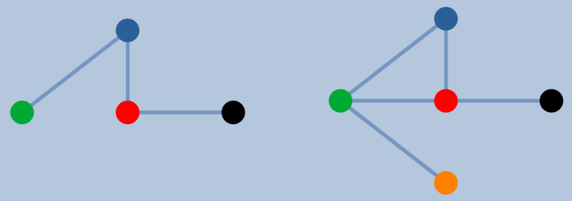
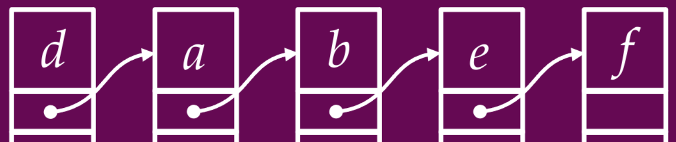
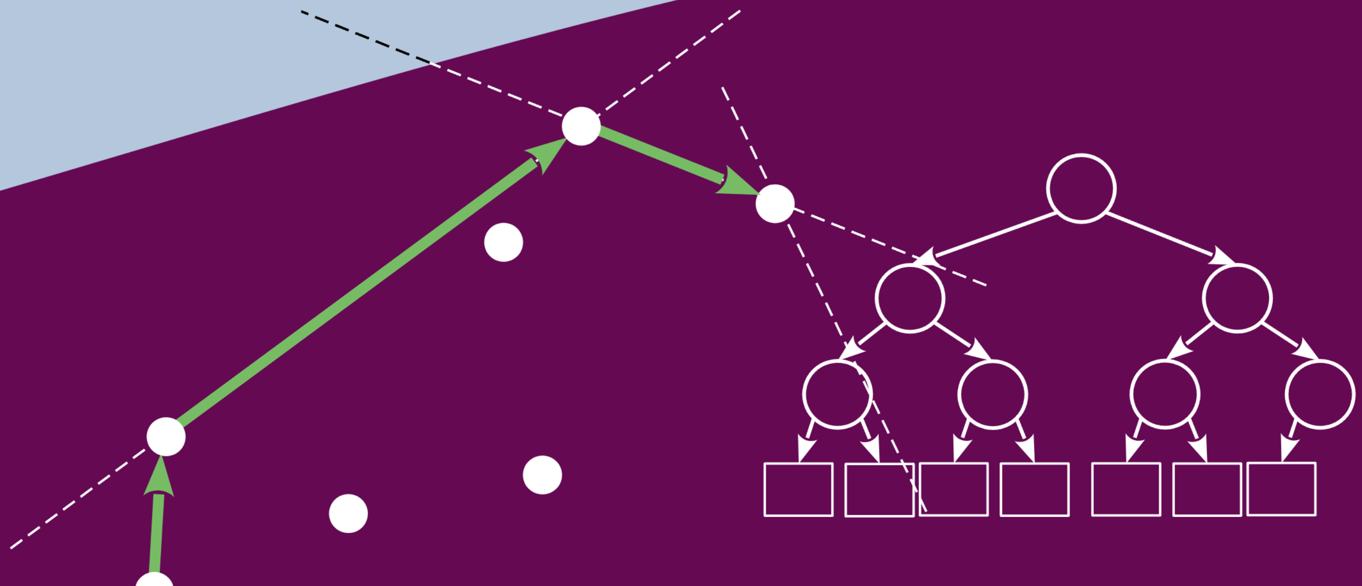
Advanced algorithms and structures data

Week 5: Linear programming

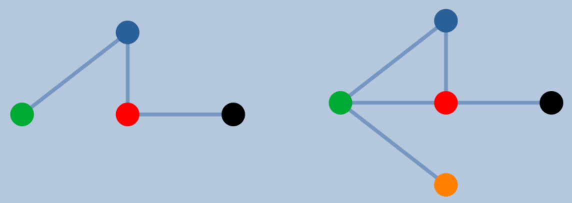


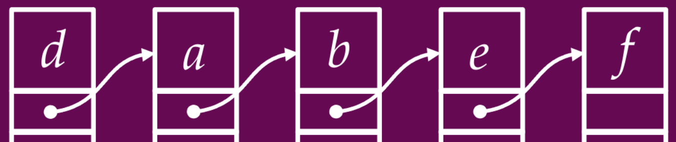


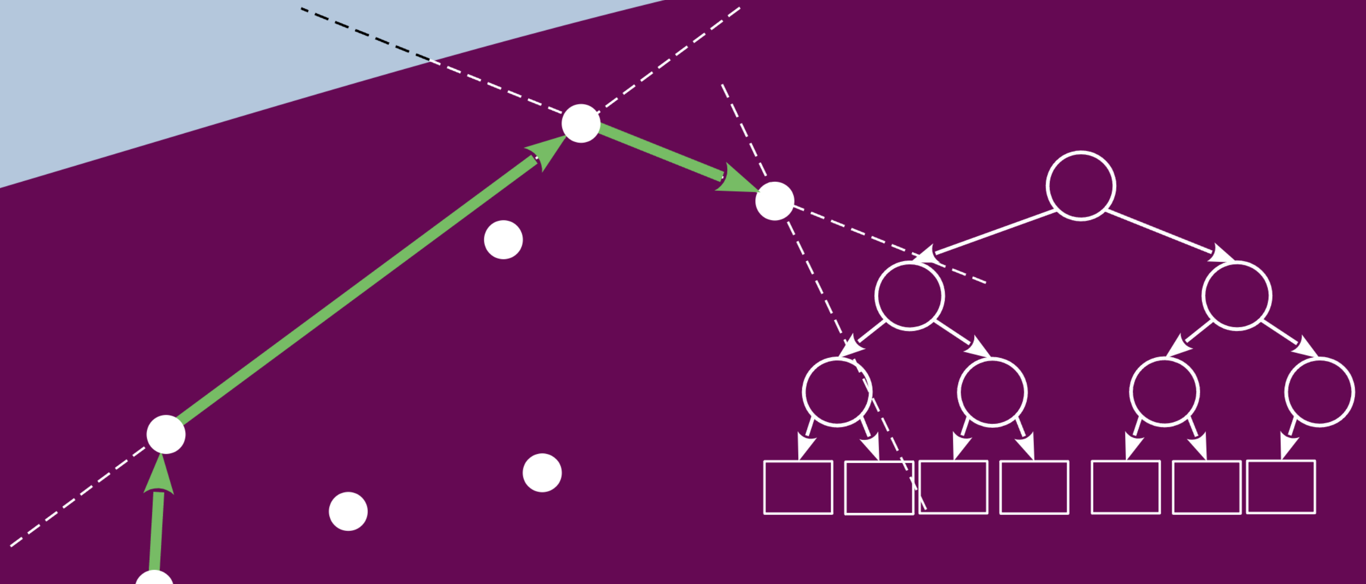


Napredni algoritmi i strukture podataka

Tjedan 5: Linearno programiranje







What is a linear program (LP)?

- Enough**general**a class of problems that can be solved efficiently

- Numerous applications in industry
  - Supply chains
  - Deployment
  - Optimizations in electrical networks

- It belongs to the category**convex optimization problems**
  - LP is the first subcategory that was effectively solved (approx. 1940s)
  - Other categories followed later, an open area of research

Što je linearni program (LP)?

- Dosta **generalna** klasa problema koja se može rješavati efikasno

- Brojne primjene u industriji
  - Lanci nabave
  - Raspoređivanje
  - Optimizacije u električnim mrežama

- Spada u kategoriju **konveksnih optimizacijskih problema**
  - LP je prva podkategorija koja je efikasno riješena (cca 1940tih)
  - Ostale kategorije su slijedile kasnije, otvoreno područje istraživanja

What is a linear program (LP)?

- Solving linear programs has been raised to the level of industrial reliability
  - Reliable and fast tools
    - Gurobi – currently the fastest solver
    - Python –**scipy.optimize.linprog**
    - Even integrated into Excel

- Essential for the design and analysis of algorithms, e.g.
  - Algorithms over graphs
  - Approximate algorithms

Što je linearni program (LP)?

- Rješavanje linearnih programa je dignuto na razinu industrijske pouzdanosti
  - Pouzdani i brzi alati
    - Gurobi – trenutno najbrži rješavač
    - Python – **scipy.optimize.linprog**
    - Čak integrirani u Excel

- Bitni i za dizajn i analizu algoritama, npr.
  - Algoritmi nad grafovima
  - Približni algoritmi

Linear program (LP)?! General...

minimize $\mathbf{c}^T\mathbf{x}$

with the condition $\mathbf{AND}\mathbf{x}\leq\mathbf{b}$

$$\mathbf{D}\mathbf{x}=\mathbf{e}$$

whereby: $\mathbf{x}\in\mathbb{R}^{n}, \mathbf{c}\in\mathbb{R}^{n}, \mathbf{b}\in\mathbb{R}^{m}, \mathbf{e}\in\mathbb{R}^{k}$,

matrix $\mathbf{AND}\in\mathbb{R}^{mxn}$, matrix $\mathbf{D}\in\mathbb{R}^{kxn}$

- Objective function f (*objective function*, *cost function*)
- Decision variables $\boldsymbol{x}$ (*control*, *structural*, *decision variables*)
- Restrictions with parameters ***A, b, D, e*** (*constraints*)

Linearni program (LP)?! Generalno...

$$\begin{array}{ll} \text{minimizirati} & \mathbf{c}^T\mathbf{x} \\ \text{uz uvjet} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{D}\mathbf{x} = \mathbf{e} \end{array}$$

pri čemu je: $\mathbf{x} \in R^n$, $\mathbf{c} \in R^n$, $\mathbf{b} \in R^m$, $\mathbf{e} \in R^k$,
matrica $\mathbf{A} \in R^{mxn}$, matrica $\mathbf{D} \in R^{kxn}$

- Ciljna funkcija *f* (*objective function*, *cost function*)
- Varijable odluke ***x*** (*control, structural, decision variables*)
- Ograničenja sa parametrima ***A,b,D,e*** (*constraints*)

Example I – humanitarian transport problem

- Pfizer produces vaccines for COVID-19 and has production facilities in three locations T1...3 with the available capacities given in the table. The clients are from four places O1...4 with needs specified in the last line of the table. Unit transport costs for all combinations of production plants and customers are listed in the table.
- How to satisfy the customer's needs in the most efficient way?

Costs of transport

	O1	O2	O3	O4	Drop.
T1	10	9	14	8	432
T2	7	11	9	11	138
T3	8	12	12	9	35
Pomegranate.	500	200	115	100	

Primjer I – humanitarni transportni problem

- Pfizer proizvodi cjepiva za COVID-19 te ima proizvodne pogone u tri mjesta T1...3 s raspoloživim kapacitetima zadanima u tablici. Naručitelji su iz četiri mjesta O1...4 sa potrebama zadanima u zadnjem retku tablice. Jedinični transportni troškovi za sve kombinacije proizvodnih pogona i naručitelja su navedeni u tablici.
- Kako na najefikasniji način zadovoljiti potrebe naručitelja?

Transportni troškovi

	O1	O2	O3	O4	Kap.
T1	10	9	14	8	432
T2	7	11	9	11	138
T3	8	12	12	9	35
Nar.	500	200	115	100	

Example I – humanitarian transport problem

xij-quantity delivered from *and* those factories *j* to that customer *cij*-

transportation cost per unit between *and* and *j*

min % !# !#

!,#

with conditions ! "!≤ " ; *and*=1, 2, 3

!

! "!= d! ; *j*=1, 2, 3, 4

"

$x_{ij} \geq 0$

Primjer I – humanitarni transportni problem

x_{ij} - količina isporučenu iz *i*-te tvornice *j*-tom naručitelju

c_{ij} - trošak transporta po jedinici između *i* i *j*

$$\min\left(\sum_{i,j} x_{ij} c_{ij}\right)$$

uz uvjete $\sum_{j} x_{ij} \leq s_i$; $i = 1, 2, 3$

$\sum_{i} x_{ij} = \mathrm{d}_j$; $j = 1, 2, 3, 4$

$x_{ij} \geq 0$

Example II – optimal matching in online dating

On the dating site, in the hetero section, there are M men and F women. Based on the completed questionnaires, the compatibilities for all potential couples were calculated using various models. Site wants to pair

all in pairs so that the total sum of normalized compatibilities would be as high as possible (utilitarianism).

	compatibility			
	M1	M2	M3	M4
F1	9	1	8	7
F2	1	2	1	7
F3	8	2	4	8
F4	2	4	6	4

Combinatorial problem

Naive solution: examine all FM combinations. Need to examine F! combination (if F=M)

Primjer II – optimalno uparivanje u online datingu

Na dating siteu u hetero rubrici postoji M muškaraca i F žena. Na temelju popunjenih upitnika raznim modelima su izračunate kompatibilnosti za sve potencijalne parove. Site želi upariti sve u parove da bi ukupna suma normaliziranih kompatibilnosti bila što veća (utilitarizam).

kompatibilnosti

	M1	M2	M3	M4
F1	9	1	8	7
F2	1	2	1	7
F3	8	2	4	8
F4	2	4	6	4

Kombinatorni problem

Naivno rješenje: ispitati sve kombinacije F-M. Treba ispitati F! kombinacija (ako je F=M)

eharmony

Example II – optimal matching in online dating

x_{ij} - 1 if it is ***and*** paired with ***j***, 0 otherwise

k_{ij} – compatibility for pairing *(i,j)*

m % !# !#

!,#

with conditions ! "! = 1, ∀ = 1,…,

!

! "! = 1, ∀ = 1,…,

"

$x_{ij} \geq 0$

Primjer II – optimalno uparivanje u online datingu

x_{ij} - 1 ako je ***i*** uparen sa ***j***, 0 inače

k_{ij} – kompatibilnost za uparivanje *(i,j)*

$$\max\left(\sum_{i,j} x_{ij} k_{ij}\right)$$

uz uvjete

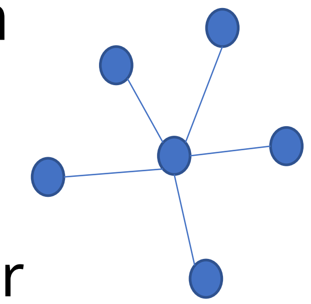
$$\sum_{j} x_{ij} = 1, \forall i = 1, \dots, F$$

$$\sum_{i} x_{ij} = 1, \forall j = 1, \dots, M$$

$$x_{ij} \geq 0$$

Example III – wireless distributed networks

The central station needs to achieve reliable wireless communication with N of distributed sensors to monitor climate change.

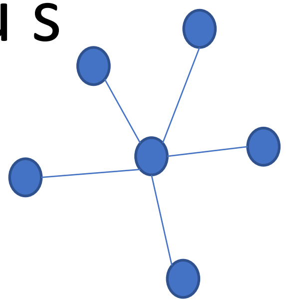


The sensors are powered by solar energy and it is important to reduce their consumption. Signal*and*-th sensor reaches the central station ***muffled.*** If *and* -that sensor emits power p_{and}, the central station receives a signal of power $\lambda_{and} \cdot p_{and}$ (λ<1). During communication with *and*- with this sensor, the signals of all other sensors that arrive at the central station represent interference and communication is possible only if the signal/noise ratio is at least ρ_{and}.

What should be the emissive power p_{and} sensors in order to achieve reliable communication with the lowest possible energy consumption?

Primjer III – bežične raspodijeljene mreže

Centralna postaja treba ostvariti pouzdanu bežičnu komunikaciju s N raspodijeljenih senzora za praćenje klimatskih promjena.



Senzori se napajaju Sunčevom energijom i važno je smanjiti njihovu potrošnju. Signal i-tog senzora do centralne postaje stiže ***prigušen.*** Ako i-ta senzor emitira snagom p_i, centralna postaja prima signal snage $\lambda_i \cdot p_i$ ($\lambda<1$). Tijekom komunikacije s i-tim senzorom, signali svih drugih senzora koji dolaze u centralnu postaju predstavljaju smetnju i komunikacija je moguća samo ako je omjer signal/šum najmanje ρ_i.

Kolike trebaju biti emitivne snage p_i senzora kako bi se ostvarila pouzdana komunikacija uz najmanju moguću potrošnju energije?

Example III – wireless distributed networks

The total consumption is proportional to the total power, so we will minimize it:

$$\min \left(\sum_{and=1}^{N} p_{and} \right)$$

with conditions

$$\frac{l_{and} p_{and}}{\sum_{j \neq and} l_j p_j} \geq r_{and} \quad ; \quad and, j = 1, \ldots, N$$

$$p_k \geq 0 \quad ; \quad k = 1, \ldots, N \quad .$$

Since the conditional (in)equations must be linear by p_k, we translate them into form

$$l_{and} p_{and} - r_{and} \sum_{j \neq and} l_j p_j \geq 0$$

Primjer III – bežične raspodijeljene mreže

Ukupna potrošnja je proporcionalna ukupnoj snazi pa ćemo to minimizirati:

$$\min\left(\sum_{i=1}^{N} p_i\right)$$

uz uvjete

$$\frac{\lambda_i p_i}{\sum_{j\neq i} \lambda_j p_j} \geq \rho_i \quad ; \quad i, j = 1, \ldots , N$$

$$p_k \geq 0 \quad ; \quad k = 1, \ldots, N \quad .$$

Budući da uvjetne (ne)jednadžbe moraju biti linearne po p_k, prevodimo ih u oblik

$$\lambda_i p_i - \rho_i \sum_{j\neq i} \lambda_j p_j \geq 0$$

LP formulations

- The general wording is "messy"

- Two formulations that we strive for in order to facilitate solving and writing algorithms
  - **The canonical form of the LP**–ideal for geometric perspective
  - **Standard LP form**–ideal for algebraic perspective

- All other LPs can be translated into both forms*

* read about transformations in the script

LP formulacije

- Općenita formulacija je „neuredna”

- Dvije formulacije kojima težimo radi lakšeg rješavanja i pisanja algoritama
  - **Kanonska forma LP** – idealna za geometrijsku perspektivu
  - **Standardna forma LP** – idealna za algebarsku perspektivu

- Svi ostali LP se mogu prevesti u obje forme*

*pročitajte o transformacijama u skripti

The canonical form of the LP

minimize $\mathbf{c}^T\mathbf{x}$

with the condition $\mathbf{Ax} \leq \mathbf{b}$

$\mathbf{x} \geq \mathbf{0}$

whereby: $\mathbf{c} \in R^n$, $\mathbf{b} \in R^m$, matrix **AND** $\in R^{mxn}$.

Kanonska forma LP

minimizirati $\mathbf{c}^T\mathbf{x}$

uz uvjet $\mathbf{A}\mathbf{x} \leq \mathbf{b}$

$\mathbf{x} \geq \mathbf{0}$

pri čemu je: $\mathbf{c} \in R^n$, $\mathbf{b} \in R^m$, matrica $\mathbf{A} \in R^{mxn}$.

