Lecture 1: Optimization problems and challenges in Lagrangian turbulence

Optimal policies for Lagrangian turbulence – Dr. Robin Heinonen

Aqtivate workshop on data-driven and model-based tools for complex flows and complex fluids

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Short course outline

- Introduce basics of Lagrangian turbulence phenomenology
- Introduce two optimization problems in Lagrangian turbulence: active microswimmers and olfactory search
- Formulate problems as Markov Decision Processes
- Discuss solution of problems using various approaches: optimal control theory, dynamic programming, reinforcement learning
- Hands-on: build heuristic policies for olfactory search

What is Lagrangian turbulence?

Eulerian description

- Study **fields** specified as functions of positions and time, e.g. $\mathbf{u}_E(\mathbf{x}, t), c(\mathbf{x}, t)$
- Fields are solutions of PDEs:

$$\partial_t \mathbf{u}_E + (\mathbf{u}_E \cdot \nabla)\mathbf{u}_E = -\nabla p + \nu \nabla^2 \mathbf{u}_E + f(\boldsymbol{x}, t)$$
$$\nabla \cdot \mathbf{u}_E = 0$$
$$\partial_t c + \nabla \cdot (c\mathbf{u}_E) = \kappa \nabla^2 c + S(\mathbf{x}, t)$$

- "Laboratory frame" description

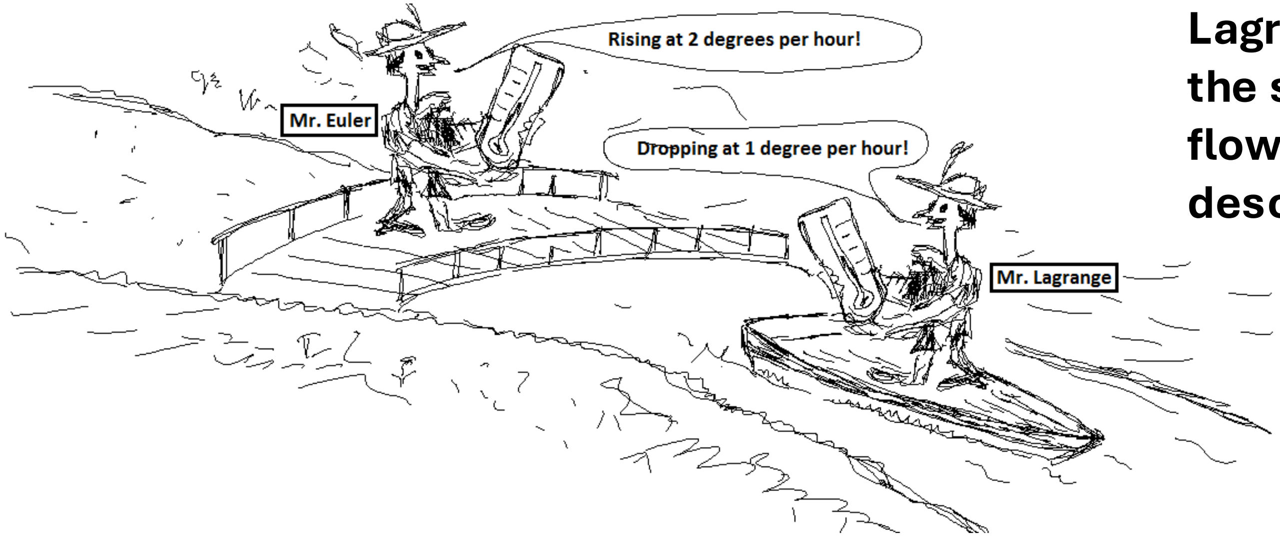
Lagrangian description

- Study properties of infinitesimal **parcels** of fluids as they are advected by the flow
- Parcel position $\mathbf{X}$ is solution to equation of motion

$$\partial_t \mathbf{X}(t; \mathbf{x}_0) = \mathbf{u}_L(t; \mathbf{x}_0)$$
with $\mathbf{u}_L(t; \mathbf{x}_0) = \mathbf{u}_E(\mathbf{X}(t; \mathbf{x}_0), t)$

- "Co-moving frame" description

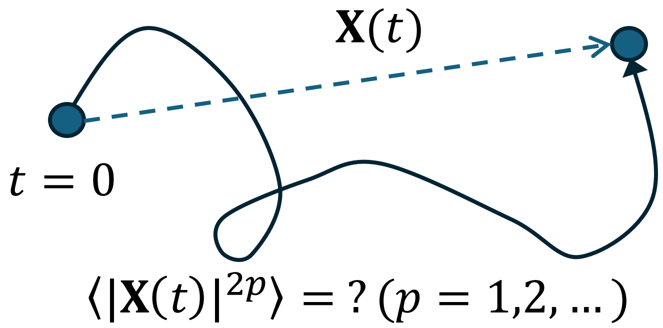
Euler is measuring $\partial_t T$ and Lagrange is measuring $\frac{DT}{Dt} = \partial_t T + (\mathbf{u} \cdot \nabla)T$

