MAT 137Y: Calculus!
Problem Set 4
Due by 3pm on Friday, November 18, 2016 online via crowdmark

1. The equation $xy^3 + x^4 + y^4 = 1$ defines implicitly a function $y = g(x)$ near $(0, 1)$. Compute $g(0)$, $g'(0)$, and $g''(0)$.

   Solution: We compute the three numbers, $g(0)$, $g'(0)$, and $g''(0)$.

   - Since $g(x)$ is defined implicitly by the equation near the point $(0, 1)$ we have that $g(0) = 1$.
   - To compute $g'(0)$, we differentiate the equation implicitly with respect to x:

   $$\begin{aligned} \frac{d}{dx}\left(xy^3 + x^4 + y^4\right) &= \frac{d}{dx}1 \\ \Rightarrow \quad y^3 + 3xy^2\frac{dy}{dx} + 4x^3 + 4y^3\frac{dy}{dx} &= 0. \end{aligned} \tag{1}$$

   Evaluating this expression at the point $(0, 1)$ and solving for $\frac{dy}{dx}$ gives us

   $$\begin{aligned} 1 + 4\left.\frac{dy}{dx}\right|_{(0,1)} &= 0 \\ \Rightarrow g'(0) = \left.\frac{dy}{dx}\right|_{(0,1)} &= -\frac{1}{4}. \end{aligned}$$

   - To compute $g''(0)$ we differentiate the equation implicitly a second time. From (1), we have

   $$\begin{aligned} \frac{d}{dx}\left(y^3 + 3xy^2\frac{dy}{dx} + 4x^3 + 4y^3\frac{dy}{dx}\right) &= 0 \\ \Rightarrow \quad 3y^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} + 6xy\left(\frac{dy}{dx}\right)^2 + 3xy^2\frac{d^2y}{dx^2} + 12x^2 + 12y^2\left(\frac{dy}{dx}\right)^2 + 4y^3\frac{d^2y}{dx^2} &= 0. \end{aligned}$$

   Evaluating at $(0, 1)$, we get

   $$-\frac{6}{4} + \frac{12}{4^2} + 4\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 0$$

   where we have substituted $x = 0$, $y = 1$, and $g'(0) = -1/4$. Rearranging this equation gives us

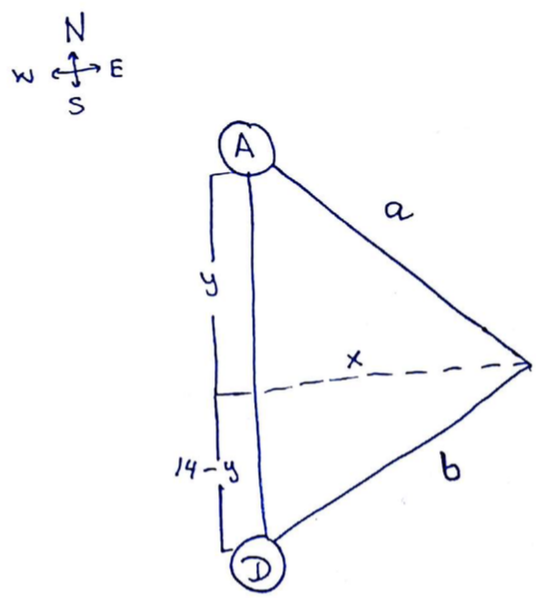
   $$g''(0) = \left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{3/2 - 3/4}{4} = 3/16.$$

2. A student has stolen the MAT137 final exam from the vault in Alfonso's home. Luckily, we had placed a microchip on the exam. There are two radar sources that track the microchip, one at Alfonso's home and one at the math department. Alfonso's home is exactly 14 km North of the math department.

   At a given time, the radar in Alfonso's home indicates that the distance from the thief to Alfonso's home is 15 km, and it is increasing at a rate of 31.2 km/h. At the same time, the second radar indicates that the distance from the thief to the math department is 13 km and it is decreasing at a rate of 20 km/h.

   How fast is the thief moving at that given time, and in which direction?

   Solution: We can sketch the situation described in the following diagram:



   In this picture:

   - A is Alfonso's house, D is the math department, and the other vertex of the big triangle is the sneaky student[1].
   - x and y are the coordinates of the position of the student, taking A as the origin. Specifically, x is the "horizontal" coordinated (defined positive to the East) and y is the "vertical" coordinate (defined positive to the South).
   - a is the distance from A to the student.

[1]Note that as illustrated in the diagram, the student is to the East of Alfonso's house and the department. Given what we know, it is entirely possible that the student is to the West of Alfonso's house and the department (i.e. x could be negative).

- b is the distance from D to the student.

