BM

March 27, 2025

[1]:
```
from IPython.core.display import HTML
HTML("""
<style>
.output_png {
    display: table-cell;
    text-align: center;
    margin:auto;
}
.prompt
    display:none;
}
</style>
""")
```

[1]:
```
<IPython.core.display.HTML object>
```

$ $

[2]:
```
! pip install rise scipy matplotlib pymc3 seaborn
```

```
DEPRECATION: Loading egg at
/home/serge/anaconda3/envs/ML/lib/python3.11/site-
packages/search_engines_new-0.5-py3.11.egg is deprecated. pip 24.3 will enforce
this behaviour change. A possible replacement is to use pip for package
installation.. Discussion can be found at
https://github.com/pypa/pip/issues/12330
DEPRECATION: Loading egg at
/home/serge/anaconda3/envs/ML/lib/python3.11/site-
packages/search_engines-0.5-py3.11.egg is deprecated. pip 24.3 will enforce this
behaviour change. A possible replacement is to use pip for package
installation.. Discussion can be found at
https://github.com/pypa/pip/issues/12330
Requirement already satisfied: rise in
```

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/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (5.7.1)
Requirement already satisfied: scipy in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (1.11.3)
Requirement already satisfied: matplotlib in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (3.8.0)
Requirement already satisfied: pymc3 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (3.11.4)
Requirement already satisfied: seaborn in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (0.13.2)
Requirement already satisfied: notebook>=6.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from rise) (6.5.4)
Requirement already satisfied: numpy<1.28.0,>=1.21.6 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from scipy) (1.26.0)
Requirement already satisfied: contourpy>=1.0.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from matplotlib)
(1.1.1)
Requirement already satisfied: cycler>=0.10 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from matplotlib)
(0.12.1)
Requirement already satisfied: fonttools>=4.22.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from matplotlib)
(4.43.1)
Requirement already satisfied: kiwisolver>=1.0.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from matplotlib)
(1.4.5)
Requirement already satisfied: packaging>=20.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from matplotlib)
(23.1)
Requirement already satisfied: pillow>=6.2.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from matplotlib)
(9.4.0)
Requirement already satisfied: pyparsing>=2.3.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from matplotlib)
(3.1.1)
Requirement already satisfied: python-dateutil>=2.7 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from matplotlib)
(2.8.2)
Requirement already satisfied: arviz>=0.11.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from pymc3) (0.17.0)
Requirement already satisfied: cachetools>=4.2.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from pymc3) (5.3.1)
Requirement already satisfied: dill in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from pymc3) (0.3.8)
Requirement already satisfied: fastprogress>=0.2.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from pymc3) (1.0.3)
Requirement already satisfied: pandas>=0.24.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from pymc3) (2.1.1)
Requirement already satisfied: patsy>=0.5.1 in
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/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from pymc3) (0.5.6)
Requirement already satisfied: semver>=2.13.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from pymc3) (3.0.2)
Requirement already satisfied: theano-pymc==1.1.2 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from pymc3) (1.1.2)
Requirement already satisfied: typing-extensions>=3.7.4 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from pymc3) (4.7.1)
Requirement already satisfied: filelock in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from theano-
pymc==1.1.2->pymc3) (3.13.1)
Requirement already satisfied: setuptools>=60.0.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
arviz>=0.11.0->pymc3) (68.0.0)
Requirement already satisfied: xarray>=0.21.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
arviz>=0.11.0->pymc3) (2024.1.1)
Requirement already satisfied: h5netcdf>=1.0.2 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
arviz>=0.11.0->pymc3) (1.3.0)
Requirement already satisfied: xarray-einstats>=0.3 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
arviz>=0.11.0->pymc3) (0.7.0)
Requirement already satisfied: jinja2 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (3.1.2)
Requirement already satisfied: tornado>=6.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (6.3.2)
Requirement already satisfied: pyzmq>=17 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (23.2.0)
Requirement already satisfied: argon2-cffi in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (21.3.0)
Requirement already satisfied: traitlets>=4.2.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (5.7.1)
Requirement already satisfied: jupyter-core>=4.6.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (5.3.0)
Requirement already satisfied: jupyter-client>=5.3.4 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (7.4.9)
Requirement already satisfied: ipython-genutils in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (0.2.0)
Requirement already satisfied: nbformat in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
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notebook>=6.0->rise) (5.9.2)
Requirement already satisfied: nbconvert>=5 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (6.5.4)
Requirement already satisfied: nest-asyncio>=1.5 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (1.5.6)
Requirement already satisfied: ipykernel in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (6.25.0)
Requirement already satisfied: Send2Trash>=1.8.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (1.8.0)
Requirement already satisfied: terminado>=0.8.3 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (0.17.1)
