McGill University | Winter 2026

MATH 455

Lecture Notes

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Based on lectures by Prof. Jessica Lin

Contents

1 Abstract Metric and Topological Spaces

1.1 Metric Spaces Review

Throughout, assume X is a non empty set.

Definition 1.1.1 (Metric). $p : X \times X \to \mathbb{R}$ is called a *metric*, and thus (X, p) a metric space, if for all $x, y, z \in X$

- $p(x, y) \geq 0$,
- $p(x, y) = 0 \iff x = y$,
- $p(x, y) = p(y, x)$,
- $p(x, y) \leq p(x, z) + p(z, y)$ (Triangle Inequality).

Definition 1.1.2 (Norm). Let X be a vector space.[1] A function $\| \cdot \| : X \to [0, \infty)$ is called a *norm*, and thus $(X, \|.\|)$ a *normed vector space*, if for all $u, v \in X$ and $\alpha \in \mathbb{R}$

- $\|u\| = 0 \iff u = 0$,
- $\|u + v\| \leq \|u\| + \|v\|$,
- $\|\alpha u\| = |\alpha| \, \|u\|$.

[1] closed under linear combinations

Remark 1.1.2. A norm induces a metric by $p(x, y) := \|x - y\|$.

Example 1.1.3. Examples of normed vector spaces:

1. $(\mathbb{R}^n, |.|)$ where $|x| = ({x_1}^2 + \ldots {x_n}^2)$
2. $L^p(E)$ for $E \subseteq \mathbb{R}^n, 1 \leq p \leq \infty$ where $\|f\|_{L^p(E)} = (\int_E |f(x)|^p \, dx)^{\frac{1}{p}}$

3. Discrete metric: if X is a non empty set, then $p(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

4. $C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b]\}$ for $a, b \subseteq \mathbb{R}$. Then, $\|f\|_\infty := \sup_{x\in[a,b]} |f(x)| = \max_{x\in[a,b]} |f(x)|$, $p(f,g) = \|f - g\|_\infty$

Definition 1.1.4. Given two metrics p, σ on X, we say they are *equivalent* if $\exists$ a $C > 0$ such that $\frac{1}{C}\sigma(x,y) \leq p(x,y) \leq C\sigma(x,y)$ for every $x, y \in X$. A similar definition follows for equivalence of norms.

Given a metric space (X, P), then, we have the notion of

- open balls $B(x,r) = \{y \in X : p(x,y) \leq r\}$
- open sets (subsets of X with the property that for every $x \in X$, there is a constant $r > 0$ such that $B(x,r) \subseteq X$), closed sets, closures, and
- *convergence*

Definition 1.1.5 (Convergence). $\{x_n\}_{n=1}^\infty \subseteq X$ *converges* to x in (X,p) if $\lim_{n\to\infty} p(x_n, x) = 0$

We have several (equivalent) notions, then, of continuity; via sequences, $\epsilon - \delta$ definition, and by pullbacks (inverse images of open sets are open).

Definition 1.1.6 (Uniform Continuity). $f : (X,p) \to (\mathbb{R}, |.|)$ *uniformly continuous* if f has a "modulus of continuity", i.e. there is a continuous function $\omega : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to 0^+} \omega(t) = 0$, and

$$|f(x) - f(y)| \leq \omega(p(x,y))$$

for every $x, y \in X$

Remark 1.1.6. For instance, we say f Lipschitz continuous if there is a constant $C > 0$ such that $\omega(.) = C(.)$. Let $\alpha \in (0,1)$. We say f α-Holder continuous if $\omega(.) = C(.)^\alpha$ for some constant C.

Definition 1.1.7 (Completeness). We say (X, p) *complete* if every Cauchy sequence in (X, p) converges to a point in X.

Remark 1.1.7. let $E \subseteq X$ and (X, p) complete metric space. Then (E, p) is complete iff $E \subseteq X$ is closed (so limits belong to E)

1.2 Compactness, Separability

Definition 1.2.1 (Open Cover, Compactness). $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$ [2], where X_λ open in X and Λ an arbitrary index set, an *open cover* of X if for every $x \in X$, $\exists \lambda \in \Lambda$ such that $x \in X_\lambda$. [3] X is *compact* if every open cover of X admits a finite subcover. We say $E \subseteq X$ compact if (E, p) compact.

[2] 2^X denotes the power set of X, i.e. the set of all subsets of X.

[3] A cover is finite if $|\Lambda| < \infty$

Remark 1.2.1. for $E \subseteq X$, $X_\lambda \subseteq E$ is open in (E, p) iff X_λ is open in (X, p) Therefore, $E \subseteq X$ is compact iff every open cover of E (in X) has a finite subcover.

Remark 1.2.1. This definition leads to another definition of compactness based on the finite intersection property.

One useful consequence of this result is if (X, p) is compact metric space, and $\{E_k\}_{k=1}^\infty \subseteq X$ closed, and $E_{k+1} \subseteq E_k \forall k$, $\cap_{k=1}^\infty E_k \neq \emptyset$.

Definition 1.2.2 (Totally Bounded, ϵ-nets). (X, p) is *totally bounded* if $\forall \epsilon > 0$, there is a finite cover of X of balls with radius $\epsilon > 0$. [4] If $E \subseteq X$, an ϵ-net of E is a collection $\{B(x_i, \epsilon)\}_{i=1}^N$ such that $E \subseteq \bigcup_{i=1}^N B(x_i, \epsilon)$ and $x_i \in X$ (note that

[4] Totally bounded implies (X, p) is bounded

x_i need not be in E).

Definition 1.2.3 (Sequentially Compact). (X, p) *sequentially compact* if every sequence in X has a convergent subsequence whose limit is in X.

Definition 1.2.4 (Relatively/Pre-Compact). $E \subseteq X$ *precompact* if $\overline{E}$ compact.

Theorem 1.2.5. TFAE:

1. X complete and totally bounded;
2. X compact;
3. X sequentially compact.

Remark 1.2.5. TFAE:

1. E is totally bdd and Cauchy Seq. converge
2. E is precompact
3. $\forall \{x_k\}_{k=1}^{\infty} \subseteq E, \exists$ a convergent subsequence

Let $f : (X, p) \to (\mathbb{R}, |.|)$ continuous with (X, p) compact. Then,

- $f(X)$ compact in $(\mathbb{R}, |.|)$;
- The max and min of f over X are attained;
- f is uniformly continuous.

Lemma 1.2.6. Any cauchy sequence [5] converges iff it has a convergent subsequence.

[5] $\forall \epsilon > 0, \exists N > 0$ s.t. $\forall m, n > N$, $\|x_n - x_m\| < \epsilon$

Proof.

(⇒) If $\{f_n\}_{n=1}^{\infty}$ converges, then $\exists f : X \to \mathbb{R}$ s.t. $\|f_n - f\|_\infty \to 0$, so all subsequences also converge to f.

(⇐) Now assume $\exists$ a subsequence $\{f_{n_k}\}_{k=1}^{\infty} \subseteq C(X)$ s.t. $\lim_{k\to\infty} f_{n_k} = f$ in $C(X) \iff \|f_{n_k} - f\|_\infty \to 0$.

Suppose for the purpose of contradiction that $f_n \not\to f$. Thus, $\exists \epsilon > 0$, and a subsequence $\{f_{n_j}\}_{j=1}^{\infty} \subseteq C(X)$ s.t. $\|f_{n_j} - f\|_\infty > \epsilon$ for every $j \geq 1$. Then,

$$\|f_{n_k} - f_{n_j}\|_\infty \geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

for k sufficiently large and for n_k, n_j large enough. But this violates $\{f_n\}_{n=1}^{\infty}$ being cauchy. (Contradiction), so we must have $f_n \to f$ in $C(X)$. □

Let $C(X) := \{f : X \in \mathbb{R} \mid f \text{ continuous}\}$ and $\|f\|_\infty := \max_{x\in X} |f(x)|$ the sup norm. Then,

Proposition 1.2.7. Let (X, p) compact. Then $(C(X), \|.\|_\infty)$ is complete.

Proof. let $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ be Cauchy. Fix $k \in \mathbb{N}$. By Cauchy defn, let $\epsilon = 2^{-k}$, so $\exists N_k$ sufficiently large s.t. $\|f_{N_k} - f_{N_k+1}\|_\infty < 2^{-k}$. We can then choose $\{n_k\}_{k=1}^{\infty}$ s.t. $n_k \to \infty$ and $\|f_{n_k} - f_{n_{k+1}}\| < 2^{-k} \quad \forall k \in \mathbb{N}$. Let $j \in \mathbb{N}$. Then

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell=k}^{k+j-1} 2^{-\ell} \leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow[k\to\infty]{} 0$$

In particular, $\forall x \in X$ fixed, let $c_k := f_{n_k}(x)$. Then $|c_{k+j} - c_k| \leq \|f_{n_{k+j}} - f_{n_k}\|_\infty \to 0 \quad \forall j \in \mathbb{N}$. Thus $\{c_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ is cauchy, so by completeness of

$\mathbb{R}$, $\exists \bar{c} \in \mathbb{R}$ s.t. $\lim_{k\to\infty} c_k = \bar{c} =: f(x)$ Doing this $\forall x \in X$, we have

$$\begin{aligned}|f_{n_k}(x) - f(x)| &= \lim_{j\to\infty} \left|f_{n_k}(x) - f_{n_{k+j}}(x)\right| \\ &\leq \lim_{j\to\infty} \|f_{n_k} - f_{n_{k+j}}\|_\infty \\ &\leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow[k\to\infty]{} 0\end{aligned}$$

$\Rightarrow \|f_{n_k} - f\|_\infty = \sup_{x\in X} |f_{n_k}(x) - f(x)| \xrightarrow[k\to\infty]{} 0$, so $f_{n_k} \to f$ in $C(X)$. Finally, by the lemma this implies $f_n \to f$ in $C(X)$, so $(C(X), \|.\|_\infty)$ is complete. □

Definition 1.2.8 (Density/Separability). A set $D \subseteq X$ is called *dense* in (X, p) if for every [6] nonempty open subset $A \subseteq X$, $D \cap A \neq \emptyset$. We say that X is *separable* if there is a countable dense subset $D \subseteq X$.

[6] If A dense in X, then $\overline{A}$ dense in X

Proposition 1.2.9. If X compact, then X is separable

Proof. Since X is compact, it is totally bounded. Therefore, for $n \in \mathbb{N}$, there is some K_n and $\{x_i^n\} \subseteq X$ such that $X \subseteq \cup_{i=1}^{K_n} B(x_i^n, \frac{1}{n})$. Then, $D = \cup_{n=1}^{\infty} \cup_{i=1}^{K_n} \{x_i^n\}$ countable and dense in X □

1.3 Arzelà-Ascoli

Goal. Given a sequence $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$, find suitable conditions for $\{f_n\}$ to have a convergent subsequence in $(C(X), \|.\|_\infty)$.

