COMPLEMENTOS de MATEMÁTICA

Aula Teórico-Prática – Ficha 7

INTEGRAIS DE SUPERFÍCIE; FLUXO

1. Dados os vectores não nulos $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ e $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, determine o integral de superfície do campo escalar $h(x,y,z) = xy$ sobre a superfície, S, parametrizada através da função vectorial a duas variáveis reais $\vec{r}(u,v) = u\vec{a} + v\vec{b}$, $(u,v) \in \Omega$, em que $\Omega = \{(u,v) \; : \; 0 \le u \le 1 \; , \; 0 \le v \le 1\}$.

2. Calcule o integral $\iint_S (2y)\, dS$ sobre a superfície, S, definida por $z = y^2/2$, $(x,y) \in \Omega$, em que $\Omega = \{(x,y) \; : \; 0 \le x \le 1 \; , \; 0 \le y \le 1\}$.

3. Calcule o integral $2\iint_S dS$ sobre a superfície, S, definida por $z = y^2/2$, $(x,y) \in \Omega$, em que $\Omega = \{(x,y) \; : \; 0 \le x \le 1 \; , \; 0 \le y \le 1\}$.

4. Calcule o integral $\iint_S 4\sqrt{x^2 + y^2}\, dS$ sobre a superfície, S, definida por $z = xy$, $(x,y) \in \Omega$, em que $\Omega = \left\{(x,y) \; : \; 0 \le x^2 + y^2 \le 1\right\}$.

5. Calcule o integral $\iint_S (xyz)\, dS$ sobre a superfície, S, que corresponde ao primeiro octante do plano $x + y + z = 1$.

6. Calcule o integral $\iint_S (x^2 z)\, dS$ sobre a superfície cilíndrica, S, definida por $x^2 + z^2 = 1$, tal que $1 \le y \le 4$ e $z \ge 0$.

10. Seja a superfície, S, parametrizada através da função vectorial a duas variáveis reais $\vec{r}(u,v)=(u+v)\vec{i}+(u-v)\vec{j}+u\vec{k}$, $(u,v)\in\Omega$, em que $\Omega=\{(u,v)\ :\ 0\le u\le 1\ ,\ 0\le v\le 1\}$. Admita que a densidade, em cada um dos seus pontos, é dada por $\lambda(x,y,z)=kz$ $(k>0)$. Calcule:

a) A sua área.

b) As coordenadas do seu centroide.

c) A sua massa.

d) As coordenadas do seu centro de massa.

e) Os momentos de inércia em relação aos eixos coordenados, I_x, I_y e I_z.

11. Seja a superfície triangular, S, com vértices nos pontos $(a,0,0)$, $(0,a,0)$ e $(0,0,a)$, tal que $a>0$. Calcule:

a) A sua área.

b) As coordenadas do seu centroide.

12. Admitindo que a densidade em cada ponto da superfície do exemplo 11 é dada por $\lambda(x,y,z)=kx^2$ $(k>0)$, calcule:

a) A sua massa.

b) As coordenadas do seu centro de massa.

16. Calcule o fluxo do campo vectorial $\vec{f}(x,y,z)=x\vec{i}+y\vec{j}+z\vec{k}$ através da superfície cilíndrica, S, parametrizada através da função vectorial a duas variáveis reais $\vec{r}(u,v)=a\cos(u)\vec{i}+a\,\mathrm{sen}(u)\vec{j}+v\vec{k}$, com $u\in[0,2\pi]$, $v\in[0,1]$ e $a>0$, no sentido de dentro para fora da superfície.

17. Calcule o fluxo do campo vectorial $\vec{f}(x,y,z)=x\vec{i}+y\vec{j}+z\vec{k}$ através da superfície do paraboloide, S, definida por $z=1-(x^2+y^2)$, $z\ge 0$, no sentido de dentro para fora da superfície.

18. Determine o fluxo do campo vectorial $\vec{f}(x,y,z)=-y\vec{i}+x\vec{j}+z\vec{k}$, através da superfície cónica, S, definida por $z=\sqrt{x^2+y^2}$, $z\le 4$, no sentido de dentro para fora da superfície.

19. Seja S a superfície parametrizada através da função vectorial a duas variáveis reais $\vec{r}(u,v)=u\cos(v)\vec{i}+u\,\mathrm{sen}(v)\vec{j}+v\vec{k}$, com $u\in[0,1]$ e $v\in[0,2\pi]$. Calcule o integral de fluxo $\iint_S x\,dy\wedge dz$ através de S, no sentido definido pelo seu produto vectorial fundamental.

20. Seja a superfície triangular, S, do exemplo 11.. Calcule o fluxo do campo vectorial $\vec{f}(x,y,z)=x^2\vec{i}-y^2\vec{j}$ através de S, no sentido definido pelo semieixo positivo dos zz.

22. Considere a superfície, S, definida por $z=xy$, $(x,y)\in\Omega$, tal que $\Omega=\{(x,y)\ :\ 0\le x\le 1\ ,\ 0\le y\le 2\}$. Determine o fluxo do campo vectorial $\vec{f}(x,y,z)=-xz\vec{j}+xy\vec{k}$ através de S, no sentido definido pelo semieixo negativo dos zz.

23. Seja a superfície fechada, S, limitada pelas superfícies $x^2+y^2=1$, $z=0$ e $z=1$. Calcule o fluxo do campo vectorial $\vec{f}(x,y,z)=x\vec{i}+2y\vec{j}+z^2\vec{k}$ através de S, no sentido de fora para dentro da superfície.

24. Considere a superfície fechada, S, que limita o cubo unitário, T, situado no quarto octante $T=\{(x,y,z)\ :\ 0\le x\le a\ ,\ -a\le y\le 0\ ,\ 0\le z\le a\}$ $(a>0)$. Em cada uma das alíneas seguintes, determine o fluxo do campo vectorial $\vec{f}(x,y,z)$ através de S, no sentido de dentro para fora da superfície.

a) $\vec{f}(x,y,z)=y\vec{i}-x\vec{j}$.

b) $\vec{f}(x,y,z)=x\vec{i}+y\vec{j}+z\vec{k}$.

c) $\vec{f}(x,y,z)=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$.

d) $\vec{f}(x,y,z)=-2x^2\vec{i}-2xz\vec{j}+z^2\vec{k}$.

e) $\vec{f}(x,y,z)=xz\vec{i}+4xyz^2\vec{j}+2yz\vec{k}$.

26. Considere a superfície fechada, S, limitada pelas superfícies $z=x^2+y^2$ e $z=4$. Determine o fluxo do campo vectorial $\vec{f}(x,y,z)=x\vec{i}+xy\vec{j}+z^2\vec{k}$ através de S, no sentido de dentro para fora da superfície.

28. Seja a superfície fechada, S, limitada pelas superfícies $(x+1)^2+y^2=1$, $z=0$ e $z=2$. Determine o fluxo do campo vectorial $\vec{f}(x,y,z)=\left(2x+ze^y\right)\vec{i}+\left(y+\operatorname{sen}(z)\right)\vec{j}+\left(3z+e^{xy}\right)\vec{k}$ através de S, no sentido de dentro para fora da superfície.

29. Considere a superfície fechada, S, situada no primeiro octante e limitada pelos planos coordenados e pela superfície $x+y+z=a$ $(a>0)$. Determine o fluxo do campo vectorial $\vec{f}(x,y,z)=3x^2\vec{i}+2xy\vec{j}-5xz\vec{k}$ através de S, no sentido de fora para dentro da superfície.

30. Calcule $\nabla\cdot\vec{f}$ (divergência) e $\nabla\times\vec{f}$ (rotacional), sendo $\vec{f}$ o campo vectorial:

a) $\vec{f}(x,y,z)=x\vec{i}+y\vec{j}+z\vec{k}$.

b) $\vec{f}(x,y,z)=-2x\vec{i}+4y\vec{j}-6z\vec{k}$.

c) $\vec{f}(x,y,z)=xyz\vec{i}+xz\vec{j}+z\vec{k}$.

d) $\vec{f}(x,y,z)=x^3y\vec{i}+y^3z\vec{j}+xy^3\vec{k}$.

e) $\vec{f}(x,y,z)=x^2y\vec{i}+(z-x-y)\vec{j}+2xy\vec{k}$.

f) $\vec{f}(x,y,z)=xz\vec{i}+4xyz^2\vec{j}+2yz\vec{k}$.

g) $\vec{f}(\vec{r})=e^{r^2}(\vec{i}+\vec{j}+\vec{k})$.

h) $\vec{f}(\vec{r})=r^{-2}\vec{r}$.

i) $\vec{f}(x,y,z)=\dfrac{\alpha x}{x^2+y^2}\vec{i}+\dfrac{\alpha y}{x^2+y^2}\vec{j}$, $\alpha\in\mathbb{R}$.

j) $\vec{f}(x,y,z)=\dfrac{\alpha y}{x^2+y^2}\vec{i}+\dfrac{\alpha x}{x^2+y^2}\vec{j}$, $\alpha\in\mathbb{R}$.

k) $\vec{f}(x,y,z)=\left(2x+ze^{y}\right)\vec{i}+\left(y+\operatorname{sen}(z)\right)\vec{j}+\left(3z+e^{xy}\right)\vec{k}$.

31. Mostre que a divergência e o rotacional são operadores lineares, isto é, se $\vec{f}$ e $\vec{g}$ são campos vectoriais e $\alpha,\beta\in\mathbb{R}$, então:

a) $\nabla\cdot(\alpha\vec{f}+\beta\vec{g})=\alpha(\nabla\cdot\vec{f})+\beta(\nabla\cdot\vec{g})$.

b) $\nabla\times(\alpha\vec{f}+\beta\vec{g})=\alpha(\nabla\times\vec{f})+\beta(\nabla\times\vec{g})$.

32. Mostre que o campo vectorial $\vec{f}(x,y,z)=2x^3y\vec{i}-y^2z\vec{j}+(yz^2-6x^2yz)\vec{k}$ é solenoidal.

33. Mostre que o campo vectorial $\vec{f}(x,y,z)=(2xy+z^2)\vec{i}+(x^2-2yz)\vec{j}+(2xz-y^2)\vec{k}$ é irrotacional.

34. Mostre que se φ é um campo escalar e $\vec{f}$ um campo vectorial, então:

a) $\nabla\cdot(\varphi\vec{f})=(\nabla\varphi)\cdot\vec{f}+\varphi(\nabla\cdot\vec{f})$.

b) $\nabla\times(\varphi\vec{f})=(\nabla\varphi)\times\vec{f}+\varphi(\nabla\times\vec{f})$.

