MAT 137Y: Calculus!
Problem Set 6
Due by 3pm on Friday, January 27, 2017 online via crowdmark

Instructions:

- For instructions on how to submit online, see `http://uoft.me/137CM`
  You will need to submit the answer to each question separately.
- Before you attempt this problem set we recommend you watch the videos on the definition of the integral and that you do the practice problems from sections 11.1, 12.1, 5.1, 5.2.

IMPORTANT NOTES ON COLLABORATION

Solving a mathematical problem set has two parts:

1. The discovery phase. This is the time you spent trying to figure out how to solve the problems, and it often takes most of the time. You are welcome and encouraged to collaborate with other students in this phase. Collaboration is a healthy practice, and this is how mathematics is done in real life.
2. The write-up phase. This consists of writing your solutions once you have an idea of how the problem can be solved. You should do this entirely by yourself. Be alone when you write your solutions. If you collaborate on this part, or you copy part of your solutions from somebody else, or you have notes written by somebody else in front of you when you write your solutions, or you use a draft or sketch that you wrote in collaboration with somebody else, you are engaging in academic misconduct.

The University of Toronto takes academic integrity very seriously. We are obligated to report all suspected instances of misconduct to OSAI. Please do not force us to do so. Every year, multiple students in this course are disciplined. For more information, see
`http://www.artsci.utoronto.ca/newstudents/transition/academic/plagiarism`.

1. Let f be a function with domain $(0,1)$. Assume f is increasing. Assume f is bounded. Let us call $L = \sup f$. Prove that $L = \lim_{x\to 1^-} f(x)$.

   Note: Drawing a picture will help. You need to prove that the number $L = \sup f$ satisfies the definition of limit. In your proof, you will probably need to use the definition of sup. Beware of assuming that $\lim_{x\to 1^-} f(x)$ exists as part of your proof.

   Solution: We want to prove the definition of the limit $L = \lim_{x\to 1^-} f(x)$, which is

   $$\forall \varepsilon > 0, \exists \delta > 0\colon\ 1-\delta < x < 1 \Rightarrow |L - f(x)| < \varepsilon.$$

   Fix $\varepsilon > 0$ arbitrary.

   - Since the supremum of f on $(0,1)$ is L, the number $L-\varepsilon$ is not an upper bound of f on $(0,1)$, and hence there exists $x_0 \in (0,1)$ such that

     $$L - \varepsilon < f(x_0).$$

   - Set $\delta = 1 - x_0$.

   Next I need to verify that

   $$1-\delta < x < 1 \Rightarrow |L - f(x)| < \varepsilon.$$

   Let x be any number such that $1-\delta < x < 1$.

   - Since f is increasing on $(0,1)$,

     $$x_0 = 1-\delta < x < 1 \Rightarrow f(x_0) < f(x).$$

   - In addition, since L is the supremum of f on $(0,1)$:

     $$f(x) \le L$$

   - Putting all the above equations together:

     $$L - \varepsilon < f(x_0) < f(x) \le L < L + \varepsilon$$

   and hence

   $$|f(x) - L| < \varepsilon$$

   as we wanted.

2. Before solving this problem, watch videos 10 and 11 again. In this problem, you are going to compute the exact value of the integral $I = \displaystyle\int_1^5 \left(x^3 - 2x\right)\, dx$ using Riemann sums. Let us call $f(x) = x^3 - 2x$. Since f is continuous on $[1, 5]$, we know it is integrable. Hence, its value can be computed using Riemann sums as video 11 explains.

For every natural number n, let us call P_n the partition that splits $[1, 5]$ into n *equal* sub-intervals. Notice that $\lim\limits_{n\to\infty} ||P_n|| = 0$. Hence, we can write $I = \lim\limits_{n\to\infty} S^\star_{P_n}(f)$ where $S^\star_{P_n}(f)$ is any Riemann sum for f and P_n. In particular, to make things simpler, we will use Riemann sums always choosing the right end-point to evaluate f on each subinterval.

(a) What is the length of each sub-interval in P_n?

$$\frac{4}{n}$$

(b) Let us write $P_n = \{x_0, x_1, \ldots, x_n\}$. Find a formula for x_i in terms of i and n.

$$x_i = 1 + \frac{4i}{n}$$

(c) Since we are using the right-endpoint, it means we are picking $x_i^\star = x_i$. Use your above answers to obtain an expression for $S^\star_{P_n}(f)$ in the form of a sum with sigma notation.