Standard LP form

minimize $\mathbf{c}^T\mathbf{x}$

with the condition $\mathbf{Ax}=\mathbf{b}$

$\mathbf{x}\geq\mathbf{0}$

whereby: $\mathbf{c} \in R^n$, $\mathbf{b} \in R^m$ and $\mathbf{b} \geq 0$, matrix **AND**$\in R^{mxn}$

It is not part of the definition, but we will assume that it is during the lecture *rank*(**AND**) = *m* and $m < n$. If the rank is lower, the linearly dependent constraints can be removed.

Standardna forma LP

$$\begin{aligned} \text{minimizirati} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{uz uvjet} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

pri čemu je: $\mathbf{c} \in \mathrm{R}^{n}$, $\mathbf{b} \in \mathrm{R}^{m}$ i $\mathbf{b} \geq 0$, matrica $\mathbf{A} \in \mathrm{R}^{mxn}$

Nije dio definicije, ali pretpostavljat ćemo u sklopu predavanja da je $rang(\mathbf{A}) = m$ i $m < n$. Ako je rang manji, linearno-zavisna ograničenja se mogu ukloniti.

Graphic method - example

max $x_1 + 5 \cdot x_2$

with conditions $\begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 30 \\ 12 \end{bmatrix}$

$x \ge 0$

GeoGebra – interactive geometry

Grafička metoda - primjer

max $x_1 + 5 \cdot x_2$

uz uvjete

$$\begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 30 \\ 12 \end{bmatrix}$$

$x \geq 0$

[GeoGebra]() – interaktivna geometrija

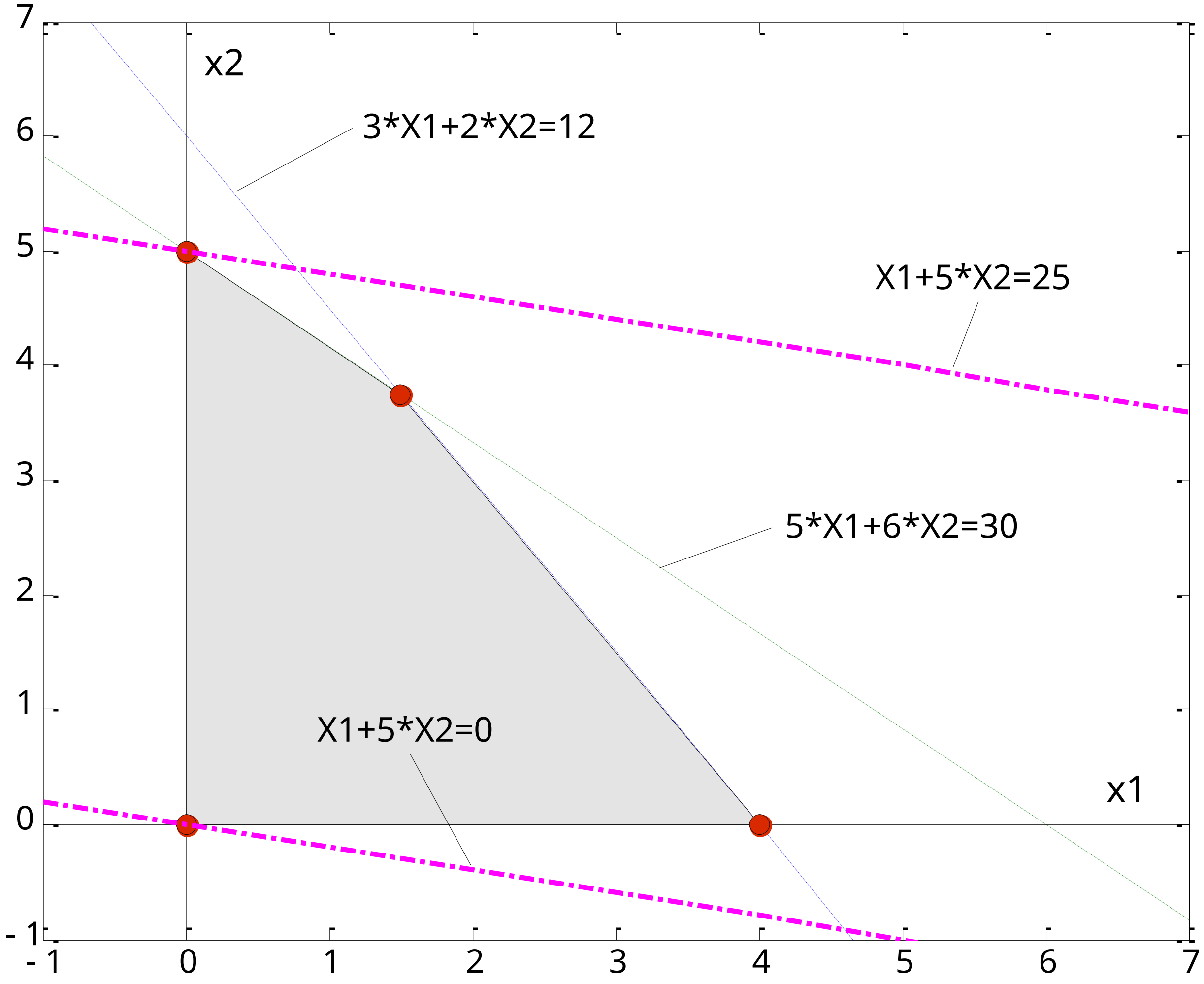
Graphic method - example

max $x_1 + 5 \cdot x_2$

with conditionseh5 6è éx_1è é30in

ê
ë3 2ú ×ê
û ëx_2ú£ê12ún

x≥0



The solution is the intersection of the x line $x_1 + 5 \cdot x_2 = f$ with the space of all possible solutions (gray polygon in the picture) for which it is f the biggest.

Solution

$x = [0, 5]^T$

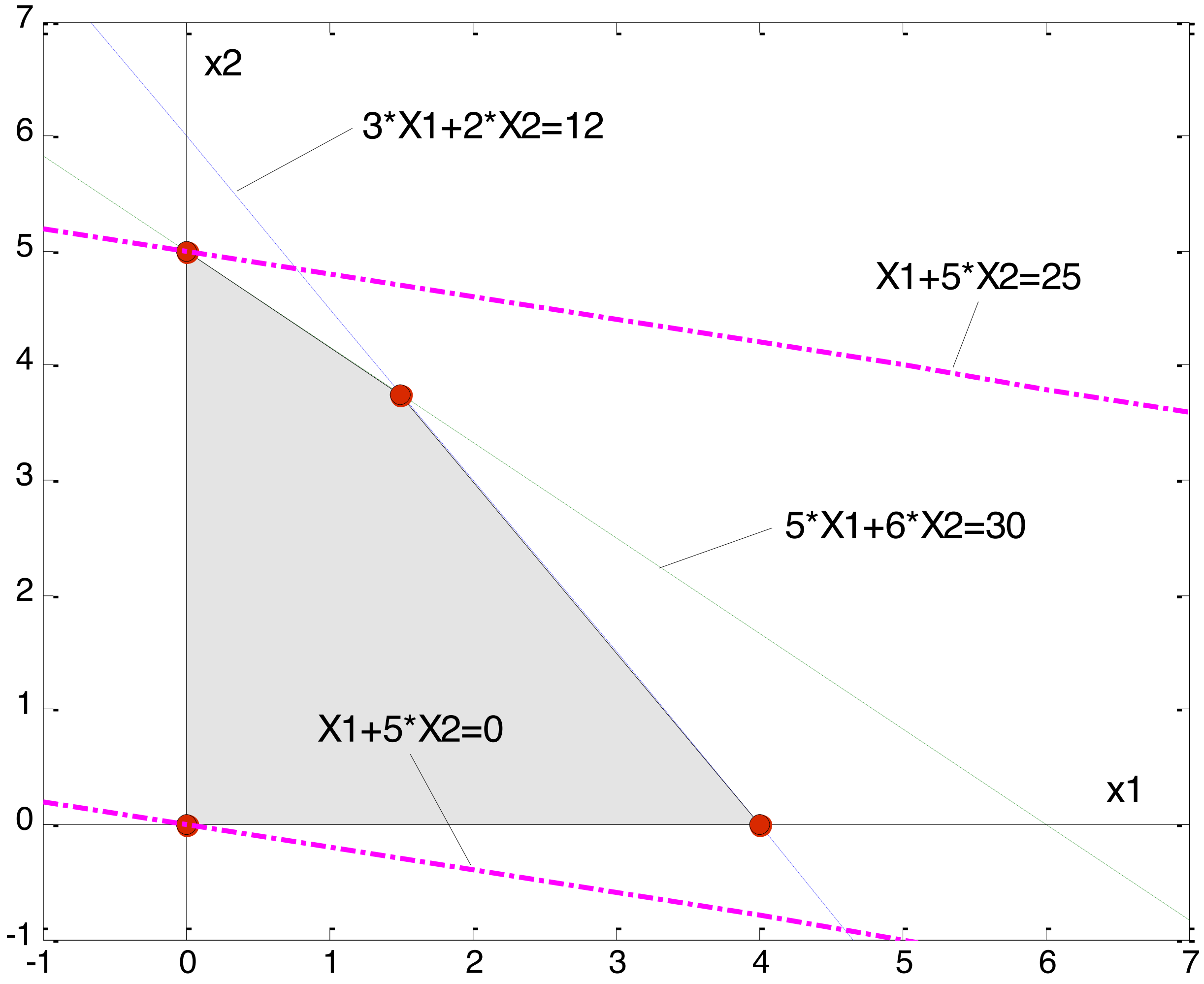
$f_{max} = 25$

Grafička metoda - primjer

max $x_1 + 5 \cdot x_2$

uz uvjete $\begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 30 \\ 12 \end{bmatrix}$

$\mathbf{x} \geq \mathbf{0}$



Rješenje je sjecište pravca $x_1 + 5 \cdot x_2 = f$ s prostorom svih mogućih rješenja (sivi poligon na slici) za koje je f najveća.

Rješenje

$x = [0, 5]^T$

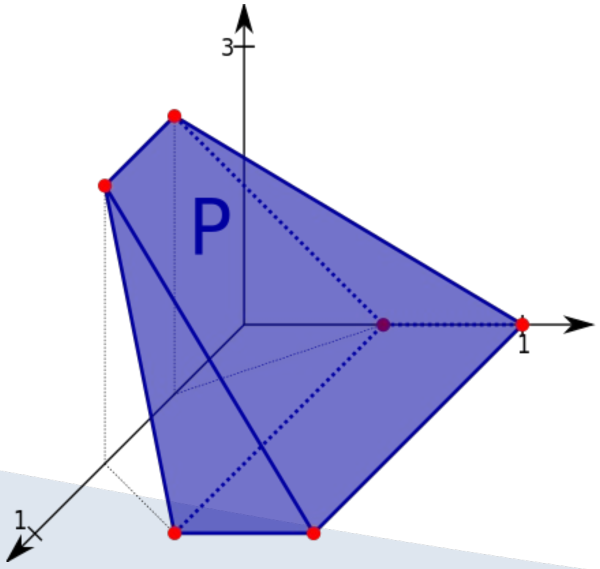
$f_{max} = 25$

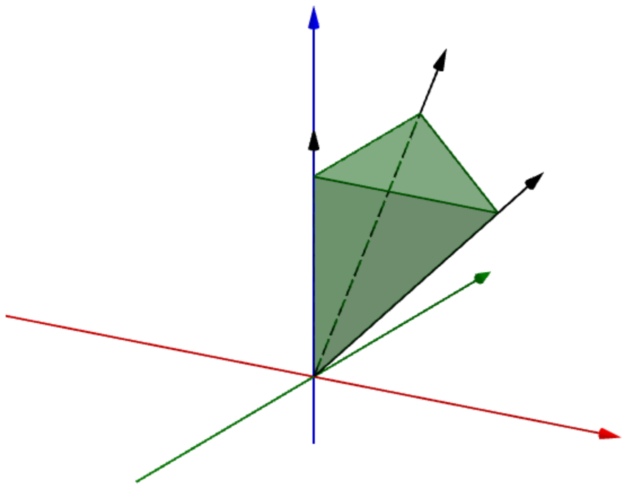
Geometric analysis

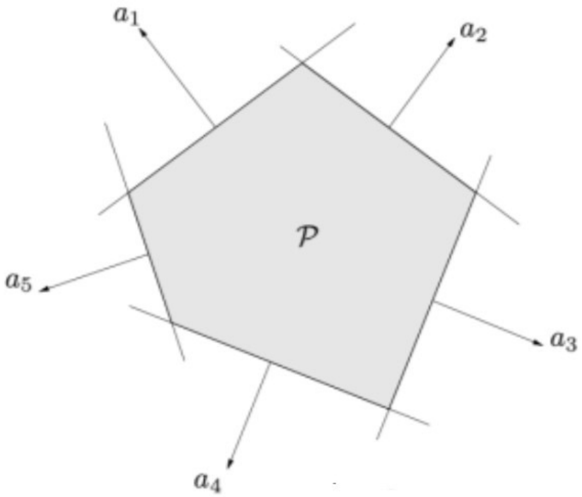
Definition. ***Convex polytope*** in n-dimensional space is a set of vectors (points):

$$\{\in \mathbb{R}^{!} \quad | \leq \quad \}$$

- Limitations of linear programs!





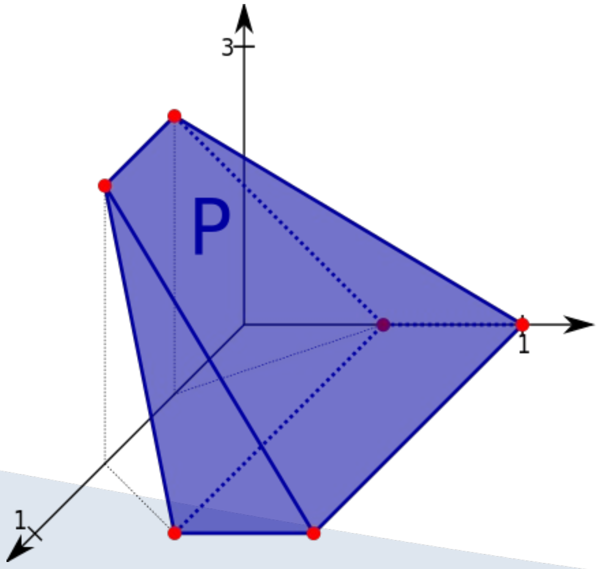


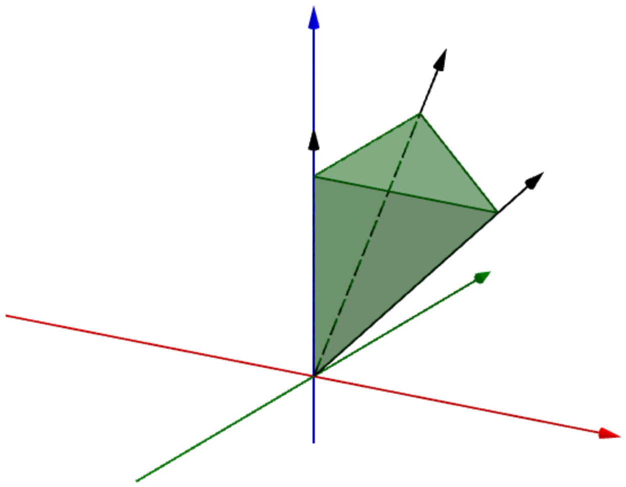
Geometrijska analiza

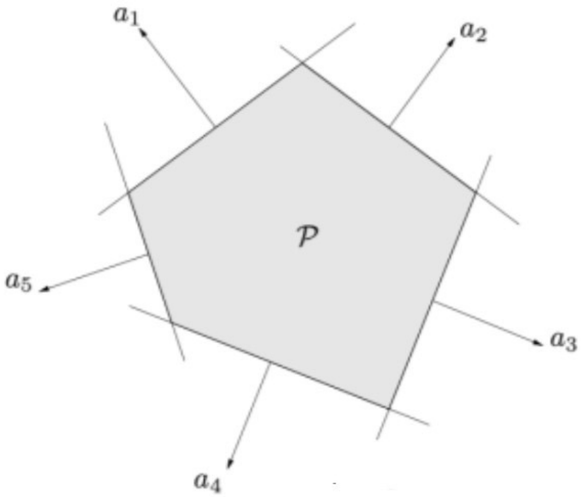
Definicija. ***Konveksni politop*** u n-dimenzionalnom prostoru jest skup vektora (točaka):

$$\{x \in \mathbb{R}^n \ \mid Ax \leq b \}$$

- Ograničenja linearnih programa!