Definition 1.3.1 (Equicontinuous). A family $\mathcal{F} \subseteq C(X)$ is called *equicontinuous* at $x \in X$ if $\forall \epsilon > 0$ there exists a $\delta_x > 0$ such that if $p(x, x') < \delta_x$ then

$|f(x) - f(x')| < \epsilon$ for every $f \in \mathcal{F}$. $\mathcal{F}$ is *pointwise equicontinuous* on X if $\mathcal{F}$ is equicontinuous at every point $x \in X$.[7]

[7] if $|\mathcal{F}| < \infty$, then $\mathcal{F}$ is pointwise equicontinuous on X.

Example 1.3.2. Fix $M > 0, [a,b] \subseteq \mathbb{R}$. $\mathcal{F} := \{f \in C([a,b]) \cap C'((a,b)) \mid |f'| \leq M\}$. By Mean Value Theorem, $|f(x) - f(y)| \leq |f'(x^*)||x-y| \leq M|x-y|$ for some $x^* \in [x,y]$, so $\forall x \in [a,b]$ if $|x-y| < \frac{\epsilon}{M}$ then $|f(x) - f(y)| < \epsilon$, $\forall f \in \mathcal{F}$, therefore $\mathcal{F}$ is pointwise equicontinuous on $[a,b]$.

Example 1.3.3. Consider $f_n(x) := x^n$ on $[0,1]$. Then $\{f_n\}_{n=1}^{\infty}$ is non equicontinuous at $x = 1$. $f_n(1) = 1\ \forall n$, but the threshold to be close to $f_n(1)$ is not uniform on n.

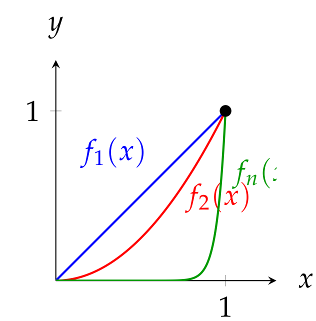

Figure 1: The sequence $f_n(x) = x^n$ is not equicontinuous.

Definition 1.3.4 (Pointwise, Uniform Boundedness). $\{f_n\}$ *pointwise bounded* if $\forall x \in X, \exists M(x) > 0$ such that $|f_n(x)| \leq M(x)\ \forall n$, and *uniformly bounded* if such an M exists independent of X.

Definition 1.3.5 (Uniform Equicontinuous). $\mathcal{F} \subseteq C(X)$ is *uniformly equicontinuous* on X if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\forall x, y \in X$ if $p(x,y) < \delta$, then $|f(x) - f(y)| < \epsilon$, $\forall f \in \mathcal{F}$.

Remark 1.3.5. $\mathcal{F}$ equicontinuous at x $\iff$ all $f \in \mathcal{F}$ share the same modulus of continuity at x, i.e. $\exists \omega_x$ s.t. $|f(x) - f(y)| \leq \omega_x |x - y|$, $\forall f \in \mathcal{F}$.

Proposition 1.3.6 (Sufficient Conditions for Uniform Equicontinuity).

1. $\mathcal{F} \subseteq C(X)$ is uniformly Lipschitz continuous, i.e. $\exists M > 0$ s.t. $|f(x) - f(y)| \leq Mp(x,y)\ \forall f \in \mathcal{F}$;

2. $\mathcal{F} \subseteq C(X) \cap C^1(X)$ has a uniform L^∞ bound on the 1st derivative (same as earlier example, by MVT);

3. If (X, p) is compact and $\mathcal{F} \subseteq C(X)$ is pointwise equicontinuous on X $\Rightarrow \mathcal{F}$ is uniformly equicontinuous (Homework).

Lemma 1.3.7 (Arzelà-Ascoli Lemma). Let X be separable and let $\{f_n\}_{n=1}^\infty \subseteq C(X)$ be pointwise bounded and equicontinuous. Then, there is a function $f \subseteq C(X)$ and a subsequence $\{f_{n_k}\}_{k=1}^\infty$ which converges pointwise to f on all of X.

Proof. Let $D = \{x_j\}_{j=1}^\infty \subseteq X$ be a countable dense subset of X. Since $\{f_n\}$ is pointwise bounded, $\{f_n(x_1)\}$ as a sequence of real numbers is bounded and so by Bolzano-Weierstrass, there is a convergent subsequence $\left\{f_{n(1,k)}(x_1)\right\}_k$ that converges to some $a_1 \in \mathbb{R}$. Consider now $\left\{f_{n(1,k)}(x_2)\right\}_k$, which is again a bounded sequence of $\mathbb{R}$ and so has a convergent subsequence, call it $\left\{f_{n(2,k)}(x_2)\right\}_k$, which converges to some $a_2 \in \mathbb{R}$. Note that $\left\{f_{n(2,k)}\right\} \subseteq \left\{f_{n(1,k)}\right\}$, so also $f_{n(2,k)}(x_1) \to a_1$ as $k \to \infty$. We can repeat this procedure, producing a sequence of real numbers $\{a_\ell\}$, and for each $j \in \mathbb{N}$ a subsequence $\left\{f_{n(j,k)}\right\}_k \subseteq \{f_n\}$ such that $f_{n(j,k)}(x_\ell) \to a_\ell$ for each $1 \le \ell \le j$. Define then

$$f : D \to \mathbb{R}, \quad f(x_j) := a_j$$

Consider now

$$f_{n_k} := f_{n(k,k)}, \quad k \ge 1$$

the "diagonal sequence", and remark that $f_{n_k}(x_j) \to a_j = f(x_j)$ as $k \to \infty$ for every $j \ge 1$. Hence, $\{f_{n_k}\}_k$ converges to f on D, pointwise.

We claim now that $\{f_{n_k}\}_k$ converges on all of X to some function $f : X \to \mathbb{R}$, pointwise. Put $g_k := f_{n_k}$ for notational convenience. Fix $x_0 \in X, \epsilon > 0$, and let $\delta_{x_0} > 0$ be such that if $x \in X$ such that $p(x, x_0) < \delta_{x_0}$, $|g_k(x) - g_k(x_0)| < \frac{\epsilon}{3}$.

Since D is dense in X, $\exists x_j \in D$ s.t. $p(x_j, x_0) < \delta_{x_0}$. Since $\{g_k(x_j)\}_k$ converges, it is thus Cauchy, and hence for every $k, \ell \geq K$, $|g_k(x_j) - g_\ell(x_j)| < \frac{\epsilon}{3}$. Therefore,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \epsilon$$

And thus $\{g_k(x_0)\}_k$ Cauchy as a sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, then $\{g_k(x_0)\}_k$ also converges, to, say, $f(x_0) \in \mathbb{R}$. Since x_0 was arbitrary, this means there is some function $f : X \to \mathbb{R}$ such that $g_k \to f$ pointwise on X as we aimed to show. □

Theorem 1.3.8 (Arzelà-Ascoli Theorem). Let X be compact and let $\{f_n\}_{n=1}^\infty \subseteq C(X)$ be uniformly bounded and uniformly equiontinuous. Then, $\exists$ subseq $\{f_{n_k}\}_{k=1}^\infty$ and $f \in C(X)$ s.t. $f_{n_k} \xrightarrow[k\to\infty]{} f$ in $C(X)$ (i.e. uniformly)

Proof. Since (X, p) is compact, it is thus separable. Also, uniform bounded/equicontinuous implies pointwise bounded/equicontinuous. Therefore, by Arzelà-Ascoli lemma, $\exists f : X \to \mathbb{R}$ and $\{f_{n_k}\}_{k=1}^\infty$ s.t. $f_{n_k} \to f$ pointwise in X. Now let $g_k := f_{n_k}$.

Claim: $\{g_k\}_{k=1}^\infty$ is uniformly Cauchy.[8]

Fix $\epsilon > 0$. By uniform equicontinuity, $\exists \delta > 0$ s.t.

$$p(x, y) < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3} \quad \forall n \in \mathbb{N}.$$

Letting $n = n_k$,

$$p(x, y) < \delta \implies |g_k(x) - g_k(y)| < \epsilon \quad \forall k \in \mathbb{N}.$$

Since X is compact, it is totally bounded, so $\exists \{x_i\}_{i=1}^N$ s.t. $X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$.

Moreover, $\forall 1 \leq i \leq N$ fixed, we know $\{g_k(x_i)\}_{k=1}^\infty \subseteq \mathbb{R}$ converges because $\{g_k\}_{k=1}^\infty$ converges pointwise, so $\{g_k(x_i)\}_{k=1}^\infty$ is a Cauchy sequence. So $\exists K_i > 0$

[8] Cauchy sequence in $(C(X), \|\cdot\|_\infty)$

s.t. $\forall k, \ell \geq K_i$,

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3}.$$

Let $K := \max_{1 \leq i \leq N} K_i$. Then, $\forall k, \ell \geq K$, we have

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3} \quad \forall 1 \leq i \leq N.$$

So $\forall x \in X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$, $\exists x_i$ s.t. $p(x, x_i) < \delta$, and $\forall k, \ell > K$,

$$|g_k(x) - g_\ell(x)| \leq |g_k(x) - g_k(x_i)| + |g_k(x_i) - g_\ell(x_i)| + |g_\ell(x_i) - g_\ell(x)| < \epsilon.$$

This implies $\forall \epsilon > 0, \exists K > 0$ s.t. $\forall k, \ell > K$,

$$\|g_k - g_\ell\|_\infty = \sup_{x \in X} |g_k(x) - g_\ell(x)| < \epsilon,$$

so $\{g_k\}_{k=1}^\infty$ is uniformly Cauchy. Since (X, p) is compact, $C(X)$ is complete, so $\{g_k\}_{k=1}^\infty = \{f_{n_k}\}_{k=1}^\infty$ converges uniformly. Since $f_{n_k} \to f$ pointwise in X, it must be that $f_{n_k} \to f$ uniformly, and thus $f \in C(X)$. □

Remark 1.3.8. How do we use the AA theorem? To extract convergent subsequence, which may give us convergence of the original sequence.

Fact. Let $\{f_n\}_{n=1}^\infty \subseteq C(X)$. If $\exists!$ f s.t. for every subsequence, $\exists$ a further subsequence $\{f_{n_{k_j}}\}_{j=1}^\infty$ s.t. $f_{n_{k_j}} \xrightarrow[j \to \infty]{} f$ uniformly, then $f_n \xrightarrow[n \to \infty]{} f$ uniformly.

Example 1.3.9 (Typical Applications of Arzelà-Ascoli).

- Verify $\{f_n\}$ satisfies hypothesis of AA;
- For every subseq $\{f_{n_k}\}$ also satisfies hypothesis of AA;
- Use AA to extract $\{f_{n_{k_j}}\}_{j=1}^\infty$ s.t. $f_{n_{k_j}} \to f$ uniformly on X.
- If you can show f is unique, then $f_n \to f$ in $C(X)$.

