MAT137Y1 - Calculus!
Test 3 Solutions
3rd February, 2017

1. *[6 points]*

   (a) Calculate $\int e^{2x}\, dx$.

   Your answer: $\dfrac{e^{2x}}{2} + c$

   (b) Find a function F satisfying

   i. $F'(x) = x + 1$ for all x.

   ii. $F(1) = 1$

   Your answer: $F(x) = \dfrac{x^2}{2} + x - \dfrac{1}{2}$

   We know that

   $$F(x) = \frac{x^2}{2} + x + c$$

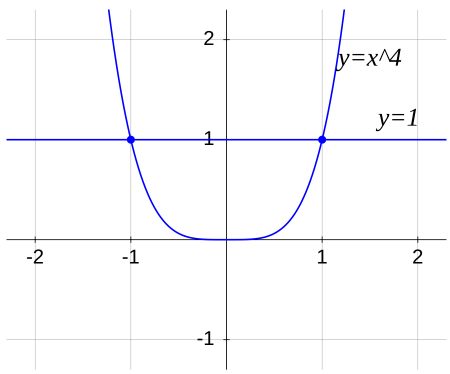
   and

   $$1 = F(1) = \frac{1}{2} + 1 + c \Rightarrow c = -\frac{1}{2}$$

2. *[6 points]*

(a) Calculate the area of the region bounded between the curves $y = x^4$ and $y = 1$.

Your answer: The area is $8/5$.



$$\text{Area} = \int_{-1}^{1} \left(1 - x^4\right) dx = \left(x - \frac{x^5}{5}\right)\Big|_{x=-1}^{x=1}$$
$$= \left(1 - \frac{1}{5}\right) - \left(-1 + \frac{1}{5}\right) = \frac{8}{5}$$

(b) Given the function $G(x) = \int_{2x^2}^{3} \frac{\sin t}{t}\, dt$, calculate $G'(1)$.

Your answer: $G'(1) = -2\sin(2)$

$G(x) = -\int_{3}^{2x^2} \frac{\sin t}{t} dt$. Using the FTC combined with chain rule:

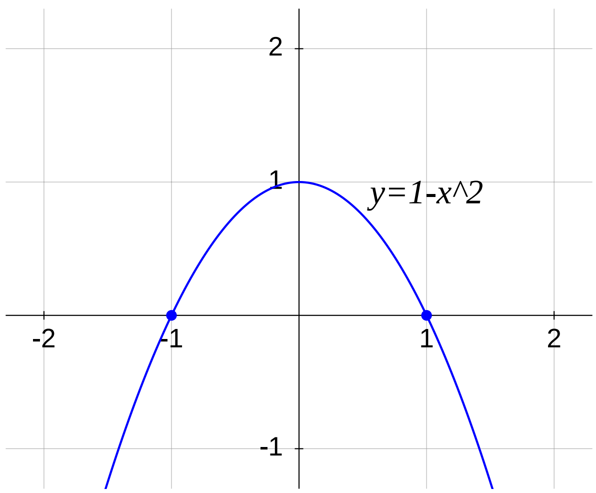
$$G'(x) = -\frac{\sin(2x^2)}{2x^2} \cdot \frac{d}{dx}\left(2x^2\right) = -\frac{\sin(2x^2)}{2x^2} \cdot 4x = \frac{-2\sin(2x^2)}{x}$$

Thus $G'(1) = -2\sin(2)$.

3. *[4 points]* Let R be the region bounded by the x-axis and the curve $y = 1 - x^2$. Calculate the volume of the solid of revolution obtained by rotating the region R around the line $y = 0$.

