

Mathematics

▼ Finding number of Digits in a Number

Given an integral number **N**. The task is to find the count of digits present in this number.\

Let's say:

N = 2019

Then answer would be 4, as 2019 has 4 digits ,i.e., 2,0,1, and 9

▼ Method 1

Dry-run of above algorithm: Consider an example, $N = 58964$. Initialize a variable **digitsCount** to zero which will store the count of digits. Keep incrementing *digitsCount* until N is not zero, and reduce it by dividing by 10 at each step.

```
Iteration 1: N not equals to 0
Increment digitsCount, digitsCount = digitsCount + 1.
digitsCount = 0 + 1 = 1.
N = N/10 = 58964/10 = 5896.
```

```
Iteration 2: N not equals to 0
Increment digitsCount, digitsCount = digitsCount + 1.
digitsCount = 1 + 1 = 2.
N = N/10 = 5896/10 = 589.
```

```
Iteration 3: N not equals to 0
Increment digitsCount, digitsCount = digitsCount + 1.
digitsCount = 2 + 1 = 3.
N = N/10 = 589/10 = 58.
```

```
Iteration 4: N not equals to 0
Increment digitsCount, digitsCount = digitsCount + 1.
digitsCount = 3 + 1 = 4.
N = N/10 = 58/10 = 5.
```

```
Iteration 5: N not equals to 0
Increment digitsCount, digitsCount = digitsCount + 1.
digitsCount = 4 + 1 = 5.
N = N/10 = 5/10 = 0.
```

```
Iteration 6: N becomes equal to 0.
Terminate any further operation.
Return value of digitsCount.
```

```
Therefore, number of digits = 5.
```

Analysis of above algorithm: You can clearly see that the number of operations performed in the above solution is equal to the count of digits present in the number. So, the time complexity of the solution is **O(digitsCount)**.

▼ Method 2

Better Solution: A better solution is to use mathematics to solve this problem. The number of digits in a number say N can be easily obtained by using the formula:

$$\text{number of digits in } N = \log_{10}(N) + 1.$$

Derivation: Suppose the number of digits in the number N is K.

$$10^{K-1} \leq N < 10^K$$

Applying base-10 logarithm to both sides in the above equation, we get:

$$K - 1 \leq \log_{10}(N) < K$$

$$\text{or, } K - 1 + 1 \leq \log_{10}(N) + 1 < K + 1$$

$$\text{or, } K \leq \log_{10}(N) + 1 < K + 1$$

Therefore,

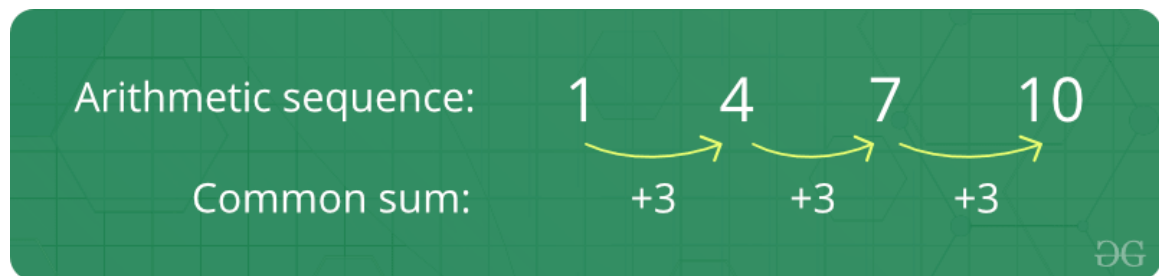
$$K = \text{floor}(\log_{10}(N) + 1)$$

Analysis of above algorithm: Since the above algorithm works in a single operation by using two mathematical operations i.e., finding logarithmic and floor value. Therefore, the time complexity of the solution is **$O(1)$** .

▼ Arithmetic and Geometric Progression

▼ Arithmetic Progression

A sequence of numbers is said to be in an **Arithmetic progression** if the difference between any two consecutive terms is always the **same**. In simple terms, it means that the next number in the series is calculated by adding a fixed number to the previous number in the series. For example, 2, 4, 6, 8, 10 is an AP because the difference between any two consecutive terms in the series (common difference) is same ($4 - 2 = 6 - 4 = 8 - 6 = 10 - 8 = 2$).

