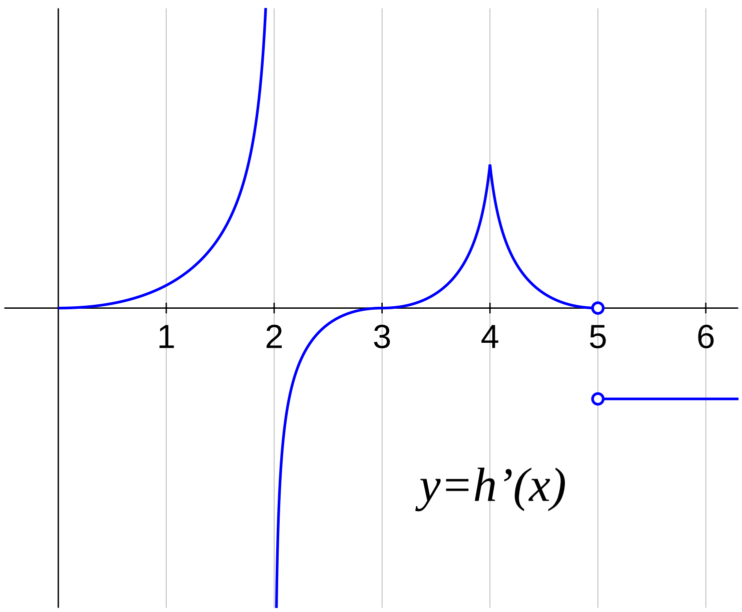
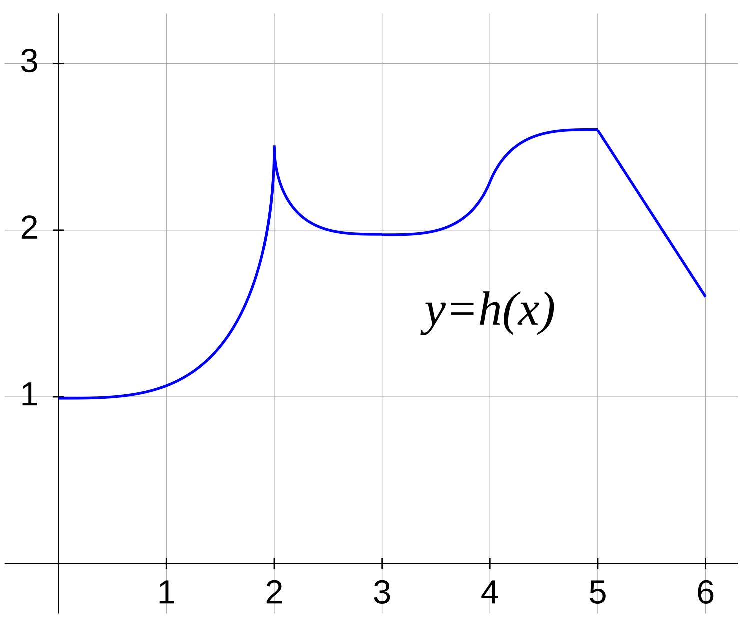
MAT 137Y: Calculus!
Problem Set 3
Due by 3pm on Friday, November 4, 2016 online via crowdmark

1. We know that the function h has domain $[0, \infty)$, satisfies $h(0) = 1$, and is continuous. Below is the graph of its derivative h':



Sketch the graph of $y = h(x)$



Explanation:

- We start with $h(0) = 1$.
- When $0 < x < 2$ the slope is positive. The graph of h has a "horizontal tangent line on the right" at 0.
- At $x = 2$ we have a point with a vertical cusp pointing upwards, because h is continuous at 2 but $\lim_{x \to 2^-} h(x) = \infty$ but $\lim_{x \to 2^+} h(x) = -\infty$.
- At $x = 3$ the slope is 0. For $2 < x < 3$ the slope is negative and for $3 < x < 4$ the slope is positive. Hence at $x = 3$ there is a local minimum.
- Notice also that h increases more when $0 < x < 2$ than it decreases when $2 < x < 3$. Hence $h(3) > h(0)$.
- At $x = 4$ the function is "smooth". It is both continuous and differentiable so nothing strange should happen. There is no vertical tangent line and no corner. (It does have an inflection point at $x = 4$, but we did not take marks off for this.) The function is increasing for $3 < x < 5$.
- At $x = 5$ we have a corner because h' has a jump discontinuity.

