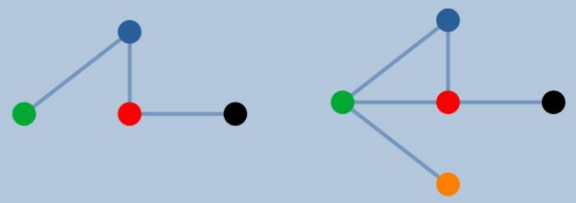
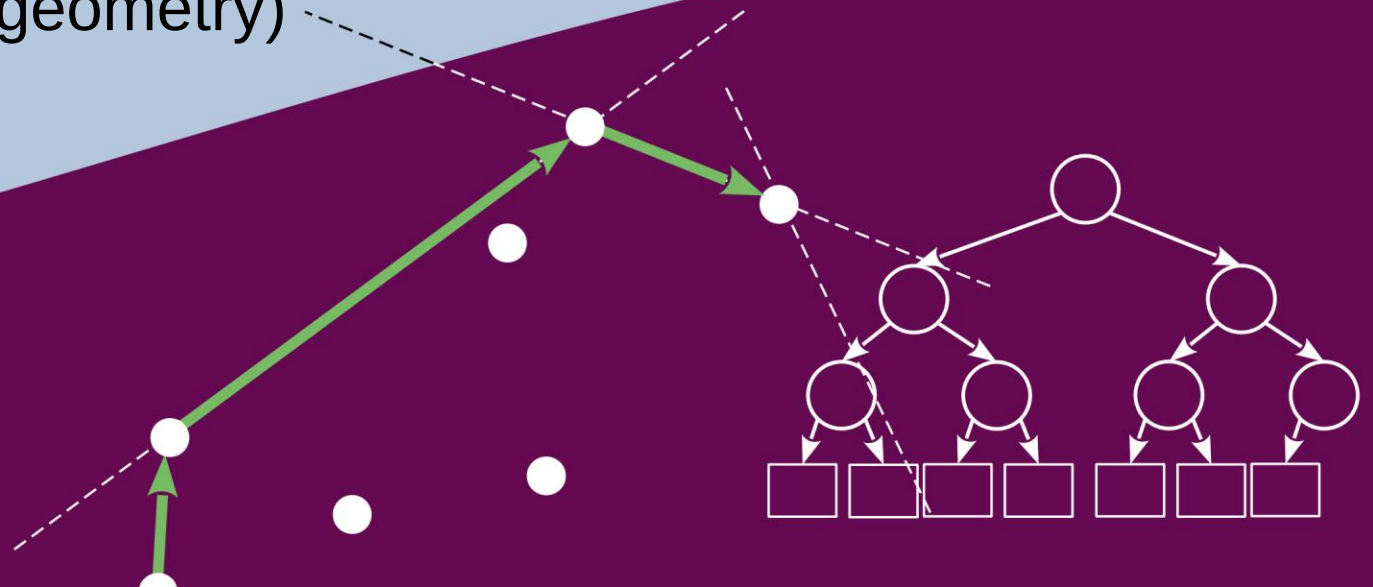
Advanced algorithms and data structures

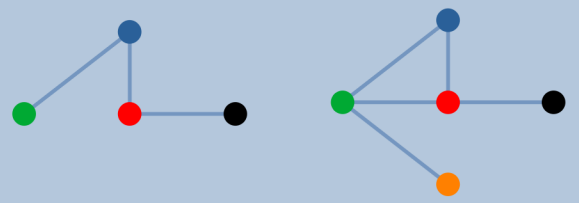
Week 4: Geometric algorithms
(computational geometry)

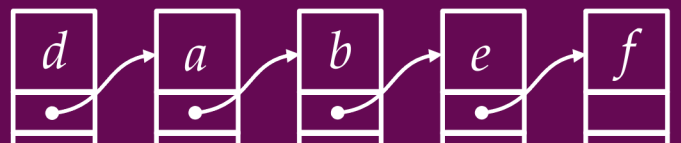


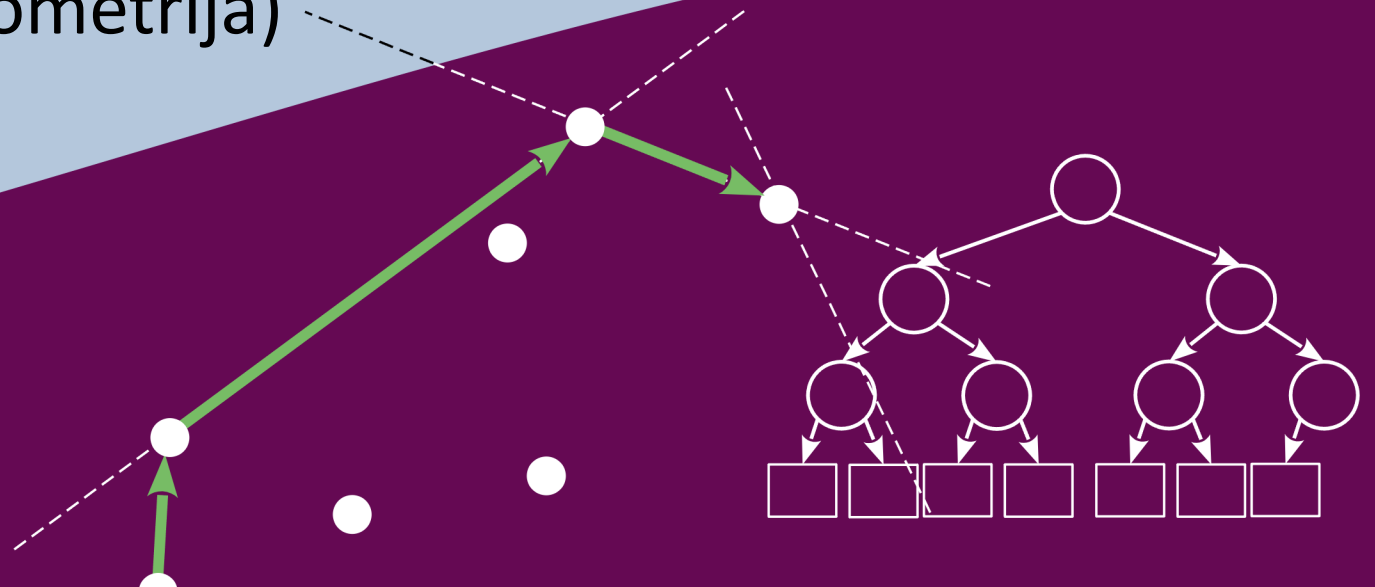


Napredni algoritmi i strukture podataka

Tjedan 4: Geometrijski algoritmi
(računalna geometrija)

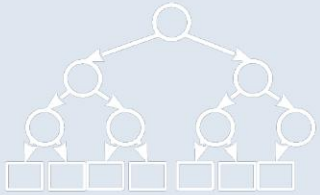






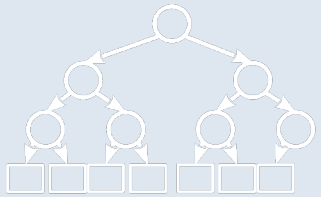
What is Computational Geometry?

- Definition: *the systematic study of data structures and algorithms in geometry*
- Examples:
  - **Computer graphics** - one of the most developed areas - computer games, visualization, modeling
  - **Spatial design of products** – layout on electronic boards, design integrated chips, packaging, **CAD/CAM**
  - **Robotics** – calculation of movements and trajectories
  - **GIS** – spatial projections, placement of objects, route calculations, object search, sections, ...

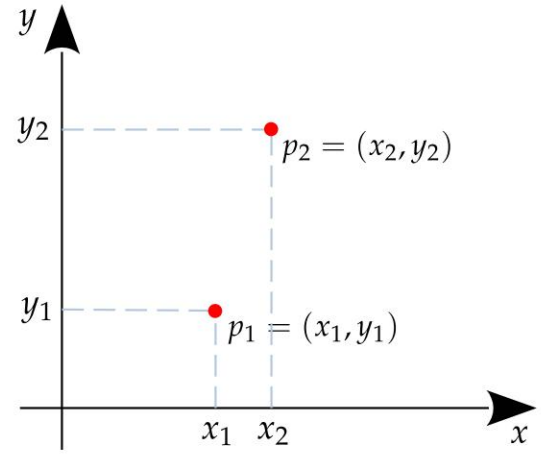


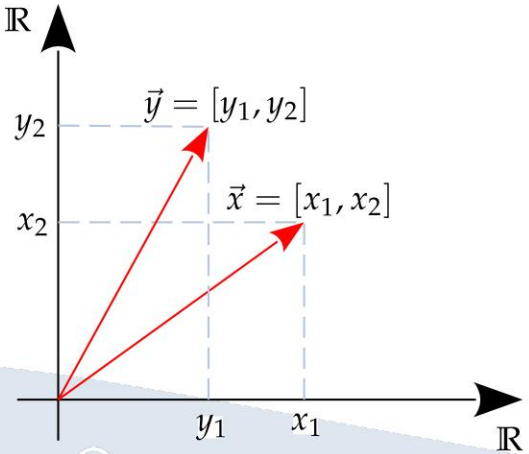
Što je računalna geometrija?

- Definicija: *sistematsko proučavanje struktura podataka i algoritama u geometriji*
- Primjeri:
  - **Računalna grafika** – jedno od najrazvijenijih područja – računalne igre, vizualizacija, modeliranje
  - **Prostorni dizajn produkata** – raspored na elektroničkim pločicama, dizajn integriranih čipova, pakiranja, **CAD/CAM**
  - **Robotika** – izračun pokreta i trajektorija
  - **GIS** – prostorne projekcije, smještanje objekata, proračuni putanja, pretraživanje objekata, presjeci, ...



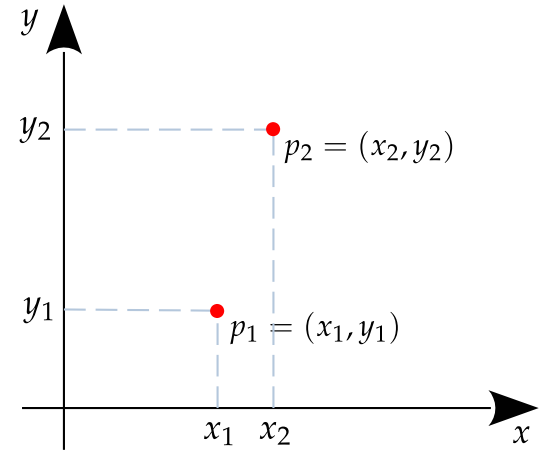
Basic geometric elements - point

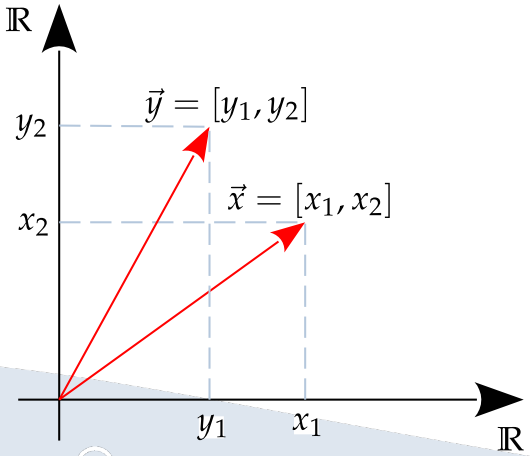




- A point is an infinitesimal element of ÿ! space
  ÿ×ÿ×ÿ× ÿ×ÿ = (", #, $, … , !) =
- Real space ÿ! is abstract - we map it to Cartesian coordinate system
- A point can also be written in vector form
  ÿ = [" # $ … !]

Osnovni geometrijski elementi - točka





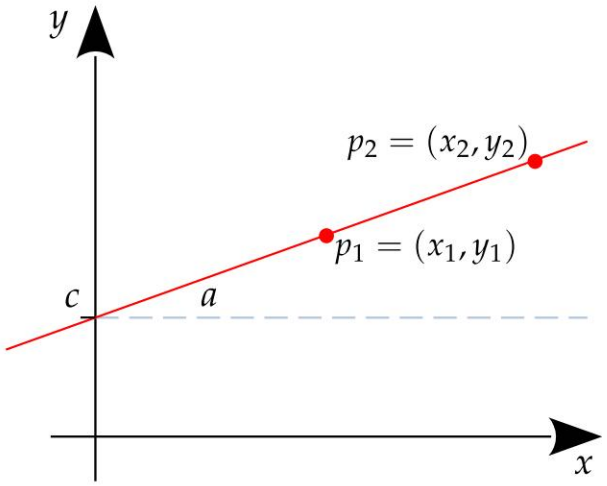
- Točka je infinitezimalni element $\mathbb{R}^n$ prostora

$$\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R} = (x_1, x_2, x_3, \ldots, x_n) = p$$

- Realni prostor $\mathbb{R}^n$ je apstraktan – preslikavamo ga u Kartezijev koordinatni sustav
- Točka se može napisati i u vektorskom obliku

$$\vec{x} = [x_1\ x_2\ x_3\ \ldots\ x_n]$$

Basic geometric elements - line



- A line is an infinite set of points in space that follow a linear equation

= {(,) : (,) ÿ ÿ# , + = }

- A linear equation can also be written as

= ÿ

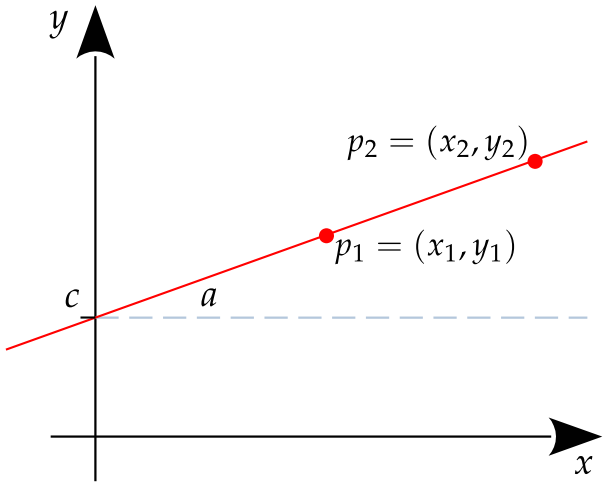
- If we have two points in space through which a line passes

" = (", " , #) = (#, #)

- The line can be written as

= " − # , = " + " − # "
" # " #

Osnovni geometrijski elementi - linija



- Linija je beskonačni skup točaka u prostoru koje slijede linearnu jednadžbu

$$L = \{(x, y): (x, y) \in \mathbb{R}^2, ax + by = c\}$$

- Linearna jednadžba može se napisati i kao

$$y = c - ax$$

- Ako imamo dvije točke u prostoru kroz koje linija prolazi

$$p_1 = (x_1, y_1), p_2 = (x_2, y_2)$$

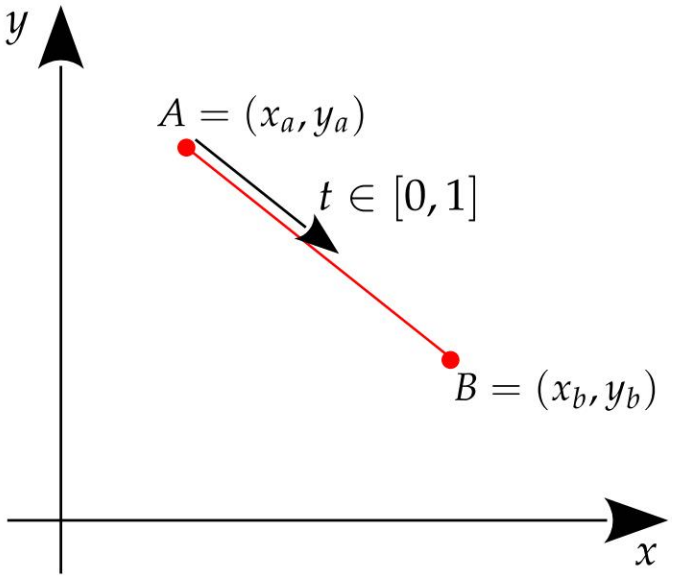
- Linija se može napisati kao

$$a = \frac{y_2 - y_1}{x_1 - x_2}, c = y_1 + \frac{y_2 - y_1}{x_1 - x_2} x_1$$

Basic geometric elements – line and segment

- The line can be parameterized

ÿ = - ", ÿ = - "
# #
= " + ÿ , = " + ÿ



- The parameter ÿ 0.1[determines the position of the point on the line:
  - for = 0 we are in !, while for = 1 we are in "
- We use the parametric approach to delimit the line
- The line passes through the (%, & ,) = (&, &)

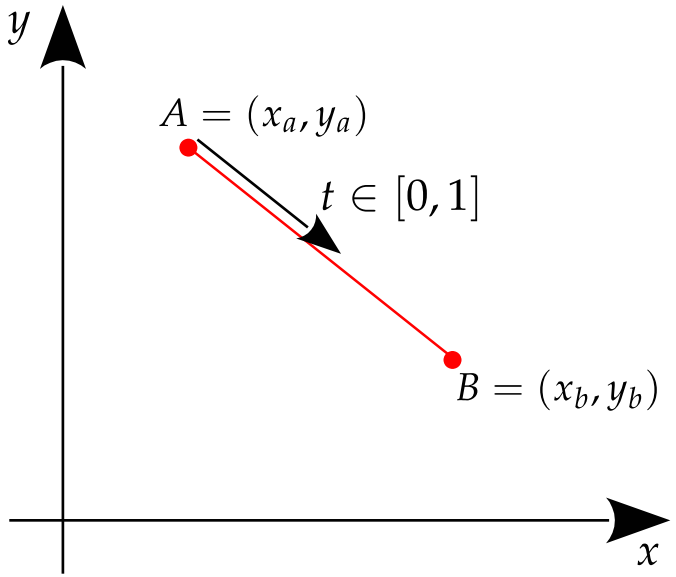
points = • If we consider that the points A and B limit it

- For parameter values ÿ 0.1, we get a set of points representing line segment or length.

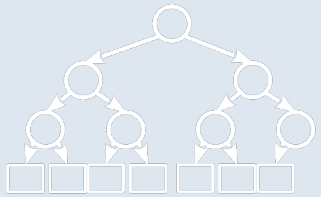
Osnovni geometrijski elementi – linija i segment

- Linija se može parametrizirati

$$\Delta y = y_2 - y_1, \Delta x = x_2 - x_1$$
$$y = y_1 + t\Delta y, x = x_1 + t\Delta x$$



- Parametar $t \in [0,1]$ određuje poziciju točke na liniji:
  - za $t = 0$ nalazimo se u p_1, dok se za $t = 1$ nalazimo u p_2
- Parametarski pristup koristimo za omeđivanje linije
- Linija prolazi točkama $A = (x_a, y_b), B = (x_b, y_b)$
- Ako smatramo da ju točke A i B omeđuju
  - Za vrijednosti parametra $t \in [0,1]$ dobivamo skup točaka koji predstavljaju segment linije ili dužinu $\overline{AB}$.

