MAT137Y1 - Calculus!
Test 4 — 24th March, 2017
Solutions

1. *[4 points]* Compute the following integral

$$K = \int_2^3 \frac{dx}{x^2 + x}$$

Your answer: $K = \ln\left(\frac{9}{8}\right)$

We can integrate by decomposing the fraction:

$$\begin{aligned}
\int_2^3 \frac{dx}{x^2+x} &= \int_2^3 \frac{dx}{x} - \int_2^3 \frac{dx}{x+1} \\
&= (\ln(x))\Big|_2^3 - (\ln(x+1))\Big|_2^3 \\
&= \ln(3) - \ln(2) - (\ln(4) - \ln(3)) \\
&= \ln\left(\frac{9}{8}\right).
\end{aligned}$$

2. *[5 points]* Compute the following integral

$$I = \int \frac{dx}{\left(\sqrt{1+x^2}\,\right)^5}$$

Your answer: $I = \dfrac{2x^3+3x}{3(\sqrt{x^2+1})^3} + C$

Note: Your final answer should be written without using any trigonometric or inverse trigonometric functions.

We perform the trigonometric substitution $\tan(\theta) = x$, $\sec^2(\theta)d\theta = dx$.

$$\begin{aligned}
\int \frac{dx}{\left(\sqrt{1+x^2}\,\right)^5} &= \int \frac{\sec^2(\theta)\,d\theta}{\left(\sqrt{1+\tan^2(\theta)}\,\right)^5} \\
&= \int \frac{\sec^2(\theta)\,d\theta}{\sec^5(\theta)} \\
&= \int \cos^3(\theta)\,d\theta \\
&= \int \left(1-\sin^2(\theta)\right)\cos(\theta)\,d\theta \\
&= \sin(\theta) - \frac{\sin^3(\theta)}{3} + C \\
&= \frac{x}{\sqrt{x^2+1}} - \frac{x^3}{3(\sqrt{x^2+1})^3} + C \\
&= \frac{2x^3+3x}{3(\sqrt{x^2+1})^3} + C.
\end{aligned}$$

In the second last line we used the fact that

$$\sin(\theta) = \frac{x}{\sqrt{x^2+1}} \text{ provided that } \tan(\theta) = x.$$

3. *[9 points]* Determine whether each of the following series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Circle your final answer and justify it.

Note: You won't get any points without a correct justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sqrt{n^2+1}}{n^3+3}$

Circle one:	AC	CC	D

Use Limit Comparison test with the p-series $\sum \frac{1}{n^2}$:

$$\lim_{n\to\infty} \left(\frac{1+\sqrt{n^2+1}}{n^3+3} \right) / \left(\frac{1}{n^2} \right) = 1$$

So the original series is convergent. Since it is positive to begin with, it must be absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{2^{2n+1}}$

Circle one:	AC	CC	D

Observe that

$$\sum_{n=1}^{\infty} \left| \frac{(-3)^n}{2^{2n+1}} \right| = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{3}{4} \right)^n .$$

Since $3/4 < 1$, this geometric series converges. Thus the original series is absolutely convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\cos^2 n}$

Circle one:	AC;	CC	D

The limit of the sequence

$$a_n = \frac{(-1)^n}{\cos^2(n)}$$

does not exist, so the series is divergent by the Necessary Condition Test.

4. *[4 points]* For each non-negative integer n, we define

$$A(n) = \int_0^\infty x^n e^{-x} dx$$

Compute the value of $A(0)$.

Your answer: $A(0) = 1$

$$\begin{aligned} A(0) &= \int_0^\infty e^{-x} dx \\ &= \lim_{b\to\infty} \int_0^b e^{-x} dx \\ &= \lim_{b\to\infty} \left(-e^{-b} + e^0\right) \\ &= 1. \end{aligned}$$

5. *[3 points]* Write the value of the integral

$$J = \int_0^\infty x^{43} e^{-2x^2} dx$$

in terms of $A(n)$ for some value of n.

Note: For the definition of $A(n)$, see Question 4.

