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# MATH 455

## Lecture Notes

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*Based on lectures by Prof. Jessica Lin*

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# 1 Abstract Metric and Topological Spaces

## 1.1 Metric Spaces Review

Throughout, assume  $X$  is a non empty set.

**Definition 1.1.1:** (Metric):  $p : X \times X \rightarrow \mathbb{R}$  is called a *metric*, and thus  $(X, p)$  a metric space, if for all  $x, y, z \in X$

- $p(x, y) \geq 0,$
- $p(x, y) = 0 \iff x = y,$
- $p(x, y) = p(y, x),$
- $p(x, y) \leq p(x, z) + p(z, y)$  (Triangle Inequality).

**Definition 1.1.2:** (Norm): Let  $X$  be a vector space.<sup>1</sup> A function  $\|\cdot\| : X \rightarrow [0, \infty)$  is called a *norm*, and thus  $(X, \|\cdot\|)$  a *normed vector space*, if for all  $u, v \in X$  and  $\alpha \in \mathbb{R}$

- $\|u\| = 0 \iff u = 0,$
- $\|u + v\| \leq \|u\| + \|v\|,$
- $\|\alpha u\| = |\alpha| \|u\|.$

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<sup>1</sup>closed under linear combinations

**Remark 1.1.3:** A norm induces a metric by  $p(x, y) := \|x - y\|.$

### Example 1.1.4

Examples of normed vector spaces:

1.  $(\mathbb{R}^n, |.|)$  where  $|x| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$
2.  $L^p(E)$  for  $E \subseteq \mathbb{R}^n, 1 \leq p \leq \infty$  where  $\|f\|_{L^p(E)} = (\int_E |f(x)|^p dx)^{\frac{1}{p}}$
3. Discrete metric: if  $X$  is a non empty set, then  $p(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
4.  $C([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$  for  $a, b \in \mathbb{R}$ . Then,  $\|f\|_\infty := \sup_{x \in [a, b]} |f(x)| = \max_{x \in [a, b]} |f(x)|$ ,  $p(f, g) = \|f - g\|_\infty$

**Definition 1.1.5:** Given two metrics  $p, \sigma$  on  $X$ , we say they are *equivalent* if  $\exists$  a  $C > 0$  such that  $\frac{1}{C}\sigma(x, y) \leq p(x, y) \leq C\sigma(x, y)$  for every  $x, y \in X$ . A similar definition follows for equivalence of norms.

Given a metric space  $(X, P)$ , then, we have the notion of

- open balls  $B(x, r) = \{y \in X : p(x, y) \leq r\}$
- open sets (subsets of  $X$  with the property that for every  $x \in X$ , there is a constant  $r > 0$  such that  $B(x, r) \subseteq X$ ), closed sets, closures, and
- *convergence*

**Definition 1.1.6 (Convergence):**  $\{x_n\}_{n=1}^{\infty} \subseteq X$  converges to  $x$  in  $(X, p)$  if  $\lim_{n \rightarrow \infty} p(x_n, x) = 0$

We have several (equivalent) notions, then, of continuity; via sequences,  $\epsilon - \delta$  definition, and by pullbacks (inverse images of open sets are open).

**Definition 1.1.7 (Uniform Continuity):**  $f : (X, p) \rightarrow (\mathbb{R}, |\cdot|)$  uniformly continuous if  $f$  has a "modulus of continuity", i.e. there is a continuous function  $\omega : [0, \infty) \rightarrow [0, \infty)$  such that  $\lim_{t \rightarrow 0^+} \omega(t) = 0$ , and

$$|f(x) - f(y)| \leq \omega(p(x, y))$$

for every  $x, y \in X$

**Remark 1.1.8:** For instance, we say  $f$  Lipschitz continuous if there is a constant  $C > 0$  such that  $\omega(\cdot) = C(\cdot)$ . Let  $\alpha \in (0, 1)$ . We say  $f$   $\alpha$ -Holder continuous if  $\omega(\cdot) = C(\cdot)^{\alpha}$  for some constant  $C$ .

**Definition 1.1.9 (Completeness):** We say  $(X, p)$  *complete* if every Cauchy sequence in  $(X, p)$  converges to a point in  $X$ .

**Remark 1.1.10:** let  $E \subseteq X$  and  $(X, p)$  complete metric space. Then  $(E, p)$  is complete iff  $E \subseteq X$  is closed (so limits belong to  $E$ )

## 1.2 Compactness, Separability

**Definition 1.2.1 (Open Cover, Compactness):**  $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$ <sup>1</sup>, where  $X_\lambda$  open in  $X$  and  $\Lambda$  an arbitrary index set, an *open cover* of  $X$  if for every  $x \in X$ ,  $\exists \lambda \in \Lambda$  such that  $x \in X_\lambda$ .<sup>2</sup>  $X$  is *compact* if every open cover of  $X$  admits a finite subcover. We say  $E \subseteq X$  compact if  $(E, p)$

compact.

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<sup>1</sup> $2^X$  denotes the power set of  $X$ , i.e. the set of all subsets of  $X$ .

<sup>2</sup>A cover is finite if  $|\Lambda| < \infty$

**Remark 1.2.2:** for  $E \subseteq X$ ,  $X_\lambda \subseteq E$  is open in  $(E, p)$  iff  $X_\lambda$  is open in  $(X, p)$ . Therefore,  $E \subseteq X$  is compact iff every open cover of  $E$  (in  $X$ ) has a finite subcover.

**Remark 1.2.3:** This definition leads to another definition of compactness based on the finite intersection property.

One useful consequence of this result is if  $(X, p)$  is compact metric space, and  $\{E_k\}_{k=1}^\infty \subseteq X$  closed, and  $E_{k+1} \subseteq E_k \forall k$ ,  $\cap_{k=1}^\infty E_k \neq \emptyset$ .

**Definition 1.2.4 (Totally Bounded,  $\epsilon$ -nets):**  $(X, p)$  is *totally bounded* if  $\forall \epsilon > 0$ , there is a finite cover of  $X$  of balls with radius  $\epsilon > 0$ .<sup>1</sup> If  $E \subseteq X$ , an  $\epsilon$ -net of  $E$  is a collection  $\{B(x_i, \epsilon)\}_{i=1}^N$  such that  $E \subseteq \bigcup_{i=1}^N B(x_i, \epsilon)$  and  $x_i \in X$  (note that  $x_i$  need not be in  $E$ ).

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<sup>1</sup>Totally bounded implies  $(X, p)$  is bounded

**Definition 1.2.5 (Sequentially Compact):**  $(X, p)$  *sequentially compact* if every sequence in  $X$  has a convergent subsequence whose limit is in  $X$ .

**Definition 1.2.6 (Relatively/Pre-Compact):**  $E \subseteq X$  *precompact* if  $\bar{E}$  compact.

### Theorem 1.2.7

TFAE:

1.  $X$  complete and totally bounded;
2.  $X$  compact;
3.  $X$  sequentially compact.

**Remark 1.2.8:** TFAE:

1.  $E$  is totally bdd and Cauchy Seq. converge
2.  $E$  is precompact
3.  $\forall \{x_k\}_{k=1}^\infty \subseteq E, \exists$  a convergent subsequence

Let  $f : (X, p) \rightarrow (\mathbb{R}, |\cdot|)$  continuous with  $(X, p)$  compact. Then,

- $f(X)$  compact in  $(\mathbb{R}, |\cdot|)$ ;
- The max and min of  $f$  over  $X$  are attained;
- $f$  is uniformly continuous.

**Lemma 1.2.9:** Any cauchy sequence<sup>1</sup> converges iff it has a convergent subsequence.

$${}^1 \forall \epsilon > 0, \exists N > 0 \text{ s.t. } \forall m, n > N, \|x_n - x_m\| < \epsilon$$

### Proof.

$\Rightarrow$  If  $\{f_n\}_{n=1}^\infty$  converges, then  $\exists f : X \rightarrow \mathbb{R}$  s.t.  $\|f_n - f\|_\infty \rightarrow 0$ , so all subsequences also converge to  $f$ .