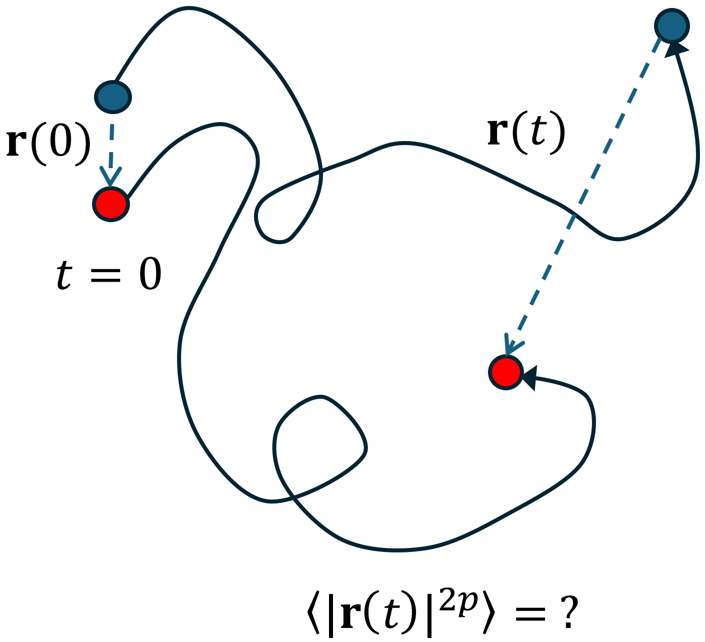


Lagrangian turbulence: the study of turbulent flows in the Lagrangian description

Typical problems in Lagrangian turbulence

- Trajectories and statistics of single tracer particles
- Dispersion of two or more tracer particles
- Passive scalar advection and plumes
- Control and navigation of microswimmers

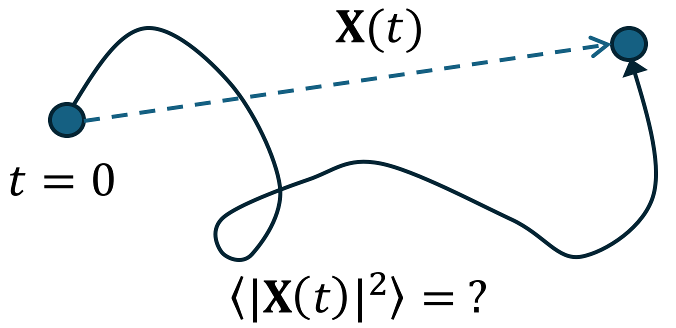




Single-particle dispersion

- G.I. Taylor (1921): long-time limit for single-particle dispersion

$$\frac{d}{dt}(\mathbf{X}\cdot\mathbf{X}) = 2\,\mathbf{X}\cdot\mathbf{u}_L$$
$$= 2\,\mathbf{u}_L(t)\cdot\int_0^t dt'\,\mathbf{u}_L(t')$$



so $\frac{d}{dt}\langle\mathbf{X}\cdot\mathbf{X}\rangle = 2\int_0^t d\tau\,\underbrace{\langle\mathbf{u}_L(\tau)\cdot\mathbf{u}_L(0)\rangle}_{\text{Two-time, longitudinal Lagrangian velocity correlation function}}$

- For $t \gg \tau_L$:

$$\frac{d}{dt}\langle\mathbf{X}\cdot\mathbf{X}\rangle \simeq 2\int_0^\infty d\tau\,\langle\mathbf{u}_L(\tau)\cdot\mathbf{u}_L(0)\rangle = \text{const.}$$

So $\langle|\mathbf{X}(t)|^2\rangle \sim t$. **Effective diffusion at long times!**

N.B. for short times $t \ll \tau_L$:

$$\frac{d}{dt}\langle\mathbf{X}\cdot\mathbf{X}\rangle \simeq 2\int_0^t d\tau\,\langle\mathbf{u}_L(0)\cdot\mathbf{u}_L(0)\rangle$$
$$= 2|\mathbf{u}_L(0)|^2 t$$

So $\langle|\mathbf{X}(t)|^2\rangle \sim t^2$. **Ballistic** motion at short times, controlled by initial velocity

