MAT137Y1 - Calculus!
Test 1. — October 21st, 2016
Time: 100 minutes

1. *[6 points]* Calculate each of the following limits. If a limit does not exist, indicate whether it is ∞, $-\infty$, or neither.

(a) $\displaystyle\lim_{x\to -1} \frac{x^2+3x+2}{x^2-1}$

Your answer: $-\frac{1}{2}$

$$\begin{aligned}
\lim_{x\to -1} \frac{x^2+3x+2}{x^2-1} &= \lim_{x\to -1} \frac{(x+1)(x+2)}{(x+1)(x-1)} && \text{Factor.}\\
&= \lim_{x\to -1} \frac{x+2}{x-1} && \text{Cancel terms.}\\
&= \frac{(-1)+2}{(-1)-1} = -\frac{1}{2}. && \text{Limit laws.}
\end{aligned}$$

(b) $\displaystyle\lim_{x\to 4^+} \frac{3+x}{4-x}$

Your answer: $-\infty$

Notice that

- $\displaystyle\lim_{x\to 4^+} (3+x) = 7 > 0$
- $\displaystyle\lim_{x\to 4^+} (4-x) = 0$
- Moreover, as $x \to 4^+$, $4-x < 0$.

Hence the limit is $-\infty$, and DNE.

(c) $\displaystyle\lim_{x\to 1} \frac{x^2-2x+1}{x^2+1}$

Your answer: 0

Since the function $\frac{x^2-2x+1}{x^2+1}$ is rational and the denominator does not vanish at $x=1$, the function is continuous at $x=1$. Thus the limit at $x=1$ is equal to the value of the function at $x=1$:

$$\lim_{x\to 1} \frac{x^2-2x+1}{x^2+1} = \frac{(1)^2-2(1)+1}{(1)^2+1} = 0.$$

2. *[4 points]* Calculate each of the following limits. If a limit does not exist, indicate whether it is ∞, $-\infty$, or neither.

(a) $\displaystyle\lim_{x\to 4} \frac{2-\sqrt{x}}{x-4}$

Your answer: $-\frac{1}{4}$

$$\begin{aligned}
\lim_{x\to 4} \frac{2-\sqrt{x}}{x-4} &= \lim_{x\to 4} \frac{2-\sqrt{x}}{(\sqrt{x}-2)(\sqrt{x}+2)} && \text{Factor.}\\
&= \lim_{x\to 4} \frac{-1}{\sqrt{x}+2} && \text{Cancel terms.}\\
&= \frac{-1}{\sqrt{4}+2} = -\frac{1}{4} && \text{Limit laws.}
\end{aligned}$$

Alternatively, we could multiply and divide by the conjugate.

(b) $\displaystyle\lim_{x\to 0} \frac{2\sin^2(2x^5)}{x^{10}}$

Your answer: 8

$$\lim_{x\to 0} \frac{2\sin^2(2x^5)}{x^{10}} = \lim_{x\to 0} 8\left(\frac{\sin(2x^5)}{2x^5}\right)^2 = 8$$

We have used the fact that since

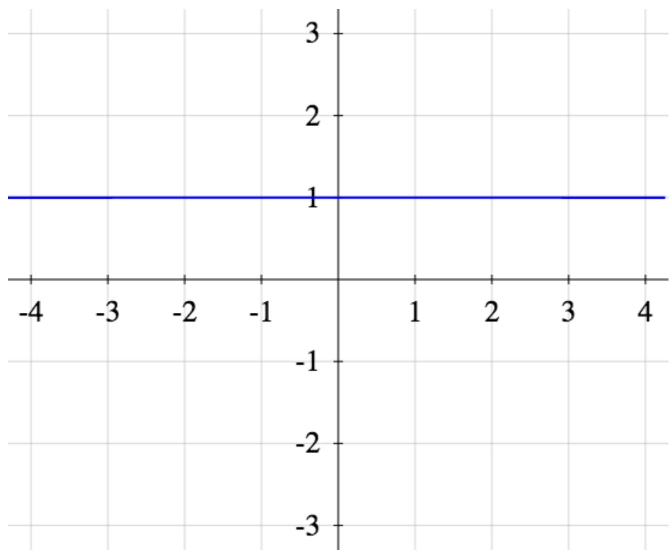
$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

