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# MATH 455

## Lecture Notes

Charles Zitella

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*Based on lectures by Prof. Jessica Lin*

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# 1 Abstract Metric and Topological Spaces

## 1.1 Metric Spaces Review

Throughout, assume  $X$  is a non empty set.

**Definition 1.1.1 (Metric).**  $p : X \times X \rightarrow \mathbb{R}$  is called a *metric*, and thus  $(X, p)$  a metric space, if for all  $x, y, z \in X$

- $p(x, y) \geq 0$ ,
- $p(x, y) = 0 \iff x = y$ ,
- $p(x, y) = p(y, x)$ ,
- $p(x, y) \leq p(x, z) + p(z, y)$  (Triangle Inequality).

**Definition 1.1.2 (Norm).** Let  $X$  be a vector space.<sup>1</sup> A function  $\|\cdot\| : X \rightarrow [0, \infty)$  is called a *norm*, and thus  $(X, \|\cdot\|)$  a *normed vector space*, if for all  $u, v \in X$  and  $\alpha \in \mathbb{R}$

- $\|u\| = 0 \iff u = 0$ ,
- $\|u + v\| \leq \|u\| + \|v\|$ ,
- $\|\alpha u\| = |\alpha| \|u\|$ .

<sup>1</sup>closed under linear combinations

**Remark 1.1.2.** A norm induces a metric by  $p(x, y) := \|x - y\|$ .

**Example 1.1.3.** Examples of normed vector spaces:

1.  $(\mathbb{R}^n, |.|)$  where  $|x| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$
2.  $L^p(E)$  for  $E \subseteq \mathbb{R}^n, 1 \leq p \leq \infty$  where  $\|f\|_{L^p(E)} = (\int_E |f(x)|^p dx)^{\frac{1}{p}}$

3. Discrete metric: if  $X$  is a non empty set, then  $p(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

4.  $C([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$  for  $a, b \subseteq \mathbb{R}$ . Then,  
 $\|f\|_\infty := \sup_{x \in [a, b]} |f(x)| = \max_{x \in [a, b]} |f(x)|$ ,  $p(f, g) = \|f - g\|_\infty$

**Definition 1.1.4.** Given two metrics  $p, \sigma$  on  $X$ , we say they are *equivalent* if  $\exists$  a  $C > 0$  such that  $\frac{1}{C}\sigma(x, y) \leq p(x, y) \leq C\sigma(x, y)$  for every  $x, y \in X$ . A similar definition follows for equivalence of norms.

Given a metric space  $(X, P)$ , then, we have the notion of

- open balls  $B(x, r) = \{y \in X : p(x, y) \leq r\}$
- open sets (subsets of  $X$  with the property that for every  $x \in X$ , there is a constant  $r > 0$  such that  $B(x, r) \subseteq X$ ), closed sets, closures, and
- *convergence*

**Definition 1.1.5 (Convergence).**  $\{x_n\}_{n=1}^\infty \subseteq X$  *converges* to  $x$  in  $(X, p)$  if  $\lim_{n \rightarrow \infty} p(x_n, x) = 0$

We have several (equivalent) notions, then, of continuity; via sequences,  $\epsilon - \delta$  definition, and by pullbacks (inverse images of open sets are open).

**Definition 1.1.6 (Uniform Continuity).**  $f : (X, p) \rightarrow (\mathbb{R}, |\cdot|)$  *uniformly continuous* if  $f$  has a "modulus of continuity", i.e. there is a continuous function  $\omega : [0, \infty) \rightarrow [0, \infty)$  such that  $\lim_{t \rightarrow 0^+} \omega(t) = 0$ , and

$$|f(x) - f(y)| \leq \omega(p(x, y))$$

for every  $x, y \in X$

**Remark 1.1.6.** For instance, we say  $f$  Lipschitz continuous if there is a constant  $C > 0$  such that  $\omega(\cdot) = C(\cdot)$ . Let  $\alpha \in (0, 1)$ . We say  $f$   $\alpha$ -Holder continuous if  $\omega(\cdot) = C(\cdot)^\alpha$  for some constant  $C$ .

**Definition 1.1.7 (Completeness).** We say  $(X, p)$  *complete* if every Cauchy sequence in  $(X, p)$  converges to a point in  $X$ .

**Remark 1.1.7.** let  $E \subseteq X$  and  $(X, p)$  complete metric space. Then  $(E, p)$  is complete iff  $E \subseteq X$  is closed (so limits belong to E)

## 1.2 Compactness, Separability

**Definition 1.2.1 (Open Cover, Compactness).**  $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$ <sup>2</sup>, where  $X_\lambda$  open in  $X$  and  $\Lambda$  an arbitrary index set, an *open cover* of  $X$  if for every  $x \in X$ ,  $\exists \lambda \in \Lambda$  such that  $x \in X_\lambda$ .<sup>3</sup>  $X$  is *compact* if every open cover of  $X$  admits a finite subcover. We say  $E \subseteq X$  compact if  $(E, p)$  compact.

<sup>2</sup> $2^X$  denotes the power set of  $X$ , i.e. the set of all subsets of  $X$ .

<sup>3</sup>A cover is finite if  $|\Lambda| < \infty$

**Remark 1.2.1.** for  $E \subseteq X$ ,  $X_\lambda \subseteq E$  is open in  $(E, p)$  iff  $X_\lambda$  is open in  $(X, p)$  Therefore,  $E \subseteq X$  is compact iff every open cover of  $E$  (in  $X$ ) has a finite subcover.

**Remark 1.2.1.** This definition leads to another definition of compactness based on the finite intersection property.

One useful consequence of this result is if  $(X, p)$  is compact metric space, and  $\{E_k\}_{k=1}^\infty \subseteq X$  closed, and  $E_{k+1} \subseteq E_k \forall k$ ,  $\cap_{k=1}^\infty E_k \neq \emptyset$ .

**Definition 1.2.2 (Totally Bounded,  $\epsilon$ -nets).**  $(X, p)$  is *totally bounded* if  $\forall \epsilon > 0$ , there is a finite cover of  $X$  of balls with radius  $\epsilon > 0$ .<sup>4</sup> If  $E \subseteq X$ , an  $\epsilon$ -net of  $E$  is a collection  $\{B(x_i, \epsilon)\}_{i=1}^N$  such that  $E \subseteq \bigcup_{i=1}^N B(x_i, \epsilon)$  and  $x_i \in X$  (note that

<sup>4</sup>Totally bounded implies  $(X, p)$  is bounded

$x_i$  need not be in  $E$ ).

**Definition 1.2.3 (Sequentially Compact).**  $(X, p)$  *sequentially compact* if every sequence in  $X$  has a convergent subsequence whose limit is in  $X$ .

**Definition 1.2.4 (Relatively/Pre-Compact).**  $E \subseteq X$  *precompact* if  $\bar{E}$  compact.

**Theorem 1.2.5.** TFAE:

1.  $X$  complete and totally bounded;
2.  $X$  compact;
3.  $X$  sequentially compact.

**Remark 1.2.5.** TFAE:

1.  $E$  is totally bdd and Cauchy Seq. converge
2.  $E$  is precompact
3.  $\forall \{x_k\}_{k=1}^{\infty} \subseteq E, \exists$  a convergent subsequence

Let  $f : (X, p) \rightarrow (\mathbb{R}, |\cdot|)$  continuous with  $(X, p)$  compact. Then,

- $f(X)$  compact in  $(\mathbb{R}, |\cdot|)$ ;
- The max and min of  $f$  over  $X$  are attained;
- $f$  is uniformly continuous.

**Lemma 1.2.6.** Any cauchy sequence<sup>5</sup> converges iff it has a convergent subsequence.

<sup>5</sup> $\forall \epsilon > 0, \exists N > 0$  s.t.  $\forall m, n > N, \|x_n - x_m\| < \epsilon$

**Proof.**

( $\Rightarrow$ ) If  $\{f_n\}_{n=1}^\infty$  converges, then  $\exists f : X \rightarrow \mathbb{R}$  s.t.  $\|f_n - f\|_\infty \rightarrow 0$ , so all subsequences also converge to  $f$ .

( $\Leftarrow$ ) Now assume  $\exists$  a subsequence  $\{f_{n_k}\}_{k=1}^\infty \subseteq C(X)$  s.t.  $\lim_{k \rightarrow \infty} f_{n_k} = f$  in  $C(X) \iff \|f_{n_k} - f\|_\infty \rightarrow 0$ .

Suppose for the purpose of contradiction that  $f_n \not\rightarrow f$ . Thus,  $\exists \epsilon > 0$ , and a subsequence  $\{f_{n_j}\}_{j=1}^\infty \subseteq C(X)$  s.t.  $\|f_{n_j} - f\|_\infty > \epsilon$  for every  $j \geq 1$ . Then,

$$\|f_{n_k} - f_{n_j}\|_\infty \geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

for  $k$  sufficiently large and for  $n_k, n_j$  large enough. But this violates  $\{f_n\}_{n=1}^\infty$  being cauchy. (Contradiction), so we must have  $f_n \rightarrow f$  in  $C(X)$ .  $\square$

Let  $C(X) := \{f : X \in \mathbb{R} \mid f \text{ continuous}\}$  and  $\|f\|_\infty := \max_{x \in X} |f(x)|$  the sup norm. Then,

**Proposition 1.2.7.** Let  $(X, p)$  compact. Then  $(C(X), \|\cdot\|_\infty)$  is complete.

**Proof.** let  $\{f_n\}_{n=1}^\infty \subseteq C(X)$  be Cauchy. Fix  $k \in \mathbb{N}$ . By Cauchy defn, let  $\epsilon = 2^{-k}$ , so  $\exists N_k$  sufficiently large s.t.  $\|f_{N_k} - f_{N_k+1}\|_\infty < 2^{-k}$ . We can then choose  $\{n_k\}_{k=1}^\infty$  s.t.  $n_k \rightarrow \infty$  and  $\|f_{n_k} - f_{n_{k+1}}\| < 2^{-k} \quad \forall k \in \mathbb{N}$ . Let  $j \in \mathbb{N}$ . Then

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell=k}^{k+j-1} 2^{-\ell} \leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow{k \rightarrow \infty} 0$$

In particular,  $\forall x \in X$  fixed, let  $c_k := f_{n_k}(x)$ . Then  $|c_{k+j} - c_k| \leq \|f_{n_{k+j}} - f_{n_k}\|_\infty \rightarrow 0 \quad \forall j \in \mathbb{N}$ . Thus  $\{c_k\}_{k=1}^\infty \subseteq \mathbb{R}$  is cauchy, so by completeness of

$\mathbb{R}, \exists \bar{c} \in \mathbb{R}$  s.t.  $\lim_{k \rightarrow \infty} c_k = \bar{c} =: f(x)$  Doing this  $\forall x \in X$ , we have

$$\begin{aligned} |f_{n_k}(x) - f(x)| &= \lim_{j \rightarrow \infty} |f_{n_k}(x) - f_{n_{k+j}}(x)| \\ &\leq \lim_{j \rightarrow \infty} \|f_{n_k} - f_{n_{k+j}}\|_\infty \\ &\leq \sum_{\ell=k}^{\infty} 2^{-\ell} \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$

$\Rightarrow \|f_{n_k} - f\|_\infty = \sup_{x \in X} |f_{n_k}(x) - f(x)| \xrightarrow{k \rightarrow \infty} 0$ , so  $f_{n_k} \rightarrow f$  in  $C(X)$ . Finally, by the lemma this implies  $f_n \rightarrow f$  in  $C(X)$ , so  $(C(X), \|\cdot\|_\infty)$  is complete.  $\square$

**Definition 1.2.8 (Density/Separability).** A set  $D \subseteq X$  is called *dense* in  $(X, p)$  if for every<sup>6</sup> nonempty open subset  $A \subseteq X$ ,  $D \cap A \neq \emptyset$ . We say that  $X$  is *separable* if there is a countable dense subset  $D \subseteq X$ .

