MAT 137Y: Calculus!
Problem Set 9
Due by 3pm on Friday, March 17, 2017 online via crowdmark

1. Let $a < b$. Let f be a continuous function on $[a, \infty)$. Prove that the following two statement are equivalent:

   - The improper integral $\displaystyle\int_a^\infty f(x)dx$ is convergent.
   - The improper integral $\displaystyle\int_b^\infty f(x)dx$ is convergent.

   Do a formal proof directly from the definition of improper integral as a limit.

   Hint: Use the limit laws. You do not need to get dirty with epsilons.

   Proof: By definition,

   - $\displaystyle\int_a^\infty f(x)dx$ is convergent if and only if the limit $\displaystyle\lim_{c\to\infty}\int_a^c f(x)dx$ exists.
   - $\displaystyle\int_b^\infty f(x)dx$ is convergent if and only if the limit $\displaystyle\lim_{c\to\infty}\int_b^c f(x)dx$ exists.

   Thus, we want to prove that:

   The limit $\displaystyle\lim_{c\to\infty}\int_a^c f(x)dx$ exists if and only if the limit $\displaystyle\lim_{c\to\infty}\int_b^c f(x)dx$ exists.

   By properties of definite integrals, we know that for any $b, c \in [a, \infty)$,

   $$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

   Taking limits on both sides of this equation, we have

   $$\lim_{c\to\infty}\int_a^c f(x)dx = \lim_{c\to\infty}\left(\int_a^b f(x)dx + \int_b^c f(x)dx\right).$$

   Since f is continuous on $[a, b]$, it is integrable, so $\displaystyle\int_a^b f(x)dx$ is a finite number. Thus, by elementary limit laws, we have

   $$\lim_{c\to\infty}\int_a^c f(x)dx = \lim_{c\to\infty}\left(\int_a^b f(x)dx + \int_b^c f(x)dx\right) = \int_a^b f(x)dx + \lim_{c\to\infty}\int_b^c f(x)dx.$$

In particular, this equation says that the limit $\lim_{c\to\infty} \int_a^c f(x)dx$ exists if and only if the limit $\lim_{c\to\infty} \int_b^c f(x)dx$ exists.

2. Consider the function $f(x) = \dfrac{\ln x}{1+x^2}$. This function is continuous on $(0, \infty)$. It is unbounded as $x \to 0^+$. In this problem, you will calculate the value of the (doubly) improper integral $\displaystyle\int_0^\infty f(x)dx$. It is doubly improper because it has a problem both as $x \to 0^+$ and as $x \to \infty$.

   (a) Prove that the improper integral $\displaystyle\int_1^\infty f(x)dx$ is convergent.

   Proof: We apply the Limit Comparison Test with $g(x) = \dfrac{\ln(x)}{x^2}$.

   - First, observe that

   $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2 \ln(x)}{(x^2+1)\ln(x)} = 1.$$

   Thus, by LCT, $\displaystyle\int_1^\infty f(x)dx$ is convergent if and only if $\displaystyle\int_1^\infty g(x)dx$ is convergent.

   - Second, $\displaystyle\int_1^\infty g(x)dx$ is convergent. Applying integration by parts with $u = \ln(x)$ and $dv = \frac{dx}{x^2}$, we get

   $$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x^2} &= \frac{-\ln(x)}{x}\bigg|_1^\infty + \int_1^\infty \frac{1}{x^2}dx.\\ &= \lim_{c\to\infty}\left(\frac{-\ln(c)}{c} + \int_1^c \frac{1}{x^2}dx\right)\\ &= 1.\end{aligned}$$

   In the last step we used both the fact that $\ln(x)/x \to 0$ as $x \to \infty$, and computed the integral $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ as in Example 2, page 566.

   Remark: Alternatively, we could use the Big Theorem and the BCT, since

   $$0 \leq \frac{\ln x}{x^2} << \frac{x^{1/2}}{x^2}$$

   and $\displaystyle\int_1^\infty \frac{x^{1/2}}{x^2}dx$ is convergent

   Thus, by LCT, we have shown that $\displaystyle\int_1^\infty f(x)dx$ is convergent.

(b) Prove that the improper integral $\int_0^1 f(x)dx$ is convergent.

