Lecture 2: Intro to Optimal Control

Optimal policies for Lagrangian turbulence – Dr. Robin Heinonen

Aqtivate workshop on data-driven and model-based tools for complex flows and complex fluids

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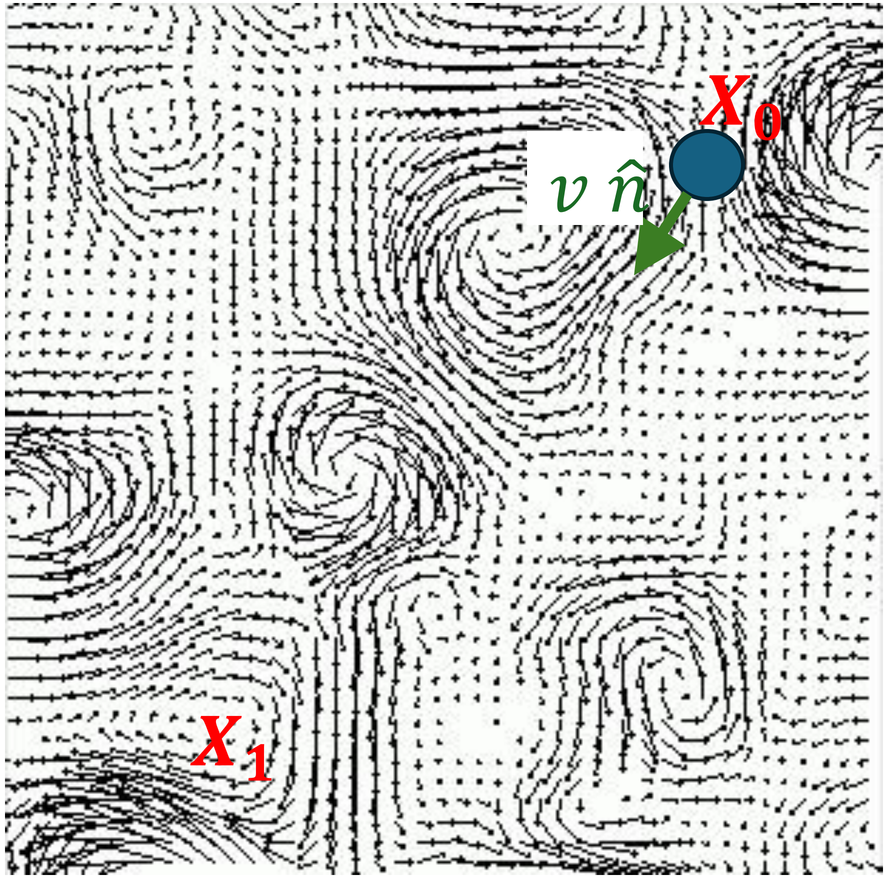
Zermelo problem in stationary flow

- To start, let's simplify the microswimmer problem a bit
- Assume the flow is **stationary** and known, $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x})$
- Have the system

$$\dot{x} = u_1\big(\mathbf{x}(t)\big) + v\cos\theta(t)$$
$$\dot{y} = u_2\big(\mathbf{x}(t)\big) + v\sin\theta(t)$$
$$\mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{x}(T) = \mathbf{x}_1$$

- Seek to choose $\theta(t)$ so as to minimize T
- This problem can be solved with **optimal control theory** methods



Generic optimal control problem

- Dynamics governed by

$$\dot{\mathbf{x}} = \mathbf{f}\big(\mathbf{x}(t), \boldsymbol{\alpha}(t)\big)$$
$$\mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{x}(T) = \mathbf{x}_1$$

- We are free to choose $\boldsymbol{\alpha}(t)$, typically with some constraints (e.g. $|\boldsymbol{\alpha}(t)| \leq A$). This is called the **control**. $\mathbf{x}_1$ is called the target
- ⚠️ States $\mathbf{x}$ need not be position coordinates! May include velocities, etc.
- **Goal**: minimize the accumulation of some function $L(\mathbf{x}, \boldsymbol{\alpha})$ and possibly some final cost $\phi(\mathbf{x}(T))$:

$$\text{minimize } J \equiv \int_0^T dt\, L\big(\mathbf{x}(\mathrm{t}), \boldsymbol{\alpha}(\mathrm{t})\big) + \phi(\mathbf{x}(T))$$

- Depending on the problem, T may be fixed or free parameter
- What are $\mathbf{f}, \boldsymbol{\alpha}, L, \phi$ for Zermelo problem?

