

Definitions

Chapter 1

Vector: A vector is a single element of a vector space.

Vector Space: A set V that is closed under linear combinations of its elements called vectors over a field \mathbb{F} .

Subspace: A subset of a vector space.

Span: The set of all linear combinations of a set of vectors

Linearly Independent: Given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of a vector space V , the set of vectors is linearly independent if all linear combinations are unique. That is, $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2, \dots + \alpha_n \mathbf{v}_n = \mathbf{0}$ if and only if $\alpha_1, \alpha_2, \dots, \alpha_n$ are all 0.

Linearly Dependent: A set of vectors is linearly dependent if they are not linearly independent.

Basis of a Vector Space: A basis of a vector space V is a linearly independent spanning set of vectors of V .

Dimension of a Vector Space: The number of elements in the basis of the vector space.

Linear Map: Given vector spaces V and W over \mathbb{C} , a linear map from V onto W is a function $f : V \rightarrow W$ such that $f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$ for all $\alpha, \beta \in \mathbb{C}$ and $\mathbf{x}, \mathbf{y} \in V$.

Matrix Transpose: Given a $m \times n$ matrix \mathbf{A} , \mathbf{A}^T is the $n \times m$ matrix obtained by swapping the rows of \mathbf{A} with the columns of \mathbf{A} .

Matrix Adjoint: Given a $m \times n$ matrix \mathbf{A} , \mathbf{A}^H is the $n \times m$ matrix obtained by swapping the rows of \mathbf{A} with the columns of the complex conjugate of \mathbf{A} .

Column Space: Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, the column space is the subspace of \mathbb{C}^m spanned by the columns of \mathbf{A} .

Rank: Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, the rank of \mathbf{A} is the dimension of the column space of \mathbf{A} or the dimension of the row space of \mathbf{A} .

Null Space: Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, the null space or kernel of \mathbf{A} is the subspace of \mathbb{C}^n spanned by the vectors $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$.

Nullity: Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, the nullity of \mathbf{A} is the dimension of the null space of \mathbf{A} .

Row Space: Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, the row space is the subspace of \mathbb{C}^n spanned by the columns of \mathbf{A} .

Left Null Space: Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, the left null space of \mathbf{A} is the subspace of \mathbb{C}^m spanned by the vectors $\mathbf{x} \in \mathbb{C}^m$ such that $\mathbf{x}\mathbf{A} = \mathbf{0}$. That is the left null space of \mathbf{A} is the null space of \mathbf{A}^T .

Eigenvalue: Given a square matrix \mathbf{A} , an eigenvalue is the scalar λ , such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution.

Eigenvector: Given a square matrix \mathbf{A} and an eigenvalue λ , an eigenvector is any nontrivial vector \mathbf{x} , satisfying $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

Eigenspace: The set of eigenvectors associated with an eigenvalue.

Invariant: A subspace S is invariant with respect to a square matrix \mathbf{A} if $\mathbf{A}S \subseteq S$.

Spectrum: Given a square matrix \mathbf{A} , the spectrum of \mathbf{A} is the set of eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of \mathbf{A} denoted $\lambda(\mathbf{A})$.

Spectral Radius: Given a square matrix \mathbf{A} , the spectral radius of \mathbf{A} is the largest absolute value of its eigenvalues denoted $\rho(\mathbf{A})$.

Similar Matrices: Given two square matrices \mathbf{A} and $\tilde{\mathbf{A}}$, they are similar if there exists a matrix \mathbf{C} such that $\mathbf{A} = \mathbf{C}\tilde{\mathbf{A}}\mathbf{C}^{-1}$.

Similarity Transform: Given two similar square matrices \mathbf{A} and $\tilde{\mathbf{A}}$, $\mathbf{A} = \mathbf{C}\tilde{\mathbf{A}}\mathbf{C}^{-1}$ is the similarity transform.

Unitarily Similar Matrices: Given two similar square matrices \mathbf{A} and $\tilde{\mathbf{A}}$, if the matrix \mathbf{C} such that $\mathbf{A} = \mathbf{C}\tilde{\mathbf{A}}\mathbf{C}^{-1}$ is a unitary matrix then \mathbf{A} and $\tilde{\mathbf{A}}$ are unitarily similar.

Nondefective: A matrix with a complete set of eigenvectors.

Diagonalization: For a nondefective matrix \mathbf{A} , the diagonalization of \mathbf{A} is $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$ where \mathbf{S} is a matrix of the eigenvectors and $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues. That is, \mathbf{A} is similar to $\mathbf{\Lambda}$.

Symmetric Matrix: A matrix \mathbf{A} such that, $\mathbf{A} = \mathbf{A}^T$.

Hermitian Matrix: A matrix \mathbf{A} such that, $\mathbf{A} = \mathbf{A}^H$.

Positive Definite Matrix: A Hermitian matrix \mathbf{A} such that, $\mathbf{x}^H \mathbf{A} \mathbf{x} > 0$ for any nonzero vector \mathbf{x} .

Orthogonal Matrix: A matrix \mathbf{A} such that, $\mathbf{A}^T \mathbf{A} = \mathbf{I}$.