39. Resolva os exercícios 23. a 29. recorrendo ao teorema da divergência.

40. Considere a superfície fechada, *S*, que limita o sólido, *V*, definido por $V=\left\{(x,y,z)\ :\ 1\geq z\geq\sqrt{x^2+y^2}\right\}$ e o campo vectorial $\vec{f}(x,y,z)=x\vec{i}+y\vec{j}+z\vec{k}$. Verifique o teorema da divergência.

41. Considere a superfície fechada, *S*, que limita o sólido, *V*, limitado pelos planos $x=0$, $y=-1$, $y=1$, $z=0$ e $x+z=2$ e o campo vectorial $\vec{f}(x,y,z)=y\vec{j}$. Verifique o teorema da divergência.

42. Recorrendo ao teorema adequado, determine o fluxo do campo vectorial $\vec{f}(x,y,z)=-x^2y\vec{i}+3y\vec{j}+2xyz\vec{k}$ através da superfície fechada, *S*, que limita o volume $V=\left\{(x,y,z)\ :\ \sqrt{x^2+y^2}\leq z\leq 2-\sqrt{x^2+y^2}\right\}$, no sentido de dentro para fora da superfície.

43. Considere o campo vectorial $\vec{f}(x,y,z)=xy^2\vec{i}+x^2y\vec{j}+z\vec{k}$ e seja a superfície fechada, *S*, limitada pelas superfícies $x^2+y^2=1$, $z=0$ e $z=1$. Calcule $\oiint_S(\vec{f}\cdot\vec{n})\,dS$:

a) Por cálculo directo do integral de fluxo. **b)** Recorrendo ao teorema da divergência.

44. Calcule o fluxo do campo vectorial $\vec{f}(x,y,z)=2xy\vec{i}+y^2\vec{j}+3yz\vec{k}$ através da superfície esférica, *S*, definida por $x^2+y^2+z^2=a^2$ $(a>0)$, no sentido de dentro para fora da superfície:

a) Por cálculo directo do integral de fluxo. **b)** Recorrendo ao teorema da divergência.

45. Sejam o campo vectorial $\vec{f}(x,y,z)=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$ e a superfície fechada, *S*, limitada pelas superfícies $x^2+y^2=2y$, $z=0$ e $z=2$. Usando o teorema adequado, determine o fluxo do campo vetorial $\vec{f}(x,y,z)$ através de *S*, no sentido de dentro para fora da superfície.

46. Seja a superfície fechada $S=\left\{(x,y,z)\ :\ (x^2+y^2+z^2=4\ ,\ z\geq 0)\ \vee\ (x^2+y^2\leq 4\ ,\ z=0)\right\}$. Recorrendo ao teorema adequado, determine o fluxo do campo vectorial $\vec{f}(x,y,z)=\dfrac{x^3}{y^2}\vec{i}+5\dfrac{x^2}{y}\vec{j}+2z\left(\dfrac{x^2}{y^2}+1\right)\vec{k}$ através de *S*, no sentido de dentro para fora da superfície.

52. Seja o campo vectorial $\vec{f}(x,y,z)=x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$. Verifique o teorema de Stokes sobre a superfície $S=\left\{(x,y,z)\ :\ z+1=x^2+y^2\ ,\ z\in[-1,0]\right\}$.

53. Considere a superfície triangular, *S*, com vértices nos pontos $A=(2,0,0)$, $B=(0,2,0)$ e $C=(0,0,2)$. Calcule o fluxo do rotacional de $\vec{f}(x,y,z)=x^3\vec{i}+2xy\vec{j}+z^2\vec{k}$ através de *S*, no sentido definido pelo semieixo positivo dos *zz*:

a) Por cálculo directo do integral de fluxo. **b**) Recorrendo ao teorema de Stokes.

54. Seja S a superfície $z=\sqrt{x^2+y^2}$, limitada por $2z=x^2+y^2$. Calcule o fluxo do rotacional de $\vec{f}(x,y,z)=z\vec{i}+x\vec{j}+2\vec{k}$ através de *S*, no sentido de fora para dentro da superfície:

a) Por cálculo directo do integral de fluxo. **b**) Recorrendo ao teorema de Stokes.

55. Seja S a superfície definida por $z=1-x^2-y^2$, $z\geq 0$. Calcule o fluxo do rotacional de $\vec{f}(x,y,z)=y\vec{i}+z\vec{j}+x\vec{k}$ através de *S*, no sentido de dentro para fora da superfície:

a) Por cálculo directo do integral de fluxo. **b**) Recorrendo ao teorema de Stokes.

56. Seja S a superfície definida por $x^2+y^2+z^2=1$, $z\geq 0$. Calcule o fluxo do rotacional de $\vec{f}(x,y,z)=z^2\vec{i}+2x\vec{j}-y^3\vec{k}$ através de *S*, no sentido de dentro para fora da superfície:

a) Por cálculo directo do integral de fluxo. **b**) Recorrendo ao teorema de Stokes.

57. Seja S a superfície $z = x^2 + y^2$, limitada superiormente pelo plano $z = 2x$. Calcule o fluxo do rotacional de $\vec{f}(x,y,z) = y^2\vec{i} - \vec{k}$ através de S, no sentido de dentro para fora da superfície:

a) Por cálculo directo do integral de fluxo. **b)** Recorrendo ao teorema de Stokes.

58. Seja S a superfície definida por $z = 4 - x^2 - y^2$, $z \geq -2$. Calcule o fluxo do rotacional de $\vec{f}(x,y,z) = (2xyz + 2z)\vec{i} + xy^2\vec{j} + xz\vec{k}$ através de S, no sentido de fora para dentro da superfície:

a) Por cálculo directo do integral de fluxo. **b)** Recorrendo ao teorema de Stokes.

59. Seja S a superfície definida por $z = x^2 + y^2$, $z \leq -2$, $y \geq 0$. Calcule o fluxo do rotacional de $\vec{f}(x,y,z) = (x^2 + xz)\vec{i} + yz\vec{j}$ através de S, no sentido de dentro para fora da superfície:

a) Por cálculo directo do integral de fluxo. **b)** Recorrendo ao teorema de Stokes.

61. Seja o campo vectorial $\vec{f}(x,y,z) = y\vec{i} + zx\vec{j} + zy\vec{k}$. Verifique o teorema de Stokes sobre a superfície $S = \left\{(x,y,z) \ : \ z = 5 - (x^2 + y^2) \ , z \geq 1\right\}$.

Soluções: Consultar o manual "Noções sobre Análise Matemática", Efeitos Gráficos, 2019. ISBN: 978-989-54350-0-5.

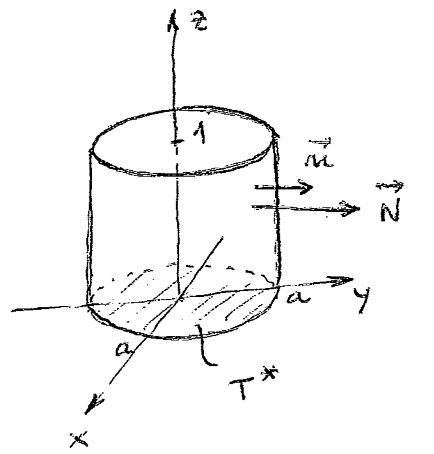
16) $\vec{F}(x,y,z) = (x, y, z)$

<u>Superfície S</u> : $\vec{r}(u,v) = (a\cos u, a\,\text{sen}\,u, v)$, $(u,v) \in T$

$T = \{(u,v) \in \mathbb{R}^2 : 0 \le u \le 2\pi \wedge 0 \le v \le 1\}$

$$\begin{cases} x = a\cos u \\ y = a\,\text{sen}\,u \\ z = v \end{cases} \Rightarrow \begin{cases} x^2+y^2 = a^2 , & (x,y) \in T^* \\ 0 \le z \le 1 \end{cases}$$

<u>Cilindro Circular</u>



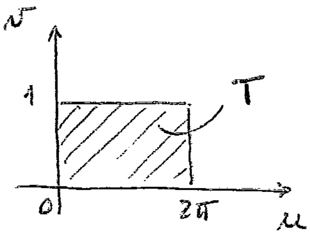
$$\vec{r}'_u = \frac{\partial \vec{r}}{\partial u} = (-a\,\text{sen}\,u, a\cos u, 0)$$

$$\vec{r}'_v = \frac{\partial \vec{r}}{\partial v} = (0, 0, 1)$$

$$\vec{N}(u,v) = \vec{r}'_u \times \vec{r}'_v = (a\cos u, a\,\text{sen}\,u, 0)$$ <u>dirigido para o exterior de S</u>

$$\vec{F}[\vec{r}(u,v)] = (a\cos u, a\,\text{sen}\,u, v)$$

$$\vec{F}[\vec{r}(u,v)] \cdot \vec{N}(u,v) = a^2\cos^2 u + a^2\,\text{sen}^2 u = a^2$$



$$\iint \vec{F}\cdot\vec{n}\, dS = \iint_T \vec{F}[\vec{r}(u,v)] \cdot \vec{n}(u,v) \underbrace{\|\vec{N}(u,v)\|\, du\, dv}_{dS} =$$

$$= \oplus \iint_T \vec{F}[\vec{r}(u,v)] \cdot \vec{N}(u,v)\, du\, dv =$$

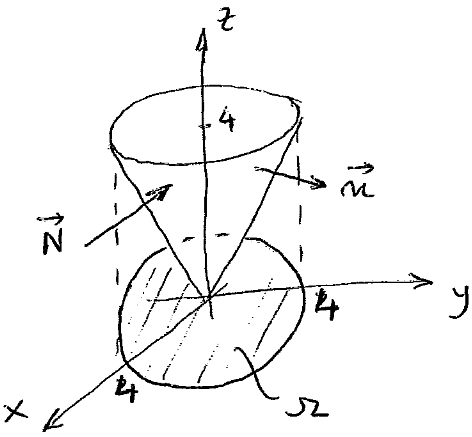
<u>fluxo de dentro para fora de S</u>

$$= + \iint_T a^2\, du\, dv = a^2 \iint_T du\, dv = a^2 A(T) = 2\pi a^2$$

Wai

18) $\vec{f}(x,y,z) = (P, Q, R) = (-y, x, z)$

Superfície S : $z = \sqrt{x^2+y^2}$, $z \leq 4$



Parametrizando :

