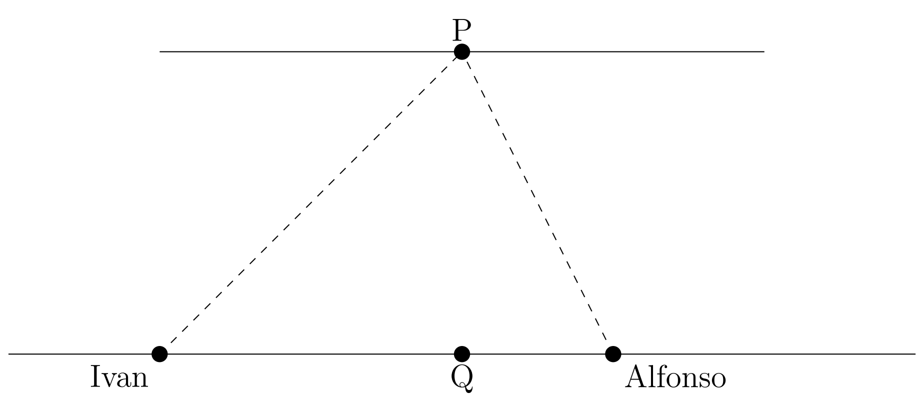
MAT 137Y: Calculus!
Problem Set B.

This problem set is intended to help you prepare for Test #2. It is not comprehensive: it only contains problems from some sections that were not included in past problem sets or in past tutorials. You do not need to turn in any of these problems.

1. Let $a \in \mathbb{R}$. Let f be a function which is differentiable at a. Assume that $f(a) > 0$.
   (a) Prove, using the definition of limit, that the function f must be positive near a. In other words, prove that there exists $\delta > 0$ such that $|x-a| < \delta \implies f(x) > 0$.
   (b) We define a new function g by $g(x) = \sqrt{f(x)}$. Prove that g is differentiable at a and that
   $$g'(a) = \frac{f'(a)}{2\sqrt{f(a)}}.$$
   Do a proof directly from the definition of derivative as a limit without using any of the other differentiation rules. For examples of this kind of proof, see the various theorems in Section 2.3 of the book.

2. Alfonso and Ivan are tied at opposite ends of a rope of fixed length. The rope is fully stretched; it starts at Ivan's foot, then passes through a fixed point P on the ceiling, and then continues to Alfonso's foot. The rope is the dashed line in the picture below. The point P is on the ceiling, $4m$ directly above the point Q on the floor. Ivan is running away from Alfonso while Alfonso is just being dragged along the floor. How fast is Alfonso moving at the time when Ivan is $4m$ away from point Q, the distance between Alfonso and Ivan is $6m$, and Ivan's velocity is $1m/s$?



3. Let
$$f(x) = \frac{(x^7+3x-3)^{10}\,\sqrt{x^2+5x+3}}{\sqrt[3]{x^2+2x+5}}.$$
Find the equation of the line tangent to the graph of f at the point with x-coordinate 1.

Hint: This is a short-ish computation if you use logarithmic differentiation.

4. Reread the formal of definition of $\lim_{x\to c} f(x) = L$ (this is Definition 2.2.1 on page 64 of the textbook). Then review Question 6 on Problem Set A. We now have new types of limit. For example, here is the definition of $\lim_{x\to\infty} f(x) = L$:

Let f be a function defined at least on an open interval (p, ∞) for some real number p. We say that
$$\lim_{x\to\infty} f(x) = L$$
when for every $\varepsilon > 0$, there exists $M \in \mathbb{R}$ such that
$$\textit{if} \quad x > M, \quad \textit{then} \quad |f(x) - L| < \varepsilon.$$

(a) Write a formal definition of the concept $\lim_{x\to-\infty} f(x) = L$.

(b) Write a formal definition of the concept $\lim_{x\to\infty} f(x) = \infty$.

(c) Write a formal definition of the concept "$\lim_{x\to\infty} f(x)$ DNE".

(d) Prove, from the formal definition, that $\lim_{x\to\infty} \frac{x}{x+1} = 1$.

5. In this problem, we will study the following two functions:
$$\diamondsuit(x) = \frac{e^x + e^{-x}}{2}, \qquad \heartsuit(x) = \frac{e^x - e^{-x}}{2}.$$

(a) Compute $\diamondsuit'(x)$ and $\heartsuit'(x)$.

(b) Simplify the expression $\diamondsuit^2(x) - \heartsuit^2(x)$.
Note: $\diamondsuit^2(x)$ means $(\diamondsuit(x))^2$.

(c) $\heartsuit$ is one-to-one. (You do not need to prove it.) Let $\spadesuit$ be its inverse function. Find an explicit formula for $\spadesuit(y)$.

Note: If you are having trouble finding an expression for the inverse, consider the following easier questions first:

- Solve for t: $t^2 + 6t + 4 = 0$.
- Solve for u: $e^{2u} + 6e^u + 4 = 0$.
- Solve for u: $e^{2u} + 6ae^u + 4 = 0$.

(d) Use your answer to Question 5c to obtain a formula for $\spadesuit'(y)$.

(e) There is a faster way to obtain a formula for $\spadesuit'(y)$ without having to obtain an explicit formula for $\spadesuit(y)$ first! Start with the identity

$$\heartsuit(\spadesuit(y)) = y,$$

take the derivative with respect to y on both sides, and use Questions 5b and 5a to obtain a formula for $\spadesuit'(y)$. This should agree with your result to Question 5d.