MAT 137
Tutorial #20– Calculating the value of a series
April 3-4, 2017

You may have noticed that, while we have all these tests to tell whether a series is convergent or divergent, we have not been able to calculate the actual value of many infinite series. Yet. This is about to change.

For your reference, these are your main Taylor series:

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots && \text{for all } x \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots && \text{for all } x \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots && \text{for all } x \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots && \text{for } |x| < 1
\end{aligned}$$

Compute the value of the following series:

1. $\displaystyle\sum_{n=1}^{\infty} n x^n$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}$
5. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+3)n!}$
6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)x^{2n+1}}{(2n+1)!}$
7. $\displaystyle\sum_{n=2}^{\infty} \frac{(4n^2+8n+3)\, 2^n}{n!}$
8. *[Very hard]* $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(4n+1)9^n}$

Hint: This is what you did in Tutorial 19, but backwards. Try to start with a series that looks "similar" to the one you want, but whose value you know. Then manipulate it algebraically and/or take derivatives and/or integrals until you get the series you want.

For example, for Question 1, notice that $\displaystyle\sum_{n=1}^{\infty} n x^{n-1} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$