Your answer: $J = \frac{1}{2^{23}} A(21)$

We make the substitution $u = 2x^2$, $du = 4x dx$.

$$\begin{aligned} J &= \int_0^\infty x^{43} e^{-2x^2} dx \\ &= \int_0^\infty \frac{(2x^2)^{21}}{2^{21}} e^{-2x^2} x dx \\ &= \frac{1}{2^{21}} \int_0^\infty u^{21} e^{-u} \frac{du}{4} \\ &= \frac{1}{2^{23}} A(21). \end{aligned}$$

6. *[4 points]* Use integration by parts to obtain a relation between $A(n+1)$ and $A(n)$.

Note: For the definition of $A(n)$, see Question 4.

Your answer: $A(n+1) = (n+1)A(n)$

We apply integration by parts to the formula for $A(n)$ with
$u = e^{-x}$, $du = -e^{-x}dx$, $dv = x^n dx$, and $v = \frac{x^{n+1}}{n+1}$.

$$\begin{aligned}
A(n) &= \int_0^\infty x^n e^{-x} dx \\
&= \left(e^{-x} \frac{x^{n+1}}{n+1} \right) \Big|_0^\infty + \int_0^\infty \frac{x^{n+1}}{n+1} e^{-x} dx \\
&= 0 + \frac{1}{n+1} \int_0^\infty x^{n+1} e^{-x} dx \\
&= \frac{1}{n+1} A(n+1).
\end{aligned}$$

where in the second last step we have used the computation

$$\left(e^{-x} \frac{x^{n+1}}{n+1} \right) \Big|_0^\infty = \lim_{x \to \infty} \frac{x^{n+1}}{e^x (n+1)} = 0.$$

We know the limit is 0 from the Big Theorem.

The resulting formula can be more simply written as

$$(n+1)A(n) = A(n+1).$$

7. *[2 points]* Use your answers to Questions 4 and 6 to compute $A(2017)$.

Your answer: $A(2017) = 2017!$

We know that $A(0) = 1$ and $A(n+1) = (n+1)A(n)$. From this recursive definition of $A(n)$ we deduce the formula

$$A(n) = n \cdot A(n-1) = \cdots = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

Plugging in $n = 2017$, we have $A(2017) = 2017!$.

8. *[8 points]* Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Write the formal definition of each of the following statements:

(a) The sequence $\{a_n\}_{n=1}^{\infty}$ is *increasing.*

$$\forall n \in \mathbb{N}, a_{n+1} > a_n.$$

(b) The sequence $\{a_n\}_{n=1}^{\infty}$ is *bounded above.*

$$\exists M \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N}, a_n \leq M.$$

(c) The sequence $\{a_n\}_{n=1}^{\infty}$ is *not bounded above.*

$$\forall M \in \mathbb{R}, \exists n \in \mathbb{N} \text{ such that } a_n > M.$$

(d) The sequence $\{a_n\}_{n=1}^{\infty}$ is *divergent to* ∞.

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow a_n > M.$$

To be more explicit, you could write

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq N \Rightarrow a_n > M$$

But it is implicit in the statement $a_n > M$ that n is a positive integer (otherwise a_n is not defined) and that the implication must be true for all n.

9. *[5 points]* Prove the following theorem:

Theorem: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

- IF the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing and *NOT* bounded above,
- THEN the sequence $\{a_n\}_{n=1}^{\infty}$ is divergent to ∞.

Do a formal proof directly from the definitions.

We must prove that the definition from question 8.d) is true.

Proof: Fix $M \in \mathbb{R}$ arbitrary.

- Since $\{a_n\}$ is not bounded above, there exists $N \in \mathbb{N}$ such that

$$a_N > M.$$

- Since $\{a_n\}$ is increasing, we know that for $n > N$,

$$a_n > a_N.$$

Thus

$$n \geq N \Rightarrow a_n \geq a_N > M.$$

More simply,

$$n \geq N \Rightarrow a_n > M.$$

Since $M \in \mathbb{R}$ was arbitrary, this completes the proof.