Proof: We apply the Basic Comparison Test with $g(x) = \ln(x)$.

- First, observe that for all $0 < x < 1$,

$$\ln(x) \leq \frac{\ln(x)}{1+x^2} \leq 0.$$

Thus, by BCT, if $\int_0^1 \ln(x)dx$ is convergent, then $\int_0^1 f(x)dx$ is convergent. Notice that I have used comparison test for negative functions.

- Second, the improper integral $\int_0^1 \ln(x)dx$ is convergent. We compute

$$\int_0^1 \ln(x)dx = \lim_{a \to 0^+} (x\ln(x) - x)\,\big|_a^1 = -1 - \lim_{a \to 0^+} a\ln(a) = -1.$$

Thus, by the Basic Comparison Test, it follows that $\int_1^\infty f(x)dx$ is convergent.

Remark: We could also do this question using the LCT with $g(x) = \ln(x)$. Since

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{\ln(x)}{(1+x^2)\ln(x)} = \lim_{x \to 0^+} \frac{1}{(1+x^2)} = 1,$$

it follows by LCT that $\int_0^1 f(x)dx$ converges if and only if $\int_0^1 g(x)dx$ converges. Notice that in this case we are comparing two negative functions.

(c) Perform the substitution $u = 1/x$ on $\int_0^1 f(x)dx$. Use the result to calculate $\int_0^\infty f(x)dx$.

Solution: Substituting $u = \frac{1}{x}$, $du = \frac{-1}{x^2}dx$, we have

$$\begin{aligned}
\int_0^1 f(x)dx &= \int_0^1 \frac{\ln(x)}{1+x^2}dx \\
&= \int_\infty^1 \frac{\ln\left(\frac{1}{u}\right)}{1+\left(\frac{1}{u}\right)^2}\left(\frac{-1}{u^2}\right)du \\
&= \int_1^\infty \frac{\ln\left(\frac{1}{u}\right)}{1+\left(\frac{1}{u}\right)^2}\frac{1}{u^2}du \\
&= \int_1^\infty \frac{\ln\left(\frac{1}{u}\right)}{1+u^2}du \\
&= -\int_1^\infty \frac{\ln(u)}{1+u^2}du \\
&= -\int_1^\infty f(x)dx
\end{aligned}$$

Thus

$$\int_0^\infty f(x)dx = \int_0^1 f(x)dx + \int_1^\infty f(x)dx = -\int_1^\infty f(x)dx + \int_1^\infty f(x)dx = 0.$$

(d) It would be incorrect to solve Question 2c before Questions 2b and 2a. Why?

In the last step of Question 2c, we wrote

$$-\int_1^\infty f(x)dx + \int_1^\infty f(x)dx = 0.$$

This can only make sense if

$$\int_1^\infty f(x)dx$$

is a finite number. Otherwise, we might be claiming that

$$\infty - \infty = 0$$

when in fact $\infty - \infty$ is undefined.

3. For each values of a and b in $\mathbb{R}$ we define the (doubly) improper integral

$$I(a,b) = \int_0^\infty \frac{x^a \ln x}{(x^2+1)^b} dx$$

For which values of a and b is the improper integral $I(a,b)$ convergent? For which is it divergent?

Solution: $I(a,b)$ converges if and only if $0 < a+1 < 2b$.

Proof: We will prove that:

(a) $\displaystyle\int_1^\infty \frac{x^a \ln x}{(x^2+1)^b} dx$ converges if and only if $2b - a > 1$.

(b) $\displaystyle\int_0^1 \frac{x^a \ln x}{(x^2+1)^b} dx$ converges if and only if $a > -1$.

By definition of doubly improper integeral

$$\int_0^\infty \frac{x^a \ln x}{(x^2+1)^b} dx \text{ converges} \iff \int_1^\infty \frac{x^a \ln x}{(x^2+1)^b} dx \text{ converges and } \int_0^1 \frac{x^a \ln x}{(x^2+1)^b} dx \text{ converges}$$

The claim then follows because:

$$0 < a+1 < 2b \qquad \iff \qquad (a > -1) \text{ and } (2b - a > 1)$$

Proof of (a): We apply the Limit Comparison Test with $g(x) = \dfrac{\ln(x)}{x^{2b-a}}$.