Pontryagin maximum principle (1956)

- Introduce a Lagrange multiplier $\boldsymbol{\lambda}(t)$ called the **costate**
- Define a **Hamiltonian**

$$H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\alpha}) \equiv \boldsymbol{\lambda} \cdot \mathbf{f}(\mathbf{x}, \boldsymbol{\alpha}) - L(\mathbf{x}, \boldsymbol{\alpha})$$

- Theorem: Let the optimal control be $\boldsymbol{\alpha}^*(t)$ and corresponding trajectory $\mathbf{x}^*(t)$. If some technical conditions hold, there is a function $\boldsymbol{\lambda}^*(t)$ such that

$$\dot{\boldsymbol{\lambda}}^* = -\partial_{\mathbf{x}} H \Big|_{\mathbf{x}=\mathbf{x}^*, \boldsymbol{\lambda}=\boldsymbol{\lambda}^*, \boldsymbol{\alpha}=\boldsymbol{\alpha}^*}$$

and the optimal control **maximizes the Hamiltonian:**

$$H(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\alpha}^*(t)) = \max_{\boldsymbol{\alpha}} H(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\alpha}) \text{ for all } t \in [0, T].$$

- Moreover, $H(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\alpha}^*(t))$ is **constant** over the trajectory.
- If T fixed, also have the **terminal condition**

$$\boldsymbol{\lambda}^*(T) = \partial_{\mathbf{x}} \phi \Big|_{\mathbf{x}=\mathbf{x}^*, t=T}$$

- If T not fixed, have instead

$$\partial_t \phi \Big|_T = H(\mathbf{x}^*(T), \boldsymbol{\alpha}^*(T), \boldsymbol{\lambda}^*(T))$$

Connection to classical mechanics

- Observe that dynamics can be combined with PMP as system

$$\dot{\mathbf{x}} = \partial_{\boldsymbol{\lambda}} H$$
$$\dot{\boldsymbol{\lambda}} = -\partial_{\mathbf{x}} H$$

- Suggests connection to Hamiltonian mechanics
- Suppose $\phi = 0, T$ fixed. Let $\dot{\mathbf{x}} = \boldsymbol{\alpha}$. Then if $L = T(\dot{\mathbf{x}}, \mathbf{x}) - V(\mathbf{x})$,

$$J = \int_0^T dt \;\; L(\mathbf{x}(t), \dot{\mathbf{x}}(t))$$

is classical action, $\boldsymbol{\lambda}$ is momentum conjugate to $\mathbf{x}$ and

$$H = \boldsymbol{\lambda} \cdot \dot{\mathbf{x}} - L$$

is classical Hamiltonian.

- H being constant is conservation of energy, and H being maximum equivalent to $\boldsymbol{\lambda} = \partial_{\dot{\mathbf{x}}} L$ and L convex in $\dot{\mathbf{x}}$

Example 1: simple model of an economy

- $x(t)$ = economic output at time t
- $\alpha(t)$ = fraction of output reinvested. $0 \leq \alpha \leq 1$
- Dynamics:

$$\dot{x} = k\alpha x$$
$$x(0) = x_0$$

for some $k > 0, x_0 > 0$.