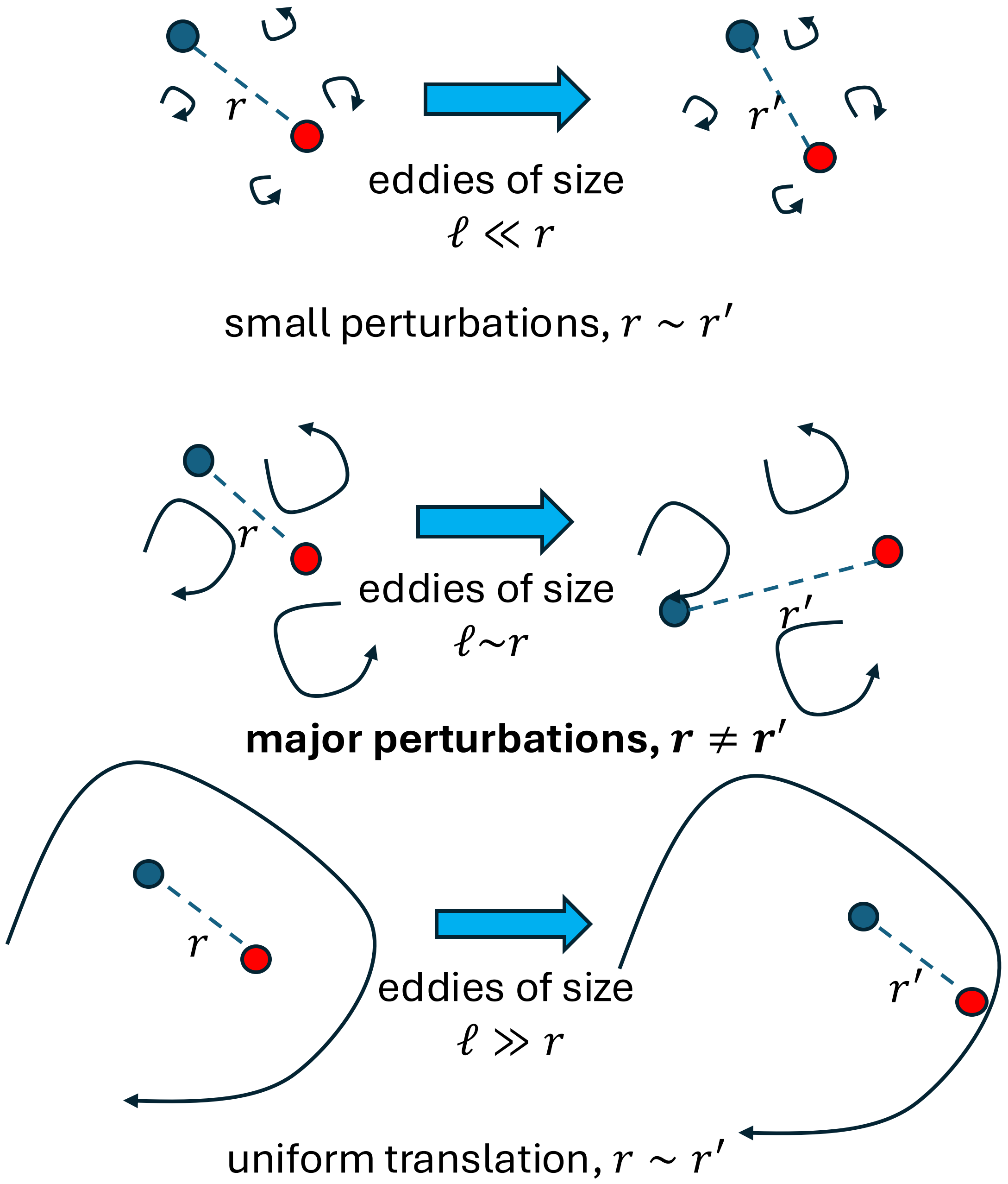
Two-particle dispersion

- If particles separated by r, only eddies of size $\ell \sim r$ effective at displacing particles
- Therefore expect $\frac{dr}{dt} = f(u_r, t)$ where u_ℓ is typical velocity at scale ℓ
- Self-similar cascade: eddies pass on their energy in one turnover time $\tau \sim \ell/u_\ell$ i.e. dissipation rate is

$$\varepsilon \sim \frac{u_\ell^2}{\ell/u_\ell} = \frac{u_\ell^3}{\ell}, \Rightarrow u_\ell \sim (\varepsilon\ell)^{1/3}$$

- Dimensional analysis: $\frac{dr}{dt} \sim u_r \sim (\varepsilon r)^{\frac{1}{3}}$

so $\frac{dr^2}{dt} \propto r^{4/3}$. **Richardson's law (1926)**



Two-particle dispersion, cont'd

- Integrates to yield $\langle r(t)^2 \rangle \sim t^3$. **Explosive separation**
- Expect to be valid only for homogeneous isotropic turbulence, long times, inertial range separation $\eta \ll r \ll L$
- Short times: ballistic $r(t) \sim t$. Leads to dependence on initial separation (Batchelor 1950)

