MAT137Y1 - Calculus!
Test 2. — November 25th, 2016
Time: 110 minutes

1. *[6 points]* Compute the following derivatives

(a) Given $P(x) = x^7 - 2x^6 + 11x^2 + e$, compute $P''(1)$.

Your answer: $P''(1) = 4$

$$
\begin{aligned}
P'(x) &= 7x^6 - 12x^5 + 22x \\
P''(x) &= 42x^5 - 60x^4 + 22 \\
P''(1) &= 42 - 60 + 22 = 4
\end{aligned}
$$

(b) Given $f(x) = x\,(\ln x)\,(\sin x)$, compute $f'(e)$.

Your answer: $f'(e) = 2\sin(e) + e\cos(e)$

$$f'(x) = \ln(x)\sin(x) + \sin(x) + x\ln(x)\cos(x)$$

$$\Longrightarrow \quad f'(e) = \sin(e) + \sin(e) + e\cos(e) = 2\sin(e) + e\cos(e).$$

(c) Given $h(x) = \tan\left(e^{1+x^2}\right)$, compute $h'(1)$.

Your answer: $h'(1) = 2e^2 \sec^2(e^2)$

$$h'(x) = \sec^2\left(e^{1+x^2}\right) e^{1+x^2}(2x)$$

$$\implies h'(1) = \sec^2\left(e^2\right) e^2(2) = 2e^2 \sec^2(e^2).$$

2. *[4 points]* Evaluate the following limits:

(a) $\displaystyle\lim_{x\to\infty} \frac{x^2+2x\sqrt{2x^2+1}}{x+1+\sqrt[3]{3x^6+6x^3}}$

Your answer: $\dfrac{1+2\sqrt{2}}{\sqrt[3]{3}}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+2x\sqrt{2x^2+1}}{x+1+\sqrt[3]{3x^6+6x^3}} &= \lim_{x\to\infty} \frac{x^2+2x\sqrt{2x^2+1}}{x+1+\sqrt[3]{3x^6+6x^3}}\left(\frac{1/x^2}{1/x^2}\right) \\
&= \lim_{x\to\infty} \frac{1+2\sqrt{\dfrac{2x^2+1}{x^2}}}{1/x+1/x^2+\sqrt[3]{\dfrac{3x^3+6}{x^3}}} \\
&= \lim_{x\to\infty} \frac{1+2\sqrt{2+\dfrac{1}{x^2}}}{1/x+1/x^2+\sqrt[3]{3+\dfrac{6}{x^3}}} \\
&= \frac{1+2\sqrt{2}}{\sqrt[3]{3}}
\end{aligned}$$

In the last step we have used that $1/x \to 0$ and $1/x^2 \to 0$ as $x \to \infty$.

(b) $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-3}\right)^{x}$

Your answer: e^6

This limit is indeterminate of the form 1^∞. Let us call

$$f(x)=\left(\frac{x+3}{x-3}\right)^{x}.$$

We first compute the limit

$$\lim_{x\to\infty}\ln f(x)=\lim_{x\to\infty}\ln\left(\frac{x+3}{x-3}\right)^{x}=\lim_{x\to\infty}x\ln\left(\frac{x+3}{x-3}\right)=\lim_{x\to\infty}\frac{\ln\left(\frac{x+3}{x-3}\right)}{1/x}.$$

This is an indeterminate form of type $\dfrac{0}{0}$. We will apply L'Hopital's rule. Before we take the derivative of the numerator, it is convenient to use properties of the logarithm to simplify the expression:

$$\begin{aligned}
\lim_{x\to\infty}\ln f(x) &\overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{d}{dx}\ln\frac{x+3}{x-3}}{\frac{d}{dx}\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{d}{dx}\left[\ln(x+3)-\ln(x-3)\right]}{\frac{d}{dx}\frac{1}{x}}\\
&=\lim_{x\to\infty}\frac{\frac{1}{x+3}-\frac{1}{x-3}}{-1/x^2}=\lim_{x\to\infty}\frac{6x^2}{(x+3)(x-3)}\\
&=\lim_{x\to\infty}\frac{6}{(1+3/x)\,(1-3/x)}=6
\end{aligned}$$