$\vec{r}(x,y) = (x, y, \sqrt{x^2+y^2})$, $(x,y) \in \Omega$

$\Omega = \{(x,y) : 0 \leq x^2+y^2 \leq 16\}$

$$\vec{r}'_x = \frac{\partial \vec{r}}{\partial x} = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right)$$

$$\vec{r}'_y = \frac{\partial \vec{r}}{\partial y} = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\vec{N}(x,y) = \vec{r}'_x \times \vec{r}'_y = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right)$$: dirigido para o interior de S

$$\vec{f}[\vec{r}(x,y)] = \left(-y, x, \sqrt{x^2+y^2}\right)$$

$$\vec{f}[\vec{r}(x,y)] \cdot \vec{N}(x,y) = \cancel{\frac{xy}{\sqrt{x^2+y^2}}} - \cancel{\frac{xy}{\sqrt{x^2+y^2}}} + \sqrt{x^2+y^2} = \sqrt{x^2+y^2}$$

$$\int_S (\vec{f}\cdot\vec{n})\, dS = -\iint_\Omega \vec{f}[\vec{r}(x,y)]\cdot\vec{N}(x,y)\, dx\, dy = -\iint_\Omega \sqrt{x^2+y^2}\, dx\, dy$$

fluxo de dentro para fora de S

Coordenadas Polares :

$x = r\cos\theta$
$y = r\,\text{sen}\,\theta$
$dx\,dy = r\,dr\,d\theta$
$\sqrt{x^2+y^2} = r$

$\Omega \rightarrow \Gamma = \{(r,\theta) : 0 \leq r \leq 4 \wedge 0 \leq \theta \leq 2\pi\}$

$$\int_S (\vec{f}\cdot\vec{n})\, dS = -\iint_\Gamma (r)\, r\, dr\, d\theta = -\int_0^4 \int_0^{2\pi} r^2\, d\theta\, dr =$$

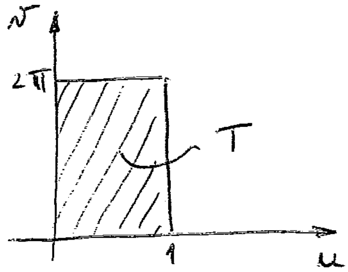
$$= -2\pi \int_0^4 r^2\, dr = -2\pi\left(\frac{64}{3}\right) = -\frac{128\pi}{3}$$

19) $\vec{F}(x,y,z) = (P, Q, R) = (x, 0, 0)$

<u>Superfície S</u> : $\vec{r}(u,v) = (u\cos v,\ u\,\text{sen}\, v,\ v)\ ,\ (u,v)\in T$

$T = \{(u,v)\in\mathbb{R}^2 : 0\le u\le 1 \wedge 0\le v\le 2\pi\}$

$$S : \begin{cases} x = u\cos v \\ y = u\,\text{sen}\, v \\ z = v \end{cases}$$



$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S \underset{\substack{\Downarrow\\ P = x}}{P}\, dy\wedge dz + \iint_S \underset{\substack{\Downarrow\\ Q=0}}{Q}\, dz\wedge dx + \iint_S \underset{\substack{\Downarrow\\ R=0}}{R}\, dx\wedge dy = \iint_S x\, dy\wedge dz$$

$$\vec{r}\,'_u = \frac{\partial\vec{r}}{\partial u} = (\cos v,\ \text{sen}\, v,\ 0)$$

$$\vec{r}\,'_v = \frac{\partial\vec{r}}{\partial v} = (-u\,\text{sen}\, v,\ u\cos v,\ 1)$$

$$\Rightarrow\ \vec{N}(u,v) = \vec{r}\,'_u \times \vec{r}\,'_v = $$

$$= (\text{sen}\, v,\ -\cos v,\ u\cos^2 v + u\,\text{sen}^2 v) =$$

$$= (\text{sen}\, v,\ -\cos v,\ u)$$

$$\vec{F}[\vec{r}(u,v)] = (u\cos v,\ 0,\ 0)$$

$$\vec{F}[\vec{r}(u,v)]\cdot\vec{N}(u,v) = u\cos v\,\text{sen}\, v$$

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S x\, dy\wedge dz = \iint_T u\cos v\,\text{sen}\, v\, du\, dv =$$

$$= \int_0^{2\pi}\int_0^1 u\cos v\,\text{sen}\, v\, du\, dv = \frac{1}{2}\int_0^{2\pi}\cos v\,\text{sen}\, v\, du\, dv =$$

$$= \frac{1}{2}\left[\frac{\text{sen}^2 v}{2}\right]_0^{2\pi} = 0$$

NOTA:

$$S: \begin{cases} x = u\cos v \\ y = u\,\text{sen}\,v \\ z = v \end{cases} \qquad \vec{N}(u,v) = \left(N_1(u,v), N_2(u,v), N_3(u,v)\right)$$

$$dy \wedge dz = \frac{\partial(y,z)}{\partial(u,v)}\,du\,dv = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du\,dv =$$

$$= \underbrace{\begin{vmatrix} \text{sen}\,v & 0 \\ u\cos v & 1 \end{vmatrix}}_{N_1(u,v)} du\,dv = \text{sen}\,v\,du\,dv$$

$$dz \wedge dx = \frac{\partial(z,x)}{\partial(u,v)}\,du\,dv = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} du\,dv =$$

$$= \underbrace{\begin{vmatrix} 0 & \cos v \\ 1 & -u\,\text{sen}\,v \end{vmatrix}}_{N_2(u,v)} du\,dv = -\cos v\,du\,dv$$

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv =$$

$$= \underbrace{\begin{vmatrix} \cos v & \text{sen}\,v \\ -u\,\text{sen}\,v & u\cos v \end{vmatrix}}_{N_3(u,v)} du\,dv = u\,du\,dv$$

30) a) $\vec{F}(x,y,z) = (x, y, z) = (P, Q, R)$

$$\nabla \cdot \vec{F} = \operatorname{div} \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x,y,z) = 1 + 1 + 1 = 3$$

$$\nabla \times \vec{F} = \operatorname{rot} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = (0,0,0)$$

$\vec{F}$ é um campo vectorial irrotacional

$\operatorname{rot} \vec{F} = (0,0,0) \Leftrightarrow \vec{F}$ é um campo conservativo (é gradiente)

$$\operatorname{rot} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

<u>NOTA</u>: $\operatorname{div}(\operatorname{rot} \vec{F}) = 0$

$\nabla \cdot (\nabla \times \vec{F}) = 0$

c) $\vec{F}(x,y,z) = (xyz, xz, z)$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (xyz, xz, z) = yz + 0 + 1 = 1 + yz$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = 1 + yz$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & xz & z \end{vmatrix} = (0 - x, xy - 0, z - xz) = (-x, xy, z - xz)$$

$$\operatorname{rot} \vec{F} = \nabla \times \vec{F} = (-x, xy, z - xz)$$

Wiy

g) $\vec{F}(\vec{r}) = e^{r^2}(1,1,1)$ Seja $\vec{r} = (x,y,z)$

Como $r^2 = x^2+y^2+z^2$ então

$$\vec{F}(x,y,z) = e^{x^2+y^2+z^2}(1,1,1)$$

$$\nabla\cdot\vec{F} = \operatorname{div}\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(e^{x^2+y^2+z^2}, e^{x^2+y^2+z^2}, e^{x^2+y^2+z^2}\right) =$$

$$= 2x\, e^{x^2+y^2+z^2} + 2y\, e^{x^2+y^2+z^2} + 2z\, e^{x^2+y^2+z^2} =$$

$$= 2e^{x^2+y^2+z^2}(x+y+z)$$

$$\nabla\times\vec{F} = \operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{x^2+y^2+z^2} & e^{x^2+y^2+z^2} & e^{x^2+y^2+z^2} \end{vmatrix} =$$

$$= \left(2y\, e^{x^2+y^2+z^2} - 2z\, e^{x^2+y^2+z^2},\ 2z\, e^{x^2+y^2+z^2} - 2x\, e^{x^2+y^2+z^2},\right.$$

$$\left. 2x\, e^{x^2+y^2+z^2} - 2y\, e^{x^2+y^2+z^2}\right) =$$

$$= 2\, e^{x^2+y^2+z^2}(y-z,\ z-x,\ x-y)$$

— " —

Forma alternativa, recorrendo às fórmulas do exercício 34)

Seja $\vec{v} = \varphi\,\vec{F}_1$ com $\varphi(x,y,z) = e^{x^2+y^2+z^2}$

$\vec{F}_1(x,y,z) = (1,1,1)$

Então

$$\nabla\cdot(\varphi\vec{F}_1) = (\nabla\varphi)\cdot\vec{F}_1 + \varphi(\nabla\cdot\vec{F}_1) \Leftrightarrow$$

$$\Leftrightarrow \operatorname{div}\vec{v} = (\operatorname{grad}\varphi)\cdot\vec{F}_1 + \varphi(\operatorname{div}\vec{F}_1)$$

v.s.f.f.