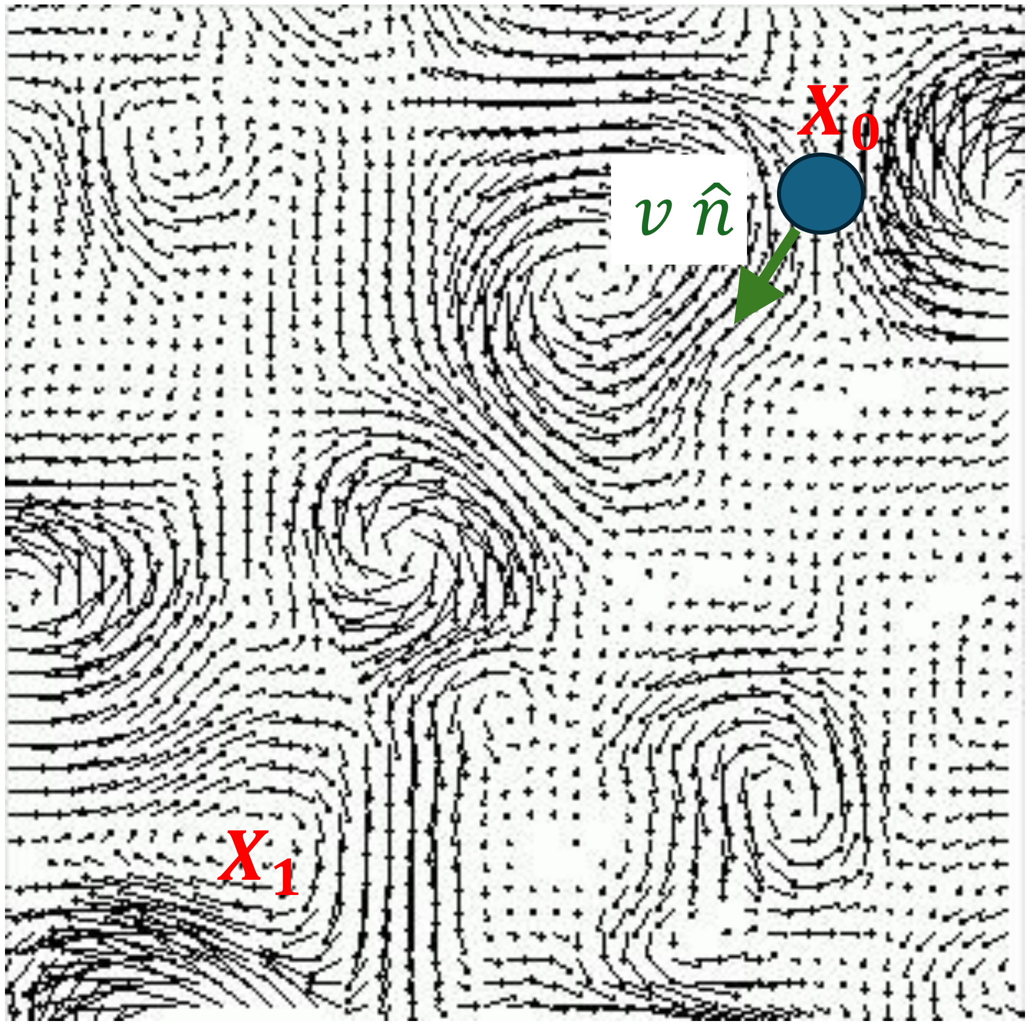
separation scale	dispersion
Sub-Kolmogorov $r \ll \eta$	$\langle r(t)^2 \rangle \sim r(0)^2 \exp(2\lambda t)$ (Batchelor 1952)
Inertial $\eta \ll r \ll L$	$t \ll \tau_L$: $\langle r(t)^2 \rangle \sim t^2$ (Batchelor 1950) $t \gg \tau_L$: $\langle r(t)^2 \rangle \sim t^3$ (Richardson 1926)
Large $r \gg L$	$\langle r(t)^2 \rangle \sim t$ (Taylor 1921)

Active microswimmers

- Generic problem: small, rigid particle moves in turbulent or otherwise **chaotic flow**
- Particle is **active**. Moves at constant speed v and we may control the direction $\hat{n}$

$$\partial_t \mathbf{x} = \mathbf{u}(\mathbf{x}(t), t) + v\,\hat{n}(t)$$

- **How to get to fixed point in minimum average time? "Zermelo's problem"**
- Assume $v \sim u_{\mathrm{rms}}$
- Note: many other variants of this problem. Different objectives, constraints, etc. (e.g. move upwards against gravity)
- Important robotics, biological applications

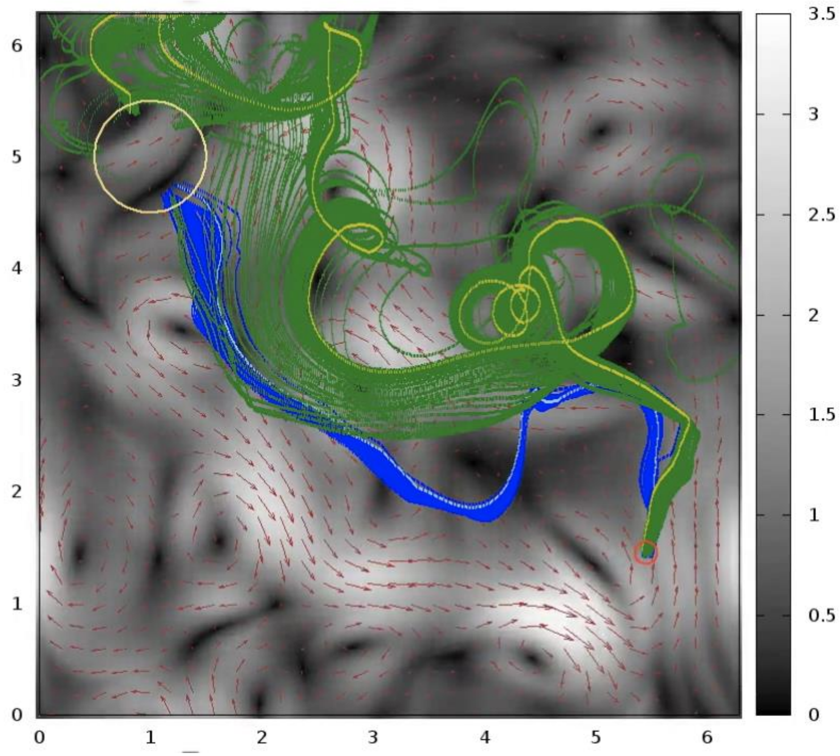


How to get from x_0 to x_1 in minimum time?

