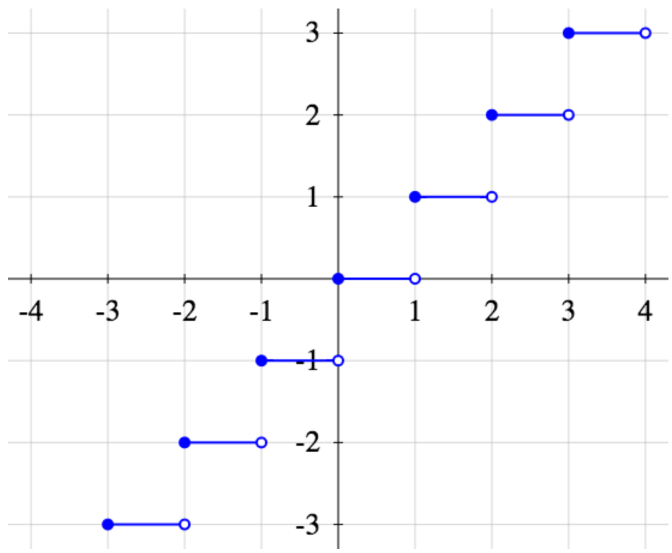
# MAT 137Y: Calculus!
# Problem Set 2 Solutions
# Due by 3pm on Friday, October 14, 2016 online via crowdmark

1. Given a real number $x$, we defined the *floor of* $x$, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to $x$. For example, $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, and $\lfloor -0.5 \rfloor = -1$ and $\lfloor -\pi \rfloor = -4$.

   (a) Sketch the graph of $y = \lfloor x \rfloor$.




   (b) Compute the following limits:

   i. $\lim_{x\to 0^+} \lfloor x \rfloor$

   ii. $\lim_{x\to 0^-} \lfloor x \rfloor$

   i. $\lim_{x\to 0} \lfloor x \rfloor$

   ii. $\lim_{x\to 0} \lfloor x^2 \rfloor$

*Solution:*

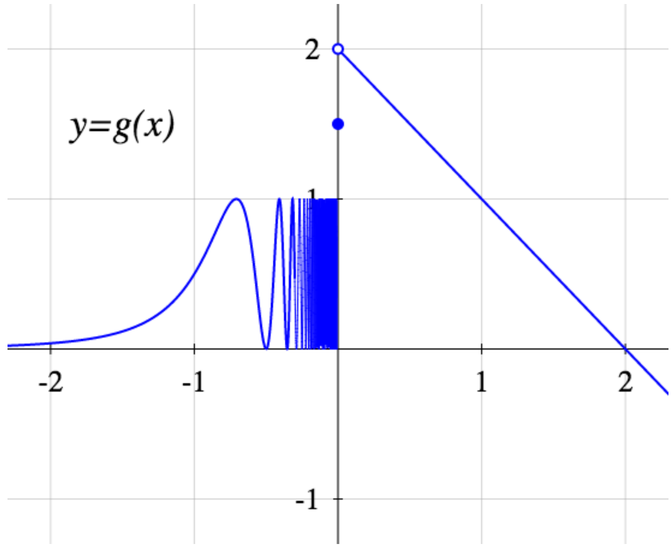
(a) $\lim_{x\to 0^+} \lfloor x \rfloor = 0$, since $\lfloor x \rfloor$ is identically 0 for x between 0 and 1.

(b) $\lim_{x\to 0^-} \lfloor x \rfloor =$ -1, since $\lfloor x \rfloor$ is identically -1 for x between -1 and 0.

(c) $\lim_{x\to 0} \lfloor x \rfloor$ does not exist, since $\lim_{x\to 0^+} \lfloor x \rfloor \neq \lim_{x\to 0^-} \lfloor x \rfloor$.

(d) $\lim_{x\to 0} \lfloor x^2 \rfloor = 0$. If $-1 < x < 1$, then $0 \leq x^2 < 1$. Thus for all $x$ such that $-1 < x < 1$, $\lfloor x^2 \rfloor = 0$.