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right) =$$

$$= \left(2x\, e^{x^2+y^2+z^2}, 2y\, e^{x^2+y^2+z^2}, 2z\, e^{x^2+y^2+z^2}\right) =$$

$$= 2\, e^{x^2+y^2+z^2} (x, y, z)$$

$$(\nabla\varphi)\cdot\vec{F}_1 = 2\, e^{x^2+y^2+z^2}(x,y,z)\cdot(1,1,1) = 2e^{x^2+y^2+z^2}(x+y+z)$$

$$\nabla\cdot\vec{F}_1 = \operatorname{div}\vec{F}_1 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(1,1,1) = 0+0+0 = 0$$

$$\varphi(\nabla\cdot\vec{F}_1) = 0$$

Tem-se então

$$\nabla\cdot\vec{v} = \operatorname{div}\vec{v} = 2\, e^{x^2+y^2+z^2}(x+y+z)$$

Por outro lado, verifica-se:

$$\nabla\times(\varphi\vec{F}_1) = (\nabla\varphi)\times\vec{F}_1 + \varphi(\nabla\times\vec{F}_1) \Leftrightarrow$$

$$\Leftrightarrow \operatorname{rot}\vec{v} = (\operatorname{grad}\varphi)\times\vec{F}_1 + \varphi(\operatorname{rot}\vec{F}_1)$$

$$(\nabla\varphi)\times\vec{F}_1 = 2\, e^{x^2+y^2+z^2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 2\, e^{x^2+y^2+z^2}(y-z, z-x, x-y)$$

$$\nabla\times\vec{F}_1 = \operatorname{rot}\vec{F}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & 1 & 1 \end{vmatrix} = (0,0,0)$$

Tem-se então

$$\nabla\times\vec{v} = \operatorname{rot}\vec{v} = 2\, e^{x^2+y^2+z^2}(y-z, z-x, x-y)$$

h) $\vec{F}(\vec{r}) = r^{-2}\,\vec{r}$ em que $\vec{r} = (x,y,z)$

$r = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$

Então

$$\vec{F}(x,y,z) = \frac{1}{x^2+y^2+z^2}\,(x,y,z)$$

$$\nabla\cdot\vec{F} = \operatorname{div}\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right) =$$

$$= \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2y^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2} =$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2} = \frac{1}{r^2}$$

$$\nabla\times\vec{F} = \operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix} =$$

$$= \left(\frac{-2yz}{(x^2+y^2+z^2)^2} + \frac{2yz}{(x^2+y^2+z^2)^2}, \frac{-2xz}{(x^2+y^2+z^2)^2} + \frac{2xz}{(x^2+y^2+z^2)^2}, \frac{-2yx}{(x^2+y^2+z^2)^2} + \frac{2yx}{(x^2+y^2+z^2)^2}\right) =$$

$$= (0,0,0)$$

Wuj

34) a) Seja $\vec{v} = \varphi \vec{F} = \varphi(x,y,z)\left(F_x(x,y,z),\, F_y(x,y,z),\, F_z(x,y,z)\right)$

Pretende-se mostrar que

$$\nabla \cdot \vec{v} = \nabla \cdot (\varphi \vec{F}) = (\nabla \varphi) \cdot \vec{F} + \varphi(\nabla \cdot \vec{F})$$

ou seja

$$\operatorname{div} \vec{v} = (\operatorname{grad} \varphi) \cdot \vec{F} + \varphi(\operatorname{div} \vec{F})$$

Tem-se então

$$\nabla \cdot \vec{v} = \nabla \cdot (\varphi \vec{F}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\varphi F_x, \varphi F_y, \varphi F_z\right) =$$

$$= \frac{\partial}{\partial x}(\varphi F_x) + \frac{\partial}{\partial y}(\varphi F_y) + \frac{\partial}{\partial z}(\varphi F_z) =$$

$$= \frac{\partial \varphi}{\partial x} F_x + \varphi \frac{\partial F_x}{\partial x} + \frac{\partial \varphi}{\partial y} F_y + \varphi \frac{\partial F_y}{\partial y} + \frac{\partial \varphi}{\partial z} F_z + \varphi \frac{\partial F_z}{\partial z} =$$

$$= \left(\frac{\partial \varphi}{\partial x} F_x + \frac{\partial \varphi}{\partial y} F_y + \frac{\partial \varphi}{\partial z} F_z\right) + \varphi\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) =$$

$$= \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) \cdot (F_x, F_y, F_z) + \varphi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z) =$$

$$= (\nabla \varphi) \cdot \vec{F} + \varphi(\nabla \cdot \vec{F}) =$$

$$= (\operatorname{grad} \varphi) \cdot \vec{F} + \varphi(\operatorname{div} \vec{F})$$

b) Seja $\vec{v} = \varphi \vec{F} = \varphi(x,y,z)\left(F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\right)$

Pretende-se mostrar que

$$\nabla \times \vec{v} = \nabla \times (\varphi \vec{F}) = (\nabla\varphi) \times \vec{F} + \varphi(\nabla \times \vec{F}).$$

ou seja

$$\text{rot}\, \vec{v} = (\text{grad}\, \varphi) \times \vec{F} + \varphi(\text{rot}\, \vec{F})$$

Tem-se então

$$\nabla \times \vec{v} = \nabla \times (\varphi \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \varphi F_x & \varphi F_y & \varphi F_z \end{vmatrix} =$$

$$= \begin{bmatrix} \frac{\partial}{\partial y}(\varphi F_z) - \frac{\partial}{\partial z}(\varphi F_y) \\ \frac{\partial}{\partial z}(\varphi F_x) - \frac{\partial}{\partial x}(\varphi F_z) \\ \frac{\partial}{\partial x}(\varphi F_y) - \frac{\partial}{\partial y}(\varphi F_x) \end{bmatrix} = \begin{bmatrix} \frac{\partial \varphi}{\partial y} F_z + \varphi \frac{\partial F_z}{\partial y} - \frac{\partial \varphi}{\partial z} F_y - \varphi \frac{\partial F_y}{\partial z} \\ \frac{\partial \varphi}{\partial z} F_x + \varphi \frac{\partial F_x}{\partial z} - \frac{\partial \varphi}{\partial x} F_z - \varphi \frac{\partial F_z}{\partial x} \\ \frac{\partial \varphi}{\partial x} F_y + \varphi \frac{\partial F_y}{\partial x} - \frac{\partial \varphi}{\partial y} F_x - \varphi \frac{\partial F_x}{\partial y} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial \varphi}{\partial y} F_z - \frac{\partial \varphi}{\partial z} F_y \\ \frac{\partial \varphi}{\partial z} F_x - \frac{\partial \varphi}{\partial x} F_z \\ \frac{\partial \varphi}{\partial x} F_y - \frac{\partial \varphi}{\partial y} F_x \end{bmatrix} + \varphi \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix} =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} + \varphi \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} =$$

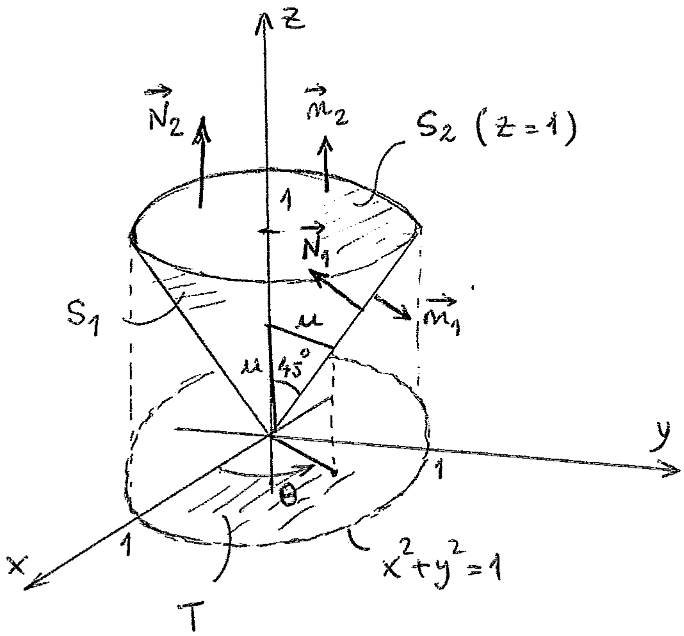
$$= (\nabla\varphi) \times \vec{F} + \varphi(\nabla \times \vec{F}) =$$

$$= (\text{grad}\, \varphi) \times \vec{F} + \varphi(\text{rot}\, \vec{F})$$

Vnig

40) Volume $V = \{(x,y,z) \in \mathbb{R}^3 : 1 \geq z \geq \sqrt{x^2+y^2}\}$

Trata-se de um <u>tronco de um cone</u>.



<u>Teorema de Gauss</u>

$$\iiint_V (\nabla \cdot \vec{F})\, dx\, dy\, dz = \oiint_S (\vec{F} \cdot \vec{n})\, dS$$

$$\vec{F}(x,y,z) = (x,y,z) \Rightarrow \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x,y,z) =$$

$$= 1 + 1 + 1 = 3$$

Então

$$\iiint_V (\nabla \cdot \vec{F})\, dx\, dy\, dz = 3 \iiint_V dx\, dy\, dz = 3\, \mathcal{V}(V) =$$

$$= 3\left[\frac{1}{3} A_{base}\, h\right] = 3\left[\frac{1}{3}\pi \times 1\right] = \pi$$

<u>Por outro lado</u>

$$\oiint_S (\vec{F} \cdot \vec{n})\, dS = \iint_{S_1} (\vec{F} \cdot \vec{n}_1)\, dS_1 + \iint_{S_2} (\vec{F} \cdot \vec{n}_2)\, dS_2$$

Wiz

<u>Superfície S_1</u> : $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$

$$\vec{r}_1(x,y) = (x, y, \sqrt{x^2+y^2}) \quad , \quad (x,y) \in T$$

$$T = \{ (x,y) \in \mathbb{R}^2 : 0 \le x^2+y^2 \le 1 \}$$

$$\vec{r}'_{1,x} = \frac{\partial \vec{r}_1}{\partial x} = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right) \qquad \vec{r}'_{1,y} = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\vec{N}_1(x,y) = \vec{r}'_{1,x} \times \vec{r}'_{1,y} = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right) \rightsquigarrow \text{dirigido para o interior de } S_1$$

Como o fluxo a considerar é de dentro para fora de S_1 (teorema de Gauss), então

$$\iint_{S_1} (\vec{F} \cdot \hat{n}_1)\, dS_1 = - \iint_T \vec{F}[\vec{r}_1(x,y)] \cdot \vec{N}_1(x,y)\, dx\, dy$$

$$\vec{F}[\vec{r}_1(x,y)] = (x, y, \sqrt{x^2+y^2})$$

$$\vec{F}[\vec{r}_1(x,y)] \cdot \vec{N}_1(x,y) = -\frac{x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2} = 0 \quad (\vec{F} \perp \vec{N}_1)$$

Assim, no caso da superfície S_1, obtém-se

$$\iint_{S_1} (\vec{F} \cdot \hat{n}_1)\, dS_1 = 0$$

<u>Superfície S_2</u> : $z = 1$

$$\vec{r}_2(x,y) = (x, y, 1) \quad , \quad (x,y) \in T$$

$$\vec{r}'_{2,x} = \frac{\partial \vec{r}_2}{\partial x} = (1,0,0) \qquad \vec{r}'_{2,y} = (0,1,0)$$

$$\vec{N}_2(x,y) = \vec{r}'_{2,x} \times \vec{r}'_{2,y} = (0,0,1) \rightsquigarrow \text{dirigido para o exterior de } S_2$$

$$\vec{F}[\vec{r}_2(x,y)] = (x, y, 1)$$

$$\vec{F}[\vec{r}_2(x,y)] \cdot \vec{N}_2(x,y) = 1$$

$$\iint_{S_2} (\vec{F} \cdot \hat{n}_2)\, dS_2 = + \iint_T \vec{F}[\vec{r}_2(x,y)] \cdot \vec{N}_2(x,y)\, dx\, dy = \iint_T dx\, dy = A(T) = \pi$$