Geometric analysis

Bodies described by constraints in LP are convex polytopes of P

$$= \in \{\mathbb{R}_+ \mid \leq \}$$

https://polyhedra.tessera.li/

Geometrijska analiza

Tijela opisana ograničenjima u LP su konveksni politopi P

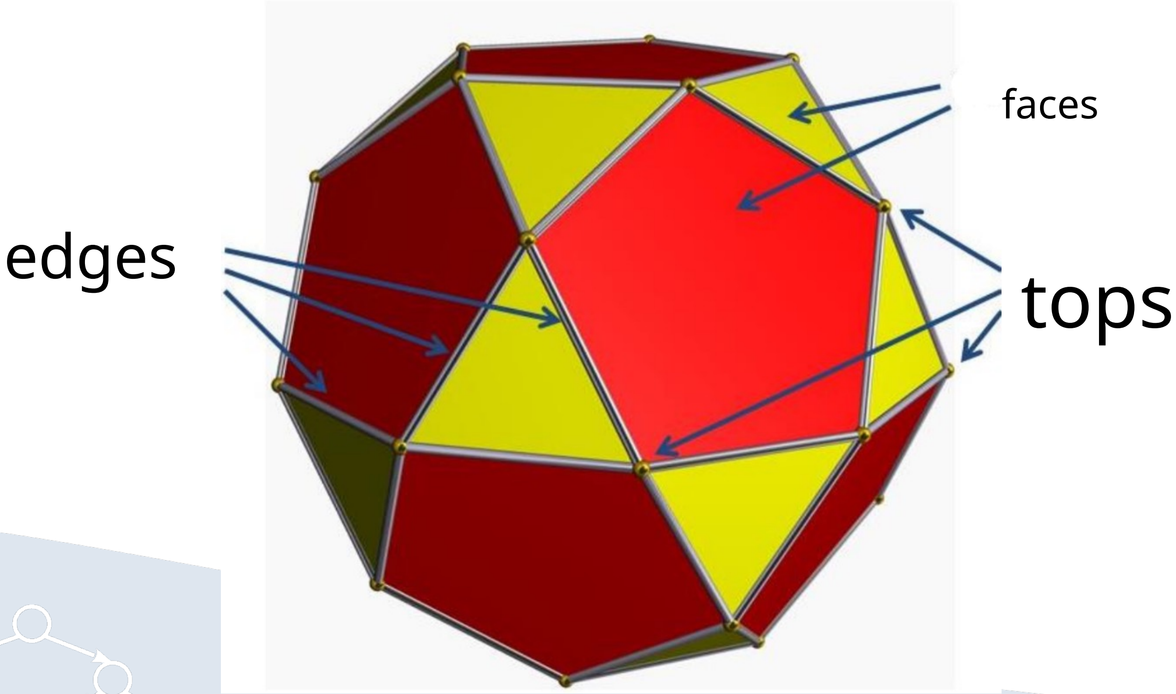
$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

https://polyhedra.tessera.li/

Geometric analysis

Bodies described by constraints in LP are convex polytopes of P

$$= \ \in\{\mathbb{R}_+ \qquad | \leq \ \}$$



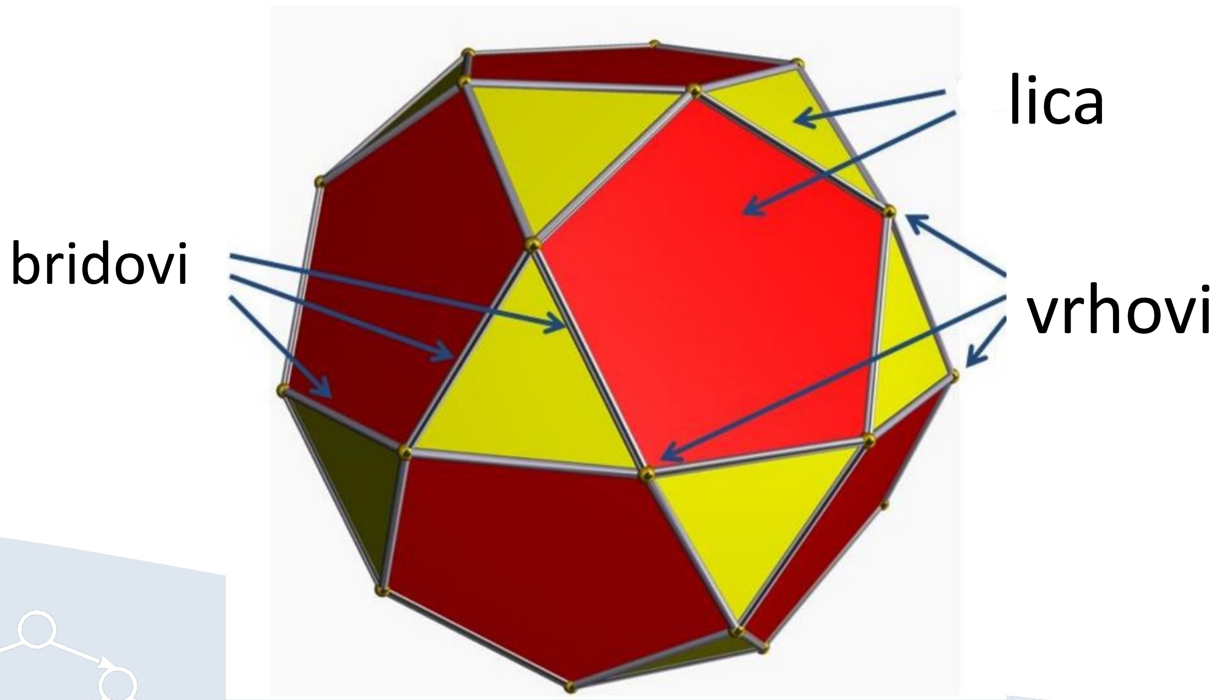
Definition. Active limitationat some point x is every constraint that is satisfied by equality at that point x.

Definition.IN*n*-dimensional space, **vertex of the polytope**is defined as the intersection of at least*n*of active linearly independent constraints, with the remaining constraints satisfied.

Geometrijska analiza

Tijela opisana ograničenjima u LP su konveksni politopi P

$$P = \{x \in \mathbb{R}^n \;\; | \, Ax \le b\,\}$$



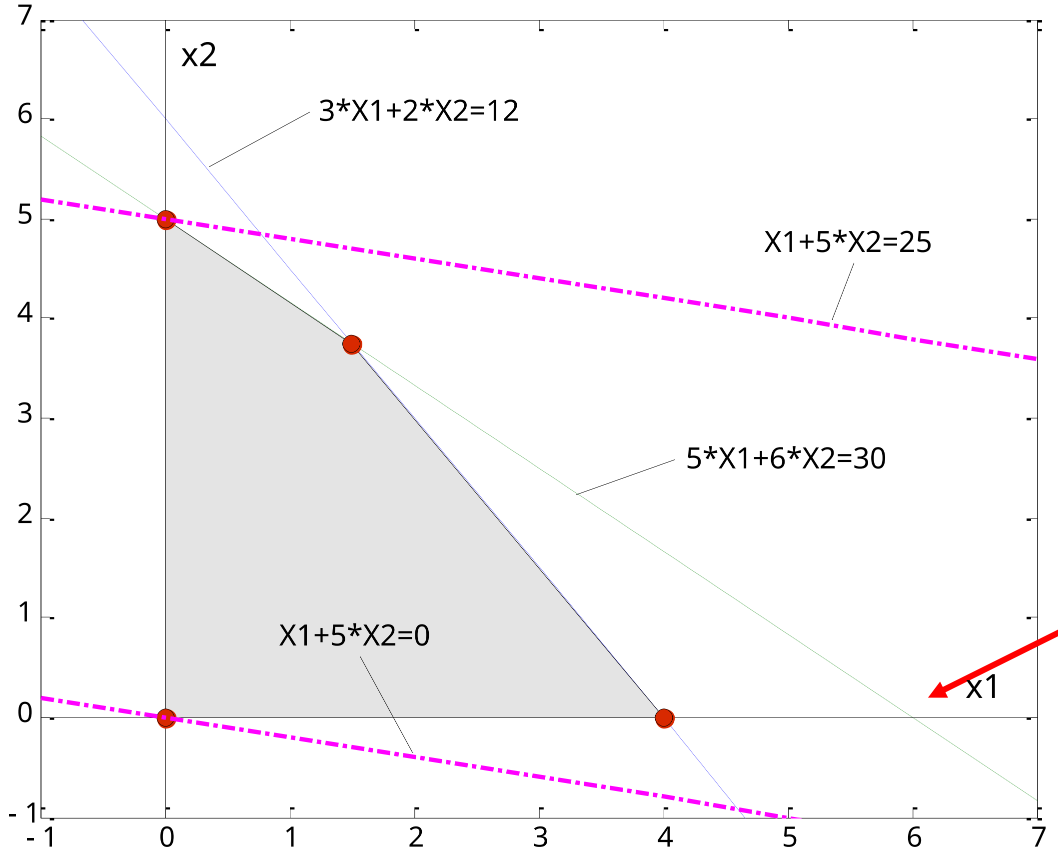
Definicija. Aktivno ograničenje u nekoj točki x je svako ograničenje koje je ispunjeno sa jednakosti u toj točki x.

Definicija. U ***n***-dimenzionalnom prostoru, **vrh politopa** je definiran kao presjecište barem ***n*** aktivnih linearno nezavisnih ograničenja pri čemu su ostala ograničenja zadovoljena.

Geometric analysis

Bodies described by constraints in LP are convex polytopes of P

= ∈{ℝ+ | ≤ }



Definition.IN*n*-dimensional space, **vertex of the polytope**is defined as the intersection of at least*n*of active linearly independent constraints, with the remaining constraints satisfied.

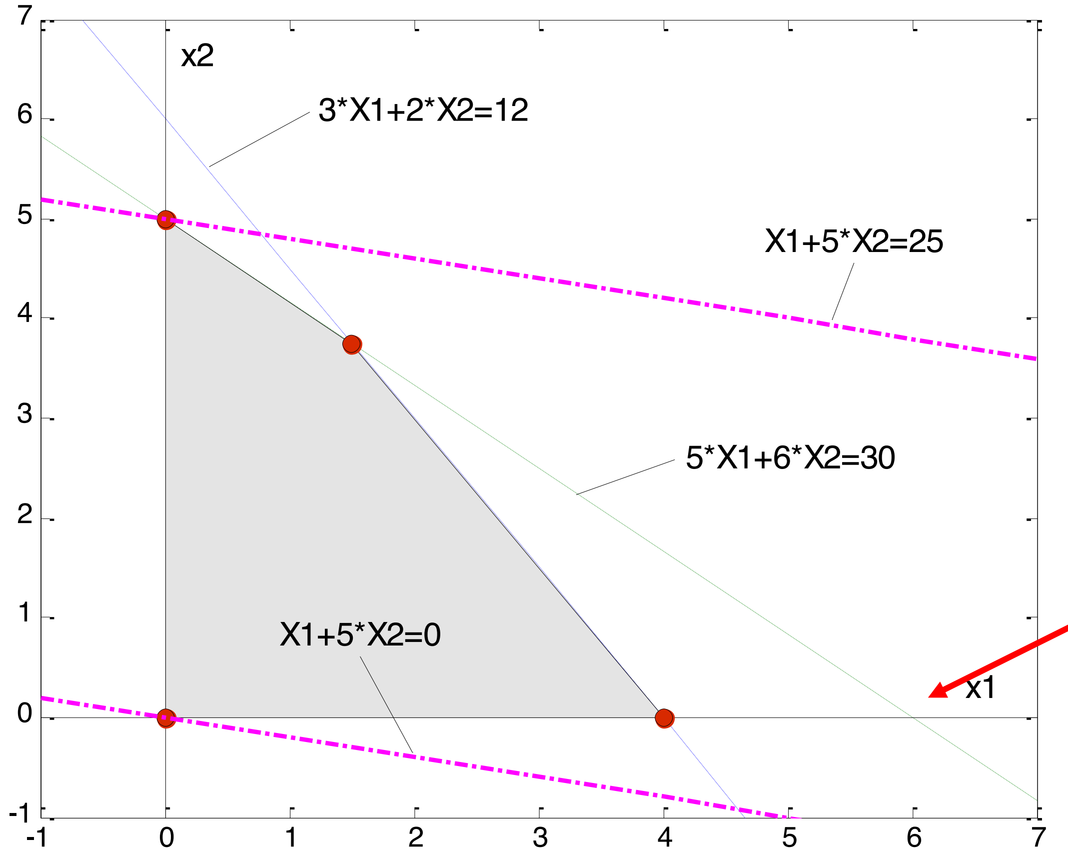
Caution!Not every intersection of n active constraints**top**! Inactive limits must be met to be in the top.

Neighbors?

Geometrijska analiza

Tijela opisana ograničenjima u LP su konveksni politopi P

$$P = \{x \in \mathbb{R}^n \ \mid Ax \leq b\}$$



Definicija. U ***n***-dimenzionalnom prostoru, **vrh politopa** je definiran kao presjecište barem ***n*** aktivnih linearno nezavisnih ograničenja pri čemu su ostala ograničenja zadovoljena.

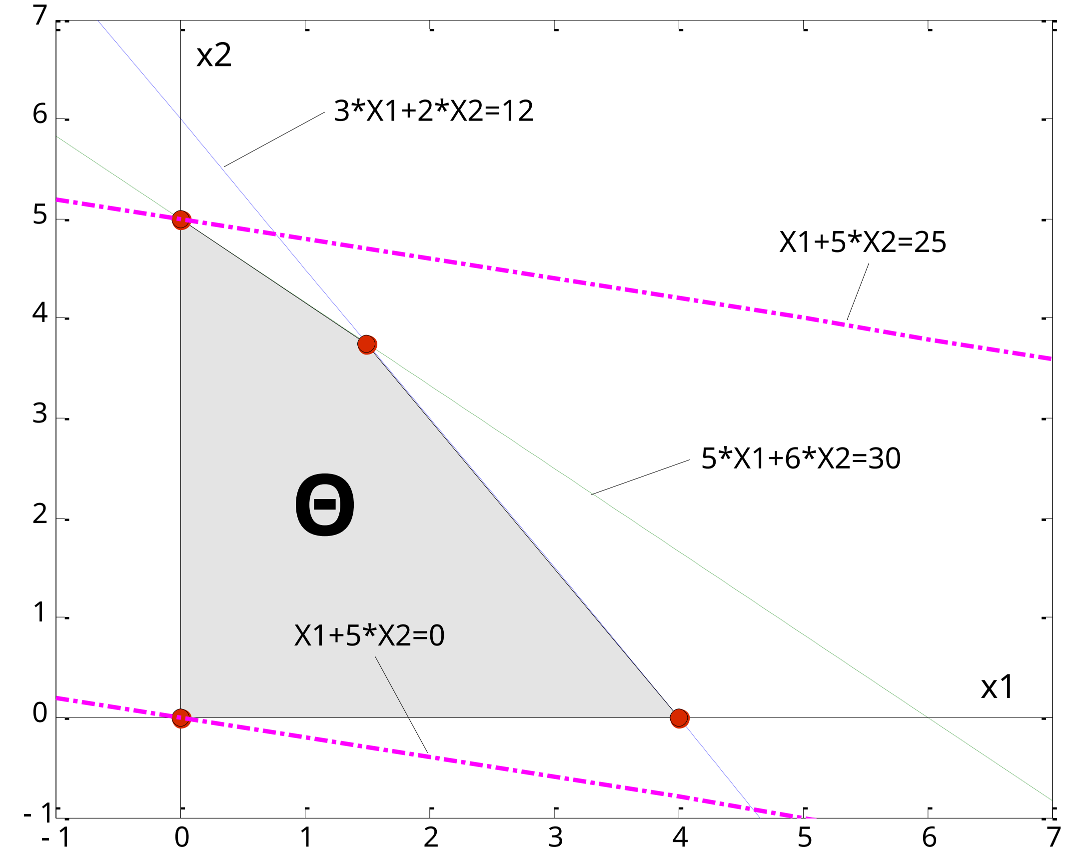
Oprez! Nije svako presjecište n aktivnih ograničenja **vrh**! Neaktivna ograničenja moraju biti ispoštovana da bismo bili u vrhu.

Susjedi?

Geometric analysis

- **Definition.**The set Θ is**convex set**if it contains all the points of the plane connection between any pair of points from the set Θ.

$$\forall x, y \in \Theta, \forall \alpha \in (0,1): \alpha x+(1-\alpha)y \in \Theta$$



- **Definition.Extreme convex set**Θis every point $\boldsymbol{x} \in \boldsymbol{\Theta}$which is not on the plane connecting any other two points of the set Θ.

$(\nexists x_1, x_2 \in \Theta \backslash \{x\}) \nexists (\in 0.1 \ (\quad))$

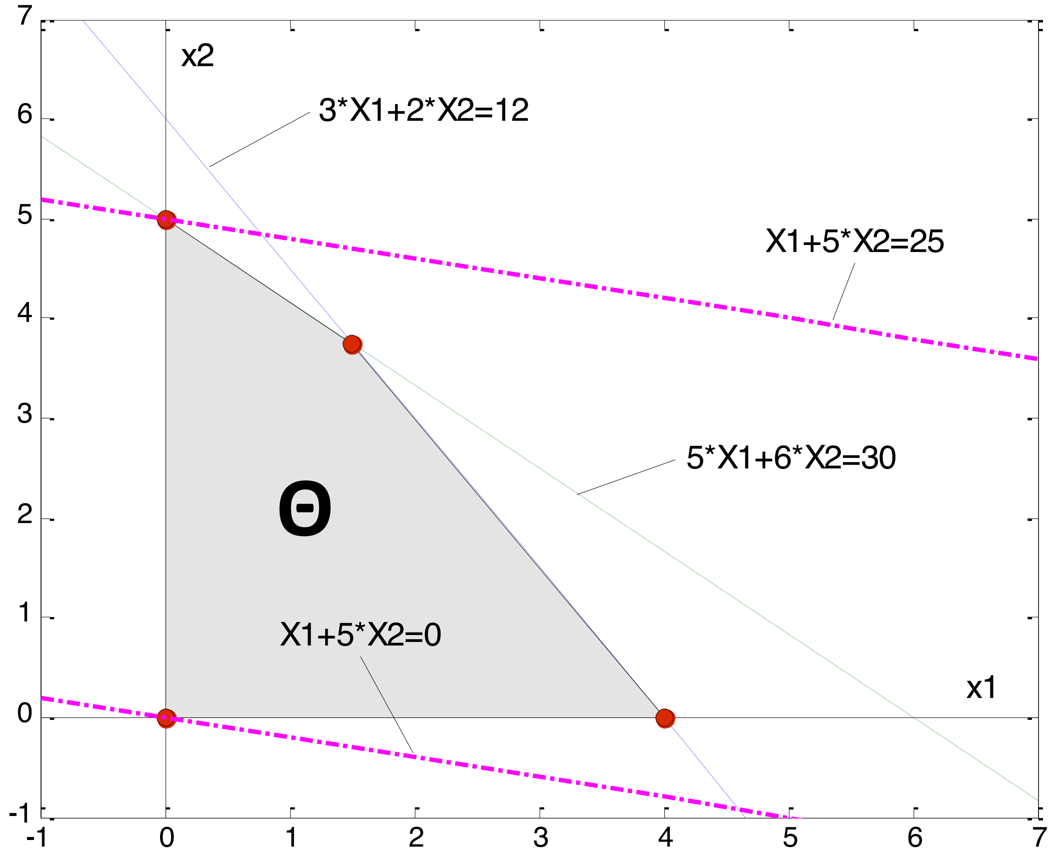
$= \quad _1 + 1 - \quad _2 \ (\quad)$

- The extrema in the polytope are**peaks -**geometric concept

Geometrijska analiza

- **Definicija.** Skup Θ je **konveksni skup** ako sadrži sve točke ravne spojnice između bilo kog para točaka iz skupa Θ.