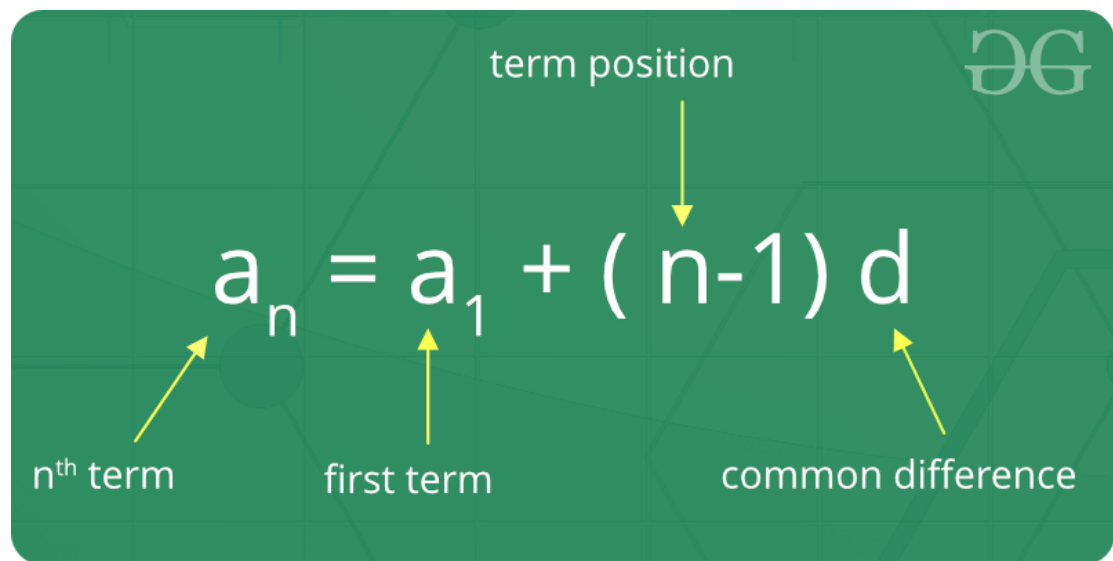


▼ Facts about Arithmetic Progression :

1. **Initial term:** In an arithmetic progression, the first number in the series is called the initial term.
2. **Common difference:** The value by which consecutive terms increase or decrease is called the common difference.
3. The behavior of the arithmetic progression depends on the common difference d . If the common difference is positive, then the members (terms) will grow towards positive infinity, but if the common difference is negative, then the members (terms) will grow towards negative infinity.

▼ **Formula of n^{th} term of an A.P:**

If 'a' is the initial term and 'd' is the common difference. Thus, the explicit formula is:

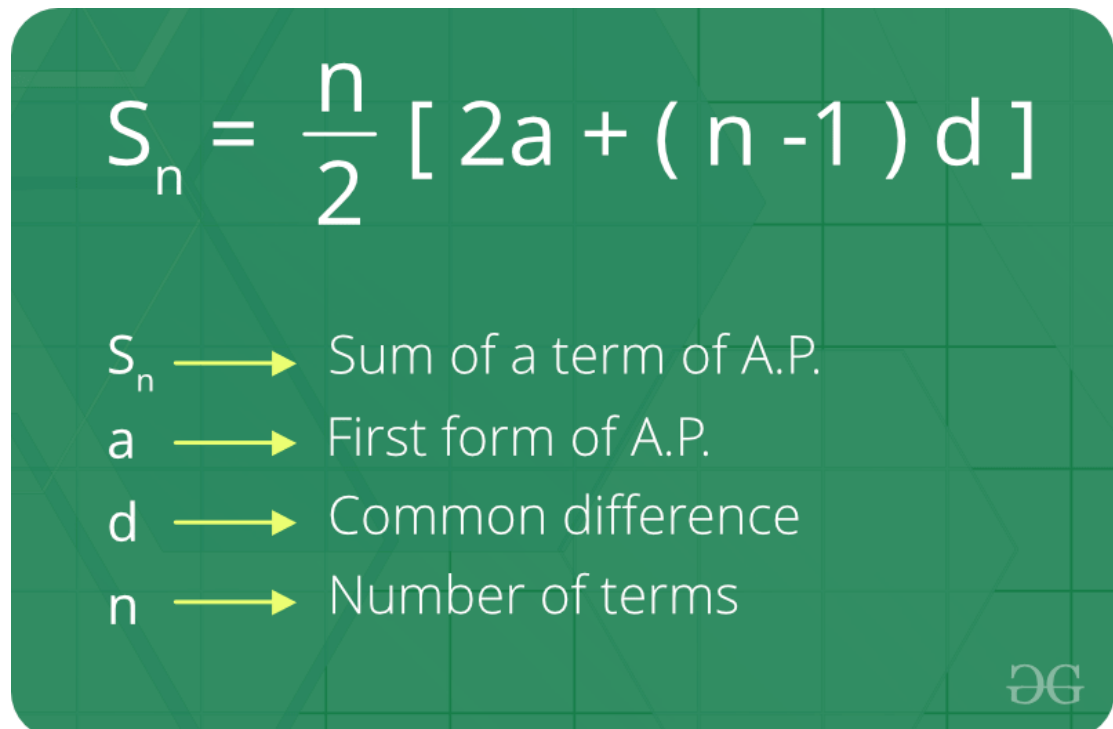


The diagram shows the formula $a_n = a_1 + (n-1)d$ on a green background. Yellow arrows point from labels to parts of the formula: 'nth term' points to a_n , 'first term' points to a_1 , 'common difference' points to d , and 'term position' points to $(n-1)$. A logo is in the top right corner.

$$a_n = a_1 + (n-1)d$$

nth term first term common difference

▼ **Formula of sum of first n term of A.P:**



The diagram shows the formula $S_n = \frac{n}{2} [2a + (n-1)d]$ on a green background. Below the formula, four definitions are listed with yellow arrows pointing from the variable to the text: S_n points to 'Sum of a term of A.P.', a points to 'First form of A.P.', d points to 'Common difference', and n points to 'Number of terms'. A logo is in the bottom right corner.

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

S_n → Sum of a term of A.P.
 a → First form of A.P.
 d → Common difference
 n → Number of terms

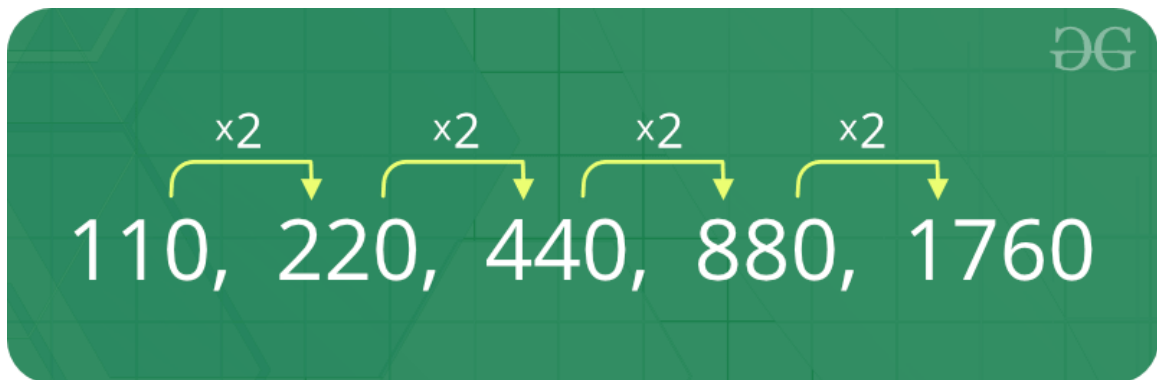
▼ General Formulas to solve problems related to Arithmetic Progressions:

If 'a' is the first term and 'd' is the common difference:

- **nth term** of an AP = $a + (n-1)d$.
- **Arithmetic Mean** = Sum of all terms in the AP / Number of terms in the AP.
- **Sum of 'n' terms** = $0.5 * n * (\text{first term} + \text{last term}) = 0.5 * n * [2a + (n-1)d]$.

▼ Geometric Progression

A sequence of numbers is said to be in a **Geometric progression** if the ratio of any two consecutive terms is always the same. In simple terms, it means that the next number in the series is calculated by multiplying a fixed number to the previous number in the series. For example, 2, 4, 8, 16 is a GP because ratio of any two consecutive terms in the series (common ratio) is the same ($4 / 2 = 8 / 4 = 16 / 8 = 2$).



▼ Facts about Geometric Progression :

1. **Initial term:** In a geometric progression, the first number is called the initial term.