2. In this problem, we ask you to obtain a formula for the derivative of $f(x) = \cot x$ in three different ways. They all should produce the same answer. **Submit only your answer to part 2c**.

   (a) **(Do not submit.)** Compute $f'(x)$ from the identity $\cot x = \dfrac{\cos x}{\sin x}$ and using differentiation rules (including the differentiation rules for sin and cos).

   Solution: We apply the quotient rule to the functions sin and cos. Note that all steps are valid since we assume that x is in the domain of csc (and thus $\sin(x) \neq 0$).

$$
\begin{aligned}
\frac{d}{dx}\cot(x) &= \frac{d}{dx}\frac{\cos(x)}{\sin(x)} & \\
&= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} & \text{Quotient rule.} \\
&= \frac{-1}{\sin^2(x)} & \text{Trig identity.} \\
&= -\csc^2(x) & \text{By Definition of } \csc.
\end{aligned}
$$

   (b) **(Do not submit.)** Compute $f'(x)$ from the identity $\cot x = \dfrac{1}{\tan x}$ and using differentiation rules (including the differentation rule for tan).

Solution: We apply the chain rule to the functions $\frac{1}{x}$ and $\tan(x)$. Note that the identity $\cot x = \dfrac{1}{\tan x}$ is only valid when $\sin(x)$ and $\cos(x)$ are not 0 (otherwise cot and tan are not defined).

$$\begin{aligned}
\frac{d}{dx}\cot(x) &= \frac{d}{dx}\frac{1}{\tan(x)} \\
&= \frac{-1}{\tan^2(x)}\sec^2(x) && \text{Chain rule + derivative of } \tan. \\
&= \frac{-\cos^2(x)}{\sin^2(x)}\frac{1}{\cos^2(x)} && \text{By } \tan = \frac{\sin}{\cos} \text{ and } \sec(x) = \frac{1}{\cos}. \\
&= \frac{-1}{\sin^2(x)} && \text{Cancelling.} \\
&= -\csc^2(x) && \text{By Definition of } \csc.
\end{aligned}$$

(c) **(Submit.)** Compute $f'(x)$ from the identity $\cot x = \dfrac{\cos x}{\sin x}$, but directly from the definition of derivative as a limit, without using any of the differentiation rules.

Solution: We apply the definition of the derivative at a point x in the domain of cot. Note that $\sin(x) \neq 0$.

$$\begin{aligned}
&\lim_{h\to 0}\frac{\cot(x+h)-\cot(x)}{h} \\
&= \lim_{h\to 0}\frac{\left(\dfrac{\cos(x+h)}{\sin(x+h)} - \dfrac{\cos(x)}{\sin(x)}\right)}{h} && \text{Applying } \cot = \cos/\sin \\
&= \lim_{h\to 0}\frac{\cos(x+h)\sin(x) - \cos(x)\sin(x+h)}{\sin(x+h)\sin(x)h} && \text{Common demoninator.}
\end{aligned}$$

Expanding $\cos(x+h)$ and $\sin(x+h)$ with trig identities, we get

$$= \lim_{h\to 0}\frac{(\cos(x)\cos(h) - \sin(x)\sin(h))\sin(x) - \cos(x)\left(\sin(x)\cos(h) + \cos(x)\sin(h)\right)}{\sin(x+h)\sin(x)h}$$

Simplifying this we get

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{-\left(\sin^2(x)+\cos^2(x)\right)\sin(h)}{\sin(x+h)\sin(x)h} \\
&= \lim_{h\to 0} \frac{-1}{\sin(x)}\frac{1}{\sin(x+h)}\frac{\sin(h)}{h} && \text{Applying } \sin^2+\cos^2=1 \\
&= \left(\lim_{h\to 0}\frac{-1}{\sin(x)}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right) && \text{Limit laws + sin is continuous.} \\
&= \frac{-1}{\sin(x)}\frac{1}{\sin(x)}\cdot 1 = \frac{-1}{\sin^2(x)} = -\csc^2(x).
\end{aligned}
$$

In the second last line: the first limit exists because it is a constant function of h, the second limit exists because sin is continuous and $\sin(x) \neq 0$, and the third limit is an identity.

