

MATH 455

Lecture Notes

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Based on lectures by Prof. Jessica Lin

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1 Abstract Metric and Topological Spaces

1.1 Metric Spaces Review

Throughout, assume X is a non empty set.

Definition 1.1.1 (Metric). $p : X \times X \rightarrow \mathbb{R}$ is called a *metric*, and thus (X, p) a metric space, if for all $x, y, z \in X$

- $p(x, y) \geq 0$,
- $p(x, y) = 0 \iff x = y$,
- $p(x, y) = p(y, x)$,
- $p(x, y) \leq p(x, z) + p(z, y)$ (Triangle Inequality).

Definition 1.1.2 (Norm). Let X be a vector space.¹ A function $\| \cdot \| : X \rightarrow [0, \infty)$ is called a *norm*, and thus $(X, \| \cdot \|)$ a *normed vector space*, if for all $u, v \in X$ and $\alpha \in \mathbb{R}$

- $\|u\| = 0 \iff u = 0$,
- $\|u + v\| \leq \|u\| + \|v\|$,
- $\|\alpha u\| = |\alpha| \|u\|$.

¹closed under linear combinations

Remark 1.1.2. A norm induces a metric by $p(x, y) := \|x - y\|$.

Example 1.1.3. Examples of normed vector spaces:

1. $(\mathbb{R}^n, |\cdot|)$ where $|x| = (x_1^2 + \dots x_n^2)^{1/2}$
2. $L^p(E)$ for $E \subseteq \mathbb{R}^n, 1 \leq p \leq \infty$ where $\|f\|_{L^p(E)} = (\int_E |f(x)|^p dx)^{1/p}$

3. Discrete metric: if X is a non empty set, then $p(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
4. $C([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$ for $a, b \subseteq \mathbb{R}$. Then,
 $\|f\|_\infty := \sup_{x \in [a, b]} |f(x)| = \max_{x \in [a, b]} |f(x)|$, $p(f, g) = \|f - g\|_\infty$

Definition 1.1.4. Given two metrics p, σ on X , we say they are *equivalent* if \exists a $C > 0$ such that $\frac{1}{C}\sigma(x, y) \leq p(x, y) \leq C\sigma(x, y)$ for every $x, y \in X$. A similar definition follows for equivalence of norms.

Given a metric space (X, P) , then, we have the notion of

- open balls $B(x, r) = \{y \in X : p(x, y) \leq r\}$
- open sets (subsets of X with the property that for every $x \in X$, there is a constant $r > 0$ such that $B(x, r) \subseteq X$), closed sets, closures, and
- *convergence*

Definition 1.1.5 (Convergence). $\{x_n\}_{n=1}^\infty \subseteq X$ *converges* to x in (X, p) if $\lim_{n \rightarrow \infty} p(x_n, x) = 0$

We have several (equivalent) notions, then, of continuity; via sequences, $\epsilon - \delta$ definition, and by pullbacks (inverse images of open sets are open).

Definition 1.1.6 (Uniform Continuity). $f : (X, p) \rightarrow (\mathbb{R}, |\cdot|)$ *uniformly continuous* if f has a "modulus of continuity", i.e. there is a continuous function $\omega : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{t \rightarrow 0^+} \omega(t) = 0$, and

$$|f(x) - f(y)| \leq \omega(p(x, y))$$

for every $x, y \in X$

Remark 1.1.6. For instance, we say f Lipschitz continuous if there is a constant $C > 0$ such that $\omega(\cdot) = C(\cdot)$. Let $\alpha \in (0, 1)$. We say f α -Holder continuous if $\omega(\cdot) = C(\cdot)^\alpha$ for some constant C .

Definition 1.1.7 (Completeness). We say (X, p) *complete* if every Cauchy sequence in (X, p) converges to a point in X .

Remark 1.1.7. let $E \subseteq X$ and (X, p) complete metric space. Then (E, p) is complete iff $E \subseteq X$ is closed (so limits belong to E)

1.2 Compactness, Separability

Definition 1.2.1 (Open Cover, Compactness). $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$, where X_λ open in X and Λ an arbitrary index set, an *open cover* of X if for every $x \in X$, $\exists \lambda \in \Lambda$ such that $x \in X_\lambda$.³ X is *compact* if every open cover of X admits a finite subcover. We say $E \subseteq X$ compact if (E, p) compact.

² 2^X denotes the power set of X , i.e. the set of all subsets of X .

³A cover is finite if $|\Lambda| < \infty$

Remark 1.2.1. for $E \subseteq X$, $X_\lambda \subseteq E$ is open in (E, p) iff X_λ is open in (X, p) . Therefore, $E \subseteq X$ is compact iff every open cover of E (in X) has a finite subcover.

Remark 1.2.1. This definition leads to another definition of compactness based on the finite intersection property.

One useful consequence of this result is if (X, p) is compact metric space, and $\{E_k\}_{k=1}^\infty \subseteq X$ closed, and $E_{k+1} \subseteq E_k \forall k$, $\bigcap_{k=1}^\infty E_k \neq \emptyset$.

Definition 1.2.2 (Totally Bounded, ϵ -nets). (X, p) is *totally bounded* if $\forall \epsilon > 0$, there is a finite cover of X of balls with radius $\epsilon > 0$.⁴ If $E \subseteq X$, an ϵ -net of E is a collection $\{B(x_i, \epsilon)\}_{i=1}^N$ such that $E \subseteq \bigcup_{i=1}^N B(x_i, \epsilon)$ and $x_i \in X$ (note that

⁴Totally bounded implies (X, p) is bounded

x_i need not be in E).

Definition 1.2.3 (Sequentially Compact). (X, p) *sequentially compact* if every sequence in X has a convergent subsequence whose limit is in X .

Definition 1.2.4 (Relatively/Pre-Compact). $E \subseteq X$ *precompact* if \bar{E} compact.

Theorem 1.2.5. TFAE:

1. X complete and totally bounded;
2. X compact;
3. X sequentially compact.

Remark 1.2.5. TFAE:

1. E is totally bdd and Cauchy Seq. converge
2. E is precompact
3. $\forall \{x_k\}_{k=1}^{\infty} \subseteq E, \exists$ a convergent subsequence

Let $f : (X, p) \rightarrow (\mathbb{R}, | \cdot |)$ continuous with (X, p) compact. Then,

- $f(X)$ compact in $(\mathbb{R}, | \cdot |)$;
- The max and min of f over X are attained;
- f is uniformly continuous.

Lemma 1.2.6. Any cauchy sequence⁵ converges iff it has a convergent subsequence.

⁵ $\forall \epsilon > 0, \exists N > 0$ s.t. $\forall m, n > N,$
 $\|x_n - x_m\| < \epsilon$

Proof.

(\Rightarrow) If $\{f_n\}_{n=1}^\infty$ converges, then $\exists f : X \rightarrow \mathbb{R}$ s.t. $\|f_n - f\|_\infty \rightarrow 0$, so all subsequences also converge to f .

(\Leftarrow) Now assume \exists a subsequence $\{f_{n_k}\}_{k=1}^\infty \subseteq C(X)$ s.t. $\lim_{k \rightarrow \infty} f_{n_k} = f$ in $C(X) \iff \|f_{n_k} - f\|_\infty \rightarrow 0$.

Suppose for the purpose of contradiction that $f_n \not\rightarrow f$. Thus, $\exists \epsilon > 0$, and a subsequence $\{f_{n_j}\}_{j=1}^\infty \subseteq C(X)$ s.t. $\|f_{n_j} - f\|_\infty > \epsilon$ for every $j \geq 1$. Then,

$$\|f_{n_k} - f_{n_j}\|_\infty \geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

for k sufficiently large and for n_k, n_j large enough. But this violates $\{f_n\}_{n=1}^\infty$ being Cauchy. (Contradiction), so we must have $f_n \rightarrow f$ in $C(X)$. \square

Let $C(X) := \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$ and $\|f\|_\infty := \max_{x \in X} |f(x)|$ the sup norm. Then,

Proposition 1.2.7. Let (X, p) compact. Then $(C(X), \|\cdot\|_\infty)$ is complete.