$\Leftarrow$  Now assume  $\exists$  a subsequence  $\{f_{n_k}\}_{k=1}^\infty \subseteq C(X)$  s.t.  $\lim_{k \rightarrow \infty} f_{n_k} = f$  in  $C(X) \iff \|f_{n_k} - f\|_\infty \rightarrow 0$ . Suppose for the purpose of contradiction that  $f_n \not\rightarrow f$ . Thus,  $\exists \epsilon > 0$ , and a subsequence  $\{f_{n_j}\}_{j=1}^\infty \subseteq C(X)$  s.t.  $\|f_{n_j} - f\|_\infty > \epsilon$  for every  $j \geq 1$ . Then,

$$\|f_{n_k} - f_{n_j}\|_\infty \geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

for  $k$  sufficiently large and for  $n_k, n_j$  large enough. But this violates  $\{f_n\}_{n=1}^\infty$  being cauchy. (Contradiction), so we must have  $f_n \rightarrow f$  in  $C(X)$ .  $\square$

Let  $C(X) := \{f : X \in \mathbb{R} \mid f \text{ continuous}\}$  and  $\|f\|_\infty := \max_{x \in X} |f(x)|$  the sup norm. Then,

**Proposition 1.2.10:** Let  $(X, p)$  compact. Then  $(C(X), \|\cdot\|_\infty)$  is complete.

**Proof.** let  $\{f_n\}_{n=1}^\infty \subseteq C(X)$  be Cauchy. Fix  $k \in \mathbb{N}$ . By Cauchy defn, let  $\epsilon = 2^{-k}$ , so  $\exists N_k$  sufficiently large s.t.  $\|f_{N_k} - f_{N_k+1}\|_\infty < 2^{-k}$ . We can then choose  $\{n_k\}_{k=1}^\infty$  s.t.  $n_k \rightarrow \infty$  and  $\|f_{n_k} - f_{n_{k+1}}\| < 2^{-k} \quad \forall k \in \mathbb{N}$ . Let  $j \in \mathbb{N}$ . Then

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell=k}^{k+j-1} 2^{-\ell} \leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow{k \rightarrow \infty} 0$$

In particular,  $\forall x \in X$  fixed, let  $c_k := f_{n_k}(x)$ . Then  $|c_{k+j} - c_k| \leq \|f_{n_{k+j}} - f_{n_k}\|_\infty \rightarrow 0 \quad \forall j \in \mathbb{N}$ . Thus  $\{c_k\}_{k=1}^\infty \subseteq \mathbb{R}$  is cauchy, so by completeness of  $\mathbb{R}$ ,  $\exists \bar{c} \in \mathbb{R}$  s.t.  $\lim_{k \rightarrow \infty} c_k = \bar{c} =: f(x)$ . Doing this  $\forall x \in X$ , we have

$$\begin{aligned} |f_{n_k}(x) - f(x)| &= \lim_{j \rightarrow \infty} |f_{n_k}(x) - f_{n_{k+j}}(x)| \\ &\leq \lim_{j \rightarrow \infty} \|f_{n_k} - f_{n_{k+j}}\|_\infty \\ &\leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$

$\Rightarrow \|f_{n_k} - f\|_\infty = \sup_{x \in X} |f_{n_k}(x) - f(x)| \xrightarrow{k \rightarrow \infty} 0$ , so  $f_{n_k} \rightarrow f$  in  $C(X)$ . Finally, by the lemma this implies  $f_n \rightarrow f$  in  $C(X)$ , so  $(C(X), \|\cdot\|_\infty)$  is complete.  $\square$

**Definition 1.2.11 (Density/Separability):** A set  $D \subseteq X$  is called *dense* in  $(X, p)$  if for every <sup>1</sup> nonempty open subset  $A \subseteq X$ ,  $D \cap A \neq \emptyset$ . We say that  $X$  is *separable* if there is a countable dense subset  $D \subseteq X$ .

<sup>1</sup>If  $A$  dense in  $X$ , then  $\overline{A}$  dense in  $X$

**Proposition 1.2.12:** If  $X$  compact, then  $X$  is separable

**Proof.** Since  $X$  is compact, it is totally bounded. Therefore, for  $n \in \mathbb{N}$ , there is some  $K_n$  and  $\{x_i^n\} \subseteq X$  such that  $X \subseteq \bigcup_{i=1}^{K_n} B(x_i^n, \frac{1}{n})$ . Then,  $D = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{K_n} \{x_i^n\}$  countable and dense in  $X$   $\square$

### 1.3 Arzelà-Ascoli

#### Goal

Given a sequence  $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ , find suitable conditions for  $\{f_n\}$  to have a convergent subsequence in  $(C(X), \|\cdot\|_\infty)$ .

**Definition 1.3.1 (Equicontinuous):** A family  $\mathcal{F} \subseteq C(X)$  is called *equicontinuous* at  $x \in X$  if  $\forall \epsilon > 0$  there exists a  $\delta_x > 0$  such that if  $p(x, x') < \delta_x$  then  $|f(x) - f(x')| < \epsilon$  for every  $f \in \mathcal{F}$ .  $\mathcal{F}$  is pointwise equicontinuous on  $X$  if  $\mathcal{F}$  is equicontinuous at every point  $x \in X$ .<sup>1</sup>

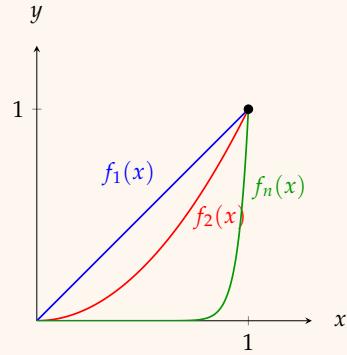
<sup>1</sup>if  $|\mathcal{F}| < \infty$ , then  $\mathcal{F}$  is pointwise equicontinuous on  $X$ .

#### Example 1.3.2

Fix  $M > 0$ ,  $[a, b] \subseteq \mathbb{R}$ .  $\mathcal{F} := \{f \in C([a, b]) \cap C'((a, b)) \mid |f'| \leq M\}$ . By Mean Value Theorem,  $|f(x) - f(y)| \leq |f'(x^*)| |x - y| \leq M |x - y|$  for some  $x^* \in [x, y]$ , so  $\forall x \in [a, b]$  if  $|x - y| < \frac{\epsilon}{M}$  then  $|f(x) - f(y)| < \epsilon$ ,  $\forall f \in \mathcal{F}$ , therefore  $\mathcal{F}$  is pointwise equicontinuous on  $[a, b]$ .

**Example 1.3.3**

Consider  $f_n(x) := x^n$  on  $[0, 1]$ . Then  $\{f_n\}_{n=1}^{\infty}$  is non equicontinuous at  $x = 1$ .  $f_n(1) = 1 \forall n$ , but the threshold to be close to  $f_n(1)$  is not uniform on  $n$ .



**Figure 1:** The sequence  $f_n(x) = x^n$  is not equicontinuous.

**Definition 1.3.4 (Pointwise, Uniform Boundedness):**  $\{f_n\}$  pointwise bounded if  $\forall x \in X, \exists M(x) > 0$  such that  $|f_n(x)| \leq M(x) \forall n$ , and uniformly bounded if such an  $M$  exists independent of  $X$ .

**Definition 1.3.5 (Uniform Equicontinuous):**  $\mathcal{F} \subseteq C(X)$  is uniformly equicontinuous on  $X$  if  $\forall \epsilon > 0 \exists \delta > 0$  s.t.  $\forall x, y \in X$  if  $p(x, y) < \delta$ , then  $|f(x) - f(y)| < \epsilon, \forall f \in \mathcal{F}$ .