Swimming is hard

- Contribution from the flow to the motion is **chaotic** at high Re — and for many flows, also at low Re
- Small perturbations in initial position have huge consequences. “Lagrangian chaos”
- At high Re, particle trajectories conjectured to be random *even when forcing vanishes*. “Spontaneous stochasticity”
- Intermittency: turbulence is strongly non-Gaussian. Large fluctuations in velocity!

In green: particles navigating with naïve policy (swim towards goal). In blue: particles navigating with policy that has learned how to exploit the flow (we will discuss how to do this later)



Pictures from Davidson's *Turbulence* (2015)

Passive scalars: time evolution

- Concentration evolves as

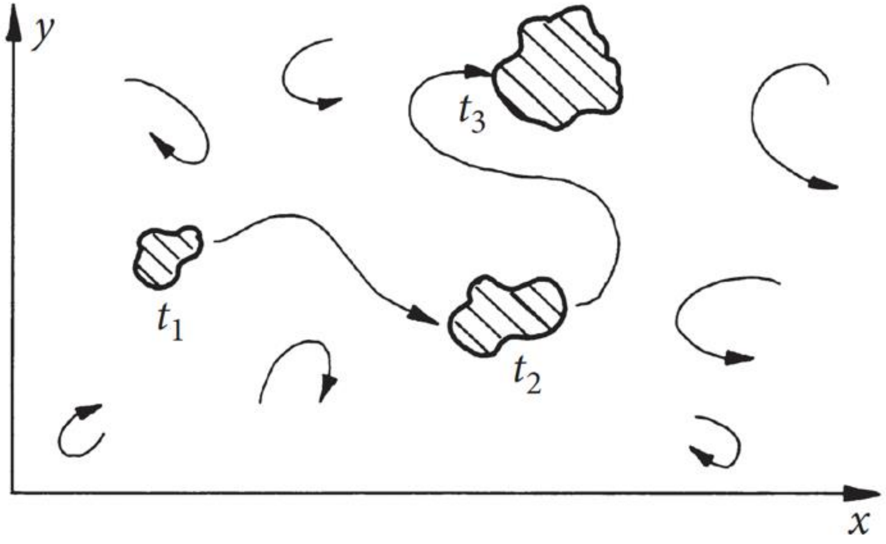
$$\partial_t c + \nabla \cdot (c\mathbf{u}) = \kappa \nabla^2 c + S(\mathbf{x}, t)$$

- Lagrangian picture: each particle evolves as

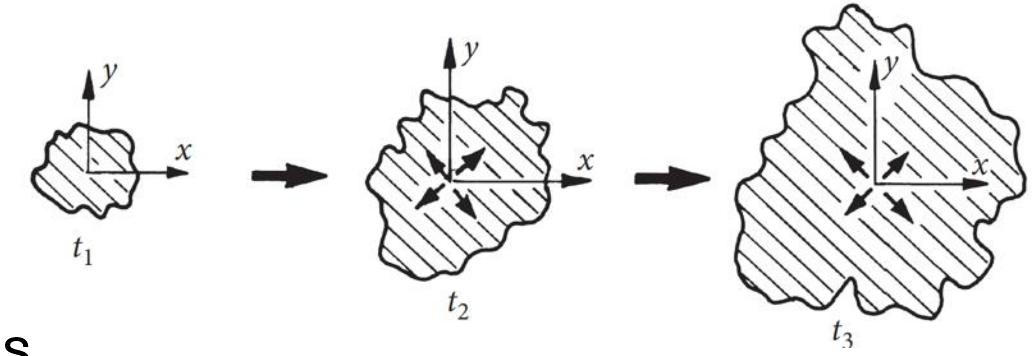
$$\partial_t \mathbf{X} = \mathbf{u}_L(t) + \boldsymbol{\eta}(t), \langle \eta_i(t)\eta_j(t')\rangle = 2\kappa\delta_{ij}\delta(t-t')$$

$$c(\mathbf{x}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{x}' p(\mathbf{X}(t) = \mathbf{x} | \mathbf{X}(t') = \mathbf{x}') S(\mathbf{x}', t')$$

- Consider patch of contaminant in HIT, κ small. Growth of patch determined by two-particle dispersion, expect $r^2 \sim t^3$
- Patch itself will move like $X^2 \sim t$
- Can also consider contaminant released from continuous source. Contamination region grows as $r^2 \sim t$



