MAT 137Y: Calculus!
Problem Set 10
Due by 3pm on Wednesday, April 5th, 2017 online via crowdmark

1. Let N be a positive integer. Consider the function $F(x) = x\sin\left(x^N\right)$. Calculate $F^{(k)}(0)$ for the smallest positive integer k such that $F^{(k)}(0) \neq 0$.

   Solution: We know that for all $x \in \mathbb{R}$,

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

   Substituting x^N, we obtain

$$\sin(x^N) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^N)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2nN+N}$$

   which is also valid for all $x \in \mathbb{R}$. Multiplying both sides of this equation by x yields

$$F(x) = x\sin(x^N) = x\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2nN+N}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2nN+N+1}.$$

   This equation holds for all $x \in \mathbb{R}$, and the power series on the RHS is centred at 0. Thus, the power series on the RHS is the Maclaurin series of F. In other words,

$$\sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} x^k = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2nN+N+1}$$

   for all $x \in \mathbb{R}$. The coefficients of these two power series must be the same, which we can write as the formula

$$\frac{F^{(k)}(0)}{k!} = \begin{cases} \dfrac{(-1)^n}{(2n+1)!} & \text{if } k = 2nN+N+1 \\ 0 & \text{else.} \end{cases}$$

   Thus, the smallest k such that $F^{(k)}(0) \neq 0$ is $k = N+1$ $(n = 0)$, and

$$F^{(N+1)}(0) = (N+1)!.$$

2. Alfonso is preparing a tricky question for the MAT137 final exam. He wants to ask students to compute a limit of the form

$$\lim_{x\to 0} \frac{Ae^{x^4} + Be^{-x^4} + C\cos(x^4) + x^n}{x^m},$$

where A, B, and C will be specific real numbers, and m and n will be specific positive integers. He wants to choose these five constants so that the limit will be very difficult for those who try to solve it using L'Hôpital's Rule (specifically, he wants the calculation to require as many uses of L'Hôpital's Rule as possible) but easier to do using Taylor series. In addition, he does not want to answer to be 0 or ∞ or $-\infty$ or "DNE", because those are all answers that students could guess by accident.

Calculate the value of the limit.

<u>*Solution:*</u> The Maclaurin series for these functions are

$$\begin{aligned} e^{x^4} &= \sum_{k=0}^{\infty} \frac{x^{4k}}{k!} \\ e^{-x^4} &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{k!} \\ \cos(x^4) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{8k}}{(2k)!} \end{aligned}$$

Let us call $f(x) = Ae^{x^4} + Be^{-x^4} + C\cos(x^4) + x^n$. We have:

$$\begin{aligned} f(x) &= x^n + \sum_{k=0}^{\infty} \left(\frac{(A+B(-1)^k)x^{4k}}{k!} \right) + \sum_{k=0}^{\infty} \frac{C(-1)^k x^{8k}}{(2k)!} \\ &= x^n + \Bigg[(A+B+C) + (A-B)x^4 + \frac{A+B-C}{2}x^8 \\ &+ \frac{A-B}{3!}x^{12} + \frac{A+B+C}{4!}x^{16} + \frac{A-B}{5!}x^{20} \\ &+ \frac{A+B-C}{6!}x^{24} + \text{higher powers of } x \Bigg] \end{aligned} \tag{1}$$

We want to choose A, B, C and n so that the first few terms in this expansion are 0. More specifically, if the smallest power with non-zero coefficient in the numerator is k, then we can take $m = k$ and solving this limit will require k iterations of L'Hôpital's Rule. We want to make k as large as possible. Thus, **we want the first**

power with non-zero coefficient in the numerator to have a exponent as large as possible.

Ignoring the term x^n for a moment, all the coefficients in the numerator are of three forms: $A+B+C$, $A-B$, or $A+B-C$. It is tempting to simply impose the conditions

$$A+B+C=0, \qquad A-B=0, \qquad A+B-C=0. \tag{2}$$

However, these three conditions imply that $A=B=C=0$, and that would make the limit very easy (with no need for a single use of L'Hôpital's Rule).