2. Below is the graph of the function $g$:




For clarification, when $-1 < x < 0$, $g(x)$ "oscillates" between 0 and 1; as $x$ approaches 0 from the left, these oscillations become faster and faster. The behaviour is similar to that of the function $f(x) = \sin(\pi/x)$, which you can see on Example 12 on section 2.1 of the book. Find the following limits:

(a) $\lim_{x\to 0^+} g(x)$

(b) $\lim_{x\to 0^+} \lfloor g(x) \rfloor$

(c) $\lim_{x\to 0^+} g(\lfloor x \rfloor)$

(d) $\lim_{x\to 0^-} g(x)$

(e) $\lim_{x\to 0^-} \lfloor g(x) \rfloor$

(f) $\lim_{x\to 0^-} \lfloor \frac{g(x)}{2} \rfloor$

(g) $\lim_{x\to 0^-} g(\lfloor x \rfloor)$

*Hint:* The correct answer is *"does not exist"* for exactly two of the seven limits. The other five limits have all different answers.

*Solution:*

(a) $\lim_{x\to 0^+} g(x) = 2.$

As $x$ approaches 0 from the right, $g(x)$ approaches 2.

(b) $\lim_{x\to 0^+} \lfloor g(x) \rfloor = 1.$

For all $x$ such that $0 \le x < 1$, $1 \le g(x) < 2$. Thus for all $x$ such that $0 \le x < 1$, $\lfloor g(x) \rfloor = 1$.

(c) $\lim_{x\to 0^+} g(\lfloor x \rfloor) = 1.5.$

For all $x$ such that $0 \le x < 1$, $\lfloor x \rfloor = 0$, so $g(\lfloor x \rfloor) = g(0) = 1.5$.

(d) $\lim_{x\to 0^-} g(x)$ does not exist.

This is for the same reason that the limit of $\sin(\frac{\pi}{x})$ as $x$ goes to 0 does not exist.

(e) $\lim_{x\to 0^-} \lfloor g(x) \rfloor$ does not exist.

As $x$ approaches 0 from the left, $g(x)$ oscillates between 0 and 1. When $g(x) < 1$, $\lfloor g(x) \rfloor = 0$. When $g(x) = 1$, $\lfloor g(x) \rfloor = 1$. Thus as $x$ approaches 0 from the right, the value of $\lfloor g(x) \rfloor$ alternates between 0 and 1, and the limit does not exist. This is similar to (but not the same as) the behaviour of the Dirichlet function.

(f) $\lim_{x\to 0^-} \lfloor \frac{g(x)}{2} \rfloor = 0.$

For $x < 0$, $0 \le g(x) \le 1$. Thus for $x < 0$, $0 \le \frac{g(x)}{2} \le \frac{1}{2}$, so $\lfloor g(x) \rfloor = 0$.

(g) $\lim_{x\to 0^-} g(\lfloor x \rfloor) = 0.5.$

For $-1 \le x < 0$, $\lfloor x \rfloor = -1$. Thus for $-1 \le x < 0$, $g(\lfloor x \rfloor) = g(-1) = 0.5$.

3. Prove that

$$\lim_{x\to 2} |2x^3| = 16.$$

Do a direct proof from the $\varepsilon$–$\delta$ definition of limit.

*Note:* Before you do this proof, read Examples 1, 2, 6, and 7 in Section 2.2 of the book.

*Proof:* Let $\varepsilon > 0$ arbitrary. Set $\delta = \min\{1, \frac{\varepsilon}{38}\}$ and assume that $0 < |x-2| < \delta$.

Note that since $|x-2| < 1$, $x > 1$, so $|2x^3| = 2x^3$.