2. **Common ratio:** The ratio of any two consecutive terms by taking the previous term in the denominator.
3. The behaviour of a geometric sequence depends on the value of the common ratio. If the common ratio is:
 - Positive, the terms will all be of the same sign as the initial term.
 - Negative, the terms will alternate between positive and negative.
 - Greater than 1, there will be exponential growth towards positive or negative infinity (depending on the sign of the initial term).
 - 1, the progression is a constant sequence.
 - Between -1 and 1 but not zero, there will be exponential decay towards zero.
 - -1, the progression is an alternating sequence.
 - Less than -1, for the absolute values there is exponential growth towards (unsigned) infinity, due to the alternating sign.

▼ Formula of nth term of a Geometric Progression

If 'a' is the first term and 'r' is the common ratio. Thus, the explicit formula is:

The diagram shows the formula $a_n = a_1 * r^{n-1}$ on a green grid background. A yellow arrow points from the text "Same General Term" to the a_n term. Another yellow arrow points from the text "General Term" to the a_n term. A third yellow arrow points from the text "First Term" to the a_1 term. A fourth yellow arrow points from the text "Common Ratio" to the r^{n-1} term. In the top right corner, there is a logo consisting of two interlocking circles.

$$a_n = a_1 * r^{n-1}$$

General Term First Term Common Ratio

▼ Formula of sum of nth term of Geometric Progression:

$$\text{Sum} = \frac{a(r^n - 1)}{r - 1}$$

r → Common ratio
 n → Number of terms
 Sum → Sum of all Geometric Progression

▼ General Formulas to solve problems related to Geometric Progressions :

If 'a' is the first term and 'r' is the common ratio:

- **nth term of a GP** $= a * r^{n-1}$
- **Geometric Mean** = n root of the product of n terms in the GPth.
- **Sum of 'n' terms** of a GP ($r < 1$) $= \frac{a(1-r^n)}{1-r}$.
- **Sum of 'n' terms** of a GP ($r > 1$) $= \frac{a(r^n-1)}{r-1}$.
- **Sum of infinite terms** of a GP ($r < 1$) $= \frac{a}{1-r}$.

▼ Quadratic Equation

A **quadratic equation** is a second-order polynomial equation of a variable say x .

The general form of a quadratic equation is given as:

$$ax^2 + bx + c = 0$$

where a, b, and c are real constants and $a \neq 0$.

▼ Roots of an Equation:

The roots of an equation are the values for which the equation satisfies the given condition. For Example, the roots of equation $x^2 - 7x + 12 = 0$ are 3 and 4 respectively. If we replace the value of x by 3 and 4 individually in the equation, the equation will evaluate to zero.

A quadratic equation has two roots. The roots of a quadratic equation can be easily obtained by using the quadratic formula:

$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2ac}$$

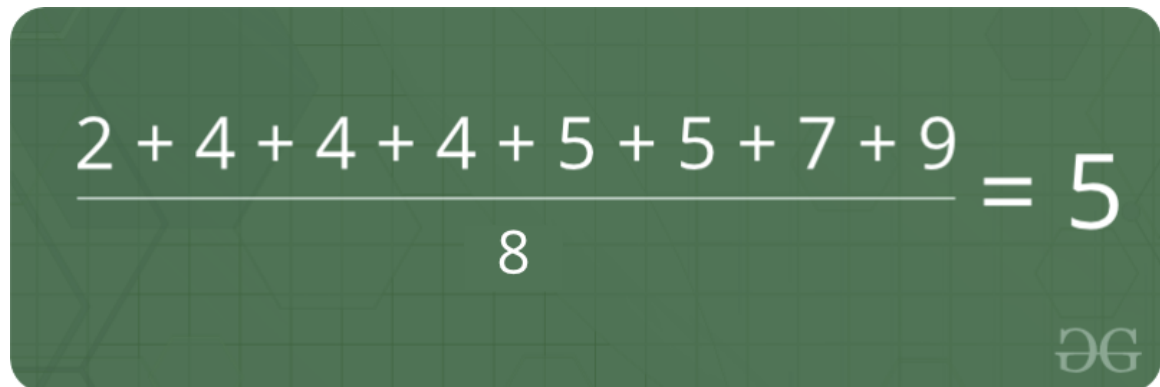
There arises **three cases** as described below while finding the roots of a quadratic equation:

1. If $b^2 < 4ac$, then the roots are imaginary.
2. If $b^2 = 4ac$, then the roots are real and equal to each other.
3. If $b^2 > 4ac$, then the roots are real and distinct.