- Goal: maximize total consumption (disclaimer: I do not endorse this view of economics)

$$J = -\int_0^T dt\,\big(1 - \alpha(t)\big)x(t)$$

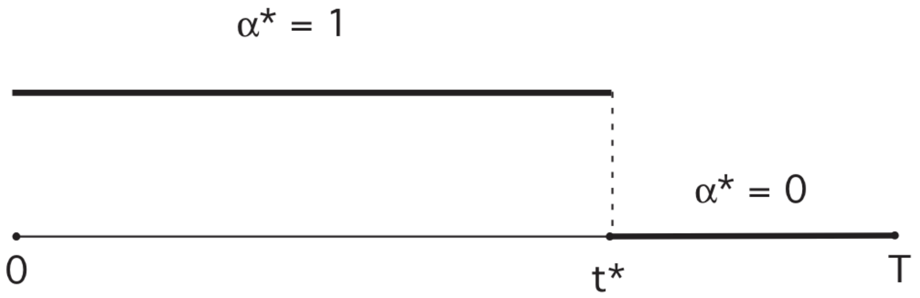
Example 1: simple model of an economy (cont'd)

- Hamiltonian $H = k\lambda\alpha x + (1-\alpha)x = x + (k\lambda - 1)\alpha x$
- PMP says

$$\dot{\lambda} = -\partial_x H = (1 - k\lambda)\alpha - 1$$
$$\lambda(T) = 0$$

Choose α to maximize H at each $t \Rightarrow$

$$\alpha^*(t) = \begin{cases} 1, & \lambda(t) > 1/k \\ 0, & \lambda(t) \le 1/k \end{cases}$$



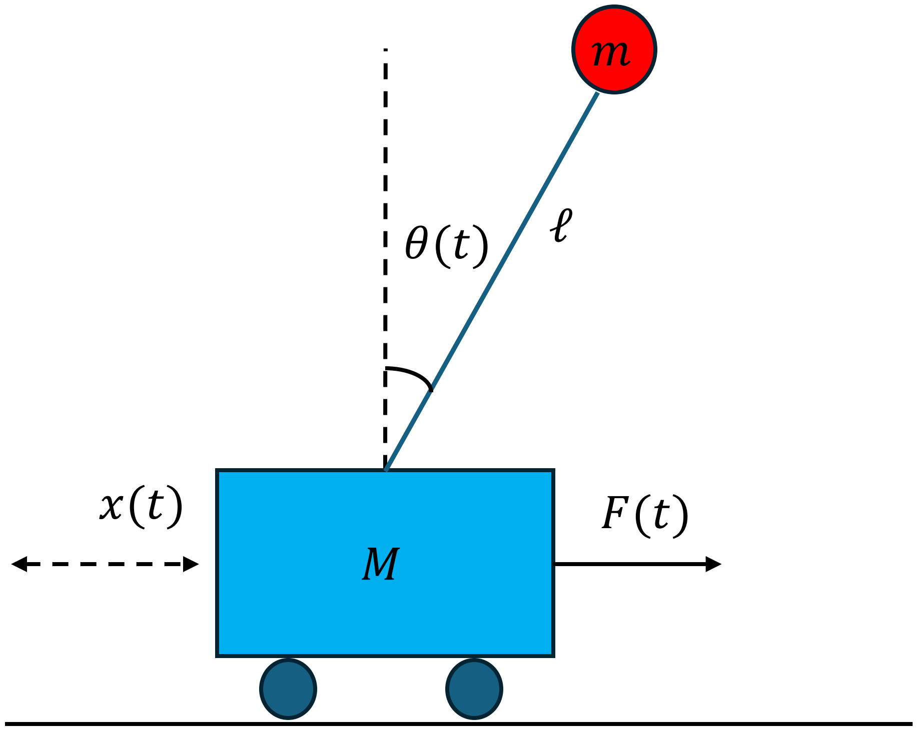
- Leads to

$$\lambda^*(t) = \begin{cases} \lambda_0 \exp(-kt), & 0 \le t < t^* \\ T - t, & t^* \le t \le T \end{cases}$$

- Exercise: solve for t^*, λ_0
- Optimal control is abrupt switching between two states. Called **bang-bang** control

Example 2: inverted pendulum

- A mass m is attached to a rigid, massless rod of length ℓ which pivots vertically from a cart of mass M
- Unstable equilibrium at $\theta = 0$
- How to apply control force F to cart in order to keep θ as close to 0 as possible?
- What's a good objective function (Lagrangian)?