Unitary Matrix: A matrix \mathbf{A} such that, $\mathbf{A}^H \mathbf{A} = \mathbf{I}$.

Permutation Matrix: An orthonormal matrix whose columns are permutations of the identity matrix.

Normal Matrix: A matrix \mathbf{A} such that, $\mathbf{A}^H \mathbf{A} = \mathbf{A} \mathbf{A}^H$. Unitary and Hermitian matrices are normal matrices.

Projection Matrix: A matrix \mathbf{A} such that, $\mathbf{A}^2 = \mathbf{A}$.

Orthogonal Projection Matrix: A matrix \mathbf{A} such that, $\mathbf{A}^2 = \mathbf{A}$ and $\mathbf{A}^T = \mathbf{A}$.

Diagonal Matrix: A matrix \mathbf{A} such that, entries $a_{ij} = 0$ for $i \neq j$.

Upper Triangular Matrix: A matrix \mathbf{A} such that, entries $a_{ij} = 0$ for $i > j$.

Tridiagonal Matrix: A matrix \mathbf{A} such that, $a_{ij} = 0$ for $|i - j| > 1$.

Banded Matrix: A matrix \mathbf{A} such that, $a_{ij} = 0$ for $i - j > m_l$ or $j - i > m_u$.

Bandwidth: For a banded matrix \mathbf{A} , the bandwidth is $m_u + m_l + 1$.

Upper Hessenberg Matrix: A matrix \mathbf{A} such that, $a_{ij} = 0$ for $i > j + 1$.

Spectral Decomposition: Given a real symmetric matrix \mathbf{A} , the spectral decomposition is $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H$ where \mathbf{Q} is a unitary matrix of the eigenvectors and $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues.

Singular Value Decomposition: Given a $m \times n$ matrix \mathbf{A} , there exists a unitary $m \times m$ matrix \mathbf{U} and a unitary $n \times n$ matrix \mathbf{V} such that $\mathbf{U}^H \mathbf{A} \mathbf{V} = \mathbf{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ which is the Singular Value Decomposition or SVD of \mathbf{A} .

Singular Values of a Matrix: Given a SVD of a matrix \mathbf{A} , each σ_i is a singular value and given by $\sigma_i(\mathbf{A}) = \sqrt{\lambda_i(\mathbf{A}^H \mathbf{A})}$.

Inner Product: Given a vector space V , an inner product is any map (\cdot, \cdot) from $V \times V$ into \mathbb{C} satisfying linearity, Hermiticity, and positive definiteness.

Orthogonal: Given two vectors \mathbf{u} and \mathbf{v} , they are orthogonal if $(\mathbf{u}, \mathbf{v}) = 0$.

Euclidean Inner Product: Given two vectors \mathbf{u} and \mathbf{v} , the Euclidean inner product is $(\mathbf{u}, \mathbf{v})_2 = \mathbf{u}^T \mathbf{v}$.

A-Energy Inner Product: Given two vectors \mathbf{u} and \mathbf{v} and a symmetric, positive definite matrix \mathbf{A} , the A-energy inner product is $(\mathbf{u}, \mathbf{v})_{\mathbf{A}} = \mathbf{u}^T \mathbf{A} \mathbf{v}$.

Vector Norm: Given a vector space V , a vector norm is any map $\|\cdot\|$ from V into \mathbb{R} satisfying positivity, homogeneity, and subadditivity.

P-Norm: Given a vector space V and a vector \mathbf{u} , $\|\mathbf{u}\|_p = (\sum_{i=1}^n |\mathbf{u}_i|^p)^{1/p}$ for $1 \leq p < \infty$.

Euclidean Norm: Given a vector space V and a vector \mathbf{u} , $\|\mathbf{u}\|_2 = (\sum_{i=1}^n |\mathbf{u}_i|^2)^{1/2} = \sqrt{(\mathbf{u}, \mathbf{u})_2}$.

∞ -Norm: Given a vector space V and a vector \mathbf{u} , $\|\mathbf{u}\|_{\infty} = \max_{i=1}^n |\mathbf{u}_i|$.

Energy Norm: Given a vector space V and a vector \mathbf{u} , $\|\mathbf{u}\|_{\mathbf{A}} = \sqrt{(\mathbf{u}, \mathbf{u})_{\mathbf{A}}}$ where \mathbf{A} is symmetric, positive definite.

Cauchy-Schwarz Inequality for a Norm: Given a vector space V and two vectors \mathbf{u} and \mathbf{v} , then we have $(\mathbf{u}, \mathbf{v}) \leq \|\mathbf{u}\| \|\mathbf{v}\|$.

Matrix Norm: Any map $\|\cdot\|$ from $\mathbb{R}^{m \times n}$ into \mathbb{R} satisfying positivity, homogeneity, and subadditivity.

Compatible: A matrix norm is compatible with a vector norm if $\|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\| \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

Induced Matrix Norm: The smallest matrix norm that is compatible with a vector norm.

Positive Definite Matrix: A matrix \mathbf{A} such that $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ for all $\mathbf{x} \neq 0$.