Wmj

Concluindo, o fluxo de dentro para fora de V é

$$\oiint_S (\vec{F}\cdot\vec{n})\, dS = 0 + \pi = \pi$$

Calculemos o fluxo através da superfície S_1 recorrendo a uma outra parametrização:

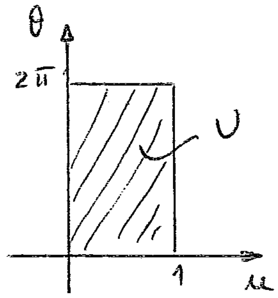
$$z = \sqrt{x^2+y^2}\ ,\quad 0 \leq z \leq 1$$

Seja $z = u$, $x = u\cos\theta$ e $y = u\,\text{sen}\,\theta$

Obtém-se

$$\vec{r}_1(u,\theta) = (u\cos\theta,\ u\,\text{sen}\,\theta,\ u)\ ,\quad (u,\theta)\in U$$

$$U = \{(u,\theta)\in\mathbb{R}^2 :\ u\in[0,1] \wedge \theta\in[0,2\pi]\}$$



$$\vec{r}\,'_{1,u} = \frac{\partial \vec{r}_1}{\partial u} = (\cos\theta,\ \text{sen}\,\theta,\ 1)$$

$$\vec{r}\,'_{1,\theta} = \frac{\partial \vec{r}_1}{\partial \theta} = (-u\,\text{sen}\,\theta,\ u\cos\theta,\ 0)$$

$$\vec{N}_1(u,\theta) = \vec{r}\,'_{1,u} \times \vec{r}\,'_{1,\theta} = (-u\cos\theta,\ -u\,\text{sen}\,\theta,\ u)$$

dirigido para o interior de S_1

$$\vec{F}[\vec{r}_1(u,\theta)] = (u\cos\theta,\ u\,\text{sen}\,\theta,\ u)$$

$$\vec{F}[\vec{r}_1(u,\theta)]\cdot\vec{N}_1(u,\theta) = -u^2\cos^2\theta - u^2\,\text{sen}^2\theta + u^2 = 0 \qquad (\vec{F}\perp\vec{N}_1)$$

Assim o fluxo de dentro para fora de S_1 é

$$\iint_{S_1} (\vec{F}\cdot\vec{n}_1)\, dS_1 = -\iint_U \vec{F}[\vec{r}_1(u,\theta)]\cdot\vec{N}_1(u,\theta)\, du\, d\theta = 0$$

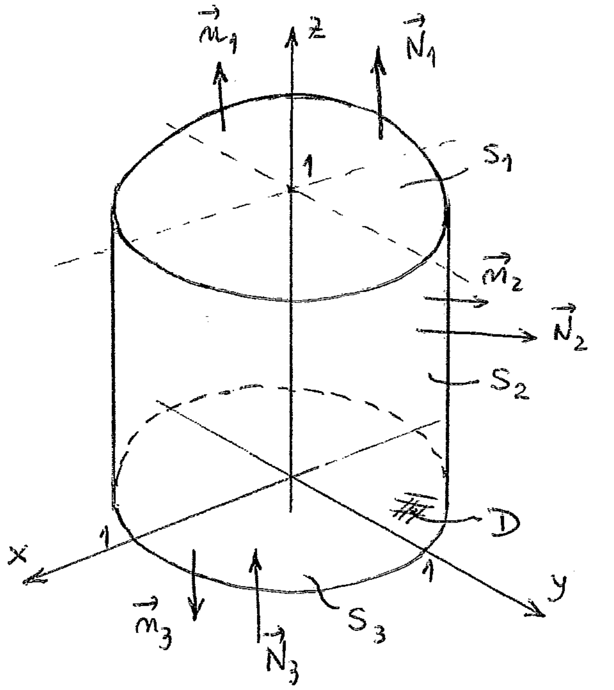
Wij

43) $\vec{F}(x,y,z) = (xy^2, x^2y, z)$

a)

Aplicando o <u>teorema de Gauss</u>

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(xy^2, x^2y, z) = y^2+x^2+1$$



$$\oiint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV =$$

$$= \iiint_V (1+x^2+y^2)\,dx\,dy\,dz =$$

$$= \iiint_V dx\,dy\,dz + \int_0^1\left[\iint_D (x^2+y^2)\,dx\,dy\right]dz =$$

$$= \underbrace{V(V)}_{\text{(volume de V)}} + \iint_D (x^2+y^2)\,dx\,dy =$$

$$= \pi + \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta =$$

$$= \pi + \frac{1}{4}(2\pi) = \frac{3\pi}{2}$$

$D = \{(x,y) : 0 \le x^2+y^2 \le 1\}$

<u>Coordenadas polares</u>:

$x = r\cos\theta$

$y = r\,\text{sen}\,\theta$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2$, $r\in[0,1] \wedge \theta\in[0,2\pi]$

b)

$$\oiint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1}\vec{F}\cdot\vec{n}_1\,dS_1 + \iint_{S_2}\vec{F}\cdot\vec{n}_2\,dS_2 + \iint_{S_3}\vec{F}\cdot\vec{n}_3\,dS_3$$

<u>Superfície S_1</u>: $z = 1$

$\vec{r}_1(x,y) = (x,y,1)$

$(x,y)\in D$

$\vec{r}\,'_{1,x} = (1,0,0) = \dfrac{\partial\vec{r}_1}{\partial x}$

$\vec{r}\,'_{1,y} = (0,1,0) = \dfrac{\partial\vec{r}_1}{\partial y}$

$\vec{N}_1(x,y) = \vec{r}\,'_{1,x}\times\vec{r}\,'_{1,y} = (0,0,1) = \vec{n}_1(x,y)$

↳ <u>dirigido para o exterior de S_1</u>

Júlio

$$\vec{F}[\vec{r}_1(x,y)] = (xy^2, x^2y, 1)$$

$$\vec{F}[\vec{r}_1(x,y)] \circ \vec{N}_1(x,y) = 1$$

$$\iint_{S_1} F \circ \vec{n}_1 \, dS_1 = \iint_D \vec{F}[\vec{r}_1(x,y)] \circ \vec{N}_1(x,y)\, dx\,dy = \iint_D dx\,dy = A(D) = \pi$$

<u>Superfície S_3</u> : $z = 0$

$\vec{r}_3(x,y) = (x, y, 0)$

$(x,y) \in D$

$$\vec{r}\,'_{3,x} = (1,0,0) = \frac{\partial \vec{r}_3}{\partial x}$$

$$\vec{r}\,'_{3,y} = (0,1,0) = \frac{\partial \vec{r}_3}{\partial y}$$

$$\vec{N}_3(x,y) = \vec{r}\,'_{3,x} \times \vec{r}\,'_{3,y} = (0,0,1) = -\vec{n}_3(x,y)$$

$\downarrow$

<u>dirigido para o interior de S_3</u>

$$\vec{F}[\vec{r}_3(x,y)] = (xy^2, x^2y, 0)$$

$$\vec{F}[\vec{r}_3(x,y)] \circ \vec{N}_3(x,y) = 0$$

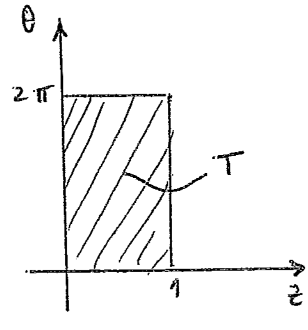
$$\iint_{S_3} \vec{F} \circ \vec{n}_3 \, dS = -\iint_D \vec{F}[\vec{r}_3(x,y)] \circ \vec{N}_3(x,y)\, dx\,dy = 0$$

<u>Superfície S_2</u> : $\begin{cases} x^2+y^2 = 1 \\ z \in [0,1] \end{cases}$

Considerando <u>coordenadas cilíndricas</u> (ver nota no final)

$$\vec{r}_2(\theta, z) = (\cos\theta, \operatorname{sen}\theta, z) \quad , (\theta,z) \in T$$

$$T = \{ (\theta, z) : \theta \in [0, 2\pi] \wedge z \in [0,1] \}$$



$$\vec{r}\,'_{2,\theta} = (-\operatorname{sen}\theta, \cos\theta, 0) = \frac{\partial \vec{r}_2}{\partial \theta}$$

$$\vec{r}\,'_{2,z} = (\ 0\ ,\ 0\ ,\ 1\ ) = \frac{\partial \vec{r}_2}{\partial z}$$

$\vec{N}_2(\theta,z) = (\cos\theta, \operatorname{sen}\theta, 0) \rightarrow$ <u>dirigido para o exterior de S_2</u>

$$\vec{F}[\vec{r}_2(\theta,z)] = (\cos\theta \operatorname{sen}^2\theta, \cos^2\theta \operatorname{sen}\theta, z)$$

$$\vec{F}[\vec{r}_2(\theta,z)] \cdot \vec{N}_2(\theta,z) = \cos^2\theta \operatorname{sen}^2\theta + \cos^2\theta \operatorname{sen}^2\theta = 2 \operatorname{sen}^2\theta \cos^2\theta =$$

$$= \frac{1}{2}\operatorname{sen}^2 2\theta = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2}\cos 4\theta\right] = \frac{1}{4} - \frac{1}{4}\cos 4\theta$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}_2\, dS_2 = \iint_T \vec{F}[\vec{r}_2(\theta,z)]\cdot\vec{N}_2(\theta,z)\; d\theta\, dz =$$

$$= \frac{1}{4}\iint_T d\theta\, dz - \frac{1}{4}\underbrace{\int_0^1\int_0^{2\pi} \cos 4\theta\, d\theta\, dz}_{=0} = \frac{1}{4}A(T) =$$

