MAT 137Y: Calculus!
Problem Set 1
Due on Friday, September 30, 2016 by 3pm

1. Negate the following statement without using any negative words (“no”, “not”, “none”, etc.):

   “*Every MAT137 student who was born in a Canadian province with a name that starts with a letter that comes alphabetically before ‘L’ obtains less than 75% on one of their term tests and more than 75% on another.*”

   Solution: There exists a MAT137 student who was born in a Canadian province with a name that starts with a letter that comes alphabetically before ‘L’ , and who obtains either 75% or more on all of their term tests, or 75% or less on all of their term tests.

2. Construct a function f that satisfies all of the following properties at once:

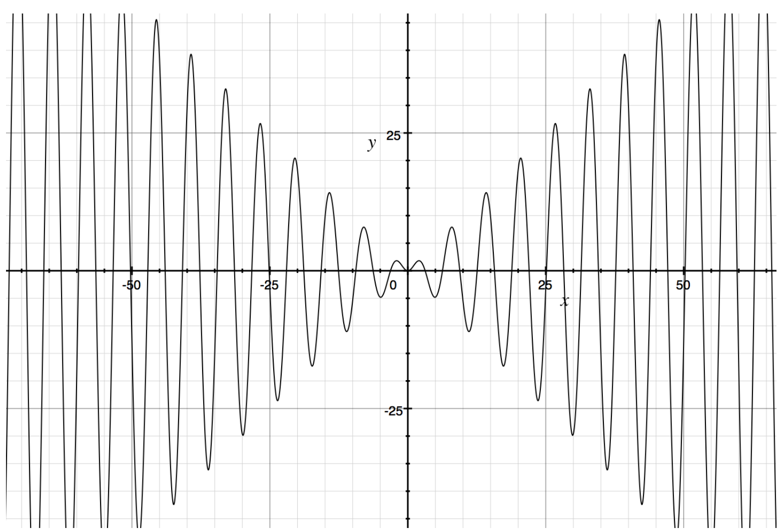
   (a) The domain of f is $\mathbb{R}$.

   (b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},$ such that $x < y$ and $f(x) < f(y)$.

   (c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},$ such that $x < y$ and $f(x) > f(y)$.

   Solution: There are many different correct solutions. Here is an example:

$$f(x) = x\sin(x)$$



To show that properties (b) and (c) are true, consider, for $n \in \mathbb{N}$, the values $y_n = (4n+1)\frac{\pi}{2}$ and $y'_n = (4n+3)\frac{\pi}{2}$.

- Notice that $f(y_n) = y_n$, and we can take y_n as large as needed, so this guarantess that property (b) is true.
- Notice that $f(y'_n) = -y'_n$, and we can take y'_n are large as needed, so this guarantess that property (c) is true.

3. Let $A \subseteq \mathbb{R}$. We define two new concepts:

- We say that A is a *blue* set when $\exists M \in \mathbb{R}$ such that for every $x \in A$, $x < M$.
- We say that A is a *sad* set when $\exists M \in \mathbb{R}$ such that for every $x \in A$, $x \leq M$.

(a) The following argument is WRONG. Explain why.

> Being blue and being sad are not the same thing. For example, take the set $A = [0, 1]$ and $M = 1$. Then the set A is sad because for every $x \in A$, $x \leq M$. However, A is not blue because $x = 1 \in A$ but $x \not< M$.

Solution: In this argument, the proof that $[0, 1]$ is sad is correct, but the proof that $[0, 1]$ is blue is wrong. We have only checked that one value of M does not work in the definition of blue, but there could be other values of M that work. Indeed, A is blue because $x < 2$ for all $x \in A$, so we can take $M = 2$. The flaw of the argument is that it falsely assumes that we must use the same M in the definitions of blue and sad sets.

(b) Prove that a blue set is exactly the same thing as a sad set.

Note: You have to prove two things. First prove that if a set is blue, then it must be sad. Second prove that if a set is sad, then it must be blue.