3. Let N be a positive integer. Consider the function $F(x) = x\sin\left(x^N\right)$. Calculate $F^{(k)}(0)$ for the smallest positive integer k such that $F^{(k)}(0) \neq 0$.

   Notes: $F^{(k)}$ is the k-th derivative of the function F. Your answer will depend on N. Please box your final answer.

   Solution:

   - If N is a positive integer then using the power rule, chain rule, and product rule we can compute the first two derivatives of F.

     $$
     \begin{aligned}
     F'(x) &= \sin(x^N) + x\cos(x^N)Nx^{N-1} = \sin(x^N) + Nx^N\cos(x^N). \\
     F''(x) &= \cos(x^N)Nx^{N-1} + N^2x^{N-1}\cos(x^N) - N^2x^{2N-1}\sin(x^N) \\
     &= (N+1)Nx^{N-1}\cos(x^N) - N^2x^{2N-1}\sin(x^N)
     \end{aligned}
     $$

   - If we keep taking derivatives, we notice that every derivative will be a sum of terms of the form
     $$(\text{number}) \times x^{(\text{number})} \times \cos(x^N)$$
     and
     $$(\text{number}) \times x^{(\text{number})} \times \sin(x^N)$$
     Of all those terms, the only one that is not zero when $x = 0$ is
     $$(\text{number}) \times x^0 \times \cos(x^N) \tag{1}$$
     Hence, we want to figure out what is the first derivative that will contain the term (1)

- Next, notice that when we take various derivatives of a product, we will get a sum of terms. Each term will be obtained when we take some derivatives of the first function and some derivatives of the second function.
- Now, let us look at the second derivative again. It is the sum of two functions.

$$F''(x) = (N+1)N\left[x^{N-1}\cos(x^N)\right] \; - \; N^2\left[x^{2N-1}\sin(x^N)\right]$$

The fastest way to get the term (1) is from the first part: $x^{N-1}\cos(x^N)$, when we take $N-1$ derivatives of x^{N-1} and no derivatives of $\cos(x^N)$. This suggests that the derivative we are looking for is the $(N-1)st$ derivative of F''. Equivalently, it is the $(N+1)$-st derivative of F. In other words

$$k = N+1.$$

- Moreover, the $(N-1)$-st derivative of x^{N-1} is

$$(N-1)(N-2)\cdots 3\cdot 2\cdot 1\cdot x^0 = (N-1)!.$$

Hence, we conclude that

$$F^{(k)}(0) = (N+1)N(N-1)!\cos(0^N) = (N+1)!.$$

4. In each of the following cases, we want to find a function satisfying various properties. For each part, is it possible to construct such a function? If you answer YES, give us an example, providing both an explicit equation for f and a sketch of its graph. If you answer NO, prove it.

(a) A function f with the following properties

- $f''(1)$ exists.
- $f'''(1)$ does not exist.

Solution: Consider the function

$$f(x) = \begin{cases} (x-1)^3 & \text{if } x \geq 1 \\ -(x-1)^3 & \text{if } x < 1 \end{cases}$$