Proof. let $\{f_n\}_{n=1}^\infty \subseteq C(X)$ be Cauchy. Fix $k \in \mathbb{N}$. By Cauchy defn, let $\epsilon = 2^{-k}$, so $\exists N_k$ sufficiently large s.t. $\|f_{N_k} - f_{N_{k+1}}\|_\infty < 2^{-k}$. We can then choose $\{n_k\}_{k=1}^\infty$ s.t. $n_k \rightarrow \infty$ and $\|f_{n_k} - f_{n_{k+1}}\|_\infty < 2^{-k} \quad \forall k \in \mathbb{N}$. Let $j \in \mathbb{N}$. Then

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell=k}^{k+j-1} 2^{-\ell} \leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow{k \rightarrow \infty} 0$$

In particular, $\forall x \in X$ fixed, let $c_k := f_{n_k}(x)$. Then $|c_{k+j} - c_k| \leq \|f_{n_{k+j}} - f_{n_k}\|_\infty \rightarrow 0 \quad \forall j \in \mathbb{N}$. Thus $\{c_k\}_{k=1}^\infty \subseteq \mathbb{R}$ is Cauchy, so by completeness of

\mathbb{R} , $\exists \bar{c} \in \mathbb{R}$ s.t. $\lim_{k \rightarrow \infty} c_k = \bar{c} =: f(x)$ Doing this $\forall x \in X$, we have

$$\begin{aligned} |f_{n_k}(x) - f(x)| &= \lim_{j \rightarrow \infty} |f_{n_k}(x) - f_{n_{k+j}}(x)| \\ &\leq \lim_{j \rightarrow \infty} \|f_{n_k} - f_{n_{k+j}}\|_{\infty} \\ &\leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$

$\Rightarrow \|f_{n_k} - f\|_{\infty} = \sup_{x \in X} |f_{n_k}(x) - f(x)| \xrightarrow{k \rightarrow \infty} 0$, so $f_{n_k} \rightarrow f$ in $C(X)$. Finally, by the lemma this implies $f_n \rightarrow f$ in $C(X)$, so $(C(X), \|\cdot\|_{\infty})$ is complete. \square

Definition 1.2.8 (Density/Separability). A set $D \subseteq X$ is called *dense* in (X, p) if for every ⁶ nonempty open subset $A \subseteq X$, $D \cap A \neq \emptyset$. We say that X is *separable* if there is a countable dense subset $D \subseteq X$.

⁶If A dense in X , then \bar{A} dense in X

Proposition 1.2.9. If X compact, then X is separable

Proof. Since X is compact, it is totally bounded. Therefore, for $n \in \mathbb{N}$, there is some K_n and $\{x_i^n\} \subseteq X$ such that $X \subseteq \bigcup_{i=1}^{K_n} B(x_i^n, \frac{1}{n})$. Then, $D = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{K_n} \{x_i^n\}$ countable and dense in X \square

1.3 Arzelà-Ascoli

Goal. Given a sequence $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$, find suitable conditions for $\{f_n\}$ to have a convergent subsequence in $(C(X), \|\cdot\|_{\infty})$.

Definition 1.3.1 (Equicontinuous). A family $\mathcal{F} \subseteq C(X)$ is called *equicontinuous* at $x \in X$ if $\forall \epsilon > 0$ there exists a $\delta_x > 0$ such that if $p(x, x') < \delta_x$ then

$|f(x) - f(x')| < \epsilon$ for every $f \in \mathcal{F}$. \mathcal{F} is *pointwise equicontinuous* on X if \mathcal{F} is equicontinuous at every point $x \in X$.⁷

⁷if $|\mathcal{F}| < \infty$, then \mathcal{F} is pointwise equicontinuous on X .

Example 1.3.2. Fix $M > 0, [a, b] \subseteq \mathbb{R}$. $\mathcal{F} := \{f \in C([a, b]) \cap C'((a, b)) \mid |f'| \leq M\}$. By Mean Value Theorem, $|f(x) - f(y)| \leq |f'(x^*)||x - y| \leq M|x - y|$ for some $x^* \in [x, y]$, so $\forall x \in [a, b]$ if $|x - y| < \frac{\epsilon}{M}$ then $|f(x) - f(y)| < \epsilon, \forall f \in \mathcal{F}$, therefore \mathcal{F} is pointwise equicontinuous on $[a, b]$.

Example 1.3.3. Consider $f_n(x) := x^n$ on $[0, 1]$. Then $\{f_n\}_{n=1}^\infty$ is non equicontinuous at $x = 1$. $f_n(1) = 1 \forall n$, but the threshold to be close to $f_n(1)$ is not uniform on n .

Definition 1.3.4 (Pointwise, Uniform Boundedness). $\{f_n\}$ *pointwise bounded* if $\forall x \in X, \exists M(x) > 0$ such that $|f_n(x)| \leq M(x) \forall n$, and *uniformly bounded* if such an M exists independent of X .

Definition 1.3.5 (Uniform Equicontinuous). $\mathcal{F} \subseteq C(X)$ is *uniformly equicontinuous* on X if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $\forall x, y \in X$ if $p(x, y) < \delta$, then $|f(x) - f(y)| < \epsilon, \forall f \in \mathcal{F}$.

Remark 1.3.5. \mathcal{F} equicontinuous at $x \iff$ all $f \in \mathcal{F}$ share the same modulus of continuity at x , i.e. $\exists \omega_x$ s.t. $|f(x) - f(y)| \leq \omega_x |x - y|, \forall f \in \mathcal{F}$.

Proposition 1.3.6 (Sufficient Conditions for Uniform Equicontinuity).

1. $\mathcal{F} \subseteq C(X)$ is uniformly Lipschitz continuous, i.e. $\exists M > 0$ s.t. $|f(x) - f(y)| \leq Mp(x, y) \forall f \in \mathcal{F}$;

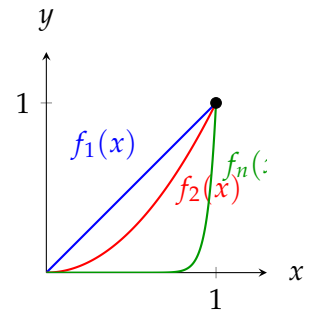


Figure 1: The sequence $f_n(x) = x^n$ is not equicontinuous.

2. $\mathcal{F} \subseteq C(X) \cap C^1(X)$ has a uniform L^∞ bound on the 1st derivative (same as earlier example, by MVT);
3. If (X, p) is compact and $\mathcal{F} \subseteq C(X)$ is pointwise equicontinuous on $X \Rightarrow \mathcal{F}$ is uniformly equicontinuous (Homework).

Lemma 1.3.7 (Arzelà-Ascoli Lemma). Let X be separable and let $\{f_n\}_{n=1}^\infty \subseteq C(X)$ be pointwise bounded and equicontinuous. Then, there is a function $f \in C(X)$ and a subsequence $\{f_{n_k}\}_{k=1}^\infty$ which converges pointwise to f on all of X .

Proof. Let $D = \{x_j\}_{j=1}^\infty \subseteq X$ be a countable dense subset of X . Since $\{f_n\}$ is pointwise bounded, $\{f_n(x_1)\}$ as a sequence of real numbers is bounded and so by Bolzano-Weierstrass, there is a convergent subsequence $\{f_{n(1,k)}(x_1)\}_k$ that converges to some $a_1 \in \mathbb{R}$. Consider now $\{f_{n(1,k)}(x_2)\}_k$, which is again a bounded sequence of \mathbb{R} and so has a convergent subsequence, call it $\{f_{n(2,k)}(x_2)\}_k$, which converges to some $a_2 \in \mathbb{R}$. Note that $\{f_{n(2,k)}\} \subseteq \{f_{n(1,k)}\}$, so also $f_{n(2,k)}(x_1) \rightarrow a_1$ as $k \rightarrow \infty$. We can repeat this procedure, producing a sequence of real numbers $\{a_\ell\}$, and for each $j \in \mathbb{N}$ a subsequence $\{f_{n(j,k)}\}_k \subseteq \{f_n\}$ such that $f_{n(j,k)}(x_\ell) \rightarrow a_\ell$ for each $1 \leq \ell \leq j$. Define then

$$f : D \rightarrow \mathbb{R}, \quad f(x_j) := a_j$$

Consider now

$$f_{n_k} := f_{n(k,k)}, \quad k \geq 1$$

the "diagonal sequence", and remark that $f_{n_k}(x_j) \rightarrow a_j = f(x_j)$ as $k \rightarrow \infty$ for every $j \geq 1$. Hence, $\{f_{n_k}\}_k$ converges to f on D , pointwise.