Proof:

i. Suppose A is a blue set. By the definition of blue set, there exists $M \in \mathbb{R}$ such that for every $x \in A$, $x < M$.
We want to show that A is a sad set. We must show there exists $N \in \mathbb{R}$ such that for every $x \in A$, $x \leq N$.
We can simply take $N = M$. Since for every $x \in A$, $x < M$, then it is also true that for every $x \in A$, $x \leq M$.

ii. Suppose A is a sad set. By the definition of sad set, there exists $M \in \mathbb{R}$ such that for every $x \in A$, $x \leq M$.
We want to show that A is a sad set. We must show there exists $N \in \mathbb{R}$ such that for every $x \in A$, $x < N$.

We can simply take $N = M + 1$. Then for every $x \in A$, $x \leq M < M + 1 = N$. Thus we have shown that for every $x \in A$, $x < N$. $\square$

(c) Is the empty set blue? Prove it.

Solution: Yes, it is blue.
Proof: By definition, $\emptyset$ is blue iff the following statement is true:

$$\exists M \in \mathbb{R} \text{ such that for every } x \in \emptyset,\ x < M.$$

This statement is true because any real number M the condition $\forall x \in \emptyset, x < M$ is vacuously true. For example, if you take $M = 1$, the statement $\forall x \in \emptyset, x < 1$ is vacuously true. $\square$

(d) Prove the following theorem.

Theorem: Let $A, B \subseteq \mathbb{R}$.
IF A and B are blue,
THEN $A \cup B$ is blue.

Proof:

- Since A is blue, there exists $M \in \mathbb{R}$ such that for every $x \in A$, $x < M$.
- Since B is blue, there exists $M' \in \mathbb{R}$ such that for every $x \in B$, $x < M'$.
- Let us call M'' the larger of the two number, M or M'. We will show that for every $x \in A \cup B$, $x < M''$. This will conclude the proof that $A \cup B$ is blue.
  Let $x \in A \cup B$. This means that $x \in A$ or $x \in B$.
  - If $x \in A$, then $x < M \leq M''$, and hence $x < M''$.
  - If $x \in B$, then $x < M' \leq M''$, and hence $x < M''$.

  $\square$

4. We want to find a formula for the sum S_N, as defined below:

$$
\begin{aligned}
S_1 &= \frac{1}{1 \cdot 3} \\
S_2 &= \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} \\
S_3 &= \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} \\
&\dots \\
S_N &= \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \ldots + \frac{1}{???} \qquad (1)
\end{aligned}
$$

(a) First, clean up the expression in Equation (1). Instead of '???', what should it say?

Solution:

$$S_N = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \ldots + \frac{1}{(2N-1)(2N+1)}$$

(b) Calculate the first few values S_1, S_2, S_3, S_4 (more if needed). Then make a conjecture for a formula for S_N.

Solution:

$$S_1 = \frac{1}{3},\ S_2 = \frac{2}{5},\ S_3 = \frac{3}{7},\ S_4 = \frac{4}{9}.$$

We conjecture that

$$S_N = \frac{N}{2N+1}.$$

(c) Prove your formula using induction.

Proof: We give a proof by induction.
Base case ($n = 1$): $S_1 = \frac{1}{3} = \frac{(1)}{2(1)+1}$.

Induction step: Let $N \geq 1$. Assume that the formula is true for N. That is, we assume that

$$S_N = \frac{N}{2N+1}$$

We want to prove that the formula is true for $N+1$. That is, we want to prove that

$$S_{N+1} = \frac{N+1}{2(N+1)+1} = \frac{N+1}{2N+3}$$

$$
\begin{aligned}
S_{N+1} &= \frac{1}{1\cdot 3} + \cdots + \frac{1}{(2N-1)(2N+1)} + \frac{1}{(2(N+1)-1)(2(N+1)+1)} \\
&= S_N + \frac{1}{(2(N+1)-1)(2(N+1)+1)} \\
&= \frac{N}{2N+1} + \frac{1}{(2N+1)(2N+3)} \text{ by the induction hypothesis.} \\
&= \frac{N(2N+3)+1}{(2N+1)(2N+3)} \\
&= \frac{(N+1)(2N+1)}{(2N+1)(2N+3)} \\
&= \frac{(N+1)}{2N+3}
\end{aligned}
$$

as desired.

Thus we have proven by induction that for all $N \in \mathbb{N}$, $S_N = \dfrac{N}{2N+1}$. $\square$