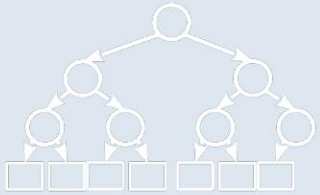


Basic geometric elements - surface

- Consider a geometric element that has one dimension less than space in which it is located
  - For example a point (0 dimension) on a line (1 dimension)
- In three-dimensional space, it is a surface
- A surface is an infinite set of points that follow a linear equation

= {(, ,):(, ,) ÿ ÿ$, + + = }

- We call a subset of points of a surface a geometric shape

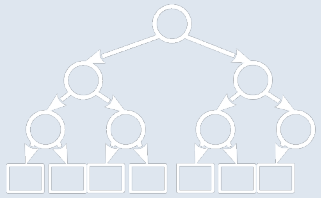


Osnovni geometrijski elementi – ploha

- Promotrimo geometrijski element koji ima jednu dimenziju manje od prostora u kojem se nalazi
  - Na primjer točka (0 dimenzija) na liniji (1 dimenzija)
- U trodimenzionalnom prostoru to je ploha
- Ploha je beskonačan skup točaka koje slijede linearnu jednadžbu

$$P = \{(x, y, z): (x, y, z) \in \mathbb{R}^3, ax + by + cz = d\}$$

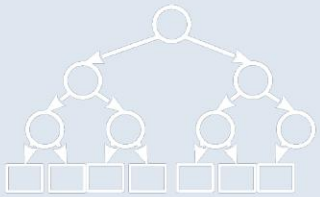
- Podskup točaka plohe nazivamo geometrijskim oblikom



Basic geometric elements – polygons

- A subset of the surface ÿ bounded by a set of points and line segments between those points is called a polygon
- A polygon composed of points is called a -gon
- The line segments bound the polygon forming the edge of the *polygon,* while the points in the polygon are called *the body of the polygon*
- When we enumerate points on the edge of a polygon, we usually do so in a clockwise direction
- The sum of the interior angles of a polygon is

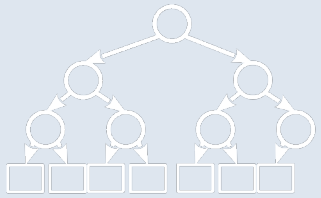
$$= ÿ(2 ÿ 180°)$$



Osnovni geometrijski elementi – poligoni

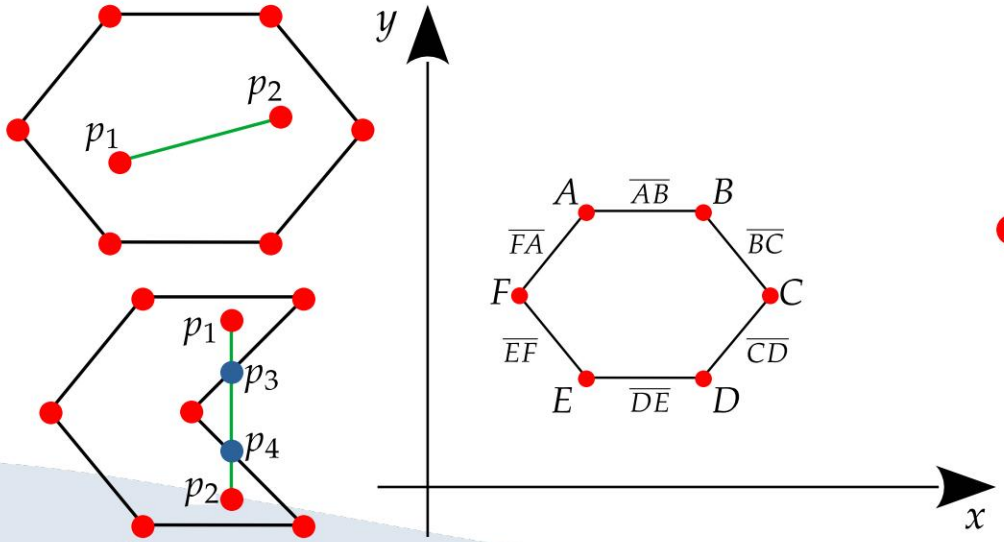
- Podskup plohe $\mathcal{P} \in P$ ograničen skupom točaka i linijskim segmentima između tih točaka naziva se poligon
- Poligon sastavljen od n točaka naziva se n-gon
- Linijski segmenti omeđuju poligon čineći brid *poligona*, dok se točke u poligonu nazivaju *tijelo poligona*
- Kada nabrajamo točke na bridu poligona to uobičajeno činimo u smjeru kazaljke na satu
- Suma unutarnjih kutova poligona je

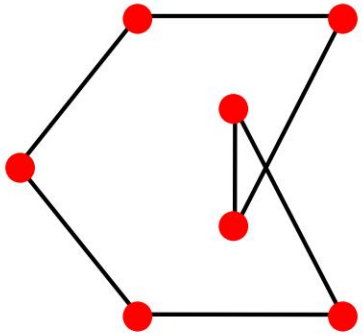
$$S = (n - 2) * 180^{\circ}$$



Basic geometric elements – polygons

- A polygon is considered **convex** if for every pair of points ", # ÿ , length " # passes inside the polygon
- A polygon is considered **simple** if it does not interact with itself, for example intersecting with its edge

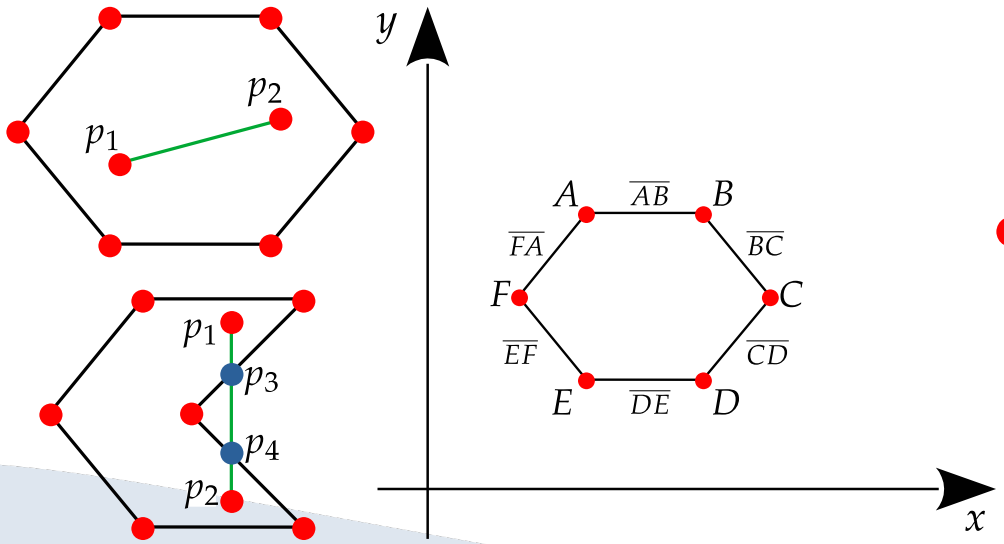


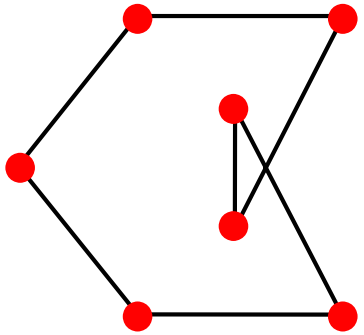


- **Regular** polygons are equiangular and equilateral, which means that their interior angles and the lengths of all line segments are equal.
  - **Example:** one-sided triangle, square, pentagon, hexagon, heptagon, ...

Osnovni geometrijski elementi – poligoni

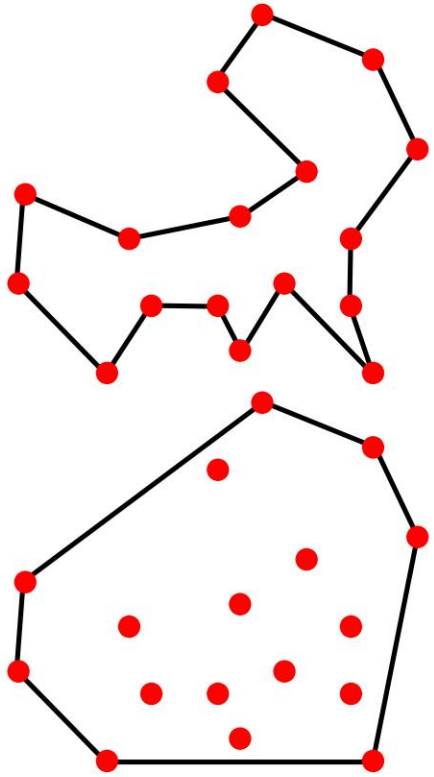
- Poligon se smatra **konveksnim** ako za svaki par točaka $p_1, p_2 \in \mathcal{P}$, dužina $\overline{p_1 p_2}$ prolazi unutar poligona
- Poligon se smatra **jednostavnim** ako nema interakcije sa samim sobom, na primjer križanje sa svojim bridom





- **Regularni** poligoni su ekviangularni i ekvilateralni, što znači da su im unutarnji kutevi i duljine svih linijskih segmenata jednaki
  - **Primjer:** jednostranični trokut, kvadrat, pentagon, hexagon, heptagon, ...

Convex Hull



• Let's imagine a set of points in ÿ#, (= { ", #, ... , ! }

• There is a subset of these points, such that it forms an edge of the which contains all points from (, polygon so that it is valid the set ÿ ÿ (:

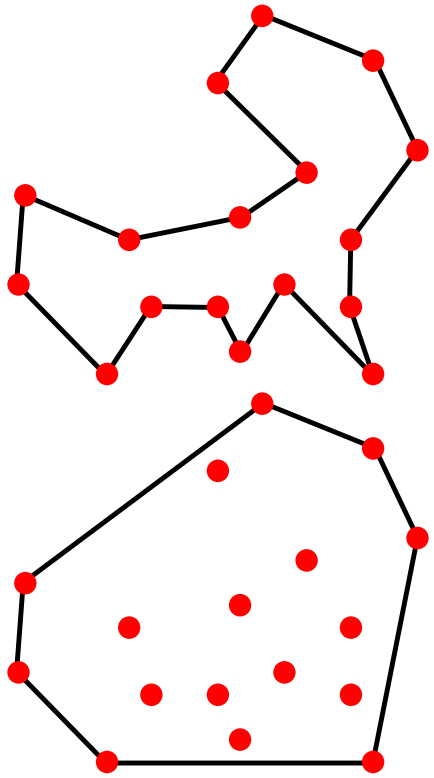
ÿ • If the polygon is convex, then it is a convex shells

• Let's imagine that • (a set of nails driven into a board

Let's put a rubber band around those nails

• The rubber band represents the edge of the polygon, which is a convex shell

Plošna konveksna ljuska (Convex Hull)



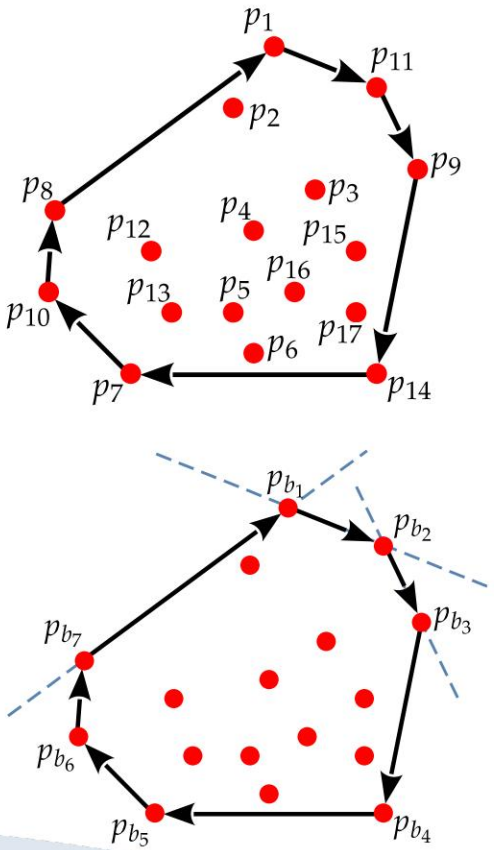
- Zamislimo skup točaka u $\mathbb{R}^2, P_p = \{p_1, p_2, \dots, p_n\}$
- Postoji podskup tih točaka, takav da čini brid poligona $\mathcal{P}$ koji sadržava sve točke iz skupa P_p, tako da vrijedi

$$\nexists p \in P_p : p \notin \mathcal{P}$$

- Ako je poligon $\mathcal{P}$ konveksan, tada se radi o konveksnoj ljusci

- Zamislimo da je P_p skup čavala zabijen u dasku
- Stavimo gumicu oko tih čavala
- Gumica predstavlja rub poligona koji je konveksna ljuska

Flat convex shell I

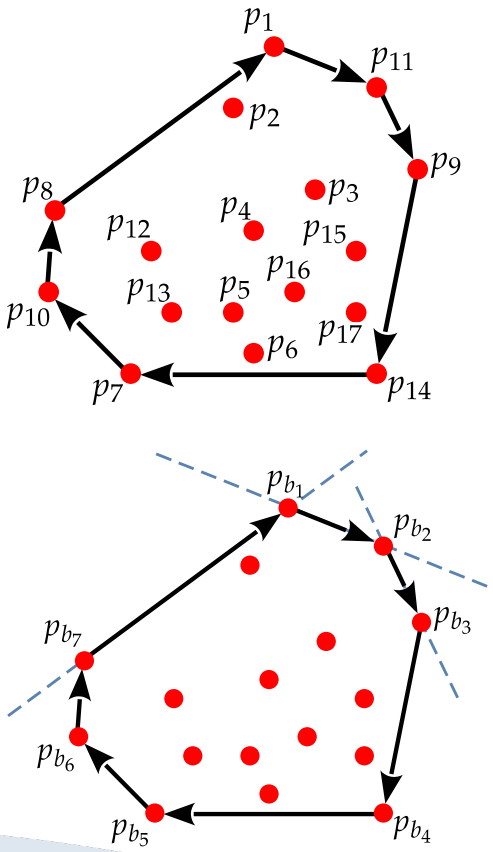


• We enumerate the points that belong to the edge of the polygon ( ) ÿ (clockwise • A simple way to determine that ( ) is a convex hull • We take each line segment of the edge of the polygon and see if they all other points in the polygon to the **right** of its line

• If yes, it is a convex shell

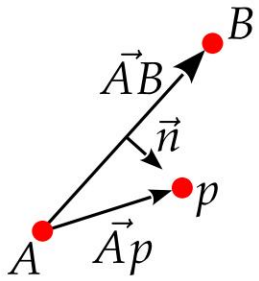
• How to determine that the point is **"to the right"** of the line? • And what does **"right" even mean?**

Plošna konveksna ljuska I



- Nabrajamo točke koje pripadaju bridu poligona $b(\mathcal{P}) \subseteq P_p$ u smjeru kazaljke na satu
- Jednostavni način utvrđivanja da je $b(\mathcal{P})$ konveksna ljuska
  - Uzmemo svaki linijski segment ruba poligona i gledamo da li su sve ostale točke u poligonu **desno** od njegove linije
  - Ako da, radi se o konveksnoj ljusci
- Kako utvrditi da je točka „**desno**” od linije?
- I što to uopće znači „**desno**”?

Flat convex shell I



- If we have two points = and then a vector can be defined

$$\longrightarrow = \&[\quad - \quad \% \quad \& \quad - \quad \%]$$

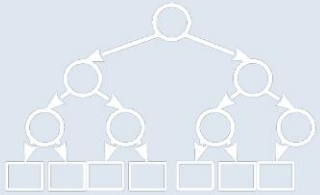
- With the system of equations

$$\longrightarrow \quad \rightarrow) \qquad \rightarrow \quad \longrightarrow)$$
$$ÿ \qquad = 0, = ÿ$$

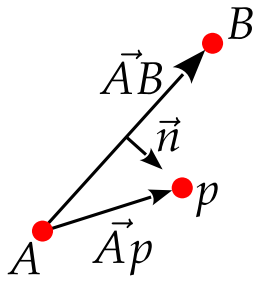
- We get

$$= \&[\quad - \quad \% \quad \% \quad - \quad \&] \begin{bmatrix} - & \% \\ - & \% \end{bmatrix}$$
$$= ÿ \quad (\quad \% \&)(\quad - \quad \%) \; - \; (\quad \%)(\quad \& \quad - \quad \%)$$

- For everything > 0, we consider that the point is to the **right** of the line segment



Plošna konveksna ljuska I



- Ako imamo dvije točke $A = (x_a, x_b), B = (x_b, y_b)$ i njihov linijski segment $\overline{AB}$, tada se može definirati vektor

$$\overrightarrow{AB} = [x_b - x_a \quad y_b - y_a]$$

- Uz sustav jednadžbi

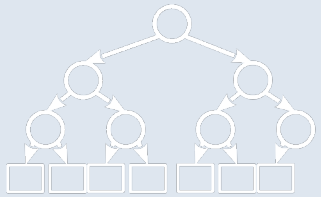
$$\overrightarrow{AB} * \vec{n}^T = 0, D = \vec{n} * \overrightarrow{Ap}^T$$

- Dobivamo

$$D = [y_b - y_a \quad x_a - x_b] \begin{bmatrix} x - x_a \\ y - y_a \end{bmatrix}$$

$$D = (x - x_a)(y_b - y_a) - (y - y_a)(x_b - x_a)$$

- Za sve $D > 0$ smatramo da je točka p **desno** od linijskog segmenta $\overline{AB}$

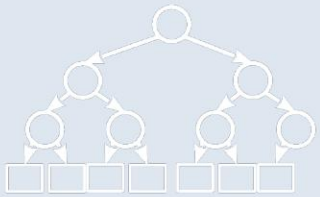


Flat convex shell I

```
function SimpleConvexHull(P_p)
    α(P_p) ← ∅
    for each pair of points p_i, p_j ∈ P_p do
        bound ← 1
        for each point p_k ∈ P_p such that p_k ∉ {p_i, p_j} do
            if p_k is left from the line segment p_i p_j then
                bound ← 0
        if bound is 1 then
            add p_i p_j to α(P_p) taking care of the clockwise order
    return α(P_p)
```

Complexity = (!)

- Given that at the beginning we only !, have to find an edge ( ) such that it represents a convex shell, which is marked with ( )
- We take pairs of points from ", # ! Ÿ and we check if they are to the right of ! all other points from of their line segment " #
- If yes, then that line segment " # belongs to ( )
- Two big problems:
  - The algorithm does not ensure that the convex hull is closed
  - High complexity of the solution

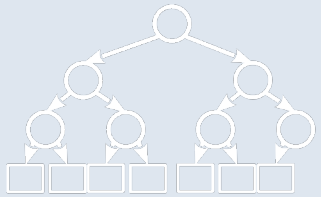


Plošna konveksna ljuska I

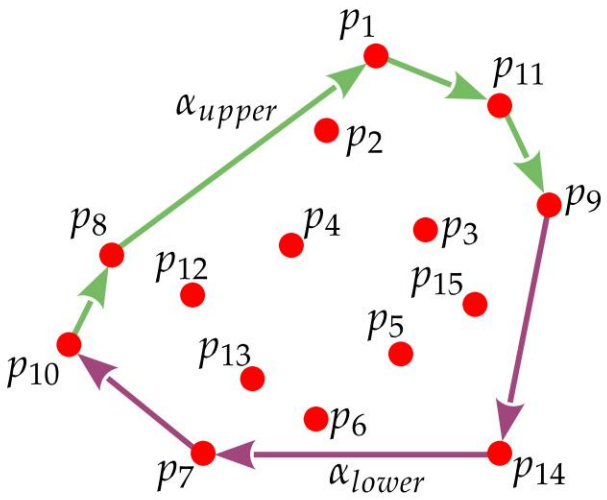
```
function SimpleConvexHull(P_p)
    α(P_p) ← ∅
    for each pair of points p_i, p_j ∈ P_p do
        bound ← 1
        for each point p_k ∈ P_p such that p_k ∉ {p_i, p_j} do
            if p_k is left from the line segment p_i p_j then
                bound ← 0
        if bound is 1 then
            add p_i p_j to α(P_p) taking care of the clockwise order
    return α(P_p)
```

Kompleksnost = $O(n^3)$

- S obzirom da na početku imamo samo P_p, moramo pronaći rub $b(\mathcal{P})$ takav da predstavlja konveksnu ljusku, što je označeno sa $\alpha(\mathcal{P})$
- Uzimamo parove točaka iz $p_i, p_j \in P_p$ i provjeravamo da li su sve ostale točke iz P_p desno od njihovog linijskog segmenta $\overline{p_i p_j}$
- Ako da, onda taj linijski segment $\overline{p_i p_j}$ pripada $\alpha(\mathcal{P})$
- Dva velika problema:
  - Algoritam ne osigurava da je konveksna ljuska zatvorena
  - Visoka kompleksnost rješenja



Flat convex shell II



- Is it possible to have a different, faster algorithm for finding the convex shells?
- Let's divide the convex shell into an upper and a lower part

( ) = %&'() ÿ( ) *++() ( )

- Horizontally sort the points in

ÿ(+) = { + : ,! ÿ ,! + }

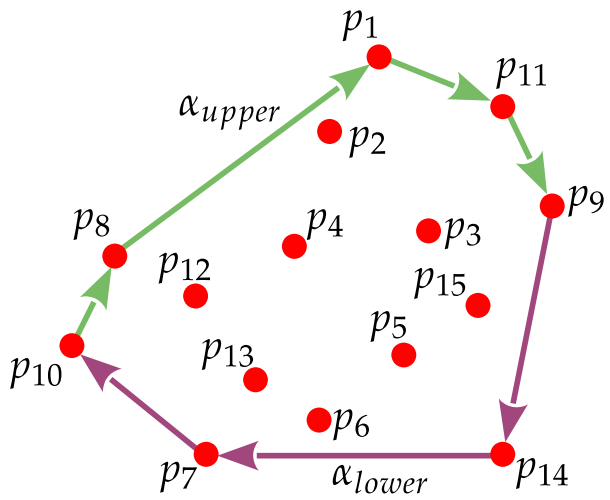
ÿ ÿ ,! , , " ÿ (+) : < , (,! ÿ • (,")

We notice that the first point in ÿ is the beginning, and the last point is the end of the upper part of the convex shell or

- Equally, only in ÿ(+) ( ) *++() reversed, the last point in ÿ is the beginning, and the first point in ÿ is the end of the lower part of the convex shell or

%&'() ( ) (+) ( )

Plošna konveksna ljuska II



- Da li je moguć drukčiji, brži algoritam za pronalazak konveksne ljuske?
- Razdijelimo konveksnu ljusku na gornji i donji dio

$$\alpha(\mathcal{P}) = \alpha_{lower}(\mathcal{P}) \cup \alpha_{upper}(\mathcal{P})$$

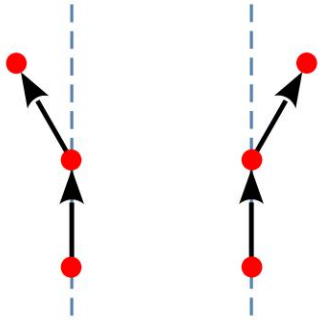
- Horizontalno sortiramo točke u P_p

$$h(P_p) = \{p_{h_i} : p_{h_i} \in P_p\}$$

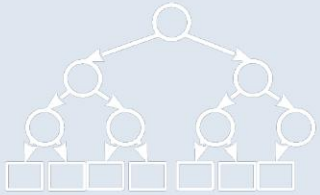
$$\forall p_{h_i}, p_{h_j} \in h(P_p) : i < j, x(p_{h_i}) \leq x(p_{h_j})$$

- Primjećujemo da je prva točka u $h(P_p)$ početak, a zadnja točka u $h(P_p)$ završetak gornjeg dijela konveksne ljuske ili $\alpha_{upper}(\mathcal{P})$
- Istovjetno, samo okrenuto, zadnja točka u $h(P_p)$ početak, a prva točka u $h(P_p)$ završetak donjeg dijela konveksne ljuske ili $\alpha_{lower}(\mathcal{P})$

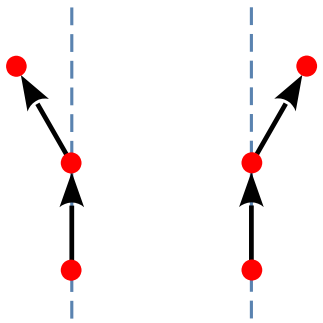
Flat convex shell II



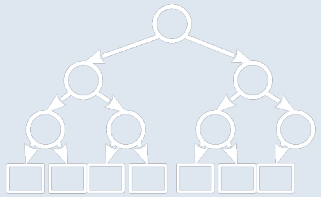
- We continue to go around both parts of the convex shell in a clockwise direction
- This means that the convex shell, and thus both of its parts, is inclined to the **right**
- In order to determine that part of the convex shell is inclined to the right, we need at least three points
- We use the same principle of determining whether the third point in a row is to the right or to the left of the line formed by the first two points



Plošna konveksna ljuska II



- I dalje oba dijela konveksne ljuske obilazimo u smjeru kazaljke na sati
- To znači da je konveksna ljuska, a time i oba njena dijela **desno naginjuća**
- Da bismo utvrdili da je dio konveksne ljuske desno naginjući, trebamo barem tri točke
- Koristimo se istim principom utvrđivanja da li je treća točka po redu desno ili lijevo od linije koju čine prve dvije točke

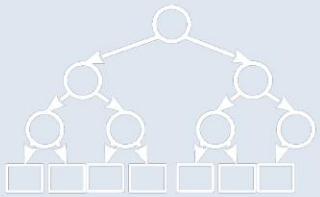


Flat convex shell II

function CONVEXHULL(P_p)
  create $h(P_p)$ as the sorted list of points from P_p
  $\alpha_{upper}(P_p) \leftarrow \{p_{h_1}, p_{h_2}\}$ from $h(P_p)$
  for $i \leftarrow 3$ **to** n **do**
    add p_{h_i} to $\alpha_{upper}(P_p)$
    while $|\alpha_{upper}(P_p)| \geq 3$ and last three points incline left **do**
      remove the point before the last from $\alpha_{upper}(P_p)$
  $\alpha_{lower}(P_p) \leftarrow \{p_{h_n}, p_{h_{n-1}}\}$ from $h(P_p)$
  for $i \leftarrow n-2$ **downto** 1 **do**
    add p_{h_i} to $\alpha_{lower}(P_p)$
    while $|\alpha_{lower}(P_p)| \geq 3$ and last three points incline left **do**
      remove the point before the last from $\alpha_{lower}(P_p)$
  return $\alpha_{upper}(P_p) \cup \alpha_{lower}(P_p)$

Complexity = (log)

- First, we sort the list of points horizontally
- We add the first two points from ÿ to (!) in the upper convex shell
- In the loop, we start from the third point
  - If the last three points lean to the left, we remove the middle one from the upper convex shell
- When we reach the last point in, we get ÿ (! ,) the upper convex one shell
- For the lower one, the same procedure is done, but in reverse order through (!)