  $\forall x,y \in \Theta,\ \forall \alpha \in (0,1):\ \alpha x+(1-\alpha)y \ \in \Theta$

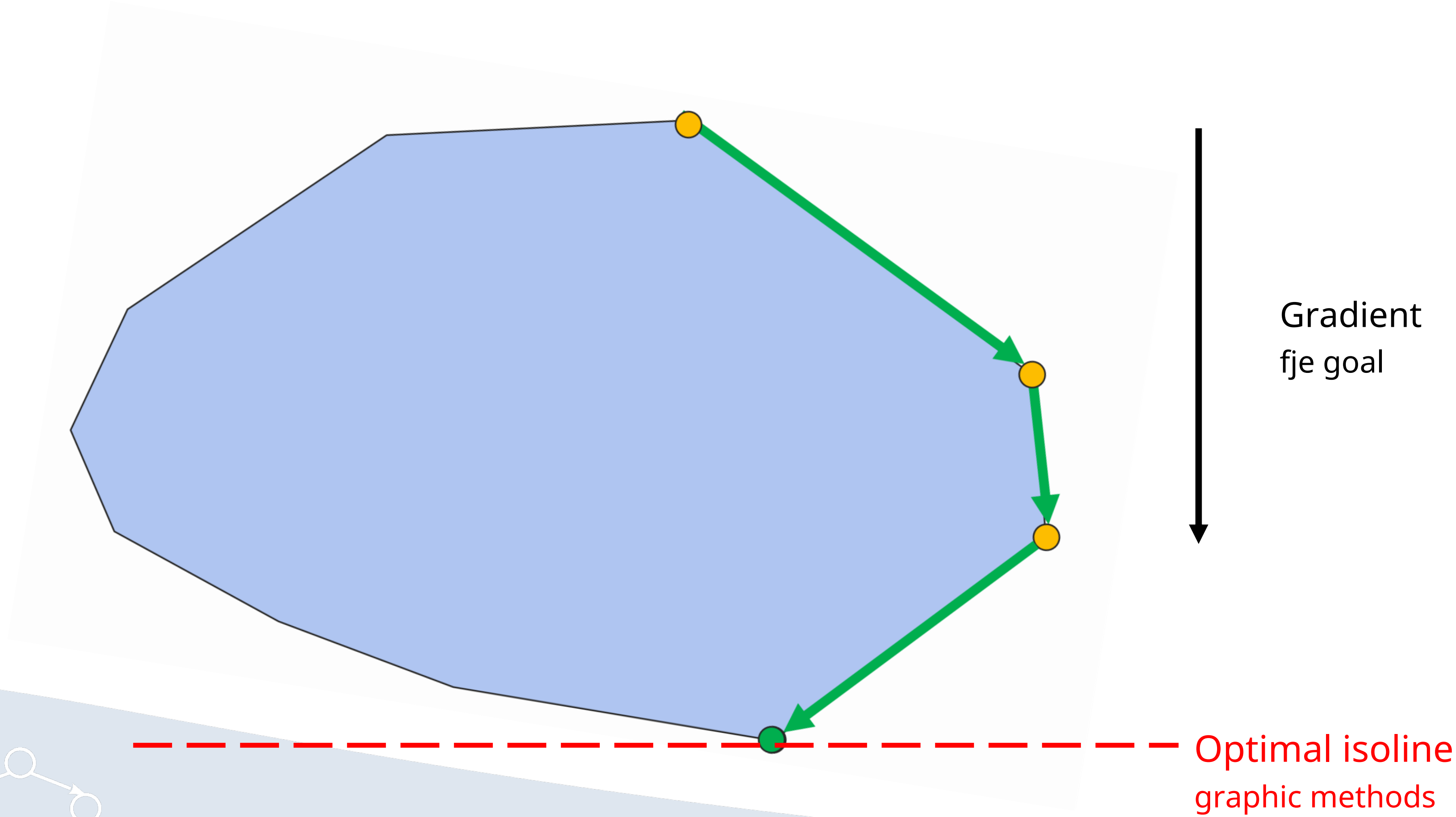


- **Definicija.** **Ekstrem konveksnog skupa** Θ je svaka točka ***x* ∈ Θ** koja nije na ravnoj spojnici ikojih drugih dviju točaka skupa Θ.

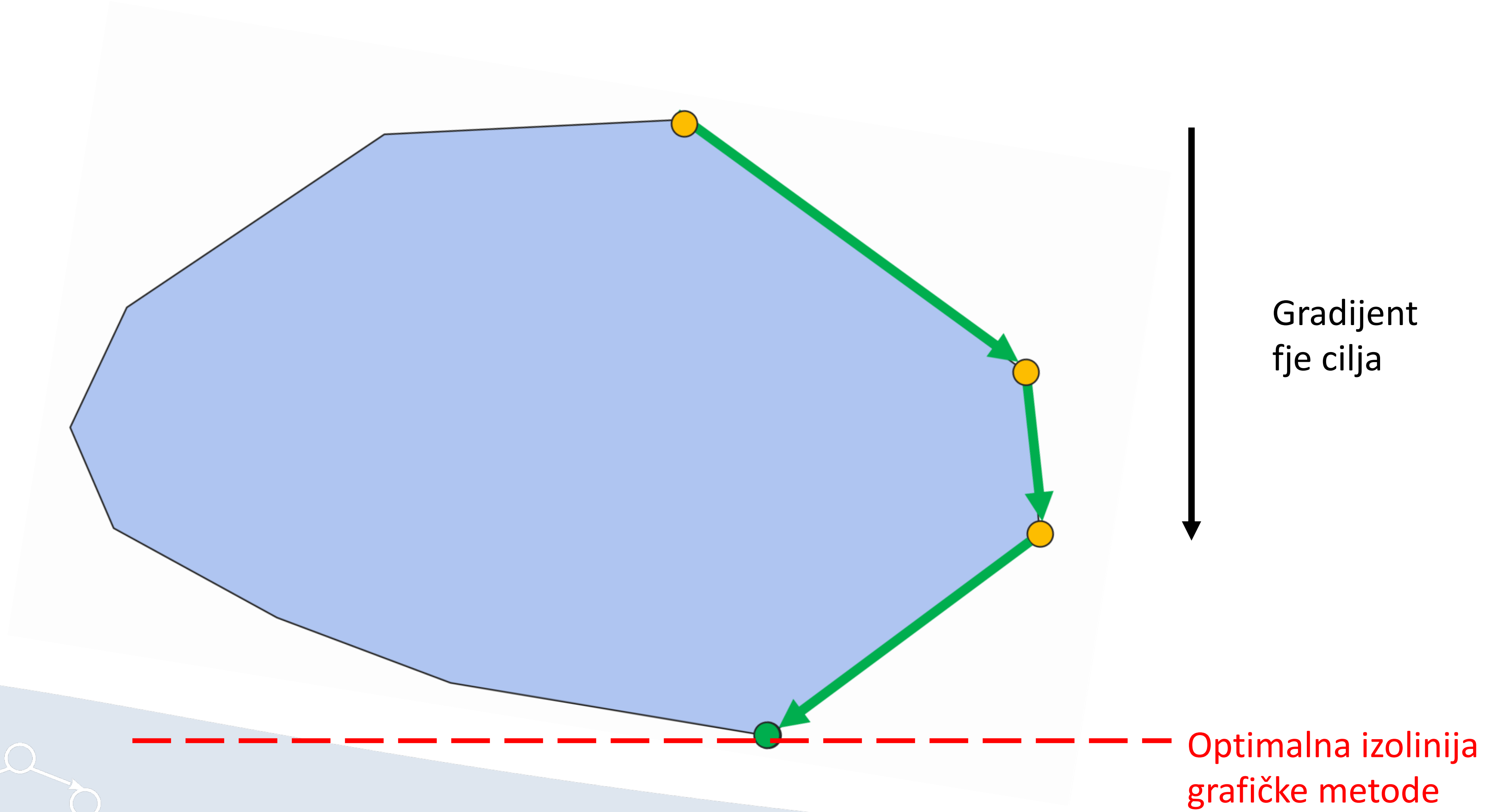
$$(\nexists x_1, x_2 \in \Theta \setminus \{x\})\left(\nexists \alpha \in (0,1)\right)$$
$$x = \alpha x_1 + (1-\alpha)x_2$$

- Ekstremi u politopu su **vrhovi –** geometrijski koncept

Sim



Simplex - ideja



Simplex – input problem

For simplex we use **LP in standard form**

minimize $\mathbf{c}^T\mathbf{x}$

with the condition $\mathbf{Ax}=\mathbf{b}$

$\mathbf{x}\geq\mathbf{0}$

whereby: $\mathbf{c}\in\mathrm{R}^n$, $\mathbf{b}\in\mathrm{R}^m$ and $\mathbf{b}\geq 0$, matrix **AND**$\in\mathrm{R}^{mxn}$, *rank*(**AND**) = m and $m < n$

We have **m** equality constraints, **n** inequality constraints ($\mathbf{x}\geq\mathbf{0}$) We are located in **n**-dimensional space (n≥m)

Tops determined with n active constraints (caution!)

Simplex – ulazni problem

Za simpleks koristimo **<u>LP u standardnoj formi</u>**

minimizirati	$\mathbf{c}^T\mathbf{x}$
uz uvjet	$\mathbf{Ax} = \mathbf{b}$
	$\mathbf{x} \geq \mathbf{0}$

pri čemu je: $\mathbf{c} \in R^n$, $\mathbf{b} \in R^m$ i $\mathbf{b} \geq 0$, matrica $\mathbf{A} \in R^{m x n}$, $rang(\mathbf{A}) = m$ i $m < n$

Imamo **m** ograničenja jednakosti, **n** ograničenja nejednakosti ($\mathbf{x} \geq \mathbf{0}$)

Nalazimo se u **n**-dimenzionalnom prostoru (n≥m)

Vrhovi određeni sa n aktivnih ograničenja (oprez!)

Simplex – the core of the method

- We are located in**n**-dimensional space (n≥m)
  - Vertices determined by**n active constraints**
  - m of active constraints is already fixed
  - **"Arbitrary"**(nm) we choose among inequalities
    - **They fix the values (nm) of variables to 0**
    - We will call these variables**non-basic**
    - When we include them in terms of limitations, we get a system of m equations with m unknowns! (we know how to solve from linear algebra) – we call the variables that we solve**basic**
- The set of all variables is partitioned into**basic**and**non-basic**

Simplex – jezgra metode

- Nalazimo se u **n**-dimenzionalnom prostoru (n≥m)
  - Vrhovi određeni sa **n aktivnih ograničenja**
  - m aktivnih ograničenja je već fiksirano
  - **„Proizvoljnih”** (n-m) biramo među nejednakostima
    - **One fiksiraju vrijednosti (n-m) varijabli na 0**
    - Te varijable ćemo nazivati **nebazičnima**
    - Kad ih uvrstimo u m ograničenja, dobijemo sustav m jednadžbi sa m nepoznanica! (znamo riješiti iz linearne algebre) – varijable koje rješavamo nazivamo **bazične**
- Particionira se skup svih varijabli na **bazične** i **nebazične**

Simplex - definitions

Definition. Basic solution system $\mathbf{A}\mathbf{x}=\mathbf{b}, \mathbf{x} \geq \mathbf{0}$ is a vector $\mathbf{x} = [\mathbf{x}_B^T\ \mathbf{0}]$, where is $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, And $\mathbf{B} \in R^{m \times m}$ the selected base (columns in the matrix **AND**) in the system of m of equations with n unknown, where is it $m < n$ and det($\mathbf{B}$) ≠ 0.

Theorem (basic theorem of linear programming): Let's consider the linear problem in standard form. The following applies: 1. If there is any solution, there is i**feasible basic solution**.

2. If there is an optimal solution, there is also a basic optimal solution.

Simplex – definicije

Definicija. Bazično rješenje sustava $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ je vektor $\mathbf{x} = [\mathbf{x}_B^T\ \mathbf{0}]$, gdje je $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, a $\mathbf{B} \in \mathrm{R}^{m \times m}$ odabrana baza (stupci u matrici **A**) u sustavu od m jednadžbi s n nepoznanica, pri čemu je $m < n$ i $\det(\mathbf{B}) \neq 0$.

Teorem (osnovni teorem linearnog programiranja):

Promatrajmo linearni problem u standardnoj formi. Vrijedi sljedeće:

1. Ako postoji bilo kakvo rješenje, postoji i **izvedivo bazično rješenje**.
2. Ako postoji optimalno rješenje, postoji i bazično optimalno rješenje.

Tabulation

Let's put all the parameters in a table:

- m+1 lines
- n+1 columns

Tabličenje

Hajdemo staviti sve parametre u tablicu:

- m+1 redaka
- n+1 stupaca

$\boldsymbol{c^T}$	$\mathbf{0}$
A	b

Tabulation

- Partitioning A by columns into basic B and non-basic N column-vectors

Tabličenje

- Particioniranje A po stupcima na bazične B i nebazične N stupac-vektore

$\boldsymbol{c_B}^{\boldsymbol{T}}$	$\boldsymbol{cN}^{\boldsymbol{T}}$	$\mathbf{0}$
B	N	b

Simplex tableau

Iterative implicit computation of the inverse ,-

- More efficient execution because adjacent vertices differ only in **one active constraint**(which sets some other variable to 0)

reduction factors by
edges

- fja aims

$c_N^T - c_B^T B^{-1} N$	$-c_B^T B^{-1} b$
B^{-1}	B^{-1}

edges to adjacent ones

base solution

Values

base var.

Simplex tableau

Iterativno implicitno izračunavanje inverza B^{-1}

- Efikasnije izvođenje jer se susjedni vrhovi razlikuju samo u **jednom aktivnom ograničenju** (koje postavlja neku drugu varijablu na 0)

faktori redukcije po bridovima

-fja cilja

$\mathbf{0}^T$	$\mathbf{c_N}^T - \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{N}$	$-c_B{}^T B^{-1}\mathbf{b}$
I	$B^{-1}N$	$B^{-1}b$

bridovi do susjednih baz.rješ

Vrijednosti baz.var.

Simplex – pseudocode

1. **The beginning** from the feasible basic solution in the simplex table

2. Optimal? If all reduction factors are non-negative, **STOP**

3. Transition to a better neighbor:

    a) Selection of the input non-basic variable corresponding to the column ***q***

    b) Selection of the output basic variable corresponding to the row ***p***. If it does not exist - the problem is unlimited, **STOP**

    c) Gauss-Jordan elimination for the pivot ***(p, q)***

4. Return to step 2

Simplex – pseudokod

1. **Početak** iz izvedivog bazičnog rješenja u simpleks tablici
2. Optimalno? Ako su svi faktori redukcije nenegativni, **STOP**
3. Tranzicija u boljeg susjeda:
   a) Odabir ulazne nebazične varijable koja odgovara stupcu ***q***
   b) Odabir izlazne bazične varijable koja odgovara retku ***p***. Ako ne postoji – problem je neograničen, **STOP**
   c) Gauss-Jordan eliminacija za pivot ***(p,q)***
4. Povratak na korak 2

Simplex – pseudocode

- Selecting the input non-basic variable corresponding to the column ***q***
  - Choose one that has **NEGATIVE** reduction factor

- Selection of output base variable $\boldsymbol{x_{[p]}}$ which corresponds to the line ***p***
  - $p = \text{argmin}_{and \in \{1,..,m\}} \{ x_{[and]} / B^{-1} AND_{iq} \mid B^{-1} AND_{iq} > 0 \}$

* [p] indicates variable selection via line reference

Simplex – pseudokod

- Odabir ulazne nebazične varijable koja odgovara stupcu ***q***
  - Odabere se neka koja ima **NEGATIVNI** faktor redukcije

- Odabir izlazne bazične varijable $\boldsymbol{x_{[p]}}$ koja odgovara retku ***p***
  - $p = \mathrm{argmin}_{i \in \{1,..,m\}}\{x_{[i]}/B^{-1}A_{iq} \mid B^{-1}A_{iq} > 0\}$

*[p] označava odabir varijable preko reference retkom

Simplex - example

$$\begin{aligned} \max \quad & 7x_1 + 6x_2 \\ \text{with} \quad & 2x_1 + x_2 \leq 3 \\ & x_1 + 4x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing two *slack* of the variable x_3 i x_4 we translate LP into standard form

$$\begin{aligned} \min \quad & -7x_1 - 6x_2 + 0x_3 + 0x_4 \\ \text{with} \quad & 2x_1 + x_2 + x_3 = 3 \\ & x_1 + 4x_2 + x_4 = 4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Simplex – primjer

$$
\begin{aligned}
\max \quad & 7x_1 + 6x_2 \\
\text{uz} \quad & 2x_1 + x_2 \leq 3 \\
& x_1 + 4x_2 \leq 4 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Uvođenjem dviju *slack* varijabli x_3 i x_4 prevodimo LP u standardnu formu

$$
\begin{aligned}
\min \quad & -7x_1 - 6x_2 + 0x_3 + 0x_4 \\
\text{uz} \quad & 2x_1 + x_2 + x_3 = 3 \\
& x_1 + 4x_2 + x_4 = 4 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

Simplex - example

Tabular record is

	And_1	And_2	And_3	And_4	b = RHS	
c^T	– 7	– 6	0	0	0	$= r^T$
	2	1	1	0	3	
	1	4	0	1	4	

- Table already valid, valid $r_{and} = c_{and}$
- the easiest is to choose the starting base $\mathbf{B}_0 = [\boldsymbol{And}_3 \boldsymbol{And}_4] = \mathbf{AND}_2$
  - basic solution $\mathbf{x}_{(0)} = [0\ 0\ 3\ 4]^T$, $f(\mathbf{x}_{(0)}) = 0$
- The zero line contains the reduction factors
  - there are negatives, so it is not optimal

Simplex – primjer

Tablični zapis je

	a_1	a_2	a_3	a_4	b = RHS	
c^T	–7	–6	0	0	0	$= r^T$
	2	1	1	0	3	
	1	4	0	1	4	

- Tablica već valjana, vrijedi $r_i = c_i$
- najlakše je odabrati početnu bazu $\mathbf{B}_0 = [\boldsymbol{a}_3\ \boldsymbol{a}_4] = \mathbf{I}_2$
  - bazično rješenje $\mathbf{x}_{(0)} = [0\ 0\ 3\ 4]^T$, $f(\mathbf{x}_{(0)})=0$
- Nulti redak sadrži faktore redukcije
  - ima negativnih pa nije optimum

Simplex - example

Tabular record is

	And_1	And_2	And_3	And_4	**RHS**	
$\mathbf{c}^T$	**– 7**	– 6	0	0	0	$=\mathbf{r}^T$
	2	1	1	0	**3**	
	1	4	0	1	**4**	

The zero line has negative reduction factors, so it is not optimal!

We choose **q=1** (the first column vector, a_1, enters the base)

We select the row p that corresponds to the output vector

$p=\text{argmin}_{and\in\{1,2\}}\{x_{[and]}/B^{-1}AND_{i1}; B^{-1}AND_{i1}>0\}=\text{argmin}\{3/2,4/1\}=1$ Pivot

(1,1): a enters the base$_1$, and outputs a_3

Gauss-Jordan elimination so that $\boldsymbol{And_1= [0,1,0]^T}$

Simplex – primjer

Tablični zapis je

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	**RHS**	
$\mathbf{c}^T$	**–7**	–6	0	0	0	$= \mathbf{r}^T$
	2	1	1	0	**3**	
	1	4	0	1	**4**	

Nulti redak ima negativne faktore redukcije pa nije optimum!