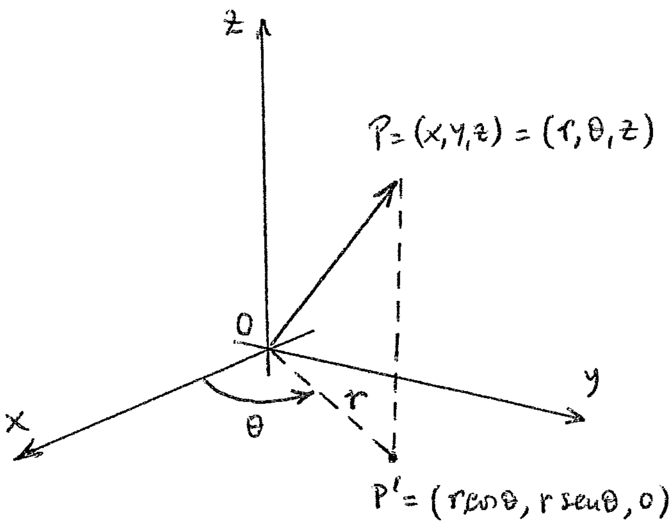
$$= \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

Concluindo:

$$\oiint_S \vec{F}\cdot\vec{n}\, dS = \pi + 0 + \frac{\pi}{2} = \frac{3\pi}{2}$$

Nota: Coordenadas Cilíndricas

$z \in \mathbb{R}$
$x = r\cos\theta$
$y = r\,\text{sen}\,\theta$



$$\vec{OP} = \vec{OP'} + \vec{P'P} =$$

$$= (r\cos\theta, r\,\text{sen}\,\theta, 0) + (0,0,z) =$$

$$= (r\cos\theta, r\,\text{sen}\,\theta, z)$$

$$dx\,dy\,dz = \underbrace{r}_{|J|}\, dr\, d\theta\, dz$$

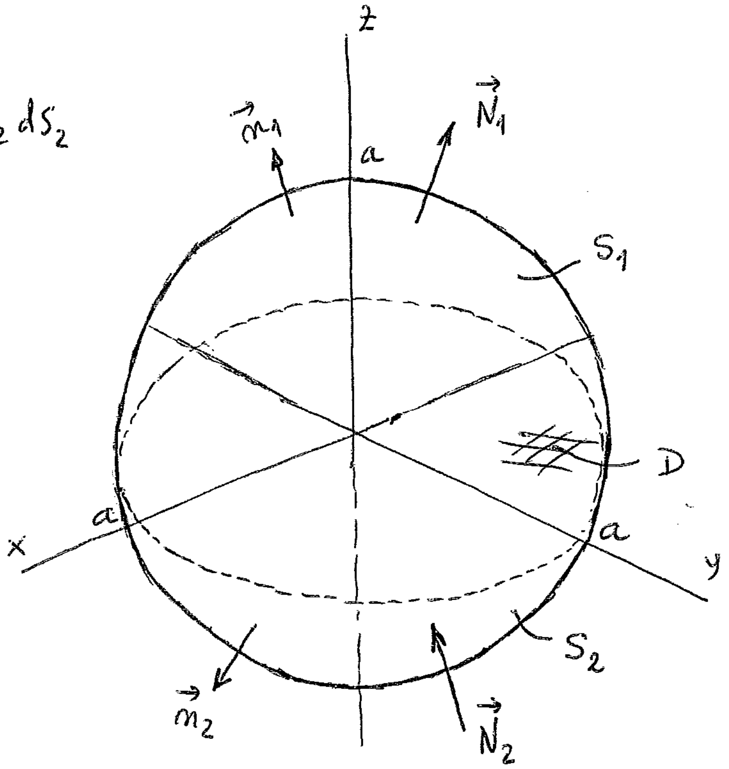
$$\underset{\text{Jacobiano}}{J} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\,\text{sen}\,\theta & 0 \\ \text{sen}\,\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\,\text{sen}^2\theta = r \Rightarrow |J| = r$$

Jacobiano

44) $\vec{F}(x,y,z) = (2xy,\ y^2,\ 3yz)$

Vamos começar por resolver o problema recorrendo a coordenadas cartesianas.

a)



$$\oiint_{S} \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}_1\,dS_1 + \iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS_2$$

Superfície S_1

(hemisfera superior)

$z = \sqrt{a^2-(x^2+y^2)}$, $z\in[0,a]$

$\vec{r}_1(x,y) = (x,\ y,\ \sqrt{a-(x^2+y^2)})$, $(x,y)\in D$

$D = \{(x,y)\in\mathbb{R}^2 : 0 \leq x^2+y^2 \leq a^2\}$

$$\vec{r}\,'_{1,x} = \left(1,\ 0,\ \frac{-x}{\sqrt{a^2-(x^2+y^2)}}\right)$$

$$\vec{r}\,'_{1,y} = \left(0,\ 1,\ \frac{-y}{\sqrt{a^2-(x^2+y^2)}}\right)$$

$$\vec{N}_1(x,y) = \vec{r}\,'_{1,x}\times\vec{r}\,'_{1,y} = \left(\frac{x}{\sqrt{a^2-(x^2+y^2)}},\ \frac{y}{\sqrt{a^2-(x^2+y^2)}},\ 1\right) \rightarrow$$ dirigido para o exterior de S_1

$$\iint_{S_1} \vec{F}\cdot\vec{n}_1\,dS_1 = \iint_D \vec{F}[\vec{r}_1(x,y)]\cdot\vec{N}_1(x,y)\,dx\,dy = (*)$$

$$\vec{F}[\vec{r}_1(x,y)] = \left(2xy,\ y^2,\ 3y\sqrt{a^2-(x^2+y^2)}\right)$$

$$\vec{F}[\vec{r}_1(x,y)]\cdot\vec{N}_1(x,y) = \frac{2x^2y}{\sqrt{a^2-(x^2+y^2)}} + \frac{y^3}{\sqrt{a^2-(x^2+y^2)}} + 3y\sqrt{a^2-(x^2+y^2)}$$

$$(*) = 2\underbrace{\iint_D \frac{x^2y}{\sqrt{a^2-(x^2+y^2)}}\,dx\,dy}_{=0} + \underbrace{\iint_D \frac{y^3}{\sqrt{a^2-(x^2+y^2)}}\,dx\,dy}_{=0} + 3\underbrace{\iint_D y\sqrt{a^2-(x^2+y^2)}\,dx\,dy}_{=0} = 0$$

(função ímpar em y) (função ímpar em y) (função ímpar em y)

$\sqrt{\cdot}\!/$

Superfície S_2

(hemisfera inferior)

$$z = -\sqrt{a^2-(x^2+y^2)} \quad , z \in [-a, 0]$$

$$\vec{r}_2(x,y) = (x, y, -\sqrt{a^2-(x^2+y^2)}), (x,y) \in D$$

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x^2+y^2 \leq a^2\}$$

$$\vec{r}\,'_{2,x} = \left(1, 0, \frac{x}{\sqrt{a^2-(x^2+y^2)}}\right)$$

$$\vec{r}\,'_{2,y} = \left(0, 1, \frac{y}{\sqrt{a^2-(x^2+y^2)}}\right)$$

dirigido para o interior de S_2

$$\vec{N}_2(x,y) = \vec{r}\,'_{2,x} \times \vec{r}\,'_{2,y} =$$

$$= \left(\frac{-x}{\sqrt{a^2-(x^2+y^2)}}, \frac{-y}{\sqrt{a^2-(x^2+y^2)}}, 1\right)$$

$$\iint_{S_2} \vec{F} \cdot \vec{n}_2 \, dS_2 = -\iint_D \vec{F}[\vec{r}_2(x,y)] \cdot N_2(x,y) \, dx\, dy = (**)$$

$$\vec{F}[\vec{r}_2(x,y)] = (2xy, y^2, -3y\sqrt{a^2-(x^2+y^2)})$$

$$\vec{F}[\vec{r}_2(x,y)] \cdot \vec{N}_2(x,y) = -\frac{2x^2y}{\sqrt{a^2-(x^2+y^2)}} - \frac{y^3}{\sqrt{a^2-(x^2+y^2)}} - 3y\sqrt{a^2-(x^2+y^2)}$$

$$(**) = 2\underbrace{\iint_D \frac{x^2y}{\sqrt{a^2-(x^2+y^2)}} \, dx\,dy}_{=0} + \underbrace{\iint_D \frac{y^3}{\sqrt{a^2-(x^2+y^2)}} \, dx\,dy}_{=0} + 3\underbrace{\iint_D y\sqrt{a^2-(x^2+y^2)} \, dx\,dy}_{=0} = 0$$

(função ímpar em y) (função ímpar em y) (função ímpar em y)

Então, o fluxo de dentro para fora de S é

$$\oiint_S \vec{F} \cdot \vec{n} \, dS = 0 + 0 = 0$$

Wir

b)

Apliquemos o teorema de Gauss

$$\oiint_S \vec{F}\cdot\vec{n}\, dS = \iiint_V \nabla\cdot\vec{F}\, dV = (***)$$

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(2xy, y^2, 3yz\right) = 2y + 2y + 3y = 7y$$

$$(***) \quad = 7 \iiint_V y\, dx\, dy\, dz = 7\,\bar{y}\,\mathcal{V}(V) = 0$$

já que

$\mathcal{V}(V)$ = volume da esfera $= \frac{4}{3}\pi a^3$

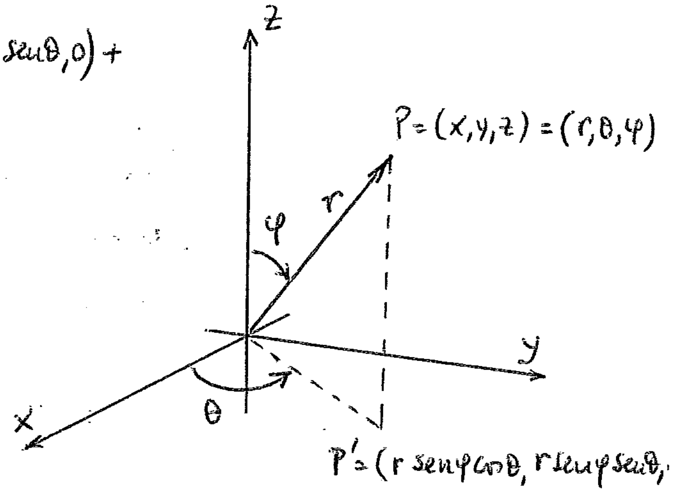
$\bar{y}$ = coordenada y do centróide da esfera $= 0$

---

44) Resolvamos agora o mesmo problema usando coordenadas esféricas.