$$\begin{aligned}
S^\star_{P_n}(f) &= \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i \\
&= \sum_{i=1}^{n} \left[\left(1 + \frac{4i}{n}\right)^3 - 2\left(1 + \frac{4i}{n}\right)\right]\cdot \frac{4}{n} \\
&= \frac{4}{n}\sum_{i=1}^{n}\left[-1 + \frac{4i}{n} + \frac{3\cdot 16\cdot i^2}{n^2} + \frac{64 i^3}{n^3}\right] \\
&= \frac{-4}{n}\sum_{i=1}^{n} 1 + \frac{16}{n^2}\sum_{i=1}^{n} i + \frac{3\cdot 16\cdot 4}{n^3}\sum_{i=1}^{n} i^2 + \frac{4\cdot 64}{n^4}\sum_{i=1}^{n} i^3.
\end{aligned}$$

(d) Using the formulas

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \quad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4}$$

if needed, add up the expression you got to obtain a nice, compact formula for $S^\star_{P_n}(f)$ without any sums or sigma symbols.

$$
\begin{aligned}
&\frac{-4}{n}\sum_{i=1}^{n} 1 + \frac{16}{n^2}\sum_{i=1}^{n} i + \frac{3\cdot 16\cdot 4}{n^3}\sum_{i=1}^{n} i^2 + \frac{4\cdot 64}{n^4}\sum_{i=1}^{n} i^3 \\
&= \frac{-4}{n}n + \frac{16}{n^2}\frac{n^2+n}{2} + \frac{3\cdot 16\cdot 4}{n^3}\frac{n(2n^2+3n+1)}{6} + \frac{4\cdot 64}{n^4}\frac{n^2(n^2+2n+1)}{4} \\
&= -4 + 8\left(1+\frac{1}{n}\right) + 32\left(2+\frac{3}{n}+\frac{1}{n^2}\right) + 64\left(1+\frac{2}{n}+\frac{1}{n^2}\right) \\
&= 132 + \frac{8+3\cdot 32+128}{n} + \frac{32+64}{n^2} \\
&= 132 + \frac{232}{n} + \frac{96}{n^2}.
\end{aligned}
$$

(e) Calculate $\lim_{n\to\infty} S^\star_{P_n}(f)$. This number will be the exact value of $\int_1^5 (x^3-2x)dx$.

Hint: Your final answer should be 132.

By (d),

$$\lim_{n\to\infty} S^\star_{P_n}(f) = \lim_{n\to\infty}\left(132 + \frac{232}{n} + \frac{96}{n^2}\right) = 132.$$

3. In this problem we want you to experience how to prove one property of integrals directly from the definition. It is long and difficult, so we have broken down the process into many small steps. You will prove that if two functions are equal everywhere except at one point, then they must have the same integral. More formally, you are going to prove the following theorem:

Theorem.
Let $a < b$. Let f and g be functions defined on $[a, b]$.
Assume f and g are bounded. For simplicity, let's say $\exists M \in \mathbb{R}$ such that for all $x \in [a, b]$, $|f(x)| \le M$ and $|g(x)| \le M$.
Assume also that f and g are equal except at one point. Let's say there exists $c \in (a, b)$ and that

$$\forall x \in [a,b],\ x \neq c \implies f(x) = g(x)$$

In this situation, f is integrable on $[a, b]$ IFF g is also integrable on $[a, b]$.
In addition, in that case

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

We are going to guide you, in small steps, to do a proof of this theorem directly from the definition of integral.

(a) Prove that $\forall\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U_f(P) < \overline{I}_a^b(f) + \varepsilon$$

Hint: Remember that the upper integral is defined as the infimum of something.

Solution: Let $\varepsilon > 0$. By definition, $\overline{I}_a^b(f)$ is the greatest lower bound of the set of all upper sums. Hence the number $\overline{I}_a^b(f) + \varepsilon$ is not a lower bound of the set of upper sums. This means there exists a partition P of $[a, b]$ such that

$$U_f(P) < \overline{I}_a^b(f) + \varepsilon$$

(b) *(Hard)* Prove that $\forall$ partition P of $[a, b]$ and $\forall\varepsilon > 0$,

$$\exists \text{ partition } Q \text{ of } [a, b] \text{ s.t. } P \subseteq Q \text{ and } |U_f(Q) - U_g(Q)| \leq \varepsilon.$$

Hint: For an arbitrary partition Q, what is $U_f(Q) - U_g(Q)$?

Solution: Let's begin by following the hint.

Let $Q = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. The functions f and g are identical on each subinterval $[x_{k-1}, x_k]$ (and hence have the same supremum and infimum) unless $c \in [x_{k-1}, x_k]$. There are one or two such subintervals (depending on whether c itself is a point in the partition or not).
First consider the case when c is not a point of the partition and $c \in (x_{k-1}, x_k)$. By definition of upper sum

$$U_f(Q) - U_g(Q) = \Delta x_k \left(\sup_{[x_{k-1},x_k]} f - \sup_{[x_{k-1},x_k]} g \right)$$

Since $|f|$ and $|g|$ are bounded by M, it follows that

$$|U_f(Q) - U_g(Q)| = \left| \Delta x_k \left(\sup_{[x_{k-1},x_k]} f - \sup_{[x_{k-1},x_k]} g \right) \right| \leq \Delta x_k \cdot 2M.$$

(The case when c is a point in the partition produces a very similar equation.)