The two smaller triangles are right triangles. We are given:

$$a = 15 \text{ km} \qquad \frac{da}{dt} = 31.2 \text{ km/h}$$
$$b = 13 \text{ km} \qquad \frac{db}{dt} = -20 \text{ km/h}.$$

From the two smaller right triangles, we have the equations

$$y^2 + x^2 = a^2 \text{ and } (14 - y)^2 + x^2 = b^2.$$

When $a = 15$ and $b = 13$, solving these equations give us that $y = 9$ and $x = \pm 12$. Differentiating with respect to time, these equations give us to new equations:

$$y\frac{dy}{dt} + x\frac{dx}{dt} = a\frac{da}{dt} \quad \text{and} \quad -(14 - y)\frac{dy}{dt} + x\frac{dx}{dt} = b\frac{db}{dt}.$$

Substituting known values into these equations, we get

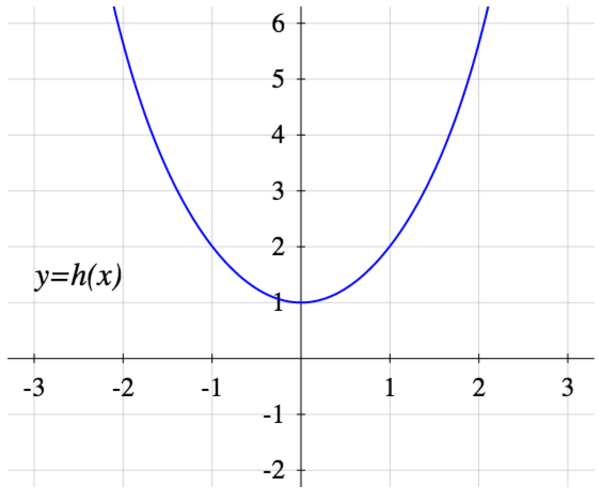
$$(9)\frac{dy}{dt} + (12)\frac{dx}{dt} = (15)(31.2) \quad \text{and} \quad -(14 - 9)\frac{dy}{dt} + (12)\frac{dx}{dt} = (13)(-20).$$

(note that we substitute $x = 12$. Substituting $x = -12$ yields the same final answer.) Solving these two equations for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ gives

$$\frac{dx}{dt} = 0 \text{ km/h and } \frac{dy}{dt} = 52 \text{ km/h}.$$

Thus at the given time the sneaky student is travelling at a speed of 52km/h directly south.

3. Consider the function $h(x) = \frac{1}{100}x^6 + x^2 + 1$. Its graph looks a bit like a parabola, like the picture below:



Based on the graph:

(a) Find the largest interval I such that $1 \in I$ and the restriction of h to I is one-to-one. Call α the inverse of this restriction. What are the domain and the range of α?

Solution: The largest interval containing 1 where h is one-to-one is $I = [0, \infty)$. h is strictly increasing on I, so its restriction to I is one-to-one.

Observe that h is an even function, which means that $h(x) = h(-x)$ for all numbers x. Since h is an even function, h is not one-to-one on any interval containing x and $-x$ for $x \neq 0$. Thus I is indeed the largest possible interval containing 1 on which h is one-to-one.

The image[2] of h restricted to I is $[1, \infty)$. If we define α to be the inverse of h restricted to I, then α has domain $[1, \infty)$ and image $[0, \infty)$.

[2] We take this as clear from the graph, as instructed. One could check this rigorously using the Intermediate Value Theorem if they were so inclined.

(b) Find the largest interval J such that $-1 \in J$ and the restriction of h to J is one-to-one. Call β the inverse of this restriction. What are the domain and the range of β?

Solution: The largest interval containing -1 where h is one-to-one is $J = (-\infty, 0]$. h is strictly decreasing on J, so its restriction to J is one-to-one.

The interval J cannot be made any larger for the same reason that the interval I could not be made any larger, which is explained above.

The image of h restricted to J is $[1, \infty)$. If we define β to be the inverse of h restricted to $(-\infty, 0]$, then β has domain $[1, \infty)$ and image $(-\infty, 0]$.

(c) Calculate the following, if defined:

i. $\alpha(h(2)) = 2$
since the functions are inverses of each other.

ii. $\alpha(h(-2)) = \alpha(h(2)) = 2$
since h is even and the functions are inverses of each other.

iii. $h(\alpha(2)) = 2$
since the functions are inverses of each other.

iv. $h(\alpha(-2))$
is not defined because -2 is not in the domain of α.

v. $\beta(h(2)) = \beta(h(-2)) = -2$
since h is even and the functions are inverse.

vi. $\beta(h(-2)) = -2$
since the functions are inverses of each other.

vii. $h(\beta(2)) = 2$
since the functions are inverses of each other.

viii. $h(\beta(-2))$
is not defined since -2 is not in the domain of β.