▼ Mean and Median

▼ Mean

Mean is defined as the average of a given set of data. Let us consider the sequence of numbers **2, 4, 4, 4, 5, 5, 7, and 9**, the mean (average) of this given sequence is 5.


$$\frac{2 + 4 + 4 + 4 + 5 + 5 + 7 + 9}{8} = 5$$

▼ Formula for finding Mean:

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$



Where, **x1, x2,...xn** denotes the terms of the given sequence and **n** is the count of numbers present in the given sequence.

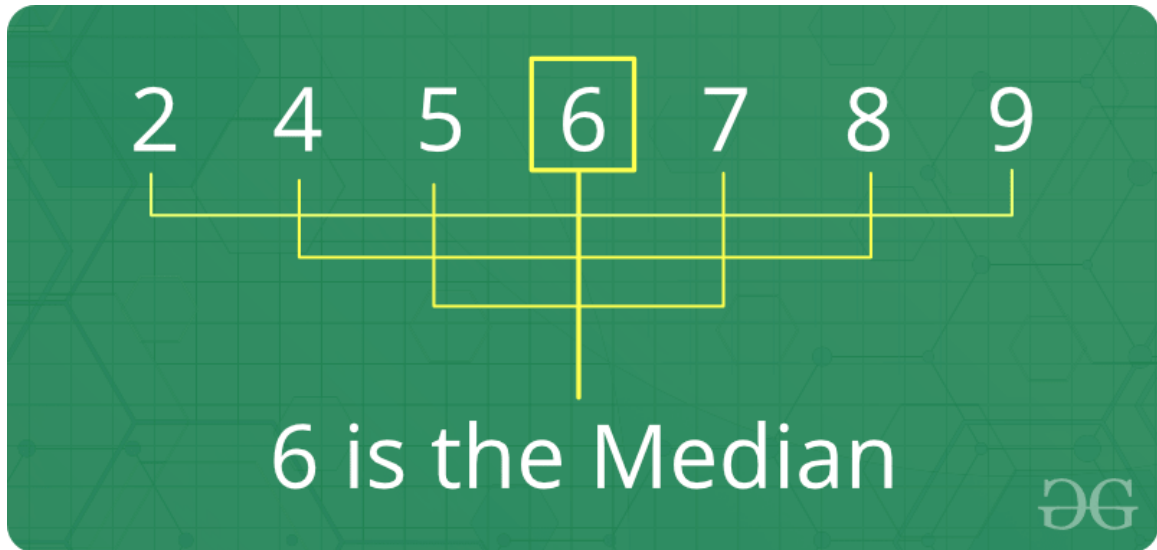
▼ Facts about Mean:

1. The mean (or average) is the most popular and well known measure of central tendency.
2. It can be used with both discrete and continuous data, although its use is most often with continuous data.
3. There are other types of means such as Geometric mean, Harmonic mean, and Arithmetic mean.
4. Mean is the only measure of central tendency where the sum of the deviations of each value from the mean is always zero.

▼ Median

Median is the middle value of a set of data. To determine the median value in a sequence of numbers, the numbers must first be arranged in an ascending order.

- If the count of numbers in the sequence is ODD, the median value is the number that is in the middle, with the same amount of numbers below and above.
- If the count of numbers in the sequence is EVEN, the median is the average of the two middle values.



▼ Facts about Median:

1. Median is an important measure (compared to mean) for distorted data, because median is not so easily distorted. For example, median of {1, 2, 2, 5, 100} is 2 and mean is 22.
2. If the user adds a constant to every value, the mean and median increases by the same constant.
3. If the user multiplies every value by a constant, the mean and the median will also be multiplied by that constant.

▼ Prime Numbers

A **prime number** is a whole number greater than 1, which is only divisible by 1 and itself. First few prime numbers are : 2, 3, 5, 7, 11, 13, 17, 19, 23.

▼ Naive Method to Check if a number is Prime:

Since a number is prime only if it is divisible by 1 and the number itself, the naive method to check for primality of a number would be to iterate from 1 to N and check if there aren't any factors of N except and 1 and N itself.

If N is less than 2 (*it is not prime*): Return False.
else:

- Iterate from 2 to $N-1$ and check if any of the numbers between 2 and $N-1$ (both inclusive) divides N or not. If yes, then N is not prime, return False.
- Otherwise, return True.

Since we are traversing linearly from 2 to $N-1$, the time complexity of the above algorithm will be linear $O(N)$.