<sup>6</sup>If  $A$  dense in  $X$ , then  $\overline{A}$  dense in  $X$

**Proposition 1.2.9.** If  $X$  compact, then  $X$  is separable

**Proof.** Since  $X$  is compact, it is totally bounded. Therefore, for  $n \in \mathbb{N}$ , there is some  $K_n$  and  $\{x_i^n\} \subseteq X$  such that  $X \subseteq \bigcup_{i=1}^{K_n} B(x_i^n, \frac{1}{n})$ . Then,  $D = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{K_n} \{x_i^n\}$  countable and dense in  $X$   $\square$

### 1.3 Arzelà-Ascoli

**Goal.** Given a sequence  $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$ , find suitable conditions for  $\{f_n\}$  to have a convergent subsequence in  $(C(X), \|\cdot\|_\infty)$ .

**Definition 1.3.1 (Equicontinuous).** A family  $\mathcal{F} \subseteq C(X)$  is called *equicontinuous* at  $x \in X$  if  $\forall \epsilon > 0$  there exists a  $\delta_x > 0$  such that if  $p(x, x') < \delta_x$  then

$|f(x) - f(x')| < \epsilon$  for every  $f \in \mathcal{F}$ .  $\mathcal{F}$  is *pointwise equicontinuous* on  $X$  if  $\mathcal{F}$  is equicontinuous at every point  $x \in X$ .<sup>7</sup>

<sup>7</sup>if  $|\mathcal{F}| < \infty$ , then  $\mathcal{F}$  is pointwise equicontinuous on  $X$ .

**Example 1.3.2.** Fix  $M > 0$ ,  $[a, b] \subseteq \mathbb{R}$ .  $\mathcal{F} := \{f \in C([a, b]) \cap C'((a, b)) \mid |f'| \leq M\}$ . By Mean Value Theorem,  $|f(x) - f(y)| \leq |f'(x^*)| |x - y| \leq M|x - y|$  for some  $x^* \in [x, y]$ , so  $\forall x \in [a, b]$  if  $|x - y| < \frac{\epsilon}{M}$  then  $|f(x) - f(y)| < \epsilon$ ,  $\forall f \in \mathcal{F}$ , therefore  $\mathcal{F}$  is pointwise equicontinuous on  $[a, b]$ .

**Example 1.3.3.** Consider  $f_n(x) := x^n$  on  $[0, 1]$ . Then  $\{f_n\}_{n=1}^\infty$  is non equicontinuous at  $x = 1$ .  $f_n(1) = 1 \forall n$ , but the threshold to be close to  $f_n(1)$  is not uniform on  $n$ .

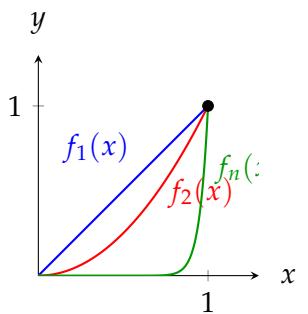
**Definition 1.3.4 (Pointwise, Uniform Boundedness).**  $\{f_n\}$  *pointwise bounded* if  $\forall x \in X, \exists M(x) > 0$  such that  $|f_n(x)| \leq M(x) \forall n$ , and *uniformly bounded* if such an  $M$  exists independent of  $X$ .

**Definition 1.3.5 (Uniform Equicontinuous).**  $\mathcal{F} \subseteq C(X)$  is *uniformly equicontinuous* on  $X$  if  $\forall \epsilon > 0 \exists \delta > 0$  s.t.  $\forall x, y \in X$  if  $p(x, y) < \delta$ , then  $|f(x) - f(y)| < \epsilon$ ,  $\forall f \in \mathcal{F}$ .

**Remark 1.3.5.**  $\mathcal{F}$  equicontinuous at  $x \iff$  all  $f \in \mathcal{F}$  share the same modulus of continuity at  $x$ , i.e.  $\exists \omega_x$  s.t.  $|f(x) - f(y)| \leq \omega_x |x - y|$ ,  $\forall f \in \mathcal{F}$ .

**Proposition 1.3.6 (Sufficient Conditions for Uniform Equicontinuity).**

1.  $\mathcal{F} \subseteq C(X)$  is uniformly Lipschitz continuous, i.e.  $\exists M > 0$  s.t.  $|f(x) - f(y)| \leq M p(x, y) \forall f \in \mathcal{F}$ ;



**Figure 1:** The sequence  $f_n(x) = x^n$  is not equicontinuous.

2.  $\mathcal{F} \subseteq C(X) \cap C^1(X)$  has a uniform  $L^\infty$  bound on the 1st derivative (same as earlier example, by MVT);
3. If  $(X, p)$  is compact and  $\mathcal{F} \subseteq C(X)$  is pointwise equicontinuous on  $X$   
 $\Rightarrow \mathcal{F}$  is uniformly equicontinuous (Homework).

**Lemma 1.3.7 (Arzelà-Ascoli Lemma).** Let  $X$  be separable and let  $\{f_n\}_{n=1}^\infty \subseteq C(X)$  be pointwise bounded and equicontinuous. Then, there is a function  $f \subseteq C(X)$  and a subsequence  $\{f_{n_k}\}_{k=1}^\infty$  which converges pointwise to  $f$  on all of  $X$ .

**Proof.** Let  $D = \{x_j\}_{j=1}^\infty \subseteq X$  be a countable dense subset of  $X$ . Since  $\{f_n\}$  is pointwise bounded,  $\{f_n(x_1)\}$  as a sequence of real numbers is bounded and so by Bolzano-Weierstrass, there is a convergent subsequence  $\{f_{n(1,k)}(x_1)\}_k$  that converges to some  $a_1 \in \mathbb{R}$ . Consider now  $\{f_{n(1,k)}(x_2)\}_k$ , which is again a bounded sequence of  $\mathbb{R}$  and so has a convergent subsequence, call it  $\{f_{n(2,k)}(x_2)\}_k$ , which converges to some  $a_2 \in \mathbb{R}$ . Note that  $\{f_{n(2,k)}\} \subseteq \{f_{n(1,k)}\}$ , so also  $f_{n(2,k)}(x_1) \rightarrow a_1$  as  $k \rightarrow \infty$ . We can repeat this procedure, producing a sequence of real numbers  $\{a_\ell\}$ , and for each  $j \in \mathbb{N}$  a subsequence  $\{f_{n(j,k)}\}_k \subseteq \{f_n\}$  such that  $f_{n(j,k)}(x_\ell) \rightarrow a_\ell$  for each  $1 \leq \ell \leq j$ . Define then

$$f : D \rightarrow \mathbb{R}, \quad f(x_j) := a_j$$

Consider now

$$f_{n_k} := f_{n(k,k)}, \quad k \geq 1$$

the "diagonal sequence", and remark that  $f_{n_k}(x_j) \rightarrow a_j = f(x_j)$  as  $k \rightarrow \infty$  for every  $j \geq 1$ . Hence,  $\{f_{n_k}\}_k$  converges to  $f$  on  $D$ , pointwise.

We claim now that  $\{f_{n_k}\}_k$  converges on all of  $X$  to some function  $f : X \rightarrow \mathbb{R}$ , pointwise. Put  $g_k := f_{n_k}$  for notational convenience. Fix  $x_0 \in X, \epsilon > 0$ , and let  $\delta_{x_0} > 0$  be such that if  $x \in X$  such that  $p(x, x_0) < \delta_{x_0}$ ,  $|g_k(x) - g_k(x_0)| < \frac{\epsilon}{3}$ .

Since  $D$  is dense in  $X$ ,  $\exists x_j \in D$  s.t.  $p(x_j, x_0) < \delta_{x_0}$ . Since  $\{g_k(x_j)\}_k$  converges, it is thus Cauchy, and hence for every  $k, \ell \geq K$ ,  $|g_k(x_j) - g_\ell(x_j)| < \frac{\epsilon}{3}$ . Therefore,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \epsilon$$

And thus  $\{g_k(x_0)\}_k$  Cauchy as a sequence in  $\mathbb{R}$ . Since  $\mathbb{R}$  is complete, then  $\{g_k(x_0)\}_k$  also converges, to, say,  $f(x_0) \in \mathbb{R}$ . Since  $x_0$  was arbitrary, this means there is some function  $f : X \rightarrow \mathbb{R}$  such that  $g_k \rightarrow f$  pointwise on  $X$  as we aimed to show.  $\square$

**Theorem 1.3.8 (Arzelà-Ascoli Theorem).** Let  $X$  be compact and let  $\{f_n\}_{n=1}^\infty \subseteq C(X)$  be uniformly bounded and uniformly equicontinuous. Then,  $\exists$  subseq  $\{f_{n_k}\}_{k=1}^\infty$  and  $f \in C(X)$  s.t.  $f_{n_k} \xrightarrow{k \rightarrow \infty} f$  in  $C(X)$  (i.e. uniformly)

**Proof.** Since  $(X, p)$  is compact, it is thus separable. Also, uniform bounded/equicontinuous implies pointwise bounded/equicontinuous. Therefore, by Arzelà-Ascoli lemma,  $\exists f : X \rightarrow \mathbb{R}$  and  $\{f_{n_k}\}_{k=1}^\infty$  s.t.  $f_{n_k} \rightarrow f$  pointwise in  $X$ .

Now let  $g_k := f_{n_k}$ .

**Claim:**  $\{g_k\}_{k=1}^\infty$  is uniformly Cauchy.<sup>8</sup>

Fix  $\epsilon > 0$ . By uniform equicontinuity,  $\exists \delta > 0$  s.t.

$$p(x, y) < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3} \quad \forall n \in \mathbb{N}.$$

Letting  $n = n_k$ ,

$$p(x, y) < \delta \implies |g_k(x) - g_k(y)| < \epsilon \quad \forall k \in \mathbb{N}.$$

Since  $X$  is compact, it is totally bounded, so  $\exists \{x_i\}_{i=1}^N$  s.t.  $X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$ .