**Remark 1.3.6:**  $\mathcal{F}$  equicontinuous at  $x \iff$  all  $f \in \mathcal{F}$  share the same modulus of continuity at  $x$ , i.e.  $\exists \omega_x$  s.t.  $|f(x) - f(y)| \leq \omega_x |x - y|, \forall f \in \mathcal{F}$ .

**Proposition 1.3.7 (Sufficient Conditions for Uniform Equicontinuity):**

1.  $\mathcal{F} \subseteq C(X)$  is uniformly Lipschitz continuous, i.e.  $\exists M > 0$  s.t.  $|f(x) - f(y)| \leq Mp(x, y) \forall f \in \mathcal{F}$ ;
2.  $\mathcal{F} \subseteq C(X) \cap C^1(X)$  has a uniform  $L^\infty$  bound on the 1st derivative (same as earlier example, by MVT);
3. If  $(X, p)$  is compact and  $\mathcal{F} \subseteq C(X)$  is pointwise equicontinuous on  $X \Rightarrow \mathcal{F}$  is uniformly equicontinuous (Homework).

**Lemma 1.3.8 (Arzelà-Ascoli Lemma):** Let  $X$  be separable and let  $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$  be pointwise bounded and equicontinuous. Then, there is a function  $f \subseteq C(X)$  and a subsequence  $\{f_{n_k}\}_{k=1}^{\infty}$  which converges pointwise to  $f$  on all of  $X$ .

**Proof.** Let  $D = \{x_j\}_{j=1}^\infty \subseteq X$  be a countable dense subset of  $X$ . Since  $\{f_n\}$  is pointwise bounded,  $\{f_n(x_1)\}$  as a sequence of real numbers is bounded and so by Bolzano-Weierstrass, there is a convergent subsequence  $\{f_{n(1,k)}(x_1)\}_k$  that converges to some  $a_1 \in \mathbb{R}$ . Consider now  $\{f_{n(1,k)}(x_2)\}_k$ , which is again a bounded sequence of  $\mathbb{R}$  and so has a convergent subsequence, call it  $\{f_{n(2,k)}(x_2)\}_k$ , which converges to some  $a_2 \in \mathbb{R}$ . Note that  $\{f_{n(2,k)}\}_k \subseteq \{f_{n(1,k)}\}_k$ , so also  $f_{n(2,k)}(x_1) \rightarrow a_1$  as  $k \rightarrow \infty$ . We can repeat this procedure, producing a sequence of real numbers  $\{a_\ell\}$ , and for each  $j \in \mathbb{N}$  a subsequence  $\{f_{n(j,k)}\}_k \subseteq \{f_n\}$  such that  $f_{n(j,k)}(x_\ell) \rightarrow a_\ell$  for each  $1 \leq \ell \leq j$ . Define then

$$f : D \rightarrow \mathbb{R}, \quad f(x_j) := a_j$$

Consider now

$$f_{n_k} := f_{n(k,k)}, \quad k \geq 1$$

the "diagonal sequence", and remark that  $f_{n_k}(x_j) \rightarrow a_j = f(x_j)$  as  $k \rightarrow \infty$  for every  $j \geq 1$ . Hence,  $\{f_{n_k}\}_k$  converges to  $f$  on  $D$ , pointwise.

We claim now that  $\{f_{n_k}\}_k$  converges on all of  $X$  to some function  $f : X \rightarrow \mathbb{R}$ , pointwise. Put  $g_k := f_{n_k}$  for notational convenience. Fix  $x_0 \in X, \epsilon > 0$ , and let  $\delta_{x_0} > 0$  be such that if  $x \in X$  such that  $p(x, x_0) < \delta_{x_0}$ ,  $|g_k(x) - g_k(x_0)| < \frac{\epsilon}{3}$ . Since  $D$  is dense in  $X$ ,  $\exists x_j \in D$  s.t.  $p(x_j, x_0) < \delta_{x_0}$ . Since  $\{g_k(x_j)\}_k$  converges, it is thus Cauchy, and hence for every  $k, \ell \geq K$ ,  $|g_k(x_j) - g_\ell(x_j)| < \frac{\epsilon}{3}$ . Therefore,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \epsilon$$

And thus  $\{g_k(x_0)\}_k$  Cauchy as a sequence in  $\mathbb{R}$ . Since  $\mathbb{R}$  is complete, then  $\{g_k(x_0)\}_k$  also converges, to, say,  $f(x_0) \in \mathbb{R}$ . Since  $x_0$  was arbitrary, this means there is some function  $f : X \rightarrow \mathbb{R}$  such that  $g_k \rightarrow f$  pointwise on  $X$  as we aimed to show.  $\square$

### Theorem 1.3.9: Arzelà-Ascoli Theorem

Let  $X$  be compact and let  $\{f_n\}_{n=1}^\infty \subseteq C(X)$  be uniformly bounded and uniformly equicontinuous. Then,  $\exists$  subseq  $\{f_{n_k}\}_{k=1}^\infty$  and  $f \in C(X)$  s.t.  $f_{n_k} \xrightarrow{k \rightarrow \infty} f$  in  $C(X)$  (i.e. uniformly)

**Proof.** Since  $(X, p)$  is compact, it is thus separable. Also, uniform bounded/equicontinuous implies pointwise bounded/equicontinuous. Therefore, by Arzelà-Ascoli lemma,  $\exists f : X \rightarrow \mathbb{R}$  and  $\{f_{n_k}\}_{k=1}^\infty$  s.t.  $f_{n_k} \rightarrow f$  pointwise in  $X$ .

Now let  $g_k := f_{n_k}$ .

**Claim:**  $\{g_k\}_{k=1}^\infty$  is uniformly Cauchy.<sup>1</sup>

Fix  $\epsilon > 0$ . By uniform equicontinuity,  $\exists \delta > 0$  s.t.

$$p(x, y) < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3} \quad \forall n \in \mathbb{N}.$$

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<sup>1</sup>Cauchy sequence in  $(C(X), \|\cdot\|_\infty)$

Letting  $n = n_k$ ,

$$p(x, y) < \delta \implies |g_k(x) - g_k(y)| < \epsilon \quad \forall k \in \mathbb{N}.$$

Since  $X$  is compact, it is totally bounded, so  $\exists \{x_i\}_{i=1}^N$  s.t.  $X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$ .

Moreover,  $\forall 1 \leq i \leq N$  fixed, we know  $\{g_k(x_i)\}_{k=1}^\infty \subseteq \mathbb{R}$  converges because  $\{g_k\}_{k=1}^\infty$  converges pointwise, so  $\{g_k(x_i)\}_{k=1}^\infty$  is a Cauchy sequence. So  $\exists K_i > 0$  s.t.  $\forall k, \ell \geq K_i$ ,

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3}.$$

Let  $K := \max_{1 \leq i \leq N} K_i$ . Then,  $\forall k, \ell \geq K$ , we have

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3} \quad \forall 1 \leq i \leq N.$$

So  $\forall x \in X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$ ,  $\exists x_i$  s.t.  $p(x, x_i) < \delta$ , and  $\forall k, \ell > K$ ,

$$|g_k(x) - g_\ell(x)| \leq |g_k(x) - g_k(x_i)| + |g_k(x_i) - g_\ell(x_i)| + |g_\ell(x_i) - g_\ell(x)| < \epsilon.$$

This implies  $\forall \epsilon > 0, \exists K > 0$  s.t.  $\forall k, \ell > K$ ,

$$\|g_k - g_\ell\|_\infty = \sup_{x \in X} |g_k(x) - g_\ell(x)| < \epsilon,$$

so  $\{g_k\}_{k=1}^\infty$  is uniformly Cauchy. Since  $(X, p)$  is compact,  $C(X)$  is complete, so  $\{g_k\}_{k=1}^\infty = \{f_{n_k}\}_{k=1}^\infty$  converges uniformly. Since  $f_{n_k} \rightarrow f$  pointwise in  $X$ , it must be that  $f_{n_k} \rightarrow f$  uniformly, and thus  $f \in C(X)$ .  $\square$

**Remark 1.3.10:** How do we use the AA theorem? To extract convergent subsequence, which may give us convergence of the original sequence.

