$$\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}.$$

Then, $\mathcal{T}_Y$ is a topology on Y, called the relative topology on Y induced by $\mathcal{T}$.
If $X = \mathbb{R}, Y = \mathbb{N}$, then $\mathcal{T}_{\mathbb{N}} = \{U \cap \mathbb{N} : U \subseteq \mathbb{R} \text{ open}\}$ So $\forall y \in \mathbb{N}, \forall x \in \mathbb{R}, r > 0$,

$$B(x, r) \cap \mathbb{N} = \begin{cases} \{y\}, & y \in B(x, r) \\ \varnothing, & y \notin B(x, r) \end{cases}$$

Thus, $\mathcal{T}_{\mathbb{N}} = P(\mathbb{N})$, the discrete topology on $\mathbb{N}$.
If $X = \mathbb{R}, Y = [0,1)$, then $\mathcal{T}_{[0,1)} = \{U \cap [0,1) : U \subseteq \mathbb{R} \text{ open}\}$ So the set $[0, \frac{1}{2}) = [0,1) \cap (-1, \frac{1}{2}) \in \mathcal{T}_{[0,1)}$. So $[0, \frac{1}{2})$ is relatively open in $Y = [0,1)$ (belings to the relative topology on Y).

In metric spaces, everything is done using balls. In a generic topological space $(X, \mathcal{T})$, what plays the role of balls?

Definition 1.5.7 (base/neighbourhood base): Let $(X, \mathcal{T})$ Topological space. Fix $x \in X$. Let $\mathcal{B}_x$ be a collection of neighborhoods of x. We call $\mathcal{B}_x$ a neighbourhood base at x if $\forall$ neighborhood of x (call it u), $\exists B \in \mathcal{B}_x$ such that $B \subseteq u$. We say $\mathcal{B}$, a collection of open sets, is a base for $\mathcal{T}$ if $\forall x \in X$, $\exists$ a neighbourhood base $\mathcal{B}_x \subseteq \mathcal{B}$ at x.

Example 1.5.8

In (X, p) a metric space, $\forall x \in X$

$$\mathcal{B}_x = \{B^p(x, r) : r > 0\} \text{ is a neighbourhood base}$$

$B = \{\text{all balls of all radii}\}$

Remark 1.5.9: Given a topology, a neighbourood base is not unique.

$$\mathcal{B}_x = \{B^p(x, \frac{1}{n})\}_{n=1}^{\infty}$$

is also a neighbourhood base at x in a metric space.

Definition 1.5.10 (first countable/second countable): Let $(X, \mathcal{T})$ be a topological space.

- We say $(X, \mathcal{T})$ is *first countable* if there is a countable neigbourhood base at each $x \in X$;
- We say $(X, \mathcal{T})$ is *second countable* if there is a countable base B of $\mathcal{T}$.

Remark 1.5.11: Any metric space is first countable, and any seperable metric space is second countable.

Remark 1.5.12: For a topology $\mathcal{T}$, $\mathcal{B} = \mathcal{T}$ is always a base for $\mathcal{T}$ (so a base always exists).

Proposition 1.5.13: If $(X, \mathcal{T})$ be a topological space. A collection of open sets $\mathcal{B}$ is a base for $\mathcal{T}$ iff every non-empty open set $u \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

Proof. ($\Rightarrow$) Suppose $\mathcal{B}$ is a base for $\mathcal{T}$. Let $u \in \mathcal{T}$. Then $\forall x \in u, \exists B_x \in \mathcal{B}_x \subseteq B$ such that

$$x \in B_x \subseteq u \Rightarrow \bigcup_{x \in u} B_x \subseteq u$$

and

$$u = \bigcup_{x \in u} \{x\} \subseteq \bigcup_{x \in u} B_x \Rightarrow u = \bigcup_{x \in u} B_x.$$

($\Leftarrow$) Suppose every non-empty open set $u \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.. Fix $x \in u$, and let $\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\} = \{B \in \mathcal{B} : \{x\} \cap B \neq \emptyset\} \subseteq \mathcal{B}$. Since $u = \cup B$, this means $\mathcal{B}_x \neq \emptyset$. So u is a neighbourhood of x, and $\exists B \in \mathcal{B}_x$ such that $B \subseteq u$. $\Rightarrow \mathcal{B}_X$ is a neighbourhood base at x. Doing that $\forall x \in X$, we get a $\mathcal{B}$ that is a base for $\mathcal{T}$. □

Given a collection $\mathcal{B}$, what does it take to be a base for some topology?