Moreover,  $\forall 1 \leq i \leq N$  fixed, we know  $\{g_k(x_i)\}_{k=1}^\infty \subseteq \mathbb{R}$  converges because  $\{g_k\}_{k=1}^\infty$  converges pointwise, so  $\{g_k(x_i)\}_{k=1}^\infty$  is a Cauchy sequence. So  $\exists K_i > 0$

<sup>8</sup>Cauchy sequence in  $(C(X), \|\cdot\|_\infty)$

s.t.  $\forall k, \ell \geq K_i$ ,

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3}.$$

Let  $K := \max_{1 \leq i \leq N} K_i$ . Then,  $\forall k, \ell \geq K$ , we have

$$|g_k(x_i) - g_\ell(x_i)| \leq \frac{\epsilon}{3} \quad \forall 1 \leq i \leq N.$$

So  $\forall x \in X \subseteq \bigcup_{i=1}^N B^p(x_i, \delta)$ ,  $\exists x_i$  s.t.  $p(x, x_i) < \delta$ , and  $\forall k, \ell > K$ ,

$$|g_k(x) - g_\ell(x)| \leq |g_k(x) - g_k(x_i)| + |g_k(x_i) - g_\ell(x_i)| + |g_\ell(x_i) - g_\ell(x)| < \epsilon.$$

This implies  $\forall \epsilon > 0, \exists K > 0$  s.t.  $\forall k, \ell > K$ ,

$$\|g_k - g_\ell\|_\infty = \sup_{x \in X} |g_k(x) - g_\ell(x)| < \epsilon,$$

so  $\{g_k\}_{k=1}^\infty$  is uniformly Cauchy. Since  $(X, p)$  is compact,  $C(X)$  is complete, so  $\{g_k\}_{k=1}^\infty = \{f_{n_k}\}_{k=1}^\infty$  converges uniformly. Since  $f_{n_k} \rightarrow f$  pointwise in  $X$ , it must be that  $f_{n_k} \rightarrow f$  uniformly, and thus  $f \in C(X)$ .  $\square$

**Remark 1.3.8.** How do we use the AA theorem? To extract convergent subsequence, which may give us convergence of the original sequence.

**Fact.** Let  $\{f_n\}_{n=1}^\infty \subseteq C(X)$ . If  $\exists! f$  s.t. for every subsequence,  $\exists$  a further subsequence  $\{f_{n_{k_j}}\}_{j=1}^\infty$  s.t.  $f_{n_{k_j}} \xrightarrow{j \rightarrow \infty} f$  uniformly, then  $f_n \xrightarrow{n \rightarrow \infty} f$  uniformly.

### Example 1.3.9 (Typical Applications of Arzelà-Ascoli).

- Verify  $\{f_n\}$  satisfies hypothesis of AA;
- For every subseq  $\{f_{n_k}\}$  also satisfies hypothesis of AA;
- Use AA to extract  $\{f_{n_{k_j}}\}_{j=1}^\infty$  s.t.  $f_{n_{k_j}} \rightarrow f$  uniformly on  $X$ .
- If you can show  $f$  is unique, then  $f_n \rightarrow f$  in  $C(X)$ .

**Corollary 1.3.10.** Let  $(X, p)$  be a compact metric space. Let  $\mathcal{F} \subseteq C(X)$  be uniformly bounded and uniformly equicontinuous. Then,  $\mathcal{F}$  is precompact in  $(C(X), \|\cdot\|_\infty)$ .

**Proof.** If  $f$  is uniformly bounded and uniformly equicontinuous, then by the AA theorem,  $\forall$  sequence  $\{f_n\}_{n=1}^\infty \subseteq \mathcal{F}$ , there is a subseq.  $\{f_{n_j}\}_{j=1}^\infty$  and  $f \in C(X)$  s.t.  $f_{n_j} \rightarrow f$  in  $C(X)$ . Note,  $f$  may not be in  $\mathcal{F}$ . So  $\mathcal{F}$  is precompact.  $\square$

**Example 1.3.11.** Let  $M > 0$ , and define

$$\mathcal{F} = \left\{ f \in C([a, b]) \cap C^1([a, b]) : \|f\|_\infty + \|f'\|_\infty < M \right\}.$$

$\mathcal{F}$  is uniformly bounded and uniformly equicontinuous.

So by AA then, for  $\{f_n\} \subseteq \mathcal{F}, \exists \{f_{n_k}\}_{k=1}^\infty$  s.t.  $f_{n_k} \rightarrow f$  uniformly. But,  $f$  may not be in  $C^1([a, b])$ . (So,  $f$  may not be in  $\mathcal{F}$ )

Extra stuff left in the assignment, go back and look at it.

## 1.4 Baire Category Theorem

**Definition 1.4.1 (Hollow/Nowhere Dense).** We say a set  $E$  is *hollow* if  $\text{Int}(E) = \emptyset$ .<sup>9</sup> We say  $E \subseteq X$  *nowhere dense* if its closure is hollow, i.e.  $\text{Int}(\overline{E}) = \emptyset$ .

<sup>9</sup>i.e.  $E$  contains no nontrivial open sets

**Remark 1.4.1.**  $E$  hollow  $\iff E^c$  dense in  $X$ , since  $\text{Int}(E) = \emptyset \iff (\text{Int}(E))^c = \overline{E^c} = X$ .

**Goal.** When can we guarantee that

- a union of hollow sets is hollow?

- an intersection of dense sets is dense?

**Theorem 1.4.2 (Baire Category Theorem).** Let  $(X, p)$  be a complete metric space.

1. Let  $\{F_n\}_{n=1}^{\infty} \subseteq X$  be a collection of closed hollow sets. Then  $\bigcup_{n=1}^{\infty} F_n$  is hollow.
2. Let  $\{\mathcal{O}_n\}_{n=1}^{\infty} \subseteq X$  be a collection of open dense sets. Then  $\bigcap_{n=1}^{\infty} \mathcal{O}_n$  is dense.

**Proof.** (2)  $\Rightarrow$  (1) by taking complements and using the previous remark, so we prove only (2).

**Claim:** Let  $G = \bigcap_{n=1}^{\infty} \mathcal{O}_n$ . Then  $G$  is dense in  $X$ .

Fix  $x \in X, r > 0$ .  $\forall n \in \mathbb{N}, \mathcal{O}_n$  is open and dense, so  $\exists y \in \mathcal{O}_n$  and  $s > 0$  s.t.

$$B(x, r) \cap \mathcal{O}_n \supseteq B(y, 2s) \supseteq \overline{B(y, s)}.$$

Now we use this fact inductively in  $n$ . Let  $x_1 \in X, r_1 < \frac{1}{2}$  s.t.  $\overline{B(x_1, r_1)} \subseteq B(x, r) \cap \mathcal{O}_1$ . Let  $x_2 \in X, r_2 < 2^{-2}$  s.t.  $\overline{B(x_2, r_2)} \subseteq B(x_1, r_1) \cap \mathcal{O}_2$ . Repeating this process, take  $x_n \in X, r_n < 2^{-n}$  s.t.  $\overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \cap \mathcal{O}_n$ .

$$\Rightarrow \overline{B(x_1, r_1)} \supseteq \overline{B(x_2, r_2)} \supseteq \dots \supseteq \overline{B(x_n, r_n)} \supseteq \dots,$$

and  $r_n \rightarrow 0$ . Therefore  $\{x_n\}_{n=1}^{\infty}$  is Cauchy, and  $(X, p)$  is complete, so  $\exists x_0 \in X$  s.t.  $x_n \rightarrow x_0$ . Thus,

$$x_0 = \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)}.$$

Since  $x_0 \in \overline{B(x_n, r_n)} \subseteq \mathcal{O}_n \forall n$ , and  $x_0 \in \overline{B(x_1, r_1)} \subseteq B(x, r) \Rightarrow x_0 \in G \cap B(x, r)$ .

$$\Rightarrow G \cap B(x, r) \neq \emptyset \quad \forall x \in X, \forall r > 0.$$

$\Rightarrow G$  is dense in  $X$ . □

Another restatement of the Baire Category Theorem is as follows: If  $(X, p)$  is complete, the countable union of nowhere dense sets is hollow.

**Proof.** Let  $\{E_n\}_{n=1}^{\infty}$  be nowhere dense sets. Then by BCT,  $\cup_{n=1}^{\infty} \overline{E_n}$  is hollow. It follows that  $\cup_{n=1}^{\infty} E_n \subseteq \cup_{n=1}^{\infty} \overline{E_n}$  so  $\cup_{n=1}^{\infty} E_n$  is also hollow.  $\square$

The main way we will use the Baire Category Theorem is the following:

**Corollary 1.4.3.** Let  $(X, p)$  be complete. Suppose  $\{F_n\}_{n=1}^{\infty}$  is a collection of closed sets. If  $X = \cup_{n=1}^{\infty} F_n$ , then  $\exists n_0$  s.t.  $\text{Int}(F_{n_0}) \neq \emptyset$ .

**Proof.** If  $\nexists n_0$ , then  $F_n$  is hollow  $\forall n$ , so by BCT  $X = \cup_{n=1}^{\infty} F_n$  is hollow, but this is a contradiction because  $X \subseteq X$  is open and nontrivial.  $\square$

**Theorem 1.4.4.** Let  $X \subseteq C(X)$  where  $(X, p)$  is complete. Suppose  $\mathcal{F}$  is pointwise bounded. Then,  $\exists$  non-empty open set  $\mathcal{O} \subseteq X$  s.t.  $\mathcal{F}$  is uniformly bounded on  $\mathcal{O}$ , i.e.  $\exists M > 0$  s.t.

$$\sup_{f \in \mathcal{F}} \sup_{x \in \mathcal{O}} |f(x)| \leq M$$

**Proof.** Let

$$E_n = \{x \in X : |f(x)| \leq n, \forall f \in \mathcal{F}\} = \bigcap_{f \in \mathcal{F}} \{x \in X : |f(x)| \leq n\}.$$

$\Rightarrow E_n$  is closed  $\forall n$ . Since  $\mathcal{F}$  is pointwise bounded,

$$\forall x \in X, \exists M_x > 0 \text{ s.t. } \sup_{f \in \mathcal{F}} |f(x)| \leq M_x.$$

Thus,  $\forall n$  s.t.  $M_x \leq n$ , then  $x \in E_n$  (since  $|f(x)| \leq M_x \leq n$ ).

So,  $X = \bigcup_{n=1}^{\infty} E_n$  and  $E_n$  is closed. By corollary,  $\exists n_0$  s.t.  $\text{Int}(E_{n_0}) \neq \emptyset$ . So  $\exists x_0 \in X, r$  s.t.  $B^p(x_0, r) \subseteq E_{n_0}$ . Letting  $\mathcal{O} = B^p(x_0, r)$ , we have

$$\sup_{x \in \mathcal{O}} |f(x)| \leq n_0 \quad \forall f \in \mathcal{F}. \quad \square$$

**Corollary 1.4.5.** Let  $(X, p)$  be a complete metric space. Suppose  $\{F_n\}_{n=1}^{\infty}$  is a collection of closed sets. Then  $\bigcup_{n=1}^{\infty} \partial F_n$  is hollow.