▼ Sieve of Eratosthenes

Using **Sieve of Eratosthenes** is the most efficient way of generating prime numbers up to a given number N . Following is the algorithm to find all the prime numbers less than or equal to a given integer n by Eratosthenes' method:

1. Create a list of consecutive integers from 2 to n : (2, 3, 4, ..., n).
2. Initially, let p equal 2, the first prime number.
3. Starting from p , count up in increments of p and mark each of these numbers greater than or equal to p^2 itself in the list. These numbers will be $p(p+1)$, $p(p+2)$, $p(p+3)$, etc..
2
4. Find the first number greater than p in the list that is not marked. If there was no such number, stop. Otherwise, let p now equal this number (which is the next prime), and repeat from step 3.

Example:

Let us take an example when $n = 50$. So we need to print all prime numbers smaller than or equal to 50.

We create a list of all numbers from 2 to 50.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

According to the algorithm we will mark all the numbers which are divisible by 2 and are greater than or equal to the square of it.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Now we move to our next unmarked number 3 and mark all the numbers which are multiples of 3 and are greater than or equal to the square of it.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

We move to our next unmarked number 5 and mark all multiples of 5 and are greater than or equal to the square of it.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

We continue this process and our final table will look like below:

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

So the prime numbers are the unmarked ones: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

▼ LCM and HCF

▼ Factors and Multiples

All numbers that divide a number completely, i.e., without leaving any remainder, are called factors of that number. For example, 24 is completely divisible by 1, 2, 3, 4, 6, 8, 12, 24. Each of these numbers is called a factor of 24 and 24 is called a multiple of each of these numbers.

▼ LCM

LCM stands for *Least Common Multiple*. The lowest number that is exactly divisible by each of the given numbers is called the least common multiple of those numbers. For example, consider the numbers 3, 31, and 62 (2×31). The LCM of these numbers would be $2 \times 3 \times 31 = 186$.

To **find the LCM of the given numbers**, express each number as its prime factors. The product of the highest power of the prime numbers that appear in the prime factorization of any of the numbers gives us the LCM.

For example, consider the numbers 2, 3, 4 (2×2), 5, 6 (2×3). The LCM of these numbers is $2 \times 2 \times 3 \times 5 = 60$. The highest power of 2 comes from prime factorization of 4, the highest power of 3 comes from prime factorization of 3 and prime factorization of 6, and the highest power of 5 comes from prime factorization of 5.

▼ HCF

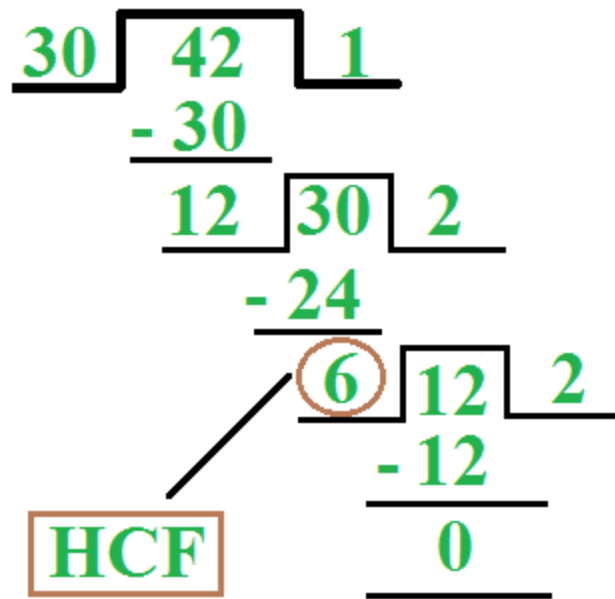
The term HCF stands for *Highest Common Factor*. The largest number that divides two or more numbers is the highest common factor (HCF) for those numbers. For example, consider the numbers 30 ($2 \times 3 \times 5$), 36 ($2 \times 2 \times 3 \times 3$), 42 ($2 \times 3 \times 7$), 45 ($3 \times 3 \times 5$). 3 is the largest number that divides each of these numbers, and hence, is the HCF for these numbers.

HCF is also known as Greatest Common Divisor (GCD).

To **find the HCF of two or more numbers**, express each number as its prime factors. The product of the minimum powers of common prime terms in both of the prime factorization gives the HCF. This is the method we illustrated in the above step.

