MAT 137Y: Calculus!
Problem Set 5
Due by 3pm on Friday, January 13, 2017 online via crowdmark

1. Find all functions f that satisfy the following three properties:

   (a) The domain of f is $\mathbb{R}$.

   (b) f is differentiable everywhere.

   (c) f' is constant.

   The answer to this question should include two proofs. First, you will need to show that all the functions that you have found satisfy the three properties. Second, you will need to prove that *there are no other such functions.* This is the difficult part. To do this second proof, you need to assume that a function f satisfies the three properties, and prove that it must be one of the functions you listed. If your proof does not use Rolle theorem or the Mean Value Theorem (or a variant of them or a consequence of them or something equivalent) then it is probably wrong. Your proof needs to be well-written as a proof. Make sure you introduce any variable you use; make sure what you write means what you want it to mean; if you use a theorem, make sure to state it and to check its hypotheses are true. If your "proof" is just a bunch of equations you will get no credit. If you just write stuff without thinking carefully about it, it will probably be nonsense.

   Solution: We claim that all functions that satisfy (a), (b), and (c) are of the form

   $$f(x) = mx + b$$

   for some $m, b \in \mathbb{R}$.

   PART 1: All functions of the form $f(x) = mx + b$ satisfy (a),(b), and (c). This part is straightforward.

   - Since $f(x) = mx + b$ is a polynomial, its domain is $\mathbb{R}$. Thus (a) is true.
   - Since $f(x) = mx + b$ is a polynomial, it is differentiable everywhere. Thus (b) is true.
   - Differentiating gives $f'(x) = m$, which is a constant function. Thus (c) is true.

   PART 2: All functions that satisfy (a),(b), and (c) are of the form $f(x) = mx + b$. We will present two different proofs of this part.

- **Method 1.** Let f be a function that satisfies (a),(b), and (c). Since f' is constant, there is a real number $m \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f'(x) = m$.

  I define a new function g via the equation $g(x) = f(x) - mx$. We see that g has domain $\mathbb{R}$ and is continuous and differentiable everywhere. Moreover, for every $x \in \mathbb{R}$,

  $$g'(x) = f'(x) - m = 0$$

  Hence, by a consequence of the MVT, it follows that the function g is constant. In other words, there exists $b \in \mathbb{R}$ such that, for every $x \in \mathbb{R}$, $g(x) = f(x) - mx = b$. In other words,

  $$f(x) = mx + g(x) = mx + b,$$

  which is what we wanted to show.

- **Method 2.**

  Let f be a function that satisfies (a),(b), and (c).

  - Since f' is constant, there is a real number $m \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f'(x) = m$.
  - Let us call $b = f(0)$.

  Now that we have said what m and b are in this case, we will prove that for all $x \in \mathbb{R}$, $f(x) = mx + b$. We do this by considering three cases separately: $x = 0$, $x > 0$, and $x < 0$.

  <u>Case I ($x = 0$):</u> This case is simple:

  $$f(0) = b = m(0) + b.$$

  <u>Case II ($x > 0$):</u> Let $x > 0$. We must show that $f(x) = mx + b$.

  Since f is differentiable everywhere (b) we know that

  - f is continuous on $[0, x]$ (differentiable functions are continuous), and
  - f is differentiable on $(0, x)$.

  Thus we may apply the Mean Value Theorem to the function f on the interval $[0, x]$ to conclude that there exists $c \in (0, x)$ such that

  $$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

  Since $f'(c) = m$ and $f(0) = b$, we have that

  $$\frac{f(x) - b}{x} = m.$$

  Rearranging this equation gives us

  $$f(x) = mx + b.$$

Thus for all $x > 0$, $f(x) = mx + b$.

Case III ($x < 0$): This case is similar to the previous case but we aply the MVT on the interval $[x, 0]$ rather than on $[0, x]$. We skip the details.

NOTE on common errors. In Part 2, you need to start with a function that satisfies (a), (b), and (c) and conclude that it is of the form $f(x) = mx + b$ for some real numbers m and b. As part of the process, you will need to explain what m and b are for this particular function. If you assume you already have values of m and b without wondering where they came from, your proof is probably not well-written or incorrect.

2. We are looking for a theorem like this:

> **Theorem.** Let n and k be positive integers. Let f be a function defined on an open interval I. Assume that f is n-times differentiable on I.
>
> - IF the equation $f^{(n)}(x) = 0$ has k solutions on I,
> - THEN the equation $f(x) = 0$ has at most ??? solutions on I.

There are various quantities we can write instead of ??? that will make the theorem true. For example, if the theorem were true when we write $??? = nk^2$ (we are not saying it is), then it would also be true when the wite $??? = nk^2 + 1$. We want you to find the *smallest* expression we can write instead of ??? that will still make the theorem true.