We claim now that $\{f_{n_k}\}_k$ converges on all of X to some function $f : X \rightarrow \mathbb{R}$, pointwise. Put $g_k := f_{n_k}$ for notational convenience. Fix $x_0 \in X, \epsilon > 0$, and let $\delta_{x_0} > 0$ be such that if $x \in X$ such that $p(x, x_0) < \delta_{x_0}$, $|g_k(x) - g_k(x_0)| < \frac{\epsilon}{3}$.

Since D is dense in X , $\exists x_j \in D$ s.t. $p(x_j, x_0) < \delta_{x_0}$. Since $\{g_k(x_j)\}_k$ converges, it is thus Cauchy, and hence for every $k, \ell \geq K$, $|g_k(x_j) - g_\ell(x_j)| < \frac{\epsilon}{3}$. Therefore,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \epsilon$$

And thus $\{g_k(x_0)\}_k$ Cauchy as a sequence in \mathbb{R} . Since \mathbb{R} is complete, then $\{g_k(x_0)\}_k$ also converges, to, say, $f(x_0) \in \mathbb{R}$. Since x_0 was arbitrary, this means there is some function $f : X \rightarrow \mathbb{R}$ such that $g_k \rightarrow f$ pointwise on X as we aimed to show. \square

Theorem 1.3.8 (Arzelà-Ascoli Theorem). Let X be compact and let $\{f_n\}_{n=1}^\infty \subseteq C(X)$ be uniformly bounded and uniformly equicontinuous. Then, \exists subseq $\{f_{n_k}\}_{k=1}^\infty$ and $f \in C(X)$ s.t. $f_{n_k} \xrightarrow[k \rightarrow \infty]{} f$ in $C(X)$ (i.e. uniformly)

Proof. Since (X, p) is compact, it is thus separable. Also, uniform bounded/equicontinuous implies pointwise bounded/equicontinuous. Therefore, by Arzelà-Ascoli lemma, $\exists f : X \rightarrow \mathbb{R}$ and $\{f_{n_k}\}_{k=1}^\infty$ s.t. $f_{n_k} \rightarrow f$ pointwise in X . Now let $g_k := f_{n_k}$.

Claim: $\{g_k\}_{k=1}^\infty$ is uniformly Cauchy.⁸

Fix $\epsilon > 0$. By uniform equicontinuity, $\exists \delta > 0$ s.t.

$$p(x, y) < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3} \quad \forall n \in \mathbb{N}.$$

Letting $n = n_k$,

$$p(x, y) < \delta \implies |g_k(x) - g_k(y)| < \epsilon \quad \forall k \in \mathbb{N}.$$

Since X is compact, it is totally bounded, so $\exists \{x_i\}_{i=1}^N$ s.t. $X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$.

Moreover, $\forall 1 \leq i \leq N$ fixed, we know $\{g_k(x_i)\}_{k=1}^\infty \subseteq \mathbb{R}$ converges because $\{g_k\}_{k=1}^\infty$ converges pointwise, so $\{g_k(x_i)\}_{k=1}^\infty$ is a Cauchy sequence. So $\exists K_i > 0$

⁸Cauchy sequence in $(C(X), \|\cdot\|_\infty)$

s.t. $\forall k, \ell \geq K_i$,

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3}.$$

Let $K := \max_{1 \leq i \leq N} K_i$. Then, $\forall k, \ell \geq K$, we have

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3} \quad \forall 1 \leq i \leq N.$$

So $\forall x \in X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$, $\exists x_i$ s.t. $p(x, x_i) < \delta$, and $\forall k, \ell > K$,

$$|g_k(x) - g_\ell(x)| \leq |g_k(x) - g_k(x_i)| + |g_k(x_i) - g_\ell(x_i)| + |g_\ell(x_i) - g_\ell(x)| < \epsilon.$$

This implies $\forall \epsilon > 0, \exists K > 0$ s.t. $\forall k, \ell > K$,

$$\|g_k - g_\ell\|_\infty = \sup_{x \in X} |g_k(x) - g_\ell(x)| < \epsilon,$$

so $\{g_k\}_{k=1}^\infty$ is uniformly Cauchy. Since (X, p) is compact, $C(X)$ is complete, so $\{g_k\}_{k=1}^\infty = \{f_{n_k}\}_{k=1}^\infty$ converges uniformly. Since $f_{n_k} \rightarrow f$ pointwise in X , it must be that $f_{n_k} \rightarrow f$ uniformly, and thus $f \in C(X)$. \square

Remark 1.3.8. How do we use the AA theorem? To extract convergent subsequence, which may give us convergence of the original sequence.

Fact. Let $\{f_n\}_{n=1}^\infty \subseteq C(X)$. If $\exists!$ f s.t. for every subsequence, \exists a further subsequence $\{f_{n_{k_j}}\}_{j=1}^\infty$ s.t. $f_{n_{k_j}} \xrightarrow{j \rightarrow \infty} f$ uniformly, then $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly.

Example 1.3.9 (Typical Applications of Arzelà-Ascoli).

- Verify $\{f_n\}$ satisfies hypothesis of AA;
- For every subseq $\{f_{n_k}\}$ also satisfies hypothesis of AA;
- Use AA to extract $\{f_{n_{k_j}}\}_{j=1}^\infty$ s.t. $f_{n_{k_j}} \rightarrow f$ uniformly on X .
- If you can show f is unique, then $f_n \rightarrow f$ in $C(X)$.

Corollary 1.3.10. Let (X, p) be a compact metric space. Let $\mathcal{F} \subseteq C(X)$ be uniformly bounded and uniformly equicontinuous. Then, \mathcal{F} is precompact in $(C(X), \|\cdot\|_\infty)$.

Proof. If f is uniformly bounded and uniformly equicontinuous, then by the AA theorem, \forall sequence $\{f_n\}_{n=1}^\infty \subseteq \mathcal{F}$, there is a subseq. $\{f_{n_j}\}_{j=1}^\infty$ and $f \in C(X)$ s.t. $f_{n_j} \rightarrow f$ in $C(X)$. Note, f may not be in \mathcal{F} . So \mathcal{F} is precompact. \square

Example 1.3.11. Let $M > 0$, and define

$$\mathcal{F} = \left\{ f \in C([a, b]) \cap C^1([a, b]) : \|f\|_\infty + \|f'\|_\infty < M \right\}.$$

\mathcal{F} is uniformly bounded and uniformly equicontinuous.

So by AA then, for $\{f_n\} \subseteq \mathcal{F}$, $\exists \{f_{n_k}\}_{k=1}^\infty$ s.t. $f_{n_k} \rightarrow f$ uniformly. But, f may not be in $C^1([a, b])$. (So, f may not be in \mathcal{F})

Extra stuff left in the assignment, go back and look at it.

1.4 Baire Category Theorem

Definition 1.4.1 (Hollow/Nowhere Dense). We say a set E is *hollow* if $\text{Int}(E) = \emptyset$.⁹ We say $E \subseteq X$ *nowhere dense* if its closure is hollow, i.e. $\text{Int}(\overline{E}) = \emptyset$.

⁹i.e. E contains no nontrivial open sets

Remark 1.4.1. E hollow $\iff E^c$ dense in X , since $\text{Int}(E) = \emptyset \iff (\text{Int}(E))^c = \overline{E^c} = X$.

Goal. When can we guarantee that

- a union of hollow sets is hollow?

- an intersection of dense sets is dense?

Theorem 1.4.2 (Baire Category Theorem). Let (X, p) be a complete metric space.

1. Let $\{F_n\}_{n=1}^{\infty} \subseteq X$ be a collection of closed hollow sets. Then $\bigcup_{n=1}^{\infty} F_n$ is hollow.
2. Let $\{\mathcal{O}_n\}_{n=1}^{\infty} \subseteq X$ be a collection of open dense sets. Then $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ is dense.

Proof. (2) \Rightarrow (1) by taking complements and using the previous remark, so we prove only (2).

Claim: Let $G = \bigcap_{n=1}^{\infty} \mathcal{O}_n$. Then G is dense in X .

