Parallel algorithms

Overhead, time, speedup

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Sequential (serial) algorithms – 1/2

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Resources

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Task

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Resources **Task**

Cost function: Money (or time)

Sequential (serial) algorithms – 2/2

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- **Cost**: # FLOPS (floating point operations), # of swaps, # bytes (amount of data) moved, . . . , execution time
- **Cost** as a **function** of the input
  - "size of the problem", "size of the input"
    length of array (n), number of vertices and edges (V, E), . . .
  - Matrix-matrix multiplication: $O(n^3)$, . . . , $O(n^{2.37286})$
    Shortest path: $O(V^2)$, $O((E + V)\log V)$, $O(E + V\log V)$

Parallel algorithms – 1/2

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Parallel algorithms – 1/2

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Problems

1. how to split the work?
2. how to coordinate the workers?

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  p *workers; naturally, we want to exploit all of them*
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  definition for "parallel execution time" needed; more later
- **Cost** as a **function** of the input
  - Shortest path: $O(V)$ processors, $O(E+V)$
    $O(\log(V))$ processors, $O(E\bar{k}(N))$

General pattern for parallel algorithms

1. Decomposition of the problem into sub-problems

2. Parallel solution of the sub-problems

3. Composition of the sub-solutions

Example: sequential program

Data: n tests, each consisting of t exercises

Task: Grade all exercises of all tests

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- The grader does everything, in whatever order the program dictates.

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$\star$: definitions might differ

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Implicit assumption: All exercises can be graded independently

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What if this is not the case? **Dependencies** — more later

Demo: Deck sorting

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- 1 Paolo vs. 3 students
- 1 Paolo vs. 4 students

- Shared memory vs. distributed memory

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1. Decomposition of the problem into sub-problems

   Examples:
   - Data distribution
   - Task decomposition
   - Domain decomposition

2. Parallel solution of the sub-problems

3. Composition of the sub-solutions

All these steps might require

$$\left.\begin{array}{l}\blacktriangleright \text{ Extra computation} \\ \blacktriangleright \text{ Synchronization} \\ \blacktriangleright \text{ Data transfer}\end{array}\right\} \quad \text{OVERHEAD}$$

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  Cumulative cost for all workers

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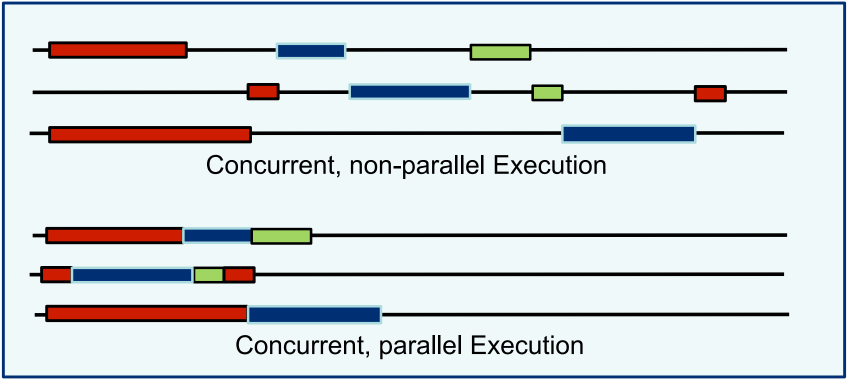
Concurrency and parallelism*



Tim Mattson

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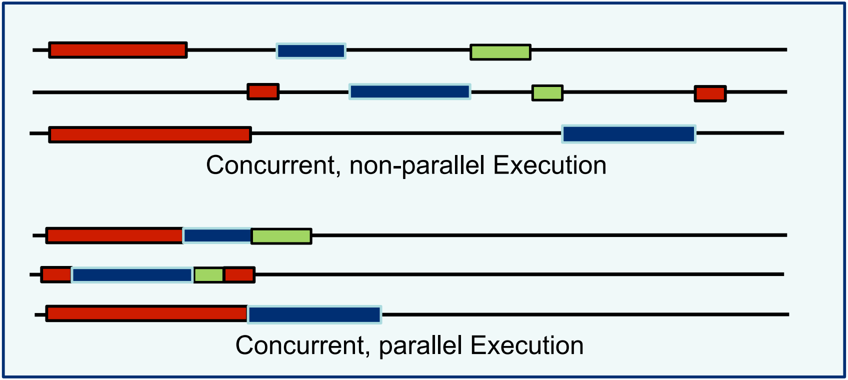

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Did you spot the mistake?

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  In practice, . . . OVERHEAD!

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Note: The sequential code could possibly implement a different algorithm than the parallel one.

Example: Eigenvalues of symmetric tridiagonal matrix

- Alg.1: `dqds` cost: $O(n^2)$, BUT inherently sequential
- Alg.2: `BX` cost: $O(n^3)$, BUT perfectly parallelizable

Speedup, Efficiency

- **What if p is large?** Then probably $T_1(n)$ is not obtainable, either because n is too large a problem to be solved sequentially, or because it would take too long to complete.