then also

$$\lim_{x\to 0} \frac{\sin(2x^5)}{2x^5)} = 1.$$

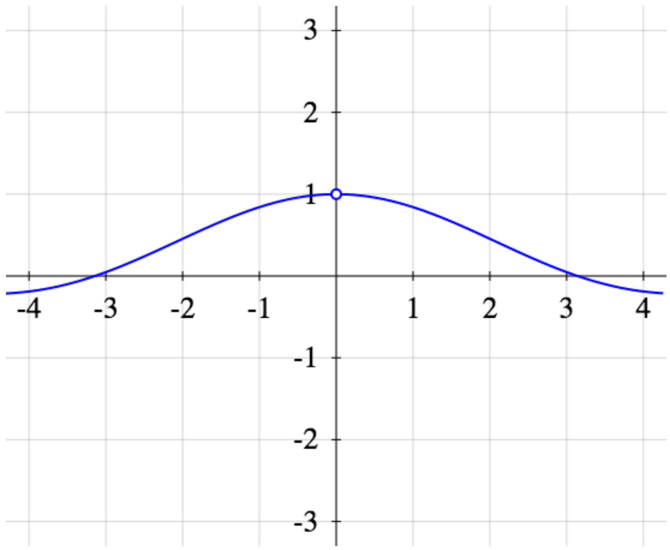
3. *[6 points]* In each of the following cases we ask you to give an example of a function with a certain property. Provide an equation for the function and sketch its graph. It is okay to use piece-wise-defined functions. You do not need to prove anything.

   (a) A function f that is continuous on $\mathbb{R}$.



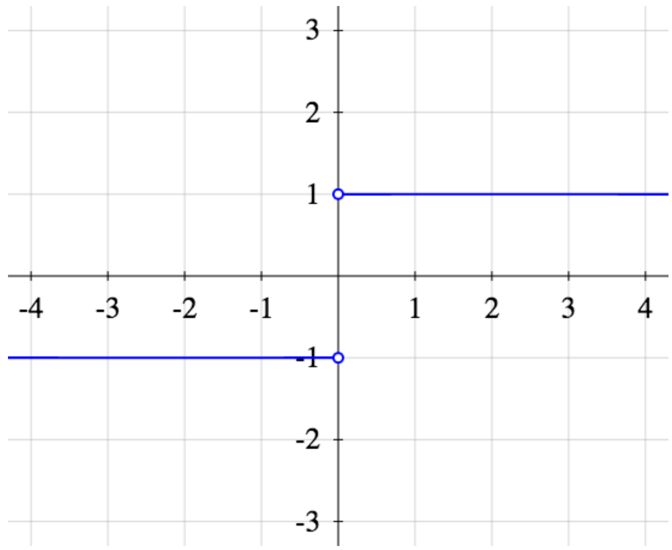
$$f(x) = 1$$

(b) A function g that is continuous on $\mathbb{R}$, except at 0, where it has a removable discontinuity.



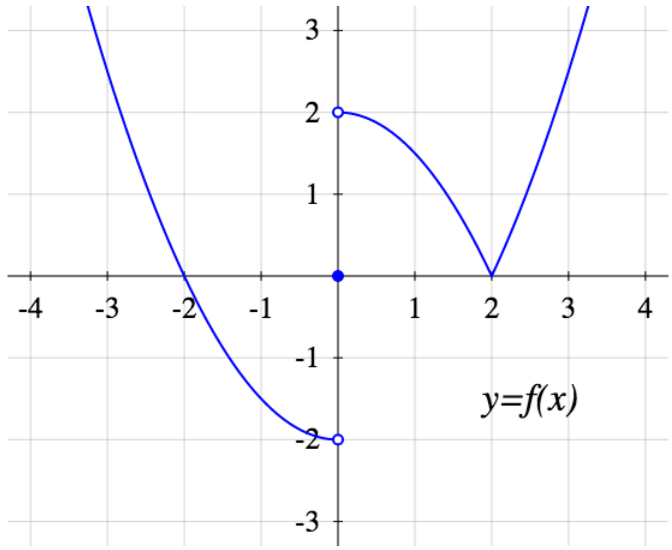
$$f(x) = \frac{\sin(x)}{x}$$

(c) A function h that is continuous on $\mathbb{R}$, except at 0, where it has a non-removable discontinuity.



$$f(x) = \frac{|x|}{x}$$

4. *[6 points]* Below is the graph of the function f:



Find the following limits for f:

(a) $\lim_{x \to 0} f(x)$

Your answer: Does not exist.

We see from the graph that

$$\lim_{x \to 0^+} f(x) = 2$$

and

$$\lim_{x \to 0^-} f(x) = -2.$$

Since the left-hand and right-hand limits are not equal, the limit does not exist.

(b) $\lim_{x\to 0} f(f(x))$

Your answer: 0

We see from the graph that

$$\lim_{x\to 0^+} f(f(x)) = \lim_{x\to 2^-} f(x) = 0$$

and

$$\lim_{x\to 0^-} f(f(x)) = \lim_{x\to -2^+} f(x) = 0.$$

Since the left-hand and right-hand limits both exist and equal 0, the limit exists and equals 0.