Instead, we can only impose two of the three conditions in (2). This will get rid of many of the terms in (1), but not all. Then we can use the term x^n to get rid of one more coefficient. We can do this in three ways:

- We could impose $A+B+C=0$ and $A-B=0$. This would leave

$$f(x)=x^n+\frac{A+B-C}{2}x^8+\frac{A+B-C}{6!}x^{24}+\text{higher powers of } x$$

  We cannot impose $A+B-C=0$. However, we can take $n=8$. Then the term with x^8 becomes

$$x^n+\frac{A+B-C}{2}x^8=\left[1+\frac{A+B-C}{2}\right]x^8$$

  We can now impose

$$1+\frac{A+B-C}{2}=0,$$

  get rid of the term x^8 altogether, and the smallest term with non-zero coefficient would be x^{24}.

- We could impose $A+B+C=0$ and $A+B-C=0$. This would leave

$$f(x)=x^n+(A-B)x^4+\frac{A-B}{3!}x^{12}+\text{higher powers of } x$$

  Then we take $n=4$, get rid of the coefficient of x^4 and the smallest term with non-zero coefficient would be x^{12}.

- We could impose $A+B-C=0$ and $A-B=0$. This would leave

$$f(x)=x^n+(A+B+C)+\frac{A+B+C}{4!}x^{16}+\text{higher powers of } x$$

  Then we take $n=0$, get rid of the coefficient of x^0 and the smallest term with non-zero coefficient would be x^{16}.

Out of these three cases the best option is the first. Thus we take $n = 8$ and

$$A + B + C = 0, \qquad A - B = 0, \qquad 1 + \frac{A + B - C}{2} = 0.$$

This solves to $C = 1$, $A = B = \frac{-1}{2}$. The numerator has now become

$$\begin{aligned} f(x) &= \frac{A + B - C}{6!} x^{24} + \text{higher powers of } x \\ &= \frac{-1}{360} x^{24} + \text{higher powers of } x \end{aligned}$$

We can finally compute the original limit:

$$\lim_{x \to 0} \frac{f(x)}{x^m} = \lim_{x \to 0} \frac{\left(\frac{-1}{360} x^{24} + \text{higher powers of } x \right)}{x^m}$$

This limit will be a non-zero, finite number if and only if $m = 24$. In this case,

$$\lim_{x \to 0} \frac{f(x)}{x^m} = \lim_{x \to 0} \left(\frac{-1}{360} + \text{higher powers of } x \right) = \frac{-1}{360}.$$

<u>*Summary:*</u> the limit is $\frac{-1}{360}$. This is accomplished by choosing:

$$\begin{aligned} A &= \frac{-1}{2} \\ B &= \frac{-1}{2} \\ C &= 1 \\ n &= 8 \\ m &= 24 \end{aligned}$$

Computing this limit will require 24 uses of L'Hôpital's Rule. Any other choice of these constants will require less than 24 iterations of L'Hôpital's Rule, or will produce 0, ∞, $-\infty$, or DNE as an answer.

3. Consider the function $f(x) = \dfrac{1}{\sqrt{1+x}}$.

(a) Find a formula for $f^{(k)}(x)$ and prove it.

Solution: We compute the first few terms and guess the pattern:

$$
\begin{aligned}
f'(x) &= \left(\frac{-1}{2}\right)(1+x)^{-3/2} \\
f''(x) &= \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(1+x)^{-5/2} \\
f^{(3)}(x) &= \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(1+x)^{-7/2} \\
&\cdots \\
f^{(k)}(x) &= \frac{(-1)^k 1\cdot 3\cdot 5\cdots(2k-1)}{2^k}(1+x)^{-(2k+1)/2} \qquad (3)
\end{aligned}
$$

This is the formula we are looking for. If you prefer, it can also be written in other ways using double factorial notation:

$$
\begin{aligned}
f^{(k)}(x) &= (-1)^k \frac{(2k-1)!!}{2^k}(1+x)^{-(2k+1)/2} \\
&= \frac{(-1)^k(2k)!}{4^k k!}(1+x)^{-(2k+1)/2}
\end{aligned}
$$

We will prove that (3) is true for all positive integers k by induction. Observe that it is correct for the base case $k = 1$ (we computed it above).