Corollary 1.3.10. Let (X, p) be a compact metric space. Let $\mathcal{F} \subseteq C(X)$ be uniformly bounded and uniformly equicontinuous. Then, $\mathcal{F}$ is precompact in $(C(X), \|.\|_\infty)$.

Proof. If f is uniformly bounded and uniformly equicontinuous, then by the AA theorem, $\forall$ sequence $\{f_n\}_{n=1}^\infty \subseteq \mathcal{F}$, there is a subseq. $\{f_{n_j}\}_{j=1}^\infty$ and $f \in C(X)$ s.t. $f_{n_j} \to f$ in $C(X)$. Note, f may not be in $\mathcal{F}$. So $\mathcal{F}$ is precompact. $\square$

Example 1.3.11. Let $M > 0$, and define

$$\mathcal{F} = \left\{ f \in C([a,b]) \cap C^1([a,b]) : \|f\|_\infty + \|f'\|_\infty < M \right\}.$$

$\mathcal{F}$ is uniformly bounded and uniformly equicontinuous.
So by AA then, for $\{f_n\} \subseteq \mathcal{F}, \exists \{f_{n_k}\}_{k=1}^\infty$ s.t. $f_{n_k} \to f$ uniformly. But, f may not be in $C^1([a,b])$. (So, f may not be in $\mathcal{F}$)

Extra stuff left in the assignment, go back and look at it.

1.4 Baire Category Theorem

Definition 1.4.1 (Hollow/Nowhere Dense). We say a set E is *hollow* if $\text{Int}(E) = \emptyset$. [9] We say $E \subseteq X$ *nowhere dense* if its closure is hollow, i.e. $\text{Int}(\overline{E}) = \emptyset$.

[9] i.e. E contains no nontrivial open sets

Remark 1.4.1. E hollow $\iff$ E^c dense in X, since $\text{Int}(E) = \emptyset \iff (\text{Int}(E))^c = \overline{E^c} = X$.

Goal. When can we guarantee that

- a union of hollow sets is hollow?

- an intersection of dense sets is dense?

Theorem 1.4.2 (Baire Category Theorem). Let (X, p) be a complete metric space.

1. Let $\{F_n\}_{n=1}^{\infty} \subseteq X$ be a collection of closed hollow sets. Then $\bigcup_{n=1}^{\infty} F_n$ is hollow.
2. Let $\{\mathcal{O}_n\}_{n=1}^{\infty} \subseteq X$ be a collection of open dense sets. Then $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ is dense.

Proof. $(2) \Rightarrow (1)$ by taking complements and using the previous remark, so we prove only (2).

Claim: Let $G = \bigcap_{n=1}^{\infty} \mathcal{O}_n$. Then G is dense in X.

Fix $x \in X, r > 0$. $\forall n \in \mathbb{N}, \mathcal{O}_n$ is open and dense, so $\exists y \in \mathcal{O}_n$ and $s > 0$ s.t.

$$B(x, r) \cap \mathcal{O}_n \supseteq B(y, 2s) \supseteq \overline{B(y, s)}.$$

Now we use this fact inductively in n. Let $x_1 \in X, r_1 < \frac{1}{2}$ s.t. $\overline{B(x_1, r_1)} \subseteq B(x, r) \cap \mathcal{O}_1$. Let $x_2 \in X, r_2 < 2^{-2}$ s.t. $\overline{B(x_2, r_2)} \subseteq B(x_1, r_1) \cap \mathcal{O}_2$. Repeating this process, take $x_n \in X$, $r_n < 2^{-n}$ s.t. $\overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \cap \mathcal{O}_n$.

$$\Rightarrow \overline{B(x_1, r_1)} \supseteq \overline{B(x_2, r_2)} \supseteq \cdots \supseteq \overline{B(x_n, r_n)} \supseteq \ldots,$$

and $r_n \to 0$. Therefore $\{x_n\}_{n=1}^{\infty}$ is Cauchy, and (X, p) is complete, so $\exists x_0 \in X$ s.t. $x_n \to x_0$. Thus,

$$x_0 = \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)}.$$

Since $x_0 \in \overline{B(x_n, r_n)} \subseteq \mathcal{O}_n \ \forall n$, and $x_0 \in \overline{B(x_1, r_1)} \subseteq B(x, r) \Rightarrow x_0 \in G \cap B(x, r)$.

$$\Rightarrow G \cap B(x, r) \neq \emptyset \ \forall x \in X, \forall r > 0.$$

$\Rightarrow G$ is dense in X. $\square$

Another restatement of the Baire Category Theorem is as follows: If (X, p) is complete, the countable union of nowhere dense sets is hollow.

Proof. Let $\{E_n\}_{n=1}^{\infty}$ be nowhere dense sets. Then by BCT, $\cup_{n=1}^{\infty}\overline{E_n}$ is hollow. It follows that $\cup_{n=1}^{\infty}E_n \subseteq \cup_{n=1}^{\infty}\overline{E_n}$ so $\cup_{n=1}^{\infty}E_n$ is also hollow. $\square$

The main way we will use the Baire Category Theorem is the following:

Corollary 1.4.3. Let (X, p) be complete. Suppose $\{F_n\}_{n=1}^{\infty}$ is a collection of closed sets. If $X = \cup_{n=1}^{\infty}F_n$, then $\exists n_0$ s.t. $\text{Int}(F_{n_0}) \neq \emptyset$.

Proof. If $\nexists n_0$, then F_n is hollow $\forall n$, so by BCT $X = \cup_{n=1}^{\infty}F_n$ is hollow, but this is a contradiction because $X \subseteq X$ is open and nontrivial. $\square$

Theorem 1.4.4. Let $X \subseteq C(X)$ where (X, p) is complete. Suppose $\mathcal{F}$ is pointwise bounded. Then, $\exists$ non-empty open set $\mathcal{O} \subseteq X$ s.t. $\mathcal{F}$ is uniformly bounded on $\mathcal{O}$, i.e. $\exists M > 0$ s.t.

$$\sup_{f\in\mathcal{F}}\sup_{x\in\mathcal{O}}|f(x)| \leq M$$

Proof. Let

$$E_n = \{x \in X : |f(x)| \leq n,\ \forall f \in \mathcal{F}\} = \bigcap_{f\in\mathcal{F}}\{x \in X : |f(x)| \leq n\}.$$

$\Rightarrow E_n$ is closed $\forall n$. Since $\mathcal{F}$ is pointwise bounded,

$$\forall x \in X, \exists M_x > 0 \text{ s.t. } \sup_{f\in\mathcal{F}}|f(x)| \leq M_x.$$

Thus, $\forall n$ s.t. $M_x \leq n$, then $x \in E_n$ (since $|f(x)| \leq M_x \leq n$).

So, $X = \bigcup_{n=1}^{\infty} E_n$ and E_n is closed. By corollary, $\exists n_0$ s.t. $\text{Int}(E_{n_0}) \neq \emptyset$. So $\exists x_0 \in X, r$ s.t. $B^p(x_0, r) \subseteq E_{n_0}$. Letting $\mathcal{O} = B^p(x_0, r)$, we have

$$\sup_{x \in \mathcal{O}} |f(x)| \leq n_0 \quad \forall f \in \mathcal{F}. \qquad \square$$

Corollary 1.4.5. Let (X, p) be a complete metric space. Suppose $\{F_n\}_{n=1}^{\infty}$ is a collection of closed sets. Then $\bigcup_{n=1}^{\infty} \partial F_n$ is hollow.

Proof. **Claim:** ∂F_n is hollow $\forall n$. Suppose for contradiction that $\exists n$ s.t. $\text{Int}(\partial F_n) \neq \emptyset$. Then $\exists x_0 \in \partial F_n, r > 0$ s.t. $B^p(x_0, r) \subseteq \partial F_n$. But then,

$$B^p(x_0, r) \cap F_n^c = B^p(x_0, r) \cap \overline{F_n^c} = B^p(x_0, r) \cap (F_n \cup \partial F_n)^c = B^p(x_0, r) \cap \partial F_n^c \cap F_n^c = \emptyset$$

and this contradicts $x_0 \in \partial F_n$ by defn. $\Rightarrow \partial F_n$ is hollow $\forall n$. Furthermore, ∂F_n is closed, since it contains all of its limit points by definition. Thus, by BCT, $\bigcup_{n=1}^{\infty} \partial F_n$ is hollow. $\square$

Now recall that in general, $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ and $f_n \to f$ pointwise, then f is not necessarily continuous.

Theorem 1.4.6. Let (X, p) be complete. Let $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ s.t. $f_n \to f$ pointwise in X. Then there is a dense subset $D \subseteq X$ where $\{f_n\}_{n=1}^{\infty}$ is pointwise equicontinuous on D and $\forall x_0 \in D$, f is continuous at x_0.

Proof. Let $m, n \in \mathbb{N}$. Define

$$E(m,n) = \left\{ x \in X : \left| f_j(x) - f_k(x) \right| \le \frac{1}{m} \ \forall j,k \ge n \right\}$$
$$= \bigcap_{j,k \ge n} \underbrace{\left\{ x \in X : \left| f_j(x) - f_k(x) \right| \le \frac{1}{m} \right\}}_{\text{closed, since } f_k, f_j \in C(X)}.$$

So $E(m,n)$ is closed $\forall m, n$. Thus, by the corollary, $\bigcup_{m,n \in \mathbb{N}} \partial E(m,n)$ is hollow. This implies that

$$D := \left(\bigcup_{m,n \in \mathbb{N}} \partial E(m,n) \right)^c = \bigcap_{m,n \in \mathbb{N}} \partial E(m,n)^c \quad \text{is dense.}$$

Claim 1: If $\exists x \in X, m, n \in \mathbb{N}$ s.t. $x \in D \cap E(m,n)$, then $x \in \operatorname{Int}(E(m,n))$.
If $x \in D$, then

$$x \in \underbrace{\partial E(m,n)^c}_{\text{open}} = \operatorname{Int}(E(m,n)) \cup \operatorname{Ext}(E(m,n)).$$

For the exterior term:

$$\operatorname{Ext}(E(m,n)) = X \setminus (\operatorname{Int}(E(m,n)) \cup \partial E(m,n))$$
$$= X \setminus E(m,n) = E(m,n)^c.$$

Since we also have $x \in E(m,n)$, this means $x \in \operatorname{Int}(E(m,n))$.
Claim 2: $\{f_n\}_{n=1}^{\infty}$ is equicontinuous on D.
Let $x_0 \in D$ and $\epsilon > 0$. Choose m s.t. $\frac{1}{m} < \frac{\epsilon}{4}$. Since $\{f_n\}_{n=1}^{\infty}$ converges, $\{f_n(x_0)\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is a Cauchy sequence. So $\exists N$ s.t. $\forall j, k \ge N$,

$$\left| f_j(x_0) - f_k(x_0) \right| \le \frac{1}{m}.$$

This means $x_0 \in E(m,n) \cap D$, so by Claim 1, $x_0 \in \operatorname{Int}(E(m,n))$. Let $B^p(x_0, r) \subseteq$