Requirement already satisfied: prometheus-client in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (0.14.1)
Requirement already satisfied: nbclassic>=0.4.7 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
notebook>=6.0->rise) (0.5.5)
Requirement already satisfied: pytz>=2020.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
pandas>=0.24.0->pymc3) (2023.3.post1)
Requirement already satisfied: tzdata>=2022.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
pandas>=0.24.0->pymc3) (2023.3)
Requirement already satisfied: six in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
patsy>=0.5.1->pymc3) (1.16.0)
Requirement already satisfied: h5py in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
h5netcdf>=1.0.2->arviz>=0.11.0->pymc3) (3.10.0)
Requirement already satisfied: entrypoints in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from jupyter-
client>=5.3.4->notebook>=6.0->rise) (0.4)
Requirement already satisfied: platformdirs>=2.5 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from jupyter-
core>=4.6.1->notebook>=6.0->rise) (3.10.0)
Requirement already satisfied: jupyter-server>=1.8 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbclassic>=0.4.7->notebook>=6.0->rise) (1.23.4)
Requirement already satisfied: notebook-shim>=0.1.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbclassic>=0.4.7->notebook>=6.0->rise) (0.2.2)
Requirement already satisfied: lxml in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
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nbconvert>=5->notebook>=6.0->rise) (4.9.3)
Requirement already satisfied: beautifulsoup4 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (4.12.2)
Requirement already satisfied: bleach in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (4.1.0)
Requirement already satisfied: defusedxml in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (0.7.1)
Requirement already satisfied: jupyterlab-pygments in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (0.1.2)
Requirement already satisfied: MarkupSafe>=2.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (2.1.1)
Requirement already satisfied: mistune<2,>=0.8.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (0.8.4)
Requirement already satisfied: nbclient>=0.5.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (0.5.13)
Requirement already satisfied: pandocfilters>=1.4.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (1.5.0)
Requirement already satisfied: pygments>=2.4.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (2.15.1)
Requirement already satisfied: tinycss2 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (1.2.1)
Requirement already satisfied: fastjsonschema in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbformat->notebook>=6.0->rise) (2.16.2)
Requirement already satisfied: jsonschema>=2.6 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbformat->notebook>=6.0->rise) (4.17.3)
Requirement already satisfied: ptyprocess in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
terminado>=0.8.3->notebook>=6.0->rise) (0.7.0)
Requirement already satisfied: argon2-cffi-bindings in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
argon2-cffi->notebook>=6.0->rise) (21.2.0)
Requirement already satisfied: comm>=0.1.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipykernel->notebook>=6.0->rise) (0.1.2)
Requirement already satisfied: debugpy>=1.6.5 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
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ipykernel->notebook>=6.0->rise) (1.6.7)
Requirement already satisfied: ipython>=7.23.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipykernel->notebook>=6.0->rise) (8.15.0)
Requirement already satisfied: matplotlib-inline>=0.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipykernel->notebook>=6.0->rise) (0.1.6)
Requirement already satisfied: psutil in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipykernel->notebook>=6.0->rise) (5.9.0)
Requirement already satisfied: backcall in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.2.0)
Requirement already satisfied: decorator in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (5.1.1)
Requirement already satisfied: jedi>=0.16 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.18.1)
Requirement already satisfied: pickleshare in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.7.5)
Requirement already satisfied: prompt-toolkit!=3.0.37,<3.1.0,>=3.0.30 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (3.0.36)
Requirement already satisfied: stack-data in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.2.0)
Requirement already satisfied: pexpect>4.3 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (4.8.0)
Requirement already satisfied: attrs>=17.4.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
jsonschema>=2.6->nbformat->notebook>=6.0->rise) (23.1.0)
Requirement already satisfied: pyrsistent!=0.17.0,!=0.17.1,!=0.17.2,>=0.14.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
jsonschema>=2.6->nbformat->notebook>=6.0->rise) (0.18.0)
Requirement already satisfied: anyio<4,>=3.1.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from jupyter-
server>=1.8->nbclassic>=0.4.7->notebook>=6.0->rise) (3.5.0)
Requirement already satisfied: websocket-client in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from jupyter-
server>=1.8->nbclassic>=0.4.7->notebook>=6.0->rise) (0.58.0)
Requirement already satisfied: cffi>=1.0.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from argon2-cffi-
bindings->argon2-cffi->notebook>=6.0->rise) (1.15.1)
Requirement already satisfied: soupsieve>1.2 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
```

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beautifulsoup4->nbconvert>=5->notebook>=6.0->rise) (2.4)
Requirement already satisfied: webencodings in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
bleach->nbconvert>=5->notebook>=6.0->rise) (0.5.1)
Requirement already satisfied: idna>=2.8 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
anyio<4,>=3.1.0->jupyter-server>=1.8->nbclassic>=0.4.7->notebook>=6.0->rise)
(3.4)
Requirement already satisfied: sniffio>=1.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