Example 2: inverted pendulum (cont'd)

- Physical Lagrangian is

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\dot{x}\dot{\theta}\cos\theta - mg\ell\cos\theta + F(t)x$$

- Euler-Lagrange eqs:

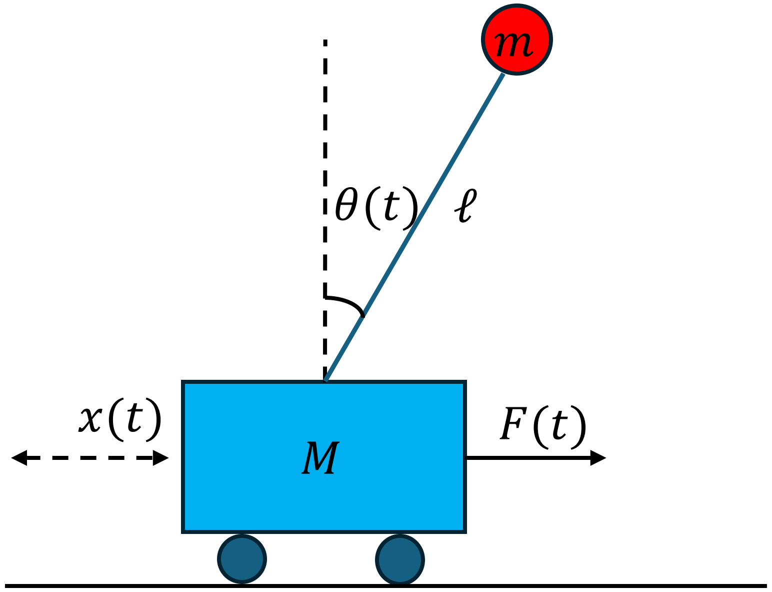
$$\begin{cases} \ddot{x}\cos\theta + \ell\ddot{\theta} = g\sin\theta \\ (m+M)\ddot{x} + m\ell\ddot{\theta}\cos\theta = m\ell\dot{\theta}^2\sin\theta + F(t) \end{cases}$$

- Eliminate $\ddot{x}$, approximate $\theta \ll 1$ to $O(\theta^2)$:

$$M\ell\ddot{\theta} \simeq (m+M)g\theta - F(t)$$

- Define $\omega \equiv \dot{\theta}, \Omega \equiv \sqrt{\frac{g}{l}\left(1+\frac{m}{M}\right)}, f = F/M\ell$:

$$\begin{cases} \dot{\omega} = \Omega^2\theta - f \\ \dot{\theta} = \omega \end{cases}$$



Example 2: inverted pendulum (cont'd)

- Dynamics: $\begin{cases} \dot{\omega} = \Omega^2\theta - f \\ \dot{\theta} = \omega \end{cases}$
- Reasonable objective: $J = \frac{1}{2}\int_0^T dt\,(f^2 + a^2\,\theta^2 + b^2\omega^2)$
- Take $b = 0, T \to \infty$
- Hamiltonian: $H = \lambda\omega + \eta(\Omega^2\theta - f) - \frac{1}{2}f^2 - \frac{1}{2}a^2\theta^2$
- PMP:

$$\dot{\lambda} = -\Omega^2\eta + a\theta$$
$$\dot{\eta} = -\lambda$$
$$\lambda(\infty) = \eta(\infty) = 0$$

Choose f to maximize H at each $t \Rightarrow$

$$f^*(t) = -\eta$$

- Leads to $\ddot{f}^* = \Omega^2 f^* + a^2\theta^*, \ddot{\theta}^* = \Omega^2\theta^* - f^*, f^*(\infty) = f^{*\prime}(\infty) = 0.$
- Exercise: show that θ^* undergoes underdamped harmonic motion.