$$\vec{OP} = \vec{OP'} + \vec{P'P} =$$
$$= (r\,\text{sen}\varphi\cos\theta, r\,\text{sen}\varphi\,\text{sen}\theta, 0) +$$
$$+ (0, 0, r\cos\varphi)$$

$$\begin{cases} x = r\,\text{sen}\varphi\cos\theta \\ y = r\,\text{sen}\varphi\,\text{sen}\theta \\ z = r\cos\varphi \end{cases}$$



$$dx\, dy\, dz = \underbrace{r^2\,\text{sen}\varphi}_{|J|}\, dr\, d\varphi\, d\theta$$

$|J| \rightarrow$ ver nota no final

a)

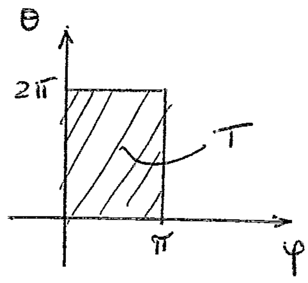
Superfície S (superfície esférica) : $x^2+y^2+z^2=a^2$

$$\Downarrow$$

$$r = a \;, \quad \theta\in[0,2\pi] \text{ e } \varphi\in[0,\pi]$$

$$\vec{r}(\theta,\varphi) = (a\,\text{sen}\varphi\cos\theta,\ a\,\text{sen}\varphi\,\text{sen}\theta,\ a\cos\varphi)\ ,\ (\theta,\varphi)\in T$$

$$T = \{(\theta,\varphi)\in\mathbb{R}^2 : \theta\in[0,2\pi] \wedge \varphi\in[0,\pi]\}$$



Wily

$$\vec{r}\,'_\theta = (-a\,\mathrm{sen}\varphi\,\mathrm{sen}\theta,\ a\,\mathrm{sen}\varphi\cos\theta,\ 0) = a(-\mathrm{sen}\varphi\,\mathrm{sen}\theta,\ \mathrm{sen}\varphi\cos\theta,\ 0)$$

$$\vec{r}\,'_\varphi = (a\cos\varphi\cos\theta,\ a\cos\varphi\,\mathrm{sen}\theta,\ -a\,\mathrm{sen}\varphi) = a(\cos\varphi\cos\theta,\ \cos\varphi\,\mathrm{sen}\theta,\ -\mathrm{sen}\varphi)$$

$$\vec{N}(\theta,\varphi) = \vec{r}\,'_\theta \times \vec{r}\,'_\varphi = a^2\left(-\mathrm{sen}^2\varphi\cos\theta,\ -\mathrm{sen}^2\varphi\,\mathrm{sen}\theta,\ -\mathrm{sen}\varphi\cos\varphi(\mathrm{sen}^2\theta+\cos^2\theta)\right) =$$

$$= a^2\,\mathrm{sen}\varphi\,(-\mathrm{sen}\varphi\cos\theta,\ -\mathrm{sen}\varphi\,\mathrm{sen}\theta,\ -\cos\varphi) \searrow$$

vector dirigido para o interior de S

NOTA: se $\varphi \in [0, \pi/2]$ e $\theta \in [0, \pi/2]$ então $\vec{N} = (a, b, c)$ em que $a<0$, $b<0$, $c<0$

$$\oiint_S \vec{F}\cdot\vec{n}\,dS = -\iint_T \vec{F}[\vec{r}(\theta,\varphi)]\cdot\vec{N}(\theta,\varphi)\,d\theta\,d\varphi$$

$$\vec{F}[\vec{r}(\theta,\varphi)] = \big(2(a\,\mathrm{sen}\varphi\cos\theta)(a\,\mathrm{sen}\varphi\,\mathrm{sen}\theta),\ a^2\,\mathrm{sen}^2\varphi\,\mathrm{sen}^2\theta,\ 3(a\,\mathrm{sen}\varphi\,\mathrm{sen}\theta)(a\cos\varphi)\big) =$$

$$= (2a^2\,\mathrm{sen}^2\varphi\,\mathrm{sen}\theta\cos\theta,\ a^2\,\mathrm{sen}^2\varphi\,\mathrm{sen}^2\theta,\ 3a^2\,\mathrm{sen}\varphi\cos\varphi\,\mathrm{sen}\theta) =$$

$$= a^2\,\mathrm{sen}\varphi\,(2\,\mathrm{sen}\varphi\,\mathrm{sen}(2\theta),\ \mathrm{sen}\varphi\,\mathrm{sen}^2\theta,\ 3\cos\varphi\,\mathrm{sen}\theta)$$

$$\vec{F}[\vec{r}(\theta,\varphi)]\cdot\vec{N}(\theta,\varphi) = a^4\,\mathrm{sen}^2\varphi\left[-2\,\mathrm{sen}^2\varphi\cos\theta\,\mathrm{sen}(2\theta) - \mathrm{sen}^2\varphi\,\mathrm{sen}^3\theta - 3\cos^2\varphi\,\mathrm{sen}\theta\right] =$$

$$= -2a^4\,\mathrm{sen}^4\varphi\cos\theta\,(2\,\mathrm{sen}\theta\cos\theta) - a^4\,\mathrm{sen}^4\varphi\,\mathrm{sen}^3\theta - 3a^4\,\mathrm{sen}^2\varphi\cos^2\varphi\,\mathrm{sen}\theta =$$

$$= -4a^4\,\mathrm{sen}^4\varphi\,\mathrm{sen}\theta\cos^2\theta - a^4\,\mathrm{sen}^4\varphi\,\mathrm{sen}^3\theta - 3a^4\,\mathrm{sen}^2\varphi\cos^2\varphi\,\mathrm{sen}\theta$$

$$\oiint_S \vec{F}\cdot\vec{n}\,dS = 4a^4\int_0^\pi \mathrm{sen}^4\varphi\left[\int_0^{2\pi}\mathrm{sen}\theta\cos^2\theta\,d\theta\right]d\varphi + a^4\int_0^\pi \mathrm{sen}^4\varphi\left[\int_0^{2\pi}\mathrm{sen}^3\theta\,d\theta\right]d\varphi +$$

$$+ 3a^4\int_0^\pi \mathrm{sen}^2\varphi\cos^2\varphi\left[\int_0^{2\pi}\mathrm{sen}\theta\,d\theta\right]d\varphi$$

Uma vez que

$$\int_0^{2\pi}\mathrm{sen}\theta\cos^2\theta\,d\theta = \left[-\frac{\cos^3\theta}{3}\right]_0^{2\pi} = \left[\frac{\cos^3\theta}{3}\right]_{2\pi}^0 = 0$$

$$\int_0^{2\pi} \operatorname{sen}^3\theta \, d\theta = \int_0^{2\pi} \operatorname{sen}\theta \operatorname{sen}^2\theta \, d\theta = \int_0^{2\pi} \operatorname{sen}\theta (1-\cos^2\theta) \, d\theta =$$

$$= \underbrace{\int_0^{2\pi} \operatorname{sen}\theta \, d\theta}_{=0} - \underbrace{\int_0^{2\pi} \operatorname{sen}\theta \cos^2\theta \, d\theta}_{=0} = 0$$

$$\int_0^{2\pi} \operatorname{sen}\theta \, d\theta = 0$$

então

$$\oiint_{S'} \vec{F} \cdot \vec{n} \, dS = 0 + 0 + 0 = 0$$

b)

Apliquemos o <u>teorema de Gauss</u>:

Sabemos já que $\nabla \cdot \vec{F} = 7y$, isto é, em coordenadas esféricas

$$\nabla \cdot \vec{F} = 7 r \operatorname{sen}\varphi \operatorname{sen}\theta$$

Então

$$\oiint_{S'} \vec{F} \cdot \vec{n} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV = 7 \int_0^a \int_0^{\pi} \int_0^{2\pi} r^3 \operatorname{sen}^2\varphi \operatorname{sen}\theta \, d\theta \, d\varphi \, dr =$$

$$= 7 \int_0^a r^3 \left[\int_0^{\pi} \operatorname{sen}^2\varphi \left[\int_0^{2\pi} \operatorname{sen}\theta \, d\theta \right] d\varphi \right] dr = 0$$

já que

$$\int_0^{2\pi} \operatorname{sen}\theta \, d\theta = 0$$

Nota: Coordenadas Esféricas

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial \varphi \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial \varphi \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial \varphi \end{vmatrix} = \begin{vmatrix} \operatorname{sen}\varphi\cos\theta & -r\operatorname{sen}\varphi\operatorname{sen}\theta & r\cos\varphi\cos\theta \\ \operatorname{sen}\varphi\operatorname{sen}\theta & r\operatorname{sen}\varphi\cos\theta & r\cos\varphi\operatorname{sen}\theta \\ \cos\varphi & 0 & -r\operatorname{sen}\varphi \end{vmatrix} =$$

$\downarrow$

Jacobiano

$$= \cos\varphi \begin{vmatrix} -r\operatorname{sen}\varphi\operatorname{sen}\theta & r\cos\varphi\cos\theta \\ r\operatorname{sen}\varphi\cos\theta & r\cos\varphi\operatorname{sen}\theta \end{vmatrix} - r\operatorname{sen}\varphi \begin{vmatrix} \operatorname{sen}\varphi\cos\theta & -r\operatorname{sen}\varphi\operatorname{sen}\theta \\ \operatorname{sen}\varphi\operatorname{sen}\theta & r\operatorname{sen}\varphi\cos\theta \end{vmatrix} =$$

$$= \cos\varphi\left(-r^2\operatorname{sen}\varphi\cos\varphi\operatorname{sen}^2\theta - r^2\operatorname{sen}\varphi\cos\varphi\cos^2\theta\right) - r\operatorname{sen}\varphi\left(r\operatorname{sen}^2\varphi\cos^2\theta + r\operatorname{sen}^2\varphi\operatorname{sen}^2\theta\right) =$$

$$= \cos\varphi\left(-r^2\operatorname{sen}\varphi\cos\varphi\right) - r\operatorname{sen}\varphi\left(r\operatorname{sen}^2\varphi\right) =$$