For the induction step, I will assume (3) is true for a fixed k, and I need to prove that the same equation is true when I substitute $k+1$ for k. We differentiate:

$$
\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx} f^{(k)}(x) \\
&= \frac{(-1)^k 1\cdot 3\cdot 5\cdots(2k-1)}{2^k}\frac{d}{dx}(1+x)^{-(2k+1)/2} \\
&= \frac{(-1)^k 1\cdot 3\cdot 5\cdots(2k-1)}{2^k}\,\frac{-(2k+1)}{2}(1+x)^{-(2k+1)/2-1} \\
&= \frac{(-1)^{k+1} 1\cdot 3\cdot 5\cdots(2k-1)\cdot(2(k+1)-1)}{2^{k+1}}(1+x)^{-(2(k+1)+1)/2}
\end{aligned}
$$

Since the $(k+1)$-st derivative has the same form, our formula is correct by induction.

(b) Write down an explicit formula for the Maclaurin series of $f(x)$. Let us call this series $S(x)$.

<u>*Solution:*</u> Plugging $x = 0$ into the derivatives from part a) and using the definition of Taylor series, we get various equivalent forms :

$$\begin{aligned} S(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k \cdot k!} x^k \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!} x^k \end{aligned}$$

The first few terms of this formula are

$$\begin{aligned} &= 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 - \cdots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 + \cdots \end{aligned}$$

(c) Calculate the radius of convergence of $S(x)$.

<u>*Solution:*</u> We apply the Ratio Test to $S(x)$.

$$\begin{aligned} \lim_{k\to\infty} \frac{\left| \frac{(-1)^{k+1} 1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1)}{2^{k+1} \cdot (k+1)!} x^{k+1} \right|}{\left| \frac{(-1)^k 1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k \cdot k!} x^k \right|} &= \lim_{k\to\infty} \frac{\left| \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1)}{2^{k+1} \cdot (k+1)!} \right|}{\left| \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k \cdot k!} \right|} |x| \\ &= \lim_{k\to\infty} \left| \frac{(2k+1)}{2 \cdot (k+1)} \right| |x| \\ &= |x| \end{aligned}$$

Thus by the Ratio Test, the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. Hence, the radius of convergence is 1.

Notice that we do not know what happens at the end-points ($x = 1$ and $x = -1$). We were only looking for the radius of convergence, not for the interval of convergence.

(d) **(Do not submit)**. Using Lagrange's formula for the remainder, prove that $f(x) = S(x)$ inside the interval of convergence.

<u>*Solution:*</u> We want to show that for all $x \in (-1, 1)$,

$$\lim_{n\to\infty} |f(x) - p_n(x)| = 0,$$

where $p_n(x)$ is the n-th Taylor polynomial centered at 0. Note that $f(0) = p_n(0)$ for all n. It remains to check that this limit is 0 for $x \neq 0$.

Case I: ($0 < x < 1$**)** This is the easier case. By the Lagrange Remainder Theorem, we know there exists $c \in (0, x)$ such that

$$\begin{aligned}
0 \leq |f(x) - p_n(x)| &= \left| \frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!} \right| \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2^{n+1}(n+1)!} \frac{|x|^{n+1}}{|1+c|^{(2n+3)/2}} \\
&< \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} |x|^{n+1} \\
&< |x|^{n+1}
\end{aligned}$$

Since $0 < x < 1$, $|x|^{n+1} \to 0$ as $n \to \infty$. Thus, by Squeeze Theorem, $\lim\limits_{n\to\infty} |f(x) - p_n(x)| = 0$.