**Proof.** **Claim:**  $\partial F_n$  is hollow  $\forall n$ . Suppose for contradiction that  $\exists n$  s.t.  $\text{Int}(\partial F_n) \neq \emptyset$ . Then  $\exists x_0 \in \partial F_n, r > 0$  s.t.  $B^p(x_0, r) \subseteq \partial F_n$ . But then,

$$B^p(x_0, r) \cap F_n^c = B^p(x_0, r) \cap \overline{F_n^c} = B^p(x_0, r) \cap (F_n \cup \partial F_n)^c = B^p(x_0, r) \cap \partial F_n^c \cap F_n^c = \emptyset$$

and this contradicts  $x_0 \in \partial F_n$  by defn.  $\Rightarrow \partial F_n$  is hollow  $\forall n$ . Furthermore,  $\partial F_n$  is closed, since it contains all of its limit points by definition. Thus, by BCT,  $\bigcup_{n=1}^{\infty} \partial F_n$  is hollow.  $\square$

Now recall that in general,  $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$  and  $f_n \rightarrow f$  pointwise, then  $f$  is not necessarily continuous.

**Theorem 1.4.6.** Let  $(X, p)$  be complete. Let  $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$  s.t.  $f_n \rightarrow f$  pointwise in  $X$ . Then there is a dense subset  $D \subseteq X$  where  $\{f_n\}_{n=1}^{\infty}$  is pointwise equicontinuous on  $D$  and  $\forall x_0 \in D, f$  is continuous at  $x_0$ .

**Proof.** Let  $m, n \in \mathbb{N}$ . Define

$$\begin{aligned} E(m, n) &= \left\{ x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \ \forall j, k \geq n \right\} \\ &= \underbrace{\bigcap_{j, k \geq n} \left\{ x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \right\}}_{\text{closed, since } f_k, f_j \in C(X)}. \end{aligned}$$

So  $E(m, n)$  is closed  $\forall m, n$ . Thus, by the corollary,  $\bigcup_{m, n \in \mathbb{N}} \partial E(m, n)$  is hollow. This implies that

$$D := \left( \bigcup_{m, n \in \mathbb{N}} \partial E(m, n) \right)^c = \bigcap_{m, n \in \mathbb{N}} \partial E(m, n)^c \quad \text{is dense.}$$

**Claim 1:** If  $\exists x \in X, m, n \in \mathbb{N}$  s.t.  $x \in D \cap E(m, n)$ , then  $x \in \text{Int}(E(m, n))$ .

If  $x \in D$ , then

$$x \in \underbrace{\partial E(m, n)^c}_{\text{open}} = \text{Int}(E(m, n)) \cup \text{Ext}(E(m, n)).$$

For the exterior term:

$$\begin{aligned} \text{Ext}(E(m, n)) &= X \setminus (\text{Int}(E(m, n)) \cup \partial E(m, n)) \\ &= X \setminus E(m, n) = E(m, n)^c. \end{aligned}$$

Since we also have  $x \in E(m, n)$ , this means  $x \in \text{Int}(E(m, n))$ .

**Claim 2:**  $\{f_n\}_{n=1}^{\infty}$  is equicontinuous on  $D$ .

Let  $x_0 \in D$  and  $\epsilon > 0$ . Choose  $m$  s.t.  $\frac{1}{m} < \frac{\epsilon}{4}$ . Since  $\{f_n\}_{n=1}^{\infty}$  converges,  $\{f_n(x_0)\}_{n=1}^{\infty} \subseteq \mathbb{R}$  is a Cauchy sequence. So  $\exists N$  s.t.  $\forall j, k \geq N$ ,

$$|f_j(x_0) - f_k(x_0)| \leq \frac{1}{m}.$$

This means  $x_0 \in E(m, n) \cap D$ , so by Claim 1,  $x_0 \in \text{Int}(E(m, n))$ . Let  $B^p(x_0, r) \subseteq$

$E(m, N)$ , so  $\forall j, k \geq N, \forall x \in B(x_0, r)$ ,

$$|f_j(x) - f_k(x)| \leq \frac{1}{m}.$$

Since  $f_N$  is continuous at  $x_0$ ,  $\exists \delta_{x_0} > 0$  (which WLOG we can choose  $< r$ ), s.t.  $\forall x \in B^p(x_0, \delta_{x_0})$ ,

$$|f_N(x) - f_N(x_0)| \leq \frac{1}{m}.$$

So  $\forall j \geq N, \forall x \in B^p(x_0, \delta_{x_0})$ ,

$$\begin{aligned} |f_j(x) - f_j(x_0)| &\leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_j(x_0)| \\ &\leq \frac{3}{m} \leq \frac{3\epsilon}{4}. \end{aligned}$$

Since this holds  $\forall j \geq N$ , this implies that  $\{f_n\}_{n=1}^\infty$  is equicontinuous at  $x_0$ . Furthermore,  $\forall x \in B^p(x_0, \delta_{x_0})$ , sending  $j \rightarrow \infty$ , we obtain that  $\forall x \in B^p(x_0, \delta_{x_0})$ ,  $|f(x) - f(x_0)| \leq \frac{3\epsilon}{4}$ , so  $f$  is continuous at  $x_0 \in D$ .  $\square$

## 1.5 Topological Spaces

We'll consider topological spaces, where we will define all concepts using open sets, and we will generalize what we have learned from Metric Spaces.

**Definition 1.5.1.** Let  $X$  be a non empty set. A *topology*  $\mathcal{T}$  on  $X$  is a collection of subsets of  $X$ , such that

- $X, \emptyset \in \mathcal{T}$ ;
- If  $\{E_n\} \subseteq \mathcal{T}, \bigcap_{n=1}^N E_n \in \mathcal{T}$  (closed under finite intersections);
- If  $\{E_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{T}, \bigcup_{\lambda \in \Lambda} E_\lambda \in \mathcal{T}$  (closed under arbitrary unions).

We say  $(X, \mathcal{T})$  is a *topological space*.

If  $E \in \mathcal{T}$ , then we call  $E$  an open set (with respect to  $\mathcal{T}$ ).

If  $x \in X$ , a set  $E \in \mathcal{T}$  containing  $x$  is called a *neighborhood* of  $x$ .

**Remark 1.5.1.** By definition of  $\mathcal{T}$ ,  $E \in \mathcal{T}$  iff  $\forall x \in E, \exists$  a neighbourhood of  $x$ , contained in  $E$ . (consistent with metric space definition of open set)

**Example 1.5.2 (Metric topology).** Let  $(X, p)$  be a metric space. Define

$$\mathcal{T} := \{\text{open sets w.r.t. } p\}.$$

Then,  $\mathcal{T}$  is a topology on  $X$ , called the metric topology induced by  $p$ .

Given a topology  $\mathcal{T}$ , if  $\exists$  a metric  $p$  s.t.  $\mathcal{T}$  is the metric topology induced by  $p$ , then we say  $\mathcal{T}$  is *metrizable*.

**Example 1.5.3 (Trivial Topology).** Let  $X$  be a non empty set. Define

$$\mathcal{T} = \{\emptyset, X\}.$$

Then,  $\mathcal{T}$  is a topology on  $X$ , called the trivial topology.

**Example 1.5.4 (Discrete Topology).** Let  $X$  be a non empty set. Let  $p(x, y)$  be the discrete metric on  $X$ . Define Then

$$B^p(x_0, r) = \begin{cases} \{x_0\}, & 0 < r \leq 1 \\ X, & r > 1 \end{cases}$$

So  $\forall E \subseteq X, \forall x \in E, B^p(x, \frac{1}{2}) = \{x\} \subseteq E \Rightarrow E$  is open. Then,

$\mathcal{T} = \mathcal{P}(X) = \{\text{All possible subsets of } X\}$  is a topology on  $X$ , called the discrete topology, and it contains all subsets of  $X$ .

**Example 1.5.5 (Relative Topology).** Let  $(X, \mathcal{T})$  be a topological space. Let  $Y \subseteq X$ . Then,

$$\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}.$$

Then,  $\mathcal{T}_Y$  is a topology on  $Y$ , called the relative topology on  $Y$  induced by  $\mathcal{T}$ . If  $X = \mathbb{R}$ ,  $Y = \mathbb{N}$ , then  $\mathcal{T}_{\mathbb{N}} = \{U \cap \mathbb{N} : U \subseteq \mathbb{R} \text{ open}\}$  So  $\forall y \in \mathbb{N}, \forall x \in \mathbb{R}, r > 0$ ,

$$B(x, r) \cap \mathbb{N} = \begin{cases} \{y\}, & y \in B(x, r) \\ \emptyset, & y \notin B(x, r) \end{cases}$$

Thus,  $\mathcal{T}_{\mathbb{N}} = \mathcal{P}(\mathbb{N})$ , the discrete topology on  $\mathbb{N}$ .

If  $X = \mathbb{R}$ ,  $Y = [0, 1)$ , then  $\mathcal{T}_{[0,1)} = \{U \cap [0, 1) : U \subseteq \mathbb{R} \text{ open}\}$  So the set  $[0, \frac{1}{2}) = [0, 1) \cap (-1, \frac{1}{2}) \in \mathcal{T}_{[0,1)}$ . So  $[0, \frac{1}{2})$  is relatively open in  $Y = [0, 1)$  (belongs to the relative topology on  $Y$ ).

In metric spaces, everything is done using balls. In a generic topological space  $(X, \mathcal{T})$ , what plays the role of balls?

**Definition 1.5.6 (base/neighbourhood base).** Let  $(X, \mathcal{T})$  Topological space. Fix  $x \in X$ . Let  $\mathcal{B}_x$  be a collection of neighborhoods of  $x$ . We call  $\mathcal{B}_x$  a *neighbourhood base* at  $x$  if  $\forall$  neighborhood of  $x$  (call it  $U_x$ ),  $\exists B \in \mathcal{B}_x$  such that  $B \subseteq U_x$ . We say  $\mathcal{B}$ , a collection of open sets, is a *base* for  $\mathcal{T}$  if  $\forall x \in X, \exists$  a neighbourhood base  $\mathcal{B}_x \subseteq \mathcal{B}$  at  $x$ .

**Example 1.5.7.** In  $(X, p)$  a metric space,  $\forall x \in X$

$$\mathcal{B}_x = \{B^p(x, r) : r > 0\} \text{ is a neighbourhood base}$$

$$B = \{\text{all balls of all radii}\}$$

**Remark 1.5.7.** Given a topology, a neighbourhood base is not unique.

$$\mathcal{B}_x = \{B^p(x, \frac{1}{n})\}_{n=1}^{\infty}$$

is also a neighbourhood base at  $x$  in a metric space.