Your answer: The volume is $\dfrac{16\pi}{15}$

The region we rotate is as in the picture



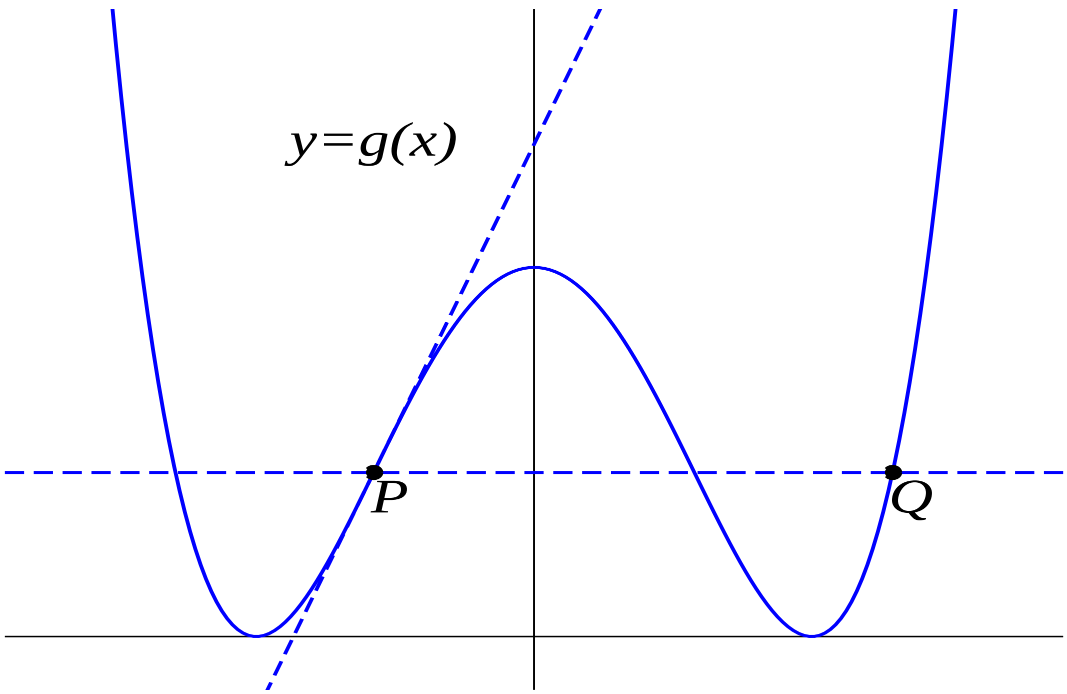
Using the method of cylindrical shells:

$$\text{Volume} = \pi \int_{-1}^{1} \left(1 - x^2\right)^2 dx = \pi \left(x - \frac{2x^3}{3} + \frac{x^5}{5} \right) \Big|_{x=-1}^{x=1} = \frac{16\pi}{15}$$

4. *[4 points]* Below is the graph of the function

$$g(x) = x^4 - 6x^2 + 9.$$

The graph is not to scale. On the graph we have marked two points P and Q. Find the coordinates of these two points.



Your answer: $P = (-1, 4)$ $Q = (\sqrt{5}, 4)$

P is an inflection point of g. The x-coordinate of P satisfies $g''(x) = 0$. We solve:

$$0 = g''(x) = 12x^2 - 12 = 12(x-1)(x+1)$$

Since $x < 0$, it must be $x = -1$. Moreover $g(-1) = 4$ so $P = (-1, 4)$. P and Q have the same y-coordinate. To find the x-coordinate of Q we must find the largest value of x such that $g(x) = 4$. We solve:

$$4 = g(x) = x^4 - 6x^2 + 9 \iff 0 = x^4 - 6x^2 + 5 = (x^2-1)(x^2-5)$$

so the x-coordinate of Q is $\sqrt{5}$.

5. *[5 points]* You are the owner of a toy company. You are considering selling the new Isaac Newton Action Figure. It costs you \$10 to build each toy. There is a fixed operating cost of \$50 per month. In addition, there is a one-time initial investment of \$1,000. Research shows that if you were to sell the toys for $\$x$ per unit, you would sell $\dfrac{10,000}{x^2}$ of them per month. Will you be able to recover the initial investment in 4 months?

Your answer: No.

The profit made in 4 months can be written as a function of x,

$$P(x) = 4(x-10)\frac{10000}{x^2} - 4 \cdot 50 - 1000$$

with domain $x \in (0, \infty)$. Let's optimize P.

$$0 = P'(x) = -\frac{40,000}{x^2} + \frac{800,000}{x^3} \implies x = \frac{800,000}{40,000} = 20$$

Thus $x = 20$ is the only critical point. Notice that $P'(x) > 0$ when $0 < x < 20$ and $P'(x) < 0$ when $x > 20$. Thus P has an absolute maximum at $x = 20$.

At $x = 20$,

$$P(20) = 4(20-10)\frac{10000}{400} - 1200 = -200$$

So the company cannot make a profit in 4 months.

6. *[5 points]* Let $a < b$. Let f be a bounded function on $[a, b]$.

(a) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Write down the definition of the P-upper sum of f.

You may use Σ-notation. If you use any other variables, define them.

Solution: $U_f(P) = \displaystyle\sum_{i=1}^{n} M_i f \cdot \Delta x_i$,

where $M_i = \displaystyle\sup_{x \in [x_{i-1}, x_i]} f(x)$, and $\Delta x_i = x_i - x_{i-1}$.