Plošna konveksna ljuska II

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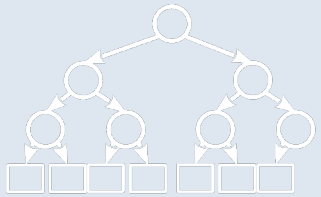
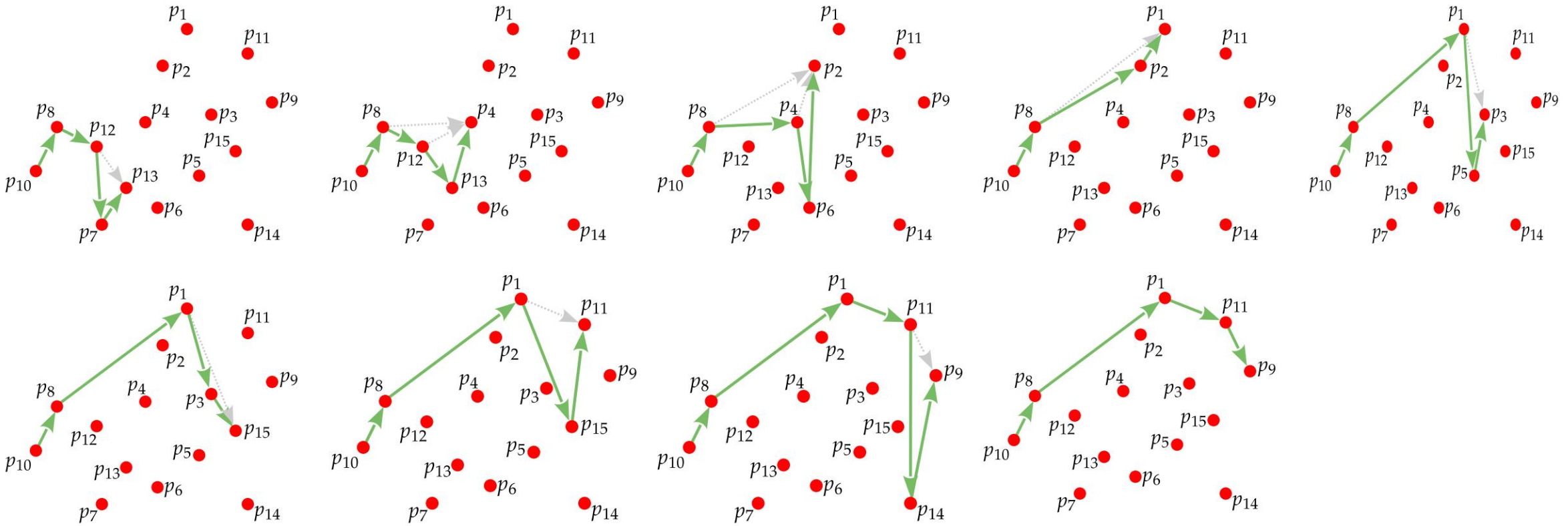
  for $i \leftarrow n-2$ **downto** 1 **do**
    add p_{h_i} to $\alpha_{lower}(P_p)$
    while $|\alpha_{lower}(P_p)| \geq 3$ and last three points incline left **do**
      remove the point before the last from $\alpha_{lower}(P_p)$
  return $\alpha_{upper}(P_p) \cup \alpha_{lower}(P_p)$

Kompleksnost = $O(n \log n)$

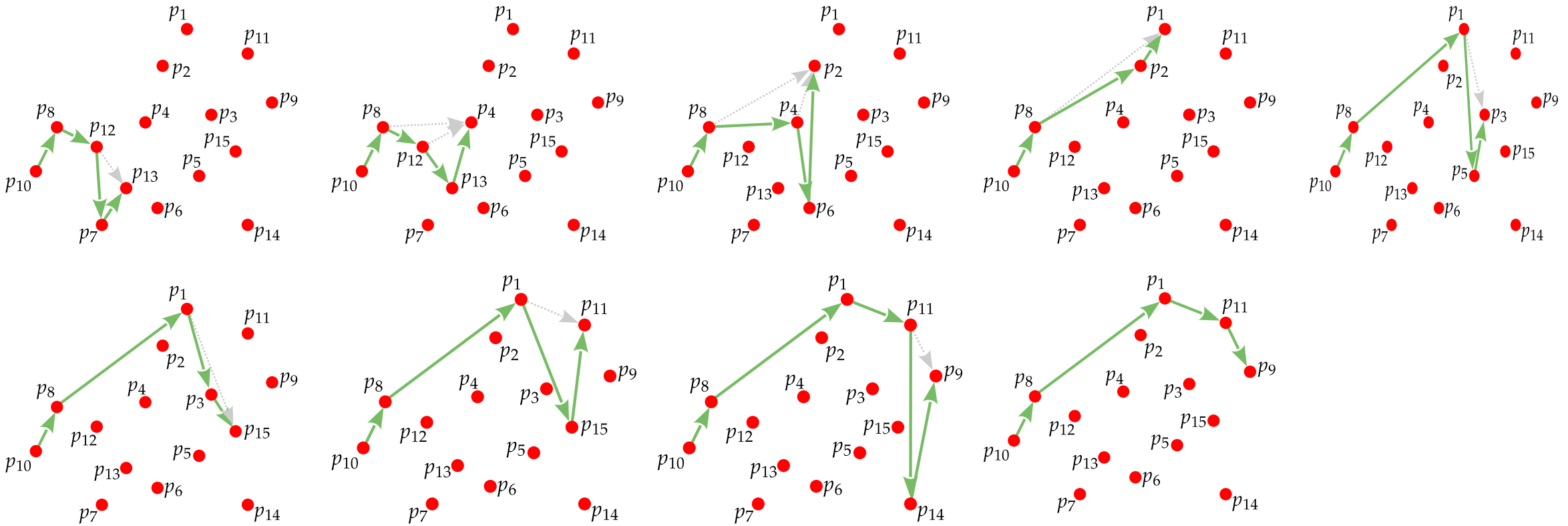
- Prvo horizontalno sortiramo listu točaka
- Dodamo prve dvije točke iz $h(P_p)$ u gornju konveksnu ljusku
- U petlji krećemo od treće točke
  - Ukoliko zadnje tri točke naginju lijevo, uklanjamo srednju od njih iz gornje konveksne ljuske
- Kada stignemo do zadnje točke u $h(P_p)$, dobili smo gornju konveksnu ljusku
- Za donju se radi isti postupak, ali obrnutim redoslijedom kroz $h(P_p)$



Flat convex shell II



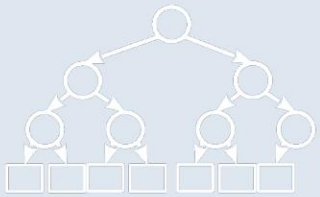
Plošna konveksna ljuska II



Flat convex shell II

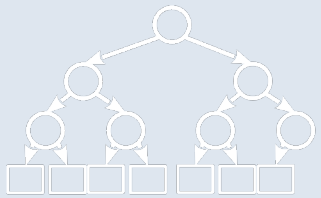
- The algorithm goes through all n points from (, which would be the complexity
  ( )
- Since we need to use one of the sorting algorithms to create ÿ, the complexity of the sorting algorithm prevails and the total complexity is (log)

- The convex hull concept is used later in linear programming

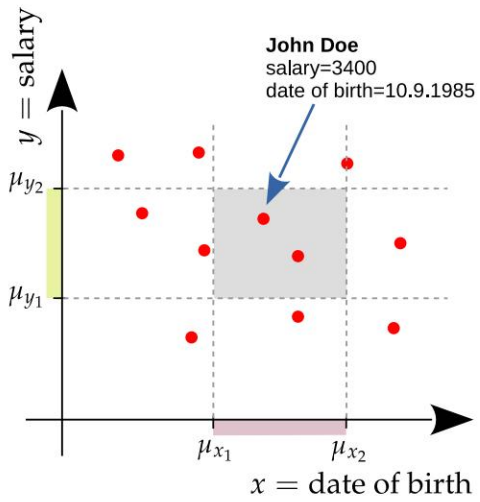


Plošna konveksna ljuska II

- Algoritam prolazi kroz svih n točaka iz P_p, čime bi kompleksnost bila $O(n)$
- Kako imamo potrebu koristiti neki od algoritama za sortiranje, za stvaranje $h(P_p)$, tako nam kompleksnost algoritma za sortiranje prevladava i ukupna kompleksnost je $O(n \log n)$
- Koncept konveksne ljuske koristi se kasnije u linearnom programiranju

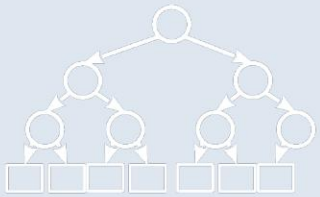


Geometric search

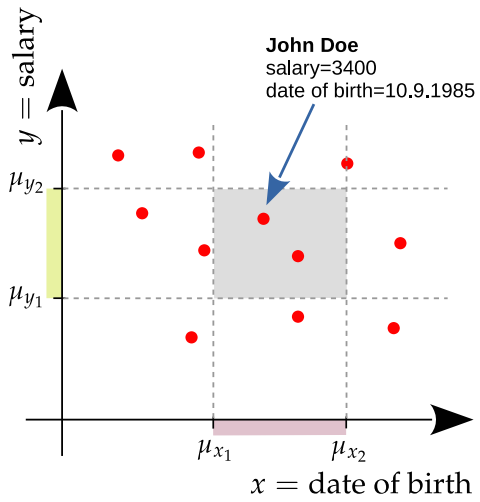


- Let's imagine that we have a set of data in the database that we can place in -dimensional space
- Some of the dimensions can be *height, weight, salary,* etc...
- Let's forget all the possibilities of the database for a moment
- B+ index trees help us retrieve data for a specific attribute value

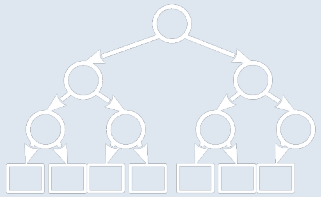
- How to retrieve a dataset that has a specific attribute in a certain range of values?
- For example: let's find all people whose height is between 165 and 184 centimeters
- In general, we have a set of values (1-dimensional) from which we want to find values in the required range as quickly as possible



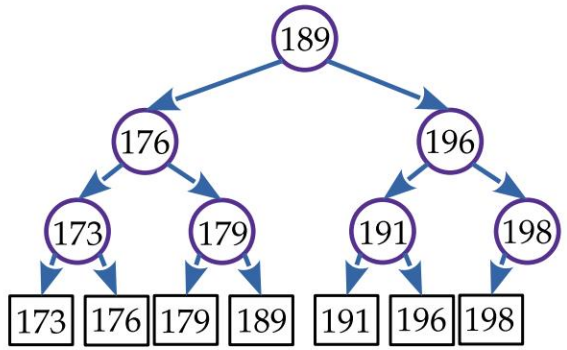
Geometrijsko pretraživanje



- Zamislimo da u bazi podataka imamo skup podataka koji možemo smjestiti u n-dimenzionalni prostor
- Neke od dimenzija mogu biti *visina*, *težina*, *plaća*, itd...
- Zaboravimo na trenutak sve mogućnosti baze podataka
- B^+ indeksna stabla nam pomažu dohvatiti podatke za točno određenu vrijednost atributa
- Kako dohvatiti skup podataka koji ima određeni atribut u određenom rasponu vrijednosti?
- Na primjer: pronađimo sve osobe koje su visine od 165 do 184 centimetara
- Generalno, imamo skup vrijednosti (1-dimenzionalan) iz kojeg što je brže moguće želimo pronaći vrijednosti u traženom rasponu



Range Search (1D)

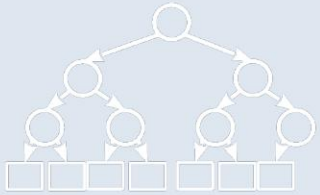


- We have a set of values , and the range we are [$, %]

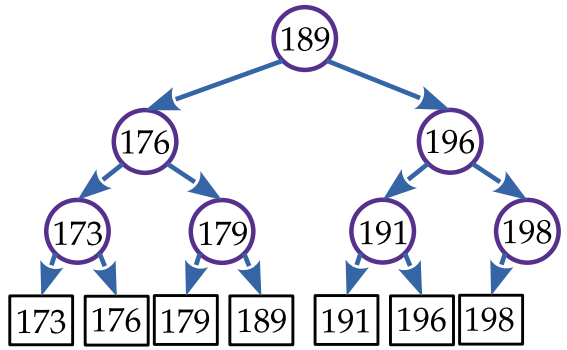
looking for • For example = {173, 176, 179, 189, 191, 196, 198}

- How to organize a set of values so that we have the ability to find the range $, % in the [most efficient way possible?
- We resort to using binary trees
- In this case, we use binary trees that have concrete values in their leaves - we call them **range trees**
  - Leaves have values from •

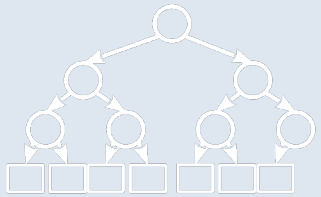
  Internal nodes are guiding *and* do not need to have the same values as leaves
  - Principle similar to that of B+ trees



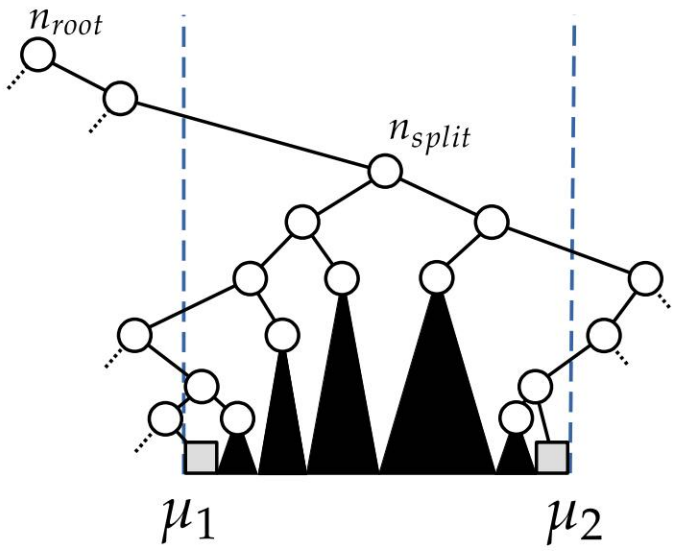
Pretraživanje raspona (1D)



- Imamo skup vrijednosti P, te raspon koji tražimo $[\mu_1, \mu_2]$
- Na primjer $P = \{173, 176, 179, 189, 191, 196, 198\}$
- Kako organizirati skup vrijednosti P tako da imamo mogućnost pronalaska raspona $[\mu_1, \mu_2]$ na najefikasniji mogući način?
- Pribjegavamo korištenju binarnih stabala
- U ovom slučaju koristimo binarna stabla koja u listovima imaju konkretne vrijednosti – zovemo ih **stabla raspona**
  - Listovi imaju vrijednosti iz P
  - Unutarnji čvorovi su navodeći (*guiding*) i ne trebaju imati iste vrijednosti kao listovi
  - Princip sličan kao i kod B^+ stabala

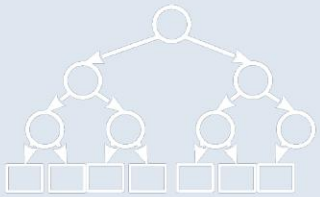


Range Search (1D)

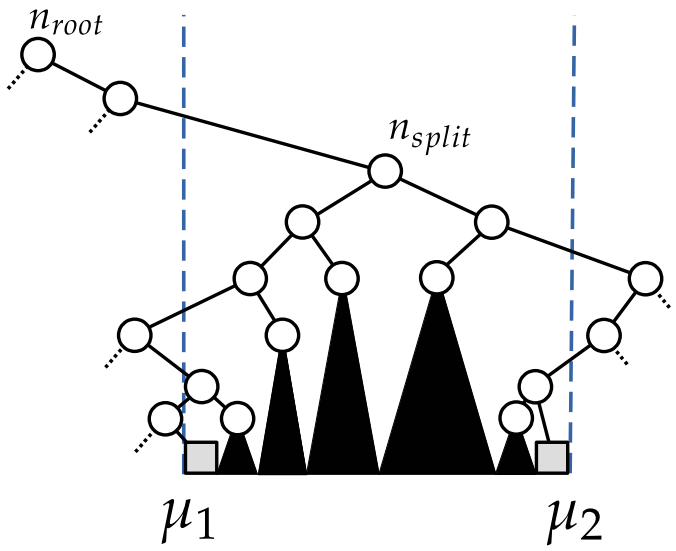


- We have a binary tree such that each internal node has a value such that:
  - In the left subtree are all values that are ÿ
  - In the right subtree are all values that are >

1. We start from the root node by looking for the first node that is in the required range [", #] the so-called *separating node* *(+,-
2. From the separating node, we move to *(+,+) to the left and to the right, thus obtaining two vertical paths from the node to the leaf: ▪ /

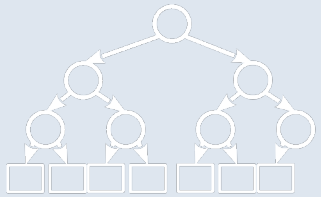


Pretraživanje raspona (1D)

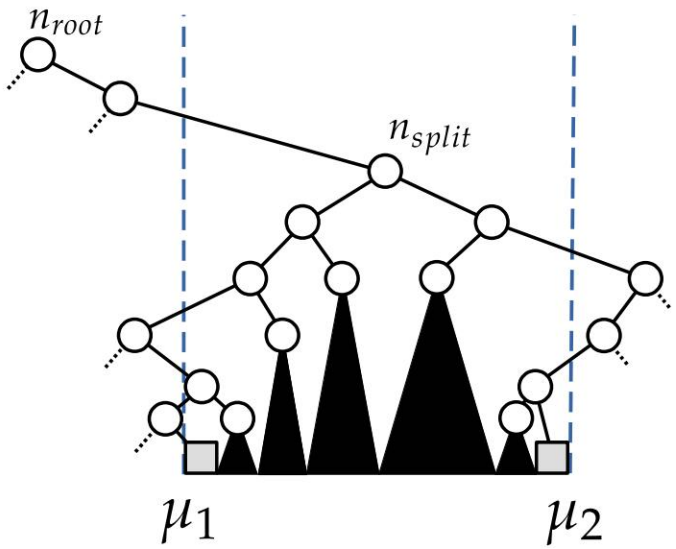


- Imamo binarno stablo τ, takvo da svaki unutarnji čvor ima vrijednost da vrijedi:
  - U lijevom podstablu su sve vrijednosti koje su $\leq$
  - U desnom podstablu su sve vrijednosti koje su $>$

1. Krenemo od korijenskog čvora tražeći prvi čvor koji je u traženom rasponu $[\mu_1, \mu_2]$, takozvani *razdvajajući čvor* n_{split}
2. Od razdvajajućeg čvora n_{split} krećemo se na lijevu i na desnu stranu, tako dobivamo dvije vertikalne putanje od čvora do lista: $\mathcal{V}_L$ i $\mathcal{V}_R$



Range Search (1D)



3. Moving to the left, we know that the nodes are smaller than and we have two options:

a) The node is within the required range !, " , which means

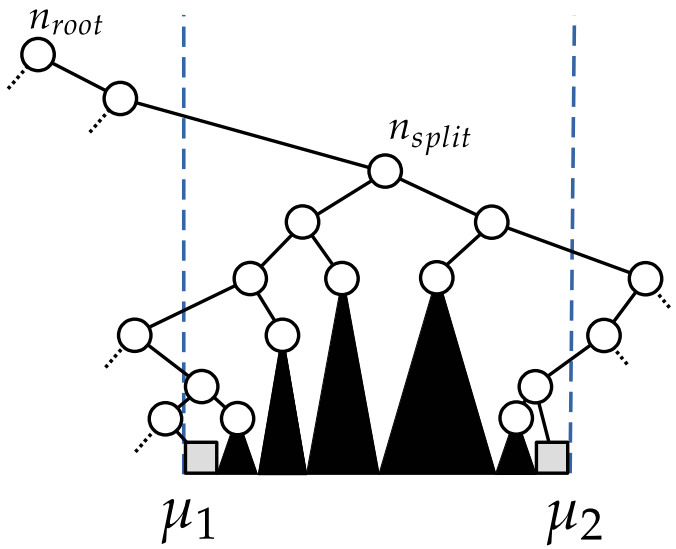
- Its right subtree is certainly in the required range and we automatically add all values to the result *(pruning).*
- We are not sure about its left subtree, so we move to left child

b) The node is not within the required range, which means

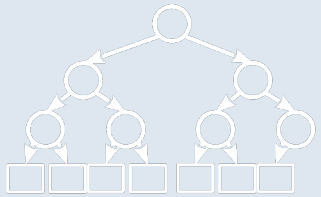
- Its left subtree is definitely not in range and we ignore it
- We are not sure about its right subtree, so we move to the right child

