MAT 137
Tutorial #4– Linear approximations and Newton's method
October 24–25, 2016

Today we are going to use derivatives to do computations and solve equations approximately.

Preparatory questions

These questions should be straightforward. They are the tools you will use on the rest of the tutorial. If you fully understand them, move to question 1 right away. Otherwise, ask your fellow students or your TA.

0. (a) What is the derivative of $f(x) = x^3 + 2x^2 - 1$?
   (b) What is the derivative of $f(x) = \sqrt[n]{x}$?
   (c) Find the equation of the line that goes through the point $(1, 2)$ and has slope 3.
   (d) Write the equation of the line tangent to $y = x^4$ at the point with x-coordinate -2. Draw the graph $y = f(x)$ and the the tangent line.
   (e) What is the intersection of the line $y = 2x + 1$ with the x-axis?

Linear approximations

1. In this problem we will estimate $\sqrt[3]{1.1}$. **For this question, you are not allowed to use a calculator.**
   (a) Consider the function $h(x) = \sqrt[3]{x}$. Notice that our goal is to compute $h(1.1)$. Write the equation of the line tangent to $y = h(x)$ at the point with x-coordinate 1. We will call this line L.
   (b) Consider the point on L with x-coordinate 1.1 and the point on the graph of h with x-coordinate 1.1. Are their y-coordinates close to each other? Use this to obtain an approximate value for $\sqrt[3]{1.1}$.
   (c) Let's say we use the same method (with the same line L) to approximate $\sqrt[3]{1.05}$ and $\sqrt[3]{1.2}$. In which case would we get a smaller error?
   (d) Without using the above method, what do you think the approximate value of $\sqrt[3]{28}$ is? Now, using the same line L as above, what value do you get? You will see that this is a very bad approximation. Why didn't it work?

2. Now you will try the whole thing by yourself. Use a similar method to the previous problem to obtain an approximate value for $\sqrt{3.9}$.

Newton's method

3. For this problem, we define the function $g(x) = x^3 + x - 1$. We want to find a number x such that $g(x) = 0$. In other words, we want to solve the equation $x^3 + x - 1 = 0$. You may use a calculator for this question.

   We suggest you draw a large picture for this question (use a full page) and make sure to label everything carefully.

   (a) First, can you think of any way to solve this equation? (There is a way to obtain the exact solution which involves some higher math. We believe you do not have any tool that would allow you to solve this yet, but we will be happy if you prove us wrong!)

   (b) Calculate $g(0)$ and $g(1)$. This guarantees there has to be some number $0 < x < 1$ such that $g(x) = 0$. Why?

   (c) We are going to make a bunch of successive guesses for the solution to the equation. None of them will be exact, but each one will be better than the previous one. Our first guess is going to be $x_1 = 1$. Write the equation of the line tangent to $y = g(x)$ at the point with x-coordinate x_1. Draw this line. We will call it L_1.

   (d) We are looking for the point of the graph $y = g(x)$ that intersects the x-axis. Since this point is not too far from $(x_1, g(x_1))$, we can look for the point where the line L_1 intersects the x-axis instead. (Convince yourself that this makes sense!) Calculate this point. Call its x–coordinate x_2. This is our second guess.

   (e) Calculuate $g(x_2)$. Notice that $g(x_2)$ is not zero yet but it is closer to zero that $g(x_1)$. We are improving!

   (f) Now repeat the process you did in the last three steps, but starting with x_2 instead of with x_1. Call the new value you obtain x_3. Calculate $g(x_3)$. Is it close enough to zero? Then obtain x_4. Is $g(x_4)$ close enough to 0?

   Keep track of your data:

n	x_n	$g(x_n)$
1	1	
2		
3		
4		
5		

4. Now you try the whole thing by yourself. Use a similar method to the previous problem to find a solution to $x^3 - x - 1 = 0$.