### Fact

Let  $\{f_n\}_{n=1}^\infty \subseteq C(X)$ . If  $\exists! f$  s.t. for every subsequence,  $\exists$  a further subsequence  $\{f_{n_{k_j}}\}_{j=1}^\infty$  s.t.  $f_{n_{k_j}} \xrightarrow{j \rightarrow \infty} f$  uniformly, then  $f_n \xrightarrow{n \rightarrow \infty} f$  uniformly.

### Example 1.3.11: Typical Applications of Arzelà-Ascoli

- Verify  $\{f_n\}$  satisfies hypothesis of AA;
- For every subseq  $\{f_{n_k}\}$  also satisfies hypothesis of AA;
- Use AA to extract  $\{f_{n_{k_j}}\}_{j=1}^\infty$  s.t.  $f_{n_{k_j}} \rightarrow f$  uniformly on  $X$ .
- If you can show  $f$  is unique, then  $f_n \rightarrow f$  in  $C(X)$ .

**Corollary 1.3.12:** Let  $(X, p)$  be a compact metric space. Let  $\mathcal{F} \subseteq C(X)$  be uniformly bounded and uniformly equicontinuous. Then,  $\mathcal{F}$  is precompact in  $(C(X), \|\cdot\|_\infty)$ .

**Proof.** If  $f$  is uniformly bounded and uniformly equicontinuous, then by the AA theorem,  $\forall$  sequence  $\{f_n\}_{n=1}^\infty \subseteq \mathcal{F}$ , there is a subseq.  $\{f_{n_j}\}_{j=1}^\infty$  and  $f \in C(X)$  s.t.  $f_{n_j} \rightarrow f$  in  $C(X)$ . Note,  $f$  may not be in  $\mathcal{F}$ . So  $\mathcal{F}$  is precompact.  $\square$

### Example 1.3.13

Let  $M > 0$ , and define

$$\mathcal{F} = \left\{ f \in C([a, b]) \cap C^1([a, b]) : \|f\|_\infty + \|f'\|_\infty < M \right\}.$$

$\mathcal{F}$  is uniformly bounded and uniformly equicontinuous.

So by AA then, for  $\{f_n\} \subseteq \mathcal{F}$ ,  $\exists \{f_{n_k}\}_{k=1}^\infty$  s.t.  $f_{n_k} \rightarrow f$  uniformly. But,  $f$  may not be in  $C^1([a, b])$ . (So,  $f$  may not be in  $\mathcal{F}$ )

**Extra stuff left in the assignment, go back and look at it.**

## 1.4 Baire Category Theorem

**Definition 1.4.1 (Hollow/Nowhere Dense):** We say a set  $E$  is *hollow* if  $\text{Int}(E) = \emptyset$ .<sup>1</sup> We say  $E \subseteq X$  *nowhere dense* if its closure is hollow, i.e.  $\text{Int}(\overline{E}) = \emptyset$ .

---

<sup>1</sup>i.e.  $E$  contains no nontrivial open sets

**Remark 1.4.2:**  $E$  hollow  $\iff E^c$  dense in  $X$ , since  $\text{Int}(E) = \emptyset \iff (\text{Int}(E))^c = \overline{E^c} = X$ .

### Goal

When can we guarantee that

- a union of hollow sets is hollow?
- an intersection of dense sets is dense?

### Theorem 1.4.3: Baire Category Theorem

Let  $(X, p)$  be a complete metric space.

1. Let  $\{F_n\}_{n=1}^\infty \subseteq X$  be a collection of closed hollow sets. Then  $\bigcup_{n=1}^\infty F_n$  is hollow.

2. Let  $\{\mathcal{O}_n\}_{n=1}^{\infty} \subseteq X$  be a collection of open dense sets. Then  $\bigcap_{n=1}^{\infty} \mathcal{O}_n$  is dense.

**Proof.** (2)  $\Rightarrow$  (1) by taking complements and using the previous remark, so we prove only (2).

**Claim:** Let  $G = \bigcap_{n=1}^{\infty} \mathcal{O}_n$ . Then  $G$  is dense in  $X$ .

Fix  $x \in X, r > 0$ .  $\forall n \in \mathbb{N}, \mathcal{O}_n$  is open and dense, so  $\exists y \in \mathcal{O}_n$  and  $s > 0$  s.t.

$$B(x, r) \cap \mathcal{O}_n \supseteq B(y, 2s) \supseteq \overline{B(y, s)}.$$

Now we use this fact inductively in  $n$ . Let  $x_1 \in X, r_1 < \frac{1}{2}$  s.t.  $\overline{B(x_1, r_1)} \subseteq B(x, r) \cap \mathcal{O}_1$ . Let  $x_2 \in X, r_2 < 2^{-2}$  s.t.  $\overline{B(x_2, r_2)} \subseteq B(x_1, r_1) \cap \mathcal{O}_2$ . Repeating this process, take  $x_n \in X, r_n < 2^{-n}$  s.t.  $\overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \cap \mathcal{O}_n$ .

$$\Rightarrow \overline{B(x_1, r_1)} \supseteq \overline{B(x_2, r_2)} \supseteq \dots \supseteq \overline{B(x_n, r_n)} \supseteq \dots,$$

and  $r_n \rightarrow 0$ . Therefore  $\{x_n\}_{n=1}^{\infty}$  is Cauchy, and  $(X, p)$  is complete, so  $\exists x_0 \in X$  s.t.  $x_n \rightarrow x_0$ . Thus,

$$x_0 = \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)}.$$

Since  $x_0 \in \overline{B(x_n, r_n)} \subseteq \mathcal{O}_n \forall n$ , and  $x_0 \in \overline{B(x_1, r_1)} \subseteq B(x, r) \Rightarrow x_0 \in G \cap B(x, r)$ .

$$\Rightarrow G \cap B(x, r) \neq \emptyset \forall x \in X, \forall r > 0.$$

$\Rightarrow G$  is dense in  $X$ . □

Another restatement of the Baire Category Theorem is as follows: If  $(X, p)$  is complete, the countable union of nowhere dense sets is hollow.

**Proof.** Let  $\{E_n\}_{n=1}^{\infty}$  be nowhere dense sets. Then by BCT,  $\bigcup_{n=1}^{\infty} \overline{E_n}$  is hollow. It follows that  $\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \overline{E_n}$  so  $\bigcup_{n=1}^{\infty} E_n$  is also hollow. □

The main way we will use the Baire Category Theorem is the following:

**Corollary 1.4.4:** Let  $(X, p)$  be complete. Suppose  $\{F_n\}_{n=1}^{\infty}$  is a collection of closed sets. If  $X = \bigcup_{n=1}^{\infty} F_n$ , then  $\exists n_0$  s.t.  $\text{Int}(F_{n_0}) \neq \emptyset$ .

**Proof.** If  $\nexists n_0$ , then  $F_n$  is hollow  $\forall n$ , so by BCT  $X = \bigcup_{n=1}^{\infty} F_n$  is hollow, but this is a contradiction because  $X \subseteq X$  is open and nontrivial. □

**Theorem 1.4.5**

Let  $X \subseteq C(X)$  where  $(X, p)$  is complete. Suppose  $\mathcal{F}$  is pointwise bounded. Then,  $\exists$  non-empty open set  $\mathcal{O} \subseteq X$  s.t.  $\mathcal{F}$  is uniformly bounded on  $\mathcal{O}$ , i.e.  $\exists M > 0$  s.t.

$$\sup_{f \in \mathcal{F}} \sup_{x \in \mathcal{O}} |f(x)| \leq M$$

**Proof.** Let

$$E_n = \{x \in X : |f(x)| \leq n, \forall f \in \mathcal{F}\} = \bigcap_{f \in \mathcal{F}} \{x \in X : |f(x)| \leq n\}.$$

$\Rightarrow E_n$  is closed  $\forall n$ . Since  $\mathcal{F}$  is pointwise bounded,

$$\forall x \in X, \exists M_x > 0 \text{ s.t. } \sup_{f \in \mathcal{F}} |f(x)| \leq M_x.$$

Thus,  $\forall n$  s.t.  $M_x \leq n$ , then  $x \in E_n$  (since  $|f(x)| \leq M_x \leq n$ ).