Example 3: Zermelo problem (stationary flow)

- Dynamics:

$$\dot{x} = u_1(\mathbf{x}(t)) + v\cos\theta(t)$$
$$\dot{y} = u_2(\mathbf{x}(t)) + v\sin\theta(t)$$
$$\mathbf{x}(0) = \mathbf{x}_0$$
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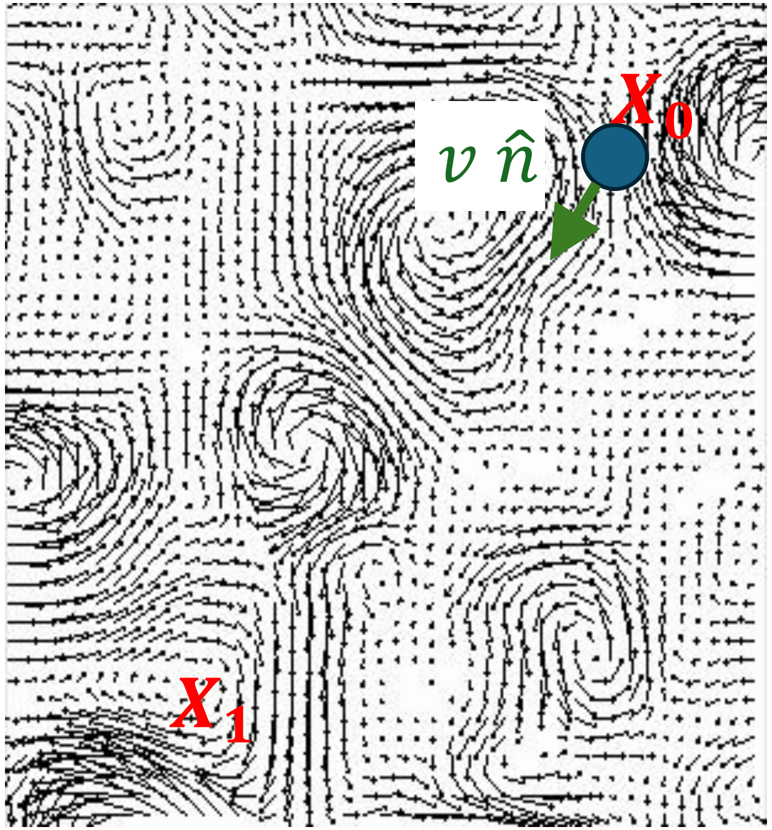


- Now T is free and $J = T \Rightarrow L = 1$
- Hamiltonian:

$$H = \lambda(u_1 + v\cos\theta) + \eta(u_2 + v\sin\theta) - 1$$

- PMP:

$$\dot{\lambda} = -\lambda\partial_x u_1 - \eta\partial_x u_2$$
$$\dot{\eta} = -\lambda\partial_y u_1 - \eta\partial_y u_2$$

Choose θ to maximize H at each $t \Rightarrow$
$\tan\theta^* = \eta/\lambda$

Example 3: Zermelo problem (stationary flow) (cont'd)

$$\begin{cases} \dot{\lambda} = -\lambda\partial_x u_1 - \eta\partial_x u_2 \\ \dot{\eta} = -\lambda\partial_y u_1 - \eta\partial_y u_2 \\ \tan\theta^* = \eta/\lambda \end{cases}$$

- Put $(\lambda, \eta) = (p\cos\theta^*, p\sin\theta^*)$, $p = p(\theta^*)$
- Leads to

$$\dot{\theta}^* = \frac{\lambda\dot{\eta} - \dot{\lambda}\eta}{\lambda^2 + \eta^2} = \frac{-\lambda^2\partial_y u_1 - \lambda\eta\partial_y u_2 + \lambda\eta\partial_x u_1 + \eta^2\partial_x u_2}{\lambda^2 + \eta^2}$$
$$= A_{12}\sin^2\theta^* - A_{21}\cos^2\theta^* + (A_{11} - A_{22})\sin\theta^*\cos\theta^*$$