anyio<4,>=3.1.0->jupyter-server>=1.8->nbclassic>=0.4.7->notebook>=6.0->rise)
(1.2.0)
Requirement already satisfied: pycparser in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
cffi>=1.0.1->argon2-cffi-bindings->argon2-cffi->notebook>=6.0->rise) (2.21)
Requirement already satisfied: parso<0.9.0,>=0.8.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
jedi>=0.16->ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.8.3)
Requirement already satisfied: wcwidth in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from prompt-toolkit!
=3.0.37,<3.1.0,>=3.0.30->ipython>=7.23.1->ipykernel->notebook>=6.0->rise)
(0.2.5)
Requirement already satisfied: executing in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from stack-
data->ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.8.3)
Requirement already satisfied: asttokens in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from stack-
data->ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (2.0.5)
Requirement already satisfied: pure-eval in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from stack-
data->ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.2.2)

[notice] A new release of pip is
available: 23.3.2 -> 24.0
[notice] To update, run:
pip install --upgrade pip

Requirement already satisfied: tinycss2 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbconvert>=5->notebook>=6.0->rise) (1.2.1)
Requirement already satisfied: fastjsonschema in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbformat->notebook>=6.0->rise) (2.16.2)
Requirement already satisfied: jsonschema>=2.6 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
nbformat->notebook>=6.0->rise) (4.17.3)
Requirement already satisfied: ptyprocess in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
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terminado>=0.8.3->notebook>=6.0->rise) (0.7.0)
Requirement already satisfied: argon2-cffi-bindings in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
argon2-cffi->notebook>=6.0->rise) (21.2.0)
Requirement already satisfied: comm>=0.1.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipykernel->notebook>=6.0->rise) (0.1.2)
Requirement already satisfied: debugpy>=1.6.5 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipykernel->notebook>=6.0->rise) (1.6.7)
Requirement already satisfied: ipython>=7.23.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipykernel->notebook>=6.0->rise) (8.15.0)
Requirement already satisfied: matplotlib-inline>=0.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipykernel->notebook>=6.0->rise) (0.1.6)
Requirement already satisfied: psutil in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipykernel->notebook>=6.0->rise) (5.9.0)
Requirement already satisfied: backcall in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.2.0)
Requirement already satisfied: decorator in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (5.1.1)
Requirement already satisfied: jedi>=0.16 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.18.1)
Requirement already satisfied: pickleshare in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.7.5)
Requirement already satisfied: prompt-toolkit!=3.0.37,<3.1.0,>=3.0.30 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (3.0.36)
Requirement already satisfied: stack-data in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.2.0)
Requirement already satisfied: pexpect>4.3 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (4.8.0)
Requirement already satisfied: attrs>=17.4.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
jsonschema>=2.6->nbformat->notebook>=6.0->rise) (23.1.0)
Requirement already satisfied: pyrsistent!=0.17.0,!=0.17.1,!=0.17.2,>=0.14.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
jsonschema>=2.6->nbformat->notebook>=6.0->rise) (0.18.0)
Requirement already satisfied: anyio<4,>=3.1.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from jupyter-
```

```
server>=1.8->nbclassic>=0.4.7->notebook>=6.0->rise) (3.5.0)
Requirement already satisfied: websocket-client in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from jupyter-
server>=1.8->nbclassic>=0.4.7->notebook>=6.0->rise) (0.58.0)
Requirement already satisfied: cffi>=1.0.1 in
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bindings->argon2-cffi->notebook>=6.0->rise) (1.15.1)
Requirement already satisfied: soupsieve>1.2 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
beautifulsoup4->nbconvert>=5->notebook>=6.0->rise) (2.4)
Requirement already satisfied: webencodings in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
bleach->nbconvert>=5->notebook>=6.0->rise) (0.5.1)
Requirement already satisfied: idna>=2.8 in
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(3.4)
Requirement already satisfied: sniffio>=1.1 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
anyio<4,>=3.1.0->jupyter-server>=1.8->nbclassic>=0.4.7->notebook>=6.0->rise)
(1.2.0)
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cffi>=1.0.1->argon2-cffi-bindings->argon2-cffi->notebook>=6.0->rise) (2.21)
Requirement already satisfied: parso<0.9.0,>=0.8.0 in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from
jedi>=0.16->ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.8.3)
Requirement already satisfied: wcwidth in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from prompt-toolkit!
=3.0.37,<3.1.0,>=3.0.30->ipython>=7.23.1->ipykernel->notebook>=6.0->rise)
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Requirement already satisfied: executing in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from stack-
data->ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.8.3)
Requirement already satisfied: asttokens in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from stack-
data->ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (2.0.5)
Requirement already satisfied: pure-eval in
/home/serge/anaconda3/envs/ML/lib/python3.11/site-packages (from stack-
data->ipython>=7.23.1->ipykernel->notebook>=6.0->rise) (0.2.2)

[notice] A new release of pip is
available: 23.3.2 -> 24.0
[notice] To update, run:
pip install --upgrade pip
```