Also, for finding the HCF of two numbers, we can proceed by the long division method. We divide the larger number by the smaller number (divisor). Now, we divide the divisor by the remainder obtained in the previous stage. We repeat the same procedure until we get zero as the remainder. At that stage, the last divisor would be the required HCF.

For example, HCF of 30 and 42:



The Euclidean algorithm is based on the below facts:

- If we subtract the smaller number from larger (we reduce larger number), GCD doesn't change. So if we keep subtracting repeatedly the larger of two, we end up with GCD.
- Now instead of subtraction, if we divide the smaller number, the algorithm stops when the remainder is found to be 0.

Below is the recursive function for finding GCD using Euclidean Algorithm:

```

gcd(a, b)
{
    if (a == 0)
        return b;

    return gcd(b % a, a);
}

```

Time Complexity: $O(\log(\min(a, b)))$

▼ Important properties of LCM and HCF:

1. For two numbers say, 'a' and 'b', $\text{LCM} \times \text{HCF} = a \times b$.
2. HCF of co-primes = 1.
3. For two fractions,
 - $\text{HCF} = \text{HCF}(\text{Numerators}) / \text{LCM}(\text{Denominators})$
 - $\text{LCM} = \text{LCM}(\text{Numerators}) / \text{HCF}(\text{Denominators})$

▼ Factorials

In mathematics, the factorial of a number say **N** is denoted by **N!**. The factorial of a number is calculated by multiplying all the integers between 1 and N(both inclusive).

For Example,

$$4! = 4 * 3 * 2 * 1 = 24$$

That is,

$$N! = N * (N-1) * (N-2) \dots * 2 * 1$$

Note: As per convention, **0! = 1**.

▼ Sample Problem:

Given a number N, the task is to count number of **trailing zeroes** in factorial of N. That is, number of zeroes at the end in the number **N!**.

```
Input: N = 5
Output: 1
Factorial of 5 is 120 which has one trailing 0.
```

```
Input: N = 20
Output: 4
```


Factorial of 20 is 2432902008176640000 which has 4 trailing zeroes.

An efficient way to solve this problem is to observe the properties of prime factorization. Consider prime factors of $N!$. A trailing zero is always produced by the prime factors

2 and 5. If we can count the number of 5s and 2s in prime factorization of $N!$, our task is done.

Consider the following examples:

- **$N = 5$:** There is one 5 and 3 2s in prime factors of $5!$ ($2 * 2 * 2 * 3 * 5$). So count of trailing 0s is 1.
- **$N = 11$:** There are two 5s and three 2s in prime factors of $11!$ ($2 * 8 * 34 * 52 * 7$). So count of trailing 0s is 2.

We can easily observe that the number of 2s in prime factors is always more than or equal to the number of 5s. So if we count 5s in prime factors, we are done.

Now, how to count the total number of 5s in prime factors of $N!$? A simple way is to calculate $\text{floor}(N/5)$. For example, $7!$ has one 5, $10!$ has two 5s. It is not done yet, there is one more thing to consider. Numbers like 25, 125, etc have more than one 5. For example, if we consider $28!$, we get one extra 5, and the number of 0s becomes 6. Handling this is simple, first divide N by 5 and remove all single 5s, then divide by 25 to remove extra 5s, and so on. Following is the summarized formula for counting trailing 0s.

Trailing 0s in $N!$ = Count of 5s in prime factors of $n!$
= $\text{floor}(n/5) + \text{floor}(n/25) + \text{floor}(n/125) + \dots$

▼ Permutation and Combinations Basics

▼ Permutation

Permutation is the different arrangements of a given number of elements taken one by one, or some, or all at a time. For example, if we have two elements A and B, then there are two possible arrangements, AB and BA.

Number of permutations when 'r' elements are arranged out of a total of 'n' elements is

$${}^n P_r = \frac{n!}{(n-r)!}.$$

For example, let $n = 4$ (A, B, C and D) and $r = 2$ (All permutations of size 2). The answer is $4!/(4-2)! = 12$. The twelve permutations are AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB and DC.

▼ Important Properties of Permutation:

1. ${}^n P_n = n * (n - 1) * (n - 2) * * 1 = n!$
2. ${}^n P_0 = n!/n! = 1$
3. ${}^n P_1 = n$
4. ${}^n P_{n-1} = n!$
5. $\frac{{}^n P_r}{{}^n P_{r-1}} = n - r + 1.$

▼ Permutation with repetition allowed:

The number of permutation or arrangements of N numbers with repetition allowed will be N^N . For Example, permutations of {1,2} with repetitions will be {{1,1}, {1,2}, {2,1},{2,2}}.