Odabiremo **q=1** (prvi stupčani vektor, a_1, ulazi u bazu)

Odabiremo redak p koji odgovara izlaznom vektoru

$p = \text{argmin}_{i\in\{1,2\}} \{x_{[i]} / B^{-1}A_{i1} \; ; B^{-1}A_{i1} > 0\} = \text{argmin}\{3/2, 4/1\} = 1$

Pivot (1,1): u bazu ulazi a_1, a izlazi a_3

Gauss-Jordanova eliminacija tako da $\boldsymbol{a_1 = [0,1,0]^T}$

Simplex - example

Tabular record is

	And$_1$	***And***$_2$	***And***$_3$	***And***$_4$	**RHS**	
$\mathbf{c}^T$	– 7	– 6	0	0	0	$=\mathbf{r}^T$
	2	1	1	0	3	
	1	4	0	1	4	

- Pivot (1,1): a enters the base$_1$, and outputs a$_3$

	And$_1$	***And***$_2$	***And***$_3$	***And***$_4$	**RHS**
$\mathbf{r}^T$	0	– 5/2	7/2	0	21/2
	1	1/2	1/2	0	3/2
	0	7/2	– 1/2	1	5/2

New base **B**$_1$= [***And***$_1$ ***And***$_4$] =**AND**$_2$

- solution $\mathbf{x}_{(1)}$= [3/2 0 0 5/2]^T, f($\mathbf{x}_{(1)}$)=-21/2

Simplex – primjer

Tablični zapis je

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	**RHS**	
$\mathbf{c}^T$	–7	–6	0	0	0	$= \mathbf{r}^T$
	2	1	1	0	3	
	1	4	0	1	4	

- Pivot (1,1): u bazu ulazi a_1, a izlazi a_3

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	**RHS**
$\mathbf{r}^T$	0	–5/2	7/2	0	21/2
	1	1/2	1/2	0	3/2
	0	7/2	–1/2	1	5/2

Nova baza $\mathbf{B}_1 = [\boldsymbol{a}_1\ \boldsymbol{a}_4] = \mathbf{I}_2$

- rješenje $\mathbf{x}_{(1)} = [3/2\ \ 0\ \ 0\ \ 5/2]^T$, $f(\mathbf{x}_{(1)})=-21/2$

Simplex - example

	And_1	And_2	And_3	And_4	RHS
r_T	0	**– 5/2**	7/2	0	21/2
	1	1/2	1/2	0	3/2
	0	7/2	– 1/2	1	5/2

- Negative reduction factor r_2 - not optimal!
- Selecting column q=2
- Line selection
  - $p = \text{argmin}_{and \in \{1,2\}} \{ x_{[and]} / B^{-1}AND_{i2} ; B^{-1}AND_{i2} > 0 \} = \text{argmin}\{\mathbf{3, 5/7}\} = 2$
- pivot (2,2)

Simplex – primjer

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	**RHS**
$\mathbf{r}^T$	0	**–5/2**	7/2	0	21/2
	1	1/2	1/2	0	3/2
	0	7/2	–1/2	1	5/2

- Negativan faktor redukcije r_2 - nije optimum!
- Odabir stupca q=2
- Odabir retka
  - $p = \mathrm{argmin}_{i\in\{1,2\}}\{x_{[i]} / B^{-1}A_{i2}\ ;\ B^{-1}A_{i2} > 0\} = \mathrm{argmin}\{\mathbf{3, 5/7}\} = 2$
- Pivot (2,2)

Simplex - example

	And_1	And_2	And_3	And_4	RHS
r_T	0	– 5/2	7/2	0	21/2
	1	1/2	1/2	0	3/2
	0	7/2	– 1/2	1	5/2

- Pivot (2,2): a enters the base$_2$, and outputs a_4

	And_1	And_2	And_3	And_4	RHS
r_T	0	0	22/7	5/7	86/7
	1	0	4/7	– 1/7	8/7
	0	1	– 1/7	2/7	5/7

Simplex – primjer

	a_1	a_2	a_3	a_4	**RHS**
$\mathbf{r}^T$	0	**–5/2**	7/2	0	21/2
	1	1/2	1/2	0	3/2
	0	7/2	–1/2	1	5/2

- Pivot (2,2): u bazu ulazi a_2, a izlazi a_4

	a_1	a_2	a_3	a_4	**RHS**
$\mathbf{r}^T$	0	0	22/7	5/7	86/7
	1	0	4/7	–1/7	8/7
	0	1	–1/7	2/7	5/7

Simplex - example

	And_1	And_2	And_3	And_4	**RHS**
$\mathbf{r}^T$	0	0	22/7	5/7	86/7
	1	0	4/7	– 1/7	8/7
	0	1	– 1/7	2/7	5/7

- there are no negative reduction factors
  - Optimum!
- Base $\mathbf{B}_2 = [And_1\ And_2] = \mathbf{AND}_2$
  - Solution $\mathbf{x}^* = [8/7\ 5/7\ 0\ 0]^T$, $f(\mathbf{x}^*) = -86/7$
- The solution to the initial problem
  - $x_1 = 8/7$ i $x_2 = 5/7$
  - $f_{max} = 86/7$

Simplex – primjer

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	**RHS**
$\mathbf{r}^T$	0	0	22/7	5/7	86/7
	1	0	4/7	–1/7	8/7
	0	1	–1/7	2/7	5/7

- nema negativnih faktora redukcije
  - Optimum!
- Baza $\mathbf{B}_2 = [\boldsymbol{a}_1\ \boldsymbol{a}_2] = \mathbf{I}_2$
  - Rješenje $\mathbf{x}^* = [8/7\ \ 5/7\ \ 0\ \ 0]^T$, $f(\mathbf{x}^*) = -86/7$
- Rješenje polaznog problema
  - $x_1 = 8/7$ i $x_2 = 5/7$
  - $f_{max} = 86/7$

Simplex – initial base problem!

- Sometimes, after converting to a standard form, the basic solution is not immediately visible!
- Two-phase simplex – 2 LPs are solved in series
  - 1ST PHASE –**auxiliary artificial step**to find the initial basic solution
    - It always has its own trivial starting base (that's how it's constructed)
  - PHASE 2 – actually resolves the LP of interest

Simplex – problem početne baze!

- Nekad se nakon pretvorbe u standardni oblik ne vidi odmah bazično rješenje!
- Dvofazni simpleks – rješavaju se 2 LPa u nizu
  - 1. FAZA – **pomoćni umjetni korak** za naći početno bazično rješenje
    - Uvijek ima svoju trivijalnu početnu bazu (tako je konstruiran)
  - 2. FAZA – zapravo riješava LP od interesa

Two-phase simplex – an artificial problem

- Prev. that we have a problem in standard form

minimize $\mathbf{c}^T\mathbf{x}$

with the condition $\mathbf{Ax}=\mathbf{b}$

$\mathbf{x}\geq\mathbf{0}$

where we have m equality constraints. We add ***m*** artificial variables to create a unit submatrix

NEW PROBLEM LP':

minimize $\mathbf{1}^T\mathbf{x}_{n+1:n+m}$

with the condition $[\mathbf{A}|\ \mathbf{AND}_m]\ [\mathbf{x}_{1:n}|\mathbf{x}_{n+1:n+m}]=\mathbf{b}$

$\mathbf{x}_{1:n+m}\geq\mathbf{0}$

Dvofazni simpleks – umjetni problem

- Pretp. da imamo problem u standardnoj formi

$$\begin{array}{ll} \text{minimizirati} & \mathbf{c}^T\mathbf{x} \\ \text{uz uvjet} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

pri čemu imamo m ograničenja jednakosti. Dodajemo ***m*** umjetnih varijabli da bismo stvorili jediničnu podmatricu

NOVI PROBLEM LP':

$$\begin{array}{ll} \text{minimizirati} & \mathbf{1}^T\mathbf{x}_{n+1:n+m} \\ \text{uz uvjet} & [\mathbf{A} \mid \mathbf{I}_m]\,[\mathbf{x}_{1:n} \mid \mathbf{x}_{n+1:n+m}] = \mathbf{b} \\ & \mathbf{x}_{1:n+m} \geq \mathbf{0} \end{array}$$

Two-phase simplex – 1ST PHASE

- Let's solve the new problem with the already defined procedure
- Three possible outcomes:

1. Optimum ∗!"#≠ 0 ⇒ !**END!**
2. Optimum ∗!"#= 0
    a) All artificial variables are non-basic. Adaptation for PHASE 2
    b) Some artificial variables are basic. Performs simplex iterations until all artificial variables are out of the database. Adaptation for PHASE 2

Dvofazni simpleks – 1. FAZA

- Riješimo novi problem već definiranim postupkom
- Tri moguća ishoda:

1. Optimum $f_{LP'}^{*} \neq 0 \Rightarrow originalni\ LP\ neizvediv$! **KRAJ!**
2. Optimum $f_{LP'}^{*} = 0$
   a) Sve umjetne varijable su nebazične. Adaptacija za 2. FAZU
   b) Neke umjetne varijable su bazične. Izvodi iteracije simpleksa dok ne izađu sve umjetne varijable iz baze. Adaptacija za 2. FAZU

Two-phase simplex – PHASE 2

- Adaptation of the table from LP' (contains the base for the original LP)

1. Delete columns of artificial variables
2. Replace the target fuse with the original one
3. Bring the first row into the reduction factors

- Solve the LP from that point on

Dvofazni simpleks – 2. FAZA

- Adaptacija tablice od LP’ (sadrži bazu za originalni LP)

1. Pobrisati kolone umjetnih varijabli
2. Zamijeniti fju cilja originalnom
3. Dovesti prvi red u faktore redukcije

- Riješiti LP od te točke nadalje

Two-phase simplex – an example

at least $2x_1 + 3x_2$

with $4x_1 + 2x_2 \geq 12$

$x_1 + 4x_2 \geq 6$

$x_1, x_2 \geq 0$

standard form:

$$
\begin{aligned}
\text{min} \quad & 2x_1 + 3x_2 \\
\text{with} \quad & 4x_1 + 2x_2 - x_3 = 12 \\
& x_1 + 4x_2 - x_4 = 6 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

Dvofazni simpleks – primjer

$$\begin{aligned} \min\ & 2x_1 + 3x_2 \\ \text{uz}\ & 4x_1 + 2x_2 \geq 12 \\ & x_1 + 4x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

standardna forma:

$$\begin{aligned} \min\ & 2x_1 + 3x_2 \\ \text{uz}\ & 4x_1 + 2x_2 - x_3 = 12 \\ & x_1 + 4x_2 - x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Two-phase simplex – an example

We add two new variables and a new target fw

And$_1$ ***And***$_2$ ***And***$_3$ ***And***$_4$ ***And***$_5$ ***And***$_6$ **b =**

cT **RHS** 0 0 0 0 1 1 0

4 2 –1 0 1 0 12

1 4 0 – 1 0 1 6

Dvofazni simpleks – primjer

Dodajemo dvije nove varijable i novu fju cilja

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\mathbf{b}$ **= RHS**
$\mathbf{c}^T$	0	0	0	0	1	1	0
	4	2	–1	0	1	0	12
	1	4	0	–1	0	1	6

Two-phase simplex – an example

After several iterations...

	And $_1$	***And*** $_2$	***And*** $_3$	***And*** $_4$	***And*** $_5$	***And*** $_6$	**RHS**
$\mathbf{r}_T$	0	0	0	0	1	1	0
	1	0	– 2/7	1/7	2/7	– 1/7	18/7
	0	1	1/14	– 2/7	– 1/14	2/7	6/7

base $\mathbf{B}_2$= [***And*** $_1$ ***And*** $_2$] = $\mathbf{AND}_2$

reduction factors are non-negative -> **OPTIMUM** of the artificial problem the artificial variables are =0 and the artificial objective function =0

Phase 1 completed

Dvofazni simpleks – primjer

Nakon nekoliko iteracija...

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	**RHS**
$\mathbf{r}^T$	0	0	0	0	1	1	0
	1	0	–2/7	1/7	2/7	–1/7	18/7
	0	1	1/14	–2/7	–1/14	2/7	6/7

baza $\mathbf{B}_2 = [\boldsymbol{a}_1\ \boldsymbol{a}_2] = \mathbf{I}_2$

faktori redukcije su nenegativni -> **OPTIMUM** umjetnog problema

umjetne varijable su =0 i umjetna ciljna funkcija =0

gotova 1. FAZA

Two-phase simplex – an example

2nd stage – the columns of artificial variables are removed from the previous table and the target fja is replaced

	And$_1$	***And***$_2$	***And***$_3$	***And***$_4$	**RHS** 2
cT		3	0	0	0
	1	0	– 2/7	1/7	7/18
	0	1	1/14	–2/7	6/7

Correction of the 0th line-by elimination above the base columns

Dvofazni simpleks – primjer

2. faza – iz prethodne tablice se izbace stupci umjetnih varijabli i zamijeni se ciljna fja

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	**RHS**
$\mathbf{c}^T$	2	3	0	0	0
	1	0	–2/7	1/7	18/7
	0	1	1/14	–2/7	6/7

Ispravak 0-tog retka - eliminacijom iznad baznih stupaca

Two-phase simplex – an example

	And$_1$	*And*$_2$	*And*$_3$	*And*$_4$	**RHS**
$\mathbf{r}^T$	0	0	5/14	4/7	– 54/7
	1	0	– 2/7	1/7	18/7
	0	1	1/14	– 2/7	6/7

- there are no negative reduction factors
  - **OPTIMUM**
  - The solution to the extended original problem is
    $\mathbf{x}$= [18/7 6/7 0 0]^T
  - solution to the original problem $\mathbf{x}$= [18/7 6/7]^T, f($\mathbf{x}$)=54/7

Dvofazni simpleks – primjer

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	**RHS**
$\mathbf{r}^T$	0	0	5/14	4/7	–54/7
	1	0	–2/7	1/7	18/7
	0	1	1/14	–2/7	6/7

- nema negativnih faktora redukcije
  - **OPTIMUM**
  - Rješenje proširenog izvornog problema je $\mathbf{x} = [18/7 \;\; 6/7 \;\; 0 \;\; 0]^T$
  - rješenje izvornog problema $\mathbf{x} = [18/7 \;\; 6/7]^T$, $f(\mathbf{x})=54/7$

Duality

- The theory created by generalizing the method of Lagrange multipliers

- Each LP (which we will call "primal") has its associated LP which we call "dual"

Dualnost

- Teorija nastala poopćenjem metode Lagrangeovih množitelja

- Svaki LP (kojeg ćemo zvati „primal”) ima svoj povezani LP kojeg zovemo „dual”

Duality – canonical form

minimize $\mathbf{c}^T\mathbf{x}$

with the condition $\mathbf{Ax \leq b}$

$\mathbf{x \geq 0}$

- Dual:

to maximize $\mathbf{b}^T\mathbf{y}$

with the condition **AND^{T}y≥c**

$\mathbf{y \geq 0}$

Dualnost – kanonska forma

$$\begin{array}{rl} \text{minimizirati} & \mathbf{c}^T\mathbf{x} \\ \text{uz uvjet} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

- Dual:

$$\begin{array}{rl} \text{maksimizirati} & \mathbf{b}^T\mathbf{y} \\ \text{uz uvjet} & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

Dual - performance

primal	dual
number of restrictions	number of variables
number of variables	number of restrictions
rhs	objective function
objective function	rhs
ANDmatrix of coefficients	**AND**T
equality	urs variable
urs variable	equality
<= limit	> = variable
> = limit	<= variable
> = variable	> = limit
<= variable	<= limit

Dual - izvođenje

primal	dual
broj ograničenja	broj varijabli
broj varijabli	broj ograničenja
rhs	funkcija cilja
funkcija cilja	rhs
A matrica koeficijenata	$\mathbf{A}^T$
jednakost	urs varijabla
urs varijabla	jednakost
<= ograničenje	>= varijabla
>= ograničenje	<= varijabla
>= varijabla	>= ograničenje
<= varijabla	<= ograničenje

Connections between duals and primals - theorems

- "Dual dual is primal"
- Weak duality
- Strong duality
- Complementarity

Veze duala i primala - teoremi

- „Dual duala je primal”
- Slaba dualnost
- Jaka dualnost
- Komplementarnost

Connections between duals and primals - theorems

- Weak duality
  - If x is a feasible primal solution and y is a feasible dual solution, then 5 ≤ 5
- Strong duality
  - If the linear program has an optimal solution, then the dual one also has one and their values are equal.

Veze duala i primala - teoremi

- Slaba dualnost
  - Ako je x izvedivo primalno rješenje i y je izvedivo dualno rješenje, tada je $y^T b \leq c^T x$
- Jaka dualnost
  - Ako linearni program ima optimalno rješenje, onda ga ima i dual i njihove vrijednosti su jednake.

- Complementarity
  - If xiy are feasible solutions of primal and dual, then they are ***optimal* if and only if** valid for:

" "(# ::= 0, ∀)

Factor
reductions
variables x_j

& &(,: − &= 0,)∀

Replenishment
i-th
limitations
the midwife

Veze duala i primala - teoremi

- **Komplementarnost**
  - Ako su x i y izvediva rješenja primala i duala, onda su ***optimalna* ako i samo ako** vrijedi:

$$x_j\underbrace{\left(c_j - y^T A_{:,j}\right)}_{\text{Faktor redukcije varijable } x_j} = 0, \forall j$$

$$y_i\underbrace{\left(A_{i,:}x - b_i\right)}_{\text{Dopunjenje i-tog ograničenja primala}} = 0, \forall i$$

Duality - utility

- Economic interpretation – prices over limited resources
- Minimax theorem in game theory
- Sensitivity analysis
- Dual simplex method
- . . .