Case II: ($-1 < x < 0$**)** By Taylor's theorem, we know that

$$\begin{aligned}
|f(x) - p_n(x)| &= \left| \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n dt \right| \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \left| \int_0^x \frac{(x-t)^n}{(1+t)^{n+3/2}} dt \right| \\
&\leq (2n+1) \int_x^0 \frac{(t-x)^n}{(1+t)^{n+3/2}} dt && \text{since } t - x \geq 0 \\
&\leq (2n+1) \int_x^0 \frac{|x|^n}{(1+t)^{3/2}} dt && \text{since } \frac{t-x}{1+t} \leq x \\
&\leq (2n+1) \frac{|x|^n}{(1+x)^{3/2}} \int_x^0 dt && \text{since } \frac{1}{1+t} \leq \frac{1}{1+x} \\
&= (2n+1) \frac{|x|^{n+1}}{(1+x)^{3/2}}
\end{aligned}$$

Since $|x| < 1$,

$$\lim_{n\to\infty}(2n+1)\frac{|x|^{n+1}}{(1+x)^{3/2}} = \frac{1}{(1+x)^{3/2}}\lim_{n\to\infty}\left((2n+1)|x|^{n+1}\right) = 0.$$

Thus, by Squeeze Theorem, $\lim_{n\to\infty}|f(x) - p_n(x)| = 0$.

(e) Use your answer to the previous questions to obtain the Maclaurin series for $g(x) = \arcsin x$ around $a = 0$. In which domain can you be certain that arcsin is equal to its Maclaurin series?

<u>*Solution:*</u> We know from 3d that

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}x^k$$

when $|x| < 1$. Plugging in $-x^2$, we obtain that

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}(-x^2)^k$$

when $|-x^2| < 1$, which is true iff $|x| < 1$. This simplifies to

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{\infty}\frac{(2k-1)!!}{(2k)!!}x^{2k}.$$

Integrating both sides of this equation, we obtain

$$\arcsin(x) = x + \sum_{k=1}^{\infty}\frac{(2k-1)!!}{(2k)!!}\frac{x^{2k+1}}{2k+1},$$

which is valid when $|x| < 1$. Note that the constant of integration is 0 since $\arcsin(0) = 0$.

The first few terms of this series are

$$\arcsin(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + C$$

We know that a power series and its derivative have the same radius of convergence. Thus, $g(x)$ is equal to its Maclaurin series when $x \in (-1, 1)$. (A priori we do not know what happens at 1 or -1, but we do know the series is divergent when $|x| > 1$).

(f) Give a formula for $g^{(n)}(0)$.

<u>*Solution:*</u> Since the power series obtained in 3e is the Maclaurin series of $g(x) = \arcsin(x)$, we know that

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = x + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{x^{2k+1}}{2k+1}$$

and the coefficients of these two power series are equal. This means that

$$g^{(n)}(0) = 0 \text{ for all even } n.$$

In addition, $g'(0) = 1$. For any other odd n, $n = 2k+1$ for some $k \in \mathbb{N}^+$ and then:

$$\frac{g^{(2k+1)}(0)}{(2k+1)!} = \frac{(2k-1)!!}{(2k)!!} \frac{1}{(2k+1)}$$

We can solve for the $(2k+1)$-st derivative, and we get various equivalent expressions:

$$\begin{aligned} g^{(2k+1)}(0) &= (2k+1)! \, \frac{(2k-1)!!}{(2k)!!} \frac{1}{(2k+1)} \\ &= (2k)! \, \frac{(2k-1)!!}{(2k)!!} \\ &= [(2k-1)!!]^2 \\ &= [1 \cdot 3 \cdot 5 \cdots (2k-1)]^2 \end{aligned}$$

This can be summarized as

$$g^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 1 \\ [1 \cdot 3 \cdot 5 \cdots (n-2)]^2 & \text{if } n \text{ is odd and } n \neq 1 \end{cases}$$