Once you have that quantity, you need to do two things. First prove that the theorem is true. Second, show with an example, that your expression is the smallest one that makes the theorem true.

Solution: For all positive integers n and k, the smallest number that we can replace ??? with that will make this statement true is $k + n$.

PART 1: ??? $= n + k$ makes the theorem true: We can prove the theorem for ??? $= n + k$ by induction on n (while treating k as a constant; in other words, we are doing a proof for all values of k at once.) (Notice that it is also possible to do a proof that is not explicitly by induction.)

Base case ($n = 1$): Let k be a positive integer and let f and I as in the statement of the theorem. When $n = 1$ the theorem says

- IF the equation $f'(x) = 0$ has k solutions on I,
- THEN the equation $f(x) = 0$ has at most $k + 1$ solutions on I

We will prove the contrapositive. We are going to assume that the equation $f(x) = 0$ has (at least) $k + 2$ distinct solutions on I, and we will conclude that the equation $f'(x) = 0$ does not have exactly k solutions on I.

Suppose that the equation $f(x) = 0$ has $k + 2$ distinct solutions on I,

$$x_1 < x_2, < \ldots < x_{k+1} < x_{k+2}.$$

Since f is differentiable on I we can apply Rolle's theorem to each of the $k + 1$ intervals $[x_1, x_2]$, $[x_2, x_3]$, $\ldots$, $[x_{k+1}, x_{k+2}]$. By Rolle's theorem, there exists $k + 1$ numbers

$$x_1 < c_1 < x_2 < c_2 < \cdots < x_{k+1} < c_{k+1} < x_{k+2}$$

such that $f'(c_i) = 0$ for $i = 1, \dots, k+1$.

Induction step: Let f and I as in the statement of the theorem. Assume we know that the theorem is true for a positive integer n. In other words, assume we know that for any positive integer k,

- IF the equation $f^{(n)}(x) = 0$ has k solutions on I,
- THEN the equation $f(x) = 0$ has at most $k + n$ solutions on I.

We want to prove that for any positive integer k,

- IF the equation $f^{(n+1)}(x) = 0$ has k solutions on I,
- THEN the equation $f(x) = 0$ has at most $k + n + 1$ solutions on I.

First, we apply the base case to the function $f^{(n)}$. For a positive integer k,

- Since the equation $f^{(n+1)}(x) = 0$ has k solutions on I,
- the equation $f^{(n)}(x) = 0$ has at least $k + 1$ solutions on I.

Now we apply the induction hypothesis to the function f.

- Since the equation $f^{(n)}(x) = 0$ has $k + 1$ solutions on I,
- the equation $f(x) = 0$ has at least $k + 1 + n$ solutions on I.

PART 2: ??? $= n+k$ is the smallest number that makes the theorem true: Fix positive integers n and k. I define the polynomial

$$f(x) = (x-1)(x-2)(x-3)\dots(x-(n+k))$$

on the interval $I = (0, n+k+1)$. Then:

- *Claim:* The equation $f^{(n)}(x) = 0$ has exactly k solutions on I. This is because:
  - On the one hand, $f^{(n)}$ is a polynomial of degree k, so it can't have more than k solutions.
  - On the other hand, as we argued in the proof of the Base Case in Part 1, between every two solutions of $f(x) = 0$ there is (at least) a solution of $f'(x) = 0$, between every two solutions of $f'(x) = 0$ there is (at least) a solution of $f''(x) = 0$, et cetera, so the equation $f^(n) = 0$ must have at least $(n+k) - n = k$ solutions.
- *Claim:* The equation $f(x) = 0$ has exactly $n + k$ solutions on I (namely, the numbers $1, 2, \dots, n+k$).

Thus this function shows that the theorem is false for any number ??? less than $n + k$.

NOTE on common errors.

- **Do explain what you are doing!** If you are doing a proof by induction, do say that. If you are proving the base case of a proof by induction, do say that. If you are going to prove something by contradiction, do say that. If you are trying to prove the statement when $n = 1$, begin by writing out the statement you want to proof (and say "I want to prove that...").

  This should be obvious, but many students ignore this. If you ignore this, it means that you are making a conscious effort to make your proof hard to read and understand. In other words, you want to write a bad proof.

- When you give an example in Part 2, you need to give an example that shows the theorem is false for $??? = n + k - 1$ for *all* values of n and k. Remember that your goal is to prove that $??? = n + k$ is the smallest value that makes the theorem true. If your example takes a specific value (for example $n = k = 2$), then you have only proven that $??? = n + k$ is the smallest value that makes the theorem true in that case, but it is still possible that $??? = n + k - 1$ works for other cases (for example, $n = 3$ and $k = 4$).