**Definition 1.5.8 (first countable/second countable).** Let  $(X, \mathcal{T})$  be a topological space.

- We say  $(X, \mathcal{T})$  is *first countable* if there is a countable neighbourhood base at each  $x \in X$ ;
- We say  $(X, \mathcal{T})$  is *second countable* if there is a countable base  $B$  of  $\mathcal{T}$ .

**Remark 1.5.8.** Any metric space is first countable, and any separable metric space is second countable.

**Remark 1.5.8.** For a topology  $\mathcal{T}$ ,  $\mathcal{B} = \mathcal{T}$  is always a base for  $\mathcal{T}$  (so a base always exists).

**Proposition 1.5.9.** If  $(X, \mathcal{T})$  be a topological space. A collection of open sets  $\mathcal{B}$  is a base for  $\mathcal{T}$  iff every non-empty open set  $U \in \mathcal{T}$  can be written as a union of elements of  $\mathcal{B}$ .

**Proof.**

$(\Rightarrow)$  Suppose  $\mathcal{B}$  is a base for  $\mathcal{T}$ . Let  $U \in \mathcal{T}$ . Then  $\forall x \in U, \exists B_x \in \mathcal{B}_x \subseteq \mathcal{B}$  such that

$$x \in B_x \subseteq U \Rightarrow \bigcup_{x \in U} B_x \subseteq U$$

and

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \Rightarrow U = \bigcup_{x \in U} B_x.$$

$(\Leftarrow)$  Suppose every non-empty open set  $U \in \mathcal{T}$  can be written as a union of elements of  $\mathcal{B}$ . Fix  $x \in U$ , and let

$$\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\} = \{B \in \mathcal{B} : \{x\} \cap B \neq \emptyset\} \subseteq \mathcal{B}.$$

Since  $U = \bigcup B$ , this means  $\mathcal{B}_x \neq \emptyset$ . So  $U$  is a neighbourhood of  $x$ , and  $\exists B \in \mathcal{B}_x$  such that  $B \subseteq U \Rightarrow \mathcal{B}_X$  is a neighbourhood base at  $x$ . Doing that  $\forall x \in X$ , we get a  $\mathcal{B}$  that is a base for  $\mathcal{T}$ .  $\square$

**Question.** Given a collection  $\mathcal{B}$ , what does it take to be a base for some topology?

**Proposition 1.5.10.** Let  $X \neq \emptyset$ . Let  $\mathcal{B} \subseteq \mathcal{P}(X)$  be a collection of sets. Then  $\mathcal{B}$  is a base for some topology  $\mathcal{T}$  iff

1.  $X = \bigcup_{B \in \mathcal{B}} B$ ;
2.  $\forall B_1, B_2 \in \mathcal{B}$ , and  $x \in B_1 \cap B_2$ , then  $\exists B \in \mathcal{B}$  such that  $x \in B \subseteq B_1 \cap B_2$ .<sup>10</sup>

<sup>10</sup>For balls, this is true

**Proof.**

$\Rightarrow$   $\mathcal{B}$  is a base for a topology  $\mathcal{T}$ . Then, since  $X \in \mathcal{T}$ , by the last result,  $X = \bigcup_{B \in \mathcal{B}} B$ , so (1) holds. Moreover, if  $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{T}$ , then  $B_1 \cap B_2 \in \mathcal{T}$ . So for  $x \in B_1 \cap B_2$ , then  $B_1 \cap B_2$  is a neighbourhood of  $x$ . Since  $\mathcal{B}$  is a base,  $\exists B \in \mathcal{B}$  s.t.  $x \in B \subseteq B_1 \cap B_2$ , so (2) holds.

$\Leftarrow$  Suppose (1) and (2) hold. Let

$$\mathcal{T} := \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}.$$

Since  $X = \bigcup_{B \in \mathcal{B}} B$ , so  $\forall x \in X, \exists B \in \mathcal{B}$  s.t.  $x \in B \subseteq X \Rightarrow x \in \mathcal{T}$ . Similarly,  $\emptyset \in \mathcal{T}$  because the condition is empty. The definition of  $\mathcal{T}$  shows us that it is closed under arbitrary unions.

Let  $U_1, U_2 \in \mathcal{T}$ , and assume  $U_1 \cap U_2 \neq \emptyset$ , so  $\forall x \in U_1 \cap U_2$ , by definition of  $\mathcal{T}$ ,  $\exists B_1 \in \mathcal{B}$  s.t.  $x \in B_1 \subseteq U_1$ , and  $\exists B_2 \in \mathcal{B}$  s.t.  $x \in B_2 \subseteq U_2 \Rightarrow x \in B_1 \cap B_2$ . By (2),  $\exists B \in \mathcal{B}$  s.t.  $x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$ . Thus,  $B_1 \cap B_2 \in \mathcal{T}$ . Inductively, we conclude  $\mathcal{T}$  is closed under finite intersections.  $\square$

Observe that the properties which define  $\mathcal{T}$  are closed under intersections, so we may define a  $\sigma$ -algebra like structure for topologies:

**Definition 1.5.11.** Let  $\mathcal{E} \subseteq \mathcal{P}(X)$  be a collection of sets. Then

$$\mathcal{T}(\mathcal{E}) = \bigcap \{\text{All topologies containing } \mathcal{E}\} = \text{topology generated by } \mathcal{E}$$

**Definition 1.5.12 (weaker/coarser vs. stronger/finer).** Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be topologies on  $X$ . If  $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ , we say  $\mathcal{T}_1$  is a *weaker/coarser* topology than  $\mathcal{T}_2$  (fewer open sets), and  $\mathcal{T}_2$  is a *stronger/finer* topology than  $\mathcal{T}_1$  (more open sets).

**Example 1.5.13.** Trivial topology is the weakest topology on  $X$  and discrete topology is the strongest topology on  $X$ . So  $\mathcal{T}(\mathcal{E})$  is the weakest topology containing  $\mathcal{E}$ .

**Proposition 1.5.14.** Let  $\mathcal{E} \subseteq \mathcal{P}(X)$ . Then

$$\mathcal{T}(\mathcal{E}) = \left\{ \emptyset, X, \bigcup \{ \text{finite intersections of sets in } \mathcal{E} \} \right\}$$

**Proof. Claim:**  $\mathcal{B} = \{\emptyset, X, \text{finite intersections of elements of } \mathcal{E}\}$  forms a base.  $\mathcal{B}$  satisfies (1) and (2) of the previous proposition, so  $\mathcal{B}$  is a base for some topology

$$\tilde{\mathcal{T}} = \left\{ \emptyset, X, \bigcup \{ \text{finite intersections of elements of } \mathcal{E} \} \right\}$$

Observe that  $\tilde{\mathcal{T}} \subseteq \{\text{any topology which contains } \mathcal{E}\} \Rightarrow \tilde{\mathcal{T}} \subseteq \mathcal{T}(\mathcal{E})$ .  $\tilde{\mathcal{T}}$  is also a topology, which contains  $\mathcal{E}$ . So  $\mathcal{T}(\mathcal{E}) \subseteq \tilde{\mathcal{T}}$ . Thus,  $\tilde{\mathcal{T}} = \mathcal{T}(\mathcal{E})$ .  $\square$

**Goal.** Topologies give us open sets, bases give ball-like sets, now we need a notion for closed sets.

**Definition 1.5.15 (limit point, closure, closed set).** If  $E \subseteq X, x \in X$  is a *limit point* if  $\forall$  neighbourhood  $U_x$  of  $x$ ,

$$U_x \cap E \neq \emptyset.$$

We say  $\bar{E} = \{\text{All limit points of } E\}$ , is the *closure* of  $E$ .

We say  $E$  is *closed* if  $E = \bar{E}$ .

**Remark 1.5.15.** We always have  $E \subseteq \bar{E}$ , so we just need  $\bar{E} \subseteq E$  to show  $E$  is closed.

**Proposition 1.5.16.** Let  $E \subseteq X$ .

1.  $\bar{E}$  is closed;
2.  $\bar{E}$  is the smallest closed set containing  $E$ , i.e. if  $\exists F$  closed s.t.  $E \subseteq F \Rightarrow \bar{E} \subseteq F$ ;
3.  $E$  is open iff  $E^c$  is closed.

**Proof of (1) + (2).**

**Claim:**  $L := \{\text{limit points of } \bar{E}\} = \bar{\bar{E}} \subseteq \bar{E}$ .

Let  $x \in L$ , and a neighbourhood  $U_x$  of  $x$ . Then by defn of  $L$ , we know  $\exists x' \in U_x \cap \bar{E}$ . This means  $x' \in \bar{E}$ , and  $U_x$  is a neighbourhood of  $x' \Rightarrow U_x \cap E \neq \emptyset$ . This holds  $\forall$  neighbourhood  $U_x$  of  $x$ , so  $x \in \bar{E} \Rightarrow L \subseteq \bar{E} \Rightarrow \bar{E}$  is closed.

Suppose  $E \subseteq F$  and  $F$  is closed. Let  $x \in \bar{E}$ , then  $\forall$  neighbourhood  $U_x$ ,  $U_x \cap E \neq \emptyset \Rightarrow U_x \cap F \neq \emptyset \Rightarrow x \in \bar{F} \Rightarrow \bar{E} \subseteq \bar{F} = F$ .  $\square$

**Proof of (3).**

( $\Rightarrow$ ) Let  $E \subseteq X$  be open. Let  $x \in \overline{E^c}$ .

**Claim:**  $x \in E^c$ .

Suppose not, so  $x \in E$ . So  $\exists$  neighbourhood  $U_x$  of  $x$  s.t.  $U_x \subseteq E \Rightarrow U_x \cap E = \emptyset$ . So  $x \notin \overline{E^c}$ . So,  $x \in \overline{E^c} \Rightarrow x \in E^c \Rightarrow \overline{E^c} \subseteq E^c \Rightarrow E^c$  closed.

( $\Leftarrow$ ) Let  $E^c$  be closed. Let  $x \in E$ .

**Claim:**  $\exists$  neighbourhood  $U_x$  s.t.  $U_x \subseteq E$ .

Suppose not, then every neighbourhood  $U_x$  we have  $U_x \cap E^c \neq \emptyset \Rightarrow x \in \overline{E^c} = E^c$  (contradicts  $x \in E$ ). By claim,  $E$  is open.  $\square$

**Remark 1.5.16.** Our proof shows,  $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$ .

**Definition 1.5.17 (Density, Separability).**  $D \subseteq X$  is *dense* if  $\forall$  non-empty open set  $U$ ,  $U \cap D \neq \emptyset \iff \overline{D} = X$ .

$(X, \mathcal{T})$  is *separable* iff  $X$  contains a countable dense set.