  In this case, $S_p(n) := \dfrac{T_{p_0}(n)}{T_p(n)}$, and $0 \le S_p(n) \le \frac{p}{p_0}$.

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- **Parallel efficiency**:

  $E_p(n) := \dfrac{S_p(n)}{p}$ $\quad \Big($ or $E_p(n) := \frac{S_p(n)}{p/p_0}\Big)$ $\quad 0 \le E_p(n) \le 1$

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- **Efficiency**: The ratio between the performance attained while solving a given problem and the TPP (or the PPP).

Speedup & Efficiency in real life

- `time0.c`
  Cholesky factorization.
  No timings. Only correctness.
- `time1.c`
  Timings through `clock()`.
  Multithreading (via LAPACK/BLAS). CPU-time.
- `time2a.c`
  Cycle accurate timer.
  Cycles, frequency. Wall time vs. CPU-time.
- `time2b.c`
  Performance (#ops/sec), efficiency.
- Ex. 6 from Report #1

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Example: $n = \bar{n}$; $p = 2^i$, with $i \in [10, \ldots, 14]$

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2) Scalability with respect to data (problem size).

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Example: $p = \bar{p}$; $n = 2^i \bar{n}$, with $i \in [0, \ldots, 4]$

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Question: Why two definitions of scalability (strong, weak)?
What are they useful for?

Weak Scalability – Example #1

- Algorithm $\mathcal{B}(n)$; input: $n \in \mathbf{N}$
  Time($\mathcal{B}(n)$) = $O(n^2)$ Space($\mathcal{B}(n)$) = $3n$ Reference: $T_{\bar{p}}(\bar{n}) = t_0$
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  - $T_{2\bar{p}}(2\bar{n}) = t_1$ $t_0 > / = / < t_1$?
    Assumption: perfect scalability wrt size, strong scalability
    $t_1 = T_{2\bar{p}}(2\bar{n}) \approx 4T_{2\bar{p}}(\bar{n}) \approx 4T_{\bar{p}}(\bar{n})/2 = 2t_0$

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    Space($\bar{n}$) = $3\bar{n}$ $\Rightarrow$ Mem/proc: $3\bar{n}/\bar{p} =: const$
    Space($2\bar{n}$) = $6\bar{n}$ $\Rightarrow$ $6\bar{n}/?? = const$ $\Rightarrow$ $?? = 2\bar{p}$
  - $T_{2\bar{p}}(2\bar{n}) = t_1$ $t_0 > / = / < t_1$?
    Assumption: perfect scalability wrt size, strong scalability
    $t_1 = T_{2\bar{p}}(2\bar{n}) \approx 4T_{2\bar{p}}(\bar{n}) \approx 4T_{\bar{p}}(\bar{n})/2 = 2t_0$
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Weak Scalability – Example #1

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    Space($\bar{n}$) = $3\bar{n} \Rightarrow$ Mem/proc: $3\bar{n}/\bar{p} =: const$
    Space($??$) = $3\,?? \Rightarrow$ Mem/proc: $3\,??/2\bar{p} = const \Rightarrow ?? = 2\bar{n}$

Weak Scalability – Example #2

- Algorithm $\mathcal{B}(n)$; input: $n \in \mathbf{N}$
  Time($\mathcal{B}(n)$): $O(n^2)$ Space($\mathcal{B}(n)$): n^2 Reference: $T_{\bar{p}}(\bar{n}) = t_0$
  - $T_{??}(2\bar{n})$ – from problem size to number of processors

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  Time($\mathcal{B}(n)$): $O(n^2)$ Space($\mathcal{B}(n)$): n^2 Reference: $T_{\bar{p}}(\bar{n}) = t_0$
  - $T_{??}(2\bar{n})$ – from problem size to number of processors
    Space($\bar{n}$) = $\bar{n}^2$ $\Rightarrow$ Mem/proc: $\bar{n}^2/\bar{p} =: const$

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  - $T_{??}(2\bar{n})$ – from problem size to number of processors
    Space($\bar{n}$) = $\bar{n}^2$ $\Rightarrow$ Mem/proc: $\bar{n}^2/\bar{p} =: const$
    Space($2\bar{n}$) = $4\bar{n}^2$ $\Rightarrow$ $4\bar{n}^2/?? = const$ $\Rightarrow$

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    Assumption: perfect scalability
    $t_1 = T_{4\bar{p}}(2\bar{n}) \approx 4T_{4\bar{p}}(\bar{n}) \approx 4T_{\bar{p}}(\bar{n})/4 = t_0$

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    $\Rightarrow$

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    $\Rightarrow\; ?? = \bar{n}\sqrt{2}$

Weak Scalability in real life

“A Parallel Eigensolver for Dense Symmetric Matrices Based on Multiple Relatively Robust Representations”

`https://hpac.cs.umu.se/~pauldj/pubs/PMR3.pdf`

Best possible parallel efficiency?