1

Aqtivate Workshop

1.0.1

Lecture 5: Bayesian Methods

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WiSe 2023/24

1.1

What are Bayesian Methods?

- Bayesian Methods are a statistical approach that applies the Bayes theorem to update the probability of a hypothesis as more evidence becomes available.
- It contrasts with classical statistics by incorporating prior knowledge alongside new data.

1.2

Probability rules

- *Product rule* : conjunction of two events A and B

$$P(A \cap B) = P(A, B) = \underbrace{P(A \mid B)}_{\text{conditional probability}} \cdot P(B) = P(B \mid A) \cdot P(A)$$

- *Sum rule* : disjunction of two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

1.3

Marginalization (third rule)

- Discrete case:

If events $B_1, ... B_N$ are mutually exclusive (cannot occur at the same time) with $\sum_{i=1}^{N} P(B_i) = 1$, then

$$P(A) = \sum_{i=1}^{N} P(A, B_i) = \sum_{i=1}^{N} P(A|B_i)P(B_i)$$

- Continous case: $\int_{\Omega} P(B)\, dB = 1$

$$P(A) = \int_{\Omega} P(A, B)\, dB = \int_{\Omega} P(A|B)P(B)$$

1.4

Marginalization example

- Factory Xs bulbs work for over 5000 hours in 99% of cases. Event: $P(A|B_x)$
- Factory Y's bulbs work for over 5000 hours in 95% of cases. Event: $P(A|B_y)$

It is known that factory$X$ supplies 60% of the total bulbs available (event B_x) and Y supplies 40% (event B_y).

What is the chance that a purchased bulb will work for longer than 5000 hours (event A)?

- Applying marginalization, we have:

$$P(A) = P(A \mid B_x)P(B_x) + P(A \mid B_y)P(B_y) =$$

$$\frac{99}{100} \cdot \frac{6}{10} + \frac{95}{100} \cdot \frac{4}{10} = \frac{594 + 380}{1000} = \frac{974}{1000} = 97.4\%$$

1.5

Bayes theorem

- Following the product rule of probabilities:

$$P(B \mid A) \cdot P(A) = P(A \mid B) \cdot P(B) \quad \Rightarrow$$

- Bayes theorem

$$\boxed{P(B \mid A) = \frac{P(A \mid B) \cdot P(B)}{P(A)}}$$

- Appying continous marginalization and product rule in the denominator:

$$P(B \mid A) = \frac{P(A \mid B) \cdot P(B)}{\int_{\Omega} P(A, B)\, dB} \Rightarrow$$

- Extended Bayes theorem

$$\boxed{P(B \mid A) = \frac{P(A \mid B) \cdot P(B)}{\int_{\Omega} P(A \mid B) \cdot P(B)\, dB}}$$

1.6

Bayes theorem

- Bayes' Theorem is a simple mathematical formula, that describes how to update the probabilities of hypoth
- Given a hypothesis H and evidence E.

$$\underbrace{P(H \mid E)}_{\text{posterior}} = \frac{\overbrace{P(E \mid H)}^{\text{likelihood}} \cdot \overbrace{P(H)}^{\text{prior}}}{\underbrace{P(E)}_{\text{evidence}}}$$

In other words:

$$\boxed{\text{posterior} \propto \text{likelihood} \times \text{prior}}$$

1.7

Bayesian analysis

- It's about updating our beliefs in the light of new evidence:
  - starting with a prior belief (prior probability)
  - then incorporating new data (likelihood)
  - to form an updated belief (posterior probability)

1.8

Example task

- Given (observed) data $\mathcal{X} = \{x_0, x_1, \ldots, x_N\}$, where $\mathcal{X} \in \mathbb{R}^N$

[3]:
```
import numpy as np
np.random.seed(42)

# ground truth prior distibutions
mu = np.random.normal(loc=0, scale=1)
sigma = 0.5 # Dirac distribution

print(f"mu: {mu:0.2f} std: {sigma}")

# likelihood ()dic
X = np.random.normal(loc=mu, scale=sigma, size=128)
X[:10]
```

```
mu: 0.50 std: 0.5
```

[3]:
```
array([0.427582  , 0.82055842, 1.25822908, 0.37963747, 0.37964567,
       1.28632056, 0.88043152, 0.26197696, 0.76799417, 0.26500531])
```

1.9

Example task cont'd

- Task is to find a distribution and its parameters, which fits the observed data by appliying the Bayes theore

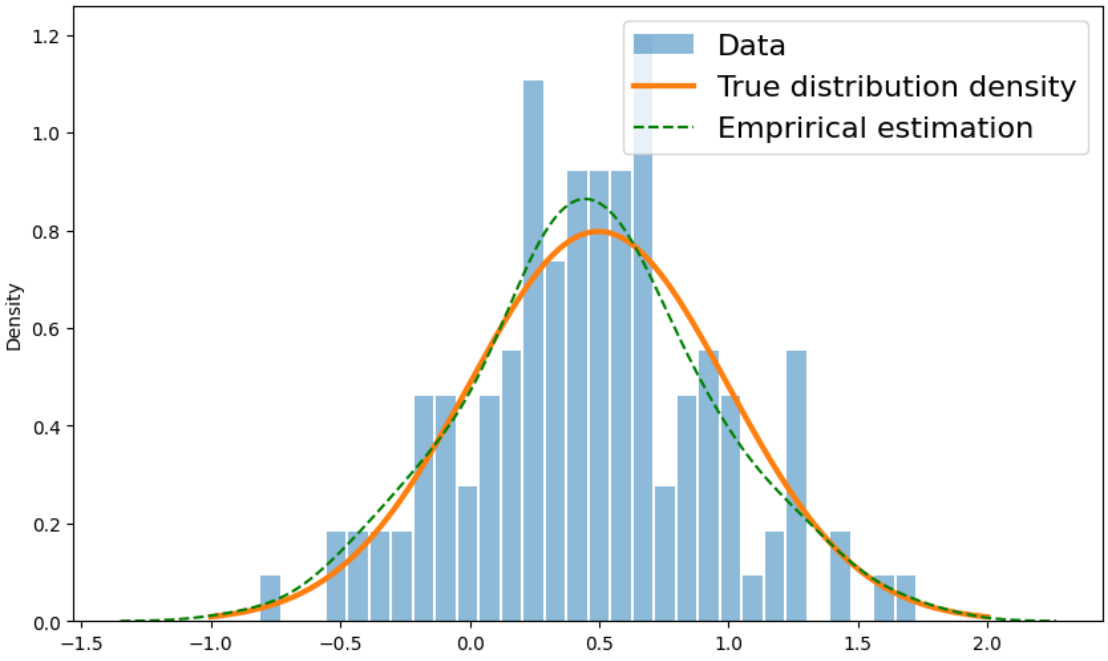
[4]:
```
import matplotlib.pyplot as plt
from scipy.stats import norm
from seaborn import kdeplot

ax = plt.figure(figsize=(10, 6)).gca()

space = np.linspace(-1, 2, 1000)
ax.hist(X, bins=30, rwidth=0.9, alpha=0.5, density=True, label='Data')

ax.plot(space, norm.pdf(space, mu, sigma), linewidth=3, label='True␣
 ↪distribution density')
kdeplot(X, ax=ax, label="Emprirical estimation", c='g', linestyle='--')

ax.legend(fontsize=16);
```