Proposition 1.5.14: Let $X \neq \emptyset$. Let $\mathcal{B} \subseteq P(X)$ be a collection of sets. Then $\mathcal{B}$ is a base for some topology $\mathcal{T}$ iff

1. $X = \bigcup_{B \in \mathcal{B}} B$;
2. $\forall B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then $\exists B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.[1]

[1]For balls, this is true

Proof. ($\Rightarrow$) $\mathcal{B}$ is a base for a topology $\mathcal{T}$. Then, since $X \in \mathcal{T}$, by the last result, $X = \bigcup_{B \in \mathcal{B}} B$, so (1) holds. Moreover, if $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{T}$, then $B_1 \cap B_2 \in \mathcal{T}$. So for $x \in B_1 \cap B_2$, then $B_1 \cap B_2$ is a neighbourhood of x. Since $\mathcal{B}$ is a base, $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq B_1 \cap B_2$, so (2) holds.

($\Leftarrow$) Suppose (1) and (2) hold. Let

$$\mathcal{T} := \{U \subseteq X : \forall x \in U \; \exists B \in \mathcal{B} \text{s.t.} x \in B \subseteq u\}$$

Since $X = \bigcup_{B \in \mathcal{B}} B$, so $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B \subseteq X, \Rightarrow x \in \mathcal{T}$. Similarly, $\emptyset \in \mathcal{T}$ because the condition is empty. The definition of $\mathcal{T}$ shows us that it is closed under arbitrary unions. Let $U_1, U_2 \in \mathcal{T}$, and assume $U_1 \cap U_2 \neq \emptyset$, so $\forall x \in U_1 \cap U_2$, by definition of $\mathcal{T}$, $\exists B_1 \in \mathcal{B}$ s.t. $x \in B_1 \subseteq U_1$, and $\exists B_2 \in \mathcal{B}$ s.t. $x \in B_2 \subseteq U_2$. $\Rightarrow x \in B_1 \cap B_2$. By (2), $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Thus, $B_1 \cap B_2 \in \mathcal{T}$. Inductively, we conclude $\mathcal{T}$ is closed under finite intersections. □

Observe that the properties which define $\mathcal{T}$ are closed under intersections. Let $\psi \subseteq P(X)$ be a collection of sets. Then

$$\mathcal{T}(\psi) = \bigcap \{\text{All topologies containing } \psi\} = \text{ topology generated by } \psi$$

Definition 1.5.15 (weaker/coarser vs. stronger/finer): Let $\mathcal{T}_1 and \mathcal{T}_2$ be topologies on X. If $\mathcal{T}_1 \subsetneq \mathcal{T}_2$, we say $\mathcal{T}_1$ is a weaker/coarser topology than $\mathcal{T}_2$ (fewer open sets), and $\mathcal{T}_2$ is a stronger/finer topology than $\mathcal{T}_1$ (more open sets).

Example 1.5.16

Trivial topology is the weakest topology on X and discrete topology is the strongest topology on X. So $\mathcal{T}(\psi)$ is the weakest topology containing ψ.

Proposition 1.5.17: Let $\psi \subseteq P(X)$. Then

$$\mathcal{T}(\psi) = \left\{\emptyset, X, \bigcup \{\text{ finite intersections of sets in } \psi\}\right\}$$

Proof. **Claim:** $\mathcal{B} = \{\emptyset, X, \text{ finite intersections of elements of } \psi\}$ form a base. $\mathcal{B}$ satisfies (1) and (2) of the previous proposition, so $\mathcal{B}$ is a base for some topology

$$\mathcal{T} = \left\{\emptyset, X, \bigcup \{\text{finite intersections of elements of } \psi\}\right\}$$

Observe that $\mathcal{T} \subseteq \{\text{any topology which contains } \psi\} \Rightarrow \mathcal{T} \subseteq \mathcal{T}(\psi)$. $\mathcal{T}$ tilda is also a topology, which contains ψ. So $\mathcal{T}(\psi) \subseteq \mathcal{T}$. Thus, $\mathcal{T} = \mathcal{T}(\psi)$. □

1.6 Separation, Countability, Separability

1.7 Continuity and Compactness