$E(m,N)$, so $\forall j,k \geq N, \forall x \in B(x_0,r)$,

$$|f_j(x) - f_k(x)| \leq \frac{1}{m}.$$

Since f_N is continuous at x_0, $\exists \delta_{x_0} > 0$ (which WLOG we can choose $< r$), s.t. $\forall x \in B^p(x_0,\delta_{x_0})$,

$$|f_N(x) - f_N(x_0)| \leq \frac{1}{m}.$$

So $\forall j \geq N, \forall x \in B^p(x_0,\delta_{x_0})$,

$$\begin{aligned}|f_j(x) - f_j(x_0)| &\leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_j(x_0)| \\ &\leq \frac{3}{m} \leq \frac{3\epsilon}{4}.\end{aligned}$$

Since this holds $\forall j \geq N$, this implies that $\{f_n\}_{n=1}^{\infty}$ is equicontinuous at x_0. Furthermore, $\forall x \in B^p(x_0,\delta_{x_0})$, sending $j \to \infty$, we obtain that $\forall x \in B^p(x_0,\delta_{x_0})$, $|f(x) - f(x_0)| \leq \frac{3\epsilon}{4}$, so f is continuous at $x_0 \in D$. □

1.5 Topological Spaces

We'll consider topological spaces, where we will define all concepts using open sets, and we will generalize what we have learned from Metric Spaces.

Definition 1.5.1. Let X be a non empty set. A *topology* $\mathcal{T}$ on X is a collection of subsets of X, such that

- $X, \emptyset \in \mathcal{T}$;
- If $\{E_n\} \subseteq \mathcal{T}, \bigcap_{n=1}^{N} E_n \in \mathcal{T}$ (closed under finite intersections);
- If $\{E_n\}_{\lambda\in\Lambda} \subseteq \mathcal{T}, \bigcup_{\lambda\in\Lambda} E_\lambda \in \mathcal{T}$ (closed under arbitrary unions).

We say $(X,\mathcal{T})$ is a *topological space*.
If $E \in \mathcal{T}$, then we call E an open set (with respect to $\mathcal{T}$).
If $x \in X$, a set $E \in \mathcal{T}$ containing x is called a *neighborhood* of x.

Remark 1.5.1. By definition of $\mathcal{T}$, $E \in \mathcal{T}$ iff $\forall x \in E, \exists$ a neighbourhood of x, contained in E. (consistent with metric space definition of open set)

Example 1.5.2 (Metric topology). Let (X, p) be a metric space. Define

$$\mathcal{T} := \{\text{open sets w.r.t. } p\} .$$

Then, $\mathcal{T}$ is a topology on X, called the metric topology induced by p.
Given a topology $\mathcal{T}$, if $\exists$ a metric p s.t. $\mathcal{T}$ is the metric topology induced by p, then we say $\mathcal{T}$ is *metrizable*.

Example 1.5.3 (Trivial Topology). Let X be a non empty set. Define

$$\mathcal{T} = \{\emptyset, X\} .$$

Then, $\mathcal{T}$ is a topology on X, called the trivial topology.

Example 1.5.4 (Discrete Topology). Let X be a non empty set. Let $p(x, y)$ be the discrete metric on X. Define Then

$$B^p(x_0, r) = \begin{cases} \{x_0\}, & 0 < r \leq 1 \\ X, & r > 1 \end{cases}$$

So $\forall E \subseteq X, \forall x \in E, B^p(x, \frac{1}{2}) = \{x\} \subseteq E \Rightarrow E$ is open. Then,
$\mathcal{T} = \mathcal{P}(X) = \{\text{All possible subsets of } X\}$ is a topology on X, called the discrete topology, and it contains all subsets of X.

Example 1.5.5 (Relative Topology). Let $(X, \mathcal{T})$ be a topological space. Let $Y \subseteq X$. Then,

$$\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\} .$$

Then, $\mathcal{T}_Y$ is a topology on Y, called the relative topology on Y induced by $\mathcal{T}$. If $X = \mathbb{R}, Y = \mathbb{N}$, then $\mathcal{T}_{\mathbb{N}} = \{U \cap \mathbb{N} : U \subseteq \mathbb{R} \text{ open}\}$ So $\forall y \in \mathbb{N}, \forall x \in \mathbb{R}, r > 0$,

$$B(x,r) \cap \mathbb{N} = \begin{cases} \{y\}, & y \in B(x,r) \\ \emptyset, & y \notin B(x,r) \end{cases}$$

Thus, $\mathcal{T}_{\mathbb{N}} = \mathcal{P}(\mathbb{N})$, the discrete topology on $\mathbb{N}$.
If $X = \mathbb{R}, Y = [0,1)$, then $\mathcal{T}_{[0,1)} = \{U \cap [0,1) : U \subseteq \mathbb{R} \text{ open}\}$ So the set $[0, \frac{1}{2}) = [0,1) \cap (-1, \frac{1}{2}) \in \mathcal{T}_{[0,1)}$. So $[0, \frac{1}{2})$ is relatively open in $Y = [0,1)$ (belongs to the relative topology on Y).

In metric spaces, everything is done using balls. In a generic topological space $(X, \mathcal{T})$, what plays the role of balls?

Definition 1.5.6 (base/neighbourhood base). Let $(X, \mathcal{T})$ Topological space. Fix $x \in X$. Let $\mathcal{B}_x$ be a collection of neighborhoods of x. We call $\mathcal{B}_x$ a *neighbourhood base* at x if $\forall$ neighborhood of x (call it U_x), $\exists B \in \mathcal{B}_x$ such that $B \subseteq U_x$. We say $\mathcal{B}$, a collection of open sets, is a *base* for $\mathcal{T}$ if $\forall x \in X$, $\exists$ a neighbourhood base $\mathcal{B}_x \subseteq \mathcal{B}$ at x.

Example 1.5.7. In (X, p) a metric space, $\forall x \in X$

$$\mathcal{B}_x = \{B^p(x,r) : r > 0\} \text{ is a neighbourhood base}$$

$B = \{\text{all balls of all radii}\}$

Remark 1.5.7. Given a topology, a neighbourood base is not unique.

$$\mathcal{B}_x = \{B^p(x, \frac{1}{n})\}_{n=1}^{\infty}$$

is also a neighbourhood base at x in a metric space.

Definition 1.5.8 (first countable/second countable). Let $(X, \mathcal{T})$ be a topological space.

- We say $(X, \mathcal{T})$ is *first countable* if there is a countable neigbourhood base at each $x \in X$;
- We say $(X, \mathcal{T})$ is *second countable* if there is a countable base B of $\mathcal{T}$.

Remark 1.5.8. Any metric space is first countable, and any seperable metric space is second countable.

Remark 1.5.8. For a topology $\mathcal{T}$, $\mathcal{B} = \mathcal{T}$ is always a base for $\mathcal{T}$ (so a base always exists).

Proposition 1.5.9. If $(X, \mathcal{T})$ be a topological space. A collection of open sets $\mathcal{B}$ is a base for $\mathcal{T}$ iff every non-empty open set $U \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

Proof.

(⇒) Suppose $\mathcal{B}$ is a base for $\mathcal{T}$. Let $U \in \mathcal{T}$. Then $\forall x \in U, \exists B_x \in \mathcal{B}_x \subseteq B$ such that

$$x \in B_x \subseteq U \Rightarrow \bigcup_{x \in U} B_x \subseteq U$$

and

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \Rightarrow U = \bigcup_{x \in U} B_x.$$

(⇐) Suppose every non-empty open set $U \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$. Fix $x \in U$, and let

$$\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\} = \{B \in \mathcal{B} : \{x\} \cap B \neq \varnothing\} \subseteq \mathcal{B}.$$

Since $U = \cup B$, this means $\mathcal{B}_x \neq \emptyset$. So U is a neighbourhood of x, and $\exists B \in \mathcal{B}_x$ such that $B \subseteq U \Rightarrow \mathcal{B}_x$ is a neighbourhood base at x. Doing that $\forall x \in X$, we get a $\mathcal{B}$ that is a base for $\mathcal{T}$. □

Question. Given a collection $\mathcal{B}$, what does it take to be a base for some topology?

Proposition 1.5.10. Let $X \neq \emptyset$. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a collection of sets. Then $\mathcal{B}$ is a base for some topology $\mathcal{T}$ iff

1. $X = \bigcup_{B \in \mathcal{B}} B$;
2. $\forall B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then $\exists B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.[10]

[10]For balls, this is true

Proof.

(⇒) $\mathcal{B}$ is a base for a topology $\mathcal{T}$. Then, since $X \in \mathcal{T}$, by the last result, $X = \bigcup_{B \in \mathcal{B}} B$, so (1) holds. Moreover, if $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{T}$, then $B_1 \cap B_2 \in \mathcal{T}$. So for $x \in B_1 \cap B_2$, then $B_1 \cap B_2$ is a neighbourhood of x. Since $\mathcal{B}$ is a base, $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq B_1 \cap B_2$, so (2) holds.

(⇐) Suppose (1) and (2) hold. Let

$$\mathcal{T} := \{U \subseteq X : \forall x \in U,\ \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}.$$

Since $X = \bigcup_{B \in \mathcal{B}} B$, so $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B \subseteq X, \Rightarrow x \in \mathcal{T}$. Similarly, $\emptyset \in \mathcal{T}$ because the condition is empty. The definition of $\mathcal{T}$ shows us that it is closed under arbitrary unions.

Let $U_1, U_2 \in \mathcal{T}$, and assume $U_1 \cap U_2 \neq \emptyset$, so $\forall x \in U_1 \cap U_2$, by definition of $\mathcal{T}$, $\exists B_1 \in \mathcal{B}$ s.t. $x \in B_1 \subseteq U_1$, and $\exists B_2 \in \mathcal{B}$ s.t. $x \in B_2 \subseteq U_2$. $\Rightarrow x \in B_1 \cap B_2$. By (2), $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Thus, $B_1 \cap B_2 \in \mathcal{T}$. Inductively, we conclude $\mathcal{T}$ is closed under finite intersections. □

Observe that the properties which define $\mathcal{T}$ are closed under intersections, so we may define a σ-algebra like structure for topologies:

Definition 1.5.11. Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of sets. Then

$$\mathcal{T}(\mathcal{E}) = \bigcap \{\text{All topologies containing } \mathcal{E}\} = \text{topology generated by } \mathcal{E}$$

Definition 1.5.12 (weaker/coarser vs. stronger/finer). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. If $\mathcal{T}_1 \subsetneq \mathcal{T}_2$, we say $\mathcal{T}_1$ is a *weaker/coarser* topology than $\mathcal{T}_2$ (fewer open sets), and $\mathcal{T}_2$ is a *stronger/finer* topology than $\mathcal{T}_1$ (more open sets).