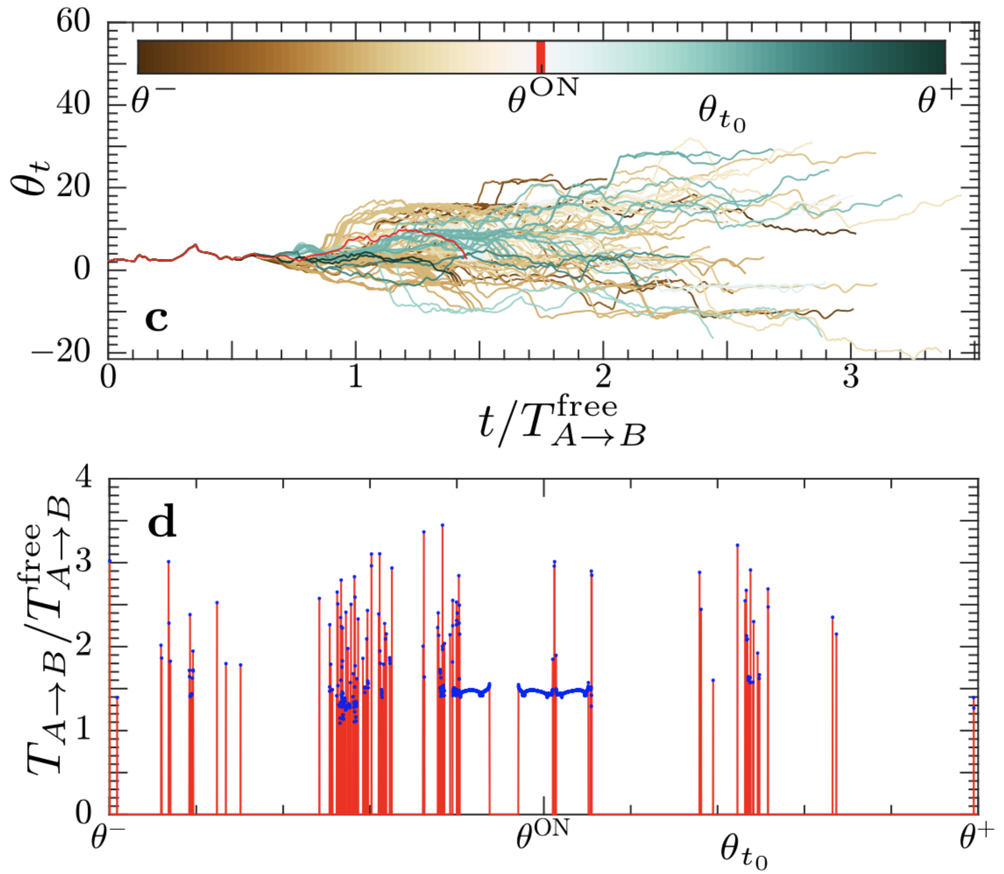
where $A_{ij} = \partial_i u_j$

Optimal control for Zermelo is unstable

- Optimal control is solution to

$$\dot{\theta}^* = A_{12}\sin^2\theta^* - A_{21}\cos^2\theta^* + (A_{11} - A_{22})\sin\theta^*\cos\theta^*$$

- Optimal choice of $\theta(0)$ is *not* fixed by the optimal control theory. Has to be obtained by trial and error
- Small perturbations from $\theta(0)$ radically change the time of arrival and can even lead to failure!



Top: divergence of trajectories with slightly differing $\theta(0)$ ($\theta^{\pm} = \theta^{ON} \pm 0.0006$). Bottom: time of arrival for same set of $\theta(0)$. $T = 0$ means failed to arrive

Summary

- Optimal control using PMP provides elegant framework for optimization in deterministic systems with known dynamics
- But questionable approach for microswimming (even in stationary flow!) beyond use as benchmark
  - Need to know full flow field
  - Unstable to perturbations in start state (Lagrangian chaos!)
  - Can't predict initial θ^* a priori
- What about olfactory search? Is PMP appropriate?
- Next: we study Markov Decision Processes