Since e is continuous:

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}e^{\ln f(x)}=e^6$$

3. *[3 points]* The function F has domain $(0, \infty)$. It is defined by

$$F(x) = \begin{cases} \arcsin\left(\frac{x^2}{2}\right) & \text{if } 0 < x < 1 \\ G(x) & \text{if } x \geq 1 \end{cases}$$

Choose a function G that will make F differentiable everywhere on its domain.

Your answer: $G(x) = \dfrac{\pi}{6} + \dfrac{2}{\sqrt{3}}(x-1)$

Let us call $h(x) = \arcsin\left(\dfrac{x^2}{2}\right)$. Probably the easiest solution is to let $y = G(x)$ be the tangent line to $y = h(x)$ at $x = 1$, since that is guaranteed to satisfy all three conditions. In other words, we will use

$$G(x) = h(1) + h'(1)(x-1)$$

Let's calculate those values.

$$\begin{aligned} h(1) &= \arcsin\frac{1}{2} = \frac{\pi}{6} \\ h'(x) &= \frac{x}{\sqrt{1 - x^4/4}} \\ h'(1) &= \frac{1}{\sqrt{3/4}} = \frac{2}{\sqrt{3}} \end{aligned}$$

4. *[5 points]* Let f be a function differentiable at 2. We define a new function g by the equation

$$g(x) = \left[f(\sqrt{x})\right]^2$$

Prove that g is differentiable at 4 and that

$$g'(4) = \frac{f(2)f'(2)}{2}.$$

Do a proof directly from the definition of derivative as a limit.

Apply the limit definition of the derivative:

$$\begin{aligned} g'(4) &= \lim_{x\to 4} \frac{g(x)-g(4)}{x-4} = \lim_{x\to 4} \frac{\left[f(\sqrt{x})\right]^2 - f(2)^2}{x-4} \\ &= \lim_{x\to 4} \frac{\left[f(\sqrt{x})-f(2)\right]\left[f(\sqrt{x})+f(2)\right]}{x-4} \\ &= \lim_{x\to 4} \frac{f(\sqrt{x})-f(2)}{\sqrt{x}-2}\,\frac{f(\sqrt{x})+f(2)}{\sqrt{x}+2} \end{aligned}$$

Using the limit laws:

$$\begin{aligned} g'(4) &= \left[\lim_{x\to 4} \frac{f(\sqrt{x})-f(2)}{\sqrt{x}-2}\right] \left[\lim_{x\to 4} \left(f(\sqrt{x})+f(2)\right)\right] \left[\lim_{x\to 4} \frac{1}{\sqrt{x}+2}\right] \\ &= f'(2)\cdot(2f(2))\cdot\frac{1}{2+2} = \frac{f(2)f'(2)}{2} \end{aligned}$$

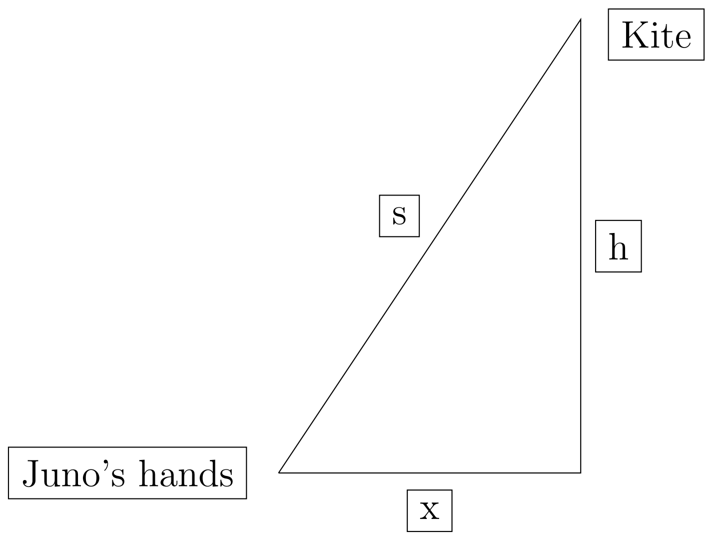
In the last line:

