Problem Set Ch-1

Problem. *1.2 Prove that similar matrices have the same spectrum of eigenvalues.*

Solution.

Problem. *1.7 Consider the matrix* $\mathbf{A}$ *that maps* $\mathbb{R}^3$ *into* $\mathbb{R}^2$*:*

$$\mathbf{A} = \begin{bmatrix} 4 & -3 \\ -2 & 6 \\ 4 & 6 \end{bmatrix}$$

a. Compute the SVD of $\mathbf{A}$ *by hand and use it to find the null space of* $\mathbf{A}$*.*

Solution. *The SVD of* $\mathbf{A}$ *is given by* $\mathbf{\Sigma} = \mathbf{U}^H \mathbf{A} \mathbf{V}$*. Where* $\mathbf{U}$ *is the matrix of eigenvectors of*

$$\mathbf{A}\mathbf{A}^H = \begin{bmatrix} 25 & -26 & -2 \\ -26 & 40 & 28 \\ -2 & 28 & 52 \end{bmatrix}$$

and $\mathbf{V}$ *is the matrix of eigenvectors of*

$$\mathbf{A}^H\mathbf{A} = \begin{bmatrix} 28 & 0 \\ 0 & 83 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}\mathbf{A}^H$ *are given by*

$$\left|\mathbf{A}\mathbf{A}^H - \lambda\mathbf{I}\right| = \begin{vmatrix} 25-\lambda & -26 & -2 \\ -26 & 40-\lambda & 28 \\ -2 & 28 & 52-\lambda \end{vmatrix} = -\lambda^3 + 117\lambda^2 - 2916\lambda = -\lambda(\lambda-36)(\lambda-81) = 0.$$

Thus, $\lambda = 0$*,* $\lambda = 36$*, or* $\lambda = 81$*. Then* $\sigma_1 = \sqrt{36} = 6$ *and* $\sigma_1 = \sqrt{81} = 9$ *and*

$$\mathbf{\Sigma} = \begin{bmatrix} 9 & 0 \\ 0 & 6 \\ 0 & 0 \end{bmatrix}.$$

When $\lambda = 0$*,*

$$(\mathbf{A}\mathbf{A}^H - \lambda\mathbf{I})\mathbf{x} = \begin{bmatrix} 25 & -26 & -2 \\ -26 & 40 & 28 \\ -2 & 28 & 52 \end{bmatrix}\mathbf{x} = \mathbf{0}.$$

Using row reduction we find $\mathbf{x} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ *is an eigenvector. When* $\lambda = 36$*,*

$$(\mathbf{A}\mathbf{A}^H - \lambda\mathbf{I})\mathbf{x} = \begin{bmatrix} -11 & -26 & -2 \\ -26 & 4 & 28 \\ -2 & 28 & 16 \end{bmatrix}\mathbf{x} = \mathbf{0}.$$

Using row reduction we find $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ *is an eigenvector. When* $\lambda = 81$*,*

$$(\mathbf{A}\mathbf{A}^H - \lambda\mathbf{I})\mathbf{x} = \begin{bmatrix} -56 & -26 & -2 \\ -26 & -41 & 28 \\ -2 & 28 & -29 \end{bmatrix}\mathbf{x} = \mathbf{0}.$$

Using row reduction we find $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ *is an eigenvector. Then* $\mathbf{U}$ *is a matrix of the unit normalized eigenvectors. Thus,* $\mathbf{U} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ *and* $\mathbf{U}^H = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$.

The eigenvalues of $\mathbf{A}^H\mathbf{A}$ *are given by*

$$\left|\mathbf{A}^H\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} 28-\lambda & 0 \\ 0 & 83-\lambda \end{vmatrix} = (28-\lambda)(83-\lambda) = 0.$$

Thus, $\lambda = 28$*, or* $\lambda = 83$*. When* $\lambda = 28$*,*

$$(\mathbf{A}^H\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 0 \\ 0 & 55 \end{bmatrix}\mathbf{x} = \mathbf{0}.$$

Thus, $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *is an eigenvector. When* $\lambda = 83$*,*

$$(\mathbf{A}^H\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \begin{bmatrix} -55 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}.$$

Thus, $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *is an eigenvector. Thus,* $\mathbf{V} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$*. Then*

$$\mathbf{\Sigma} = \mathbf{U}^H\mathbf{A}\mathbf{V} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -2 & 6 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As $\mathbf{V} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{V}\mathbf{x} = \mathbf{0}$ *has one solution namely* $\mathbf{x} = \mathbf{0}$*. Thus the null space of* $\mathbf{A}$ *is* $\mathbf{0}$.

Problem Set Ch-2

Problem. *2.5 Write a program to implement the reverse Cuthill-McKee algorithm for Symmetric matrices.*

Solution. *Solution can be found in CuthillMcKee.ipynb.*

Problem. *2.6 Use the Cuthill-McKee algorithm to reorder the verticies in the circular graph layout of the bottlenose dolphin network in the example on page 49.*

Solution. *Solution can be found in CuthillMcKee.ipynb.*