**Proposition 1.5.18.** Every second countable space is separable.

**Proof.** Let  $(X, \mathcal{T})$  be second countable, so  $\exists$  base  $\mathcal{B} = \{B_i\}_{i=1}^\infty$ . Pick  $x_i \in B_i$  (need axiom of choice). Let  $D = \{x_i\}_{i=1}^\infty$ . Then  $\forall U \subseteq X$  open, since  $\mathcal{B}$  is a base,  $\exists B_i \subseteq U$ ,

$$D \cap U \supseteq \{x_i\} \cap B_i \neq \emptyset$$

$\Rightarrow D \cap U \neq \emptyset$ , so  $D$  is dense.  $\square$

**Definition 1.5.19 (Convergence in Topology).** Given  $(X, \mathcal{T})$  a topological space, let  $\{x_n\}_{n=1}^\infty \subseteq X$ . We say  $x_n \rightarrow x$  in  $\mathcal{T}$  if  $\forall$  neighbourhood  $U_x$  of  $x$ ,  $\exists N$

s.t.  $\forall n \geq N, x_n \in U_x$ .

**Proposition 1.5.20.** Suppose  $(X, \mathcal{T})$  is first countable, and  $E \subseteq X$ . Then,  $x \in \bar{E}$  iff  $\exists \{x_n\} \subseteq E$  s.t.  $x_n \rightarrow x$  in  $\mathcal{T}$ .

**Proof.**

$\Rightarrow$ ) Let  $\mathcal{B}_x = \{B_j\}_{j=1}^{\infty}$  be a neighbourhood base at  $x \in \bar{E}$ . WLOG, can assume  $B_{j+1} \subseteq B_j \quad \forall j$ . Since  $x \in \bar{E}$  and  $B_j$  is a neighbourhood of  $x$ ,  $B_j \cap E \neq \emptyset \quad \forall j$ . Let  $x_i \in B_i \cap E$ . Then,  $\forall$  neighbourhood  $U_x$  of  $x$ ,  $\exists B_J \in \mathcal{B}_x$  s.t.  $B_J \subseteq U_x$ . But since  $\{B_j\}$  are nested,  $\forall j \geq J$ ,

$$U_x \supseteq B_j \cap U_x = B_j \supseteq B_i \cap E \supseteq \{x_j\}$$

$$\Rightarrow x_j \rightarrow x \text{ in } \mathcal{T}.$$

$\Leftarrow$ ) If  $\exists \{x_j\}_{j=1}^{\infty} \subseteq E$  s.t.  $x_j \rightarrow x$  in  $\mathcal{T}$ , suppose  $x \notin \bar{E}$ . Then  $x \in \bar{E}^c$  and  $\bar{E}^c$  open, so  $\bar{E}^c$  is a neighbourhood of  $x$  s.t.  $\{x_j\}_{j=1}^{\infty} \cap \bar{E}^c \neq \emptyset$ . Therefore  $x_j \not\rightarrow x$  in  $\mathcal{T}$ .  $\square$

## 1.6 Separation Properties

While  $(X, \mathcal{T})$  allows us to consider a very general framework, weird stuff can happen because of it, for example:

**Example 1.6.1.** Let  $\mathcal{T} = \{\emptyset, X\}$ . So the only non-empty neighbourhood is  $X$ , so any sequence  $\{x_n\}_{n=1}^{\infty} \subseteq X$  converges to any point  $x \in X$ .

To avoid cases like this, we require topologies with more structure.

**Definition 1.6.2 (neighbourhood of a set, Separating sets by disjoint neighbourhoods).** Let  $(X, \mathcal{T})$  be a topological space, and  $K, A, B \subseteq X$ . A neigh-

bourhood of  $K$  is an open set  $U$  s.t.  $K \subseteq U$ . We say  $A, B$  can be separated by disjoint neighbourhoods if  $\exists U \supseteq A, V \supseteq B$  neighbourhoods s.t.  $U \cap V = \emptyset$ .

**Definition 1.6.3 (Separation Notions).** Let  $(X, \mathcal{T})$  be a topological space.  $(X, \mathcal{T})$  is

1. *Tychonoff* (T1) if  $\forall x \neq y \in X, \exists$  neighbourhood  $U_x$  s.t.  $y \notin U_x$ , and  $\exists$  neighbourhood  $U_y$  s.t.  $x \notin U_y$ ;
2. *Hausdorff* (T2) if  $\forall x \neq y \in X, \{x\}, \{y\}$ , can be separated by disjoint neighbourhoods, i.e.  $\exists U_x \supseteq \{x\}, U_y \supseteq \{y\}$  s.t.  $U_x \cap U_y = \emptyset$ ;
3. *Regular* (T3) if  $(X, \mathcal{T})$  is Tychonoff and  $\forall x \in X, \forall F \subseteq X$  closed, with  $x \notin F, \{x\}$  and  $F$  can be separated by disjoint neighbourhoods;
4. *Normal* (T4) if  $(X, \mathcal{T})$  is Tychonoff and  $\forall A, B \subseteq X$  closed and disjoint,  $A$  and  $B$  can be separated by disjoint neighbourhoods.

**Remark 1.6.3.** Metric  $\subseteq$  Normal  $\subseteq$  Regular  $\subseteq$  Hausdorff  $\subseteq$  Tychonoff.

**Example 1.6.4.** Consider  $\mathbb{R}$  and  $\mathcal{T} = \{\emptyset, (-\infty, c) \text{ for } c \in \mathbb{R}\}$ . Then,  $\forall x \in \mathbb{R}$ , a neighbourhood of  $x$  is of the form  $(-\infty, c)$  for some  $c > x$ . Let  $x \neq y \in \mathbb{R}$ , WLOG assume  $x < y$ . Then  $x \in U_y \forall$  neighbourhood  $U_y$  of  $y$ . So  $(\mathbb{R}, \mathcal{T})$  is not Tychonoff.

**Example 1.6.5.** Let  $X = \mathbb{R}$  and let  $K := \{\frac{1}{n} : n \in \mathbb{Z}\}$ . Define the collection  $\mathcal{B}$  as:

$$\mathcal{B} = \{(a, b) : a < b\} \cup \{(a, b) \setminus K : a < b\}.$$

We verify the properties of this space:

1. **Basis Check:** Clearly,  $\mathbb{R} = \bigcup_{B \in \mathcal{B}} B$ . Now, suppose  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ . Since  $B_1$  and  $B_2$  are intersections of standard intervals with

either  $\mathbb{R}$  or  $\mathbb{R} \setminus K$ , their intersection is also of the form  $(a, b)$  or  $(a, b) \setminus K$ . Thus, there exists a  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ . Therefore,  $\mathcal{B}$  is a basis for a topology  $\mathcal{T}$  on  $\mathbb{R}$ , called the **K-topology**.

2. **Hausdorff ( $T_2$ )**: Suppose  $x, y \in X$  with  $x \neq y$ . Since the standard topology is Hausdorff, there exist standard disjoint intervals  $(a, b)$  and  $(c, d)$  separating  $x$  and  $y$ . These intervals are also in  $\mathcal{B}$ . Thus,  $U_x \cap U_y = \emptyset \implies (X, \mathcal{T})$  is Hausdorff.
3. **Not Regular ( $T_3$ )**: The set  $K$  is closed in  $X$  because its complement  $K^c = \mathbb{R} \setminus K$  is open (every point in  $K^c$ , including 0, has a neighborhood disjoint from  $K$ ).

However, observe that  $0 \notin K$ . We claim 0 and  $K$  cannot be separated. Suppose  $U$  and  $V$  are disjoint open neighborhoods such that  $0 \in U$  and  $K \subseteq V$ .

- Since  $0 \in U$ , there exists a basis element  $(-\delta, \delta) \setminus K \subseteq U$ .
- Since  $K \subseteq V$ , for each  $n$ , there exists an interval  $(a_n, b_n)$  containing  $\frac{1}{n}$  such that  $(a_n, b_n) \subseteq V$ .

For sufficiently large  $n$ , we have  $\frac{1}{n} \in (-\delta, \delta)$ . The interval  $(a_n, b_n)$  around  $\frac{1}{n}$  necessarily contains points strictly between terms of  $K$ . These points are present in  $(-\delta, \delta) \setminus K$ .

Therefore,  $U \cap V \neq \emptyset$ . Thus  $(X, \mathcal{T})$  is Hausdorff but not regular.

**Proposition 1.6.6.** If  $(X, \mathcal{T})$  is Hausdorff, then for  $x_n \rightarrow x$  in  $\mathcal{T}$ ,  $x$  is unique.

**Proof.** If  $x_n \rightarrow x$  and  $x_n \rightarrow y$  in  $\mathcal{T}$  and  $x \neq y$ , then  $\exists U_x \supseteq \{x\}, U_y \supseteq \{y\}$  s.t.  $U_x \cap U_y \neq \emptyset$ . So we cannot have  $x_n \in U_x \cap U_y \Rightarrow x = y$   $\square$

**Proposition 1.6.7.**  $(X, \mathcal{T})$  is Tychonoff iff  $\forall x \in X, \{x\}$  is closed.

**Proof.**

$$\begin{aligned}\{x\} \text{ is closed} &\iff \{x\}^c \text{ is open} \\ &\iff \forall y \in \{x\}^c, \exists \text{ neighbourhood } U_y \subseteq \{x\}^c \\ &\iff x \notin U_y\end{aligned}$$

□

**Remark 1.6.7.**  $(X, \mathcal{T})$  normal  $\Rightarrow (X, \mathcal{T})$  regular.

**Proposition 1.6.8 (Nested neighbourhood property).** Let  $(X, \mathcal{T})$  be Tychonoff. Then  $X$  is normal iff  $\forall F \subseteq X$  closed,  $\forall U$  neighbourhood of  $F$ ,  $\exists O \subseteq X$  open s.t.  $F \subseteq O \subseteq \overline{O} \subseteq U$ .

**Proof.**

( $\Rightarrow$ ) Suppose  $X$  is normal. Consider  $F, U^c$  are two closed disjoint sets. By normality,  $\exists O, V$  open s.t.  $F \subseteq O, U^c \subseteq V$  and  $O \cap V = \emptyset$ .  $\Rightarrow V^c \subseteq U$  and  $\Rightarrow O \subseteq V^c$ .

$$\Rightarrow F \subseteq O \subseteq V^c \subseteq U$$

Since  $O \subseteq V^c \Rightarrow \overline{O} \subseteq \overline{V^c} = V^c$  because  $V$  is open,  $\Rightarrow F \subseteq O \subseteq \overline{O} \subseteq V^c \subseteq U$

( $\Leftarrow$ ) Suppose the nested neighbourhood property holds. Let  $A, B \subseteq X$  be closed,  $A \cap B = \emptyset$ .  $\Rightarrow A \subseteq B^c$  and  $B^c$  open. By assumption,  $\exists O$  open s.t.  $A \subseteq O \subseteq \overline{O} \subseteq B^c$ ,  $\Rightarrow B \subseteq \overline{O}^c$ .  $A \subseteq O, B \subseteq \overline{O}^c$  and  $O \cap \overline{O}^c = \emptyset$ . □

**Corollary 1.6.9.** Every metric space  $(X, p)$  is normal.

**Proof.** By last result, just need to prove the nested neighbourhood property. Let  $F \subseteq X$  closed,  $U \subseteq X$  open s.t.  $F \subseteq U \Rightarrow F \cap U^c = \emptyset$ ,  $U^c$  closed. Let

$$\text{dist}(F, U^c) = \inf_{x \in F} \text{dist}(x, U^c) = \inf_{x \in F} \inf_{y \in U^c} p(x, y)$$

Observe,

$$\text{dist}(x, U^c) = \begin{cases} 0 & x \in U^c \\ > 0 & x \notin U^c \end{cases}$$

For  $x \notin U^c$ ,  $\forall \epsilon > 0$ ,  $\exists x' \in U^c$  s.t.  $\text{dist}(x, U^c) + \epsilon \geq p(x, x')$ .