So,  $X = \bigcup_{n=1}^{\infty} E_n$  and  $E_n$  is closed. By corollary,  $\exists n_0$  s.t.  $\text{Int}(E_{n_0}) \neq \emptyset$ . So  $\exists x_0 \in X, r$  s.t.  $B^p(x_0, r) \subseteq E_{n_0}$ . Letting  $\mathcal{O} = B^p(x_0, r)$ , we have

$$\sup_{x \in \mathcal{O}} |f(x)| \leq n_0 \quad \forall f \in \mathcal{F}. \quad \square$$

**Corollary 1.4.6:** Let  $(X, p)$  be a complete metric space. Suppose  $\{F_n\}_{n=1}^{\infty}$  is a collection of closed sets. Then  $\bigcup_{n=1}^{\infty} \partial F_n$  is hollow.

**Proof.** **Claim:**  $\partial F_n$  is hollow  $\forall n$ . Suppose for contradiction that  $\exists n$  s.t.  $\text{Int}(\partial F_n) \neq \emptyset$ . Then  $\exists x_0 \in \partial F_n, r > 0$  s.t.  $B^p(x_0, r) \subseteq \partial F_n$ . But then,

$$B^p(x_0, r) \cap F_n^c = B^p(x_0, r) \cap \overline{F_n^c} = B^p(x_0, r) \cap (F_n \cup \partial F_n)^c = B^p(x_0, r) \cap \partial F_n^c \cap F_n^c = \emptyset$$

and this contradicts  $x_0 \in \partial F_n$  by defn.  $\Rightarrow \partial F_n$  is hollow  $\forall n$ . Furthermore,  $\partial F_n$  is closed, since it contains all of its limit points by definition. Thus, by BCT,  $\bigcup_{n=1}^{\infty} \partial F_n$  is hollow.  $\square$

Now recall that in general,  $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$  and  $f_n \rightarrow f$  pointwise, then  $f$  is not necessarily continuous.

**Theorem 1.4.7**

Let  $(X, p)$  be complete. Let  $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$  s.t.  $f_n \rightarrow f$  pointwise in  $X$ . Then there is a dense subset  $D \subseteq X$  where  $\{f_n\}_{n=1}^{\infty}$  is pointwise equicontinuous on  $D$  and  $\forall x_0 \in D, f$  is continuous at  $x_0$ .

**Proof.** Let  $m, n \in \mathbb{N}$ . Define

$$E(m, n) = \left\{ x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \forall j, k \geq n \right\} = \underbrace{\bigcap_{j, k \geq n} \left\{ x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \right\}}_{\text{closed, since } f_k, f_j \in C(X)}.$$

So  $E(m, n)$  is closed  $\forall m, n$ . Thus, by the corollary,  $\bigcup_{m, n \in \mathbb{N}} \partial E(m, n)$  is hollow. This implies that

$$D := \left( \bigcup_{m, n \in \mathbb{N}} \partial E(m, n) \right)^c = \bigcap_{m, n \in \mathbb{N}} \partial E(m, n)^c \quad \text{is dense.}$$

**Claim 1:** If  $\exists x \in X, m, n \in \mathbb{N}$  s.t.  $x \in D \cap E(m, n)$ , then  $x \in \text{Int}(E(m, n))$ .

If  $x \in D$ , then

$$x \in \underbrace{\partial E(m, n)^c}_{\text{open}} = \text{Int}(E(m, n)) \cup \text{Ext}(E(m, n)).$$

For the exterior term:

$$\begin{aligned} \text{Ext}(E(m, n)) &= X \setminus (\text{Int}(E(m, n)) \cup \partial E(m, n)) \\ &= X \setminus E(m, n) = E(m, n)^c. \end{aligned}$$

Since we also have  $x \in E(m, n)$ , this means  $x \in \text{Int}(E(m, n))$ .

**Claim 2:**  $\{f_n\}_{n=1}^{\infty}$  is equicontinuous on  $D$ .

Let  $x_0 \in D$  and  $\epsilon > 0$ . Choose  $m$  s.t.  $\frac{1}{m} < \frac{\epsilon}{4}$ . Since  $\{f_n\}_{n=1}^{\infty}$  converges,  $\{f_n(x_0)\}_{n=1}^{\infty} \subseteq \mathbb{R}$  is a Cauchy sequence. So  $\exists N$  s.t.  $\forall j, k \geq N$ ,

$$|f_j(x_0) - f_k(x_0)| \leq \frac{1}{m}.$$

This means  $x_0 \in E(m, n) \cap D$ , so by Claim 1,  $x_0 \in \text{Int}(E(m, n))$ . Let  $B^p(x_0, r) \subseteq E(m, N)$ , so  $\forall j, k \geq N, \forall x \in B(x_0, r)$ ,

$$|f_j(x) - f_k(x)| \leq \frac{1}{m}.$$

Since  $f_N$  is continuous at  $x_0$ ,  $\exists \delta_{x_0} > 0$  (which WLOG we can choose  $< r$ ), s.t.  $\forall x \in B^p(x_0, \delta_{x_0})$ ,

$$|f_N(x) - f_N(x_0)| \leq \frac{1}{m}.$$

So  $\forall j \geq N, \forall x \in B^p(x_0, \delta_{x_0})$ ,

$$|f_j(x) - f_j(x_0)| \leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_j(x_0)| \leq \frac{3}{m} \leq \frac{3\epsilon}{4}.$$

Since this holds  $\forall j \geq N$ , this implies that  $\{f_n\}_{n=1}^{\infty}$  is equicontinuous at  $x_0$ . Furthermore,  $\forall x \in B^p(x_0, \delta_{x_0})$ , sending  $j \rightarrow \infty$ , we obtain that  $\forall x \in B^p(x_0, \delta_{x_0})$ ,

$|f(x) - f(x_0)| \leq \frac{3\epsilon}{4}$ , so  $f$  is continuous at  $x_0 \in D$ .  $\square$

## 1.5 Topological Spaces

We'll consider topological spaces, where we will define all concepts using open sets, and we will generalize what we have learned from Metric Spaces.

**Definition 1.5.1:** Let  $X$  be a non empty set. A *topology*  $\mathcal{T}$  on  $X$  is a collection of subsets of  $X$ , such that

- $X, \emptyset \in \mathcal{T}$ ;
- If  $\{E_n\} \subseteq \mathcal{T}, \bigcap_{n=1}^N E_n \in \mathcal{T}$  (closed under finite intersections);
- If  $\{E_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{T}, \bigcup_{\lambda \in \Lambda} E_\lambda \in \mathcal{T}$  (closed under arbitrary unions).

We say  $(X, \mathcal{T})$  is a *topological space*.

If  $E \in \mathcal{T}$ , then we call  $E$  an open set (with respect to  $\mathcal{T}$ ).

If  $x \in X$ , a set  $E \in \mathcal{T}$  containing  $x$  is called a *neighborhood* of  $x$ .

**Remark 1.5.2:** By definition of  $\mathcal{T}$ ,  $E \in \mathcal{T}$  iff  $\forall x \in E, \exists$  a neighbourhood of  $x$ , contained in  $E$ . (consistent with metric space definition of open set)

### Example 1.5.3: Metric topology

Let  $(X, p)$  be a metric space. Define

$$\mathcal{T} := \{\text{open sets w.r.t. } p\}.$$

Then,  $\mathcal{T}$  is a topology on  $X$ , called the metric topology induced by  $p$ .