  If you made this error, it is probably due to reading the word "example" and not pausing to think what you are trying to do.

3. We are looking for a theorem like this:

> **Theorem.** Let n and k be positive integers. Let f be a function defined on an open interval I. Assume that f is n-times differentiable on I.
>
> - IF the equation $f^{(n)}(x) = 0$ has k solutions on I,
> - THEN the equation $f(x) = 0$ has at least ??? solutions on I.

There are various quantities we can write instead of ??? that will make the theorem true. For example, if the theorem were true when we write $??? = nk^2$ (we are not saying it is), then it would also be true when the wite $??? = nk^2 - 1$. We want you to find the *largest* expression we can write instead of ??? that will still make the theorem true.

Once you have that quantity, you need to do two things. First prove that the theorem is true. Second, show with an example, that your expression is the largest one that makes the theorem true.

Solution: For all positive integers n and k, the largest number that we can replace ??? with that will make this statement true is 0.

PART 1: ??? $= 0$ makes the theorem true: If we set $??? = 0$ then the statement is true since any equation $f(x) = 0$ on any interval has at least 0 solutions. There is nothing to prove here!

PART 2: ??? $= 0$ is the largest number that makes the theorem true: Consider the function $f(x) = \sin(x) + 2$. Given positive integers n and k,

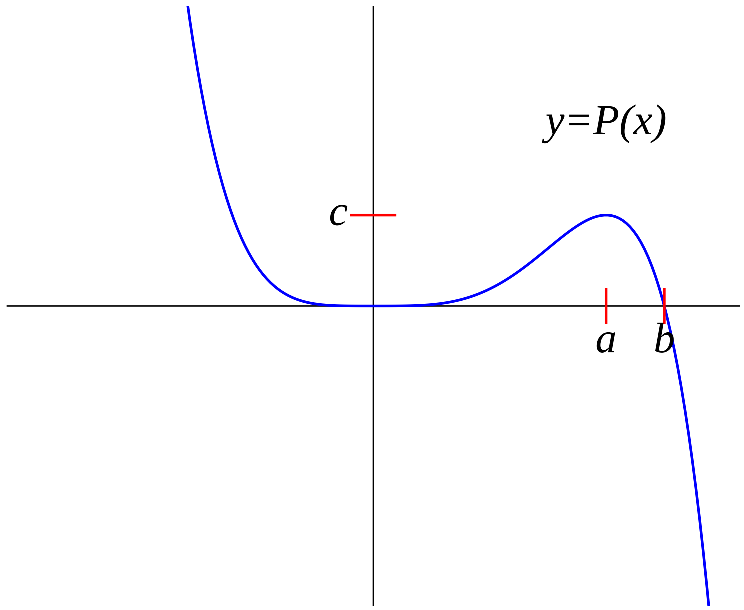
- we can find an open interval I so that the equation $f^{(n)}(x) = 0$ has k solutions on I.
  - If n is odd then the equation $f^{(n)}(x) = 0$ is $\pm\cos(x) = 0$. This equation has k solutions on the interval $I = (0, k\pi)$.
  - If n is even then the equation $f^{(n)}(x) = 0$ is $\pm\sin(x) = 0$. This equation has k solutions on the open interval $I = (0, k\pi + \pi)$.
- AND the equation $f(x) = 0$ has 0 solutions on any open interval I.

Thus the function $f(x) = \sin(x) + 2$ demonstrates that the theorem is false if we let ??? be a number larger than 0.

NOTE on common errors: When you give an example in Part 2, you need to give an example that shows the the equation $f(x) = 0$ may have 0 solutions for *any* values of n and k. If your example takes a specific value (for example $n = k = 2$), then it still possible for the theorem to be true with $??? = 1$ for other values of n and k.

If you made this error, it is probably due to reading the word "example" and not pausing to think what you are trying to do.

4. Below is the graph of a polynomial P:



Notice that it is not at scale. The coordinates in the graph are $a = 24$, $b = 25$, and $c = 1$. Find the equation of P.

In addition to verifying that your answer works, explain how you came up with it. A correct answer obtained by divine inspiration will not get any points.

Solution: Let's begin with some observations:

- Since the polynomial has a root at $b = 25$, it should have a factor of $(x - 25)$.
- Since the polynomial has a root at 0, it should have a factor of x.
- 0 is a local minimum, so the order of the factor x should be an even number.
- Combining these observations, we guess that P has the form

$$P(x) = Mx^{2n}(x - 25)$$

  for some positive integer n and some $M \in \mathbb{R}$ that we have yet to determine.
- Since P has a local maximum at $x = 24$, $P'(24) = 0$. This gives us

$$0 = P'(24) = M(2n)(24)^{2n-1}(24 - 25) + M(24)^{2n} = M(24)^{2n}\left(\frac{-2n}{24} + 1\right).$$

  Thus $2n = 24$.

