University of Toronto
Faculty of Arts and Sciences
Final Examinations - April 2015

MAT137Y1 – Calculus!
Time: 3 hours

Instructors: A. Belotto da Silva, A. Gracia-Saz, T. Holden, I. Khatchatourian, V. Mouquin

Fill with **ALL CAPITAL LETTERS**:

Last Name:_____ Student ID:_____

First Name:_____

INSTRUCTIONS:

- This exam booklet contains 16 pages including this one. It consists of 13 questions. The maximum score is 50 points.
- Show your work for every question. We may disallow answers that have no supporting work.
- **FOR QUESTIONS 1-6, TRANSFER YOUR FINAL ANSWERS TO PAGE 2.** You will get no credit if you do not do this.
- No aids of any kind are allowed or needed. In particular, no calculators and no extra scrap paper.
- If you run out of space, you may use the back of the pages or the last page. If you do so, on the problem page, clearly indicate where your solution continues.
- Do not turn over this page until the invigilators tell you to do so. Good luck!

FOR MARKERS ONLY:

Questions	Marks	Value
1 to 6		23
7		3
8		3
9		3

Questions	Marks	Value
10		6
11		4
12		4
13		4
Total		**50**

COPY YOUR FINAL ANSWERS FROM QUESTIONS 1–6 TO THIS PAGE.

1. *[6 points]*

   (a) $f'(x) =$

   (b) $F(x) =$

   (c) The limit is

2. *[4 points]*

   (a) The limit is

   (b) The limit is

3. *[3 points]* The sum of the series is

4. *[3 points]* The interval of convergence is

5. *[5 points]* Circle the correct answer for each statement:

   (a) T F N
   (b) T F N
   (c) T F N
   (d) T F N
   (e) T F N

6. *[2 points]* The limit is

1. *[6 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

(a) Calculate the derivative of the function $f(x) = x^2 + 5x + 1$.

(b) Find a function F such that $F'(x) = \sin x$ and $F(0) = 2$.

(c) Calculate $\lim_{n\to\infty} \dfrac{n! + 2}{3(n!) + 4}$.

2. *[4 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Calculate the following limits. If a limit does not exist, indicate whether it is ∞, $-\infty$, or neither.

(a) $\lim_{x\to\infty}\left[x-\sqrt{x^2+7x}\right]$

(b) $\lim_{x\to\infty} x^{-x}$

3. *[3 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

The series

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n+1)!}$$

is convergent. (You do not need to prove this.) Notice that the series starts at $n = 1$. Calculate its value.

4. *[3 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Find the interval of convergence of the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n \, 2^{n+1}}{3n} \, x^n.$$

5. *[5 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Let f be a bounded function with domain the interval $[0,1]$. Assume we have found two partitions P and Q of $[0,1]$ whose lower and upper sums satisfy

$$L_f(Q) = 2, \quad U_f(Q) = 5,$$
$$L_f(P) = 3, \quad U_f(P) = 6.$$

For each of the following statements, decide whether it is necessarily true (T), it is necessarily false (F), or it cannot be determined (N). Circle the correct answers on Page 2.

(a) There exists a partition R of $[0,1]$ such that $L_f(R) \geq 3$ and $U_f(R) \leq 5$.

(b) Every partition R of $[0,1]$ satisfies that $L_f(R) \geq 3$ and $U_f(R) \leq 5$.

(c) The lower integral satisfies $\underline{I}_0^1(f) \geq 3$.

(d) The lower integral satisfies $\underline{I}_0^1(f) \leq 5$.

(e) f is integrable on $[0,1]$.

6. *[2 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Calculate

$$\lim_{x \to 0} \frac{[\sin x - x]\,[\cos(3x) - 1]}{x\,[e^x - 1]^4}.$$

Suggestion: Do not use L'Hôpital.

7. *[3 points]* Calculate $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$.

Your answer:

8. *[3 points]* Calculate $\int_{-2}^{0} \sqrt{4-x^2}\, dx$.

Your answer:

9. *[3 points]* Let g be a function that has all of its derivatives everywhere. Assume that $g(0) = 0$. Calculate

$$\lim_{x \to 0} \frac{\int_x^{2x} g(t)dt}{x^2}.$$

Your answer:

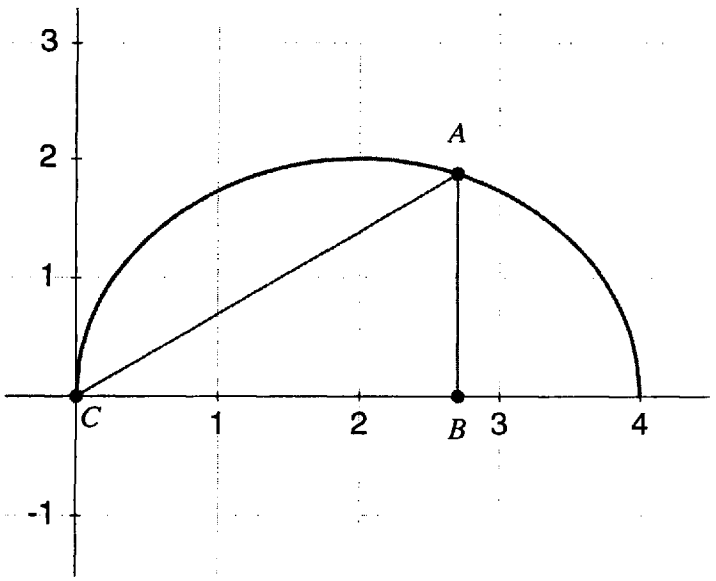
10. *[6 points]* Let f be a function with domain $\mathbb{R}$. Let $a, L \in \mathbb{R}$.

(a) Give the formal definition of the statement "$\lim_{x \to a} f(x) = L$".

(b) Give the formal definition of the statement "f is continuous at a".

(c) Define $f'(a)$.

11. *[4 points.]* In the picture below, we show the graph of $y = \sqrt{4x - x^2}$. We construct the triangle ABC, where A is any point on the graph of $y = \sqrt{4x - x^2}$, B is the point on the x-axis directly below B, and C is the origin. What is the largest possible area of the triangle ABC?



12. *[4 points]* Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences. Assume that

$$\lim_{n\to\infty} a_n = L \qquad \text{and} \qquad \lim_{n\to\infty} b_n = M.$$

Prove that

$$\lim_{n\to\infty} (a_n + 2b_n) = L + 2M.$$

Give a direct proof from the formal definition of the limit of a sequence. Do not use any of the limit laws.

13. *[4 points.]* Find one polynomial P that satisfies the following three properties simultaneously:

    (a) It has an inflection point at $x = 3$.

    (b) It has a local maximum or a local minimum at $x = 2$.

    (c) Its graph intersects the x-axis at $x = 1$.

    Notes: It does not matter if your polynomial has other inflection points and local extrema in addition to the ones we ask for. If you cannot find a polynomial satisfying all three properties, give us a polynomial satisfying some of them. Write your final answer in the box below, and show your work underneath.

Your answer: $P(x) =$

This page is intended for extra work in case you run out of space. If you use it for any problem, clearly indicate so on the corresponding problem page.