Example 1.5.13. Trivial topology is the weakest topology on X and discrete topology is the strongest topology on X. So $\mathcal{T}(\mathcal{E})$ is the weakest topology containing $\mathcal{E}$.

Proposition 1.5.14. Let $\mathcal{E} \subseteq \mathcal{P}(X)$. Then

$$\mathcal{T}(\mathcal{E}) = \left\{\emptyset, X, \bigcup \{ \text{ finite intersections of sets in } \mathcal{E}\}\right\}$$

Proof. **Claim:** $\mathcal{B} = \{\emptyset, X, \text{ finite intersections of elements of } \mathcal{E}\}$ forms a base. $\mathcal{B}$ satisfies (1) and (2) of the previous proposition, so $\mathcal{B}$ is a base for some topology

$$\tilde{\mathcal{T}} = \left\{\emptyset, X, \bigcup \{\text{finite intersections of elements of } \mathcal{E}\}\right\}$$

Observe that $\tilde{\mathcal{T}} \subseteq \{\text{any topology which contains } \mathcal{E}\} \Rightarrow \tilde{\mathcal{T}} \subseteq \mathcal{T}(\mathcal{E})$. $\tilde{\mathcal{T}}$ is also a topology, which contains $\mathcal{E}$. So $\mathcal{T}(\mathcal{E}) \subseteq \tilde{\mathcal{T}}$. Thus, $\tilde{\mathcal{T}} = \mathcal{T}(\mathcal{E})$. $\square$

Goal. Topologies give us open sets, bases give ball-like sets, now we need a notion for closed sets.

Definition 1.5.15 (limit point, closure, closed set). If $E \subseteq X, x \in X$ is a *limit point* if $\forall$ neighbourhood U_x of x,

$$U_x \cap E \neq \emptyset.$$

We say $\overline{E} = \{\text{All limit points of E}\}$, is the *closure* of E.
We say E is *closed* if $E = \overline{E}$.

Remark 1.5.15. We always have $E \subseteq \overline{E}$, so we just need $\overline{E} \subseteq E$ to show E is closed.

Proposition 1.5.16. Let $E \subseteq X$.

1. $\overline{E}$ is closed;
2. $\overline{E}$ is the smallest closed set containing E, i.e. if $\forall F$ closed s.t. $E \subseteq F \Rightarrow \overline{E} \subseteq F$;
3. E is open iff E^c is closed.

Proof of (1) + (2).
Claim: $L := \left\{\text{limit points of } \overline{E}\right\} = \overline{\overline{E}} \subseteq \overline{E}$.

Let $x \in L$, and a neighbourhood U_x of x. Then by defn of L, we know $\exists x' \in U_x \cap \overline{E}$. This means $x' \in \overline{E}$, and U_x is a neighbourhood of $x' \Rightarrow U_x \cap E \neq \emptyset$. This holds $\forall$ neighbourhood U_x of x, so $x \in \overline{E} \Rightarrow L \subseteq \overline{E} \Rightarrow \overline{E}$ is closed.

Suppose $E \subseteq F$ and F is closed. Let $x \in \overline{E}$, then $\forall$ neighbourhood U_x, $U_x \cap E \neq \emptyset \Rightarrow U_x \cap F \neq \emptyset \Rightarrow x \in \overline{F} \Rightarrow \overline{E} \subseteq \overline{F} = F$. $\square$

Proof of (3).

($\Rightarrow$) Let $E \subseteq X$ be open. Let $x \in \overline{E^c}$.

Claim: $x \in E^c$.

Suppose not, so $x \in E$. So $\exists$ neighbourhood U_x of x s.t. $U_x \subseteq E$. $\Rightarrow$ $U_x \cap E = \emptyset$. So $x \notin \overline{E^c}$. So, $x \in \overline{E^c} \Rightarrow x \in E^c \Rightarrow \overline{E^c} \subseteq E^c \Rightarrow E^c$ closed.

($\Leftarrow$) Let E^c be closed. Let $x \in E$.

Claim: $\exists$ neighbourhood U_x s.t. $U_x \subseteq E$.

Suppose not, then every neighbourhood U_x we have $U_x \cap E^c \neq \emptyset \Rightarrow$ $x \in \overline{E^c} = E^c$ (contradicts $x \in E$). By claim, E is open. □

Remark 1.5.16. Our proof shows, $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.

Definition 1.5.17 (Density, Separability). $D \subseteq X$ is *dense* if $\forall$ non-empty open set U, $U \cap D \neq \emptyset$. $\iff$ $\overline{D} = X$.
$(X, \mathcal{T})$ is *separable* iff X contains a countable dense set.

Proposition 1.5.18. Every second countable space is separable.

Proof. Let $(X, \mathcal{T})$ be second countable, so $\exists$ base $\mathcal{B} = \{B_i\}_{i=1}^{\infty}$. Pick $x_i \in B_i$ (need axiom of choice). Let $D = \{x_i\}_{i=1}^{\infty}$. Then $\forall U \subseteq X$ open, since $\mathcal{B}$ is a base, $\exists B_i \subseteq U$,

$$D \cap U \supseteq \{x_i\} \cap B_i \neq \emptyset$$

$\Rightarrow D \cap U \neq \emptyset$, so D is dense. □

Definition 1.5.19 (Convergence in Topology). Given $(X, \mathcal{T})$ a topological space, let $\{x_n\}_{n=1}^{\infty} \subseteq X$. We say $x_n \to x$ in $\mathcal{T}$ if $\forall$ neighbourhood U_x of x, $\exists N$

s.t. $\forall n \geq N$, $x_n \in U_x$.

Proposition 1.5.20. Suppose $(X, \mathcal{T})$ is first countable, and $E \subseteq X$. Then, $x \in \overline{E}$ iff $\exists \{x_n\} \subseteq E$ s.t. $x_n \to x$ in $\mathcal{T}$.

Proof.

(⇒) Let $\mathcal{B}_x = \{B_j\}_{j=1}^{\infty}$ be a neighbourhood base at $x \in \overline{E}$. WLOG, can assume $B_{j+1} \subseteq B_j \quad \forall j$. Since $x \in \overline{E}$ and B_j is a neighbourhood of x, $B_j \cap E \neq \emptyset \quad \forall j$. Let $x_i \in B_i \cap E$. Then, $\forall$ neighbourhood U_x of x, $\exists B_J \in \mathcal{B}_x$ s.t. $B_J \subseteq U_x$. But since $\{B_j\}$ are nested, $\forall j \geq J$,

$$U_x \supseteq B_j \cap U_x = B_j \supseteq B_j \cap E \supseteq \{x_j\}$$

$\Rightarrow x_j \to x$ in $\mathcal{T}$.

(⇐) If $\exists \{x_j\}_{j=1}^{\infty} \subseteq E$ s.t. $x_j \to x$ in $\mathcal{T}$, suppose $x \notin \overline{E}$. Then $x \in \overline{E}^c$ and $\overline{E}^c$ open, so $\overline{E}^c$ is a neighbourhood of x s.t. $\{x_j\}_{j=1}^{\infty} \cap \overline{E}^c = \emptyset$. Therefore $x_j \not\to x$ in $\mathcal{T}$. □

1.6 Separation Properties

While $(X, \mathcal{T})$ allows us to consider a very general framework, weird stuff can happen because of it, for example:

Example 1.6.1. Let $\mathcal{T} = \{\emptyset, X\}$. So the only non-empty neighbourhood is X, so any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges to any point $x \in X$.

To avoid cases like this, we require topologies with more structure.

Definition 1.6.2 (neighbourhood of a set, Separating sets by disjoint neighbourhoods). Let $(X, \mathcal{T})$ be a topological space, and $K, A, B \subseteq X$. A neighbourhood of K is an open set U s.t. $K \subseteq U$. We say A, B can be separated by disjoint neighbourhoods if $\exists U \supseteq A, V \supseteq B$ neighbourhoods s.t. $U \cap V = \emptyset$.

Definition 1.6.3 (Separation Notions). Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is

1. *Tychonoff* (T1) if $\forall x \neq y \in X$, $\exists$ neighbourhood U_x s.t. $y \notin U_x$, and $\exists$ neighbourhood U_y s.t. $x \notin U_y$;
2. *Hausdorff* (T2) if $\forall x \neq y \in X$, $\{x\}, \{y\}$, can be separated by disjoint neighbourhoods, i.e. $\exists U_x \supseteq \{x\}, U_y \supseteq \{y\}$ s.t. $U_x \cap U_y = \emptyset$;
3. *Regular* (T3) if $(X, \mathcal{T})$ is Tychonoff and $\forall x \in X$, $\forall F \subseteq X$ closed, with $x \notin F$, $\{x\}$ and F can be separated by disjoint neighbourhoods;
4. *Normal* (T4) if $(X, \mathcal{T})$ is Tychonoff and $\forall A, B \subseteq X$ closed and disjoint, A and B can be separated by disjoint neighbourhoods.

Remark 1.6.3. Metric $\subseteq$ Normal $\subseteq$ Regular $\subseteq$ Hausdorff $\subseteq$ Tychonoff.

Example 1.6.4. Consider $\mathbb{R}$ and $\mathcal{T} = \{\emptyset, (-\infty, c)$ for $c \in \mathbb{R}\}$ Then, $\forall x \in \mathbb{R}$, a neighbourhood of x is of the form $(-\infty, c)$ for some $c > x$. Let $x \neq y \in \mathbb{R}$, WLOG assume $x < y$. Then $x \in U_y \forall$ neighbourhood U_y of y. So $(\mathbb{R}, \mathcal{T})$ is not Tychonoff.

Example 1.6.5. Let $X = \mathbb{R}$ and let $K := \{\frac{1}{n} : n \in \mathbb{Z}\}$. Define the collection $\mathcal{B}$ as:

$$\mathcal{B} = \{(a, b) : a < b\} \cup \{(a, b) \setminus K : a < b\}.$$

We verify the properties of this space:

1. **Basis Check:** Clearly, $\mathbb{R} = \bigcup_{B \in \mathcal{B}} B$. Now, suppose $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Since B_1 and B_2 are intersections of standard intervals with

either $\mathbb{R}$ or $\mathbb{R} \setminus K$, their intersection is also of the form (a, b) or $(a, b) \setminus K$. Thus, there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Therefore, $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$, called the **K-topology**.

2. **Hausdorff (T_2):** Suppose $x, y \in X$ with $x \neq y$. Since the standard topology is Hausdorff, there exist standard disjoint intervals (a, b) and (c, d) separating x and y. These intervals are also in $\mathcal{B}$. Thus, $U_x \cap U_y = \emptyset \implies (X, \mathcal{T})$ is Hausdorff.