Patch of contaminant moves and grows over time



Growth of contaminant region from continuous point source

Passive scalars: fluctuations

- As passive scalar mixes into a turbulent flow, concentration is **not uniform**. How to characterize fluctuations?
- Define $\Delta c = c(\mathbf{x}+\mathbf{r},t) - c(\mathbf{x},t)$. K41-type reasoning leads (Obukhov 1949, Corrsin 1951) to

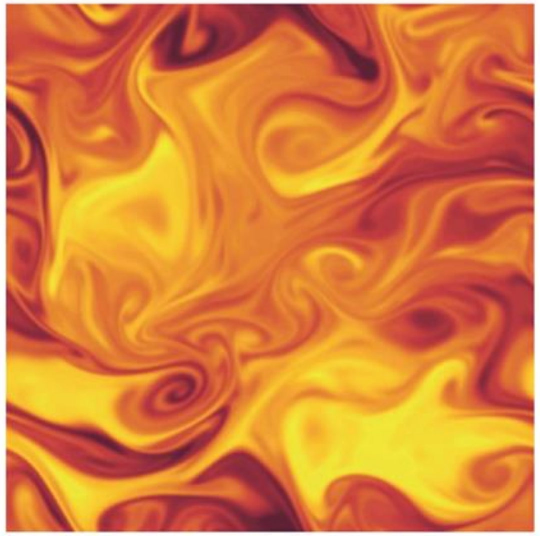
$$\langle(\Delta c)^2\rangle \sim \varepsilon_c \varepsilon^{-1/3} r^{2/3}$$

where $\varepsilon_c = \kappa\langle|\nabla c|^2\rangle$ (rate of destruction of contaminant fluctuations)

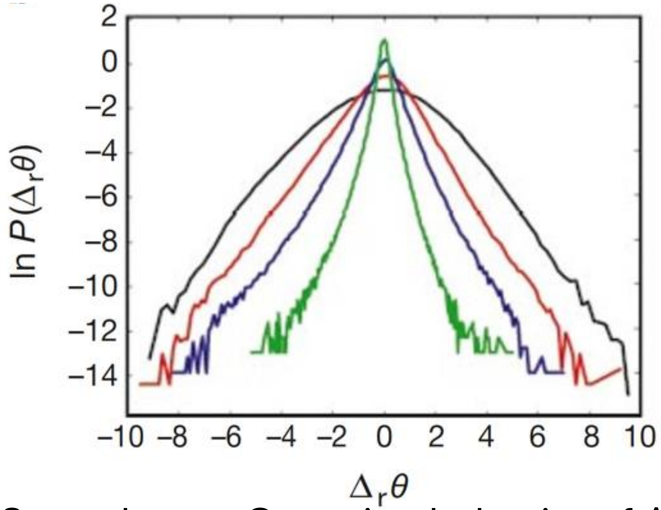
- Extending this reasoning leads to

$$\langle c^p\rangle \sim \varepsilon_c^{p/2} \varepsilon^{-p/6} \ell^{p/3}$$

- Turns out that for $p > 2$, KOC theory is very inaccurate. “anomalous scaling” — a signal of intermittency



Non-uniform mixing of passive scalar. From Davidson (2015)



Strongly non-Gaussian behavior of Δc at $r = 230\eta$ (black), $r = 96\eta$ (red), $r = 26\eta$ (purple), $r = 6.5\eta$ (green). From Shraiman and Siggia (2000)

Odor plumes

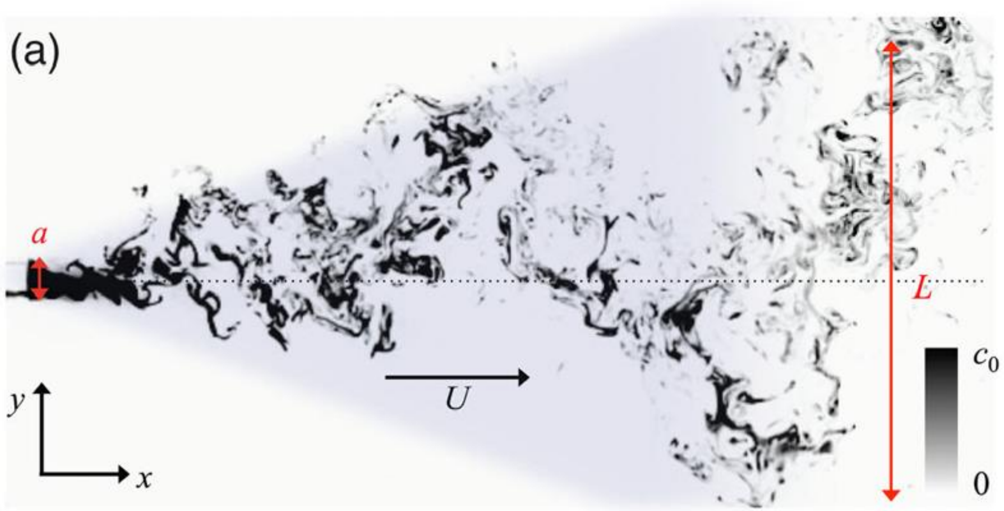
- Now consider the plume from a fixed, continuously emitting point source with uniform mean flow $U > u_{\mathrm{rms}}$
- $\langle c \rangle$ has well known Gaussian structure in space (Sutton 1932):