4. By moving to the right, we perform mirror operations

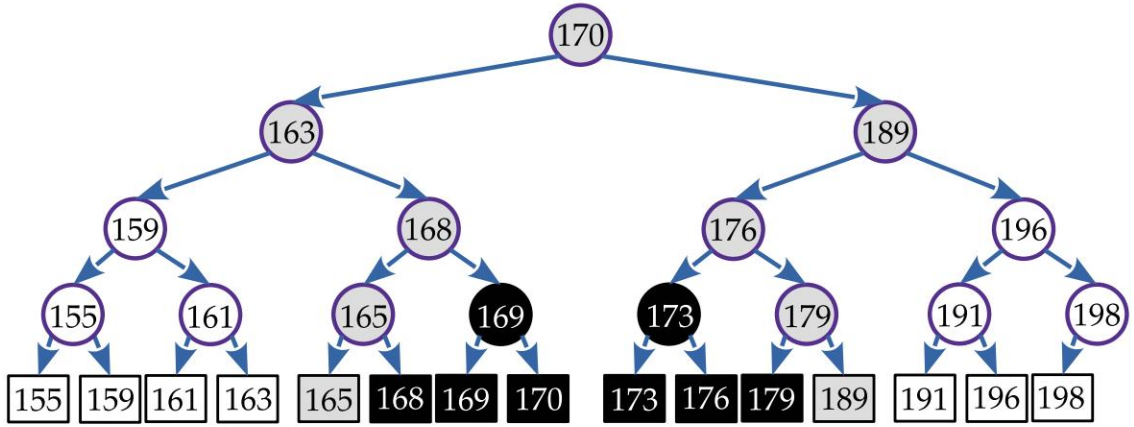
Pretraživanje raspona (1D)



3. Krećući se na lijevo, znamo da su čvorovi manji od n_{split}, te imamo dvije opcije:
   a) Čvor je unutar traženog raspona $[\mu_1, \mu_2]$, što znači
      - Njegovo desno podstablo je sigurno u traženom rasponu i sve vrijednosti automatski dodajemo rezultatu (*pruning*)
      - Za njegovo lijevo podstablo nismo sigurni, pa se krećemo na lijevo dijete
   b) Čvor nije unutar traženog raspona, što znači
      - Njegovo lijevo podstablo sigurno nije u rasponu i ignoriramo ga
      - Za njegovo desno podstablo nismo sigurni, pa se krećemo na desno dijete
4. Kretanjem u desno izvodimo zrcalne operacije



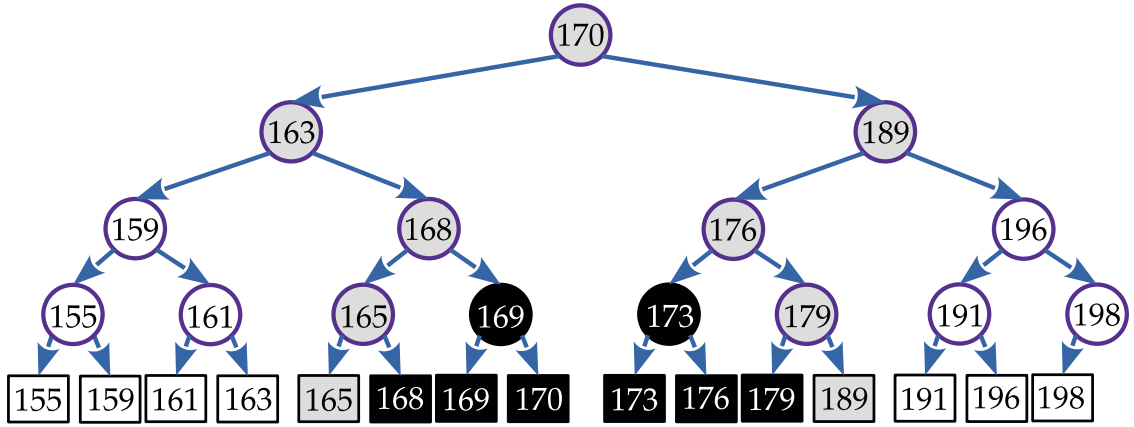
Range Search (1D)



Complexity = (+ log)

- Example of tree and search range 165,184 [ ]
- The root node is also • The gray nodes are tested #$%&'
- Black nodes are automatically added to the result *(pruned)*
- White nodes are discarded as out of range
- Search complexity is (+ log), where is the number of automatically added values, and log is two passes through the binary tree

Pretraživanje raspona (1D)



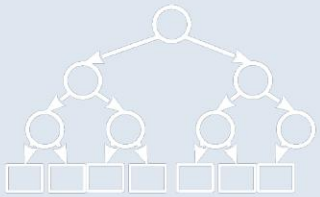
Kompleksnost = $O(k + \log n)$

- Primjer stabla i pretraživanja raspona $[165,184]$
- Korijenski čvor je ujedno i n_{split}
- Sivi čvorovi su testirani
- Crni čvorovi su automatski dodani u rezultat *(pruned)*
- Bijeli čvorovi su odbačeni kao van raspona
- Kompleksnost pretraživanja je $O(k + \log n)$, gdje je k broj automatski dodanih vrijednosti, a $\log n$ su dva prolaska kroz binarno stablo

Range Search (1D)

```
function 1DRangeQuery(τ, μ1, μ2)
    rv ← ∅
    n_split ← FindSplittingNode(τ, μ1, μ2)
    if n_split is leaf then
        if v(n_split) is in the range [μ1, μ2] then
            add n_split to rv
    else
                                                ▷ left traversal
        n ← leftChild(n_split)
        while n is not leaf do
            if μ1 ≤ v(n) then
                add ReportSubtree(rightChild(n)) to rv
                n ← leftChild(n)
            else
                n ← rightChild(n)
        if n is leaf and v(n) is in the range [μ1, μ2] then
            add n to rv
                                                ▷ right traversal
        n ← rightChild(n_split)
        while n is not leaf do
            if v(n) ≤ μ2 then
                add ReportSubtree(leftChild(n)) to rv
                n ← rightChild(n)
            else
                n ← leftChild(n)
        if n is leaf and v(n) is in the range [μ1, μ2] then
            add n to rv
    return rv
```

```
function FindSplittingNode(τ, μ1, μ2)
    n ← root(τ)
    while n is not leaf and (μ1 ≥ v(n) or μ2 ≤ v(n)) do
        if μ2 ≤ v(n) then
            n ← leftChild(n)
        else
            n ← rightChild(n)
    return n
```



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    else
                                                ▷ left traversal
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        while n is not leaf do
            if μ1 ≤ v(n) then
                add ReportSubtree(rightChild(n)) to rv
                n ← leftChild(n)
            else
                n ← rightChild(n)
        if n is leaf and v(n) is in the range [μ1, μ2] then
            add n to rv
                                                ▷ right traversal
        n ← rightChild(n_split)
        while n is not leaf do
            if v(n) ≤ μ2 then
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                n ← rightChild(n)
            else
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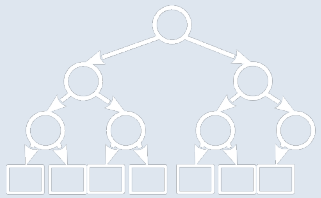
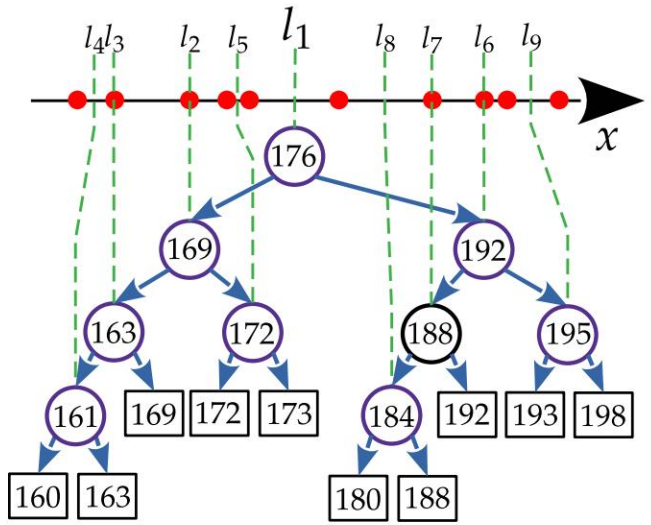
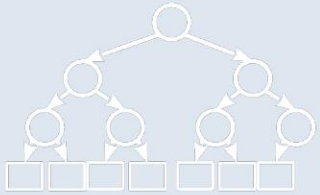


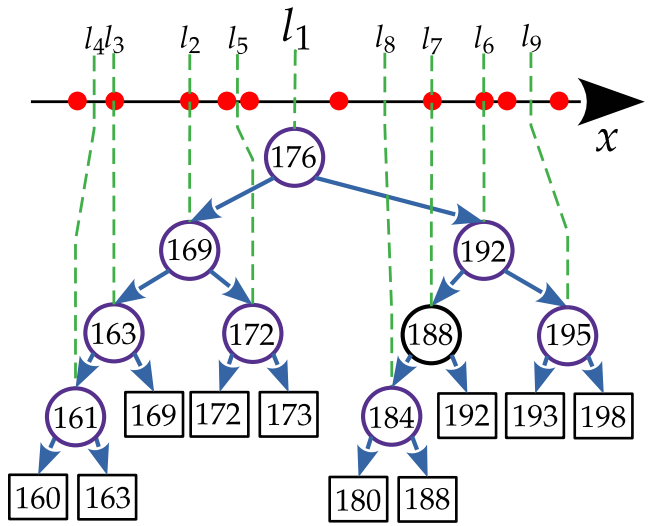
Range Search (1D)



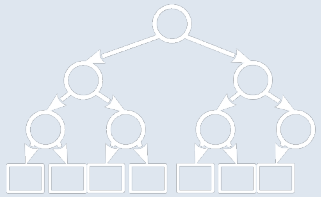
- How to create a binary tree that reflects a specific set of values ?
- With binary trees, we learned how to create a tree based on a sorted set of values - thoughtfully
- A similar principle is used here
  - The created node contains the median of all values that are in to its subtree - we want the tree to be balanced
  - Its left subtree contains only values that are ÿ of the value of the node
  - Its right subtree contains only values that are > of the value of the node
  - Using a recursive procedure, we create a binary tree up to its leaves - concrete values



Pretraživanje raspona (1D)



- Kako stvoriti binarno stablo koje reflektira određeni skup vrijednosti P?
- Kod binarnih stabala učili smo kako stvoriti stablo na temelju sortiranog skupa vrijednosti - promišljeno
- Sličan princip koristi se i ovdje
  - Stvoreni čvor sadrži medijan svih vrijednosti koje su u njegovom podstablu – želimo da stablo bude uravnoteženo
  - Njegovo lijevo podstablo sadrži samo vrijednosti koje su $\leq$ od vrijednosti čvora
  - Njegovo desno podstablo sadrži samo vrijednosti koje su $>$ od vrijednosti čvora
  - Rekurzivnim postupkom stvaramo binarno stablo sve do njegovih listova – konkretnih vrijednosti

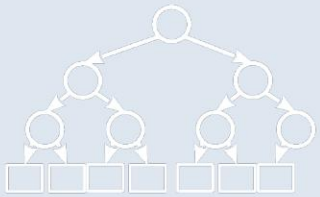


Range Search (1D)

```
function CREATE1DRANGETREE(P_p)
    if |P_p| = 1 then
        return n ← leaf having the value from P_p
    else
        v_m ← median(P_p)
        P_{p_left} ← {p_i : p_i ∈ P_p ∧ p_i ≤ v_m}
        P_{p_right} ← {p_i : p_i ∈ P_p ∧ p_i > v_m}
        n_left ← CREATE1DRANGETREE(P_{p_left})
        n_right ← CREATE1DRANGETREE(P_{p_right})
        n ← create node having value v_m
        leftChild(n) ← n_left
        rightChild(n) ← n_right
    return n
```

1. If we got only one value
   a) Let's return the node with that value (leaf)
2. If we got more values
   a) We calculate the median for the root node
   b) Divide the values into left (ÿ) and right (>)
   c) Recursively create left and right subtrees
   d) Place the created subtrees as left and right a child of the root node

- The complexity of creating a range tree is (log) for sorting due to searching for the median