Given a topology  $\mathcal{T}$ , if  $\exists$  a metric  $p$  s.t.  $\mathcal{T}$  is the metric topology induced by  $p$ , then we say  $\mathcal{T}$  is *metrizable*.

### Example 1.5.4: Trivial Topology

Let  $X$  be a non empty set. Define

$$\mathcal{T} = \{\emptyset, X\}.$$

Then,  $\mathcal{T}$  is a topology on  $X$ , called the trivial topology.

**Example 1.5.5: Discrete Topology**

Let  $X$  be a non empty set. Let  $p(x, y)$  be the discrete metric on  $X$ . Define Then

$$B^p(x_0, r) = \begin{cases} \{x_0\}, & 0 < r \leq 1 \\ X, & r > 1 \end{cases}$$

So  $\forall E \subseteq X, \forall x \in E, B^p(x, \frac{1}{2}) = \{x\} \subseteq E \Rightarrow E$  is open. Then,

$\mathcal{T} = \mathcal{P}(X) = \{\text{All possible subsets of } X\}$  is a topology on  $X$ , called the discrete topology, and it contains all subsets of  $X$ .

**Example 1.5.6: Relative Topology**

Let  $(X, \mathcal{T})$  be a topological space. Let  $Y \subseteq X$ . Then,

$$\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}.$$

Then,  $\mathcal{T}_Y$  is a topology on  $Y$ , called the relative topology on  $Y$  induced by  $\mathcal{T}$ .

If  $X = \mathbb{R}, Y = \mathbb{N}$ , then  $\mathcal{T}_{\mathbb{N}} = \{U \cap \mathbb{N} : U \subseteq \mathbb{R} \text{ open}\}$  So  $\forall y \in \mathbb{N}, \forall x \in \mathbb{R}, r > 0$ ,

$$B(x, r) \cap \mathbb{N} = \begin{cases} \{y\}, & y \in B(x, r) \\ \emptyset, & y \notin B(x, r) \end{cases}$$

Thus,  $\mathcal{T}_{\mathbb{N}} = \mathcal{P}(\mathbb{N})$ , the discrete topology on  $\mathbb{N}$ .

If  $X = \mathbb{R}, Y = [0, 1]$ , then  $\mathcal{T}_{[0,1]} = \{U \cap [0, 1] : U \subseteq \mathbb{R} \text{ open}\}$  So the set  $[0, \frac{1}{2}] = [0, 1] \cap (-1, \frac{1}{2}) \in \mathcal{T}_{[0,1]}$ . So  $[0, \frac{1}{2}]$  is relatively open in  $Y = [0, 1]$  (belongs to the relative topology on  $Y$ ).

In metric spaces, everything is done using balls. In a generic topological space  $(X, \mathcal{T})$ , what plays the role of balls?

**Definition 1.5.7 (base/neighbourhood base):** Let  $(X, \mathcal{T})$  Topological space. Fix  $x \in X$ . Let  $\mathcal{B}_x$  be a collection of neighborhoods of  $x$ . We call  $\mathcal{B}_x$  a neighbourhood base at  $x$  if  $\forall$  neighborhood of  $x$  (call it  $u$ ),  $\exists B \in \mathcal{B}_x$  such that  $B \subseteq u$ . We say  $\mathcal{B}$ , a collection of open sets, is a base for  $\mathcal{T}$  if  $\forall x \in X, \exists$  a neighbourhood base  $\mathcal{B}_x \subseteq \mathcal{B}$  at  $x$ .

**Example 1.5.8**

In  $(X, p)$  a metric space,  $\forall x \in X$

$\mathcal{B}_x = \{B^p(x, r) : r > 0\}$  is a neighbourhood base

$B = \{\text{all balls of all radii}\}$

**Remark 1.5.9:** Given a topology, a neighbourhood base is not unique.

$$\mathcal{B}_x = \left\{ B^p\left(x, \frac{1}{n}\right) \right\}_{n=1}^{\infty}$$

is also a neighbourhood base at  $x$  in a metric space.

**Definition 1.5.10 (first countable/second countable):** Let  $(X, \mathcal{T})$  be a topological space.

- We say  $(X, \mathcal{T})$  is *first countable* if there is a countable neighbourhood base at each  $x \in X$ ;
- We say  $(X, \mathcal{T})$  is *second countable* if there is a countable base  $B$  of  $\mathcal{T}$ .

**Remark 1.5.11:** Any metric space is first countable, and any separable metric space is second countable.

**Remark 1.5.12:** For a topology  $\mathcal{T}$ ,  $\mathcal{B} = \mathcal{T}$  is always a base for  $\mathcal{T}$  (so a base always exists).

**Proposition 1.5.13:** If  $(X, \mathcal{T})$  be a topological space. A collection of open sets  $\mathcal{B}$  is a base for  $\mathcal{T}$  iff every non-empty open set  $U \in \mathcal{T}$  can be written as a union of elements of  $\mathcal{B}$ .

**Proof.**

$(\Rightarrow)$  Suppose  $\mathcal{B}$  is a base for  $\mathcal{T}$ . Let  $U \in \mathcal{T}$ . Then  $\forall x \in U, \exists B_x \in \mathcal{B}_x \subseteq \mathcal{B}$  such that

$$x \in B_x \subseteq U \Rightarrow \bigcup_{x \in U} B_x \subseteq U$$

and

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \Rightarrow U = \bigcup_{x \in U} B_x.$$

$(\Leftarrow)$  Suppose every non-empty open set  $U \in \mathcal{T}$  can be written as a union of elements of  $\mathcal{B}$ . Fix  $x \in U$ , and let

$$\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\} = \{B \in \mathcal{B} : \{x\} \cap B \neq \emptyset\} \subseteq \mathcal{B}.$$

Since  $U = \bigcup B$ , this means  $\mathcal{B}_x \neq \emptyset$ . So  $U$  is a neighbourhood of  $x$ , and  $\exists B \in \mathcal{B}_x$  such that

$B \subseteq U \Rightarrow \mathcal{B}_X$  is a neighbourhood base at  $x$ . Doing that  $\forall x \in X$ , we get a  $\mathcal{B}$  that is a base for  $\mathcal{T}$ .  $\square$

### Question

Given a collection  $\mathcal{B}$ , what does it take to be a base for some topology?

**Proposition 1.5.14:** Let  $X \neq \emptyset$ . Let  $\mathcal{B} \subseteq \mathcal{P}(X)$  be a collection of sets. Then  $\mathcal{B}$  is a base for some topology  $\mathcal{T}$  iff

1.  $X = \bigcup_{B \in \mathcal{B}} B$ ;
2.  $\forall B_1, B_2 \in \mathcal{B}$ , and  $x \in B_1 \cap B_2$ , then  $\exists B \in \mathcal{B}$  such that  $x \in B \subseteq B_1 \cap B_2$ .<sup>1</sup>

<sup>1</sup>For balls, this is true

### Proof.

( $\Rightarrow$ )  $\mathcal{B}$  is a base for a topology  $\mathcal{T}$ . Then, since  $X \in \mathcal{T}$ , by the last result,  $X = \bigcup_{B \in \mathcal{B}} B$ , so (1) holds. Moreover, if  $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{T}$ , then  $B_1 \cap B_2 \in \mathcal{T}$ . So for  $x \in B_1 \cap B_2$ , then  $B_1 \cap B_2$  is a neighbourhood of  $x$ . Since  $\mathcal{B}$  is a base,  $\exists B \in \mathcal{B}$  s.t.  $x \in B \subseteq B_1 \cap B_2$ , so (2) holds.