3. **Not Regular (T_3):** The set K is closed in X because its complement $K^c = \mathbb{R} \setminus K$ is open (every point in K^c, including 0, has a neighborhood disjoint from K).

   However, observe that $0 \notin K$. We claim 0 and K cannot be separated. Suppose U and V are disjoint open neighborhoods such that $0 \in U$ and $K \subseteq V$.

   - Since $0 \in U$, there exists a basis element $(-\delta, \delta) \setminus K \subseteq U$.
   - Since $K \subseteq V$, for each n, there exists an interval (a_n, b_n) containing $\frac{1}{n}$ such that $(a_n, b_n) \subseteq V$.

   For sufficiently large n, we have $\frac{1}{n} \in (-\delta, \delta)$. The interval (a_n, b_n) around $\frac{1}{n}$ necessarily contains points strictly between terms of K. These points are present in $(-\delta, \delta) \setminus K$.

   Therefore, $U \cap V \neq \emptyset$. Thus $(X, \mathcal{T})$ is Hausdorff but not regular.

Proposition 1.6.6. If $(X, \mathcal{T})$ is Hausdorff, then for $x_n \to x$ in $\mathcal{T}$, x is unique.

Proof. If $x_n \to x$ and $x_n \to y$ in $\mathcal{T}$ and $x \neq y$, then $\exists U_x \supseteq \{x\}, U_y \supseteq \{y\}$ s.t. $U_x \cap U_y \neq \emptyset$. So we cannot have $x_n \in U_x \cap U_y$, $\Rightarrow x = y$ $\square$

Proposition 1.6.7. $(X, \mathcal{T})$ is Tychonoff iff $\forall x \in X$, $\{x\}$ is closed.

Proof.

$$\begin{aligned}\{x\} \text{ is closed} &\iff \{x\}^c \text{ is open}\\ &\iff \forall y \in \{x\}^c, \exists \text{ neighbourhood } U_y \subseteq \{x\}^c\\ &\iff x \notin U_y\end{aligned}$$

□

Remark 1.6.7. $(X, \mathcal{T})$ normal $\Rightarrow (X, \mathcal{T})$ regular.

Proposition 1.6.8 (Nested neighbourhood property). Let $(X, \mathcal{T})$ be Tychonoff. Then X is normal iff $\forall F \subseteq X$ closed, $\forall U$ neighbourhood of F, $\exists O \subseteq X$ open s.t. $F \subseteq O \subseteq \overline{O} \subseteq U$.

Proof.

(⇒) Suppose X is normal. Consider F, U^c are two closed disjoint sets. By normality, $\exists O, V$ open s.t. $F \subseteq O$, $U^c \subseteq V$ and $O \cap V = \emptyset$. $\Rightarrow V^c \subseteq U$ and $\Rightarrow O \subseteq V^c$.

$$\Rightarrow F \subseteq O \subseteq V^c \subseteq U$$

Since $O \subseteq V^c \Rightarrow \overline{O} \subseteq \overline{V^c} = V^c$ because V is open, $\Rightarrow F \subseteq O \subseteq \overline{O} \subseteq V^c \subseteq U$

(⇐) Suppose the nested neighbourhood property holds. Let $A, B \subseteq X$ be closed, $A \cap B = \emptyset$. $\Rightarrow A \subseteq B^c$ and B^c open. By assumption, $\exists O$ open s.t. $A \subseteq O \subseteq \overline{O} \subseteq B^c$, $\Rightarrow B \subseteq \overline{O}^c$. $A \subseteq O, B \subseteq \overline{O}^c$ and $O \cap \overline{O}^c = \emptyset$. □

Corollary 1.6.9. Every metric space (X, p) is normal.

Proof. By last result, just need to prove the nested neighbourhood property. Let $F \subseteq X$ closed, $U \subseteq X$ open s.t. $F \subseteq U \Rightarrow F \cap U^c = \emptyset$, U^c closed. Let

$$dist(F, U^c) = \inf_{x \in F} dist(x, U^c) = \inf_{x \in F} \inf_{y \in U^c} p(x, y)$$

Observe,

$$dist(x, U^c) = \begin{cases} 0 & x \in U^c \\ > 0 & x \notin U^c \end{cases}$$

For $x \notin U^c, \forall \epsilon > 0, \exists x' \in U^c$ s.t. $dist(x, U^c) + \epsilon \geq p(x, x')$.

So, $\forall y$ s.t. $p(x, y) < \epsilon$,

$$dist(y, U^c) - dist(x, U^c) \leq p(y, x') - p(x, x') - \epsilon \leq p(y, x) - \epsilon \leq \epsilon$$

A symmetric argument gets that $x \mapsto dist(x, U^c)$ is continuous.

Since $dist(x, U^c) \geq 0$, $\inf_{x \in F} dist(x, U^c) \geq 0$. if $\inf_{x \in F} dist(x, U^c) = 0$, then by continuity and F closed, $F \cap U^c \neq \emptyset$. $\Rightarrow$ $dist(F, U^c) = \epsilon > 0$. Let $O := \bigcup_{x \in F} B^p(x, \frac{\epsilon}{2})$ and $\overline{O} = \overline{\bigcup_{x \in F} B^p(x, \frac{\epsilon}{2})} = \bigcup_{x \in F} \overline{B^p(x, \frac{\epsilon}{2})} \Rightarrow F \subseteq O \subseteq \overline{O} \subseteq U$. $\square$

1.7 Compact Topological Spaces

Definition 1.7.1 (Compact Topological Space). $(X, \mathcal{T})$ is a topological space.

- $\{E\}_{\lambda \in \Lambda}$ is an *open cover* if $X \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$ and each E_λ is open.
- $(X, \mathcal{T})$ is a *compact topological space* if every open cover has a finite subcover.
- For $K \subseteq X$, K is *compact* if $(K, \mathcal{T}_k)$ is compact where

$$\mathcal{T}_k := \{K \cap U : U \in \mathcal{T}\}$$

Remark 1.7.1. As before, for $K \subseteq X$, by defn of $\mathcal{T}_k$, $O \in \mathcal{T}_k$ iff $O = K \cap U$ for $U \in \mathcal{T}$. Therefore, $K \subseteq X$ compact iff $\forall$ open cover of K (in X) has a finite subcover.

Proposition 1.7.2 (properties identical to metric spaces).

1. If $F \subseteq X$ closed and $(X, \mathcal{T})$ is compact, then F is compact;
2. $(X, \mathcal{T})$ compact $\Rightarrow \forall \{F_k\}_{k=1}^{\infty} \subseteq X$ closed, nested and non-empty, $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$;

Proof. In the lecture notes (exercise). $\square$

In metric spaces, K compact $\Rightarrow$ K is closed and bounded.

Proposition 1.7.3. Let $(X, \mathcal{T})$ be Hausdorff. If $K \subseteq X$ is compact, then K is closed in X.

Proof. **Claim:** K^c is open.
Fix $y \in K^c$. $\forall x \in K$, $\exists U_{xy}, O_{xy}$ open, disjoint s.t. $y \in U_{xy}$ and $x \in O_{xy}$. So $\{O_{xy}\}_{x \in K}$ is an open cover of K, but K compact, so

$$K \subseteq \bigcup_{i=1}^{N} O_{x_i y} \Rightarrow \bigcap_{i=1}^{N} O_{x_i y}^c \subseteq K^c$$

Let $E := \bigcap_{i=1}^{N} U_{x_i y}$ is open. So E is a neighbourhood of y, and $E \cap O_{xy} = \emptyset \quad \forall i = 1, \ldots, N$. $\Rightarrow E \subseteq O_{x_i y}^c \quad \forall i = 1, \ldots, N$, $\Rightarrow E \subseteq \bigcap_{i=1}^{N} O_{x_i y}^c \subseteq K^c$. $\Rightarrow K^c$ is open. $\square$

Definition 1.7.4 (sequential compactness). $(X, \mathcal{T})$ is *sequentially compact* if every sequence in X has a convergent subsequence, whose limit is in X.

Proposition 1.7.5 (equivalence of compactness). Let $(X, \mathcal{T})$ be second countable. Then X compact iff X is sequentially compact.

Proof.

(⇒) Let X be compact. Let $\{x_k\}_{k=1}^{\infty} \subseteq X$. Let $F_n := \overline{\{x_k : k \geq n\}}$. So F_n is closed $\forall n$, and $F_n \supseteq F_{n+1} \supseteq \dots$, so since X is compact $\exists x_o \in \bigcap_{n=1}^{\infty} F_n$. Observe X second countable $\Rightarrow$ X first countable. So let $\mathcal{B}_{x_o} = \{B_j\}_{j=1}^{\infty}$ be a neighbourhood base at x_0. WLOG assume $B_{j+1} \subseteq B_j \quad \forall j$. Since $x_0 \in \bigcap_{n=1}^{\infty} F_n$, and B_j is a neighbourhood of x_0, then $B_j \cap F_n \neq \emptyset \quad \forall n$.
Claim: $\exists x_k, (k \geq n)$ s.t. $x_k \in B_j \cap F_n$ (i.e. $B_j \cap \text{Int}(F_n) \neq \emptyset$)
We know $B_j \cap F_n \neq \emptyset$, so $\exists y \in B_j \cap F_n$. Then B_j is a neighbourhood of y and $y \in F_n$, so by defn of F_n

$$B_j \cap \{x_k : k \geq n\} \neq \emptyset$$

Let this element be $\{x_{n_j}\} \in B_j$. So $\{x_{n_j}\}_{j=1}^{\infty} \subseteq \{x_k\}_{k=1}^{\infty}$ and $x_{n_j} \in B_j$ with $B_j \supseteq B_{j+1}$. Thus $\forall$ neighbourhood U_{x_0} of x_0, $\exists B_N \subseteq U_{x_0}$ and if $j \geq N$, $x_{n_j} \in B_j \subseteq B_N \subseteq U_{x_0}$. $\Rightarrow x_{n_j} \to x_0$ in $\mathcal{T}$.

(⇐) Let X be sequentially compact. X second countable $\Rightarrow$ every open cover has a countable subcover, $(X = \bigcup_{B \in \mathcal{B}} B$)
Claim: Every countable cover of X has a finite subcover.
Let $X \subseteq \bigcup_{j=1}^{\infty} E_j$, E_j open $\forall j$. Assume there is no finite subcover. So $\forall n$, $\exists m(n) > n$ s.t. $E_{m(n)} \setminus \bigcup_{j=1}^{n} E_j \neq \emptyset$. Let $x_n \in E_{m(n)} \setminus \bigcup_{j=1}^{n} E_j$. X sequentially compact means $\exists \{x_{n_k}\}$ s.t. $x_{n_k} \to x_o \in X$. Since $x_0 \in X$, $\exists E_N$ s.t. $x_o \in E_N$. But $x_{n_k} \in E_{m(n_k)} \setminus \bigcup_{j=1}^{n_k} E_j$, so $\forall n_k \geq N$, $x_{n_k} \not\to x_0$ contradiction.