Thus

$$\begin{aligned} ||2x^3| - 16| &= |2x^3 - 16| \\ &= 2|x^3 - 8| \\ &= 2|(x-2)(x^2+2x+4)| \\ &= 2|x-2||x^2+2x+4| \\ &\le 2|x-2|\left(|x|^2 + 2|x| + 4\right) \text{ by the triangle inequality.} \end{aligned}$$

By the triangle inequality and our assumption that $|x-2| < \delta \leq 1$, we have

$$|x| = |x-2+2| \leq |x-2| + 2 < 1 + 2 = 3.$$

Thus

$$\begin{aligned} 2|x-2|\left(|x|^2 + 2|x| + 4\right) &< 2|x-2|\left(3^2 + 2\cdot 3 + 4\right) \\ &= 38|x-2| \\ &< 38\delta \\ &\leq 38\frac{\varepsilon}{38} = \varepsilon. \end{aligned}$$

□

4. The following theorem is false and the proof is incorrect.

**(Bad) Theorem:** Let $a \in \mathbb{R}$.
Let $f$ and $g$ be functions with domain $\mathbb{R}$, except perhaps $a$.
IF $\lim\limits_{x\to a} f(x) = 0$,
THEN $\lim\limits_{x\to a} f(x)g(x) = 0$.

**(Bad) Proof:** $\lim\limits_{x\to a} f(x)g(x) = \left[\lim\limits_{x\to a} f(x)\right] \cdot \left[\lim\limits_{x\to a} g(x)\right] = 0 \cdot \left[\lim\limits_{x\to a} g(x)\right] = 0$,
because 0 times anything is 0.

(a) Show that the (Bad) Theorem is false by providing a counterexample.

*Solution:* There are many possible solutions. For example, let $f(x) = x - a$ and $g(x) = \frac{1}{x-a}$. Note that the domain of $g(x)$ is all of $\mathbb{R}$ except the point $a$. Then we have

$$\lim_{x\to a} f(x) = \lim_{x\to a}(x-a) = 0.$$

However,

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a}(x-a)\frac{1}{x-a} = \lim_{x\to a} 1 = 1 \neq 0.$$

(b) Explain why the above proof is incorrect.

*Solution:* The proof is incorrect because we are only allowed to use the first equality,

$$\lim_{x\to a} f(x)g(x) = \left[\lim_{x\to a} f(x)\right] \cdot \left[\lim_{x\to a} g(x)\right],$$

(known as the Limit Law for products) if we already know that both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. Moreover, our counterexample above demonstrates how the limit law may fail if one of the limits does not exist.

5. Prove the following theorem:

   **Theorem:** Let $a \in \mathbb{R}$.
   Let $f$ and $g$ be functions with domain $\mathbb{R}$, except perhaps $a$.
   IF

   - $\lim\limits_{x \to a} f(x) = 0$, and
   - $g$ is bounded.
     (This means that there exists $M > 0$ such that for all $x \in \mathbb{R}$, $|g(x)| \leq M$, except perhaps when $x = a$.)

   THEN $\lim\limits_{x \to a} f(x)g(x) = 0$.

   *Proof:* Let $\varepsilon > 0$ arbitrary.

   (a) Since $g$ is bounded, there exists $M > 0$ such that for all $x \in \mathbb{R}$ except $x = a$, $|g(x)| \leq M$.
   (b) Since $\lim\limits_{x \to a} f(x) = 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x)| < \frac{\varepsilon}{M}$. Let this be our choice of $\delta$.

   Assume that $0 < |x - a| < \delta$. Then

$$\begin{aligned} |f(x)g(x)| &= |f(x)||g(x)| \\ &\leq |f(x)|M \text{ by (a)} \\ &< \frac{\varepsilon}{M} M = \varepsilon \text{ by (b)}. \end{aligned}$$

   □

6. Use the theorem from Question 5 to prove that $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0$.

   *Proof:* Let $f(x) = x$ and $g(x) = \sin(\frac{1}{x})$. We check the hypotheses of the Theorem, for the functions $f$, $g$, and $a = 0$.

   - The function $f$ has domain $\mathbb{R}$. The function $g$ has domain $\mathbb{R}$ except 0.
   - $\lim\limits_{x \to 0} x = 0$
   - $g(x)$ is bounded: for all $x$ in the domain of $g$, $|\sin\left(\frac{1}{x}\right)| \leq 1$.

   Thus all of the hypotheses of the theorem hold, so we may apply the theorem to conclude that

$$\lim_{x \to 0} x \sin \frac{1}{x} = \lim_{x \to 0} f(x)g(x) = 0.$$

   □