Fix $x \in X, r > 0$. $\forall n \in \mathbb{N}, \mathcal{O}_n$ is open and dense, so $\exists y \in \mathcal{O}_n$ and $s > 0$ s.t.

$$B(x, r) \cap \mathcal{O}_n \supseteq B(y, 2s) \supseteq \overline{B(y, s)}.$$

Now we use this fact inductively in n . Let $x_1 \in X, r_1 < \frac{1}{2}$ s.t. $\overline{B(x_1, r_1)} \subseteq B(x, r) \cap \mathcal{O}_1$. Let $x_2 \in X, r_2 < 2^{-2}$ s.t. $\overline{B(x_2, r_2)} \subseteq B(x_1, r_1) \cap \mathcal{O}_2$. Repeating this process, take $x_n \in X, r_n < 2^{-n}$ s.t. $\overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \cap \mathcal{O}_n$.

$$\Rightarrow \overline{B(x_1, r_1)} \supseteq \overline{B(x_2, r_2)} \supseteq \cdots \supseteq \overline{B(x_n, r_n)} \supseteq \cdots,$$

and $r_n \rightarrow 0$. Therefore $\{x_n\}_{n=1}^{\infty}$ is Cauchy, and (X, p) is complete, so $\exists x_0 \in X$ s.t. $x_n \rightarrow x_0$. Thus,

$$x_0 = \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)}.$$

Since $x_0 \in \overline{B(x_n, r_n)} \subseteq \mathcal{O}_n \forall n$, and $x_0 \in \overline{B(x_1, r_1)} \subseteq B(x, r) \Rightarrow x_0 \in G \cap B(x, r)$.

$$\Rightarrow G \cap B(x, r) \neq \emptyset \forall x \in X, \forall r > 0.$$

$\Rightarrow G$ is dense in X . □

Another restatement of the Baire Category Theorem is as follows: If (X, p) is complete, the countable union of nowhere dense sets is hollow.

Proof. Let $\{E_n\}_{n=1}^\infty$ be nowhere dense sets. Then by BCT, $\bigcup_{n=1}^\infty \overline{E_n}$ is hollow. It follows that $\bigcup_{n=1}^\infty E_n \subseteq \bigcup_{n=1}^\infty \overline{E_n}$ so $\bigcup_{n=1}^\infty E_n$ is also hollow. \square

The main way we will use the Baire Category Theorem is the following:

Corollary 1.4.3. Let (X, p) be complete. Suppose $\{F_n\}_{n=1}^\infty$ is a collection of closed sets. If $X = \bigcup_{n=1}^\infty F_n$, then $\exists n_0$ s.t. $\text{Int}(F_{n_0}) \neq \emptyset$.

Proof. If $\nexists n_0$, then F_n is hollow $\forall n$, so by BCT $X = \bigcup_{n=1}^\infty F_n$ is hollow, but this is a contradiction because $X \subseteq X$ is open and nontrivial. \square

Theorem 1.4.4. Let $X \subseteq C(X)$ where (X, p) is complete. Suppose \mathcal{F} is pointwise bounded. Then, \exists non-empty open set $\mathcal{O} \subseteq X$ s.t. \mathcal{F} is uniformly bounded on \mathcal{O} , i.e. $\exists M > 0$ s.t.

$$\sup_{f \in \mathcal{F}} \sup_{x \in \mathcal{O}} |f(x)| \leq M$$

Proof. Let

$$E_n = \{x \in X : |f(x)| \leq n, \forall f \in \mathcal{F}\} = \bigcap_{f \in \mathcal{F}} \{x \in X : |f(x)| \leq n\}.$$

$\Rightarrow E_n$ is closed $\forall n$. Since \mathcal{F} is pointwise bounded,

$$\forall x \in X, \exists M_x > 0 \text{ s.t. } \sup_{f \in \mathcal{F}} |f(x)| \leq M_x.$$

Thus, $\forall n$ s.t. $M_x \leq n$, then $x \in E_n$ (since $|f(x)| \leq M_x \leq n$).

So, $X = \bigcup_{n=1}^{\infty} E_n$ and E_n is closed. By corollary, $\exists n_0$ s.t. $\text{Int}(E_{n_0}) \neq \emptyset$. So $\exists x_0 \in X, r$ s.t. $B^p(x_0, r) \subseteq E_{n_0}$. Letting $\mathcal{O} = B^p(x_0, r)$, we have

$$\sup_{x \in \mathcal{O}} |f(x)| \leq n_0 \quad \forall f \in \mathcal{F}. \quad \square$$

Corollary 1.4.5. Let (X, p) be a complete metric space. Suppose $\{F_n\}_{n=1}^{\infty}$ is a collection of closed sets. Then $\bigcup_{n=1}^{\infty} \partial F_n$ is hollow.

Proof. Claim: ∂F_n is hollow $\forall n$. Suppose for contradiction that $\exists n$ s.t. $\text{Int}(\partial F_n) \neq \emptyset$. Then $\exists x_0 \in \partial F_n, r > 0$ s.t. $B^p(x_0, r) \subseteq \partial F_n$. But then,

$$B^p(x_0, r) \cap F_n^c = B^p(x_0, r) \cap \overline{F_n^c} = B^p(x_0, r) \cap (F_n \cup \partial F_n)^c = B^p(x_0, r) \cap \partial F_n^c \cap F_n^c = \emptyset$$

and this contradicts $x_0 \in \partial F_n$ by defn. $\Rightarrow \partial F_n$ is hollow $\forall n$. Furthermore, ∂F_n is closed, since it contains all of its limit points by definition. Thus, by BCT, $\bigcup_{n=1}^{\infty} \partial F_n$ is hollow. \square

Now recall that in general, $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ and $f_n \rightarrow f$ pointwise, then f is not necessarily continuous.

Theorem 1.4.6. Let (X, p) be complete. Let $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ s.t. $f_n \rightarrow f$ pointwise in X . Then there is a dense subset $D \subseteq X$ where $\{f_n\}_{n=1}^{\infty}$ is pointwise equicontinuous on D and $\forall x_0 \in D, f$ is continuous at x_0 .

Proof. Let $m, n \in \mathbb{N}$. Define

$$\begin{aligned} E(m, n) &= \left\{ x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \forall j, k \geq n \right\} \\ &= \bigcap_{j, k \geq n} \underbrace{\left\{ x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \right\}}_{\text{closed, since } f_k, f_j \in C(X)}. \end{aligned}$$

So $E(m, n)$ is closed $\forall m, n$. Thus, by the corollary, $\bigcup_{m, n \in \mathbb{N}} \partial E(m, n)$ is hollow. This implies that

$$D := \left(\bigcup_{m, n \in \mathbb{N}} \partial E(m, n) \right)^c = \bigcap_{m, n \in \mathbb{N}} \partial E(m, n)^c \text{ is dense.}$$

Claim 1: If $\exists x \in X, m, n \in \mathbb{N}$ s.t. $x \in D \cap E(m, n)$, then $x \in \text{Int}(E(m, n))$.

If $x \in D$, then

$$x \in \underbrace{\partial E(m, n)^c}_{\text{open}} = \text{Int}(E(m, n)) \cup \text{Ext}(E(m, n)).$$

For the exterior term:

$$\begin{aligned} \text{Ext}(E(m, n)) &= X \setminus (\text{Int}(E(m, n)) \cup \partial E(m, n)) \\ &= X \setminus E(m, n) = E(m, n)^c. \end{aligned}$$

Since we also have $x \in E(m, n)$, this means $x \in \text{Int}(E(m, n))$.

Claim 2: $\{f_n\}_{n=1}^\infty$ is equicontinuous on D .

Let $x_0 \in D$ and $\epsilon > 0$. Choose m s.t. $\frac{1}{m} < \frac{\epsilon}{4}$. Since $\{f_n\}_{n=1}^\infty$ converges, $\{f_n(x_0)\}_{n=1}^\infty \subseteq \mathbb{R}$ is a Cauchy sequence. So $\exists N$ s.t. $\forall j, k \geq N$,

$$|f_j(x_0) - f_k(x_0)| \leq \frac{1}{m}.$$

This means $x_0 \in E(m, n) \cap D$, so by Claim 1, $x_0 \in \text{Int}(E(m, n))$. Let $B^p(x_0, r) \subseteq$

$E(m, N)$, so $\forall j, k \geq N, \forall x \in B(x_0, r)$,

$$|f_j(x) - f_k(x)| \leq \frac{1}{m}.$$

Since f_N is continuous at x_0 , $\exists \delta_{x_0} > 0$ (which WLOG we can choose $< r$), s.t. $\forall x \in B^p(x_0, \delta_{x_0})$,

$$|f_N(x) - f_N(x_0)| \leq \frac{1}{m}.$$

So $\forall j \geq N, \forall x \in B^p(x_0, \delta_{x_0})$,

$$\begin{aligned} |f_j(x) - f_j(x_0)| &\leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_j(x_0)| \\ &\leq \frac{3}{m} \leq \frac{3\epsilon}{4}. \end{aligned}$$

Since this holds $\forall j \geq N$, this implies that $\{f_n\}_{n=1}^\infty$ is equicontinuous at x_0 . Furthermore, $\forall x \in B^p(x_0, \delta_{x_0})$, sending $j \rightarrow \infty$, we obtain that $\forall x \in B^p(x_0, \delta_{x_0})$, $|f(x) - f(x_0)| \leq \frac{3\epsilon}{4}$, so f is continuous at $x_0 \in D$. \square

1.5 Topological Spaces

We'll consider topological spaces, where we will define all concepts using open sets, and we will generalize what we have learned from Metric Spaces.