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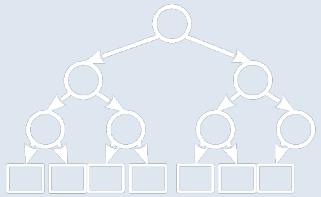
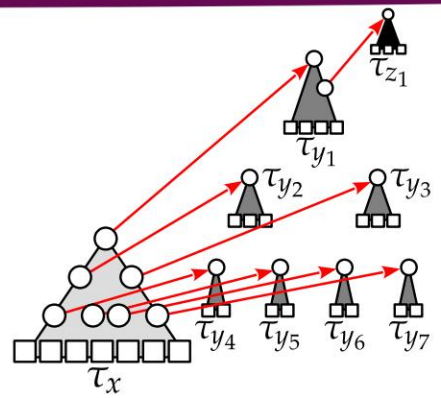
        n ← create node having value v_m
        leftChild(n) ← n_left
        rightChild(n) ← n_right
    return n
```

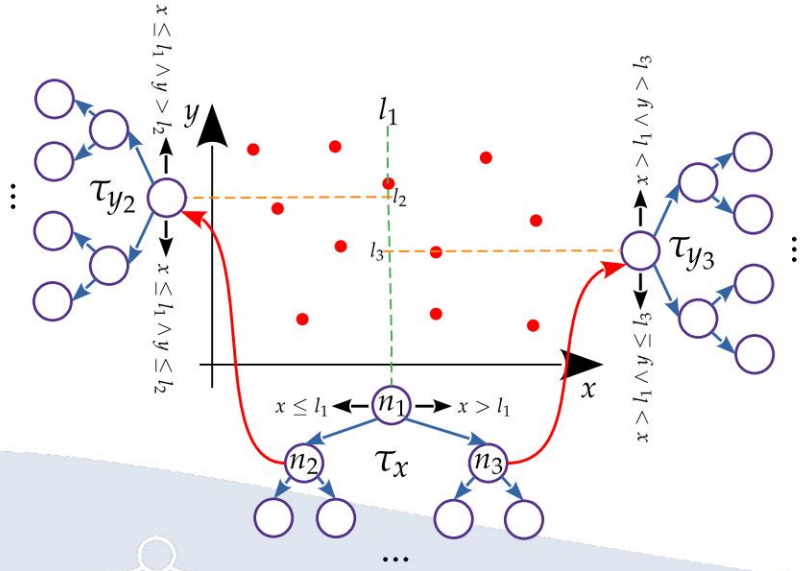
1. Ako smo dobili samo jednu vrijednost
   a) Vratimo čvor s tom vrijednošću (list)
2. Ako smo dobili više vrijednosti
   a) Izračunamo medijan za korijenski čvor
   b) Podijelimo vrijednosti na lijeve ($\leq$) i desne ($>$)
   c) Rekurzivno stvorimo lijevo i desno podstablo
   d) Postavimo stvorena podstabla kao lijevo i desno dijete korijenskog čvora

- Kompleksnost stvaranja stabla raspona je $O(n \log n)$ radi sortiranja zbog traženja medijana



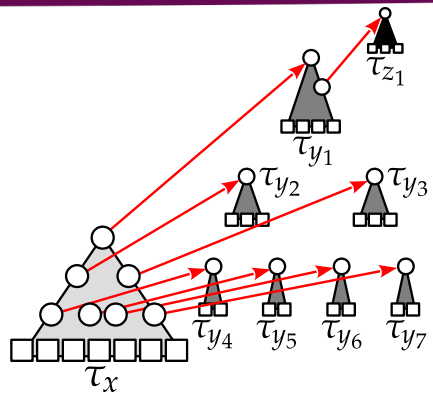
Range Search (2D)

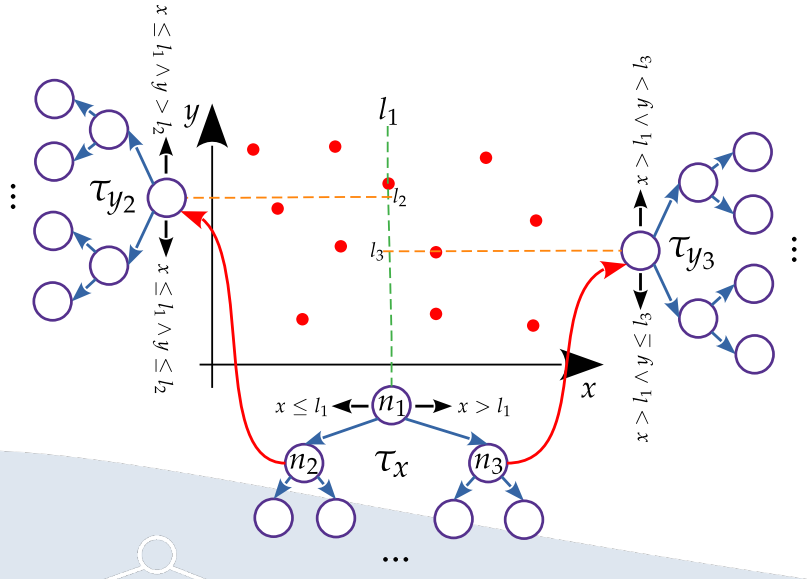




- To each node of the range tree, we add an subtree • %**01 additional dimension ( )
  This chaining can be continued for other dimensions
- %**01 represents a range tree for all points represented by a node but for a higher dimension ,, if , for -os, then it is formed for -os %**01
- Search complexity for two dimensions is now (+ log#) for chaining trees
- The complexity of creation is still (log) for sorting

Pretraživanje raspona (2D)



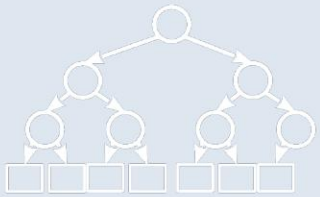


- Svakom čvoru stabla raspona dodajemo podstablo τ_{assoc} za dodatnu dimenziju (y)
  - Ovo ulančavanje se može nastaviti za ostale dimenzije
- τ_{assoc} predstavlja stablo raspona za sve točke koje predstavlja čvor n_i, ali za višu dimenziju – ako je n_i za x-os, tada se τ_{assoc} formira za y-os
- Kompleksnost pretraživanja za dvije dimenzije je sada $O(k + \log^2 n)$ radi ulančavanja stabala
- Kompleksnost stvaranja je i dalje $O(n \log n)$ radi sortiranja

Interval search (1D)

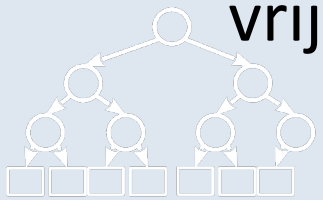
- Let's imagine a set of intervals = { , = ([,!, ," ,] ,) ,!, ,"", , ÿ ÿ } which are parallel to the -axis – actually horizontal line segments
- Dots (,!, ,) (,", , are called the endpoints of the interval
- We would like to have a structure in which we save a set of intervals, so that we can quickly and efficiently find the intervals that are in the query window

  2 = [3!, 3"] × [4!, 4"]
- The solution is in a composite binary tree that stores values in nodes - we call them **interval trees**
- In this case, we do not have sheets representing specific values

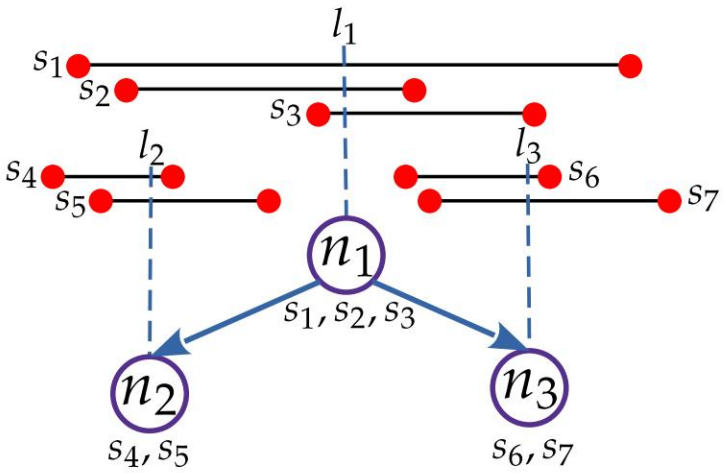


Pretraživanje intervala (1D)

- Zamislimo skup intervala $I = \{s_i = ([x_{i_1}, x_{i_2}], y_i) : x_{i_1}, x_{i_2}, y_i \in \mathbb{R}\}$ koji su paralelni sa x-osi – zapravo horizontalni linijski segmenti
- Točke (x_{i_1}, y_i) i (x_{i_2}, y_i) zovemo krajnjim točkama intervala
- Želja nam je imati strukturu u koju spremamo skup intervala, tako da brzo i efikasno možemo pronaći intervale koji se nalaze u prozoru upita $W_q = [q_{x_1}, q_{x_2}] \times [q_{y_1}, q_{y_2}]$
- Rješenje se nalazi u kompozitnom binarnom stablu koje vrijednosti sprema u čvorovima – zovemo ih **stabla intervala**
- U ovom slučaju nemamo listove koji predstavljaju konkretne vrijednosti

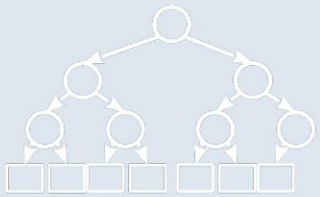


Interval search (1D)

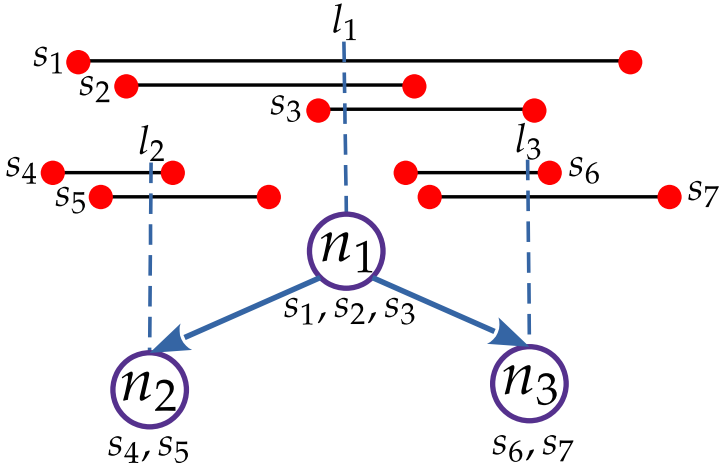


- We calculate the median) of all endpoints of the interval from the set of intervals for which we create a tree
  - For the root node the set of intervals is •

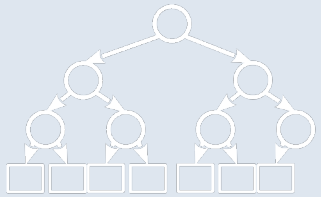
  For each node below the root it is a subset of – the set of intervals stored in the parent
- Let's create a node with the value of the calculated median)
  - Let's save all the intervals that intersect • U in the node (! this is how we save the intervals I(!) = { !, ",) }
- We recursively create left and right subtrees
  - All intervals that are both extreme go into the left subtree points < (
  - All intervals that are both extreme go into the right subtree points > (



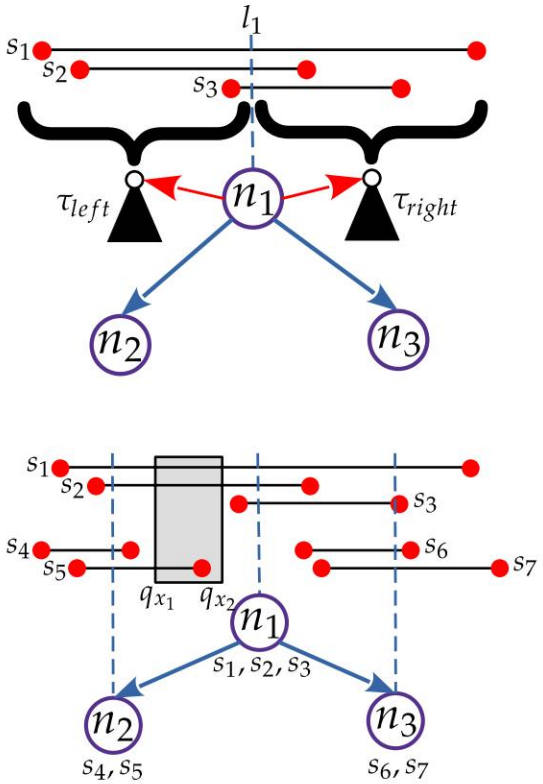
Pretraživanje intervala (1D)



- Izračunamo medijan v_m svih krajnjih točaka intervala iz skupa intervala za koji stvaramo stablo
  - Za korijenski čvor skup intervala je I
  - Za svaki čvor ispod korijenskog to je podskup od I – skupa intervala spremljenog u roditelja
- Stvorimo čvor s vrijednošću izračunatog medijana v_m
  - U čvor spremimo sve intervale koji presijecaju v_m
  - U n_1 tako spremamo intervale $\mathrm{I}(n_1) = \{s_1, s_2, s_3\}$
- Rekurzivno stvaramo lijevo i desno podstablo
  - U lijevo podstablo idu svi intervali čije su obje krajnje točke $< v_m$
  - U desno podstablo idu svi intervali čije su obje krajnje točke $> v_m$

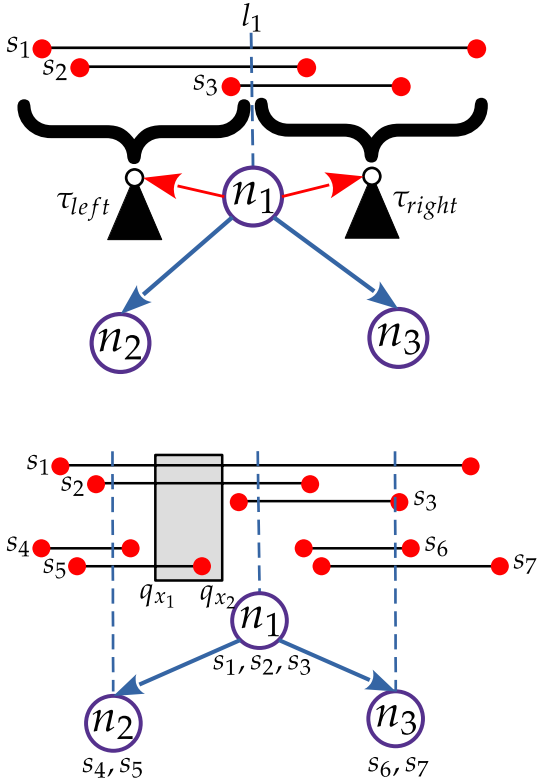


Interval search (1D)

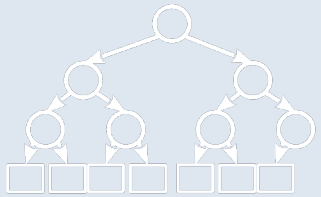


- Two additional data structures are added to each node (2D range trees for example)
  - In the left #$%& , the left endpoints of the intervals saved in that node (ÿ) are stored
  - In the right node (>) the right endpoints of the intervals saved in that '()*&
  - Endpoints in '()*& trees have back references to intervals in nodes to avoid any additional searching
- We do this because the search results in two possible scenarios:
  1. The interval completely intersects, which means that its left endpoint is < and its right endpoint is +!, [+!, +"]
  2. An interval has a beginning or end at [+!, +" ,] which means that it is left or right an end point in that range
  3. We should also make sure that the interval is in the range [,!, ,"]
- This can be determined through additional structures %*+' or ,&-.'
  - That is why the interval tree is called composite

Pretraživanje intervala (1D)



- Svakom čvoru se dodaju dvije dodatne strukture podataka (2D stabla raspona recimo)
  - U lijevom τ_{left} se čuvaju lijeve krajnje točke intervala spremljenih u taj čvor ($\leq$)
  - U desnom τ_{right} se čuvaju desne krajnje točke intervala spremljenih u taj čvor ($>$)
  - Krajnje točke u stablima τ_{left} i τ_{right} imaju povratne reference na intervale u čvorovima kako bi se izbjeglo bilo kakvo dodatno pretraživanje
- Ovo radimo zato što pretraživanje rezultira sa dva moguća scenarija:
  1. Interval u potpunosti presijeca $[q_{x_1}, q_{x_2}]$, što znači da mu je lijeva krajnja točka $< q_{x_1}$, a desna $> q_{x_2}$
  2. Interval ima početak ili kraj u $[q_{x_1}, q_{x_2}]$, što znači da mu je lijeva ili desna krajnja točka u tom rasponu
  3. Trebamo paziti i da je interval u rasponu $[q_{y_1}, q_{y_2}]$
- Ovo se može utvrditi kroz dodatne strukture τ_{left} ili τ_{right}
  - Zato se stablo intervala i naziva kompozitnim

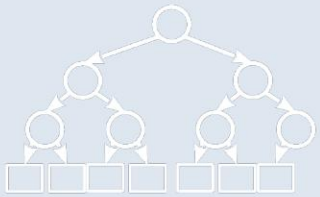


Interval search (1D)

function CREATEINTERVALTREE(I)
  if $I = \emptyset$ **then**
    $n \leftarrow$ empty leaf
  else
    $x_{med} \leftarrow median(endpoints(I))$
    $I_{left} \leftarrow \{s_i : s_i \in I \wedge x_{i_1}(s_i) < x_{med} \wedge x_{i_2}(s_i) < x_{med}\}$
    $I_{right} \leftarrow \{s_i : s_i \in I \wedge x_{i_1}(s_i) > x_{med} \wedge x_{i_2}(s_i) > x_{med}\}$
    $I_{med} \leftarrow I \setminus (I_{left} \cup I_{right})$
    $n \leftarrow$ create a node having value x_{med}
    $intervals(n) \leftarrow I_{med}$
    $leftChild(n) \leftarrow$ CREATEINTERVALTREE(I_{left})
    $rightChild(n) \leftarrow$ CREATEINTERVALTREE(I_{right})
    if $I_{med} \neq \emptyset$ **then**
      $P_{left} \leftarrow \{((x_{i_k}(s_i), y_i(s_i)), s_i) : s_i \in I_{mid} \wedge x_{i_k}(s_i) \leq x_{med}\}$
      $P_{right} \leftarrow \{((x_{i_k}(s_i), y_i(s_i)), s_i) : s_i \in I_{mid} \wedge x_{i_k}(s_i) > x_{med}\}$
        ▷ We include the back reference to s_i
      $\tau_{left}(n) \leftarrow$ CREATE2DRANGETREE(P_{left})
      $\tau_{right}(n) \leftarrow$ CREATE2DRANGETREE(P_{right})
  return n

- Due to the need for sorting, the basic interval tree has log (complexity • If we add to that

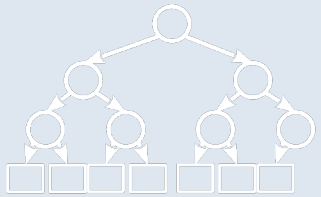
two range trees per log 2 log node, is ultimately log# (((we get))(t • What))
( )



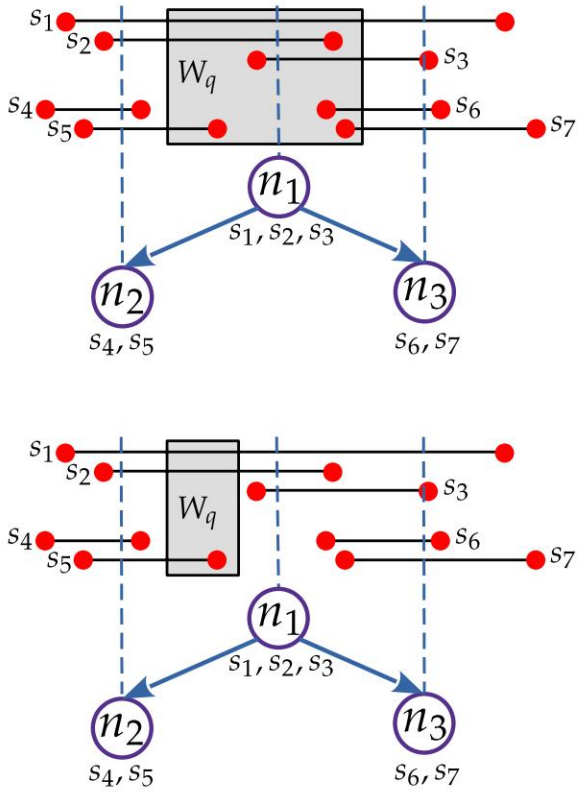
Pretraživanje intervala (1D)

function CreateIntervalTree(I)
  if $I = \emptyset$ **then**
    $n \leftarrow$ empty leaf
  else
    $x_{med} \leftarrow median(endpoints(I))$
    $I_{left} \leftarrow \{s_i : s_i \in I \wedge x_{i_1}(s_i) < x_{med} \wedge x_{i_2}(s_i) < x_{med}\}$
    $I_{right} \leftarrow \{s_i : s_i \in I \wedge x_{i_1}(s_i) > x_{med} \wedge x_{i_2}(s_i) > x_{med}\}$
    $I_{med} \leftarrow I \setminus (I_{left} \cup I_{right})$
    $n \leftarrow$ create a node having value x_{med}
    $intervals(n) \leftarrow I_{med}$
    $leftChild(n) \leftarrow$ CreateIntervalTree(I_{left})
    $rightChild(n) \leftarrow$ CreateIntervalTree(I_{right})
    if $I_{med} \neq \emptyset$ **then**
      $P_{left} \leftarrow \{((x_{i_k}(s_i), y_i(s_i)), s_i) : s_i \in I_{mid} \wedge x_{i_k}(s_i) \leq x_{med}\}$
      $P_{right} \leftarrow \{((x_{i_k}(s_i), y_i(s_i)), s_i) : s_i \in I_{mid} \wedge x_{i_k}(s_i) > x_{med}\}$
        ▷ We include the back reference to s_i
      $\tau_{left}(n) \leftarrow$ Create2DRangeTree(P_{left})
      $\tau_{right}(n) \leftarrow$ Create2DRangeTree(P_{right})
  return n

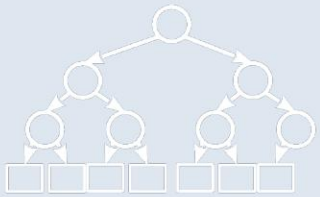
- Zbog potrebe za sortiranjem, osnovno stablo intervala ima kompleksnost $O(n \log n)$
- Dodamo li tome dva stabla raspona po čvoru, dobivamo $O\big((n \log n)(2 \log n)\big)$
- Što je u konačnici $O(n \log^2 n)$



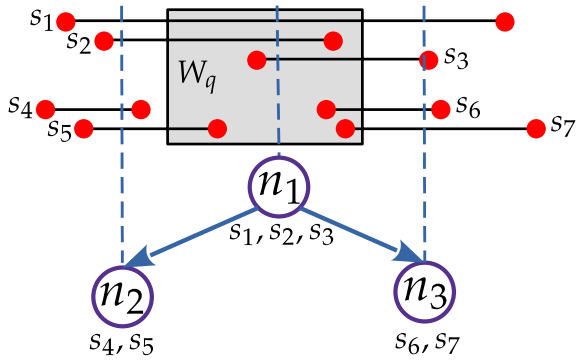
Interval search (1D)

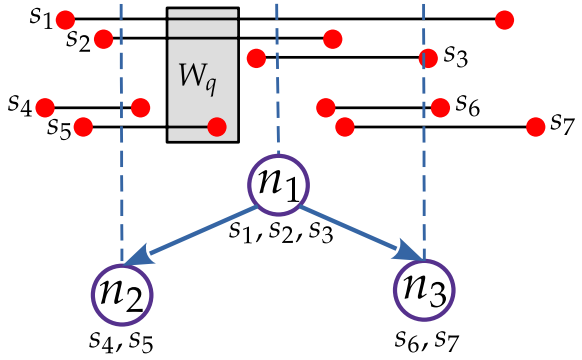


- Searching an interval tree is similar to searching a range tree. If we are currently in node 2 (starting from node 1), then we have three basic cases:
  - the median of node !"# $ is within the range %!, %" [ ]
    - both trees are consulted in order to select intervals that '()*&, and #$%& end in window +!, +" [ ] × [,!, ,"]
  - the median node !"# $( ) is on the left side of the range %!, %" [ ]
    - the right additional tree is consulted in order to select the intervals that '()*&, end on the right side in the window +!, +" [ ] × [,!, ,"]
    - for the left subtree of the node, we consider that there are no required intervals
    - we continue in the right subtree
  - the median !"# $( ) is to the right of the range %!, %" [ ] node • the left additional tree is %&, × in order to select the intervals that #$ consulted, the left side ends in the window]+!, +" [,!, ,"
    - for the right subtree of the node (we consider that there are no required intervals
    - we continue in the left subtree