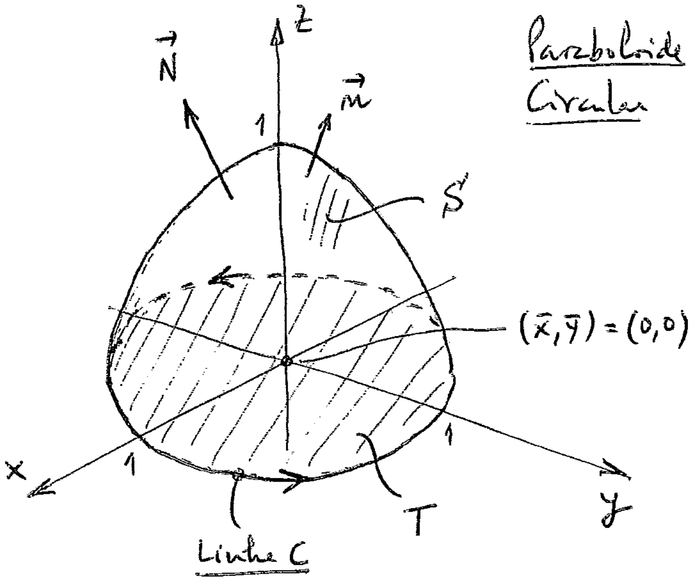
$$= -r^2\operatorname{sen}\varphi\cos^2\varphi - r^2\operatorname{sen}^3\varphi = -r^2\operatorname{sen}\varphi\left(1-\operatorname{sen}^2\varphi\right) - r^2\operatorname{sen}^3\varphi =$$

$$= -r^2\operatorname{sen}\varphi \quad \Longrightarrow \quad |J| = r^2\operatorname{sen}\varphi$$

$$\begin{cases} x = r\operatorname{sen}\varphi\cos\theta \\ y = r\operatorname{sen}\varphi\operatorname{sen}\theta \\ z = r\cos\varphi \end{cases} \Longrightarrow \quad dx\,dy\,dz = |J|\,dr\,d\theta\,d\varphi$$

55) $\vec{F}(x,y,z) = (P,Q,R) = (y,z,x)$

Superfície S : $z = 1-x^2-y^2$, $z \geq 0$



a)

Parametrização da Superfície S :

$\vec{r}(x,y) = (x, y, 1-x^2-y^2)$, $(x,y) \in T$

$T = \{(x,y) \in \mathbb{R}^2 : 0 \leq x^2+y^2 \leq 1\}$

$\vec{r}\,'_x = \dfrac{\partial \vec{r}}{\partial x} = (1, 0, -2x)$

$\vec{r}\,'_y = \dfrac{\partial \vec{r}}{\partial y} = (0, 1, -2y)$

$\Rightarrow \vec{N}(x,y) = \vec{r}\,'_x \times \vec{r}\,'_y = (2x, 2y, 1)$

$\hookrightarrow$ dirigido para o exterior de S

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = (-1,-1,-1)$$

$(\nabla \times \vec{F})[\vec{r}(x,y)] = (-1,-1,-1)$

$(\nabla \times \vec{F})[\vec{r}(x,y)] \cdot \vec{N}(x,y) = -2x-2y-1$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \iint_T (\nabla \times \vec{F})[\vec{r}(x,y)] \cdot \vec{N}(x,y)\, dx\, dy =$$

$$= -\iint_T (2x+2y+1)\, dx\, dy = -2\iint_T x\, dx\, dy - 2\iint_T y\, dx\, dy -$$

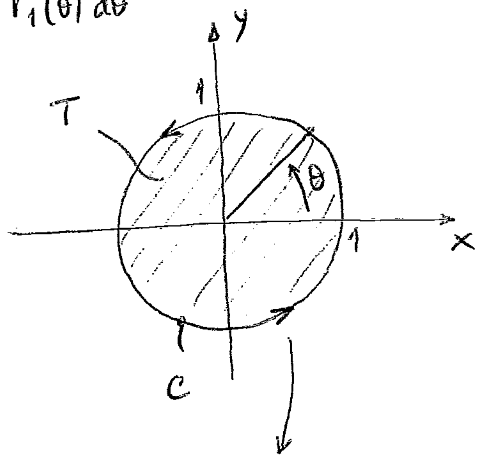
$$- \iint_T dx\, dy = -2(\underset{0}{\underset{\shortparallel}{\bar{x}}})\, A(T) - 2(\underset{0}{\underset{\shortparallel}{\bar{y}}})\, A(T) - A(T) =$$

$$= -A(T) = -\pi$$

Wir

b) Aplicando o teorema de Stokes ao caso presente

$$\iint_{S} (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \oint_C F(\vec{r}_1) \cdot d\vec{r}_1 = \oint_C \vec{F}[\vec{r}_1(\theta)] \cdot \vec{r}_1'(\theta) \, d\theta$$



Linha C

$$x^2 + y^2 = 1$$

Parametrizando

$$\vec{r}_1(\theta) = (\cos\theta, \operatorname{sen}\theta, 0) \quad , \theta \in [0, 2\pi]$$

$$\vec{r}_1'(\theta) = (-\operatorname{sen}\theta, \cos\theta, 0)$$

$$\vec{F}[\vec{r}_1(\theta)] = (\operatorname{sen}\theta, 0, \cos\theta)$$

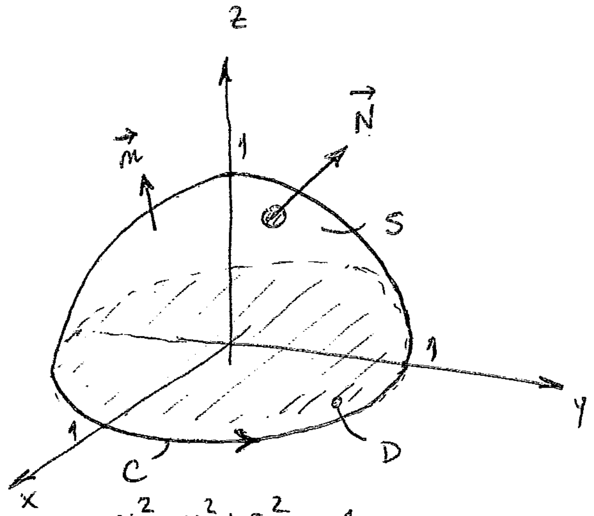
$$\vec{F}[\vec{r}_1(\theta)] \cdot \vec{r}_1'(\theta) = -\operatorname{sen}^2\theta = -\frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

Sentido directo: o vector $\vec{n}$ está orientado no sentido do semieixo positivo do eixo dos ZZ (fluxo de dentro para fora de S)

Então

$$\iint_{S} (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \frac{1}{2}\underbrace{\int_0^{2\pi} \cos(2\theta) \, d\theta}_{=0} - \frac{1}{2}\int_0^{2\pi} d\theta = -\pi$$

56) $\vec{F}(x,y,z) = (z^2, 2x, -y^3)$



a)

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 2x & -y^3 \end{vmatrix} =$$

$$= (-3y^2, 2z, 2)$$

$$x^2+y^2+z^2 = 1$$

$$\vec{r}(x,y) = \left(x, y, \sqrt{1-(x^2+y^2)}\right)$$

$$(x,y) \in D$$

$$D = \{(x,y) : 0 \leq x^2+y^2 \leq 1\}$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS =$$

$$= \iint_D (\nabla\times\vec{F})[\vec{r}(x,y)]\cdot\vec{N}(x,y)\, dx\, dy$$

$$\vec{r}\,'_x = \left(1, 0, \frac{-x}{\sqrt{1-(x^2+y^2)}}\right) = \frac{\partial\vec{r}}{\partial x}$$

$$(\nabla\times\vec{F})[\vec{r}(x,y)] = \left(-3y^2, 2\sqrt{1-(x^2+y^2)}, 2\right)$$

$$\vec{r}\,'_y = \left(0, 1, \frac{-y}{\sqrt{1-(x^2+y^2)}}\right) = \frac{\partial\vec{r}}{\partial y}$$

$$\vec{N}(x,y) = \vec{r}\,'_x \times \vec{r}\,'_y =$$

$$= \left(\frac{x}{\sqrt{1-(x^2+y^2)}}, \frac{y}{\sqrt{1-(x^2+y^2)}}, 1\right)$$

dirigido para o exterior de S

$$(\nabla\times\vec{F})[\vec{r}(x,y)]\cdot\vec{N}(x,y) = \frac{-3xy^2}{\sqrt{1-(x^2+y^2)}} + 2y + 2$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = \underbrace{\iint_D \frac{-3xy^2}{\sqrt{1-(x^2+y^2)}}\, dx\, dy}_{=0} + 2\iint_D y\, dx\, dy + 2\iint_D dx\, dy =$$

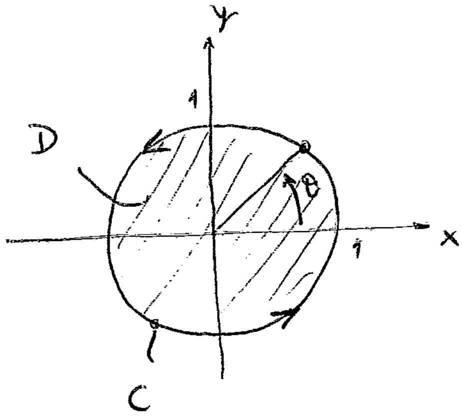
função ímpar em x, sendo a região D simétrica em relação a y.

$$= 2\underbrace{\bar{y}}_{=0}A(D) + 2A(D) = 2\pi \qquad (\bar{y} = 0)$$

Coordenada y do centróide do círculo.

b) Aplicando o teorema de Stokes ao caso presente

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \oint_C \vec{F}[\vec{r}_1] \cdot d\vec{r}_1 = \oint_C \vec{F}[\vec{r}_1(\theta)] \cdot \vec{r}_1'(\theta)\, d\theta$$



Linha C

$x^2 + y^2 = 1$

Parametrizando

$\vec{r}_1(\theta) = (\cos\theta, \operatorname{sen}\theta, 0)$, $\theta \in [0, 2\pi]$

$\vec{r}_1'(\theta) = (-\operatorname{sen}\theta, \cos\theta, 0)$

$\vec{F}[\vec{r}_1(\theta)] = (0, 2\cos\theta, -\operatorname{sen}^3\theta)$

$$\vec{F}[\vec{r}_1(\theta)] \cdot \vec{r}_1'(\theta) = 2\cos^2\theta = 2\left[\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right] =$$

$$= 1 + \cos(2\theta)$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \int_0^{2\pi} d\theta + \underbrace{\int_0^{2\pi} \cos(2\theta)\, d\theta}_{=0} = 2\pi$$