1.10

Bayesian modeling (example)

- **Model distribution** (likelihood) is assumed to be an univariate normal (Gaussian).

$$p(x_i \mid \mu, \sigma) = \mathcal{N}(x_i \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \frac{(x-\mu)^2}{2\sigma^2} \Big)$$

- $**Parameters:** w = { ,> },> where R, R_{+} $
- All data points assumed to be iid given the parameters.

$$p(\mathcal{X} \mid w) = \prod_{i=1}^{N} p(x_i \mid \mu, \sigma = 1) = \prod_{i=1}^{N} \mathcal{N}(x_i \mid \mu, \sigma = 1)$$

- $**Prior distribution** (over parameters) $

$$p(\mu) = \mathcal{N}(\mu \mid \underbrace{m = 0, s = 1}_{\text{hyperparameters}}), \quad p(\sigma) = Dirac(\sigma \mid \underbrace{m = 0.5}_{\text{hyperparameter}})$$

1.11

Bayes posterior distribution

$$\overbrace{\underbrace{p(w \mid \mathcal{X})}_{\text{must compute}} \cdot \overbrace{p(\mathcal{X})}^{\text{marginal}}}^{\text{posterior}} = \overbrace{\underbrace{p(w, \mathcal{X})}_{\text{given}}}^{\text{joint}} = \overbrace{p(\mathcal{X} \mid w)}^{\text{likelihood}} \cdot \overbrace{p(w)}^{\text{prior}}$$

1.12

Marginal likelihood

- Normalization factor of a posterior distribution

$$p(\mathcal{X}) = \int p(w, \mathcal{X})\, dw = \int p(\mathcal{X} \mid w) \cdot p(w)\, dw =$$

$$\prod_{i=1}^{N} \int p(x_i \mid \mu, \sigma) \cdot p(\mu) \cdot p(\sigma)\, d\mu\, d\sigma =$$

$$\prod_{i=1}^{N} \int \mathcal{N}(x_i \mid \mu, \sigma) \cdot \mathcal{N}(\mu | m = 0, s = 1) \cdot Dirac(\sigma \mid m = 1)\, d\mu\, d\sigma$$

1.13

Bayes estimator

- Expected value of the posterior distribution

$$\boxed{\widehat{w} = \mathbb{E}_{w \sim p(w \mid \mathcal{X})}[w] = \int w \cdot p(w \mid \mathcal{X})\, dw}$$

- Discrete example: Expected value of eyes throw for a perfect dice

$$\hat{x} = \mathbb{E}_{x \sim p(x \mid \mathcal{X})}[x] = \sum_{n=1}^{6} n \cdot \underbrace{p(n \mid \mathcal{X})}_{\text{perfect dice}} = \frac{1}{6} \sum_{n=1}^{N} n = 3,5$$

1.14

Predictive distribution

$$p(x_{new} \mid \mathcal{X}) = \int p(x_{new}, w \mid \mathcal{X})\, dw = \underbrace{\int p(x_{new}, \mid w, \mathcal{X})}_{x_{new} \perp\!\!\!\perp \mathcal{X}} \cdot \underbrace{p(w \mid \mathcal{X})}_{\text{posterior}}\, dw$$

$$p(x_{new} \mid \mathcal{X}) = \int \underbrace{p(x_{new} \mid w)}_{\text{likelihood}} \cdot \underbrace{p(w \mid \mathcal{X})}_{\text{posterior}}\, dw$$

1.15

Coin flipping posterior example

- $Observed data $

[5]:
```
theta_real = 0.35 # true p value for the Binomial distribution

ns = [0, 1, 2, 3, 4, 8, 16, 32, 50, 150, 210, 270, 330] # growing number of␣
↪trials

heads = [0, 1, 1, 1, 1, 4, 6, 9, 13, 48, 78, 96, 118]  # number of the observed␣
↪heads (k)
```

- Model distribution (likelihood)

$$p(h \mid n, \theta) = \text{Binomial}(h \mid n, \theta) = \binom{n}{h} \theta^h (1-\theta)^{n-h}$$

- Prior distribution

$$p(\theta|\alpha, \beta) = \text{Beta}(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

1.16

Binomial distribution (discrete)

- Number of successes in a sequence of n independent experiments, where p is the probability of a success.