So,  $\forall y$  s.t.  $p(x, y) < \epsilon$ ,

$$\text{dist}(y, U^c) - \text{dist}(x, U^c) \leq p(y, x') - p(x, x') - \epsilon \leq p(y, x) - \epsilon \leq \epsilon$$

A symmetric argument gets that  $x \mapsto \text{dist}(x, U^c)$  is continuous.

Since  $\text{dist}(x, U^c) \geq 0$ ,  $\inf_{x \in F} \text{dist}(x, U^c) \geq 0$ . if  $\inf_{x \in F} \text{dist}(x, U^c) = 0$ , then by continuity and  $F$  closed,  $F \cap U^c \neq \emptyset \Rightarrow \text{dist}(F, U^c) = \epsilon > 0$ . Let  $O := \bigcup_{x \in F} B^p(x, \frac{\epsilon}{2})$  and  $\overline{O} = \overline{\bigcup_{x \in F} B^p(x, \frac{\epsilon}{2})} = \bigcup_{x \in F} \overline{B^p(x, \frac{\epsilon}{2})} \Rightarrow F \subseteq O \subseteq \overline{O} \subseteq U$ .  $\square$

## 1.7 Compact Topological Spaces

**Definition 1.7.1 (Compact Topological Space).**  $(X, \mathcal{T})$  is a topological space.

- $\{E\}_{\lambda \in \Lambda}$  is an *open cover* if  $X \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$  and each  $E_\lambda$  is open.
- $(X, \mathcal{T})$  is a *compact topological space* if every open cover has a finite subcover.
- For  $K \subseteq X$ ,  $K$  is *compact* if  $(K, \mathcal{T}_k)$  is compact where

$$\mathcal{T}_k := \{K \cap U : U \in \mathcal{T}\}$$

**Remark 1.7.1.** As before, for  $K \subseteq X$ , by defn of  $\mathcal{T}_k$ ,  $O \in \mathcal{T}_k$  iff  $O = K \cap U$  for  $U \in \mathcal{T}$ . Therefore,  $K \subseteq X$  compact iff  $\forall$  open cover of  $K$  (in  $X$ ) has a finite subcover.

**Proposition 1.7.2 (properties identical to metric spaces).**

1. If  $F \subseteq X$  closed and  $(X, \mathcal{T})$  is compact, then  $F$  is compact;
2.  $(X, \mathcal{T})$  compact  $\Rightarrow \forall \{F_k\}_{k=1}^{\infty} \subseteq X$  closed, nested and non-empty,  
 $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$ ;

**Proof.** In the lecture notes (exercise). □

In metric spaces,  $K$  compact  $\Rightarrow K$  is closed and bounded.

**Proposition 1.7.3.** Let  $(X, \mathcal{T})$  be Hausdorff. If  $K \subseteq X$  is compact, then  $K$  is closed in  $X$ .

**Proof. Claim:**  $K^c$  is open.

Fix  $y \in K^c$ .  $\forall x \in K, \exists U_{xy}, O_{xy}$  open, disjoint s.t.  $y \in U_{xy}$  and  $x \in O_{xy}$ . So  $\{O_{xy}\}_{x \in K}$  is an open cover of  $K$ , but  $K$  compact, so

$$K \subseteq \bigcup_{i=1}^N O_{x_i y} \Rightarrow \bigcap_{i=1}^N O_{x_i y}^c \subseteq K^c$$

Let  $E := \bigcap_{i=1}^N U_{x_i y}$  is open. So  $E$  is a neighbourhood of  $y$ , and  $E \cap O_{xy} = \emptyset \quad \forall i = 1, \dots, N \Rightarrow E \subseteq O_{x_i y}^c \quad \forall i = 1, \dots, N \Rightarrow E \subseteq \bigcap_{i=1}^N O_{x_i y}^c \subseteq K^c \Rightarrow K^c$  is open. □

**Definition 1.7.4 (sequential compactness).**  $(X, \mathcal{T})$  is *sequentially compact* if every sequence in  $X$  has a convergent subsequence, whose limit is in  $X$ .

**Proposition 1.7.5 (equivalence of compactness).** Let  $(X, \mathcal{T})$  be second countable. Then  $X$  compact iff  $X$  is sequentially compact.

**Proof.**

( $\Rightarrow$ ) Let  $X$  be compact. Let  $\{x_k\}_{k=1}^{\infty} \subseteq X$ . Let  $F_n := \overline{\{x_k : k \geq n\}}$ . So  $F_n$  is closed  $\forall n$ , and  $F_n \supseteq F_{n+1} \supseteq \dots$ , so since  $X$  is compact  $\exists x_0 \in \bigcap_{n=1}^{\infty} F_n$ . Observe  $X$  second countable  $\Rightarrow X$  first countable. So let  $\mathcal{B}_{x_0} = \{B_j\}_{j=1}^{\infty}$  be a neighbourhood base at  $x_0$ . WLOG assume  $B_{j+1} \subseteq B_j \quad \forall j$ . Since  $x_0 \in \bigcap_{n=1}^{\infty} F_n$ , and  $B_j$  is a neighbourhood of  $x_0$ , then  $B_j \cap F_n \neq \emptyset \quad \forall n$ .

**Claim:**  $\exists x_k, (k \geq n)$  s.t.  $x_k \in B_j \cap F_n$  (i.e.  $B_j \cap \text{Int}(F_n) \neq \emptyset$ )

We know  $B_j \cap F_n \neq \emptyset$ , so  $\exists y \in B_j \cap F_n$ . Then  $B_j$  is a neighbourhood of  $y$  and  $y \in F_n$ , so by defn of  $F_n$

$$B_j \cap \{x_k : k \geq n\} \neq \emptyset$$

Let this element be  $\{x_{n_j}\} \in B_j$ . So  $\{x_{n_j}\}_{j=1}^{\infty} \subseteq \{x_k\}_{k=1}^{\infty}$  and  $x_{n_j} \in B_j$  with  $B_j \supseteq B_{j+1}$ . Thus  $\forall$  neighbourhood  $U_{x_0}$  of  $x_0$ ,  $\exists B_N \subseteq U_{x_0}$  and if  $j \geq N$ ,  $x_{n_j} \in B_j \subseteq B_N \subseteq U_{x_0} \Rightarrow x_{n_j} \rightarrow x_0$  in  $\mathcal{T}$ .

( $\Leftarrow$ ) Let  $X$  be sequentially compact.  $X$  second countable  $\Rightarrow$  every open cover has a countable subcover, ( $X = \bigcup_{B \in \mathcal{B}} B$ )

**Claim:** Every countable cover of  $X$  has a finite subcover.

Let  $X \subseteq \bigcup_{j=1}^{\infty} E_j$ ,  $E_j$  open  $\forall j$ . Assume there is no finite subcover. So  $\forall n$ ,  $\exists m(n) > n$  s.t.  $E_{m(n)} \setminus \bigcup_{j=1}^n E_j \neq \emptyset$ . Let  $x_n \in E_{m(n)} \setminus \bigcup_{j=1}^n E_j$ .  $X$  sequentially compact means  $\exists \{x_{n_k}\}$  s.t.  $x_{n_k} \rightarrow x_0 \in X$ . Since  $x_0 \in X$ ,  $\exists E_N$  s.t.  $x_0 \in E_N$ . But  $x_{n_k} \in E_{m(n_k)} \setminus \bigcup_{j=1}^{n_k} E_j$ , so  $\forall n_k \geq N$ ,  $x_{n_k} \not\rightarrow x_0$  contradiction.

□

**Theorem 1.7.6.** A compact Hausdorff space is normal

**Proof.** Let  $(X, \mathcal{T})$  be compact Hausdorff.

**Claim:**  $(X, \mathcal{T})$  is regular.

Let  $F \subseteq X$  closed,  $x \notin F$ . Let  $y \in F$ . Since  $X$  is Hausdorff,  $\exists U_{xy}, O_{xy}$  open,

disjoint s.t.  $y \in U_{xy}$  and  $x \in O_{xy}$ . So  $\{U_{xy}\}_{y \in F}$  is an open cover of  $F$ , but  $F$  closed  $\Rightarrow F$  compact.<sup>11</sup> So  $F \subseteq \bigcup_{i=1}^N U_{xy_i} =: U$ . Let  $N := \bigcap_{i=1}^N O_{xy_i}$ , so  $N$  is open,  $x \in N$  and  $U \cap N = \emptyset$ .  $\Rightarrow (X, \mathcal{T})$  is regular. We just rerun the same argument to get that  $(X, \mathcal{T})$  is normal.  $\square$

<sup>11</sup>Because closed subsets of compact subspaces are themselves compact

## 1.8 Continuity and Urysohn's Lemma

**Definition 1.8.1 (continuous map).** If  $(X, \mathcal{T})$  and  $(Y, \mathcal{S})$  are topological spaces, then  $f : X \rightarrow Y$  continuous at  $x_0 \in X$  if  $\forall$  neighbourhood  $O_{f(x_0)} \subseteq Y$ ,  $\exists$  a neighbourhood  $U_{x_0} \subseteq X$  s.t.  $f(U_{x_0}) \subseteq O_{f(x_0)}$ . We say  $f$  is *continuous* if  $f$  is continuous at every  $x \in X$ .

### Proposition 1.8.2.

1.  $f : X \rightarrow Y$  continuous iff  $\forall$  open set  $O \subseteq Y$ ,  $f^{-1}(O)$  is open in  $X$ ;
2. Composition of continuous functions is continuous;
3.  $X$  is compact and  $f$  continuous, then  $f(X)$  is compact in  $Y$ ;
4. If  $f : X \rightarrow \mathbb{R}$ ,  $X$  compact,  $f$  continuous, then max/min of  $f(x)$  are achieved.

#### Proof of (1).

$\Rightarrow$  Let  $O \subseteq Y$  be open, and let  $x \in f^{-1}(O) \Rightarrow f(x) \in O$ . Since  $f$  is continuous, and  $O$  is a neighbourhood of  $f(x)$ ,  $\exists U_x \subseteq X$  s.t.  $f(U_x) \subseteq O$ . So  $U_x \subseteq f^{-1}(O)$  and thus  $f^{-1}(O)$  is open in  $X$ .