Definition 1.5.1. Let X be a non empty set. A *topology* \mathcal{T} on X is a collection of subsets of X , such that

- $X, \emptyset \in \mathcal{T}$;
- If $\{E_n\} \subseteq \mathcal{T}, \bigcap_{n=1}^N E_n \in \mathcal{T}$ (closed under finite intersections);
- If $\{E_n\}_{n \in \Lambda} \subseteq \mathcal{T}, \bigcup_{n \in \Lambda} E_n \in \mathcal{T}$ (closed under arbitrary unions).

We say (X, \mathcal{T}) is a *topological space*.

If $E \in \mathcal{T}$, then we call E an open set (with respect to \mathcal{T}).

If $x \in X$, a set $E \in \mathcal{T}$ containing x is called a *neighborhood* of x .

Remark 1.5.1. By definition of \mathcal{T} , $E \in \mathcal{T}$ iff $\forall x \in E, \exists$ a neighbourhood of x , contained in E . (consistent with metric space definition of open set)

Example 1.5.2 (Metric topology). Let (X, p) be a metric space. Define

$$\mathcal{T} := \{\text{open sets w.r.t. } p\}.$$

Then, \mathcal{T} is a topology on X , called the metric topology induced by p .

Given a topology \mathcal{T} , if \exists a metric p s.t. \mathcal{T} is the metric topology induced by p , then we say \mathcal{T} is *metrizable*.

Example 1.5.3 (Trivial Topology). Let X be a non empty set. Define

$$\mathcal{T} = \{\emptyset, X\}.$$

Then, \mathcal{T} is a topology on X , called the trivial topology.

Example 1.5.4 (Discrete Topology). Let X be a non empty set. Let $p(x, y)$ be the discrete metric on X . Define Then

$$B^p(x_0, r) = \begin{cases} \{x_0\}, & 0 < r \leq 1 \\ X, & r > 1 \end{cases}$$

So $\forall E \subseteq X, \forall x \in E, B^p(x, \frac{1}{2}) = \{x\} \subseteq E \Rightarrow E$ is open. Then,

$\mathcal{T} = \mathcal{P}(X) = \{\text{All possible subsets of } X\}$ is a topology on X , called the discrete topology, and it contains all subsets of X .

Example 1.5.5 (Relative Topology). Let (X, \mathcal{T}) be a topological space. Let $Y \subseteq X$. Then,

$$\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}.$$

Then, \mathcal{T}_Y is a topology on Y , called the relative topology on Y induced by \mathcal{T} .
 If $X = \mathbb{R}, Y = \mathbb{N}$, then $\mathcal{T}_{\mathbb{N}} = \{U \cap \mathbb{N} : U \subseteq \mathbb{R} \text{ open}\}$ So $\forall y \in \mathbb{N}, \forall x \in \mathbb{R}, r > 0$,

$$B(x, r) \cap \mathbb{N} = \begin{cases} \{y\}, & y \in B(x, r) \\ \emptyset, & y \notin B(x, r) \end{cases}$$

Thus, $\mathcal{T}_{\mathbb{N}} = \mathcal{P}(\mathbb{N})$, the discrete topology on \mathbb{N} .

If $X = \mathbb{R}, Y = [0, 1)$, then $\mathcal{T}_{[0,1)} = \{U \cap [0, 1) : U \subseteq \mathbb{R} \text{ open}\}$ So the set $[0, \frac{1}{2}) = [0, 1) \cap (-1, \frac{1}{2}) \in \mathcal{T}_{[0,1)}$. So $[0, \frac{1}{2})$ is relatively open in $Y = [0, 1)$ (belongs to the relative topology on Y).

In metric spaces, everything is done using balls. In a generic topological space (X, \mathcal{T}) , what plays the role of balls?

Definition 1.5.6 (base/neighbourhood base). Let (X, \mathcal{T}) Topological space. Fix $x \in X$. Let \mathcal{B}_x be a collection of neighborhoods of x . We call \mathcal{B}_x a *neighbourhood base* at x if \forall neighborhood of x (call it U_x), $\exists B \in \mathcal{B}_x$ such that $B \subseteq U_x$. We say \mathcal{B} , a collection of open sets, is a *base* for \mathcal{T} if $\forall x \in X, \exists$ a neighbourhood base $\mathcal{B}_x \subseteq \mathcal{B}$ at x .

Example 1.5.7. In (X, p) a metric space, $\forall x \in X$

$$\mathcal{B}_x = \{B^p(x, r) : r > 0\} \text{ is a neighbourhood base}$$

$$\mathcal{B} = \{\text{all balls of all radii}\}$$

Remark 1.5.7. Given a topology, a neighbourhood base is not unique.

$$\mathcal{B}_x = \{B^p(x, \frac{1}{n})\}_{n=1}^{\infty}$$

is also a neighbourhood base at x in a metric space.

Definition 1.5.8 (first countable/second countable). Let (X, \mathcal{T}) be a topological space.

- We say (X, \mathcal{T}) is *first countable* if there is a countable neighbourhood base at each $x \in X$;
- We say (X, \mathcal{T}) is *second countable* if there is a countable base \mathcal{B} of \mathcal{T} .

Remark 1.5.8. Any metric space is first countable, and any separable metric space is second countable.

Remark 1.5.8. For a topology \mathcal{T} , $\mathcal{B} = \mathcal{T}$ is always a base for \mathcal{T} (so a base always exists).

Proposition 1.5.9. If (X, \mathcal{T}) be a topological space. A collection of open sets \mathcal{B} is a base for \mathcal{T} iff every non-empty open set $U \in \mathcal{T}$ can be written as a union of elements of \mathcal{B} .

Proof.

(\Rightarrow) Suppose \mathcal{B} is a base for \mathcal{T} . Let $U \in \mathcal{T}$. Then $\forall x \in U, \exists B_x \in \mathcal{B}_x \subseteq B$ such that

$$x \in B_x \subseteq U \Rightarrow \bigcup_{x \in U} B_x \subseteq U$$

and

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \Rightarrow U = \bigcup_{x \in U} B_x.$$

(\Leftarrow) Suppose every non-empty open set $U \in \mathcal{T}$ can be written as a union of elements of \mathcal{B} . Fix $x \in U$, and let

$$\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\} = \{B \in \mathcal{B} : \{x\} \cap B \neq \emptyset\} \subseteq \mathcal{B}.$$

Since $U = \cup B$, this means $\mathcal{B}_x \neq \emptyset$. So U is a neighbourhood of x , and $\exists B \in \mathcal{B}_x$ such that $B \subseteq U \Rightarrow \mathcal{B}_x$ is a neighbourhood base at x . Doing that $\forall x \in X$, we get a \mathcal{B} that is a base for \mathcal{T} . \square

Question. Given a collection \mathcal{B} , what does it take to be a base for some topology?

Proposition 1.5.10. Let $X \neq \emptyset$. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a collection of sets. Then \mathcal{B} is a base for some topology \mathcal{T} iff

1. $X = \bigcup_{B \in \mathcal{B}} B$;
2. $\forall B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then $\exists B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.¹⁰

¹⁰For balls, this is true

Proof.

(\Rightarrow) \mathcal{B} is a base for a topology \mathcal{T} . Then, since $X \in \mathcal{T}$, by the last result, $X = \bigcup_{B \in \mathcal{B}} B$, so (1) holds. Moreover, if $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{T}$, then $B_1 \cap B_2 \in \mathcal{T}$. So for $x \in B_1 \cap B_2$, then $B_1 \cap B_2$ is a neighbourhood of x . Since \mathcal{B} is a base, $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq B_1 \cap B_2$, so (2) holds.