Dualnost - korisnost

- Ekonomska interpretacija – cijene nad ograničenim resursima
- Minimax teorem u teoriji igara
- Analiza osjetljivosti
- Dualna simpleksna metoda
- …

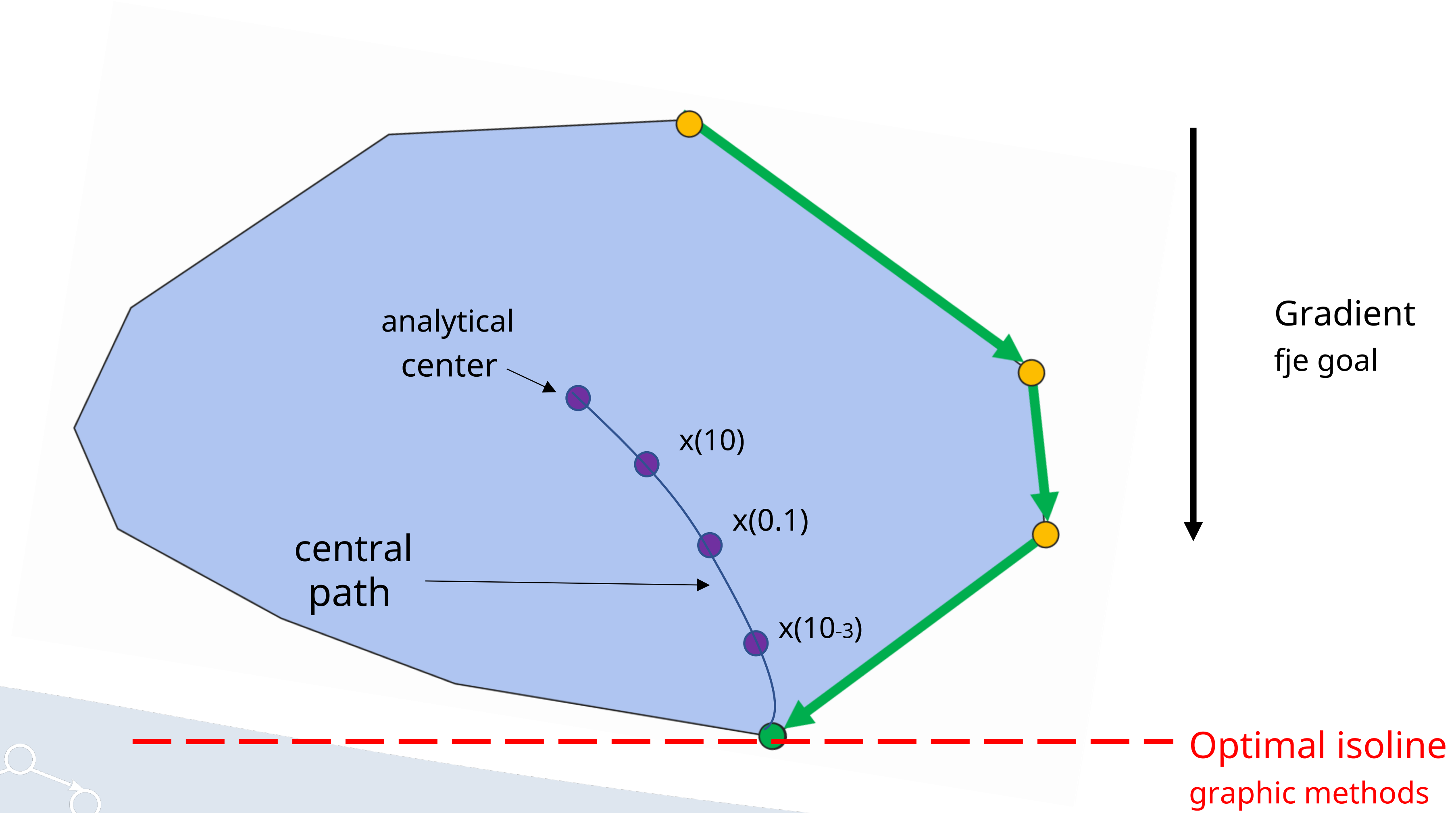
Simplex - problem!

- Klee-Minty 1972 – perturbed unit hypercube construction for popular pivot selection rules
- Simplex has exponential complexity in the worst case
- LP is within the class of problems P

Simplex – problem!

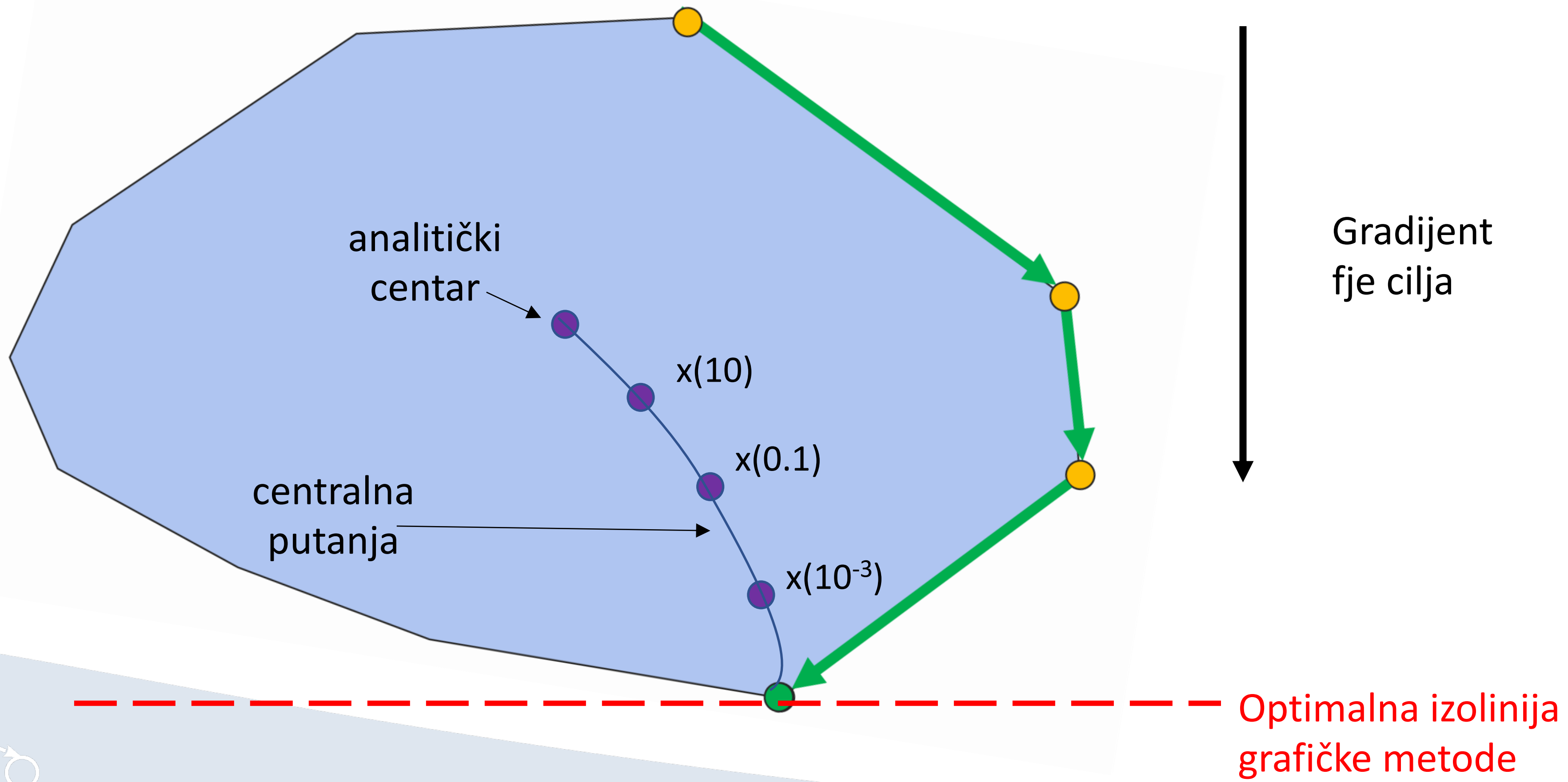
- Klee-Minty 1972. – konstrukcija perturbirane jedinične hiperkocke za popularna pravila biranja pivota
- Simplex u najgorem slučaju ima eksponencijalnu složenost
- LP je unutar klase problema P

Me e



analytical center
x(10)
x(0.1)
central path
x(10-3)
Gradient
fje goal
Optimal isoline
graphic methods

Metoda unutarnjih točaka - ideja



Method of internal points - idea 1/3

min $\mathbf{c}^T\mathbf{x}$

with the condition $\mathbf{Ax=b}$

$\mathbf{x \geq 0}$

---

max $\mathbf{b}^T\mathbf{y}$

with the condition **AND^{T}y+with=c**

with≥0

Metoda unutarnjih točaka – ideja 1/3

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{uz uvjet} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbf{b}^T\mathbf{y} \\ \text{uz uvjet} \quad & \mathbf{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c} \\ & \mathbf{s} \geq \mathbf{0} \end{aligned}$$

Method of interior points – idea 2/3

min $c^T x - \mu\, 1^T \log(x)$

with the condition $Ax=b$

Barrier problems!

Logarithmic barrier

---

max $b^T y + \mu\, 1^T \log(s)$ A^T

with the condition y+with=c

Metoda unutarnjih točaka – ideja 2/3

$$\min \quad \mathbf{c}^T\mathbf{x} - \mu\, \mathbf{1}^T\log(\mathbf{x})$$
$$\text{uz uvjet} \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

Barijerni problemi!

Logaritamska barijera

$$\max \quad \mathbf{b}^T\mathbf{y} + \mu\, \mathbf{1}^T\log(\mathbf{s})$$
$$\text{uz uvjet} \quad \mathbf{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c}$$

Method of interior points – idea 3/3

KKT conditions for μ-barrier problems

Ax(μ)=b

x(μ)≥0

AND^{T}y (μ)+with (μ)

=cs(μ)≥0

X(μ)S(μ)1=1μ

where X(μ)=diag(x(μ)), S(μ)=diag(s(μ))

Metoda unutarnjih točaka – ideja 3/3

KKT uvjeti za μ-barijerne probleme

$$\mathbf{Ax(\mu) = b}$$

$$\mathbf{x(\mu) \geq 0}$$

$$\mathbf{A^T y\,(\mu) + s\,(\mu) = c}$$

$$\mathbf{s(\mu) \geq 0}$$

$$\mathbf{X(\mu)S(\mu)1=1\mu}$$

pri čemu $X(\mu)=\mathrm{diag}(x(\mu))$, $S(\mu)=\mathrm{diag}(s(\mu))$

A primal trajectory tracking algorithm

- Barrier problem "too difficult" from KKT

- The Taylor expansion of the barrier fje aims to the quadratic term

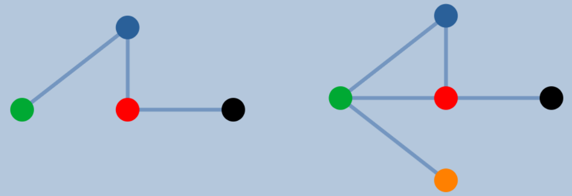
- Optimization of the Taylor approximation by the method of Lagrange multipliers for finding of the minimizing direction from the current point

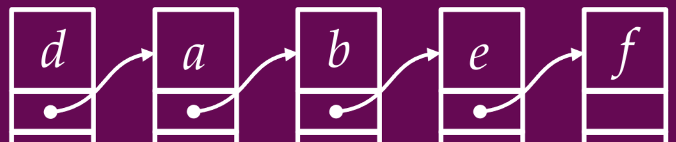
Primalni algoritam praćenja putanje

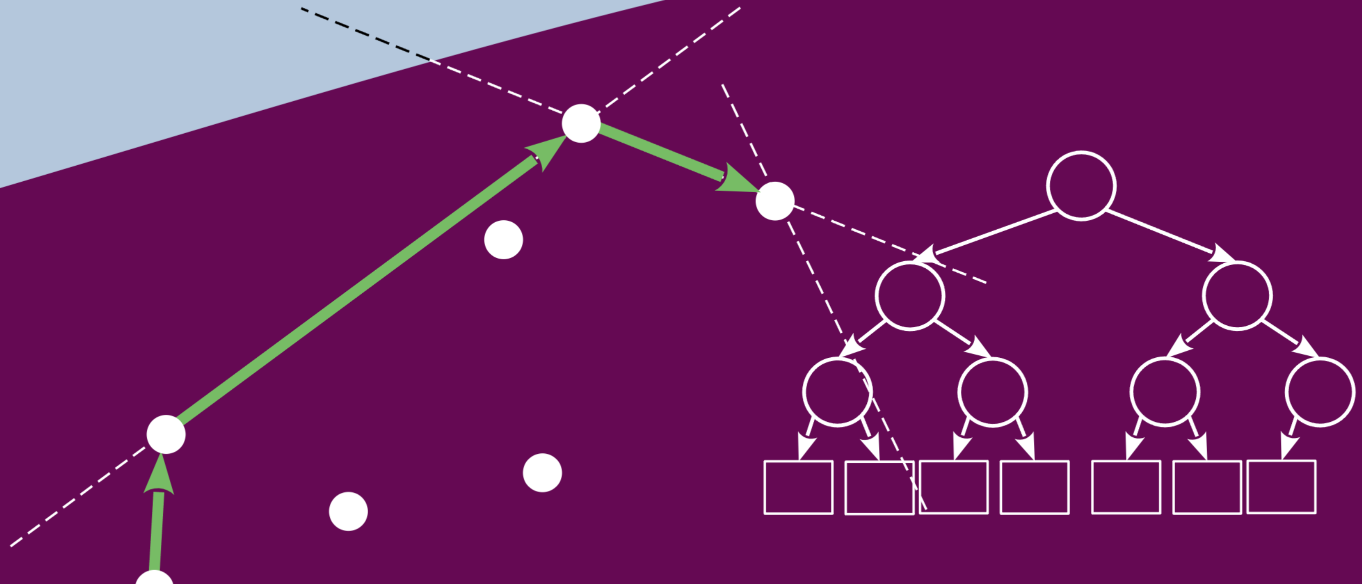
- Barijerni problem „pretežak” iz KKT
- Taylorov raspis barijerne fje cilja do kvadratnog člana
- Optimizacija Taylorove aproksimacije metodom Lagrangeovih množitelja za pronalazak minimizirajućeg smjera iz trenutne točke

Advanced algorithms and structures data

Addition: Simplex – graphic analogy

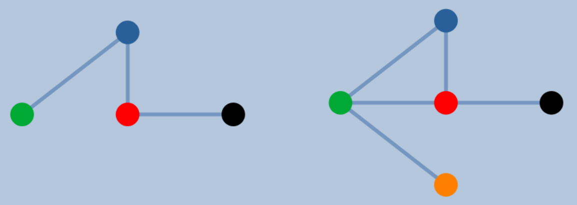


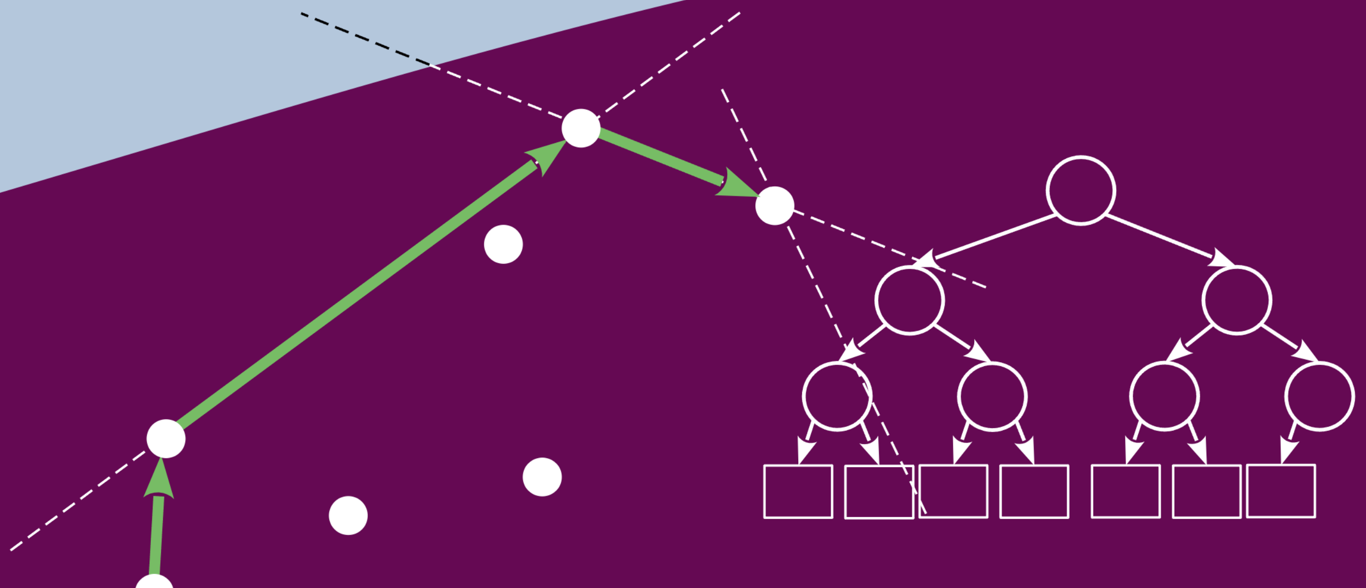


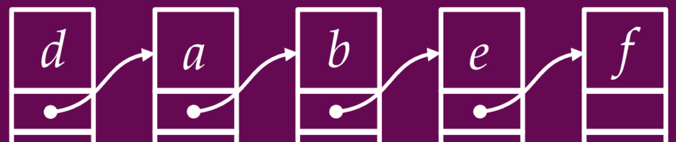


Napredni algoritmi i strukture podataka

Dodatak: Simplex – grafička analogija







An issue

Solve:

$$\max\ 3x_1 + 5x_2$$

$$\begin{aligned} \text{with}\quad x_1 + 5x_2 &\leq 40 \\ 2x_1 + x_2 &\leq 20 \\ x_1 + x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Problem