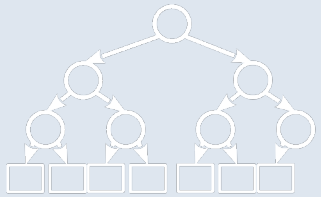


Pretraživanje intervala (1D)

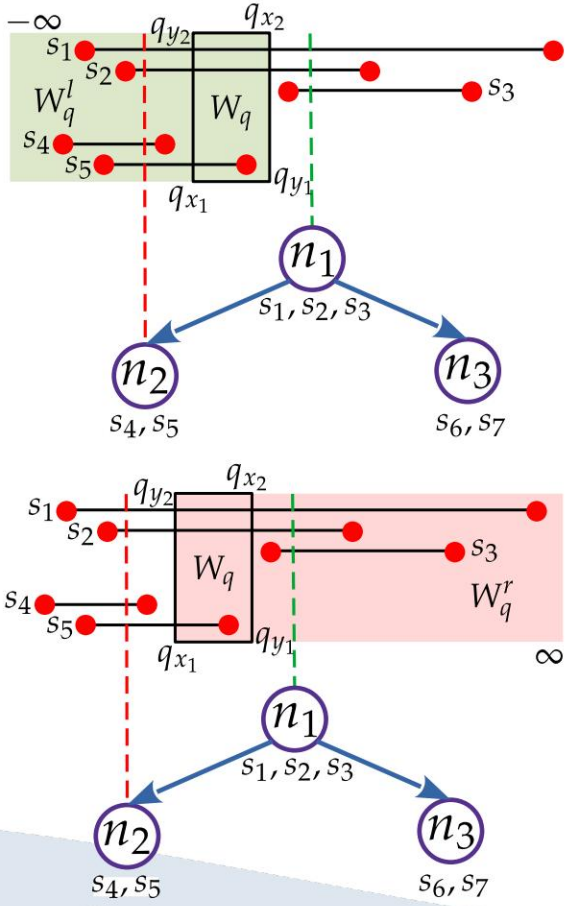




- Pretraživanje stabla intervala slično je pretraživanju stabla raspona. Ako se trenutno nalazimo u čvoru n_i (krećemo od korijenskog čvora), tada imamo tri osnovna slučaja:
  - medijan čvora $x_{med}(n_i)$ je unutar raspona $[q_{x_1}, q_{x_2}]$
    - konzultiraju se oba stabla τ_{left} i τ_{right}, kako bi se selektirali intervali koji završavaju u prozoru $[q_{x_1}, q_{x_2}]\times[q_{y_1}, q_{y_2}]$
  - medijan čvora $x_{med}(n_i)$ s lijeve je strane raspona $[q_{x_1}, q_{x_2}]$
    - konzultira se desno dodatno stablo τ_{right}, kako bi se selektirali intervali koji desnom stranom završavaju u prozoru $[q_{x_1}, q_{x_2}]\times[q_{y_1}, q_{y_2}]$
    - za lijevo podstablo čvora n_i smatramo da nema traženih intervala
    - nastavljamo u desnom podstablu
  - medijan čvora $x_{med}(n_i)$ s desne je strane raspona $[q_{x_1}, q_{x_2}]$
    - konzultira se lijevo dodatno stablo τ_{left}, kako bi se selektirali intervali koji lijevom stranom završavaju u prozoru $[q_{x_1}, q_{x_2}]\times[q_{y_1}, q_{y_2}]$
    - za desno podstablo čvora n_i smatramo da nema traženih intervala
    - nastavljamo u lijevom podstablu

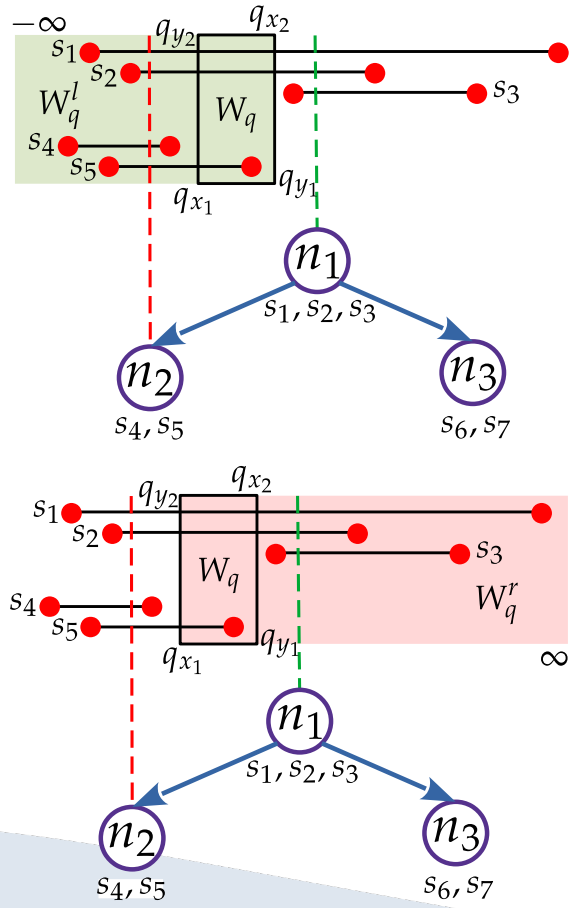


Interval search (1D)



- But what happens to intervals that completely intersect the range, like say # in the example? "
- Instead of searching endpoints exclusively in the range 3!, 3" , let's expand the search [ ]
  - We search for the left endpoints in the window (ÿÿ, 3-]× [4., 4-]
  - We search for the right endpoints in the window [3rd, ÿ)× [4., 4-]

Pretraživanje intervala (1D)



- No, što se dešava s intervalima koji u potpunosti presijecaju raspon, kao recimo s_1 i s_2 u primjeru?
- Umjesto da krajnje točke pretražujemo isključivo u rasponu $[q_{x_1}, q_{x_2}]$, pretraživanje proširimo
  - Lijeve krajnje točke pretražujemo u prozoru $(-\infty, q_{x_2}] \times [q_{y_1}, q_{y_2}]$
  - Desne krajnje točke pretražujemo u prozoru $[q_{x_1}, \infty) \times [q_{y_1}, q_{y_2}]$

Interval search (1D)

function INTERVALQUERY($n, [q_{x_1}, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
 $iv \leftarrow \emptyset$
 if $q_{x_1} \leq x_{med}(n) \leq q_{x_2}$ **then**
  $p \leftarrow$ 2DRANGEQUERY($\tau_{left}(n), (-\infty, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
  $p \leftarrow p \cup$ 2DRANGEQUERY($\tau_{right}(n), [q_{x_1}, \infty) \times [q_{y_1}, q_{y_2}]$)
  $iv \leftarrow$ all intervals of the endpoints in p
  $move \leftarrow both$
 else if $x_{med}(n) < q_{x_1}$ **then**
  $p \leftarrow$ 2DRANGEQUERY($\tau_{right}(n), [q_{x_1}, \infty) \times [q_{y_1}, q_{y_2}]$)
  $iv \leftarrow$ all intervals of the endpoints in p
  $move \leftarrow right$
 else if $q_{x_2} < x_{med}(n)$ **then**
  $p \leftarrow$ 2DRANGEQUERY($\tau_{left}(n), (-\infty, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
  $iv \leftarrow$ all intervals of the endpoints in p
  $move \leftarrow left$
 if $move \in \{both, left\}$ and exists $lc \leftarrow leftChild(n)$ **then**
  $iv \leftarrow iv \cup$ INTERVALQUERY($lc, [q_{x_1}, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
 if $move \in \{both, right\}$ and exists $rc \leftarrow rightChild(n)$ **then**
  $iv \leftarrow iv \cup$ INTERVALQUERY($rc, [q_{x_1}, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
 return iv

- Given that we have two levels of trees, the first level is an interval tree, and the second is two two-dimensional range trees, search complexity 2 log# log =

((log$ • For some other data)(structures, it can)

be different. i +9:- ;,<=- Let's say

is a logfor an ordinary list it

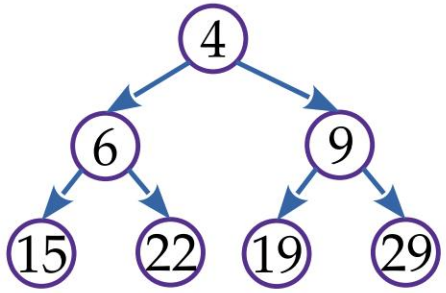
( )

Pretraživanje intervala (1D)

function IntervalQuery($n, [q_{x_1}, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
  $iv \leftarrow \emptyset$
  if $q_{x_1} \leq x_{med}(n) \leq q_{x_2}$ **then**
    $p \leftarrow$ 2DRangeQuery($\tau_{left}(n), (-\infty, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
    $p \leftarrow p \cup$ 2DRangeQuery($\tau_{right}(n), [q_{x_1}, \infty) \times [q_{y_1}, q_{y_2}]$)
    $iv \leftarrow$ all intervals of the endpoints in p
    $move \leftarrow both$
  else if $x_{med}(n) < q_{x_1}$ **then**
    $p \leftarrow$ 2DRangeQuery($\tau_{right}(n), [q_{x_1}, \infty) \times [q_{y_1}, q_{y_2}]$)
    $iv \leftarrow$ all intervals of the endpoints in p
    $move \leftarrow right$
  else if $q_{x_2} < x_{med}(n)$ **then**
    $p \leftarrow$ 2DRangeQuery($\tau_{left}(n), (-\infty, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
    $iv \leftarrow$ all intervals of the endpoints in p
    $move \leftarrow left$
  if $move \in \{both, left\}$ and exists $lc \leftarrow leftChild(n)$ **then**
    $iv \leftarrow iv \cup$ IntervalQuery($lc, [q_{x_1}, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
  if $move \in \{both, right\}$ and exists $rc \leftarrow rightChild(n)$ **then**
    $iv \leftarrow iv \cup$ IntervalQuery($rc, [q_{x_1}, q_{x_2}] \times [q_{y_1}, q_{y_2}]$)
  return iv

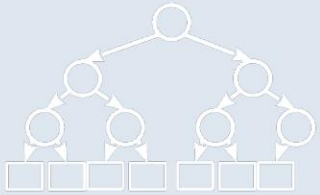
- S obzirom da imamo dvije razine stabala, u prvoj razini je stablo intervala, a u drugoj su dva dvodimenzionalna stabla raspona, kompleksnost pretraživanja
$$O\big((2\log^2 n)(\log n)\big) = O(\log^3 n)$$
- Za neke druge strukture podataka u τ_{left} i τ_{right} to može biti drukčije. Recimo za običnu listu je to $O(n \log n)$

Priority *tree*

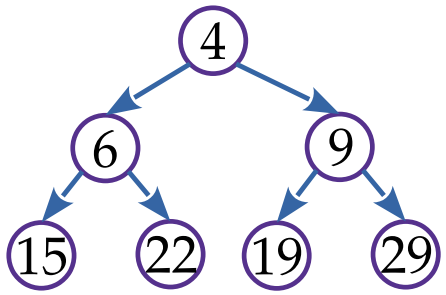


- Using a two-dimensional span tree in an interval tree seems inefficient – a two-dimensional span tree has two layers (three in total with the interval tree)
- Interval tree queries are specific: (ÿÿ, 0/]× 10, 1/ [ ] [00, ÿ)× [10, 1/]
- A *heap* can be used as data for this purpose structure
  - A heap is a sorted structure, where the root of the heap is the smallest or the largest member
  - We are talking about an ascending or descending pile
- If we sort the points along the -axis, we can store them in a pile
  - This solves the horizontal part of the query (ÿÿ, 1"] and [• 1#, ÿ)

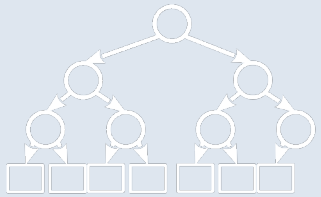
For (ÿÿ, 1"] we need an ascending sorted heap – we descend from the root of the pile, until we come across ( ) > 1"



Stablo prioriteta *(priority tree)*

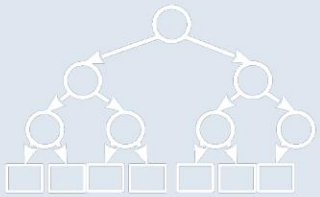


- Korištenje dvodimenzionalnog stabla raspona u stablu intervala čini se neefikasno – dvodimenzionalno stablo raspona ima dva sloja (tri ukupno sa stablom intervala)
- Upiti za stablo intervala su specifični: $(-\infty, q_{x_2}]\times[q_{y_1}, q_{y_2}]$ i $[q_{x_1}, \infty)\times[q_{y_1}, q_{y_2}]$
- Za ovu se namjenu može koristiti gomila *(heap)* kao podatkovna struktura
  - Gomila je sortirana struktura, gdje je korijen gomile najmanji ili najveći član
  - Govorimo o uzlazno ili silazno sortiranoj gomili
- Ukoliko točke sortiramo po x-osi, možemo ih spremiti u gomilu
  - To nam rješava horizontalni dio upita $(-\infty, q_{x_2}]$ i $[q_{x_1}, \infty)$
  - Za $(-\infty, q_{x_2}]$ trebamo uzlazno sortiranu gomilu – spuštamo se od korijena gomile, pa sve do dok ne naiđemo na $x(n) > q_{x_2}$



Priority tree

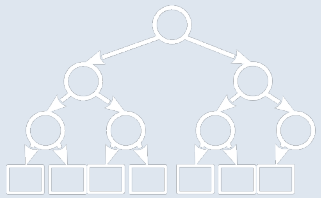
- What about the -axis? The crowd alone will not support that second dimension.
- We use the fact that heap partitioning can be arbitrary as long as the heap is sorted - parent smaller or larger than children
- We incorporate the concept of a binary tree - **a priority** tree - into the heap
  - We always take ( ) represents a point stored in a node - it follows the heap principle, the value of the node from the next point with the smallest ( ) value
  - We add the parameter ( ) to the node
    - The structure of the priority tree by parameter ( ) follows the principle of a binary tree
      - Left child • Right child – has parameter - has ( ) ÿ ( ) - less than or equal to the parent parameter . ( ) > ( ) - greater than the parent
    - When we create a priority tree and decide in which subtree we put the points
      - We remove the node point from the set of points
      - We calculate the median /$0 value of the remaining points - we set the parameter (n) to /$0
      - Remaining points with ( ) ÿ • /$0 they go to the left subtree
        Remaining points with > ( ) /$0 they go to the right subtree



BE CAREFUL! (( )) ÿ ( ) – the value of the point in the node and the parameter of the node are not the same!

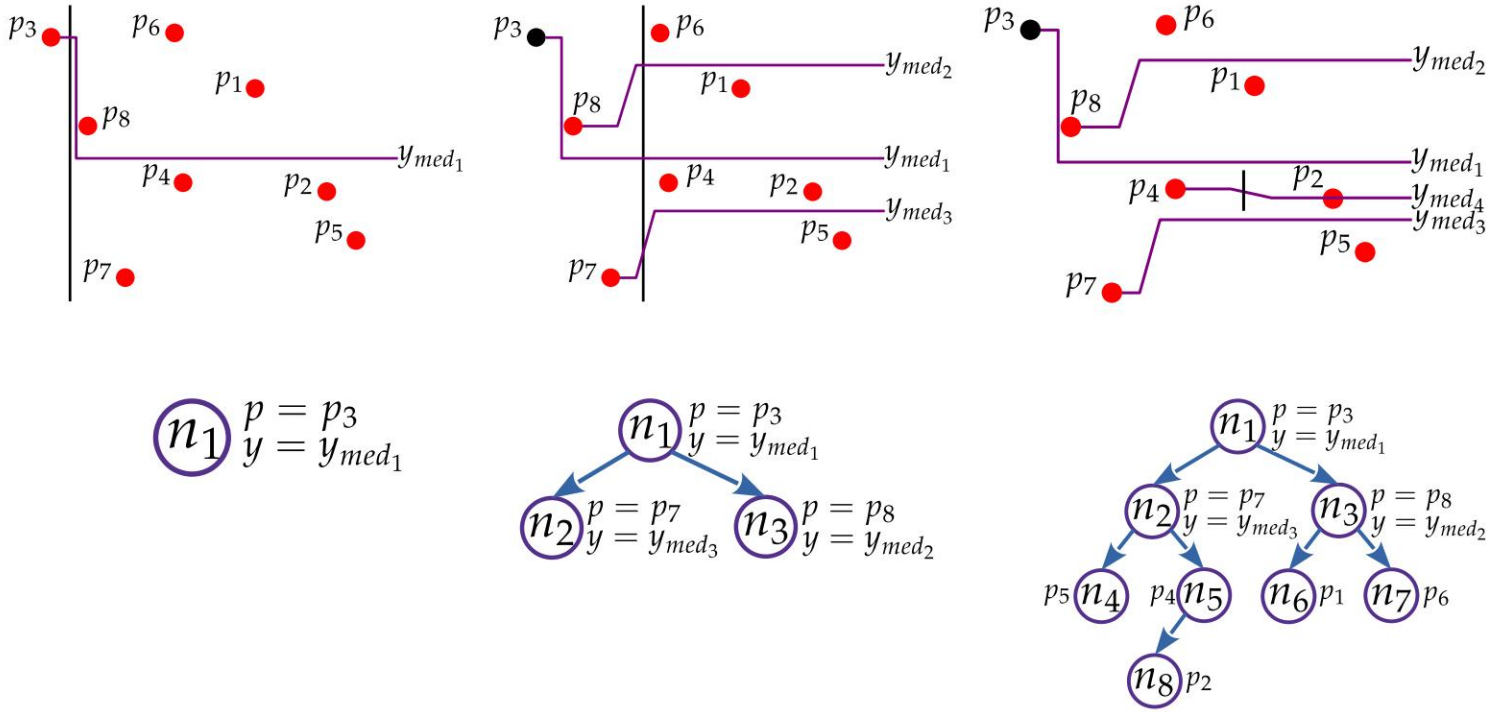
Stablo prioriteta

- Što je s y-osi? Gomila sama po sebi neće podržati tu drugu dimenziju.
- Koristimo činjenicu da particioniranje gomile može biti proizvoljno tako dugo dok je gomila sortirana – roditelj manji ili veći od djece
- U gomilu ugrađujemo koncept binarnog stabla – **stablo prioriteta**
  - Vrijednost čvora $p(n)$ predstavlja točku spremljenu u čvor n – slijedi princip gomile, uvijek uzimamo slijedeću točku s najmanjom $x(p)$ vrijednosti
  - Čvoru dodajemo parametar $y(n)$
    - Struktura stabla prioriteta po parametru $y(n)$ slijedi princip binarnog stabla
      - Lijevo dijete n_L ima parametar $y(n_L) \leq y(n)$ - manji ili jednak od roditelja
      - Desno dijete n_R ima parametar $y(n_R) > y(n)$ - veći od roditelja
    - Kada stvaramo stablo prioriteta i odlučujemo u koje podstablo stavljamo točke
      - Iz skupa točaka maknemo točku čvora
      - Izračunamo medijan y_{med} vrijednosti preostalih točaka – parametar $y(\mathrm{n})$ postavimo na y_{med}
      - Preostale točke s $y(p) \leq y_{med}$ idu u lijevo podstablo
      - Preostale točke s $y(p) > y_{med}$ idu u desno podstablo

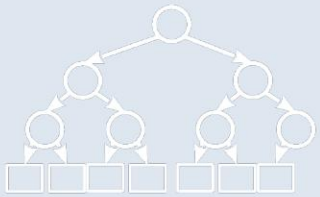


PAZITI! $y(p(n)) \neq y(n)$ – y vrijednost točke u čvoru i y parametar čvora nisu isto!

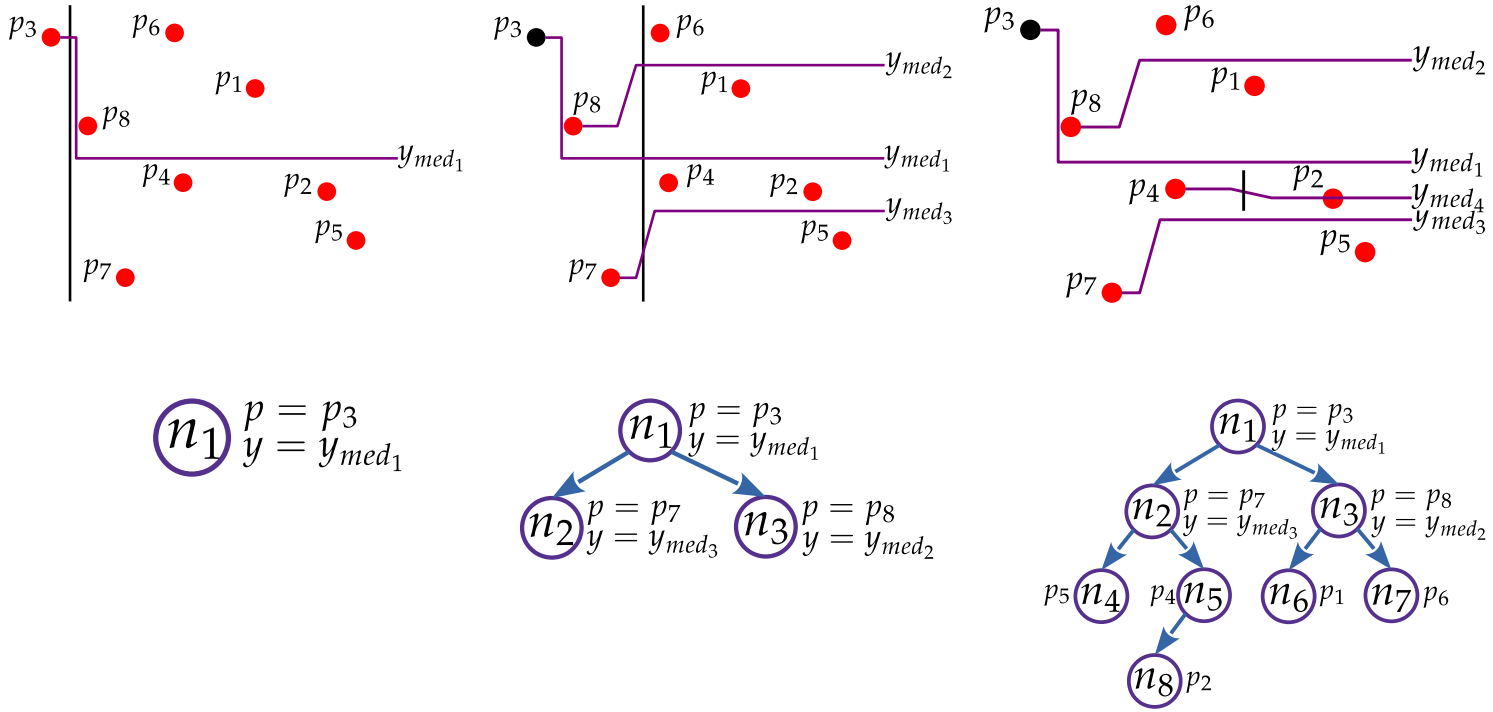
Priority tree



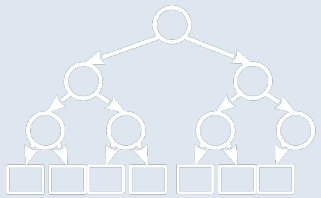
function CREATEPRIORITYTREE(P)
  $p_{min} = \underset{p_i \in P}{arg\ min}\ x(p_i)$
  $n \leftarrow$ new node
  $p(n) \leftarrow p_{min}$
  if $P \setminus p_{min} \neq \emptyset$ **then**
    $y_{med} \leftarrow median(P \setminus p_{min})$
    $P_L \leftarrow \{p : p \in P \setminus p_{min}, y(p) \leq y_{med}\}$
    $P_R \leftarrow \{p : p \in P \setminus p_{min}, y(p) > y_{med}\}$
    $y(n) \leftarrow y_{med}$
    if $P_L \neq \emptyset$ **then**
      $leftChild(n) \leftarrow$ CREATEPRIORITYTREE(P_L)
    if $P_R \neq \emptyset$ **then**
      $rightChild(n) \leftarrow$ CREATEPRIORITYTREE(P_R)
  return n



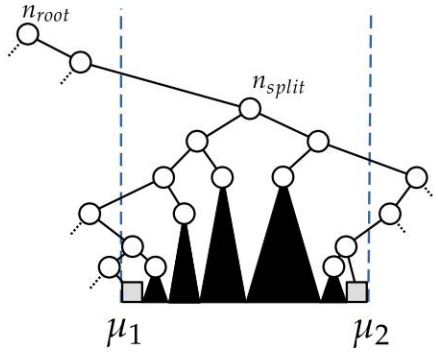
Stablo prioriteta