□

Theorem 1.7.6. A compact Hausdorff space is normal

Proof. Let $(X, \mathcal{T})$ be compact Hausdorff.
Claim: $(X, \mathcal{T})$ is regular.
Let $F \subseteq X$ closed, $x \notin F$. Let $y \in F$. Since X is Hausdorff, $\exists U_{xy}, O_{xy}$ open,

disjoint s.t. $y \in U_{xy}$ and $x \in O_{xy}$. So $\{U_{xy}\}_{y\in F}$ is an open cover of F, but F closed $\Rightarrow F$ compact. [11] So $F \subseteq \bigcup_{i=1}^{N} U_{xy_i} =: U$. Let $N := \bigcap_{i=1}^{N} O_{xy_i}$, so N is open, $x \in N$ and $U \cap N = \emptyset$. $\Rightarrow (X, \mathcal{T})$ is regular. We just rerun the same argument to get that $(X, \mathcal{T})$ is normal. □

[11] Because closed subsets of compact subspaces are themselves compact

1.8 Continuity and Urysohn's Lemma

Definition 1.8.1 (continuous map). If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological spaces, then $f : X \to Y$ continuous at $x_0 \in X$ if $\forall$ neighbourhood $O_{f(x_0)} \subseteq Y$, $\exists$ a neighbourhood $U_{x_0} \subseteq X$ s.t. $f(U_{x_0}) \subseteq O_{f(x_0)}$. We say f is *continuous* if f is continuous at every $x \in X$.

Proposition 1.8.2.

1. $f : X \to Y$ continuous iff $\forall$ open set $O \subseteq Y$, $f^{-1}(O)$ is open in X;
2. Composition of continuous functions is continuous;
3. X is compact and f continuous, then $f(X)$ is compact in Y;
4. If $f : X \to \mathbb{R}$, X compact, f continuous, then max/min of $f(x)$ are achieved.

Proof of (1).

($\Rightarrow$) Let $O \subseteq Y$ be open, and let $x \in f^{-1}(O) \Rightarrow f(x) \in O$. Since f is continuous, and O is a neighbourhood of $f(x)$, $\exists U_x \subseteq X$ s.t. $f(U_x) \subseteq O$. So $U_x \subseteq f^{-1}(O)$ and thus $f^{-1}(O)$ is open in X.

($\Leftarrow$) Suppose $O \subseteq Y$ open and $f^{-1}(O)$ is open in X. Let $U := f^{-1}(O)$. Then U is open and $f(U) \subseteq O$. So f is continuous. □

Definition 1.8.3 (weak-topology induced by $\mathcal{F}$). Let

$$\mathcal{F} := \{f_\lambda : X \to X_\lambda\}_{\lambda \in \Lambda}$$

where $(X_\lambda, \mathcal{T}_\lambda)$ is a topological space $\forall \lambda \in \Lambda$. Let

$$S := \left\{ f_\lambda^{-1}(O_\lambda) : f_\lambda \in \mathcal{F}, O_\lambda \in \mathcal{T}_\lambda \right\}$$

Then $\mathcal{T}(S)$ is called the *weak-topology* induced by $\mathcal{F}$.

Remark 1.8.3. $\mathcal{T}(S) = \bigcap \{\text{topologies containing } S\}$ and if $f_\lambda^{-1}(O_\lambda)$ belongs to the topology, then f_λ is continuous $\forall \lambda \in \Lambda$. Thus this topology makes every f_λ continuous.

Corollary 1.8.4. $\mathcal{T}(S)$ is the weakest topology amongst all topologies on X for which $f_\lambda : X \to X_\lambda$ is continuous $\forall \lambda \in \Lambda$.

Example 1.8.5. $\Lambda = \{1, 2\}$. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces. Consider $X := X_1 \times X_2 = \prod_{i=1}^{2} X_i$. Let

$$\mathcal{F} := \begin{Bmatrix} \pi_1 : X \to X_1, & \pi_1(x_1, x_2) = x_1 \\ \pi_2 : X \to X_2, & \pi_2(x_1, x_2) = x_2 \end{Bmatrix}.$$

Let $S = \{\pi_i^{-1}(O_i) : O_i \in \mathcal{T}_i\}$. Then $\mathcal{T}(S)$ is called the product topology. [12] Recall, we have learned that

$$\mathcal{T}(S) = \left\{ \emptyset, X, \bigcup \{\text{finite intersections of elements of } S\} \right\}.$$

So, a base for $\mathcal{T}(S)$ is given by

$$\mathcal{B} := \left\{ \bigcap_{i=1}^{2} \pi_i^{-1}(O_i) : O_i \in \mathcal{T}_i \right\}$$

[12] This is similar to how the product σ-algebra is the smallest σ-algebra that makes π_i measurable. The product topology is the weakest topology that makes π_i continuous.

and we note that $\pi_1^{-1}(O_1) \cap \pi_1^{-1}(\tilde{O}_1) = \pi_1^{-1}(O_1 \cap \tilde{O}_1)$.
Also, since $\pi_1^{-1}(O_1) = O_1 \times X_2$ and $\pi_2^{-1}(O_2) = X_1 \times O_2$, we have

$$\bigcap_{i=1}^{2} \pi_i^{-1}(O_i) = O_1 \times O_2$$

$$\implies \mathcal{B} = \left\{ \prod_{i=1}^{2} O_i : O_i \in \mathcal{T}_i \right\}$$

is a base for the product topology.

Example 1.8.6. Λ infinite.[13] Let $(X_\lambda, \mathcal{T}_\lambda)$ be a topological space. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ and let $\pi_\lambda : X \to X_\lambda$ be the projection map. Consider the product topology on X, and a base is given by

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}(O_{\lambda_i}) : O_{\lambda_i} \in \mathcal{T}_{\lambda_i}, n \in \mathbb{N} \right\}$$

which equals

$$= \left\{ \prod_{\lambda \in \Lambda} O_\lambda : O_\lambda = X_\lambda \text{ for all but finitely many } \lambda \right\}$$

So open in the product topology means a base is given by finite products of open sets.

[13] Could even be uncountable

Motivation. Let (X, p) be a metric space. Let A, B closed and disjoint. Let

$$f(x) := \frac{dist(x, A)}{dist(x, A) + dist(x, B)}$$

Note,

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

$0 \leq f(x) \leq 1$. f is continuous because $dist(., A)$ and $dist(., B)$ are continuous, and denominator is non-zero. Urysohn's lemma does this on any normal topological space.

Lemma 1.8.7 (Urysohn's Lemma). Let $(X, \mathcal{T})$ be normal. Let $A, B \subseteq X$ closed and disjoint. Then $\exists f : X \to \mathbb{R}$ s.t.

- f is continuous;
- $0 \leq f(x) \leq 1$;
- $f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$

Remark 1.8.7. Infact, we can replace $\{0, 1\} \to \{\alpha, \beta\}$ $\forall \alpha < \beta$

Definition 1.8.8 (normally ascending). Let $(X, \mathcal{T})$ and $\Lambda \subseteq \mathbb{R}$. We say $\{O_\lambda\}_{\lambda \in \Lambda}$ with O_λ open is *normally ascending* if $\forall \lambda_1, \lambda_2 \in \Lambda$, $\overline{O_{\lambda_1}} \subseteq O_{\lambda_2}$ whenever $\lambda_1 < \lambda_2$.

Lemma 1.8.9. Let $(X, \mathcal{T})$ be normal. Let $F \subseteq X$ be closed, U a neighbourhood of F. There exists a dense set $\Lambda \subseteq (0, 1)$ and a normally ascending colleciton of open sets $\{O_\lambda\}_{\lambda \in \Lambda}$ such that

$$F \subseteq O_\lambda \subseteq \overline{O_\lambda} \subseteq U \quad \forall \lambda \in \Lambda$$

Proof. Consider $\Lambda := \left\{\frac{m}{2^n} : m, n \in \mathbb{N}, 1 \leq m \leq 2^n - 1\right\}$. Clearly, Λ is dense in $(0, 1)$. Let

$$\Lambda_n := \left\{\frac{m}{2^n} : m \in \mathbb{N}, 1 \leq m \leq 2^n - 1\right\} \Rightarrow \Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$$

We will define $\{O_\lambda\}_{\lambda \in \Lambda}$ inductively. Since X is normal, by nested neighbourhood property, let $O_{\frac{1}{2}}$ be s.t.

$$F \subseteq O_{\frac{1}{2}} \subseteq \overline{O_{\frac{1}{2}}} \subseteq U$$

We now define $O_{\frac{1}{4}}, O_{\frac{3}{4}}$ by

$$F \subseteq O_{\frac{1}{4}} \subseteq \overline{O_{\frac{1}{4}}} \subseteq O_{\frac{1}{2}} \subseteq \overline{O_{\frac{1}{2}}} \subseteq O_{\frac{3}{4}} \subseteq \overline{O_{\frac{3}{4}}} \subseteq U.$$

We proceed inductively to build $\{O_\lambda\}_{\lambda \in \Lambda}$, which are necessarily normally ascending. □

Lemma 1.8.10. Let $(X, \mathcal{T})$ be a topological space s.t $\exists \Lambda \subseteq (0, 1)$ and a normally ascending collection of open sets $\{O_\lambda\}_{\lambda \in \Lambda}$. Let

$$f(x) := \begin{cases} 1 & \text{if } x \in \left(\bigcup_{\lambda \in \Lambda} O_\lambda\right)^c \\ \inf\{\lambda \in \Lambda : x \in O_\lambda\} & \text{if } x \in \bigcup_{\lambda \in \Lambda} O_\lambda \end{cases}$$

Then $0 \leq f(x) \leq 1$ and f is continuous.

Proof. Notice $0 \leq f \leq 1$ because $\Lambda \subseteq (0, 1)$. Observe

$$\mathcal{D} := \{(-\infty, c), (d, \infty) : c, d \in \mathbb{R}\}$$

$$\mathcal{B} := \{\text{finite intersections of elements of } \mathcal{D}\}$$

is a base for $(\mathbb{R}, \mathcal{T}_{|.|})$. So,

$$\mathcal{T}_{|.|} = \left\{\emptyset, X, \bigcup B : B \in \mathcal{B}\right\}$$

So if $f^{-1}(-\infty, c)$ and $f^{-1}(d, \infty)$ are open, then $\forall O \subseteq \mathbb{R}$ open, $f^{-1}(O)$ is open.[14]

[14]because it is just finite intersections and arbitrary unions

Claim: $f^{-1}(-\infty, c)$ and $f^{-1}(d, \infty)$ are open $\forall c, d \in \mathbb{R}$.
$f(x) < c$ iff $x \in O_\lambda$ for $\lambda < c$ iff $x \in \bigcup_{\lambda < c} O_\lambda$. $f^{-1}((-\infty, c)) = \bigcup_{\lambda < c} O_\lambda$ is open.