[6]:
```
from scipy.stats import binom

n = 40 # num trials
p = 0.5 # probability of a success
n_experimensts = 10
```

```
k = binom.rvs(n, p , size=n_experimensts) # number of successes e.g coin flips head
k
```

[6]:
```
array([21, 13, 20, 18, 21, 17, 22, 19, 25, 17])
```

1.17

Binomial probability mass function

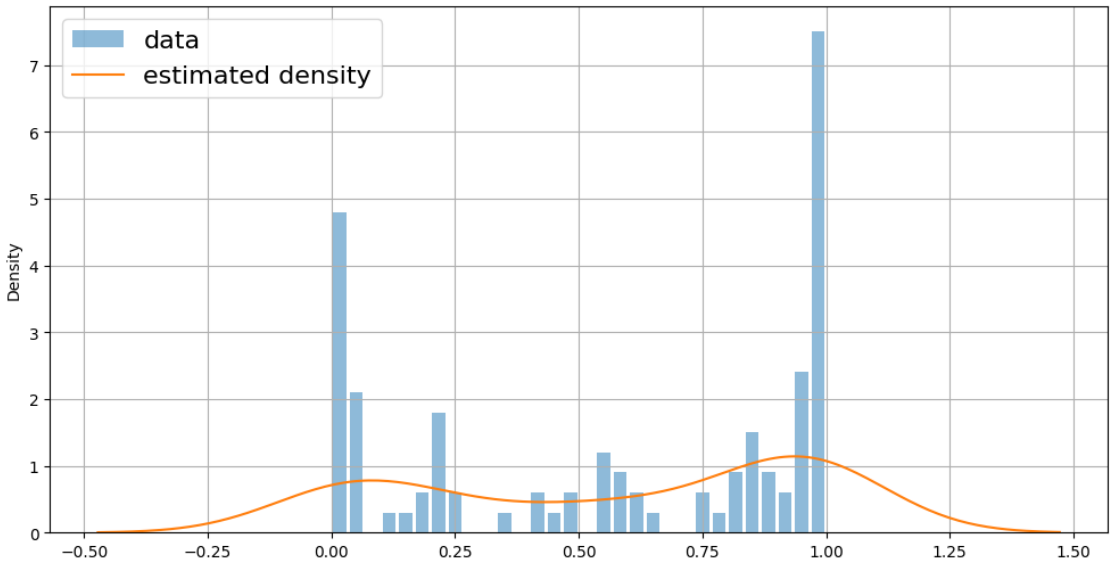
$$p(k \mid n, p) = \text{Binomial}(k \mid n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

1.18

Beta distribution

[27]:
```
from scipy.stats import beta

a, b = .4, .3
samples = beta.rvs(a, b, size=100)
ax = plt.figure(figsize=(12, 6)).gca()
ax.hist(samples, bins=30, rwidth=.8, alpha=0.5, density=True, label='data')
kdeplot(samples, label='estimated density', ax=ax)
ax.legend(fontsize=16)
ax.grid()
```



1.19

Beta distribution

- Beta PDF: $\alpha > 0$ and $\beta > 0$ are shape parameters

$$p(x \mid \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathcal{B}(\alpha, \beta)}, \quad x \in (0, 1)$$