This function is differentiable everywhere and

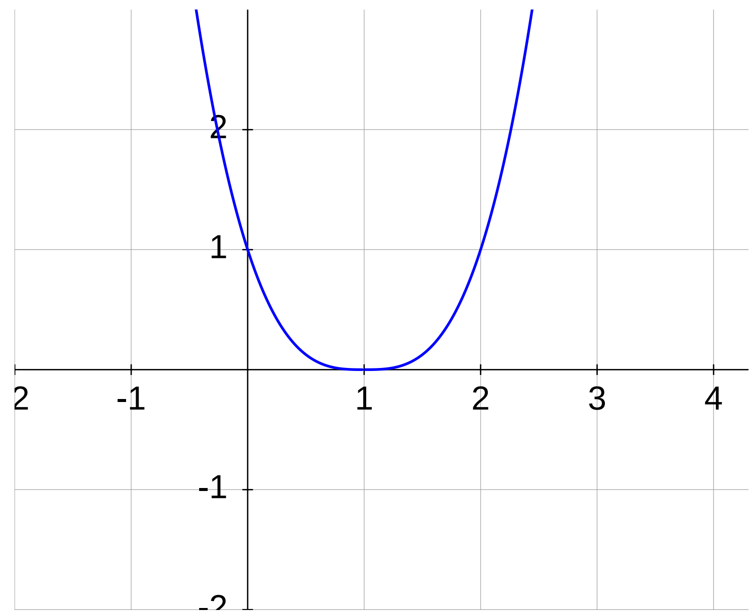
$$f'(x) = \begin{cases} 3(x-1)^2 & \text{if } x \geq 1 \\ -3(x-1)^2 & \text{if } x < 1 \end{cases}$$

This function is again differentiable everywhere and

$$f''(x) = \begin{cases} 6(x-1) & \text{if } x \geq 1 \\ -6(x-1) & \text{if } x < 1 \end{cases}$$

This is just the function $f''(x) = 6|x-1|$ which is not differentiable at 1 because the absolute value function is not differentiable at 0.

Thus $f''(1)$ exists, but $f'''(1)$ does not exist.

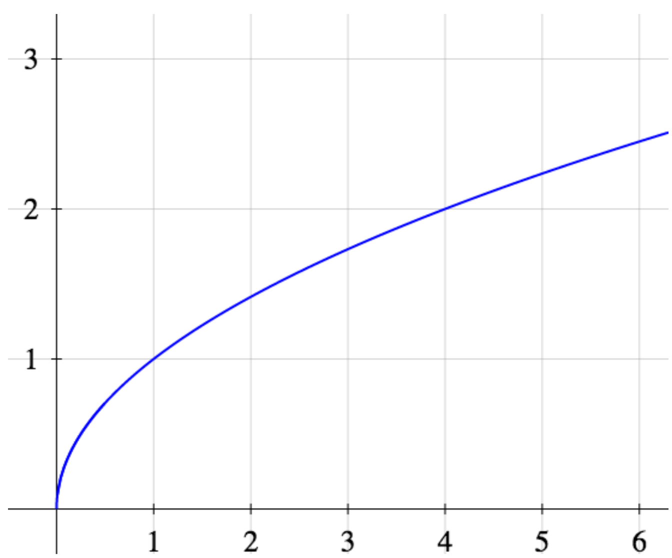


Notice that the fact that the third derivative does not exists is not noticeable in the graph at all.

(b) A function g with the following properties

- g is differentiable, at least, on some interval of the form $(0, c)$.
- $\lim_{x \to 0^+} g'(x) = \infty$.
- $\lim_{x \to 0^+} g(x)$ is not ∞.
- $\lim_{x \to 0^+} g(x)$ is not $-\infty$.

Solution:



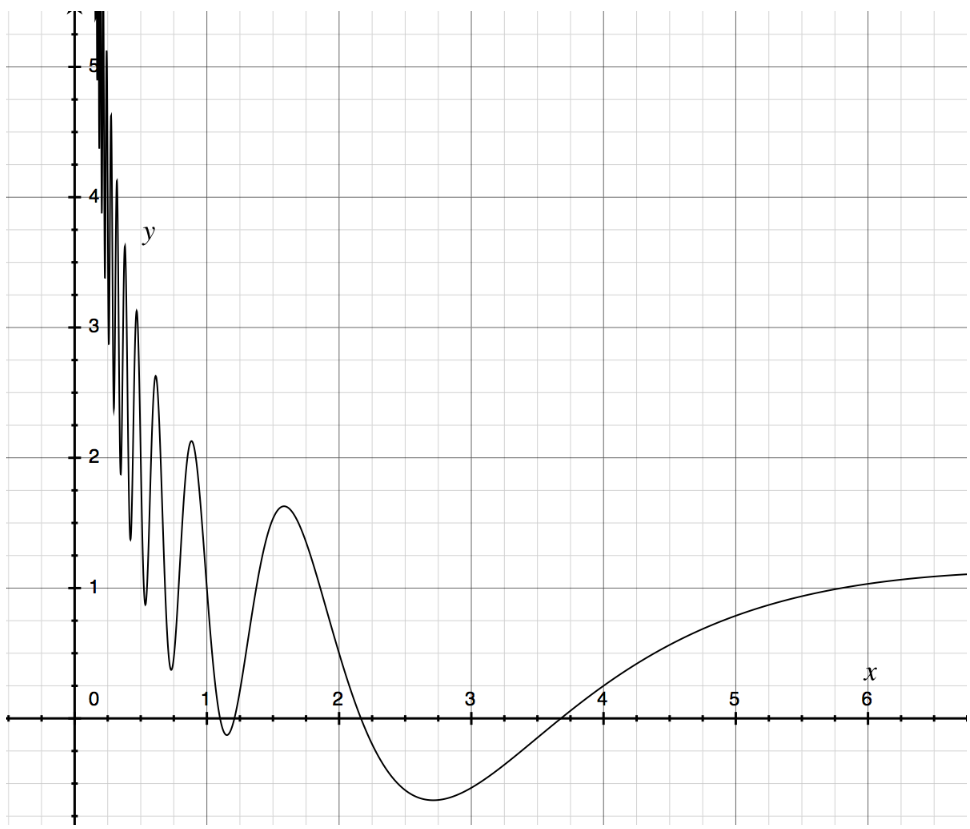
Consider the function $g(x) = \sqrt{x}$.

- $\lim_{x \to 0^+} g(x) = 0$. So the limit exists. In particular it is not ∞ or $-\infty$.
- $g'(x) = \dfrac{1}{2\sqrt{x}}$ for all $x > 0$.
- $\lim_{x \to 0^+} g'(x) = \infty$.

(c) A function h with the following properties

- h is differentiable, at least, on some interval of the form $(0, c)$.
- $\lim_{x \to 0^+} h(x) = \infty$.
- $\lim_{x \to 0^+} h'(x)$ is not ∞.
- $\lim_{x \to 0^+} h'(x)$ is not $-\infty$.

Solution: It is perhaps easier to guess what the graph should look like than to come up with the equation. The key is that $h(x)$ will approach ∞ but will keep both increasing and decreasing as $x \to 0^+$. It is like climbing up a mountain by taking two steps up and one step down. Here is the graph of one such function:



Let $h(x) = \frac{1}{x} + \sin\left(\frac{4\pi}{x}\right)$. Observe that

- $\lim_{x\to 0^+} h(x) = \infty$. This is due to the Squeeze Theorem for infinite limits. We know that

$$h(x) \geq \frac{1}{x} - 1$$

for all $x > 0$ and $\lim_{x\to 0^+} h(x) = \infty$

- Using differentiation rules, for $x > 0$,

$$h'(x) = \frac{-1}{x^2} - \frac{4\pi}{x^2}\cos\left(\frac{4\pi}{x}\right) = \frac{1}{x^2}\left[-1 - 4\pi\cos\frac{4\pi}{x}\right].$$

- $\lim_{x\to 0^+} h'(x)$ does not exist. As x goes to 0 from the right:
  - the function $\cos\left(\frac{4\pi}{x}\right)$ oscillates between -1 and 1,
  - hence the function $\left[-1 - 4\pi\cos\frac{4\pi}{x}\right]$ oscillates between $-1-4\pi$ and $-1+4\pi$,
  - hence $h'(x)$ oscillates between ∞ and $-\infty$. Thus $\lim_{x\to 0^+} h(x)$ cannot be ∞ or $-\infty$ (or a number).