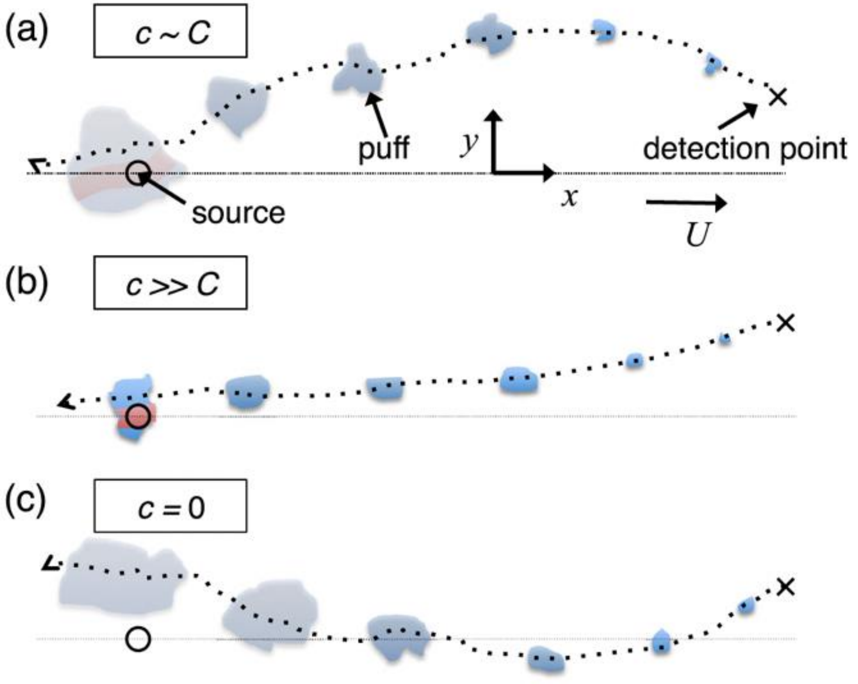
$$\langle c \rangle \sim \exp\left(-\frac{y^2 + z^2}{2\sigma(x)^2}\right)$$

But what about fluctuations/tails?

- Celani et al (2014) computed tail of pdf of $c(\mathbf{x}, t)$. Idea: follow puff backward in time
- At $t = x/U$, puff must overlap with source or else $c = 0$. Can compute $\Pr(c > 0) \sim \exp\left[-\left(\frac{Uy}{u_{\mathrm{rms}} x}\right)^2\right]$

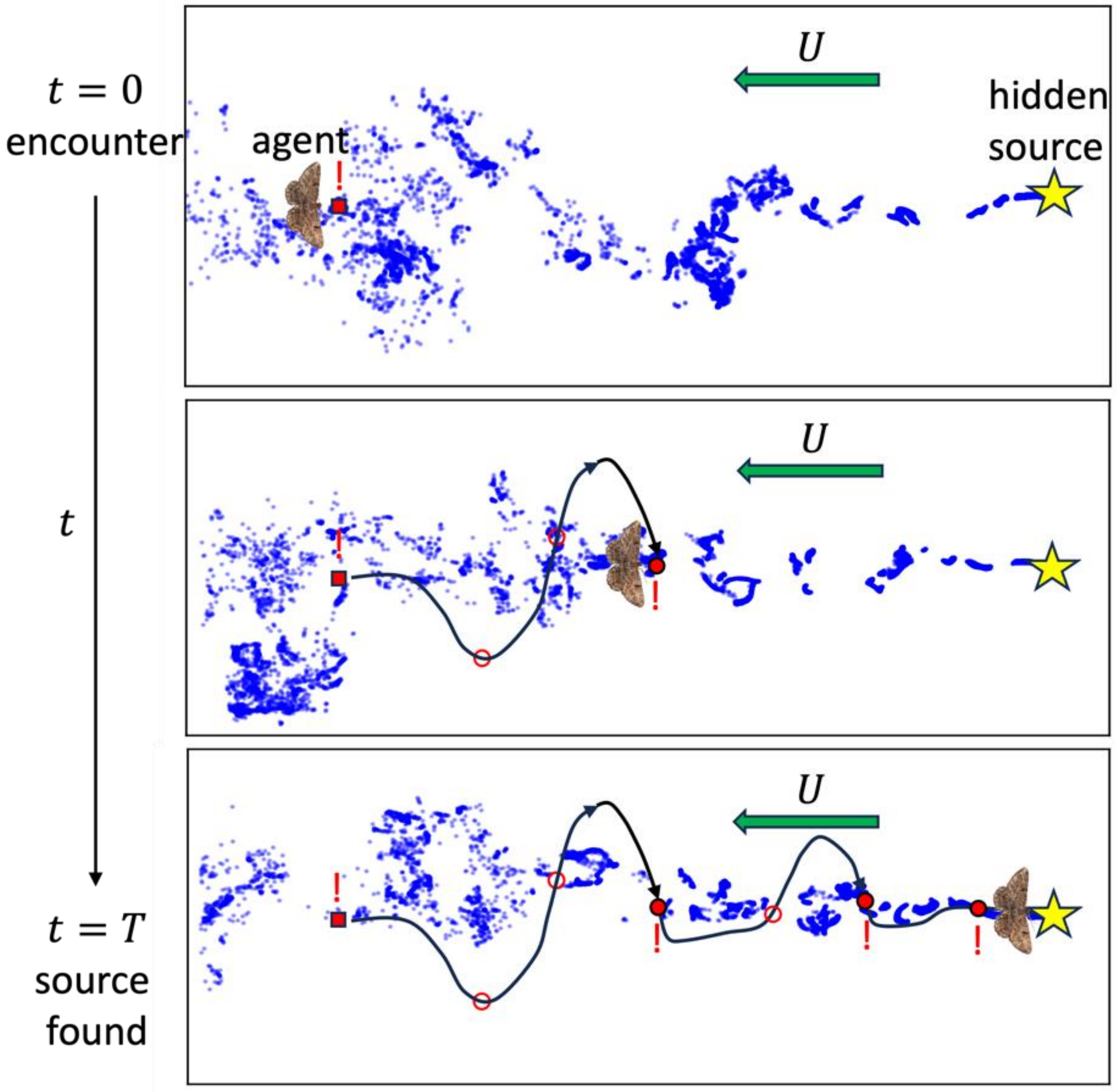
c large means puff is small at time of overlap. Can show $\Pr(c > c_{\mathrm{thr}} | c > 0) \sim e^{-x/x_0}$