Riješite:

$$\max\ 3x_1 + 5x_2$$

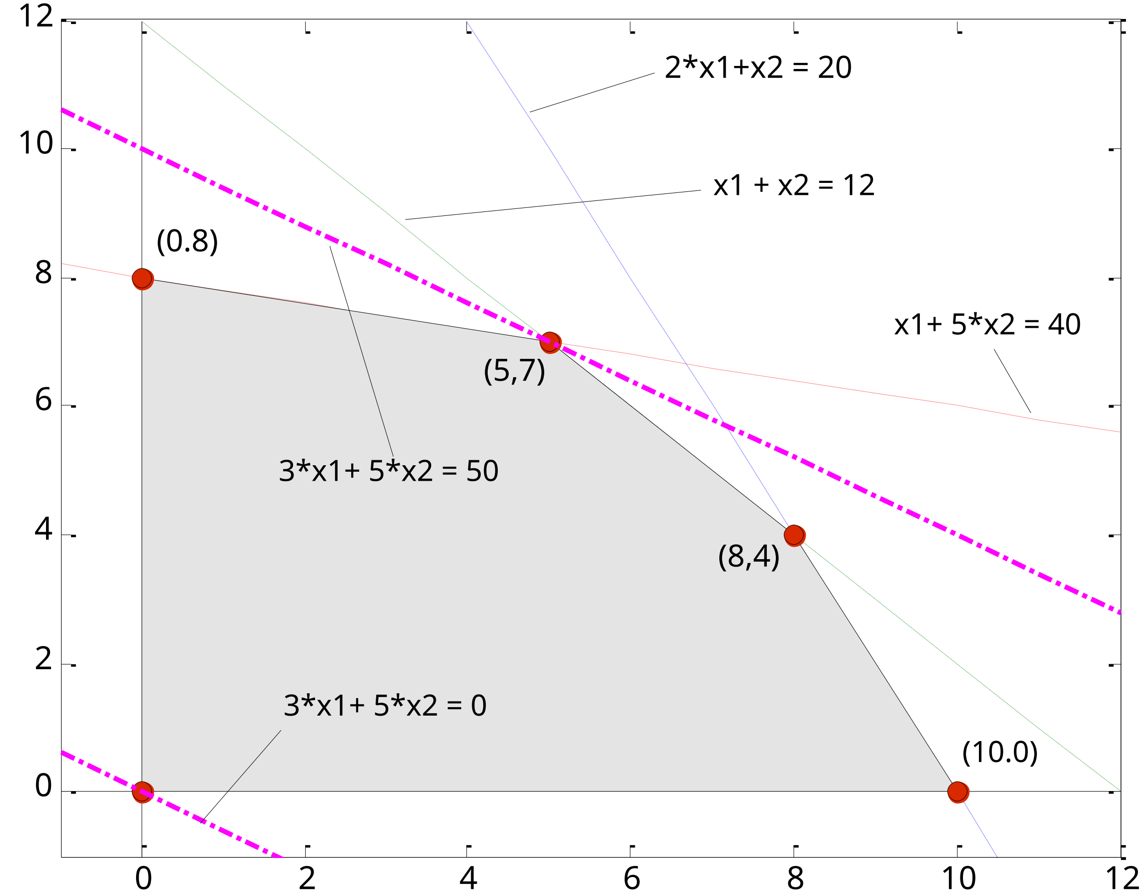
$$\begin{aligned} \text{uz} \quad x_1 + 5x_2 &\le 40 \\ 2x_1 + x_2 &\le 20 \\ x_1 + x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Graphical solution

Graphic solution

$f_{min} = 0$; $x_1 = 0$, $x_2 = 0$ $f_{max} = 50$; $x_1 = 5$, $x_2 = 7$

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{with} \quad & x_1 + 5x_2 \leq 40 \\ & 2x_1 + x_2 \leq 20 \\ & x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$



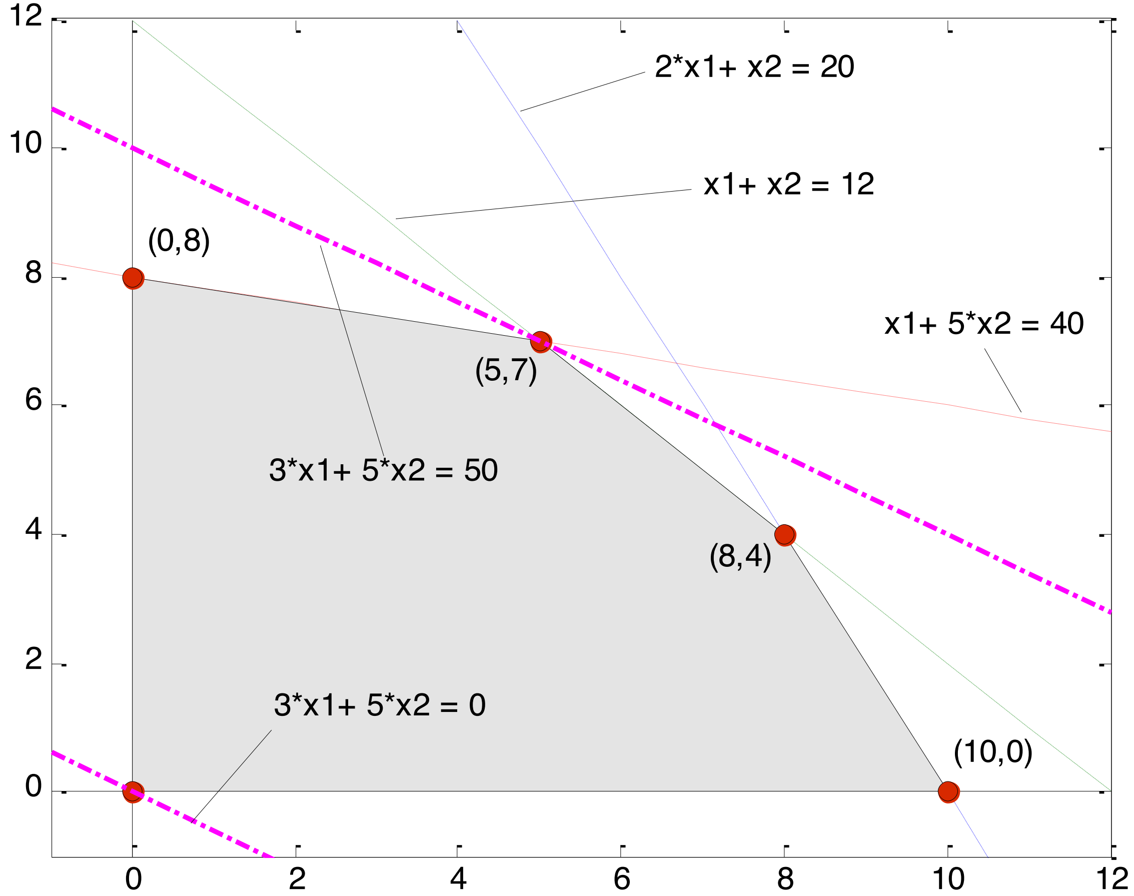
Rješenje grafičkom metodom

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{uz} \quad & x_1 + 5x_2 \le 40 \\ & 2x_1 + x_2 \le 20 \\ & x_1 + x_2 \le 12 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Grafičko rješenje

$f_{min} = 0; \quad x_1 = 0, x_2 = 0$

$f_{max} = 50; x_1 = 5, x_2 = 7$



Simplex – init (iteration 0)

Conversion to standard form:

c^T $\qquad\qquad$ b

$$
\begin{aligned}
\min \quad & \boxed{-3x_1 - 5x_2} \\
\text{with} \quad & \boxed{\begin{aligned} x_1 + 5x_2 + x_3 &= \\ 2x_1 + x_2 + x_4 &= \\ x_1 + x_2 + x_5 &= \end{aligned}} \boxed{\begin{matrix} 40 \\ 20 \\ 12 \end{matrix}} \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

- Expressing the solution in the base of column vectors of the matrix A

$$
\begin{array}{cccccc}
\textbf{And}_1 & \textbf{And}_2 & \textbf{And}_3 & \textbf{And}_4 & \textbf{And}_5 & \textbf{b}
\end{array}
$$

$$
x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 40 \\ 20 \\ 12 \end{bmatrix}
$$

- As a starting solution, we take the extreme $\mathbf{x}_0 = [0, 0, 40, 20, 12]^T$. $f(x_0)=0$
- Basic variables (x_3, x_4, x_5)

Simplex – init (iteracija 0)

Pretvorba u standardnu formu:

$$\begin{array}{lrl} & \mathbf{c}^T & \\ \min & \boxed{-3x_1 - 5x_2} & \mathbf{b} \\ \text{uz} & \begin{array}{rrrrr} x_1 + & 5x_2 + & x_3 & & \\ 2x_1 + & x_2 + & & x_4 & \\ x_1 + & x_2 + & & & x_5 \end{array} & \begin{array}{l} = 40 \\ = 20 \\ = 12 \end{array} \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 & \end{array}$$

- Izražavanje rješenja u bazi stupčanih vektora matrice A

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & b \end{array}$$

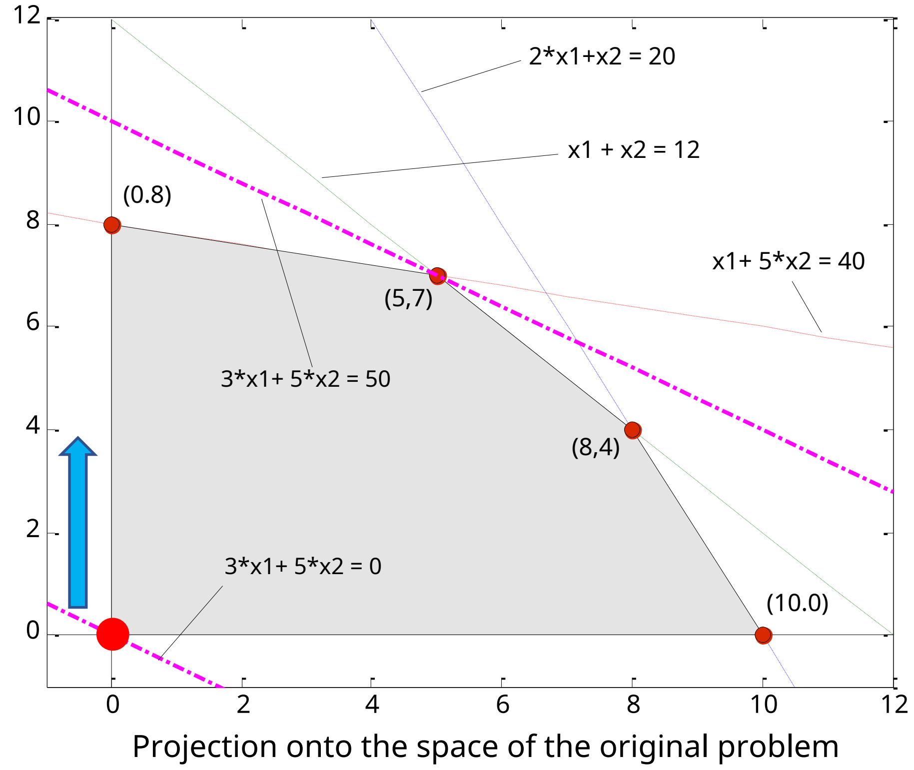
$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 40 \\ 20 \\ 12 \end{bmatrix}$$

- Kao polazno rješenje uzimamo ekstrem $\mathbf{x}_0 = [0, 0, 40, 20, 12]^T$. $f(x_0)=0$
- Bazične varijable (x_3,x_4,x_5)

Simplex – iteration 1

And1	And2	And3	And4	And5	RHS
– 3	– 5	0	0	0	0
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

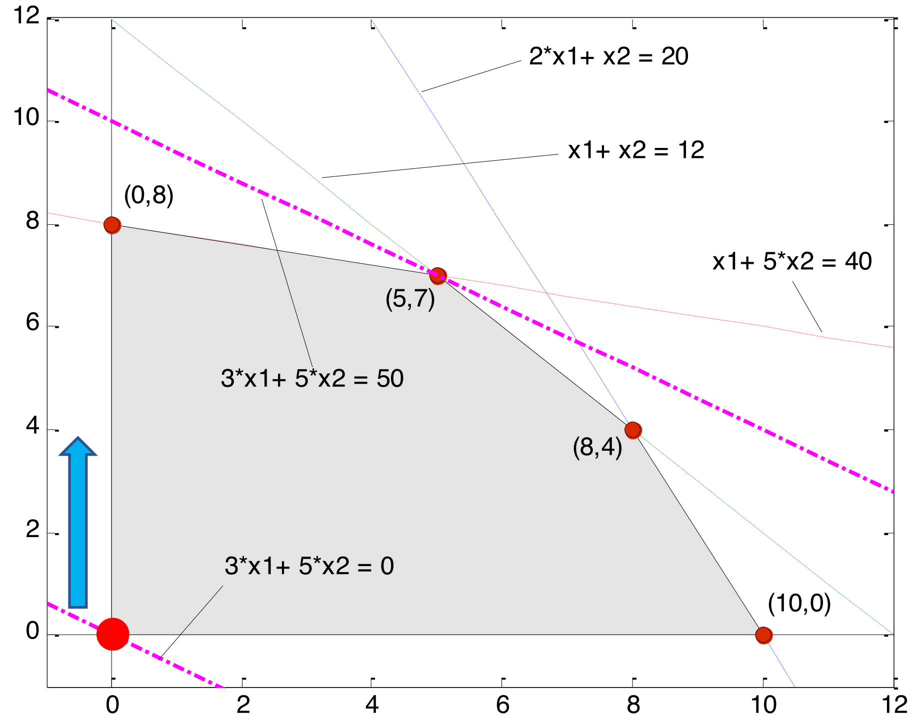
$x_{(0)}=[0,0,40,20,12]^T$, $f(x_{(0)})=0$



Simplex – iteracija 1

a_1	a_2	a_3	a_4	a_5	RHS
–3	–5	0	0	0	0
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

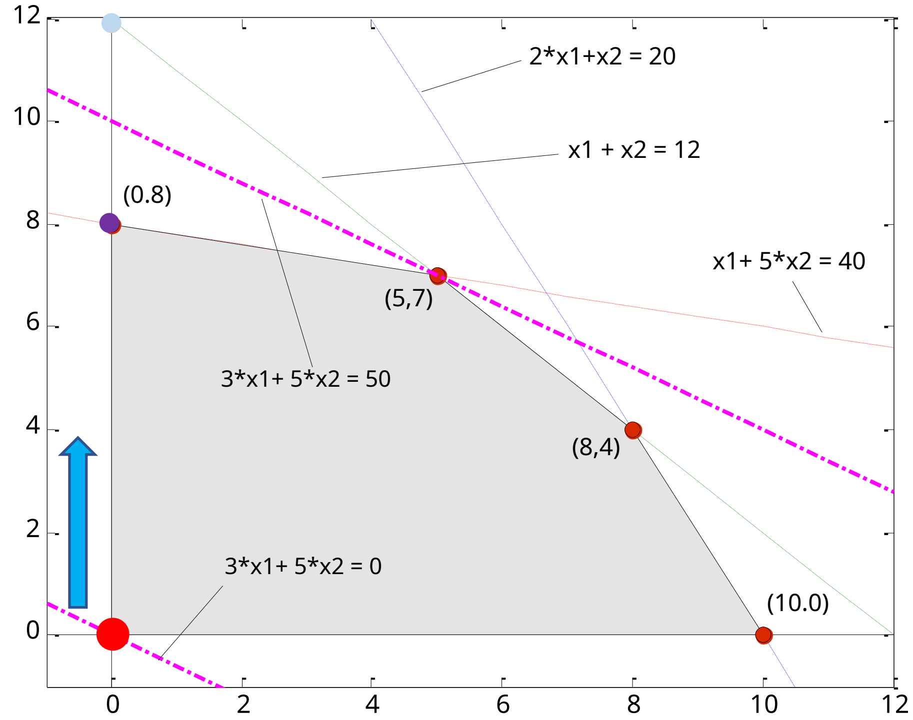
$x_{(0)}=[0,0,40,20,12]^T$, $f(x_{(0)})=0$



Projekcija na prostor originalnog problema

Simplex – iteration 1

And$_1$	And$_2$	And$_3$	And$_4$	And$_5$	RHS	Q
– 3	– 5	0	0	0	0	
1	5	1	0	0	40	8
2	1	0	1	0	20	20
1	1	0	0	1	12	12



Projection onto the space of the original problem

Pivot: (1,2)

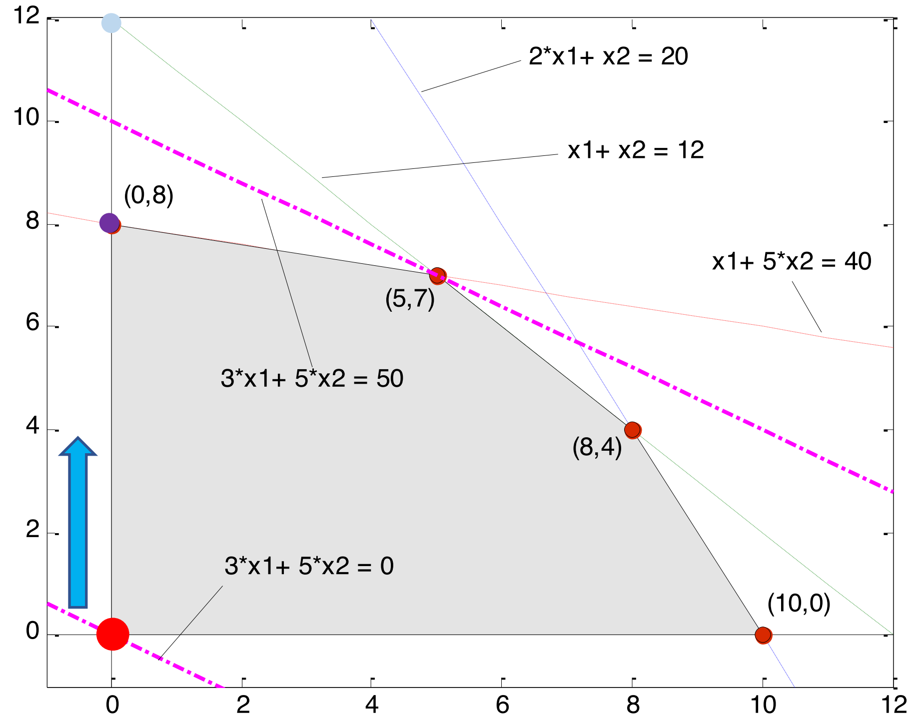
* The iteration ends with pivoting

Simplex – iteracija 1

a_1	a_2	a_3	a_4	a_5	RHS	Q
–3	–5	0	0	0	0	
1	5	1	0	0	40	8
2	1	0	1	0	20	20
1	1	0	0	1	12	12

Pivot: (1,2)

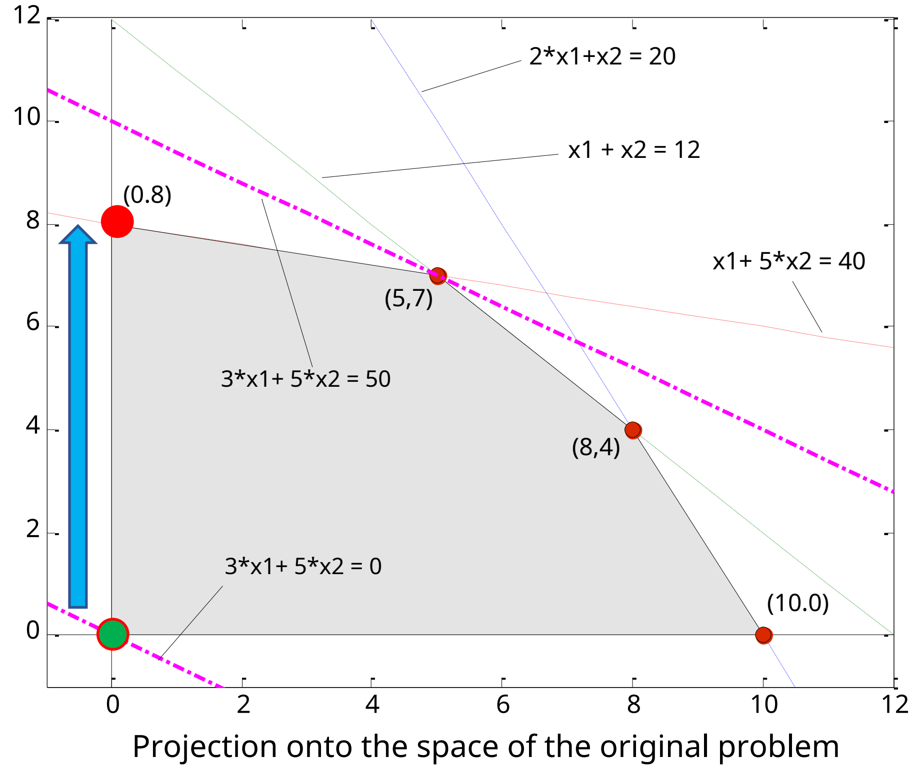
*Iteracija se završava pivotiranjem



Projekcija na prostor originalnog problema

Simplex – iteration 1

And$_1$	And$_2$	And$_3$	And$_4$	And$_5$	RHS
– 2	0	1	0	0	40
1/5	1	1/5	0	0	8
9/5	0	- 1/5	1	0	12
4/5	0	- 1/5	0	1	4