(\Leftarrow) Suppose (1) and (2) hold. Let

$$\mathcal{T} := \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}.$$

Since $X = \bigcup_{B \in \mathcal{B}} B$, so $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B \subseteq X \Rightarrow x \in \mathcal{T}$. Similarly, $\emptyset \in \mathcal{T}$ because the condition is empty. The definition of \mathcal{T} shows us that it is closed under arbitrary unions.

Let $U_1, U_2 \in \mathcal{T}$, and assume $U_1 \cap U_2 \neq \emptyset$, so $\forall x \in U_1 \cap U_2$, by definition of \mathcal{T} , $\exists B_1 \in \mathcal{B}$ s.t. $x \in B_1 \subseteq U_1$, and $\exists B_2 \in \mathcal{B}$ s.t. $x \in B_2 \subseteq U_2 \Rightarrow x \in B_1 \cap B_2$. By (2), $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Thus, $B_1 \cap B_2 \in \mathcal{T}$. Inductively, we conclude \mathcal{T} is closed under finite intersections. \square

Observe that the properties which define \mathcal{T} are closed under intersections, so we may define a σ -algebra like structure for topologies:

Definition 1.5.11. Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of sets. Then

$$\mathcal{T}(\mathcal{E}) = \bigcap \{ \text{All topologies containing } \mathcal{E} \} = \text{topology generated by } \mathcal{E}$$

Definition 1.5.12 (weaker/coarser vs. stronger/finer). Let \mathcal{T}_1 and \mathcal{T}_2 be topologies on X . If $\mathcal{T}_1 \subsetneq \mathcal{T}_2$, we say \mathcal{T}_1 is a *weaker/coarser* topology than \mathcal{T}_2 (fewer open sets), and \mathcal{T}_2 is a *stronger/finer* topology than \mathcal{T}_1 (more open sets).

Example 1.5.13. Trivial topology is the weakest topology on X and discrete topology is the strongest topology on X . So $\mathcal{T}(\mathcal{E})$ is the weakest topology containing \mathcal{E} .

Proposition 1.5.14. Let $\mathcal{E} \subseteq \mathcal{P}(X)$. Then

$$\mathcal{T}(\mathcal{E}) = \left\{ \emptyset, X, \bigcup \{ \text{finite intersections of sets in } \mathcal{E} \} \right\}$$

Proof. Claim: $\mathcal{B} = \{ \emptyset, X, \text{finite intersections of elements of } \mathcal{E} \}$ forms a base. \mathcal{B} satisfies (1) and (2) of the previous proposition, so \mathcal{B} is a base for some topology

$$\tilde{\mathcal{T}} = \left\{ \emptyset, X, \bigcup \{ \text{finite intersections of elements of } \mathcal{E} \} \right\}$$

Observe that $\tilde{\mathcal{T}} \subseteq \{ \text{any topology which contains } \mathcal{E} \} \Rightarrow \tilde{\mathcal{T}} \subseteq \mathcal{T}(\mathcal{E})$. $\tilde{\mathcal{T}}$ is also a topology, which contains \mathcal{E} . So $\mathcal{T}(\mathcal{E}) \subseteq \tilde{\mathcal{T}}$. Thus, $\tilde{\mathcal{T}} = \mathcal{T}(\mathcal{E})$. \square

Goal. Topologies give us open sets, bases give ball-like sets, now we need a notion for closed sets.

Definition 1.5.15 (limit point, closure, closed set). If $E \subseteq X, x \in X$ is a *limit point* if \forall neighbourhood U_x of x ,

$$U_x \cap E \neq \emptyset.$$

We say $\bar{E} = \{\text{All limit points of } E\}$, is the *closure* of E .

We say E is *closed* if $E = \bar{E}$.

Remark 1.5.15. We always have $E \subseteq \bar{E}$, so we just need $\bar{E} \subseteq E$ to show E is closed.

Proposition 1.5.16. Let $E \subseteq X$.

1. \bar{E} is closed;
2. \bar{E} is the smallest closed set containing E , i.e. if $\exists F$ closed s.t. $E \subseteq F \Rightarrow \bar{E} \subseteq F$;
3. E is open iff E^c is closed.

Proof of (1) + (2).

Claim: $L := \{\text{limit points of } \bar{E}\} = \bar{\bar{E}} \subseteq \bar{E}$.

Let $x \in L$, and a neighbourhood U_x of x . Then by defn of L , we know $\exists x' \in U_x \cap \bar{E}$. This means $x' \in \bar{E}$, and U_x is a neighbourhood of $x' \Rightarrow U_x \cap E \neq \emptyset$. This holds \forall neighbourhood U_x of x , so $x \in \bar{E} \Rightarrow L \subseteq \bar{E} \Rightarrow \bar{E}$ is closed.

Suppose $E \subseteq F$ and F is closed. Let $x \in \bar{E}$, then \forall neighbourhood U_x , $U_x \cap E \neq \emptyset \Rightarrow U_x \cap F \neq \emptyset \Rightarrow x \in \bar{F} \Rightarrow \bar{E} \subseteq \bar{F} = F$. \square

Proof of (3).

(\Rightarrow) Let $E \subseteq X$ be open. Let $x \in \overline{E^c}$.

Claim: $x \in E^c$.

Suppose not, so $x \in E$. So \exists neighbourhood U_x of x s.t. $U_x \subseteq E$. $\Rightarrow U_x \cap E^c = \emptyset$. So $x \notin \overline{E^c}$. So, $x \in \overline{E^c} \Rightarrow x \in E^c \Rightarrow \overline{E^c} \subseteq E^c \Rightarrow E^c$ closed.

(\Leftarrow) Let E^c be closed. Let $x \in E$.

Claim: \exists neighbourhood U_x s.t. $U_x \subseteq E$.

Suppose not, then every neighbourhood U_x we have $U_x \cap E^c \neq \emptyset \Rightarrow x \in \overline{E^c} = E^c$ (contradicts $x \in E$). By claim, E is open. \square

Remark 1.5.16. Our proof shows, $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.

Definition 1.5.17 (Density, Separability). $D \subseteq X$ is *dense* if \forall non-empty open set U , $U \cap D \neq \emptyset$. $\iff \overline{D} = X$.

(X, \mathcal{T}) is *separable* iff X contains a countable dense set.

Proposition 1.5.18. Every second countable space is separable.

Proof. Let (X, \mathcal{T}) be second countable, so \exists base $\mathcal{B} = \{B_i\}_{i=1}^\infty$. Pick $x_i \in B_i$ (need axiom of choice). Let $D = \{x_i\}_{i=1}^\infty$. Then $\forall U \subseteq X$ open, since \mathcal{B} is a base, $\exists B_i \subseteq U$,

$$D \cap U \supseteq \{x_i\} \cap B_i \neq \emptyset$$

$\Rightarrow D \cap U \neq \emptyset$, so D is dense. \square

Definition 1.5.19 (Convergence in Topology). Given (X, \mathcal{T}) a topological space, let $\{x_n\}_{n=1}^\infty \subseteq X$. We say $x_n \rightarrow x$ in \mathcal{T} if \forall neighbourhood U_x of x , $\exists N$

s.t. $\forall n \geq N, x_n \in U_x$.

Proposition 1.5.20. Suppose (X, \mathcal{T}) is first countable, and $E \subseteq X$. Then, $x \in \bar{E}$ iff $\exists \{x_n\} \subseteq E$ s.t. $x_n \rightarrow x$ in \mathcal{T} .

Proof.