Now suppose P is an arbitrary partition of $[a, b]$ and fix $\varepsilon > 0$ arbitrary. Let Q be the partition obtained by adding the points $c - \frac{\varepsilon}{4M}$ and $c + \frac{\varepsilon}{4M}$ to P (these points may already be in the partition P, which is fine).

By definition, $P \subseteq Q$.

Since Q contains the points $c - \dfrac{\varepsilon}{4M}$ and $c + \dfrac{\varepsilon}{4M}$, the subinterval $[x_{k-1}, x_k]$ of Q that contains c is either

$$\left[c - \frac{\varepsilon}{4M}, c + \frac{\varepsilon}{4M}\right].$$

or something smaller. Either way, $\Delta x_k \leq \dfrac{\varepsilon}{2M}$. Applying this to the equation above, we get

$$|U_f(Q) - U_g(Q)| \leq \Delta x_k \cdot 2M \leq \frac{\varepsilon}{2M} \cdot 2M = \varepsilon.$$

(c) *(Hard)* Use parts 3a and 3b to prove that

$$\forall \varepsilon > 0,\ \exists \text{ partition } R \text{ of } [a, b] \text{ s.t. } U_g(R) \leq \overline{I}_a^b(f) + \varepsilon$$

Hint: When you use part 3b, remember: what is the relation between $U_f(Q)$ and $U_f(P)$?
Solution: Fix $\varepsilon > 0$ arbitrary.

By part (a), using $\varepsilon/2$ instead of ε, there is a partition P of $[a, b]$ such that

$$U_f(P) < \overline{I}_a^b(f) + \frac{\varepsilon}{2}.$$

By part (b), using $\varepsilon/2$ instead of ε, there is a partition R such that $P \subseteq R$ and

$$|U_f(R) - U_g(R)| \leq \frac{\varepsilon}{2}.$$

Combining these two facts, we have that

$$\begin{aligned} U_g(R) &= U_f(R) - U_f(R) + U_g(R) \\ &\leq U_f(P) - U_f(R) + U_g(R) \text{ since } R \subseteq P \\ &\leq U_f(P) + |U_f(R) - U_g(R)| \\ &< \overline{I}_a^b(f) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \overline{I}_a^b(f) + \varepsilon \text{ applying (a) and (b).} \end{aligned}$$

(d) Use part 3c to prove that

$$\forall \varepsilon > 0,\ \overline{I}_a^b(g) \leq \overline{I}_a^b(f) + \varepsilon$$

Hint: Remember the definition of upper integral.

Solution: By (c), for all $\varepsilon > 0$, there exists a partition R of $[a, b]$ such that

$$U_g(R) \leq \overline{I}_a^b(f) + \varepsilon.$$

Since $\overline{I}_a^b(g)$ is a lower bound for the set of upper sums of g, $\overline{I}_a^b(g) \leq U_g(R)$.

Combining these two facts, we have that for all $\varepsilon > 0$,

$$\overline{I}_a^b(g) \leq \overline{I}_a^b(f) + \varepsilon.$$

(e) Prove that

$$\overline{I}_a^b(g) = \overline{I}_a^b(f)$$

Solution: We know that

$$\forall \varepsilon > 0,\ \overline{I}_a^b(g) \leq \overline{I}_a^b(f) + \varepsilon.$$

Hence

$$\overline{I}_a^b(g) \leq \overline{I}_a^b(f).$$

In addition, the roles of f and g are entirely interchangeable in the statement of the theorem. Hence, we could repeat everything we have done by swapping f and g and conclude that

$$\overline{I}_a^b(f) \leq \overline{I}_a^b(g).$$

The two inequalities together tell us that

$$\overline{I}_a^b(g) = \overline{I}_a^b(f).$$

(f) **(Do not submit)**. Repeat the previous questions with lower sums and lower integrals instead of upper sums and upper integrals to conclude that

$$\underline{I}_a^b(g) = \underline{I}_a^b(f)$$

(g) Prove the theorem.

Solution: From (e) we have that

$$\overline{I}_a^b(g) = \overline{I}_a^b(f).$$

From (f) we have that

$$\underline{I}_a^b(g) = \underline{I}_a^b(f).$$

By definition of integrable functions, f is integrable if and only if

$$\overline{I}_a^b(f) = \underline{I}_a^b(f)$$

which by (e) and (f) is true if and only if

$$\overline{I}_a^b(g) = \underline{I}_a^b(g)$$

which by definition of integrable functions, is true if and only if g is integrable.

Finally, if f and g are integrable then by definition of the definite integral and (e), we have that

$$\int_a^b f = \overline{I}_a^b(f) = \overline{I}_a^b(g) = \int_a^b g.$$