```
function CreatePriorityTree(P)
    p_min = arg min_{p_i ∈ P} x(p_i)
    n ← new node
    p(n) ← p_min
    if P \ p_min ≠ ∅ then
        y_med ← median(P \ p_min)
        P_L ← {p : p ∈ P \ p_min, y(p) ≤ y_med}
        P_R ← {p : p ∈ P \ p_min, y(p) > y_med}
        y(n) ← y_med
        if P_L ≠ ∅ then
            leftChild(n) ← CreatePriorityTree(P_L)
        if P_R ≠ ∅ then
            rightChild(n) ← CreatePriorityTree(P_R)
    return n
```

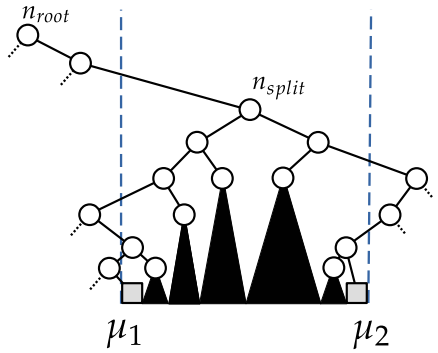


Priority tree

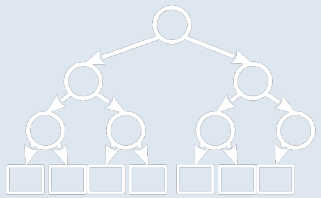


- Given that the value of the node point (( )) of the tree is not the same as the node parameter (( )) – the node parameter is used in forming a binary tree
  - (( )) is used when checking whether a specific point is in the query window, that is, inside [&!, &"]
  - ( ) is used to navigate the priority tree
- Let's look for the separation node :+%2; – the first one below the root that intervals 4. is within ÿ (:+%2;) 4- ÿ • We separate the search into two vertical paths
  - The concept is the same as range searches
- All the time we observe that (( )) ÿ - the principle of the heap 3-
- For the nodes of all paths that we pass from the root, including the nodes of all subtrees that we automatically add, we check each point of the node to see if it is within the query window(ÿÿ, 3-]× [4., 4-]

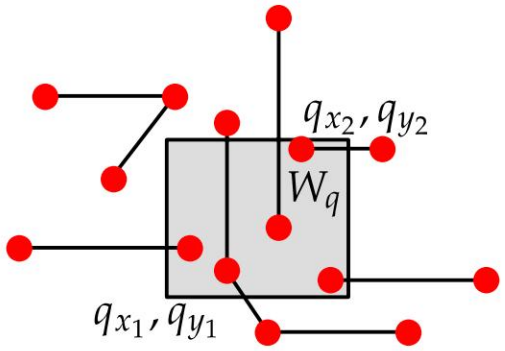
Stablo prioriteta



- S obzirom da y vrijednost točke čvora ($y(p(n))$) stabla nije isto što i y parametar čvora ($y(n)$) – y parametar čvora je korišten u formiranju binarnog stabla
  - $y(p(n))$ se koristi kada se provjerava da li je konkretna točka u prozoru upita, to jest unutar $[q_{y_1}, q_{y_2}]$
  - $y(n)$ se koristi za kretanje po stablu prioriteta
- Potražimo čvor razdvajanja n_{split} – prvi ispod korijena koji je unutar intervala $q_{y_1} \leq y(n_{split}) \leq q_{y_2}$
- Pretraživanje razdvajamo u dvije vertikalne putanje
  - Koncept je isti kao i kao pretraživanja raspona
- Cijelo vrijeme pratimo da je $x(p(n)) \leq q_{x_2}$ - princip gomile
- Za čvorove svih putanja koje prolazimo od korijena, uključujući i čvorove svih podstabala koje automatski dodajemo, svaku točku čvora provjeravamo da li je unutar prozora upita$(-\infty, q_{x_2}] \times [q_{y_1}, q_{y_2}]$



Searching for line segments



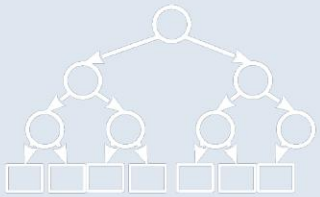
- Let's imagine a two-dimensional example where instead of intervals we have line segments

$$= \{ , = (\quad ,!, ,! \quad)(\quad ,", ," \quad) : (\quad ,/, ,/ \quad) \text{ ÿ ÿ\#} \}$$

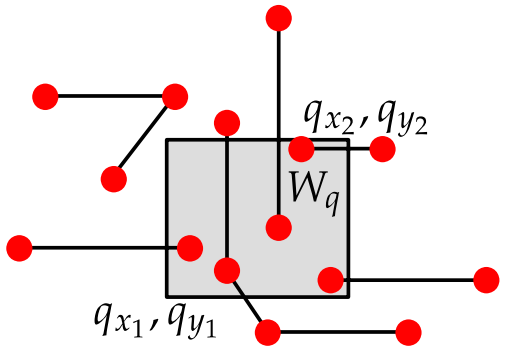
- Then we define the search area

$$= [\quad 3!, 3" \quad] \times [\quad 4!, 4" \quad]$$

  - An example of the printed circuit board we want find all lines that pass through a certain area • An example of a city plan where you want to find streets that are within a search area



Pretraživanje linijskih segmenata



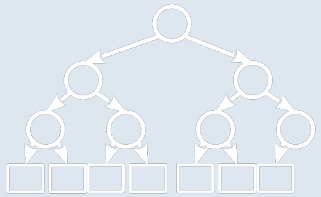
- Zamislimo dvodimenzionalni primjer u kojem umjesto intervala imamo linijske segmente

$$S = \left\{ s_i = \overline{(x_{i_1}, y_{i_1})(x_{i_2}, y_{i_2})} : (x_{i_k}, y_{i_k}) \in \mathbb{R}^2 \right\}$$

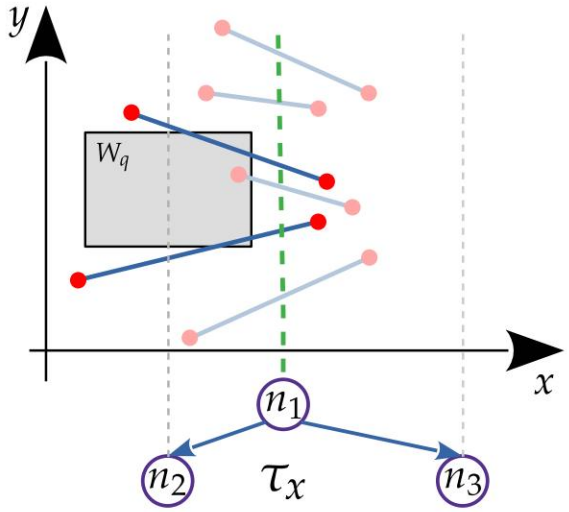
- Zatim definiramo područje pretraživanja

$$W = [q_{x_1}, q_{x_2}] \times [q_{y_1}, q_{y_2}]$$

  - Primjer štampane elektroničke pločice na kojoj želimo pronaći sve vodove koji prolaze kroz određeno područje
  - Primjer plana grada na kojem se žele pronaći ulice koje se nalaze unutar nekog područja pretraživanja

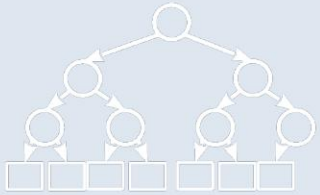


Searching for line segments

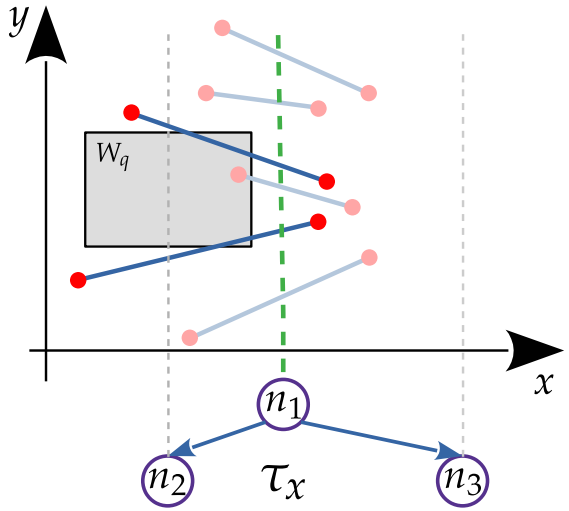


- Can we use a tree for such a search? interval?
- Let's imagine that we have a query window that is to the left of the median node (" in the example).
  - We check whether the left ends of the line segments of the node are
    - within ÿÿ,
  - For example, we see at least two line segments with their left ends not in the required area, but they still intersect the query window

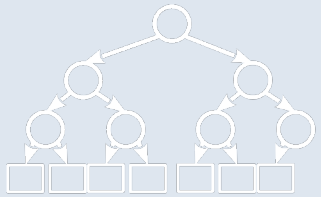
- Although an interval tree will work correctly for most cases, some will still produce a *false negative* result



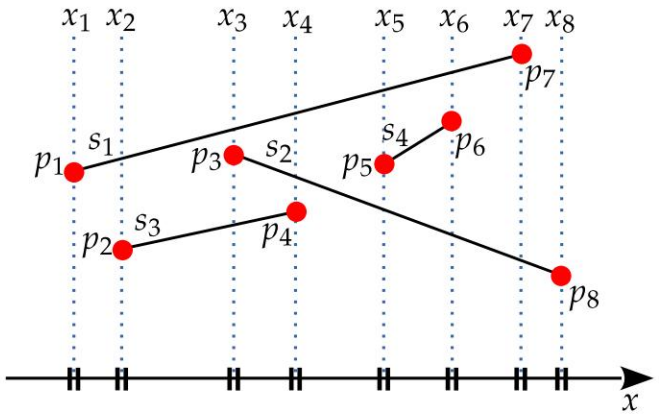
Pretraživanje linijskih segmenata



- Da li za takvo pretraživanje možemo koristiti stablo intervala?
- Zamislimo da imamo prozor upita koji je lijevo od medijana čvora (n_1 na primjeru).
  - Provjeravamo da li su lijevi krajevi linijskih segmenata čvora n_1 unutar $(-\infty, q_{x_2}]\times[q_{y_1}, q_{y_2}]$
  - Na primjeru vidimo barem dva linijska segmenta svojim lijevim krajevima nisu u traženom području, no ipak presijecaju prozor upita W_q - 😕
- Iako će stablo intervala ispravno raditi za većinu slučajeva, neki će ipak proizvesti *false negative* rezultat



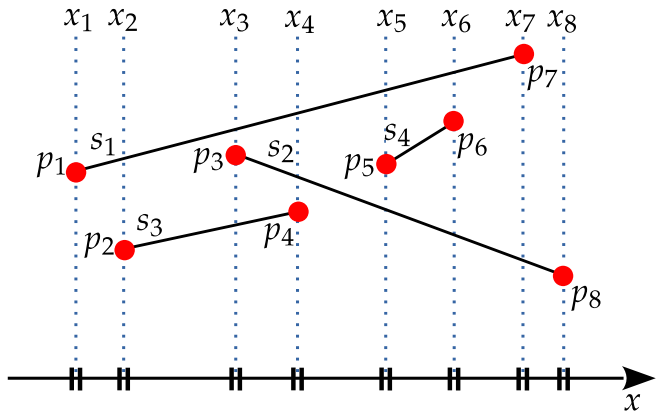
Searching for line segments



- Instead, let's apply the positional approach *(locus approach)*
- We start from the -axis
  - We break the set of line segments into elementary ones intervals - these are intervals along the -axis in which there are no changes in the number of line segments
  - Changes occur at the end points of the linear ones of segments – we use end point projections on - axis : $_{\&}$ = ($_{\&}$)
  - We break the example in the picture into the following elementary intervals

(ÿÿ, ! ,) [!, ! ,] (!, " ,) [", " , ...] (2, 3 , 3, 3 , (3, ÿ]

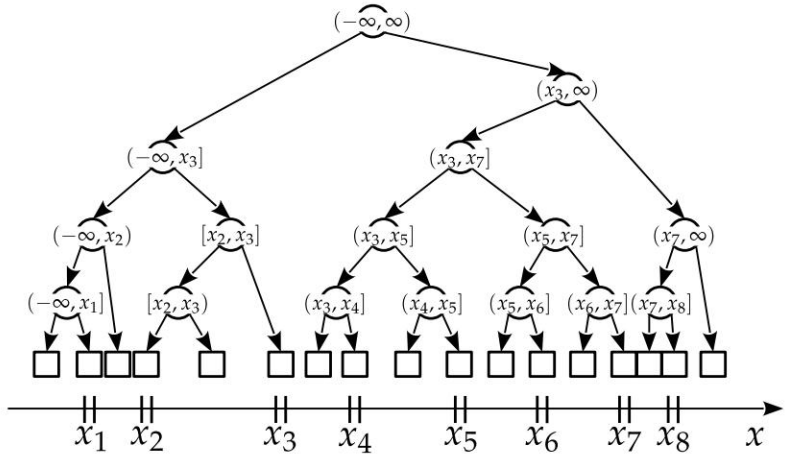
Pretraživanje linijskih segmenata



- Umjesto toga primijenimo pozicijski pristup (*locus approach*)
- Krenemo od x-osi
  - Skup linijskih segmenata razbijemo u elementarne intervale – to su intervali po x-osi u kojima nema promjena broja linijskih segmenata
  - Promjene se dešavaju u krajnjim točkama linijskih segmenata – koristimo se projekcijama krajnjih točaka na x-os : $x_i = x(p_i)$
  - Primjer na slici razbijamo u slijedeće elementarne intervale

$$(-\infty, x_1), [x_1, x_1], (x_1, x_2), [x_2, x_2], \dots, (x_7, x_8), [x_8, x_8], (x_8, \infty)$$

Searching for line segments



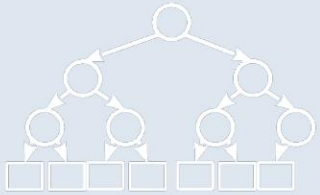
(ÿÿ, ! ,) [!, ! ,] (!, " ,) [", " , ...] (2, 3 , 3, 3 , (3, ÿ]

- Let's turn each of the elementary intervals into one leaf of a balanced binary tree - the leaves are values (concept of B+ index tree), and the internal nodes are just for reference

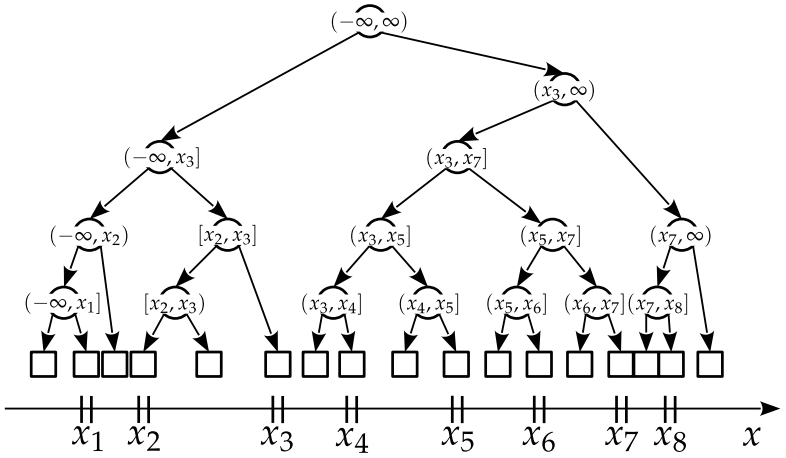
- We make the tree above those leaves in a thoughtful way
  - We want the tree to be balanced so that the search is + log/ - ( ) again coming back to the concept of searching for span

- The internal nodes of the tree aggregate the intervals of their subtrees = (4) ÿ (5)
  - This results in intervals in nodes of the same level not overlapping and having no gaps

- The root node aggregates all elementary intervals, which results in ÿÿ, ÿ( )

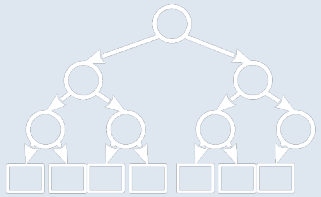


Pretraživanje linijskih segmenata

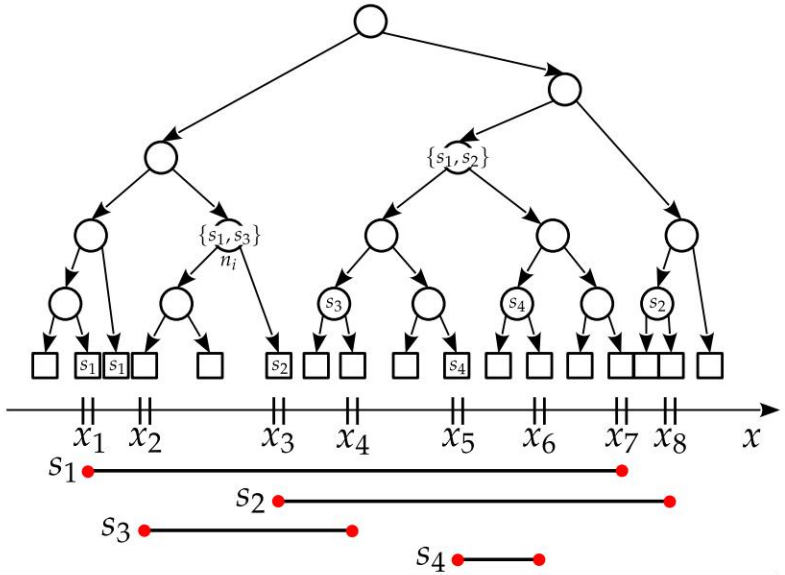


$(-\infty, x_1), [x_1, x_1], (x_1, x_2), [x_2, x_2], \ldots, (x_7, x_8), [x_8, x_8], (x_8, \infty)$

- Svaki od elementarnih intervala pretvorimo u jedan list uravnoteženog binarnog stabla – listovi su vrijednosti (koncept B⁺ indeksnog stabla), a unutarnji čvorovi samo su za navođenje
- Stablo iznad tih listova napravimo na promišljeni način
  - Želimo da stablo bude uravnoteženo kako bi pretraživanje bilo $O(k + \log^2 n)$ - opet se vraćamo na koncept pretraživanja za raspon
- Unutarnji čvorovi stabla agregiraju intervale svojih podstabala $I(n) = I(n_L) \cup I(n_R)$
  - Ovo rezultira time da se intervali u čvorovima iste razine ne preklapaju i nemaju razmaka
- Korijenski čvor agregira sve elementarne intervale, što rezultira sa $(-\infty, \infty)$



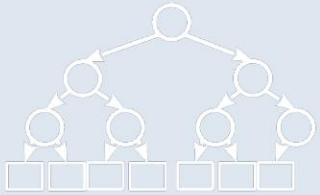
Searching for line segments



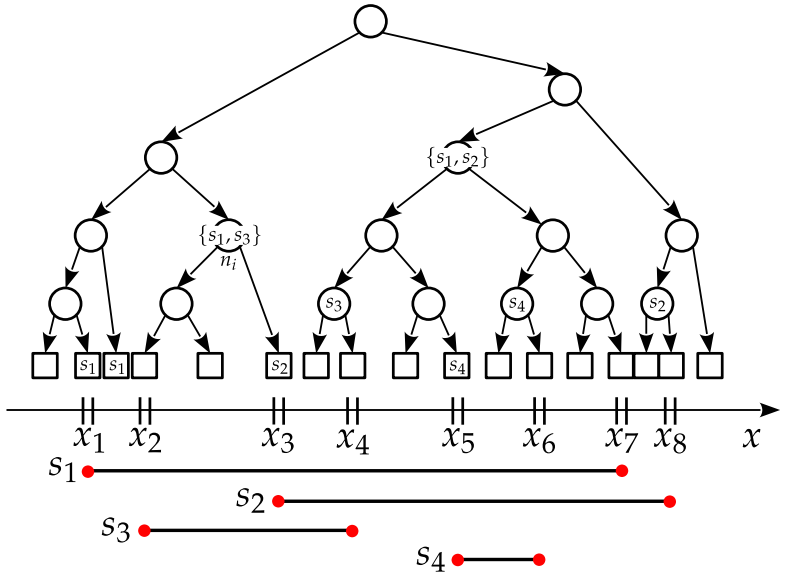