Negative Definite Matrix: A matrix \mathbf{A} such that $\mathbf{x}^T \mathbf{A} \mathbf{x} < 0$ for all $\mathbf{x} \neq 0$.

Congruent Matrices: Given two matrices \mathbf{A} and \mathbf{B} , \mathbf{A} is congruent to \mathbf{B} if there exists an invertible matrix \mathbf{S} such that $\mathbf{B} = \mathbf{S}^T \mathbf{A} \mathbf{S}$.

Quadratic Form of a Matrix: Given a real, Symmetric, square matrix \mathbf{A} , $q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$.

Well-Posed: A problem is well-posed if a solution for the problem exists, the solution is unique, and the solution is stable under small perturbations in the data.

Ill-Posed: A problem that is not well-posed.

Chapter 2

Gaussian Elimination: The process of using elementary row operations to convert a system of equations into an equivalent upper triangular representation of the system of equations.

LU Decomposition: For a matrix \mathbf{A} , an LU decomposition is a unit lower triangular matrix \mathbf{L} and an upper triangular matrix \mathbf{U} such that $\mathbf{A} = \mathbf{L}\mathbf{U}$.

Linear Programming: An optimization technique used on a system of linear constraints and a linear objective function to find the optimal solution.

Basic Variables: A variable in the basic solution. The value is not typically 0.

Nonbasic Variables: A variable not in the basic solution. Set the value to 0 for the basic feasible solution.

Primal Linear Programming Problem: The standard form of a linear programming problem. The general form is the following: Find the maximum of the objective function $z = \mathbf{c}^T \mathbf{x}$ subject to constraint $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and nonnegativity restriction $\mathbf{x}, \mathbf{b} \geq 0$.

Dual Linear Programming Problem: Every variable of the primal problem is a constraint in the dual, and every constraint of the primal is a variable in the dual. The general form is the following: Find the minimum of the objective function $z = \mathbf{b}^T \mathbf{y}$ subject to constraint $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ and nonnegativity restriction $\mathbf{y}, \mathbf{c} \geq 0$.

Sparse Matrix: A matrix with enough 0s that it is computationally efficient to take advantage of them.

Compressed Sparse Column: A data structure that only saves pointers to where new columns begin.

Laplacian Matrix: A matrix such that $\mathbf{L} = \mathbf{D} - \mathbf{A}$ where \mathbf{D} is the degree matrix and \mathbf{A} is the adjacency matrix.

Theorems

Chapter 1

Theorem (1). *Schur Decomposition. Every square matrix is unitarily Similar to an upper triangular matrix. That is, given \mathbf{A} , there exists a unitary matrix \mathbf{U} and an upper triangular matrix \mathbf{T} such that $\mathbf{T} = \mathbf{U}^H \mathbf{A} \mathbf{U}$. Further, the diagonal elements of \mathbf{T} are the eigenvalues of \mathbf{A} .*

Theorem (2). *Spectral theorem. If \mathbf{A} is a normal matrix ($\mathbf{A}^H \mathbf{A} = \mathbf{A} \mathbf{A}^H$), then $\mathbf{A} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^H$ where \mathbf{U} is a unitary matrix whose columns are eigenvalues of \mathbf{A} and $\mathbf{\Lambda}$ is a diagonal matrix whose elements are the eigenvalues of \mathbf{A} . Further, the eigenvalues of a Hermitian matrix are real.*

Theorem (3). *An induced norm is compatible with its associated vector norm. An induced norm of the identity matrix is one. An induced norm is submultiplicative.*

Theorem (4). *$\|\mathbf{A}\|_2$ equals the largest Singular value of \mathbf{A} . If \mathbf{A} is Hermitian, then $\|\mathbf{A}\|_2$ equals the spectral radius of \mathbf{A} . If \mathbf{A} is unitary then $\|\mathbf{A}\|_2 = 1$.*

Theorem (5). *$\rho(\mathbf{A}) \leq \|\mathbf{A}\|$ where $\|\cdot\|$ is an induced norm. Further, for symmetric matrices, $\|\mathbf{A}\|_{\max} \leq \rho(\mathbf{A})$ where the max norm $\|\mathbf{A}\|_{\max} = \max_{i,j} |a_{i,j}|$.*

Theorem (6). *A symmetric matrix is postive definite if and only if it has only positive eigenvalues.*

Chapter 2

Theorem (7). *A symmetric, postive definite matrix \mathbf{A} has the decomposition $\mathbf{R}^T \mathbf{R}$ where $\mathbf{R} = \mathbf{L} \mathbf{D}^{1/2}$.*

Theorem (8). *The maxima of a linear functional over a convex polytope occur at its verticies. If the values are at k verticies, then they must be along the k -cell between them.*

Theorem (9). *Given a linear programming problem there exists two related formulations, the primal problem and the dual problem, with four possibilities. The possibilities are the following: both the primal and the dual are infeasible, the primal is infeasible and the dual is unbounded, the dual is infeasible and the primal is unbounded, or both the primal and the dual have feasible solutions and their values are the same.*