- the first limit is the definition of $f'(2)$ after the change of variable $u=\sqrt{x}$:

$$\lim_{x\to 4} \frac{f(\sqrt{x})-f(2)}{\sqrt{x}-2} = \lim_{u\to 2} \frac{f(u)-f(2)}{u-2} = f'(2)$$

- In the second limit we have used that f is continuous at 2 (because f is differentiable at 2).

5. *[4 points]* Juno is flying a kite. The kite is 21 meters above the ground and it is being blown horizontally by the wind at 2 m/s. Juno's hands are 1 meter above the ground. Right now 30 meters of string are out. At what rate is the string being released from Juno's hands?



Let s be the length of the string between Juno's hands and the kite, x the horizontal displacement of the kite from Juno's hands, and h the vertical displacement of the kite from Juno's hands.

- We are given $s = 30$m, , $\frac{dx}{dt} = 2$m/s and $h = 20$. Notice that x and s depend on time, but h is a constant.
- The three quantities are related by Pythagoras's theorem

$$x^2 + h^2 = s^2.$$

  At the given time, this equation gives us $x = \sqrt{30^2 - 20^2} = 10\sqrt{5}$.
- Differentiating gives the equation

$$x\frac{dx}{dt} + 0 = s\frac{ds}{dt}$$

- At the given time, we have

$$\sqrt{30^2 - 20^2}(2) = (30)\frac{ds}{dt} \implies \frac{ds}{dt} = \frac{2\sqrt{5}}{3} \text{ m/s}$$

Since the rate of change of the length of string between Juno's hands and the kite is $+2\sqrt{5}/3$ m/s, the string is being released from his hands at this rate,

$$\frac{2\sqrt{5}}{3} \text{ m/s.}$$

6. *[4 points]* Let f be a function with domain $\mathbb{R}$.
Write the formal definitions of the following statements:

(a) $\lim_{x \to -\infty} f(x) = 0$

$$\forall \varepsilon > 0, \exists M \in \mathbb{R} \text{ such that if } x < M \text{ then } |f(x)| < \varepsilon.$$

(b) $\lim_{x \to \infty} f(x)$ DNE

This means that for all $L \in \mathbb{R}$, the limit $\lim_{x \to \infty} f(x)$ is not equal to L. This statement can be written as follows:

$$\forall L \in \mathbb{R}, \exists \varepsilon > 0 \text{ such that } \forall M \in \mathbb{R}, \exists x > M \text{ such that } |f(x) - L| \geq \varepsilon.$$

7. *[7 points]* On the next page there are some axes. Sketch on them the graph of $y = g(x)$ for a function g that satisfies all of the following properties:

   (a) *[1 point]* The domain of g is $\mathbb{R}$.

   (b) *[1 point]* g is continuous everywhere except at -2.

   (c) *[1 point]* g is differentiable everywhere except at -2 and 1.

   (d) *[1 point]* g is one-to-one or injective.
   Note: "one-to-one" and "injective" mean the same thing. Your instructor may have only used one of the two terms in class.

   (e) *[1 point]* $(g^{-1})'(-4) = 2$

   (f) *[2 points]* $\lim_{x\to 0} \dfrac{g(x) - 2}{\sin(4x)} = \dfrac{1}{2}$

   If you cannot draw the graph of a function with all the properties, just try to include as many of them as possible.

   For this question only your final graph will be graded. You do not need to show your work.