Similarly, $f(x) > d$ iff $x \notin O_\lambda$ for some $\lambda > d$ iff $x \notin \overline{O}_{\lambda-\epsilon}$ for some $\lambda - \epsilon > d$ iff $x \in \bigcup_{\lambda > d}(\overline{O}_\lambda)^c$. So, $f^{-1}(d, \infty) = \bigcup_{\lambda > d}(\overline{O}_\lambda)^c$ is open. So, by prior argument, f is continuous. □

Proof of Urysohn's Lemma. Let $(X, \mathcal{T})$ be normal, $A, B \subseteq X$ closed and disjoint. Consider $A \subseteq B^c$ open. By prior lemma, $\exists \Lambda \subseteq (0, 1)$ dense and $\{O_\lambda\}_{\lambda \in \Lambda}$ normally ascending open sets s.t.

$$A \subseteq O_\lambda \subseteq \overline{O_\lambda} \subseteq B^c.$$

Let f be as in the last lemma, so $0 \leq f \leq 1$ and f is continuous. If $x \in B$, $\bigcup_{\lambda \in \Lambda} O_\lambda \subseteq B^c \Rightarrow B \subseteq (\bigcup_{\lambda \in \Lambda} O_\lambda)^c \Rightarrow f(x) = 1$. Similarly, if $x \in A$, $x \in O_\lambda \ \forall l\lambda \in \Lambda$, so $f(x) = \inf\{\lambda \in \Lambda\} = 0$. □

1.9 Connected Topological Spaces

Definition 1.9.1 (Separating X by open sets). Two (non-empty) open sets (O_1, O_2) *separate* $(X, \mathcal{T})$ if $X = O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$

Definition 1.9.2 (connected). $(X, \mathcal{T})$ is *connected* if X cannot be separated by non-empty open sets.

Remark 1.9.2. On $(X, \mathcal{T})$, if (O_1, O_2) open and separate X, then O_1 and O_2 are also closed. [15] So $(X, \mathcal{T})$ is connected iff the only "clopen" sets of X are $X, \varnothing$.

As a consequence of this, we also have that if (O_1, O_2) open and separate X,

$$O_1 \cap O_2 = \varnothing \Rightarrow O_1 \cap \overline{O_2} = \overline{O_1} \cap O_2 = \varnothing$$

[15] b/c $X = O_1 \cup O_2$

Remark 1.9.2. Recall

$$\mathcal{T}_E := \{E \cap U : U \in \mathcal{T}\}$$

So $E \subseteq X$ is connected if $\nexists$ open, non-empty sets (O_1, O_2) s.t. $O_1 \cap E \neq \varnothing$, $O_2 \cap E \neq \varnothing$, $E \subseteq O_1 \cup O_2$, $E \cap O_1 \cap O_2 = \varnothing$, $E = (E \cap O_1) \cup (E \cap O_2)$ and $(E \cap O_1) \cap (E \cap O_2) = \varnothing$

Connectedness works best with "contradiction". It also works well with continuous maps.

Proposition 1.9.3. Let $f : X \to Y$ where $(X, \mathcal{T})$ is connected and f is continuous w.r.t. $(X, \mathcal{T})$ and $(Y, \mathcal{S})$. Then $f(X)$ is connected in $(Y, \mathcal{S})$.

Proof. Suppose $f(X)$ is not connected. Then $\exists (O_1, O_2)$ non-empty open sets in Y s.t. $f(X) \cap O_1 \neq \varnothing$, $f(X) \cap O_2 \neq \varnothing$, $f(X) \subseteq O_1 \cup O_2$, $f(X) \cap O_1 \cap O_2 = \varnothing$. Since f is continuous, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are open in $(X, \mathcal{T})$, and they are non-empty.

So we have $X \cap f^{-1}(O_1) \neq \varnothing$, $X \cap f^{-1}(O_2) \neq \varnothing$, $X \subseteq f^{-1}(O_1) \cup f^{-1}(O_2)$ and $X \cap f^{-1}(O_1) \cap f^{-1}(O_2) = \varnothing$. $\Rightarrow (X, \mathcal{T})$ is not connected. $\square$

Proposition 1.9.4. On $\mathbb{R}$, for $E \subseteq \mathbb{R}$, TFAE:

1. E is connected
2. E is an interval
3. E is convex

Definition 1.9.5 (Intermediate Value Property (IVP)). $(X, \mathcal{T})$ has the *intermediate value property* provided $\forall f \in C(X)$, then $f(X) \subseteq \mathbb{R}$

Proposition 1.9.6. X has the IVP iff X is connected.

Proof.

(⇐) By last result, X is connected, $f \in C(X) \Rightarrow f(X)$ is connected $\Rightarrow f(X)$ an interval.

(⇒) Suppose X not connected. Then $\exists (O_1, O_2)$ open, non-empty and disjoint s.t. $X = O_1 \cup O_2$. Let $f : X \to \mathbb{R}$ be s.t.

$$f(x) = \mathcal{X}_{O_2}(x) = \begin{cases} 1 & x \in O_2 \\ 0 & x \in O_1 \end{cases}$$

$\forall A \subseteq \mathbb{R}$,

$$f^{-1}(A) = \begin{cases} \emptyset & \{0,1\} \notin A \\ O_1 & 0 \in A, 1 \notin A \\ O_2 & 1 \in A, 0 \notin A \\ X & \{0,1\} \in A \end{cases}$$

Note these are all open sets, so $f^{-1}(A)$ is open $\forall A \subseteq \mathbb{R}$ open, therefore f is continuous. But $f(X) = \{0,1\}$ is not an interval, so X does not have the IVP. □

Definition 1.9.7 (path-connectedness/arcwise connectedness). X is *path-connected* if $\forall x, y \in X, \exists f : [0,1] \to X$ continuous s.t. $f(0) = x, f(1) = y$

Proposition 1.9.8. X path-connected $\Rightarrow X$ connected.

Proof. Suppose X is not connected. Then $\exists(O_1, O_2)$ non-empty, open, disjoint s.t. $X = O_1 \cup O_2$. Suppose $\exists f : [0,1] \to X$ s.t. f is continuous, and $f(0) = x, f(1) = y$. So $f^{-1}(O_1)$ and $f^{-1}(O_2)$ open, non-empty, disjoint and $[f^{-1}(O_1) \cup f^{-1}(O_2) = f^{-1}(X) = [0,1]$. Thus $[0,1]$ is not connected, contradiction. $\square$

1.10 Stone-Weierstrass Theorem

Goal. We want to find sufficient conditions for a collection of sets $\mathcal{A} \subseteq C(X)$ to be dense in $(C(X), \|.\|_\infty)$

Stone-Weierstrass is a generalization of the following result, which will be proved in part 3 of the course.

Theorem 1.10.1 (Weierstrass Approximation Theorem). Let $[a,b] \subseteq \mathbb{R}$ and let $f \in C([a,b])$. Then $\forall \epsilon > 0$, $\exists$ a polynomial $p(x)$ s.t. $\|p - f\|_\infty = \sup_{x\in[a,b]} |p(x) - f(x)| < \epsilon$.

In the above, $\mathcal{A} = \{\text{polynomials}\}$, so $\mathcal{A}$ is dense in $C([a,b])$. Stone Weierstrass is for the case when X is compact and Hausdorff $\Rightarrow X$ is normal.

Definition 1.10.2 (algebra of functions, separating points). A collection $\mathcal{A} \subseteq C(X)$ is an *algebra* if

- $\mathcal{A}$ is closed under linear combinations;
- $\mathcal{A}$ is closed under products ($f, g \in \mathcal{A} \Rightarrow fg \in \mathcal{A}$).

A collection $\mathcal{A} \subseteq C(X)$ *separates points* if $\forall x, y \in X, x \neq y, \exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$. [16]

[16] Separating points really only makes sense in X is Hausdorff

Example 1.10.3. If $X = [a, b]$, then $\mathcal{A} = \{\text{polys}\}$ is an algebra that separates points.

- $\mathcal{A}$ is a linear vector space;
- Product of polynomials is a polynomial;
- Let $f(x) = x \in \mathcal{A}$, then $x \neq y \Rightarrow f(x) \neq f(y)$.

Remark 1.10.3. If X is compact and Hausdorff, how do we generally separate points? Use Urysohn's lemma.

Let $\{x\}, \{y\}$ closed, $x \neq y \Rightarrow$ disjoint. By Urysohn's lemma, $\exists f \in C(X)$ s.t. $f(x) = 0$, $f(y) = 1$.

So, $C(X)$ is an algebra that separates points. (in any compact Hausdorff space).

Theorem 1.10.4 (Stone-Weierstrass). Let X be compact and Hausdorff.[17] Suppose $\mathcal{A} \subseteq C(X)$ is an algebra that separates points and contains the constant functions. Then $\mathcal{A}$ is dense in $(C(X), \|.\|_\infty)$.

[17] Hausdorff only used in the only if part

Remark 1.10.4. If $\mathcal{A} = C(X)$ the result holds true.

Remark 1.10.4. In fact, the theorem is an iff.

Proof Idea. Fix $f \in C(X)$. Then $f(X)$ is compact in $\mathbb{R}$, so $f(X)$ is bounded. WLOG assume $0 \leq f \leq 1$. We decompose

$$X = \bigcup_{k=1}^{n} \left\{ x : \frac{k-1}{n} \leq f(x) \leq \frac{k}{n} \right\}$$

Suppose $\forall 1 \leq k \leq n$, $\exists g_k \in \mathcal{A}$ s.t.

$$g_k(x) = \begin{cases} 1 & \text{if } f(x) \geq \frac{k}{n} \\ 0 & \text{if } f(x) \leq \frac{k-1}{n} \end{cases} = \mathcal{X}_{\left\{f \geq \frac{k}{n}\right\}}(x)$$

We will consider

$$g(x) = \frac{1}{n}\sum_{k=1}^{n} g_k(x)$$

So if $x \in X$, $\frac{\bar{k}-1}{n} \leq f(x) \leq \frac{\bar{k}}{n}$, then if $\bar{k} - 1 \geq k \Rightarrow g_k(x) = 1$, $\bar{k} \leq k - 1 \Rightarrow g_k(x) = 0$.

$$\Rightarrow g_k(x) = \begin{cases} 1 & k \leq \bar{k} - 1 \\ 0 & k \geq \bar{k} + 1 \end{cases}$$

$$\Rightarrow g(x) = \frac{1}{n}\sum_{k=1}^{n} g_k(x) \approx \frac{\bar{k}-1}{n} \approx f(x)$$

This is how we'll build $g \in \mathcal{A}$ s.t. $g \approx f$. The hard part is showing $g_k \in \mathcal{A}$. $\square$

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