(\Rightarrow) Let $\mathcal{B}_x = \{B_j\}_{j=1}^\infty$ be a neighbourhood base at $x \in \bar{E}$. WLOG, can assume $B_{j+1} \subseteq B_j \quad \forall j$. Since $x \in \bar{E}$ and B_j is a neighbourhood of x , $B_j \cap E \neq \emptyset \quad \forall j$. Let $x_j \in B_j \cap E$. Then, \forall neighbourhood U_x of x , $\exists B_J \in \mathcal{B}_x$ s.t. $B_J \subseteq U_x$. But since $\{B_j\}$ are nested, $\forall j \geq J$,

$$U_x \supseteq B_j \cap U_x = B_j \supseteq B_j \cap E \supseteq \{x_j\}$$

$$\Rightarrow x_j \rightarrow x \text{ in } \mathcal{T}.$$

(\Leftarrow) If $\exists \{x_j\}_{j=1}^\infty \subseteq E$ s.t. $x_j \rightarrow x$ in \mathcal{T} , suppose $x \notin \bar{E}$. Then $x \in \bar{E}^c$ and \bar{E}^c open, so \bar{E}^c is a neighbourhood of x s.t. $\{x_j\}_{j=1}^\infty \cap \bar{E}^c \neq \emptyset$. Therefore $x_j \not\rightarrow x$ in \mathcal{T} . \square

1.6 Separation Properties

While (X, \mathcal{T}) allows us to consider a very general framework, weird stuff can happen because of it, for example:

Example 1.6.1. Let $\mathcal{T} = \{\emptyset, X\}$. So the only non-empty neighbourhood is X , so any sequence $\{x_n\}_{n=1}^\infty \subseteq X$ converges to any point $x \in X$.

To avoid cases like this, we require topologies with more structure.

Definition 1.6.2 (neighbourhood of a set, Separating sets by disjoint neighbourhoods). Let (X, \mathcal{T}) be a topological space, and $K, A, B \subseteq X$. A neigh-

neighbourhood of K is an open set U s.t. $K \subseteq U$. We say A, B can be separated by disjoint neighbourhoods if $\exists U \supseteq A, V \supseteq B$ neighbourhoods s.t. $U \cap V = \emptyset$.

Definition 1.6.3 (Separation Notions). Let (X, \mathcal{T}) be a topological space. (X, \mathcal{T}) is

1. **Tychonoff** (T1) if $\forall x \neq y \in X, \exists$ neighbourhood U_x s.t. $y \notin U_x$, and \exists neighbourhood U_y s.t. $x \notin U_y$;
2. **Hausdorff** (T2) if $\forall x \neq y \in X, \{x\}, \{y\}$, can be separated by disjoint neighbourhoods, i.e. $\exists U_x \supseteq \{x\}, U_y \supseteq \{y\}$ s.t. $U_x \cap U_y = \emptyset$;
3. **Regular** (T3) if (X, \mathcal{T}) is Tychonoff and $\forall x \in X, \forall F \subseteq X$ closed, with $x \notin F, \{x\}$ and F can be separated by disjoint neighbourhoods;
4. **Normal** (T4) if (X, \mathcal{T}) is Tychonoff and $\forall A, B \subseteq X$ closed and disjoint, A and B can be separated by disjoint neighbourhoods.

Remark 1.6.3. Metric \subseteq Normal \subseteq Regular \subseteq Hausdorff \subseteq Tychonoff.

Example 1.6.4. Consider \mathbb{R} and $\mathcal{T} = \{\emptyset, (-\infty, c) \text{ for } c \in \mathbb{R}\}$. Then, $\forall x \in \mathbb{R}$, a neighbourhood of x is of the form $(-\infty, c)$ for some $c > x$. Let $x \neq y \in \mathbb{R}$, WLOG assume $x < y$. Then $x \in U_y \forall$ neighbourhood U_y of y . So $(\mathbb{R}, \mathcal{T})$ is not Tychonoff.

Example 1.6.5. Let $X = \mathbb{R}$ and let $K := \{\frac{1}{n} : n \in \mathbb{Z}\}$. Define the collection \mathcal{B} as:

$$\mathcal{B} = \{(a, b) : a < b\} \cup \{(a, b) \setminus K : a < b\}.$$

We verify the properties of this space:

1. **Basis Check:** Clearly, $\mathbb{R} = \bigcup_{B \in \mathcal{B}} B$. Now, suppose $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Since B_1 and B_2 are intersections of standard intervals with

either \mathbb{R} or $\mathbb{R} \setminus K$, their intersection is also of the form (a, b) or $(a, b) \setminus K$. Thus, there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Therefore, \mathcal{B} is a basis for a topology \mathcal{T} on \mathbb{R} , called the **K-topology**.

2. **Hausdorff (T_2):** Suppose $x, y \in X$ with $x \neq y$. Since the standard topology is Hausdorff, there exist standard disjoint intervals (a, b) and (c, d) separating x and y . These intervals are also in \mathcal{B} . Thus, $U_x \cap U_y = \emptyset \implies (X, \mathcal{T})$ is Hausdorff.

3. **Not Regular (T_3):** The set K is closed in X because its complement $K^c = \mathbb{R} \setminus K$ is open (every point in K^c , including 0, has a neighborhood disjoint from K).

However, observe that $0 \notin K$. We claim 0 and K cannot be separated. Suppose U and V are disjoint open neighborhoods such that $0 \in U$ and $K \subseteq V$.

- Since $0 \in U$, there exists a basis element $(-\delta, \delta) \setminus K \subseteq U$.
- Since $K \subseteq V$, for each n , there exists an interval (a_n, b_n) containing $\frac{1}{n}$ such that $(a_n, b_n) \subseteq V$.

For sufficiently large n , we have $\frac{1}{n} \in (-\delta, \delta)$. The interval (a_n, b_n) around $\frac{1}{n}$ necessarily contains points strictly between terms of K . These points are present in $(-\delta, \delta) \setminus K$.

Therefore, $U \cap V \neq \emptyset$. Thus (X, \mathcal{T}) is Hausdorff but not regular.

Proposition 1.6.6. If (X, \mathcal{T}) is Hausdorff, then for $x_n \rightarrow x$ in \mathcal{T} , x is unique.

Proof. If $x_n \rightarrow x$ and $x_n \rightarrow y$ in \mathcal{T} and $x \neq y$, then $\exists U_x \supseteq \{x\}, U_y \supseteq \{y\}$ s.t. $U_x \cap U_y \neq \emptyset$. So we cannot have $x_n \in U_x \cap U_y, \Rightarrow x = y$ \square

Proposition 1.6.7. (X, \mathcal{T}) is Tychonoff iff $\forall x \in X, \{x\}$ is closed.

Proof.

$$\begin{aligned}
 \{x\} \text{ is closed} &\iff \{x\}^c \text{ is open} \\
 &\iff \forall y \in \{x\}^c, \exists \text{ neighbourhood } U_y \subseteq \{x\}^c \\
 &\iff x \notin U_y
 \end{aligned}$$

□

Remark 1.6.7. (X, \mathcal{T}) normal $\Rightarrow (X, \mathcal{T})$ regular.

Proposition 1.6.8 (Nested neighbourhood property). Let (X, \mathcal{T}) be Tychonoff. Then X is normal iff $\forall F \subseteq X$ closed, $\forall U$ neighbourhood of F , $\exists O \subseteq X$ open s.t. $F \subseteq O \subseteq \overline{O} \subseteq U$.

Proof.

(\Rightarrow) Suppose X is normal. Consider F, U^c are two closed disjoint sets. By normality, $\exists O, V$ open s.t. $F \subseteq O, U^c \subseteq V$ and $O \cap V = \emptyset$. $\Rightarrow V^c \subseteq U$ and $\Rightarrow O \subseteq V^c$.

$$\Rightarrow F \subseteq O \subseteq V^c \subseteq U$$

Since $O \subseteq V^c \Rightarrow \overline{O} \subseteq \overline{V^c} = V^c$ because V is open, $\Rightarrow F \subseteq O \subseteq \overline{O} \subseteq V^c \subseteq U$

(\Leftarrow) Suppose the nested neighbourhood property holds. Let $A, B \subseteq X$ be closed, $A \cap B = \emptyset$. $\Rightarrow A \subseteq B^c$ and B^c open. By assumption, $\exists O$ open s.t. $A \subseteq O \subseteq \overline{O} \subseteq B^c$, $\Rightarrow B \subseteq \overline{O}^c$. $A \subseteq O, B \subseteq \overline{O}$ and $O \cap \overline{O}^c = \emptyset$.

□

Corollary 1.6.9. Every metric space (X, p) is normal.

Proof. By last result, just need to prove the nested neighbourhood property. Let $F \subseteq X$ closed, $U \subseteq X$ open s.t. $F \subseteq U \Rightarrow F \cap U^c = \emptyset$, U^c closed. Let

$$\text{dist}(F, U^c) = \inf_{x \in F} \text{dist}(x, U^c) = \inf_{x \in F} \inf_{y \in U^c} p(x, y)$$

Observe,

$$\text{dist}(x, U^c) = \begin{cases} 0 & x \in U^c \\ > 0 & x \notin U^c \end{cases}$$

For $x \notin U^c$, $\forall \epsilon > 0$, $\exists x' \in U^c$ s.t. $\text{dist}(x, U^c) + \epsilon \geq p(x, x')$.