▼ Permutation with duplicates

The number of permutations or arrangements of N objects of which p_1 are of one kind, p_2 are of second kind, ..., p_k are of k-th kind and the rest if any, are of different kinds is:

$$\frac{N!}{(p_1! * p_2! * * p_k!)}$$

▼ Combination

Combination is the different selections of a given number of elements taken one by one, or some, or all at a time. For example, if we have two elements A and B, then there is only one way to select two items, we select both of them.

Number of combinations when 'r' elements are selected out of a total of 'n' elements is

$${}^nC_r = n! / [(r!) * (n - r)!].$$

For example, let n = 4 (A, B, C and D) and r = 2 (All combinations of size 2). The answer is $4! / ((4-2)! * 2!) = 6$. The six combinations are AB, AC, AD, BC, BD, CD.

▼ Important Properties of Combination:

1. ${}^nC_0 = {}^nC_n = 1$.
2. ${}^nC_r = {}^nC_{n-r}$
3. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
4. $n * {}^{n-1}C_{r-1} = (n - r + 1) * {}^nC_{r-1}$

▼ Modular Arithmetic

Let us take a look at some of the **basic rules and properties** that can be applied in Modular Arithmetic (Addition, Subtraction, Multiplication etc.). Consider numbers **a** and **b** operated under modulo **M**.

1. $(a + b) \bmod M = ((a \bmod M) + (b \bmod M)) \bmod M$.
2. $(a - b) \bmod M = ((a \bmod M) - (b \bmod M)) \bmod M$.
3. $(a * b) \bmod M = ((a \bmod M) * (b \bmod M)) \bmod M$.

The above three expressions are valid and can be performed as stated. But when it comes to modular division, there are some limitations.

There isn't any formula to calculate $(a/b) \bmod M$. For this we have to learn **modular inverse**.

▼ Modular Inverse

The modular inverse is an integer 'x' such that:

$$a * x \equiv 1 \pmod{M}$$

The value of x should be in $\{0, 1, 2, \dots, M-1\}$, i.e., in the ring of integer modulo M.

The multiplicative inverse of "a modulo M" exists if and only if a and M are relatively prime (i.e., if $\gcd(a, M) = 1$).

```
Input:  a = 3, M = 11
Output: 4
Since  $(4*3) \bmod 11 = 1$ , 4 is modulo inverse of 3
One might think, 15 also as a valid output as " $(15*3) \bmod 11$ "
is also 1, but 15 is not in ring  $\{0, 1, 2, \dots, 10\}$ , so not
valid.

Input:  a = 10, M = 17
Output: 12
Since  $(10*12) \bmod 17 = 1$ , 12 is modulo inverse of 10
```

Methods of finding Modular Inverse: There are two very popular methods of finding modular inverse of any number **a** under modulo **M**.

1. **Extended Euclidean Algorithm:** This method can be used when **a** and **M** are co-prime.
2. **Fermat Little Theorem:** This method can be used when **M** is prime.

▼ Extended Euclidean algorithm

Extended Euclidean algorithm that takes two integers 'a' and 'b', finds their gcd and also find 'x' and 'y' such that, $ax + by = \gcd(a, b)$.

To find the modulo inverse of 'a' under 'M', we put $b = M$ in the above formula. Since we know that a and M are relatively prime, we can put value of gcd as 1.

So, the formula becomes - $ax + My = 1$.

If we take modulo M on both sides, we get - $ax + My \equiv 1 \pmod{M}$.

We can remove the second term on the left side, as 'My (mod M)' would always be 0 for an integer y. Therefore,

$$ax \equiv 1 \pmod{M}$$

▼ Fermat Little Algorithm

The Fermat's little theorem states that if M is a prime number, then for any integer a, the number $a^M - a$ is an integer multiple of M.

That is, $a^M \equiv a \pmod{M}$.

Since, a and M are co-prime to each other then a^{M-1} is an integral multiple of M.

That is, $a^{M-1} \equiv 1 \pmod{M}$.

If we multiply both sides by a^{-1} , we get $a^{-1} \equiv a^{M-2} \pmod{M}$.

Therefore, if M is a prime number to find modulo inverse of a under M, find modular exponentiation of a^{M-2} under modulo M.