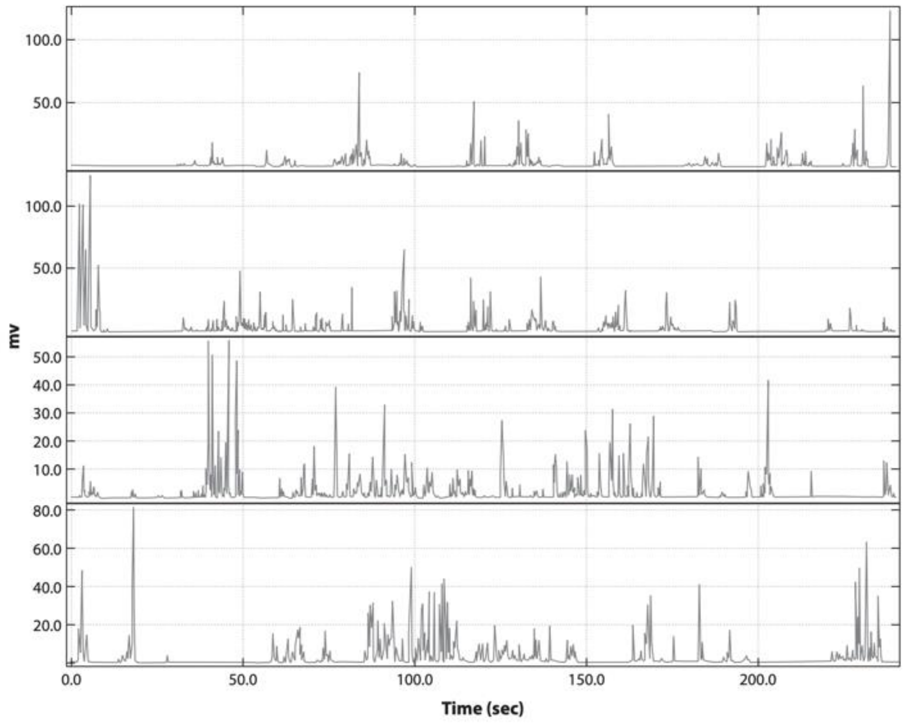
Olfactory search

- Unseen, fixed point source emits a passive scalar continuously
- At each time step, agent “sniffs,” detects odor if $c > c_{\text{thr}}$.
- Agent moves at constant speed. Strong swimmer/flyer (not advected by flow)
- **How to use observations to choose trajectory s.t. expect time of arrival is minimum?**
- Important behavior for many animals (flying insects, plankton, etc.)



Olfactory search is hard

- Observation signal is highly **intermittent**. We have already seen e.g. that $c = 0$ with finite probability
- As a result, local concentration gradients frequently do not exist instantaneously. If they do, they may point in “wrong” direction
- Odor encounters are **random**, and **rare** when far from source
- Thus the observation sequence supplies only **limited, imperfect information about the source**



Concentration time series at fixed point 50 m downwind of propylene source (Yee et al 1994)

Summary of Lecture 1

- We have introduced two problems (microswimmers, olfactory search) in Lagrangian turbulence
- We have discussed the physics of why these problems are difficult (e.g. intermittency and Lagrangian chaos)
- Soon we will formalize both problems as "Markov decision processes" as a first step towards solving them.
- But first: a brief introduction to optimal control theory, and why it is not the ideal choice for these problems

Questions?