where

$$\mathcal{B}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

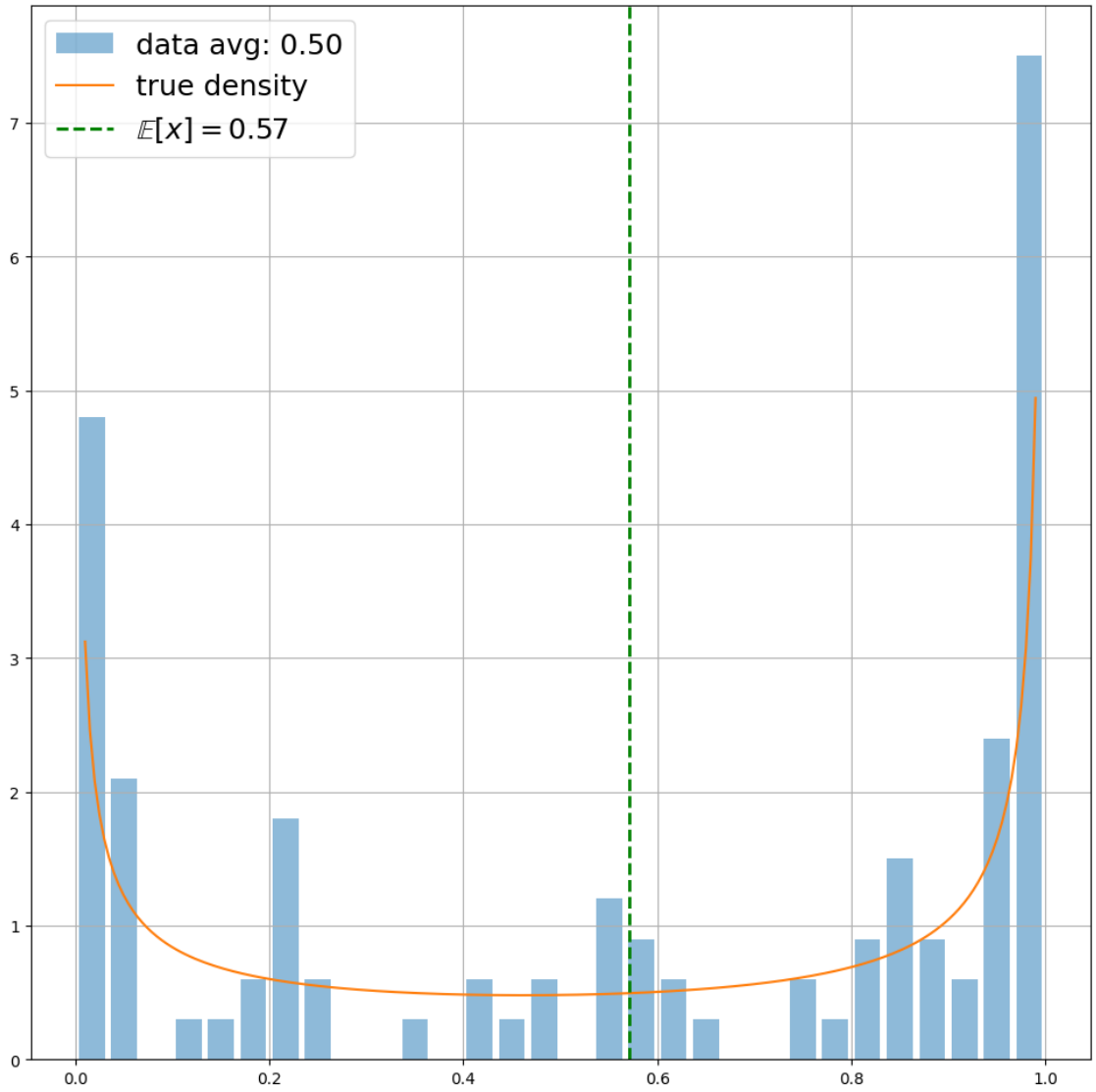
[28]:
```
from scipy.special import loggamma

def robust_beta_pdf(x, a, b):
    ln_B = loggamma(a) + loggamma(b) - loggamma(a + b)
    ln_ = (a - 1)*np.log(x) + (b - 1)*np.log(1 - x)
    return np.exp(ln_- ln_B)

np.allclose(robust_beta_pdf(rvs, a, b), beta.pdf(rvs, a, b))
```

[28]:
```
True
```

[29]:
```
ax = plt.figure(figsize=(12, 12)).gca()
ax.hist(samples, bins=30, rwidth=.8, alpha=0.5, density=True, label='data avg:␣
 ↪'+f'{rvs.mean():0.2f}')
ax.plot(np.linspace(0.01, 0.99, 200), robust_beta_pdf(space, a, b), label='true␣
 ↪density')
E = a / (a + b)
plt.axvline(E, linestyle='--', linewidth=2, color='g', label='$\\mathbb{E}[x]␣
 ↪=$'+f'{E:0.2f}')
plt.legend(fontsize=18)
plt.grid()
```



1.20

Coin flipping posterior

$$\overbrace{p(\theta|h,\alpha,\beta)}^{\text{posterior}} = \overbrace{\theta^h(1-\theta)^{n-h}}^{\text{likelihood}} \overbrace{\theta^{\alpha-1}(1-\theta)^{\beta-1}}^{\text{prior}} \cdot const$$

- where

$$const = \frac{\binom{n}{h}}{B(\alpha,\beta)}$$

- after rearranging the term looks like a Beta distribution.

$$p(\theta \mid h, \alpha, \beta) \propto \theta^{h+\alpha-1}(1-\theta)^{n-h+\beta-1}$$

$$p(\theta \mid h, \alpha, \beta) = \text{Beta}(\hat{\alpha}, \hat{\beta}), \quad \text{where} \quad \hat{\alpha} = h + \alpha, \quad \hat{\beta} = n - h + \beta$$

1.21

Coin flipping

[30]:
```
beta_prior_params = [(1, 1), (0.1, 0.1), (10, 10)] # multiple a and b
```

[31]:
```
def posterior(h, n, a, b):
    a_hat = a + h
    b_hat = b + n - h
    return beta.pdf(space, a_hat, b_hat), a_hat, b_hat
```

