University of Toronto
Faculty of Arts and Sciences
Final Examinations - April 2016

MAT137Y1 – Calculus!
Time: 3 hours

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Fill with **ALL CAPITAL LETTERS**:

Last Name:______________________ Student ID:________________

First Name:______________________

INSTRUCTIONS:

- This exam booklet contains 16 pages including this one. It consists of 13 questions. The maximum score is 62 points.
- Show your work for every question. We may disallow answers that have no supporting work.
- **FOR QUESTIONS 1-8, TRANSFER YOUR FINAL ANSWERS TO PAGE 2.** You will get no credit if you do not do this.
- No aids of any kind are allowed or needed. In particular, no calculators and no extra scrap paper.
- **If you run out of space, you may use the back of the pages or the last page.** If you do so, on the problem page, clearly indicate where your solution continues.
- Do not turn over this page until the invigilators tell you to do so. Good luck!

FOR MARKERS ONLY:

Questions	Marks	Value
1 to 7		26
8		8
9		4
10		4

Questions	Marks	Value
11		4
12		8
13		8
Total		**62**

COPY YOUR FINAL ANSWERS FROM QUESTIONS 1–8 TO THIS PAGE.

1. *[6 points]*
   (a) The equation of the tangent line is
   (b) The value of the integral is
   (c) The limit is
2. *[4 points]*
   (a) The limit is
   (b) The limit is
3. *[2 points]* $h'(1) =$
4. *[4 points]* The integral is
5. *[3 points]* The first non-zero term is
6. *[3 points]* $S =$
7. *[4 points]* Your power series is
8. *[8 points]* Circle the correct answer for each statement:

(a)	T	F	E
(b)	T	F	E
(c)	T	F	E
(d)	T	F	E
(e)	T	F	E
(f)	T	F	E
(g)	T	F	E
(h)	T	F	E

1. *[6 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

(a) Find the equation of the line tangent to $y = 5x - x^2 + 2$ at the point with x-coordinate 1.

(b) Compute $\displaystyle\int_0^1 e^x \, dx$.

(c) Calculate $\displaystyle\lim_{x \to 0} \frac{\sin(2x)}{\ln(x+1)}$.

2. *[4 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Compute the following limits or say they do not exist. If a limit does not exist, indicate whether it is ∞, $-\infty$, or neither.

(a) $\lim_{n\to\infty} \dfrac{2^n + n^{2016}}{2^{n+2} + (n+3)^2}$

(b) $\lim_{x\to 0} \dfrac{e^{-x^2} - x\sin x - \cos(x^3)}{\ln|x|}$

3. *[2 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

The equation

$$x^3y + 2xy^3 = 3$$

defines a function $y = h(x)$ implicitly near the point $(1, 1)$. Compute $h'(1)$.

4. *[4 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Compute

$$\int \left[x^3 e^{-x^2} + 1\right] dx$$

5. *[3 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Find the first non-zero term of the Taylor series of the function

$$f(x) = x^6 \sin\left(e^{2x^4} - 1\right) + 4\cos\left(x^{10}\right) - 4.$$

about $a = 0$.

Suggestion: Do not try to do this taking derivatives. Instead, manipulate the Taylor series you already know.

6. *[3 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Calculate the sum of this series:

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n \, n}{3^{2n+1}}$$

7. *[4 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Construct a power series whose interval of convergence is $(0, 4]$

Note: If you cannot make the interval of convergence to be exactly $(0, 4]$, try to get it as close as possible.

8. *[8 points]* **TRANSFER YOUR FINAL ANSWERS TO PAGE 2!**

Let g be a function with domain $[0, \infty)$. Assume we know the following:

- g is positive and continuous on all of its domain.
- $\lim_{x \to \infty} g(x) = 0$.

We also define the function $G(x) = \int_0^x g(t)dt$. With only this information, which ones of the following statements are necessarily true (T), necessarily false (F), or could be either (E)? Circle the correct answers on Page 2. You do not need to provide a justification.

Important note: Your score for this question will equal your number of correct answers **minus one half** of your number of wrong answers. This is to discourage random guessing; we are not trying to trick you. You may leave any answer blank, and it won't affect your score. The minimum score for this question is 0.

(a) $\lim_{x \to 1} g(x)$ exists.

(b) The function g is bounded on $[0, 1]$.

(c) The function g has a maximum on $[0, \infty)$.

(d) The function g is integrable on $[0, 1]$.

(e) The function G is differentiable at $x = 1$.

(f) The function G is increasing on $(0, 1)$.

(g) The sequence $\{g(n)\}_{n=1}^{\infty}$ is convergent.

(h) The series $\sum_{n=2}^{\infty} g(n)$ is convergent.

9. *[4 points]* Calculate the area of the region bounded by the curves $y = 5 - 3x^2$ and $y = |2x|$.

10. *[4 points]* Find the points in the curve $y = x^2$ which are closest to the point $(0, 3/2)$.

11. *[4 points]* Peter, Iva, and Trefor tell us which sequences they like. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

- Peter likes the sequence when for every $n \in \mathbb{N}$, $a_n < a_{n+1}$.
- Iva likes the sequence when there exists $n_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, if $n \geq n_0$ then $a_n < a_{n+1}$.
- Trefor likes the sequence when for every $n \in \mathbb{N}$, $a_n < a_{n+2}$.

(a) Construct a sequence that Iva likes but Peter does not like.
You do not need to prove that your sequence works.

(b) Construct a sequence that Trefor likes but Iva does not like.
You do not need to prove that your sequence works.

12. *[8 points]*

(a) Let $L, a \in \mathbb{R}$. Let f be a function defined near a, except possibly at a. Give the formal definition of the following statements:

i. $\lim_{x \to a} f(x) = L$

ii. $\lim_{x \to a} f(x) = \infty$

(b) Prove the following theorem:

> Let f be a positive function defined near a, except possibly at a.
> IF $\lim_{x \to a} f(x) = \infty$,
> THEN $\lim_{x \to a} \frac{1}{f(x)} = 0$.

Do a formal proof directly from the definitions. Do not use the limit laws.

13. *[8 points]*

(a) State the Mean Value Theorem.

(b) Define what it means for a function f to be decreasing on an interval (a, b).

(c) Use your above two answers to prove the following theorem:

Let f be a differentiable function on the interval (a, b).
If $f'(x) < 0$ for every $x \in (a, b)$, then f is decreasing on (a, b).

This page is intended for extra work in case you run out of space. If you use it for any problem, clearly indicate so on the corresponding problem page.