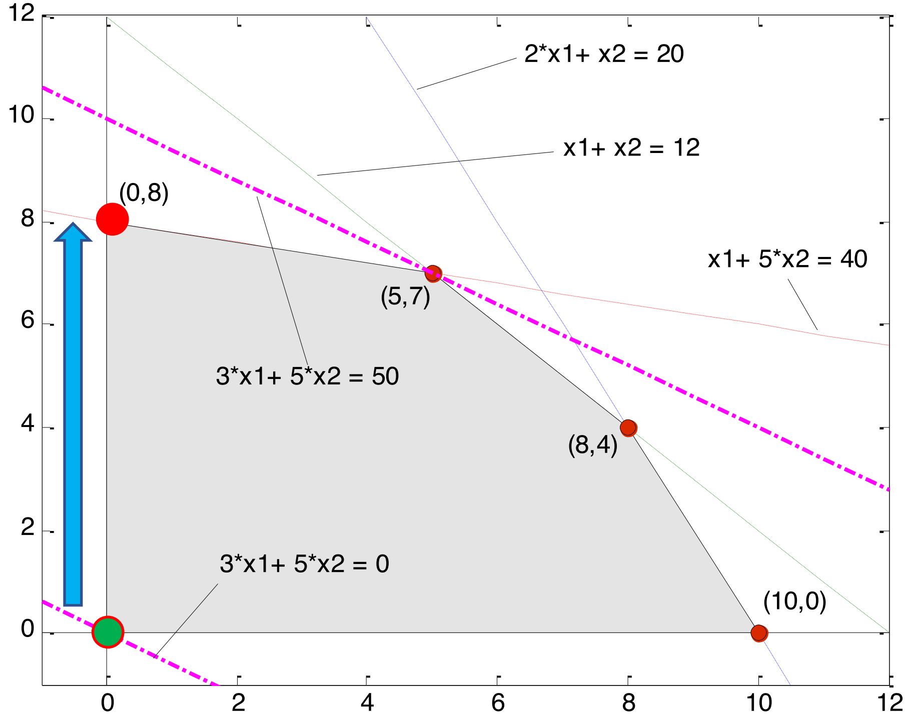


Jump from $x_{(0)}$ in:

$x_{(1)}=[0,8,0,12,4]^T$, $f(x_{(1)})=-40$

Simplex – iteracija 1

a_1	a_2	a_3	a_4	a_5	RHS
–2	0	1	0	0	40
1/5	1	1/5	0	0	8
9/5	0	-1/5	1	0	12
4/5	0	-1/5	0	1	4



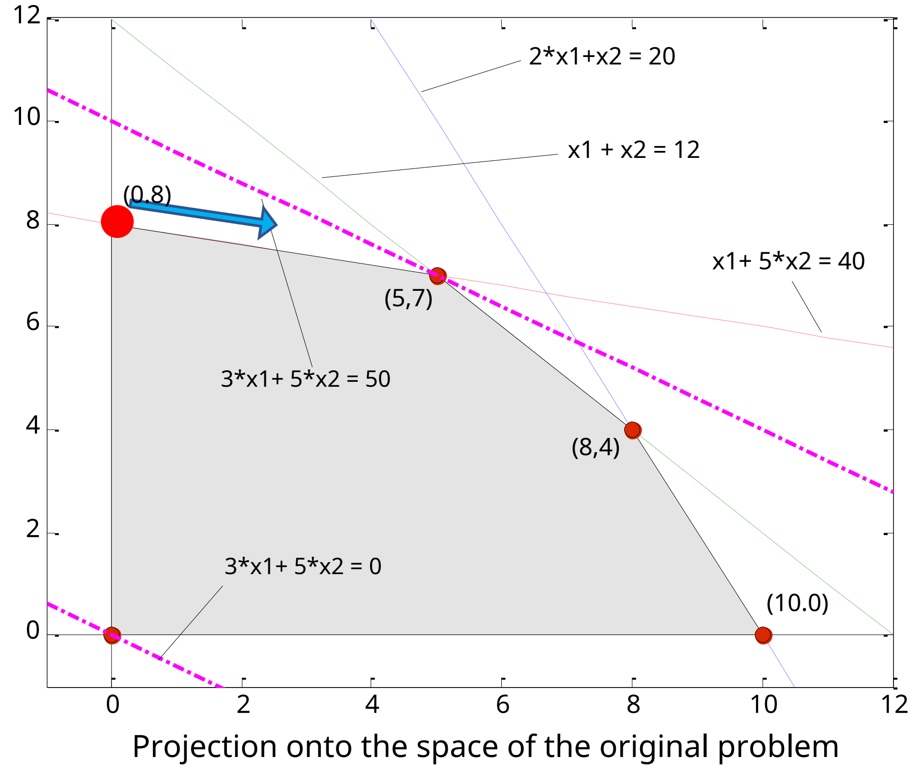
Projekcija na prostor originalnog problema

Skok iz $x_{(0)}$ u:

$x_{(1)}=[0,8,0,12,4]^T$, $f(x_{(1)})=-40$

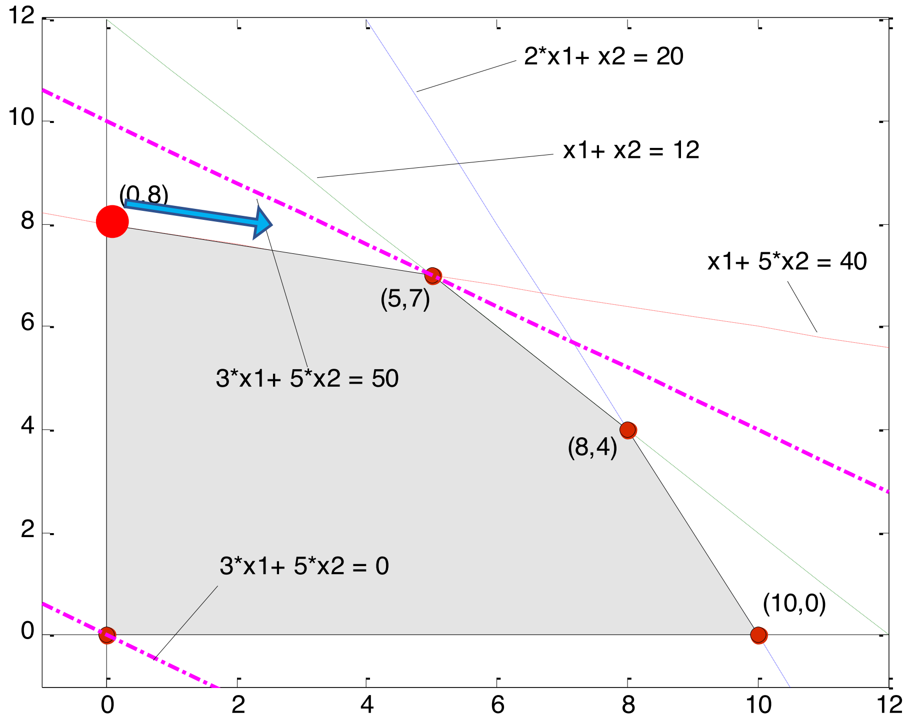
Simplex – iteration 2

And1	And2	And3	And4	And5	RHS
– 2	0	1	0	0	40
1/5	1	1/5	0	0	8
9/5	0	- 1/5	1	0	12
4/5	0	- 1/5	0	1	4



Simplex – iteracija 2

a_1	a_2	a_3	a_4	a_5	RHS
–2	0	1	0	0	40
1/5	1	1/5	0	0	8
9/5	0	-1/5	1	0	12
4/5	0	-1/5	0	1	4

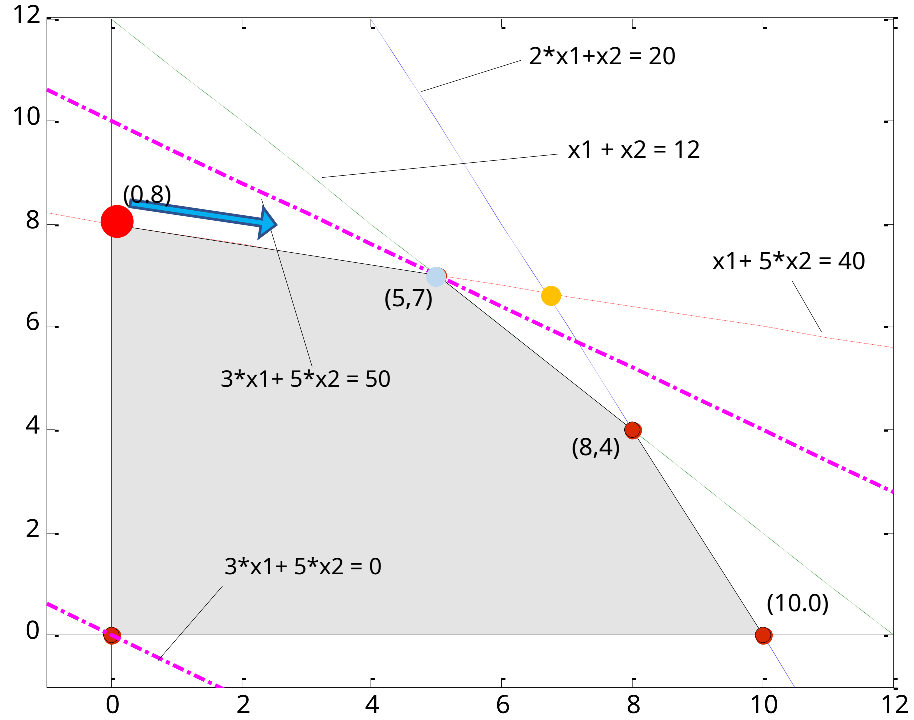


Projekcija na prostor originalnog problema

Simplex – iteration 2

And1	And2	And3	And4	And5	RHS	Q
– 2	0	1	0	0	40	
1/5	1	1/5	0	0	8	40
9/5	0	- 1/5	1	0	12	60/9
4/5	0	- 1/5	0	1	4	5

Pivot: (3,1)

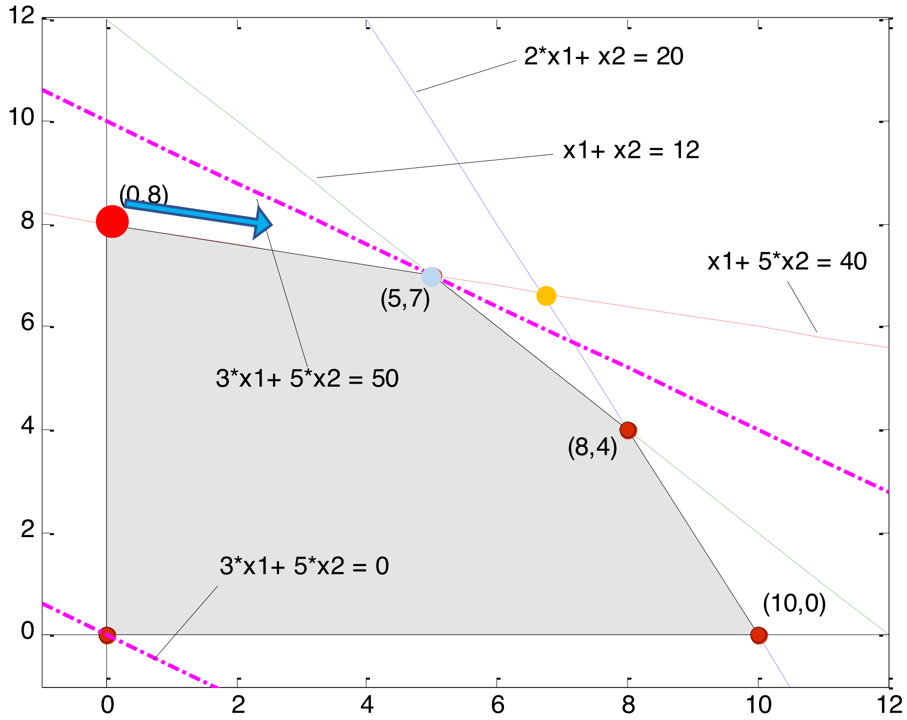


Projection onto the space of the original problem

Simplex – iteracija 2

a_1	a_2	a_3	a_4	a_5	RHS	Q
−2	0	1	0	0	40	
1/5	1	1/5	0	0	8	40
9/5	0	-1/5	1	0	12	60/9
4/5	0	-1/5	0	1	4	5

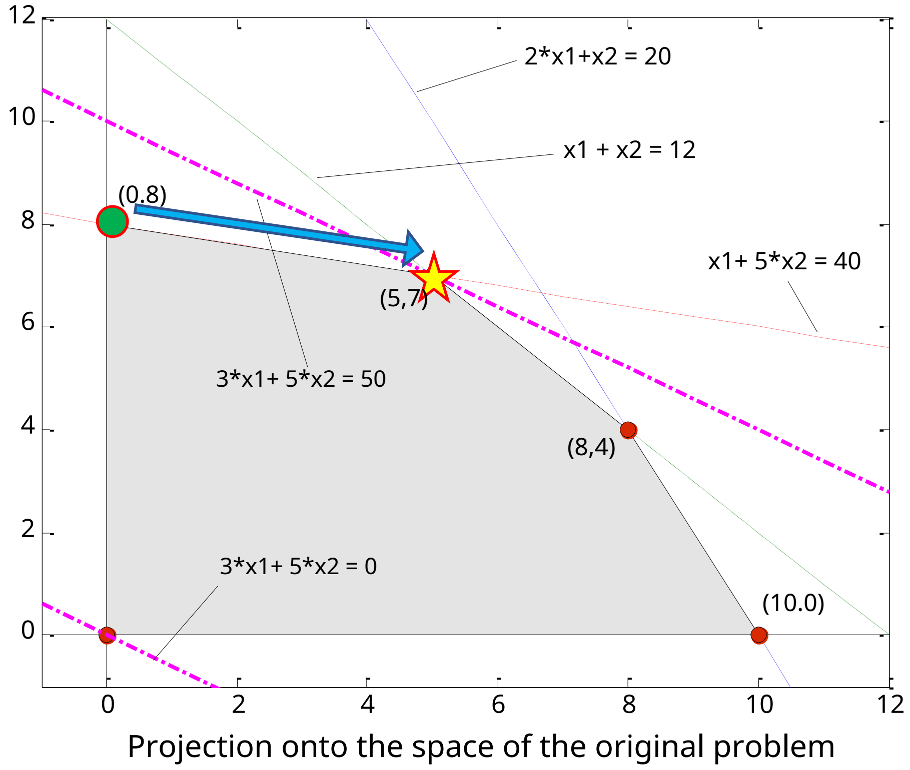
Pivot: (3,1)



Projekcija na prostor originalnog problema

Simplex – iteration 2

And1	And2	And3	And4	And5	RHS
0	0	1/2	0	10/4	50
0	1	1/20	0	- 1/4	7
0	0	1/4	1	- 9/4	3
1	0	- 1/4	0	5/4	5



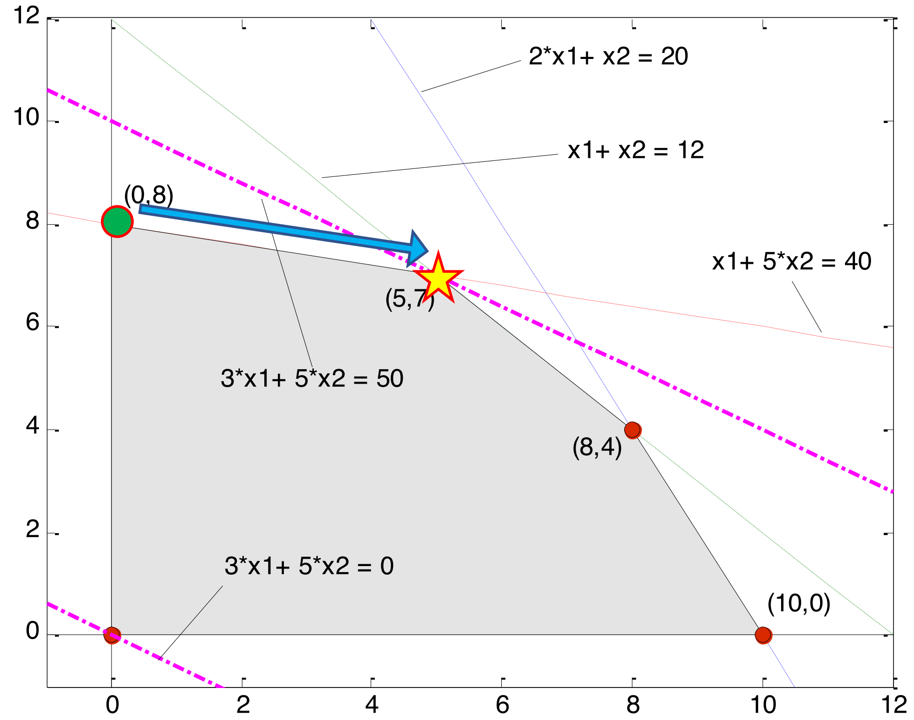
Jump from $x_{(1)}$ in:

$x_{(2)}=[5,7,0,3,0]^T$, $f(x_{(2)})=50$

OPTIMUM!

Simplex – iteracija 2

a_1	a_2	a_3	a_4	a_5	RHS
0	0	1/2	0	10/4	50
0	1	1/20	0	-1/4	7
0	0	1/4	1	-9/4	3
1	0	-1/4	0	5/4	5



Projekcija na prostor originalnog problema

Skok iz $x_{(1)}$ u:

$x_{(2)}=[5,7,0,3,0]^T$, $f(x_{(2)})=-50$

OPTIMUM!