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1. #flops/p no dependencies

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2. Amdahl's law — seq. vs. parallel portions

Best possible parallel efficiency?

1. #flops/p — no dependencies
2. Amdahl's law — seq. vs. parallel portions
3. Length of the critical path — sequence of deps

Amdahl's law

Maximum possible speedup when only a portion of the code scales

- T_{seq} portion of the algorithm (in secs) which is strictly sequential

Amdahl's law

Maximum possible speedup when only a portion of the code scales

- T_{seq} portion of the algorithm (in secs) which is strictly sequential
- T_{par} portion of the algorithm (in secs) that can be parallelized

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- T_{seq} portion of the algorithm (in secs) which is strictly sequential
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- $\beta := \dfrac{T_{\text{seq}}}{T_{\text{seq}} + T_{\text{par}}}$ fraction of the algorithm that is strictly sequential

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- $T_1(n) := T_{\text{seq}} + T_{\text{par}}$ by definition

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- $T_p(n) := T_{\text{seq}} + T_{\text{par}}/p$ ideal parallelisation

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- $T_1(n) := T_{\text{seq}} + T_{\text{par}}$ by definition
- $T_p(n) := T_{\text{seq}} + T_{\text{par}}/p$ ideal parallelisation
- Speedup: $S_p(n) := \dfrac{T_1(n)}{T_p(n)} = \dfrac{T_{\text{seq}} + T_{\text{par}}}{T_{\text{seq}} + T_{\text{par}}/p}$
- Ideal Speedup: $\lim\limits_{p\to\infty} S_p(n) = \dfrac{1}{\beta}$

Amdahl's law

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- Speedup: $S_p(n) := \dfrac{T_1(n)}{T_p(n)} = \dfrac{T_{\text{seq}} + T_{\text{par}}}{T_{\text{seq}} + T_{\text{par}}/p}$
- Ideal Speedup: $\lim_{p \to \infty} S_p(n) = \dfrac{1}{\beta}$

Note: Counterpart of Amdahl's law for weak scalability: Gustafson's law.

Amdahl's law in real life

- `time3.c`
  Timings breakdown: Malloc, init, compute, test.
  Scalability. Serial vs. parallel code. Amdahl's law.

- "A Scalable, Linear-Time Dynamic Cutoff Algorithm for MD"

  `https://arxiv.org/pdf/1701.05242.pdf`

- Ex. 3 from Report #1

Critical path

Longest sequence of dependent tasks

$\Rightarrow$ Dependencies

On dependencies and parallel execution

- Two independent instructions can be executed "concurrently" = "in parallel" (by different resources)

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  When two instructions are executed in parallel,
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- Dependencies $\Rightarrow$ ordering
  The "right" ordering is dictated by the semantics of the program

- Some dependencies can be removed by duplicating data

True / Flow dependency

```
               {x = 1, y = 2, a = 3}


...
y := a * x + y
w := 3 * y
...

{y = 5, w = 15}
```

True / Flow dependency

```
               {x = 1, y = 2, a = 3}

...
y := a * x + y
w := 3 * y
...

{y = 5, w = 15}
```

- The value of `w` depends on the updated value of `y`

True / Flow dependency

```
            {x = 1, y = 2, a = 3}

...                        ...
y := a * x + y       |     w := 3 * y
w := 3 * y           |     y := a * x + y
...                        ...

{y = 5, w = 15}            {y = 5, w = 6}
```

- The value of `w` depends on the updated value of `y`
- The semantics of the program depends on the **order** of the statements

Anti dependency

```
                 {x = 1, y = 2, a = 3}


...
w := 3 * y
y := a * x + y
...

{y = 5, w = 6}
```

- The value of `w` depends on the initial value of `y`

Anti dependency

```
              {x = 1, y = 2, a = 3}

...                           ...
w := 3 * y             |     y := a * x + y
y := a * x + y         |     w := 3 * y
...                           ...

{y = 5, w = 6}               {y = 5, w = 15}
```

- The value of `w` depends on the initial value of `y`
- The semantics of the program depends on the **order** of the statements

Output dependency

```
              {x = 1, y = 2, a = 3}


...
w := 3 * y
w := a * x
...

{w = 3}
```

- The value of `w` depends on the order of the statements

Output dependency

```
                 {x = 1, y = 2, a = 3}

  ...                                ...
  w := 3 * y              |          w := a * x
  w := a * x              |          w := 3 * y
  ...                                ...

  {w = 3}                            {w = 6}
```

- The value of `w` depends on the order of the statements
- The semantics of the program depends on the **order** of the statements