( $\Leftarrow$ ) Suppose (1) and (2) hold. Let

$$\mathcal{T} := \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}.$$

Since  $X = \bigcup_{B \in \mathcal{B}} B$ , so  $\forall x \in X, \exists B \in \mathcal{B}$  s.t.  $x \in B \subseteq X \Rightarrow x \in \mathcal{T}$ . Similarly,  $\emptyset \in \mathcal{T}$  because the condition is empty. The definition of  $\mathcal{T}$  shows us that it is closed under arbitrary unions.

Let  $U_1, U_2 \in \mathcal{T}$ , and assume  $U_1 \cap U_2 \neq \emptyset$ , so  $\forall x \in U_1 \cap U_2$ , by definition of  $\mathcal{T}$ ,  $\exists B_1 \in \mathcal{B}$  s.t.  $x \in B_1 \subseteq U_1$ , and  $\exists B_2 \in \mathcal{B}$  s.t.  $x \in B_2 \subseteq U_2 \Rightarrow x \in B_1 \cap B_2$ . By (2),  $\exists B \in \mathcal{B}$  s.t.  $x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$ . Thus,  $B_1 \cap B_2 \in \mathcal{T}$ . Inductively, we conclude  $\mathcal{T}$  is closed under finite intersections.  $\square$

Observe that the properties which define  $\mathcal{T}$  are closed under intersections, so we may define a  $\sigma$ -algebra like structure for topologies:

**Definition 1.5.15:** Let  $\mathcal{E} \subseteq \mathcal{P}(X)$  be a collection of sets. Then

$$\mathcal{T}(\mathcal{E}) = \bigcap \{\text{All topologies containing } \mathcal{E}\} = \text{topology generated by } \mathcal{E}$$

**Definition 1.5.16 (weaker/coarser vs. stronger/finer):** Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be topologies on  $X$ . If  $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ , we say  $\mathcal{T}_1$  is a weaker/coarser topology than  $\mathcal{T}_2$  (fewer open sets), and  $\mathcal{T}_2$  is a

stronger/finer topology than  $\mathcal{T}_1$  (more open sets).

### Example 1.5.17

Trivial topology is the weakest topology on  $X$  and discrete topology is the strongest topology on  $X$ . So  $\mathcal{T}(\mathcal{E})$  is the weakest topology containing  $\mathcal{E}$ .

**Proposition 1.5.18:** Let  $\mathcal{E} \subseteq \mathcal{P}(X)$ . Then

$$\mathcal{T}(\mathcal{E}) = \left\{ \emptyset, X, \bigcup \{ \text{finite intersections of sets in } \mathcal{E} \} \right\}$$

**Proof.** **Claim:**  $\mathcal{B} = \{\emptyset, X, \text{finite intersections of elements of } \mathcal{E}\}$  forms a base.  $\mathcal{B}$  satisfies (1) and (2) of the previous proposition, so  $\mathcal{B}$  is a base for some topology

$$\tilde{\mathcal{T}} = \left\{ \emptyset, X, \bigcup \{ \text{finite intersections of elements of } \mathcal{E} \} \right\}$$

Observe that  $\tilde{\mathcal{T}} \subseteq \{\text{any topology which contains } \mathcal{E}\} \Rightarrow \tilde{\mathcal{T}} \subseteq \mathcal{T}(\mathcal{E})$ .  $\tilde{\mathcal{T}}$  is also a topology, which contains  $\mathcal{E}$ . So  $\mathcal{T}(\mathcal{E}) \subseteq \tilde{\mathcal{T}}$ . Thus,  $\tilde{\mathcal{T}} = \mathcal{T}(\mathcal{E})$ .  $\square$

### Goal

Topologies give us open sets, bases give ball-like sets, now we need a notion for closed sets.

**Definition 1.5.19 (limit point, closure, closed set):** If  $E \subseteq X, x \in X$  is a limit point if  $\forall$  neighbourhood  $U_x$  of  $x$ ,

$$U_x \cap E \neq \emptyset.$$

We say  $\bar{E} = \{\text{All limit points of } E\}$ , is the closure of  $E$ .

We say  $E$  is closed if  $E = \bar{E}$ .

**Remark 1.5.20:** We always have  $E \subseteq \bar{E}$ , so we just need  $\bar{E} \subseteq E$  to show  $E$  is closed.

**Proposition 1.5.21:** Let  $E \subseteq X$ .

1.  $\bar{E}$  is closed;
2.  $\bar{E}$  is the smallest closed set containing  $E$ , i.e. if  $\exists F$  closed s.t.  $E \subseteq F \Rightarrow \bar{E} \subseteq F$ ;
3.  $E$  is open iff  $E^c$  is closed.

**Proof of (1) + (2).**

**Claim:**  $L := \{\text{limit points of } \bar{E}\} = \bar{\bar{E}} \subseteq \bar{E}$ .

Let  $x \in L$ , and a neighbourhood  $U_x$  of  $x$ . Then by defn of  $L$ , we know  $\exists x' \in U_x \cap \bar{E}$ . This means  $x' \in \bar{E}$ , and  $U_x$  is a neighbourhood of  $x'$   $\Rightarrow U_x \cap E \neq \emptyset$ . This holds  $\forall$  neighbourhood  $U_x$  of  $x$ , so  $x \in \bar{E} \Rightarrow L \subseteq \bar{E} \Rightarrow \bar{E}$  is closed.

Suppose  $E \subseteq F$  and  $F$  is closed. Let  $x \in \bar{E}$ , then  $\forall$  neighbourhood  $U_x$ ,  $U_x \cap E \neq \emptyset \Rightarrow U_x \cap F \neq \emptyset \Rightarrow x \in \bar{F} \Rightarrow \bar{E} \subseteq \bar{F} = F$ .  $\square$

**Proof of (3).**

( $\Rightarrow$ ) Let  $E \subseteq X$  be open. Let  $x \in \bar{E}^c$ .

**Claim:**  $x \in E^c$ .

Suppose not, so  $x \in E$ . So  $\exists$  neighbourhood  $U_x$  of  $x$  s.t.  $U_x \subseteq E \Rightarrow U_x \cap E = \emptyset$ . So  $x \notin \bar{E}^c$ . So,  $x \in \bar{E}^c \Rightarrow x \in E^c \Rightarrow \bar{E}^c \subseteq E^c \Rightarrow E^c$  closed.

( $\Leftarrow$ ) Let  $E^c$  be closed. Let  $x \in E$ .

**Claim:**  $\exists$  neighbourhood  $U_x$  s.t.  $U_x \subseteq E$ .

Suppose not, then every neighbourhood  $U_x$  we have  $U_x \cap E^c \neq \emptyset \Rightarrow x \in \bar{E}^c = E^c$  (contradicts  $x \in E$ ). By claim,  $E$  is open.  $\square$

**Remark 1.5.22:** Our proof shows,  $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$ .

**Definition 1.5.23 (Density, Separability):**  $D \subseteq X$  is dense if  $\forall$  non-empty open set  $U$ ,  $U \cap D \neq \emptyset$ .  $\iff \bar{D} = X$ .

$(X, \mathcal{T})$  is separable iff  $X$  contains a countable dense set.

**Proposition 1.5.24:** Every second countable space is separable.

**Proof.** Let  $(X, \mathcal{T})$  be second countable, so  $\exists$  base  $\mathcal{B} = \{B_i\}_{i=1}^\infty$ . Pick  $x_i \in B_i$  (need axiom of choice). Let  $D = \{x_i\}_{i=1}^\infty$ . Then  $\forall U \subseteq X$  open, since  $\mathcal{B}$  is a base,  $\exists B_i \subseteq U$ ,

$$D \cap U \supseteq \{x_i\} \cap B_i \neq \emptyset$$

$\Rightarrow D \cap U \neq \emptyset$ , so  $D$  is dense.  $\square$

**Definition 1.5.25 (Convergence in Topology):** Given  $(X, \mathcal{T})$  a topological space, let  $\{x_n\}_{n=1}^\infty \subseteq X$ . We say  $x_n \rightarrow x$  in  $\mathcal{T}$  if  $\forall$  neighbourhood  $U_x$  of  $x$ ,  $\exists N$  s.t.  $\forall n \geq N$ ,  $x_n \in U_x$ .

