MAT 137Y: Calculus!
Bonus Problem Set
Infinite Sums are not commutative

This problem set contains two questions to lead you to one of the most surprising facts about series: infinite sums are not commutative! This is a bonus problem set: you do not need to work on it if you are not interested, but we know some of you will find these problems very rewarding.

First, in Question 1 you will derive a stronger version of the Integral Test. Then, in Question 2 you will use this new theorem to compute the value of various reorderings of the same series.

These two questions appear long, but it is only because we have broken the problem into very small steps with lots of hints.

1. Let us fix a continuous function f with domain at least $[1, \infty)$. For every natural number N we define $S_N = \sum_{n=1}^{N} f(n)$ and $I_N = \int_1^{N+1} f(x)\, dx$.

   (a) What is the relation between the series $\sum_{n=1}^{\infty} f(n)$ and the sequence $\{S_N\}_{N=1}^{\infty}$?

   (b) What is the relation between the improper integral $\int_1^{\infty} f(x)\, dx$ and the sequence $\{I_N\}_{N=1}^{\infty}$?

   (c) Assume the function f is positive and decreasing. Draw a picture that shows I_N as an area. In the same picture, draw the Riemann sum for this integral when we divide the interval of integration into subintervals with length 1 each and we use either the left endpoint or the right endpoint. One of these two Riemann sums is S_N. Which one?

   (d) For the rest of the problem, assume the function f is decreasing. We now define the sequence $\{b_N\}_{N=1}^{\infty}$ by the formula $b_N = S_N - I_N$. Prove that the sequence $\{b_N\}_{N=1}^{\infty}$ is increasing. (You may want to use the picture in Question 1c.)

   (e) For the rest of the problem, assume the function f is positive. Prove that $b_N \leq f(1)$ for every $N \geq 1$. (You may want to use the picture in Question 1c again.)

(f) Prove that the sequence $\{b_N\}_{N=1}^{\infty}$ is convergent. Make sure to cite exactly the Theorem you are using.

(g) Now let us call $C = \lim_{N\to\infty} b_N$. We know this is a number, but we do not know its value. Let us also define a new sequence $\{\varepsilon_N\}_{N=1}^{\infty}$ by the equation $\varepsilon_N = b_N - C$ for all $N \geq 1$. What is $\lim_{N\to\infty} \varepsilon_N$?

If you put all of the above together, you have proven

> **Theorem (Strong Integral Test).** *Let f be a continuous, positive, decreasing function with domain at least $[1,\infty)$. For every natural number N we define $S_N = \sum_{n=1}^{N} f(n)$ and $I_N = \int_1^{N+1} f(x)\,dx$.*
>
> *Then there exists a real number C and a sequence $\{\varepsilon_N\}_{N=1}^{\infty}$ such that $\lim_{N\to\infty} \varepsilon_N =???$ and $S_N = I_N + C + \varepsilon_N$ for all N.*

(You need to replace the question marks with your answer to Question 1g). The Integral Test (Theorem 12.3.2 on page 586) is a direct consequence of this new theorem.

2. In this question, we are going to study the following two series:

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots \tag{1}$$

$$B = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \frac{1}{17} + \frac{1}{19} - \frac{1}{10} + \dots \tag{2}$$

If you look carefully, you will notice that both series contain the exact same terms, but added in a different order. The surprising result is that the two series are convergent, but they are convergent to different numbers! This is what you are going to prove.

(a) First, we need some notation. For every natural number N, we define the N-th harmonic sum as

$$H_N = \sum_{n=1}^{N} \frac{1}{n}$$

Notice that H_N is what we called S_N in question 1 for the function $f(x) = \dfrac{1}{x}$. For this particular function, apply the Strong Integral Test to prove that there exists a sequence $\{\varepsilon_N\}_{N=1}^{\infty}$ and number C such that

$$H_N = \ln(N+1) + C + \varepsilon_N \tag{3}$$

and such that $\lim_{N\to\infty} \varepsilon_N = 0$.

(b) Now consider the following sums:

$$E_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2N}$$
$$O_N = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \ldots + \frac{1}{2N-1}$$

Find a formula for E_N and O_N in terms of H_N and H_{2N}.

(c) Now we turn our attention to the infinite sum in Equation (1). We interpret this as a series. Let us call A_N the N-th partial sum of this series (that is, the sum of the first N terms). With the help from your answer to Question 2b, find a formula for A_{2N} in terms of H_N and H_{2N}.

(d) Now use Equation (3) in your answer to Question 2c to compute $\lim_{N\to\infty} A_{2N}$. Obtain the value of the infinite sum A from it.

(e) Do a similar (albeit a bit more convoluted) process to calculate the value of the infinite sum B in Equation (2).