MAT 137Y: Calculus!
Problem Set E

This problem set is contains some extra questions about the material covered in the last three weeks of classes. You do not need to turn in any of these problems.

1. Calculate the radius of convergence, and the interval of convergence, of the following power series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2} x^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (n-1)}{n^3} (x-2)^n$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{n!} x^n$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} (x+1)^n$

2. Consider the function $F(x) = \displaystyle\int_0^x e^{-t^4} dt$. It is impossible to find an "elementary antiderivative" for the function $f(t) = e^{-t^4}$, so we use series instead to understand this function.

   (a) Obtain the Taylor series of $f(t) = e^{-t^4}$ around $t = 0$.

   (b) Use the previous answer to represent the function $F(x) = \displaystyle\int_0^x e^{-t^4} dt$ as a power series.

   (c) Estimate $\displaystyle\int_0^1 e^{-t^4} dt$ with an error smaller than 0.001.
   Hint: Notice the series is alternating.

3. Compute the following limits by using Taylor series:

   (a) $\displaystyle\lim_{x \to 0} \frac{6 \sin x - 6x - x^3}{x^5}$

   (b) $\displaystyle\lim_{x \to 0} \frac{e^{x^2} - \cos(2x) - 3x^2}{x^2 \sin(x^2)}$

4. Use Taylor series to estimate the following quantities with an error smaller than 0.001.

   (a) $1/e$

   (b) $\sin 0.3$

   (c) $\ln 1.1$

5. This question is an example of an application of Taylor series to physics, but you do not need to know any physics to solve it.

   Charged particles create a property in space called *electric potential*. If we have a particle with charge q at a point P, it will create an electric potential V at a point Q given by the equation

   $$V = \frac{kq}{r}$$

   where k is a constant and r is the distance between P and Q. We say that the electric potential created by one particle is inversely proportional to the distance.

   (a) Let's say that we have a charge with value q at the point $x = a$ in the x-axis and a charge with value $-q$ at the point with $x = -a$. (We call this a *dipole*.) We want to study how the total electric potential (that is, the sum of the electric potentials created by both particles) depends on the distance at points very far away from both charges. The total electric potential at a point $x > a$ in the x-axis will be

   $$V = \frac{kq}{x-a} - \frac{kq}{x+a}. \tag{1}$$

   Since we are looking at points very far away from both charges, we may assume that x is much bigger than a. Let us call $u = \frac{a}{x}$. Then the quantity u is very small.

   Express Equation 1 in terms of k, q, x, and u (but not a). Then write it as a Taylor series using u as the variable.

   Hint: All you need is the geometric series, which you already know. You do not need to take any derivatives.

   (b) Since u is very small, it makes sense to keep only the first non-zero term of the Taylor series you obtained in Question 5a. Do so. If you do this correctly, you will have proven that the potential created by these two charges together is directly proportional to a and inversely proportional to the square of the distance.

   (c) This time assume that we have a charge with value $-q$ at $x = a$, a charge with value $2q$ at $x = 0$ and a charge with value $-q$ at $x = -a$. Then the total electric potential created by these charges is

   $$V = -\frac{kq}{x-a} + \frac{2kq}{x} - \frac{kq}{x+a}.$$

   Do a calculation similar to the above to answer the following question: For values of x far away from these charges, the total electric potential is inversely proportional to which power of the distance?