- First, observe that
$$\lim_{x\to\infty} \frac{\left(\dfrac{x^a \ln x}{(x^2+1)^b}\right)}{\left(\dfrac{\ln(x)}{x^{2b-a}}\right)} = \lim_{x\to\infty} \frac{x^{2b}}{(1+x^2)^b} = \lim_{x\to\infty} \left(\frac{1}{x^2}+1\right)^{-b} = 1.$$
Thus by LCT, $\displaystyle\int_1^\infty \frac{x^a \ln x}{(x^2+1)^b}dx$ converges if and only if $\displaystyle\int_1^\infty g(x)dx$ converges.

- Second, $\displaystyle\int_1^\infty g(x)dx$ converges if and only if $2b-a>1$. There are two cases:
  - **Case 1:** $(2b-a=1)$ In this case we apply u-substitution to evaluate
  $$\int_1^\infty g(x)dx = \int_1^\infty \frac{\ln(x)}{x}dx = \lim_{c\to\infty} \frac{\ln(c)^2}{2} = \infty.$$
  Thus, this improper integral is divergent.
  - **Case 2:** $(2b-a\neq 1)$ In this case, we may apply integration by parts with $u=\ln(x)$ and $dv = x^{a-2b}dx$. This yields
  $$\int_1^\infty \frac{\ln(x)}{x^{2b-a}}dx = \lim_{c\to\infty} \frac{\ln(c)c^{a-2b+1}}{a-2b+1} - \frac{1}{a-2b+1}\int_1^\infty \frac{1}{x^{2b-a}}dx.$$
  - By the big theorem, the limit on the RHS converges if and only if $2b-a>1$.
  - By the computation in Example 2, page 566, the improper integral on the RHS converges if and only if $2b-a>1$.

  Thus, this improper integral converges if and only if $2b-a>1$.

  Remark: We can also make an argument using big theorem and the BCT. $\forall \varepsilon>0$:
  $$\frac{1}{x^{2b-a}} << \frac{\ln x}{x^{2b-a}} << \frac{x^\varepsilon}{x^{2b-a}}$$
  Then we can use the first inequality to show that $\displaystyle\int_1^\infty g(x)dx$ diverges when $2b-a\leq 1$ and the second inequality (choosing an appropriate value of ε) to show that $\displaystyle\int_1^\infty g(x)dx$ converges when $2b-a>1$.

Combining these two observations, it follows by the Limit Comparison Test that $\displaystyle\int_1^\infty \frac{x^a \ln x}{(x^2+1)^b}dx$ converges if and only if $2b-a>1$.

Proof of (b): We apply the Limit Comparison Test with $g(x) = x^a \ln(x)$. Notice that we are using LCT for two negative functions.

- First, observe that

$$\lim_{x\to 0^+} \frac{\left(\dfrac{x^a \ln(x)}{(1+x^2)^b}\right)}{x^a \ln(x)} = \lim_{x\to 0^+} \frac{1}{(1+x^2)^b} = 1.$$

  Thus by LCT, $\displaystyle\int_0^1 \frac{x^a \ln x}{(x^2+1)^b} dx$ converges if and only if $\displaystyle\int_0^1 g(x)dx$ converges.

- Second, $\displaystyle\int_0^1 x^a \ln(x)dx$ converges if and only if $a > -1$. Again, there are two cases,
  - **Case 1:** $(a = -1)$ Again, we apply u-substitution to evaluate

$$\int_0^1 \frac{\ln(x)}{x} dx = \lim_{c\to 0^+} \frac{\ln(c)^2}{2} = \infty.$$

    Thus, this improper integral is divergent.

  - **Case 2:** $(a \neq -1)$ In this case, we may apply integration by parts with $u = \ln(x)$ and $dv = x^a dx$. This yields

$$\int_0^1 x^a \ln(x)dx = \lim_{c\to 0^+} \frac{c^{a+1}\ln(c)}{a+1} - \frac{1}{a+1}\int_0^1 x^a dx.$$

    - By the big theorem, the limit on the RHS converges if and only if $a > -1$.
    - The improper integral on the RHS converges if and only if $a > -1$.

Thus, this improper integral converges if and only if $a > -1$.