(b) As you know, the lower integral of f from a to b is defined as

$$\underline{I_a^b}(f) \;=\; \sup \{ \text{ lower sums of } f \ \}.$$

As you know, not every set has a supremum. But in this case, the supremum always exists and the lower integral is well-defined. Why?

Solution. By the Least Upper Bound Principle, if a set is bounded above and non-empty, then it has a supremum. Let $S = \{$ lower sums of $f\}$. We need to check that S is bounded above and non-empty.

- Call $P_0 = \{a, b\}$ the trivial partition. Then every lower sum P satisfies that

  $$L_f(P) \leq U_{P_0}(f)$$

  Thus S is bounded above.
- The set S is non-empty because $L_f(P_0) \in S$.

7. *[2 points]* Let f be a function defined on an interval I. We say that f is a *green function on I* when it satisfies the following property:

> For every three points P, Q, and R on the graph of f that come in this order from left to right, the slope of the secant line through P and Q is smaller than the slope of the secant line through Q and R.

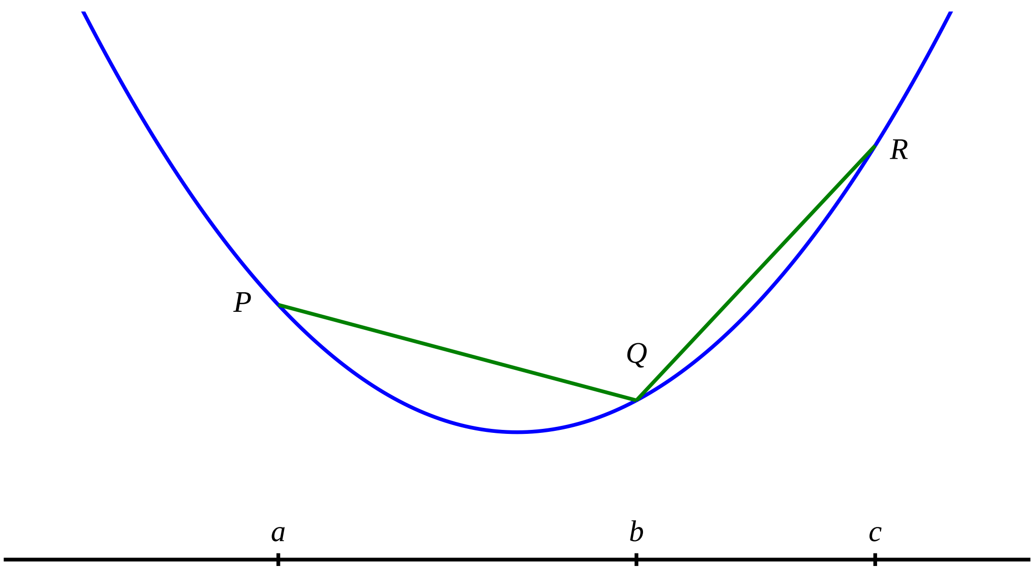
This is equivalent to saying that

$$\forall a, b, c \in I, \quad a < b < c \implies \text{ INEQUALITY}$$

Figure out what the INEQUALITY is.

Hint: Drawing a picture will help.

Your answer: $\dfrac{f(b) - f(a)}{b - a} < \dfrac{f(c) - f(b)}{c - b}$



Note: In some books, "green function" is actually the definition of "concave-up function".

8. *[8 points]*

(a) State the Mean Value Theorem

Let f be a function defined on $[a, b]$. IF

- f is continuous on $[a, b]$ AND
- f is differentiable on (a, b).

THEN there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) Let g be a function defined on an interval I.
Define what it means for g to be *increasing* on the interval I.

g is *increasing on* I if

$$\forall x_1, x_2 \in I, x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$$

(c) Prove the following theorem.

Theorem. Let f be a differentiable function on an open interval I.
IF f' is increasing on I,
THEN f is a green function on I.

Suggestion: Use questions 7, 8a, and 8b.

Let $a, b, c \in I$ with $a < b < c$.

- Since f is differentiable on I, f is continuous on $[a, b]$ and differentiable on (a, b). The same is true for $[b, c]$.
- By the MVT on $[a, b]$, $\exists c_1 \in (a, b)$ such that

$$f'(c_1) = \frac{f(b) - f(a)}{b - a}$$

- By the MVT on $[b, c]$, $\exists c_2 \in (b, c)$ such that

$$f'(c_2) = \frac{f(c) - f(b)}{c - b}$$

- Since $c_1 < b < c_2$ and f' is increasing on I, it follows that

$$\frac{f(b) - f(a)}{b - a} = f'(c_1) < f'(c_2) = \frac{f(c) - f(b)}{c - b}$$

So f is a green function.