4. The function arcsec is defined as the inverse of the restriction of sec to the domain $[0, \pi/2) \cup (\pi/2, \pi]$. The function arcsec has domain $(-\infty, -1] \cup [1, \infty)$. Its derivative is

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2 - 1}}$$

Derive this formula in two different ways:

(a) First, prove the formula by using that $\operatorname{arcsec} x = \arccos \dfrac{1}{x}$ and a formula for $\dfrac{d}{dx} \arccos x$

Solution: Since $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$, we apply the chain rule and the formula for the derivative of arccos to get

$$\begin{aligned}
\frac{d}{dx} \operatorname{arcsec}(x) &= \frac{d}{dx} \arccos\left(\frac{1}{x}\right) \\
&= \frac{-1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \left(\frac{-1}{x^2}\right) \\
&= \frac{1}{x^2 \left(\dfrac{\sqrt{x^2 - 1}}{\sqrt{x^2}}\right)} \\
&= \frac{1}{|x|\sqrt{x^2 - 1}}.
\end{aligned}$$

In the last step we use the fact that $\sqrt{x^2} = |x|$ and $x^2 = |x|^2$.

(b) Second, prove the formula by differentiating both sides of

$$\sec \operatorname{arcsec} t = t$$

with respect to t.

Solution: We present three correct ways to write this solution and comment on some wrong ones.

If $y = \operatorname{arcsec}(x)$ then we have the equation $\sec(y) = x$. Differentiating this equation with respect to x we get

$$\begin{aligned}
\frac{d}{dx} \sec(y) &= \frac{d}{dx} x \\
\Rightarrow \sec(y) \tan(y) \frac{dy}{dx} &= 1 \qquad (2) \\
\Rightarrow \frac{dy}{dx} &= \frac{1}{\sec(y) \tan(y)}
\end{aligned}$$

We know that $\sec(y) = x$ and we need to find $\tan(y)$. We can proceed in various ways.

- **Good Solution #1.** We use the identity $\sec^2 y = 1 + \tan^2 y$ to obtain

$$\tan y = \pm\sqrt{1 - \sec^2 y} = \pm\sqrt{1 - x^2}$$

  Substituting into (2) we get

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{dy}{dx} = \frac{1}{\pm x\sqrt{1 - x^2}} \tag{3}$$

  and we need to decide which sign to use. Next we notice that
  - when $x > 0$, $0 < y < \dfrac{\pi}{2}$ and $\tan y > 0$, so

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{x\sqrt{1 - x^2}}$$

  - when $x < 0$, $\dfrac{\pi}{2} < y < \pi$ and $\tan y < 0$, so

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{-x\sqrt{1 - x^2}}$$

  The two equations can be written as a single one as

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{|x|\sqrt{1 - x^2}}$$

- **Good solution #2.** We can start as in Option 1 and get the equation (3). Now we look at the graph of arcsec and noticed that it is always increasing, so the derivative must always be positive. Hence we conclude that

$$\frac{d}{dx}\operatorname{arcsec}(x) = \left|\frac{1}{\pm x\sqrt{1 - x^2}}\right| = \frac{1}{|x|\sqrt{1 - x^2}}$$

- **Good solution #3.** Since $\sec(y) = x$, we have that $\frac{1}{x} = \cos(y)$ so by Pythagoras' theorem,

$$1 = \frac{1}{x^2} + \sin^2(y) \Rightarrow \sin(y) = \sqrt{1 - \frac{1}{x^2}}.$$

  because $\sin y \geq 0$ when $0 \leq y \leq \pi$. Thus

$$\tan(y) = \frac{\sin(y)}{\cos(y)} = x\sqrt{1 - \frac{1}{x^2}}.$$

Substituting these formulas into (2), we get

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{dy}{dx} = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{x^2\left(\frac{\sqrt{x^2-1}}{\sqrt{x^2}}\right)} = \frac{1}{|x|\sqrt{x^2-1}}.$$

where again we have used that $\sqrt{x^2} = |x|$ and $x^2 = |x|^2$.

- **Bad solutions.**
  - If you choose Option 3 and do not explain why $\sin y \geq 0$ on this interval, your solution is not complete. You just ignored the signs and got to the correct final expression by accident.
  - If you choose Options 1 and 2 but write something like $\tan y = \sqrt{1-t^2}$ or $\tan y = \sqrt{1-\sec^2 y}$, your solution is wrong, as both these statements are incorrect. It is not okay to later "fix" them by adding the absolute value at the end, if you have been working with incorrect equations. Two wrongs do not make a right.
  - If at any point you write $\operatorname{arcsec}' t = \dfrac{1}{t\sqrt{1-t^2}}$ and later you "fix" it to $\operatorname{arcsec}' t = \dfrac{1}{|t|\sqrt{1-t^2}}$, your answer is wrong. You should not have gotten a wrong answer in the first place. Your should either get a right answer from the beginning, or notice *from the beginning* that you do not have decided on the sign yet.
  - It is not true that $\sec \operatorname{arcsec} t = |t|$.
  - In general, do not bluff. Most students wrote solutions to this question not understanding where the absolute value comes from, and they simply threw in some comments hoping for luck. It is better to write a partially-correct solution and point out at the part you are missing, than to bluff. Bluffing amounts to lying. Your grader will be merciless when they realize you are trying to trick them.