**Proposition 1.5.26:** Suppose  $(X, \mathcal{T})$  is first countable, and  $E \subseteq X$ . Then,  $x \in \overline{E}$  iff  $\exists \{x_n\} \subseteq E$  s.t.  $x_n \rightarrow x$  in  $\mathcal{T}$ .

**Proof.**

$\Rightarrow$  Let  $\mathcal{B}_x = \{B_j\}_{j=1}^{\infty}$  be a neighbourhood base at  $x \in \overline{E}$ . WLOG, can assume  $B_{j+1} \subseteq B_j \quad \forall j$ . Since  $x \in \overline{E}$  and  $B_j$  is a neighbourhood of  $x$ ,  $B_j \cap E \neq \emptyset \quad \forall j$ . Let  $x_i \in B_i \cap E$ . Then,  $\forall$  neighbourhood  $U_x$  of  $x$ ,  $\exists B_J \in \mathcal{B}_x$  s.t.  $B_J \subseteq U_x$ . But since  $\{B_j\}$  are nested,  $\forall j \geq J$ ,

$$U_x \supseteq B_j \cap U_x = B_j \supseteq B_i \cap E \supseteq \{x_i\}$$

$$\Rightarrow x_i \rightarrow x \text{ in } \mathcal{T}.$$

$\Leftarrow$  If  $\exists \{x_j\}_{j=1}^{\infty} \subseteq E$  s.t.  $x_j \rightarrow x$  in  $\mathcal{T}$ , suppose  $x \notin \overline{E}$ . Then  $x \in \overline{E}^c$  and  $\overline{E}^c$  open, so  $\overline{E}^c$  is a neighbourhood of  $x$  s.t.  $\{x_j\}_{j=1}^{\infty} \cap \overline{E}^c \neq \emptyset$ . Therefore  $x_j \not\rightarrow x$  in  $\mathcal{T}$ .  $\square$

While  $(X, \mathcal{T})$  allows us to consider a very general framework, weird stuff can happen.

## 1.6 Separation Properties

### Example 1.6.1

Let  $\mathcal{T} = \{\emptyset, X\}$ . So the only non-empty neighbourhood is  $X$ , so any sequence  $\{x_n\}_{n=1}^{\infty} \subseteq X$  converges to any point  $x \in X$ .

We need topologies with more structure.

**Definition 1.6.2 (neighbourhood of a set, Separating sets by disjoint neighbourhoods):** Let  $(X, \mathcal{T})$  be a topological space, and  $K, A, B \subseteq X$ . A neighbourhood of  $K$  is an open set  $U$  s.t.  $K \subseteq U$ . We say  $A, B$  can be separated by disjoint neighbourhoods if  $\exists U \supseteq A, V \supseteq B$  neighbourhoods s.t.  $U \cap V = \emptyset$ .

**Definition 1.6.3 (Separation Notions):** Let  $(X, \mathcal{T})$  be a topological space.  $(X, \mathcal{T})$  is

1. Tychonoff (T1) if  $\forall x \neq y \in X, \exists$  neighbourhood  $U_x$  s.t.  $y \notin U_x$ , and  $\exists$  neighbourhood  $U_y$  s.t.  $x \notin U_y$ ;
2. Hausdorff (T2) if  $\forall x \neq y \in X, \{x\}, \{y\}$ , can be separated by disjoint neighbourhoods, i.e.  $\exists U_x \supseteq \{x\}, U_y \supseteq \{y\}$  s.t.  $U_x \cap U_y = \emptyset$ ;
3. Regular (T3) if  $(X, \mathcal{T})$  is Tychonoff and  $\forall x \in X, \forall F \subseteq X$  closed, with  $x \notin F$ ,  $\{x\}$  and  $F$  can be separated by disjoint neighbourhoods;

4. Normal (T4) if  $(X, \mathcal{T})$  is Tychonoff and  $\forall A, B \subseteq X$  closed and disjoint,  $A$  and  $B$  can be separated by disjoint neighbourhoods.

#### Example 1.6.4

Consider  $\mathbb{R}$  and  $\mathcal{T} = \{\emptyset, (-\infty, c) \text{ for } c \in \mathbb{R}\}$ . Then,  $\forall x \in \mathbb{R}$ , a neighbourhood of  $x$  is of the form  $(-\infty, c)$  for some  $c > x$ . Let  $x \neq y \in \mathbb{R}$ , WLOG assume  $x < y$ . Then  $x \in U_y \forall$  neighbourhood  $U_y$  of  $y$ . So  $(\mathbb{R}, \mathcal{T})$  is not Tychonoff.

#### Example 1.6.5

Let  $X = \mathbb{R}$ . Let  $K := \{\frac{1}{n} : n \in \mathbb{Z}\}$ . Let  $B = \{(a, b) : a < b\} \cup \{(a, b) \setminus K : a < b\}$ . We check

- $\mathbb{R} = \bigcup_{B \in \mathcal{B}} B$
- if  $B_1, B_2 \in \mathcal{B}$  and  $B_1 \cap B_2 \neq \emptyset$ ,  $B_1 = (a_1, b_1)$  or  $(a_1, b_1) \setminus K$ ,  $B_2 = (a_2, b_2)$  or  $(a_2, b_2) \setminus K$ . So  $B_1 \cap B_2 \neq \emptyset \Rightarrow a_1 < a_2 < b_1 < b_2 \Rightarrow (a_2, b_1) \text{ or } (a_2, b_1) \setminus K \subseteq B_1 \cap B_2$ . So  $\mathcal{B}$  is a base for some topology  $\mathcal{T}$  on  $\mathbb{R}$  called the K-topology.

Suppose  $x \neq y$  and  $x < y$ .  $\exists U_x$  and  $U_y \in \mathcal{B}$  s.t.  $U_x \cap U_y = \emptyset \Rightarrow (X, \mathcal{T})$  is Hausdorff. But,  $K \subseteq X$  is closed because  $K^c = \mathbb{R} \setminus K = (-\infty, -2) \cup (-2, 2) \setminus K \cup (2, \infty)$  is open. But  $0 \notin K$ . So  $\{0\} \in K$  if  $U, V$  are neighbourhoods s.t.  $\{0\} \in U, K \subseteq V, K \subseteq V = \cup(a, b)$ .  $\{0\} \in U = (c, d) \setminus K \Rightarrow V \cap U \neq \emptyset$ .

$(X, \mathcal{T})$  is thus Hausdorff but not regular.

**Proposition 1.6.6:** If  $(X, \mathcal{T})$  is Hausdorff, then for  $x_n \rightarrow x$  in  $\mathcal{T}$ ,  $x$  is unique.

**Proof.** If  $x_n \rightarrow x$  and  $x_n \rightarrow y$  in  $\mathcal{T}$  and  $x \neq y$ , then  $\exists U_x \supseteq \{x\}, U_y \supseteq \{y\}$  s.t.  $U_x \cap U_y \neq \emptyset$ . So we cannot have  $x_n \in U_x \cap U_y \Rightarrow x = y$   $\square$

**Proposition 1.6.7:**  $(X, \mathcal{T})$  is Tychonoff iff  $\forall x \in X, \{x\}$  is closed.

**Proof.**

$$\begin{aligned} \{x\} \text{ is closed} &\iff \{x\}^c \text{ is open} \\ &\iff \forall y \in \{x\}^c, \exists \text{ neighbourhood } U_y \subseteq \{x\}^c \\ &\iff x \notin U_y \end{aligned}$$

$\square$

**Remark 1.6.8:** so  $(X, \mathcal{T})$  normal  $\Rightarrow (X, \mathcal{T})$  regular.