So, $\forall y$ s.t. $p(x, y) < \epsilon$,

$$\text{dist}(y, U^c) - \text{dist}(x, U^c) \leq p(y, x') - p(x, x') - \epsilon \leq p(y, x) - \epsilon \leq \epsilon$$

A symmetric argument gets that $x \mapsto \text{dist}(x, U^c)$ is continuous.

Since $\text{dist}(x, U^c) \geq 0$, $\inf_{x \in F} \text{dist}(x, U^c) \geq 0$. if $\inf_{x \in F} \text{dist}(x, U^c) = 0$, then by continuity and F closed, $F \cap U^c \neq \emptyset \Rightarrow \text{dist}(F, U^c) = \epsilon > 0$. Let $O := \bigcup_{x \in F} B^p(x, \frac{\epsilon}{2})$ and $\overline{O} = \overline{\bigcup_{x \in F} B^p(x, \frac{\epsilon}{2})} = \bigcup_{x \in F} \overline{B^p(x, \frac{\epsilon}{2})} \Rightarrow F \subseteq O \subseteq \overline{O} \subseteq U$. \square

1.7 Compact Topological Spaces

Definition 1.7.1 (Compact Topological Space). (X, \mathcal{T}) is a topological space.

- $\{E\}_{\lambda \in \Lambda}$ is an *open cover* if $X \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$ and each E_λ is open.
- (X, \mathcal{T}) is a *compact topological space* if every open cover has a finite subcover.
- For $K \subseteq X$, K is *compact* if (K, \mathcal{T}_k) is compact where

$$\mathcal{T}_k := \{K \cap U : U \in \mathcal{T}\}$$

Remark 1.7.1. As before, for $K \subseteq X$, by defn of \mathcal{T}_k , $O \in \mathcal{T}_k$ iff $O = K \cap U$ for $U \in \mathcal{T}$. Therefore, $K \subseteq X$ compact iff \forall open cover of K (in X) has a finite subcover.

Proposition 1.7.2 (properties identical to metric spaces).

1. If $F \subseteq X$ closed and (X, \mathcal{T}) is compact, then F is compact;
2. (X, \mathcal{T}) compact $\Rightarrow \forall \{F_k\}_{k=1}^{\infty} \subseteq X$ closed, nested and non-empty, $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$;

Proof. In the lecture notes (exercise). □

In metric spaces, K compact $\Rightarrow K$ is closed and bounded.

Proposition 1.7.3. Let (X, \mathcal{T}) be Hausdorff. If $K \subseteq X$ is compact, then K is closed in X .

Proof. Claim: K^c is open.

Fix $y \in K^c$. $\forall x \in K$, $\exists U_{xy}, O_{xy}$ open, disjoint s.t. $y \in U_{xy}$ and $x \in O_{xy}$. So $\{O_{xy}\}_{x \in K}$ is an open cover of K , but K compact, so

$$K \subseteq \bigcup_{i=1}^N O_{x_i y} \Rightarrow \bigcap_{i=1}^N O_{x_i y}^c \subseteq K^c$$

Let $E := \bigcap_{i=1}^N U_{x_i y}$ is open. So E is a neighbourhood of y , and $E \cap O_{x_i y} = \emptyset \quad \forall i = 1, \dots, N. \Rightarrow E \subseteq O_{x_i y}^c \quad \forall i = 1, \dots, N, \Rightarrow E \subseteq \bigcap_{i=1}^N O_{x_i y}^c \subseteq K^c. \Rightarrow K^c$ is open. □

Definition 1.7.4 (sequential compactness). (X, \mathcal{T}) is *sequentially compact* if every sequence in X has a convergent subsequence, whose limit is in X .

Proposition 1.7.5 (equivalence of compactness). Let (X, \mathcal{T}) be second countable. Then X compact iff X is sequentially compact.

Proof.

(\Rightarrow) Let X be compact. Let $\{x_k\}_{k=1}^\infty \subseteq X$. Let $F_n := \overline{\{x_k : k \geq n\}}$. So F_n is closed $\forall n$, and $F_n \supseteq F_{n+1} \supseteq \dots$, so since X is compact $\exists x_0 \in \bigcap_{n=1}^\infty F_n$. Observe X second countable $\Rightarrow X$ first countable. So let $\mathcal{B}_{x_0} = \{B_j\}_{j=1}^\infty$ be a neighbourhood base at x_0 . WLOG assume $B_{j+1} \subseteq B_j \quad \forall j$. Since $x_0 \in \bigcap_{n=1}^\infty F_n$, and B_j is a neighbourhood of x_0 , then $B_j \cap F_n \neq \emptyset \quad \forall n$.

Claim: $\exists x_k, (k \geq n)$ s.t. $x_k \in B_j \cap F_n$ (i.e. $B_j \cap \text{Int}(F_n) \neq \emptyset$)

We know $B_j \cap F_n \neq \emptyset$, so $\exists y \in B_j \cap F_n$. Then B_j is a neighbourhood of y and $y \in F_n$, so by defn of F_n

$$B_j \cap \{x_k : k \geq n\} \neq \emptyset$$

Let this element be $\{x_{n_j}\} \in B_j$. So $\{x_{n_j}\}_{j=1}^\infty \subseteq \{x_k\}_{k=1}^\infty$ and $x_{n_j} \in B_j$ with $B_j \supseteq B_{j+1}$. Thus \forall neighbourhood U_{x_0} of x_0 , $\exists B_N \subseteq U_{x_0}$ and if $j \geq N$, $x_{n_j} \in B_j \subseteq B_N \subseteq U_{x_0} \Rightarrow x_{n_j} \rightarrow x_0$ in \mathcal{T} .

(\Leftarrow) Let X be sequentially compact. X second countable \Rightarrow every open cover has a countable subcover, ($X = \bigcup_{B \in \mathcal{B}} B$)

Claim: Every countable cover of X has a finite subcover.

Let $X \subseteq \bigcup_{j=1}^\infty E_j$, E_j open $\forall j$. Assume there is no finite subcover. So $\forall n, \exists m(n) > n$ s.t. $E_{m(n)} \setminus \bigcup_{j=1}^n E_j \neq \emptyset$. Let $x_n \in E_{m(n)} \setminus \bigcup_{j=1}^n E_j$. X sequentially compact means $\exists \{x_{n_k}\}$ s.t. $x_{n_k} \rightarrow x_0 \in X$. Since $x_0 \in X$, $\exists E_N$ s.t. $x_0 \in E_N$. But $x_{n_k} \in E_{m(n_k)} \setminus \bigcup_{j=1}^{n_k} E_j$, so $\forall n_k \geq N$, $x_{n_k} \notin E_N$ contradiction.

□

Theorem 1.7.6. A compact Hausdorff space is normal

Proof. Let (X, \mathcal{T}) be compact Hausdorff.

Claim: (X, \mathcal{T}) is regular.

Let $F \subseteq X$ closed, $x \notin F$. Let $y \in F$. Since X is Hausdorff, $\exists U_{xy}, O_{xy}$ open,

disjoint s.t. $y \in U_{xy}$ and $x \in O_{xy}$. So $\{U_{xy}\}_{y \in F}$ is an open cover of F , but F closed $\Rightarrow F$ compact.¹¹ So $F \subseteq \bigcup_{i=1}^N U_{xy_i} =: U$. Let $N := \bigcap_{i=1}^N O_{xy_i}$, so N is open, $x \in N$ and $U \cap N = \emptyset$. $\Rightarrow (X, \mathcal{T})$ is regular. We just rerun the same argument to get that (X, \mathcal{T}) is normal. \square

¹¹Because closed subsets of compact subspaces are themselves compact

1.8 Continuity and Urysohn's Lemma

Definition 1.8.1 (continuous map). If (X, \mathcal{T}) and (Y, \mathcal{S}) are topological spaces, then $f : X \rightarrow Y$ continuous at $x_0 \in X$ if \forall neighbourhood $O_{f(x_0)} \subseteq Y$, \exists a neighbourhood $U_{x_0} \subseteq X$ s.t. $f(U_{x_0}) \subseteq O_{f(x_0)}$. We say f is *continuous* if f is continuous at every $x \in X$.

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