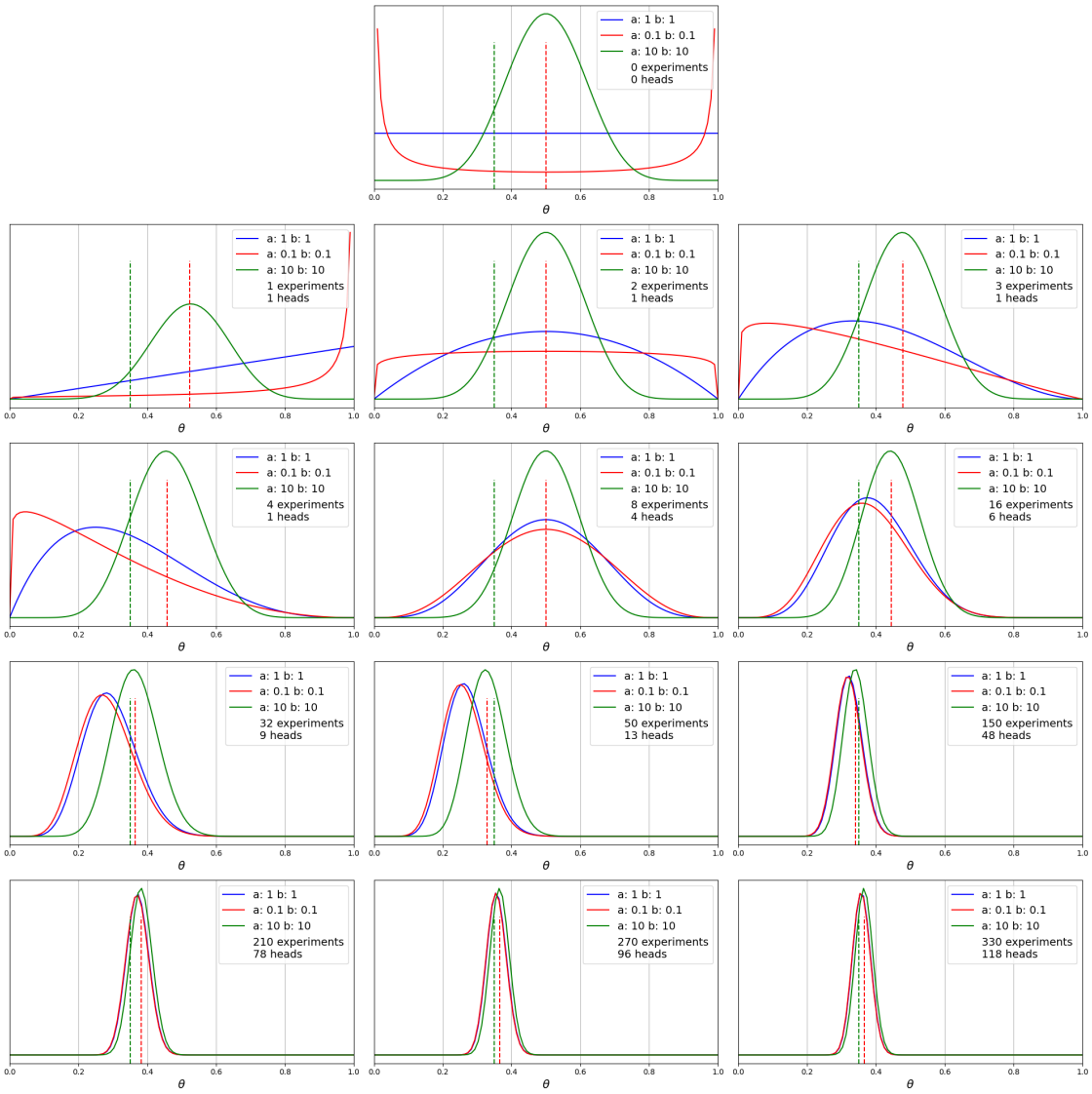
[32]:
```
space = np.linspace(0, 1, 100)

fig = plt.figure(figsize=(20,20))
for idx, n in enumerate(ns):
    if idx == 0:
        plt.subplot(5, 3, 2)
    else:
        plt.subplot(5, 3, idx+3)
    h = heads[idx]
    for (a, b), c in zip(beta_prior_params, ('b', 'r', 'g')):
        post, a_hat, b_hat = posterior(h, n, a, b)
        plt.plot(space, post, c, label=f'a: {a} b: {b}')
        plt.grid(axis='x')

    plt.axvline(a_hat/(a_hat + b_hat), ymax=.8, c='r', linestyle='--')
    plt.axvline(theta_real, ymax=0.8, color='g', linestyle='--')
    plt.plot(0, 0, label="{:d} experiments\n{:d} heads".format(n,h), alpha=0)
    plt.xlim(0,1)
    plt.xlabel(r'$\theta$', fontsize=15)
    plt.legend(fontsize=14)
    plt.yticks([])
    plt.tight_layout()

E_value = a_hat/(a_hat + b_hat) # aka Bayes estimator
print()
```

[32]:
```
0.3657142857142857
```



1.22

Conjugacy

- In the coin example the selection of the prior distribution was not made arbitrary.
- In order to get a closed-form for your posterior you've to pick a distribution, which depends on your model distribution.
- Such distribution is called conjugate.
- Then your posterior will have the same form as your conjugate prior.

$$\overbrace{Beta}^{\text{posterior}} = \overbrace{Bernoulli}^{\text{model (likelihood)}} \cdot \overbrace{Beta}^{\text{conj. prior}}$$

Check by yourself:

table of conjugate distributions

Thank you for your attention !