```
procedure InsertSegment(n, [x1, x2])
    if I(n) ⊆ [x1, x2] then
        add [x1, x2] to S(n)
    else
        if I(leftChild(n)) ∩ [x1, x2] ≠ ∅ then
            InsertSegment(leftChild(n), [x1, x2])
        if I(rightChild(n)) ∩ [x1, x2] ≠ ∅ then
            InsertSegment(rightChild(n), [x1, x2])
```

- Then we assign line segments to nodes trees
  - Logic dictates that each line segment in assign your sheet to the elementary interval
  - It is possible (and probable) that several leaves contain the same line segment & - thus we cause high complexity of saving the tree *(storage complexity)*
  - We assign such a line segment to an internal node that contains all leaves that have that line segment &
  - Using node intervals for such allocation we can start from the root node
    &, interval is • For contained while is completely an
  - represents a canonical set of line segments that completely pass through the

interval • We obtain **a tree of segments**



Pretraživanje linijskih segmenata




```
procedure InsertSegment(n, [x1, x2])
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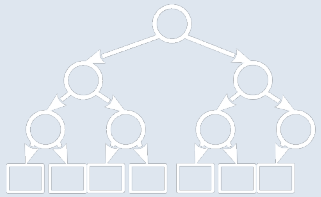
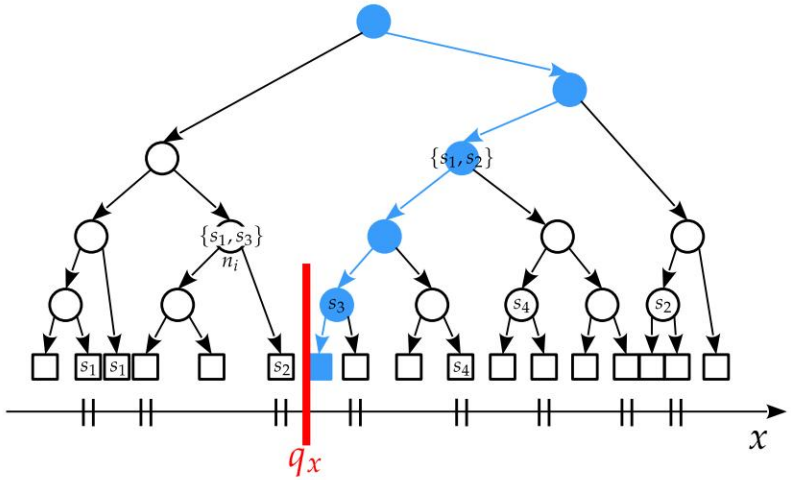
        add [x1, x2] to S(n)
    else
        if I(leftChild(n)) ∩ [x1, x2] ≠ ∅ then
            InsertSegment(leftChild(n), [x1, x2])
        if I(rightChild(n)) ∩ [x1, x2] ≠ ∅ then
            InsertSegment(rightChild(n), [x1, x2])
```

- Zatim pridjeljujemo linijske segmente čvorovima stabla
  - Logika nalaže da se svaki linijski segment u elementarnom intervalu pridijeli svojem listu
  - Moguće je (i vjerojatno) da više listova sadrži isti linijski segment s_i - time uzrokujemo visoku kompleksnost spremanja stabla (*storage complexity*)
  - Takav linijski segment pridijelimo na unutarnji čvor koji sadrži sve listove koji imaju taj linijski segment s_i
  - Korištenjem intervala čvorova $I(n)$, za takvo pridjeljivanje možemo krenuti od korijenskog čvora
  - Za n_i, interval je $I(n_i) = [x_2, x_3]$, što u je u potpunosti sadržano u linijskim segmentima s_1 i s_3
  - $S(n_i) = \{s_1, s_3\}$ predstavlja kanonski skup linijskih segmenata koji u potpunosti prolaze intervalom $I(n_i)$
- Dobivamo **stablo segmenata**



Searching for line segments

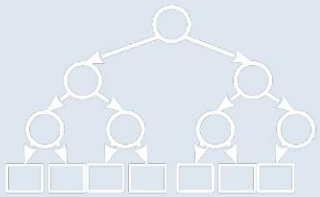


- If we want to find line segments through which they pass
through < = 3× 4.[, 4-]
- We need to find the elementary interval through which it passes
3
- We perform a classic traversal of the tree, passing through the nodes where ÿ ( )
3
- The sheet represents the requested elementary interval
- We denote the vertical path from the root node to the leaf as • The set of all line segments that

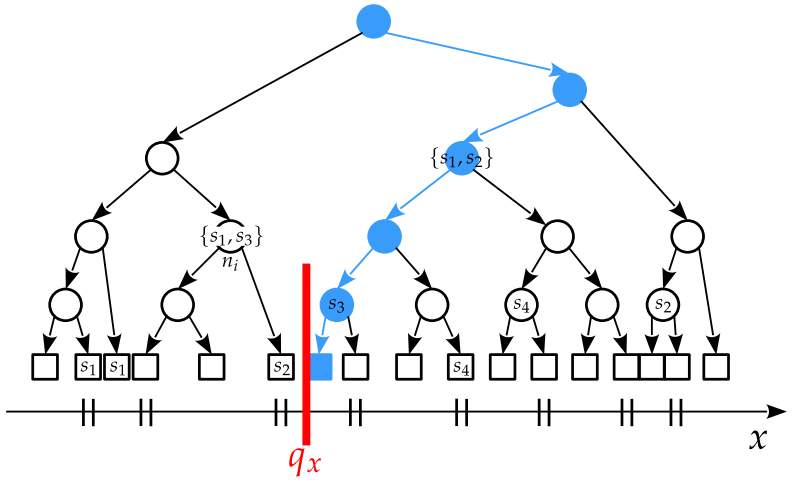
pass through 3 are

E ( ) ÿ
=ÿ

- In our example !, ",{@ }



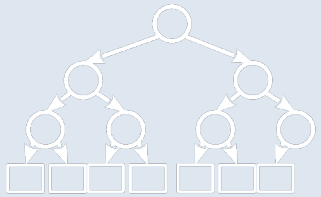
Pretraživanje linijskih segmenata



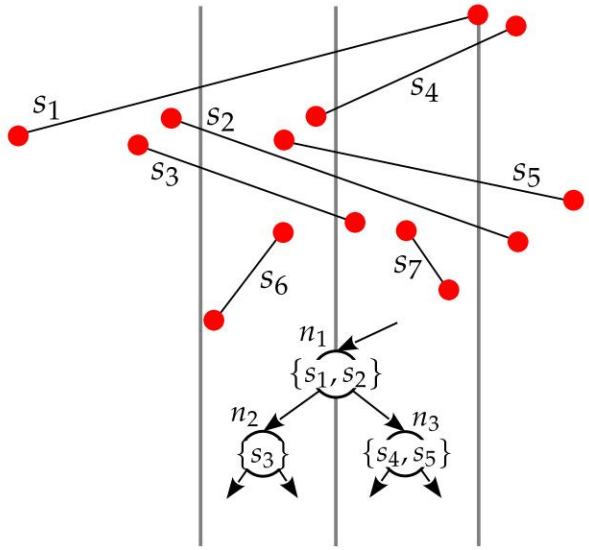
- Ukoliko želimo pronaći linijske segmente iz S koji prolaze kroz $I_q = q_x \times [q_{y_1}, q_{y_2}]$
- Trebamo pronaći elementarni interval kroz koji prolazi q_x
- Izvodimo klasični prolaz kroz stablo, prolazeći kroz čvorove gdje je $q_x \in I(n)$
- List predstavlja traženi elementarni interval
- Vertikalnu putanju od korijenskog čvora do lista označimo kao $\mathcal{V}$
- Skup svih linijskih segmenata koji prolaze kroz q_x su

$$\bigcup_{n \in \mathcal{V}} S(n) \subseteq S$$

- U našem primjeru $\{s_1, s_2, s_3\}$



Searching for line segments



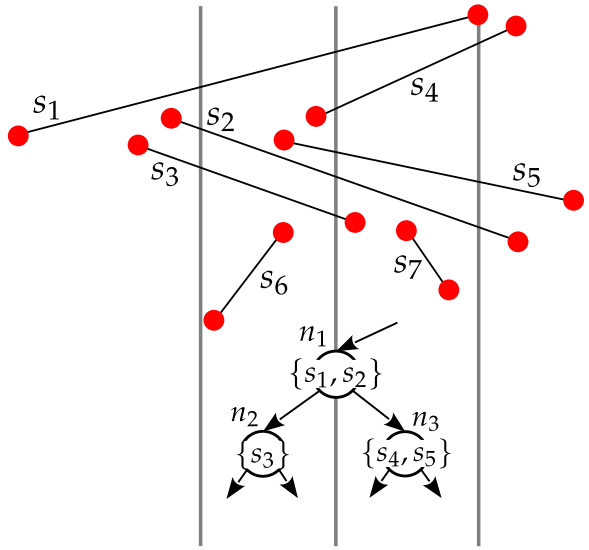
- If we take the interval of a node ( ) on = ( )×(ÿÿ, defines a vertical block & ( ) ÿ)
- All line segments in ( ) pass completely horizontally through the entire & • (In) the example in the picture, we

notice another property that is related to the principle of interval aggregation

- AND !) = AND ") ÿ A @( )
- Line segments thus (! completely pass through and A! ,) and through • Line ") - AND @)
  segments that completely pass through A (,) 2

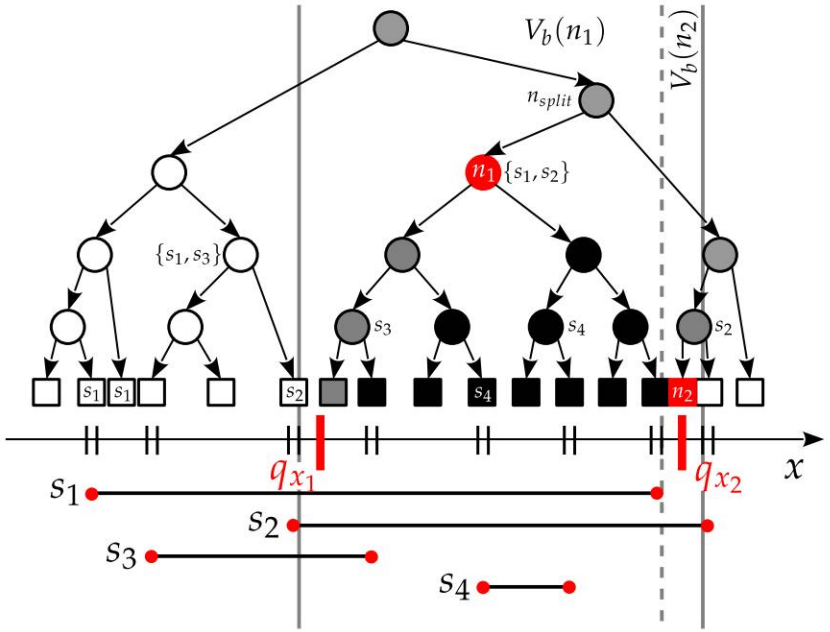
  where 2 is in the subtree whose root is B, **they partially pass** through AB ( )

Pretraživanje linijskih segmenata



- Ukoliko uzmemo interval nekog čvora $I(n)$ on definira vertikalni blok $V_b(n) = I(n) \times (-\infty, \infty)$
- Svi linijski segmenti u $S(n)$ u potpunosti horizontalno prolaze kroz cijeli $V_b(n)$
- U primjeru na slici primjećujemo i jedno drugo svojstvo koje je vezano uz princip agregacije intervala
  - $V_b(n_1) = V_b(n_2) \cup V_b(n_3)$
  - Linijski segmenti $S(n_1)$ u potpunosti prolaze kroz $V_b(n_1)$, a time i kroz $V_b(n_2)$ i $V_b(n_3)$
  - Linijski segmenti koji u potpunosti prolaze kroz $V_b(n_i)$, gdje je n_i u podstablu čiji je korijen n_j, **djelomično prolaze** kroz $V_b(n_j)$

Searching for line segments



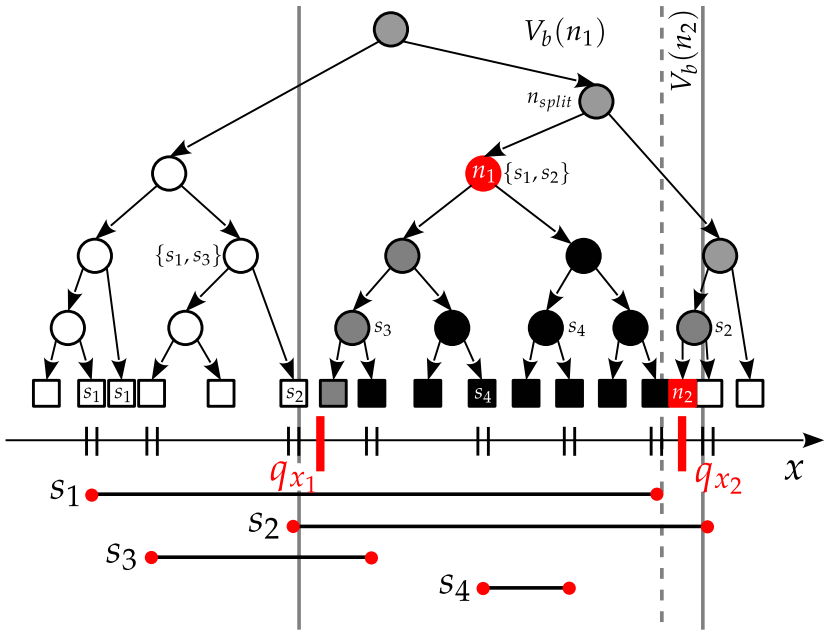
- Let's go back to window search < [10, 1/] = [00, 0/] ×
- The concept is the same as the range search
- We find the splitting node &!'" which is in this case:
  - Contains the entire $6 interval = [1#, 1"]
  - Left child 4 contains • 1# ÿ (4)
  - Right child 5 contains • After 1" ÿ (5)

that we get:

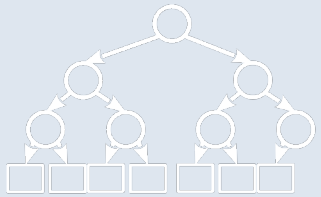
- Two vertical paths, the left 4 and the right 5 - in example gray nodes
- The set of subtrees 7 that are surely in the interval $6 - in for example nodes of black color

- The result is a set of vertical blocks

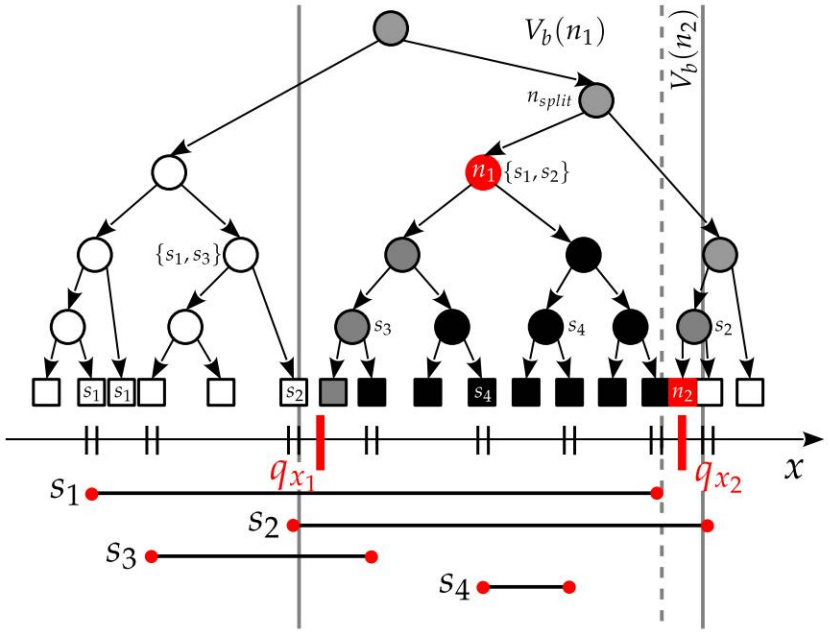
Pretraživanje linijskih segmenata



- Vratimo se pretraživanju prozora $W_q = [q_{x_1}, q_{x_2}] \times [q_{y_1}, q_{y_2}]$
- Koncept je isti kao i kod pretraživanja raspona
- Pronađemo čvor razdvajanja n_{split} koji je u ovom slučaju:
  - Sadrži cijeli interval $I_{q_x} = [q_{x_1}, q_{x_2}]$
  - Lijevo dijete n_L sadrži $q_{x_1} \in I(n_L)$
  - Desno dijete n_R sadrži $q_{x_2} \in I(n_R)$
- Nakon toga dobivamo:
  - Dvije vertikalne putanje, lijevu $\mathcal{V}_L$ i desnu $\mathcal{V}_R$ - u primjeru čvorovi sive boje
  - Skup podstabala N_T koje su sigurno u intervalu I_{q_x} - u primjeru čvorovi crne boje
- Rezultat je skup vertikalnih blokova



Searching for line segments

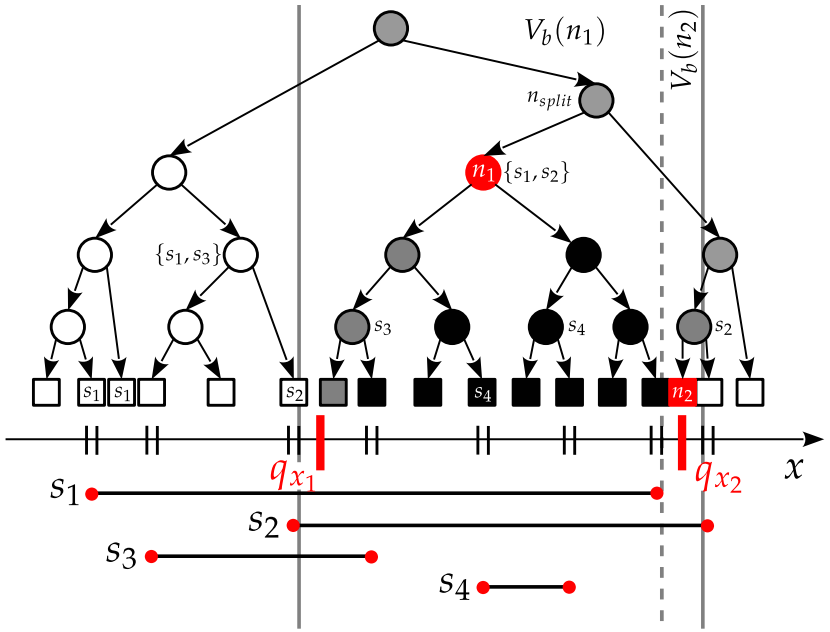


- In a set of vertical blocks that are the result of:
  - We can only have one vertical block
    - When 01#(& is in the elementary interval – in the sheet
    - When 01#(& is the root of a subtree, all of which are elementary intervals in 2#
  - The first and last vertical blocks contain 1# and 1"
- A set of line segments that completely or partially intersects sets of line segments of <8 is the segment tree canonical subtrees that form a set of vertical blocks - see red nodes

E ( ) ÿ

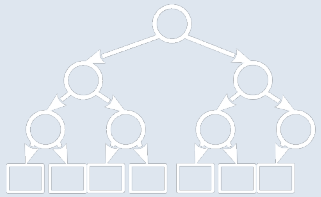
>ÿ 9ÿ :ÿB;

Pretraživanje linijskih segmenata

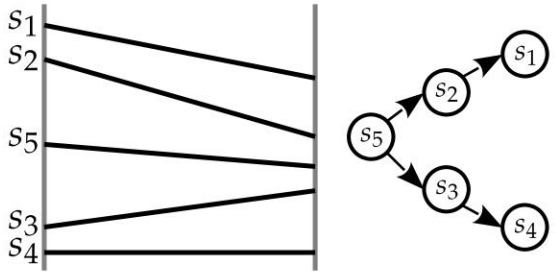


- U skupu vertikalnih blokova koji su rezultat:
  - Možemo imati samo jedan vertikalni blok
    - Kada je n_{split} u elementarnom intervalu – u listu
    - Kada je n_{split} korijen podstabla čiji su svi elementarni intervali u I_{q_x}
  - Prvi i posljednji vertikalni blok sadrže q_{x_1} i q_{x_2}
- Skup linijskih segmenata koji u potpunosti ili djelomično presijeca I_{q_x} je unija svih kanonskih skupova linijskih segmenata podstabala koja čine skup vertikalnih blokova – vidi crvene čvorove

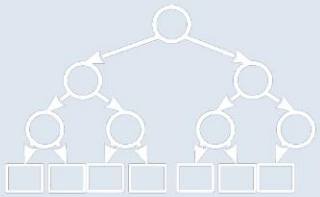
$$\bigcup_{n \in \mathcal{V}_L \cup \mathcal{V}_R \cup N_T} S(n) \subseteq S$$



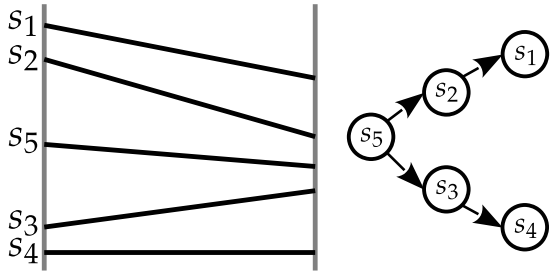
Searching for line segments



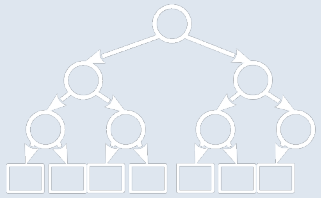
- What about vertical search, how we find
  = [ ] ?
  << 10, 1/
- So far we know that the canonical set of line segments ( ) contains line segments that completely horizontally intersect the vertical block C( )
- If we organize the canonical set of segments in balanced binary tree, we can use the same principle of geometric search as with ranges
  - Problem: In order to clearly know the vertical arrangement of the lines segments, they **must not intersect !**
- We associate such a tree with a node that has ( ) ÿ ÿ and denote it with ( )



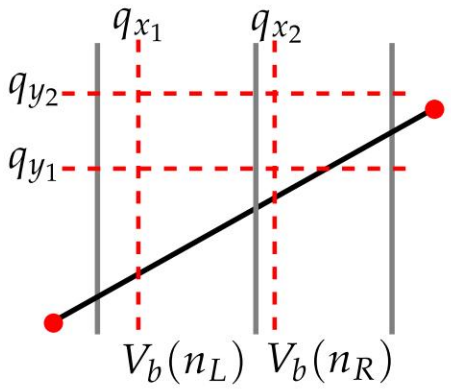
Pretraživanje linijskih segmenata



- Što je s vertikalnim pretraživanjem, kako pronalazimo $I_{q_y} = [q_{y_1}, q_{y_2}]$?
- Do sada znamo da kanonski skup linijski segmenata $S(n)$ sadrži linijske segmente koji u potpunosti horizontalno presijecaju vertikalni blok $V_b(n)$
- Ako kanonski skup segmenata organiziramo u uravnoteženo binarno stablo, možemo koristiti isti princip geometrijskog pretraživanja kao i kod raspona
  - Problem: Da bismo jasno znali vertikalni raspored linijskih segmenata, oni se **ne smiju presijecati** !
- Takvo stablo pridružimo čvoru koji ima $S(n) \neq \emptyset$ i označujemo sa $\tau(n)$



Searching for line segments

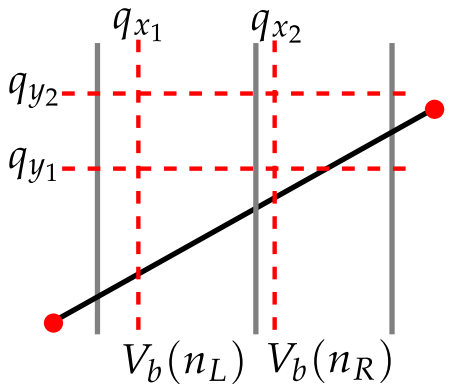


- The first and last vertical blocks contain 3" and 3"
- In these vertical blocks, we must not only look at the place where the line segment intersects the edges of the vertical block

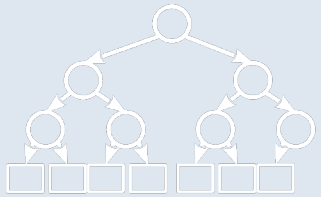
- The intersection of the line segment with must be taken into account in order to avoid *false positive* addition that is, of the line segment to the final result

3", 3!,

Pretraživanje linijskih segmenata



- Prvi i posljednji vertikalni blok sadrže q_{x_1} i q_{x_2}
- U tim vertikalnim blokovima ne smijemo gledati samo mjesto gdje linijski segment presijeca rubove vertikalnog bloka
- Mora se uzeti u obzir sjecište linijskog segmenta sa q_{x_1}, odnosno q_{x_2}, kako ne bi došlo do *false positive* pribrajanja linijskog segmenta konačnom rezultatu

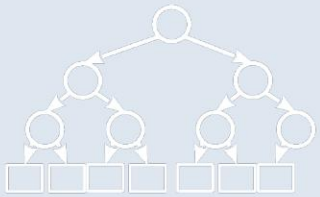


Searching for line segments

```
function ReportSubtree(n, W_q = [q_x1, q_x2] × [q_y1, q_y2])
    rv ← ∅
    if n is not nil then
        add VerticalQuery(τ(n), n, W_q) to rv
        add ReportSubtree(leftChild(n), W_q) to rv
        add ReportSubtree(rightChild(n), W_q) to rv
    return rv
function FindSplittingNode(τ, I_q = [q_x1, q_x2])
    n ← root(τ)
    while n is not leaf do
        lc ← leftChild(n), rc ← rightChild(n)
        if q_x1 ∈ I(lc) and q_x2 ∈ I(rc) then
            return n
        else
            if I_q ⊆ I(lc) then
                n ← lc
            else
                n ← rc
    return n
```

```
function SegmentTreeQuery(τ, W_q = [q_x1, q_x2] × [q_y1, q_y2])
    rv ← ∅
    n_split ← FindSplittingNode(τ, [q_x1, q_x2])
    if n_split is leaf then
        if S(n_split) ≠ ∅ then
            add VerticalQuery(τ(n_split), n_split, W_q) to rv
    else
        n ← leftChild(n_split)
        while n is not leaf do
            add VerticalQuery(τ(n), n, W_q) to rv
            lc ← leftChild(n), rc ← rightChild(n)
            if q_x1 ∈ I(lc) then
                add ReportSubtree(rc, W_q) to rv
                n ← lc
            else
                n ← rc
        add VerticalQuery(τ(n), n, W_q) to rv
        n ← rightChild(n_split)
        while n is not leaf do
            add VerticalQuery(τ(n), n, W_q) to rv
            lc ← leftChild(n), rc ← rightChild(n)
            if q_x2 ∈ I(rc) then
                add ReportSubtree(lc, W_q) to rv
                n ← rc
            else
                n ← lc
        add VerticalQuery(τ(n), n, W_q) to rv
    return rv
```

Questions



Pretraživanje linijskih segmenata

```
function ReportSubtree(n, W_q = [q_x1, q_x2] × [q_y1, q_y2])
    rv ← ∅
    if n is not nil then
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    return rv
function FindSplittingNode(τ, I_q = [q_x1, q_x2])
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                n ← lc
            else
                n ← rc
    return n
```

```
function SegmentTreeQuery(τ, W_q = [q_x1, q_x2] × [q_y1, q_y2])
    rv ← ∅
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    if n_split is leaf then
        if S(n_split) ≠ ∅ then
            add VerticalQuery(τ(n_split), n_split, W_q) to rv
    else
        n ← leftChild(n_split)
        while n is not leaf do
            add VerticalQuery(τ(n), n, W_q) to rv
            lc ← leftChild(n), rc ← rightChild(n)
            if q_x1 ∈ I(lc) then
                add ReportSubtree(rc, W_q) to rv
                n ← lc
            else
                n ← rc
        add VerticalQuery(τ(n), n, W_q) to rv
        n ← rightChild(n_split)
        while n is not leaf do
            add VerticalQuery(τ(n), n, W_q) to rv
            lc ← leftChild(n), rc ← rightChild(n)
            if q_x2 ∈ I(rc) then
                add ReportSubtree(lc, W_q) to rv
                n ← rc
            else
                n ← lc
        add VerticalQuery(τ(n), n, W_q) to rv
    return rv
```

Pitanja