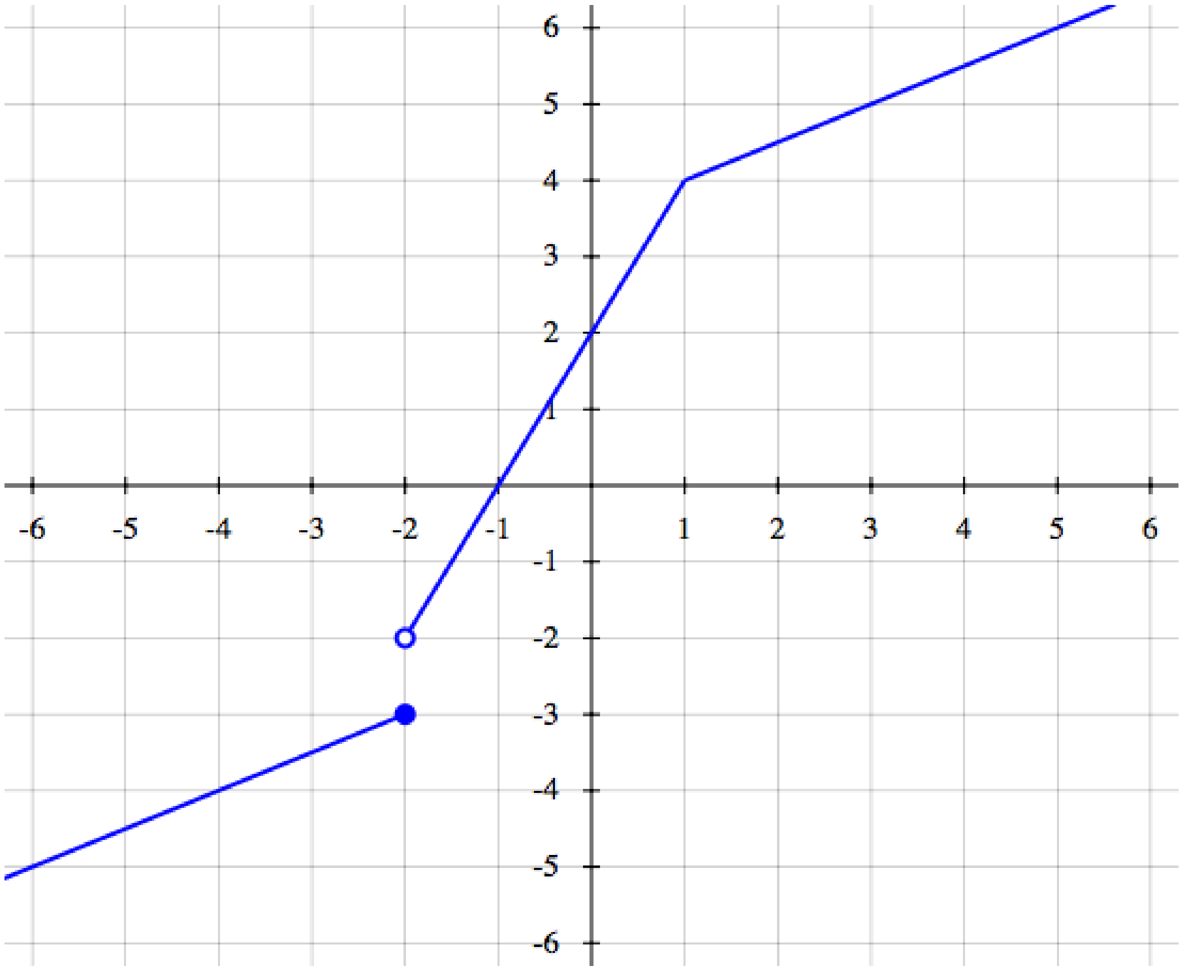
Explanations:

Part (e) requires having a value of x_0 such that

- $g(x_0) = -4$, and
- $g'(x_0) = \dfrac{1}{2}$

Using L'Hopital's rule, the easiest way to satisfy part (f) is to having a smooth function g such that

- $g(0) = 2$ and
- $g'(0) = 2$


6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6