- Since $P(24) = 1$, we have

$$1 = P(24) = M(24)^{24}(24 - 25) = -M(24)^{24}.$$

  Thus

$$M = \frac{-1}{(24)^{24}}.$$

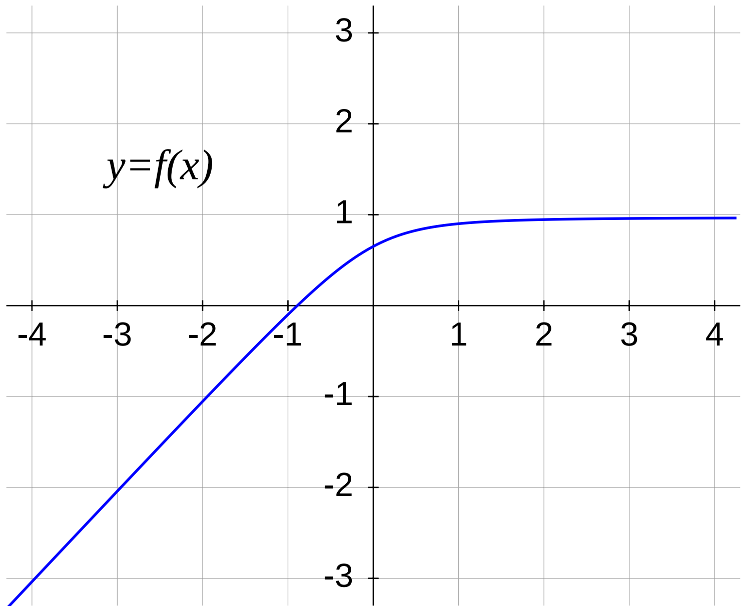
- Thus the polynomial

$$P(x) = -\left(\frac{x}{24}\right)^{24}(x - 25)$$

  has all the desired properties.

5. An elementary function is any function you can write with sum, product, quotient, and composition of polynomials, roots, exponentials, logarithms, trigonometric functions, and inverse trigonometric functions. An elementary function cannot be defined piece-wise.

   Construct an elementary function f with the following graph



   Notice that this function has two asymptotes. Make sure to prove that your solution has the right two asymptotes.

   Hint: It may be best to first try to construct a function with two asymptotes, even if they are not the right ones, and only then to try to get the right asymptotes. One possible way to solve this problem (although not the only one) is to first think about what the derivative should be.

   Solution: Here are several possible answers:

   (a)
   $$f(x) = \frac{1}{2}\left[x - \sqrt{x^2 + a}\right] + 1,$$
   for any $a > 0$

   (b)
   $$f(x) = x\left[\frac{1}{2} - \frac{1}{\pi}\arctan x\right] - \frac{1}{\pi} + 1$$

   (c)
   $$f(x) = x - \ln(e^x + 1) + 1$$

To check the asymptotes for your function, you must show two things:

- $\lim_{x\to\infty} f(x) = 1$, and
- $\lim_{x\to-\infty} (f(x) - (x+1)) = 0$.

Let's check that (c) has the correct asymptotes:

- **Asymptote at ∞.** We must show that

  $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x - \ln(e^x+1) + 1) = 1.$$

  Let's do some algebra:

  $$\begin{aligned}\lim_{x\to\infty} (x - \ln(e^x+1)+1) &= \lim_{x\to\infty} (\ln e^x - \ln(e^x+1)+1) = \lim_{x\to\infty} \left[\ln \frac{e^x}{e^x+1} + 1\right] \\ &= \lim_{x\to\infty} \left[\ln \frac{1}{1+\frac{1}{e^x}} + 1\right] = \ln 1 + 1 = 1\end{aligned}$$

- **Asymptote at ∞.** We must show that

  $$\lim_{x\to-\infty} [f(x) - (x+1)] = 0$$

  This follows directly:

  $$\lim_{x\to-\infty} [f(x) - (x+1)] = \lim_{x\to-\infty} [(x - \ln(e^x+1)+1) - (x+1)] = -\lim_{x\to-\infty} \ln(e^x+1)$$

  As $x \to -\infty$, $e^x \to 0$. Thus as $x \to -\infty$, $\ln(e^x+1) \to 0$. In other words,

  $$\lim_{x\to-\infty} -\ln(e^x+1) = 0.$$