(c) $\lim_{x\to 2} f(f(x))$

Your answer: 2

We see from the graph that

$$\lim_{x\to 2^+} f(f(x)) = \lim_{x\to 0^+} f(x) = 2$$

and

$$\lim_{x\to 2^-} f(f(x)) = \lim_{x\to 0^+} f(x) = 2$$

Since the left-hand and right-hand limits both exist and equal 2, the limit exists and equals 2.

5. *[6 points]* Let f be a function with domain $(-\infty, \infty)$. Let $a, L \in \mathbb{R}$. Write the ε–δ definition of the following statements:

(a) $\lim_{x \to a} f(x) = L$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that, if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

(b) $\lim_{x \to a^+} f(x) = L$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that, if } x \in (a, a + \delta), \text{ then } |f(x) - L| < \varepsilon.$$

(c) $\lim_{x \to a^-} f(x) = -\infty$

$$\forall M \in \mathbb{R}, \exists \delta > 0 \text{ such that, if } x \in (a - \delta, a), \text{ then } f(x) < M.$$

6. *[4 points]* Let A, B be subsets of $\mathbb{R}$. Consider the following two statements:

(I) $\forall x \in A, \exists y \in B$ such that $x < y$.

(II) $\forall y \in B, \exists x \in A$ such that $x < y$.

Are statements (I) and (II) equivalent?

Answer: No

We provide a counter-example. Let $A = \{0\}$ and let $B = \{-1, 1\}$. Then (I) is true and (II) is false:

- **(I) is true:** We need to prove that

$$\forall x \in \{0\}, \exists y \in \{-1, 1\} \text{ such that } x < y.$$

  The only element in A is 0. For $x = 0$, we can take $y = 1 \in B$, and it is true that $0 < 1$.

- **(II) is false:** We want to show that

$$\forall y \in \{-1, 1\}, \exists x \in \{0\} \text{ such that } x < y$$

  is false. But for $y = -1 \in B$, the only possible element $x \in A$ is $x = 0$, and it is not true that $0 < -1$.

  Alternatively, if you prefer, you can write the negation of (II) and prove that the negation is true in this case.

Thus (I) and (II) are be equivalent.

7. *[4 points]*

(a) Write the precise statement of the Intermediate Value Theorem.

IF f is continuous on $[a, b]$ AND K is a number such that

- $f(a) < K < f(b)$ or
- $f(b) < K < f(a)$,

THEN there is at least one number $c \in (a, b)$ such that

$$f(c) = K.$$

(b) Use the Intermediate Value Theorem to prove that the equation

$$x^5 - 2x = 100$$

has a positive solution.

Proof: Let $f(x) = x^5 - 2x$. This function is a polynomial, so it is continuous on any interval.

We also see that $f(0) = 0$ and $f(10) = 100000 - 20 = 99980$.

Thus, f is continuous on the interval $[0, 10]$, and the number $K = 100$ satisfies the condition

$$f(0) < 100 < f(10).$$

By the Intermediate Value Theorem, it follows that there is at least one number $0 < c < 10$ such that $f(c) = c^5 - 2c = 100$.

In other words, we have proven that there exists a positive number c that solves the equation

$$x^5 - 2x = 100.$$

8. *[4 points]* Let f be a function with domain $(-\infty, \infty)$. Assume that

$$\lim_{x \to a} f(x) = 2$$

Prove that $\lim_{x \to a} (2f(x) - 1) = 3$.

Do a formal proof directly from the ε–δ definition of limit. Do not use any of the limit laws.

Proof: We will prove that the $\varepsilon - \delta$ definition of the limit is true. We want to show that:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that, if } 0 < |x - a| < \delta \text{ then } |\,(2f(x) - 1) - 3| < \varepsilon.$$

Let $\varepsilon > 0$. Let us use $\frac{\varepsilon}{2}$ in the definition of $\lim_{x \to a} f(x) = 2$. We know that there exists a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - 2| < \frac{\varepsilon}{2}. \tag{1}$$

That is precisely the value of δ we need.

It follows that if $0 < |x - a| < \delta$, then

$$\begin{aligned}
|(2f(x) - 1) - 3| &= |2f(x) - 4| \\
&= 2|f(x) - 2| && \text{Properties of } |\cdot|. \\
&< 2 \cdot \frac{\varepsilon}{2} && \text{By (1).} \\
&= \varepsilon.
\end{aligned}$$

Thus we have shown that $\lim_{x \to a} (2f(x) - 1) = 3$.