$\Leftarrow$  Suppose  $O \subseteq Y$  open and  $f^{-1}(O)$  is open in  $X$ . Let  $U := f^{-1}(O)$ . Then  $U$  is open and  $f(U) \subseteq O$ . So  $f$  is continuous.  $\square$

**Definition 1.8.3 (weak-topology induced by  $\mathcal{F}$ ).** Let

$$\mathcal{F} := \{f_\lambda : X \rightarrow X_\lambda\}_{\lambda \in \Lambda}$$

where  $(X_\lambda, \mathcal{T}_\lambda)$  is a topological space  $\forall \lambda \in \Lambda$ . Let

$$S := \left\{ f_\lambda^{-1}(O_\lambda) : f_\lambda \in \mathcal{F}, O_\lambda \in \mathcal{T}_\lambda \right\}$$

Then  $\mathcal{T}(S)$  is called the *weak-topology* induced by  $\mathcal{F}$ .

**Remark 1.8.3.**  $\mathcal{T}(S) = \cap \{\text{topologies containing } S\}$  and if  $f_\lambda^{-1}(O_\lambda)$  belongs to the topology, then  $f_\lambda$  is continuous  $\forall \lambda \in \Lambda$ . Thus this topology makes every  $f_\lambda$  continuous.

**Corollary 1.8.4.**  $\mathcal{T}(S)$  is the weakest topology amongst all topologies on  $X$  for which  $f_\lambda : X \rightarrow X_\lambda$  is continuous  $\forall \lambda \in \Lambda$ .

**Example 1.8.5.**  $\Lambda = \{1, 2\}$ . Let  $(X_1, \mathcal{T}_1)$  and  $(X_2, \mathcal{T}_2)$  be topological spaces. Consider  $X := X_1 \times X_2 = \prod_{i=1}^2 X_i$ . Let

$$\mathcal{F} := \left\{ \begin{array}{l} \pi_1 : X \rightarrow X_1, \quad \pi_1(x_1, x_2) = x_1 \\ \pi_2 : X \rightarrow X_2, \quad \pi_2(x_1, x_2) = x_2 \end{array} \right\}.$$

Let  $S = \{\pi_i^{-1}(O_i) : O_i \in \mathcal{T}_i\}$ . Then  $\mathcal{T}(S)$  is called the product topology.<sup>12</sup> Recall, we have learned that

$$\mathcal{T}(S) = \left\{ \emptyset, X, \bigcup \{\text{finite intersections of elements of } S\} \right\}.$$

So, a base for  $\mathcal{T}(S)$  is given by

$$\mathcal{B} := \left\{ \bigcap_{i=1}^2 \pi_i^{-1}(O_i) : O_i \in \mathcal{T}_i \right\}$$

<sup>12</sup>This is similar to how the product  $\sigma$ -algebra is the smallest  $\sigma$ -algebra that makes  $\pi_i$  measurable. The product topology is the weakest topology that makes  $\pi_i$  continuous.

and we note that  $\pi_1^{-1}(O_1) \cap \pi_1^{-1}(\tilde{O}_1) = \pi_1^{-1}(O_1 \cap \tilde{O}_1)$ .  
Also, since  $\pi_1^{-1}(O_1) = O_1 \times X_2$  and  $\pi_2^{-1}(O_2) = X_1 \times O_2$ , we have

$$\begin{aligned} \bigcap_{i=1}^2 \pi_i^{-1}(O_i) &= O_1 \times O_2 \\ \implies \mathcal{B} &= \left\{ \prod_{i=1}^2 O_i : O_i \in \mathcal{T}_i \right\} \end{aligned}$$

is a base for the product topology.

**Example 1.8.6.**  $\Lambda$  infinite.<sup>13</sup> Let  $(X_\lambda, \mathcal{T}_\lambda)$  be a topological space. Let  $X = \prod_{\lambda \in \Lambda} X_\lambda$  and let  $\pi_\lambda : X \rightarrow X_\lambda$  be the projection map. Consider the product topology on  $X$ , and a base is given by

$$\mathcal{B} = \left\{ \bigcap_{j=1}^n \pi_{\lambda_j}^{-1}(O_{\lambda_j}) : O_{\lambda_j} \in \mathcal{T}_{\lambda_j}, n \in \mathbb{N} \right\}$$

which equals

$$= \left\{ \prod_{\lambda \in \Lambda} O_\lambda : O_\lambda = X_\lambda \text{ for all but finitely many } \lambda \right\}$$

So open in the product topology means a base is given by finite products of open sets.

<sup>13</sup>Could even be uncountable

**Motivation.** Let  $(X, p)$  be a metric space. Let  $A, B$  closed and disjoint. Let

$$f(x) := \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}$$

Note,

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

$0 \leq f(x) \leq 1$ .  $f$  is continuous because  $\text{dist}(., A)$  and  $\text{dist}(., B)$  are continuous, and denominator is non-zero. Urysohn's lemma does this on any normal topological space.

**Lemma 1.8.7 (Urysohn's Lemma).** Let  $(X, \mathcal{T})$  be normal. Let  $A, B \subseteq X$  closed and disjoint. Then  $\exists f : X \rightarrow \mathbb{R}$  s.t.

- $f$  is continuous;

- $0 \leq f(x) \leq 1$ ;

- $f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$

**Remark 1.8.7.** Infact, we can replace  $\{0, 1\} \rightarrow \{\alpha, \beta\}$   $\forall \alpha < \beta$

**Definition 1.8.8 (normally ascending).** Let  $(X, \mathcal{T})$  and  $\Lambda \subseteq \mathbb{R}$ . We say  $\{O_\lambda\}_{\lambda \in \Lambda}$  with  $O_\lambda$  open is *normally ascending* if  $\forall \lambda_1, \lambda_2 \in \Lambda$ ,  $\overline{O_{\lambda_1}} \subseteq O_{\lambda_2}$  whenever  $\lambda_1 < \lambda_2$ .

**Lemma 1.8.9.** Let  $(X, \mathcal{T})$  be normal. Let  $F \subseteq X$  be closed,  $U$  a neighbourhood of  $F$ . There exists a dense set  $\Lambda \subseteq (0, 1)$  and a normally ascending collection of open sets  $\{O_\lambda\}_{\lambda \in \Lambda}$  such that

$$F \subseteq O_\lambda \subseteq \overline{O_\lambda} \subseteq U \quad \forall \lambda \in \Lambda$$

**Proof.** Consider  $\Lambda := \left\{ \frac{m}{2^n} : m, n \in \mathbb{N}, 1 \leq m \leq 2^n - 1 \right\}$ . Clearly,  $\Lambda$  is dense in  $(0, 1)$ . Let

$$\Lambda_n := \left\{ \frac{m}{2^n} : m \in \mathbb{N}, 1 \leq m \leq 2^n - 1 \right\} \Rightarrow \Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$$

We will define  $\{O_\lambda\}_{\lambda \in \Lambda}$  inductively. Since  $X$  is normal, by nested neighbourhood property, let  $O_{\frac{1}{2}}$  be s.t.

$$F \subseteq O_{\frac{1}{2}} \subseteq \overline{O_{\frac{1}{2}}} \subseteq U$$

We now define  $O_{\frac{1}{4}}, O_{\frac{3}{4}}$  by

$$F \subseteq O_{\frac{1}{4}} \subseteq \overline{O_{\frac{1}{4}}} \subseteq O_{\frac{1}{2}} \subseteq \overline{O_{\frac{1}{2}}} \subseteq O_{\frac{3}{4}} \subseteq \overline{O_{\frac{3}{4}}} \subseteq U.$$

We proceed inductively to build  $\{O_\lambda\}_{\lambda \in \Lambda}$ , which are necessarily normally ascending.  $\square$

**Lemma 1.8.10.** Let  $(X, \mathcal{T})$  be a topological space s.t  $\exists \Lambda \subseteq (0, 1)$  and a normally ascending collection of open sets  $\{O_\lambda\}_{\lambda \in \Lambda}$ . Let

$$f(x) := \begin{cases} 1 & \text{if } x \in (\bigcup_{\lambda \in \Lambda} O_\lambda)^c \\ \inf \{\lambda \in \Lambda : x \in O_\lambda\} & \text{if } x \in \bigcup_{\lambda \in \Lambda} O_\lambda \end{cases}$$

Then  $0 \leq f(x) \leq 1$  and  $f$  is continuous.

**Proof.** Notice  $0 \leq f \leq 1$  because  $\Lambda \subseteq (0, 1)$ . Observe

$$\mathcal{D} := \{(-\infty, c), (d, \infty) : c, d \in \mathbb{R}\}$$

$$\mathcal{B} := \{\text{finite intersections of elements of } \mathcal{D}\}$$

is a base for  $(\mathbb{R}, \mathcal{T}_{|.|})$ . So,

$$\mathcal{T}_{|.|} = \left\{ \emptyset, X, \bigcup B : B \in \mathcal{B} \right\}$$

So if  $f^{-1}(-\infty, c)$  and  $f^{-1}(d, \infty)$  are open, then  $\forall O \subseteq \mathbb{R}$  open,  $f^{-1}(O)$  is open.<sup>14</sup>

**Claim:**  $f^{-1}(-\infty, c)$  and  $f^{-1}(d, \infty)$  are open  $\forall c, d \in \mathbb{R}$ .

$f(x) < c$  iff  $x \in O_\lambda$  for  $\lambda < c$  iff  $x \in \bigcup_{\lambda < c} O_\lambda$ .  $f^{-1}((-\infty, c)) = \bigcup_{\lambda < c} O_\lambda$  is open.

Similarly,  $f(x) > d$  iff  $x \notin O_\lambda$  for some  $\lambda > d$  iff  $x \notin \overline{O}_{\lambda-\epsilon}$  for some  $\lambda - \epsilon > d$  iff  $x \in \bigcup_{\lambda > d} (\overline{O}_\lambda)^c$ . So,  $f^{-1}(d, \infty) = \bigcup_{\lambda > d} (\overline{O}_\lambda)^c$  is open. So, by prior argument,  $f$  is continuous.  $\square$

<sup>14</sup>because it is just finite intersections and arbitrary unions

**Proof of Urysohn's Lemma.** Let  $(X, \mathcal{T})$  be normal,  $A, B \subseteq X$  closed and disjoint. Consider  $A \subseteq B^c$  open. By prior lemma,  $\exists \Lambda \subseteq (0, 1)$  dense and  $\{O_\lambda\}_{\lambda \in \Lambda}$  normally ascending open sets s.t.

$$A \subseteq O_\lambda \subseteq \overline{O_\lambda} \subseteq B^c.$$

Let  $f$  be as in the last lemma, so  $0 \leq f \leq 1$  and  $f$  is continuous. If  $x \in B$ ,  $\bigcup_{\lambda \in \Lambda} O_\lambda \subseteq B^c \Rightarrow B \subseteq (\bigcup_{\lambda \in \Lambda} O_\lambda)^c \Rightarrow f(x) = 1$ . Similarly, if  $x \in A$ ,  $x \in O_\lambda \forall l \lambda \in \Lambda$ , so  $f(x) = \inf \{\lambda \in \Lambda\} = 0$ .  $\square$

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