UNIVERSITY OF TORONTO
Faculty of Arts and Science

AUGUST 2016 EXAMINATIONS
MAT137Y1 – Calculus!

Duration – 3 hours

NO AIDS ALLOWED

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Fill with **ALL CAPITAL LETTERS**:

Last Name:____________________ Student ID:________________

First Name:____________________

<u>**INSTRUCTIONS:**</u> (READ CAREFULLY!)

- This exam booklet contains 23 pages including this one. It consists of 16 questions. The maximum score is 100 points.
- **For questions where you are provided with a box labelled "Your answer:", write your final answer in the box.** Write any justifications, explanations, or calculations you need underneath.
- **Always justify your answers.** An incorrect final answer supported by correct calculations and explanations might still receive partial credit.
- **If you need extra space for a question, you may use either the back of the pages or page 23.** If you do so, clearly indicate it on the corresponding problem page. You can also use the back of the pages for rough work.
- **Organize your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- **No aids are permitted on this examination.** Examples of illegal aids include, but are not limited to textbooks, notes, cheatsheets, calculators, cellphones, or any other electronic device.
- **Do not turn this page over until the invigilators instruct you to do so.**

Good luck!

<u>**FOR MARKERS ONLY:**</u>

Question	Marks	Value
1		15
2		9
3		3
4		6
5		5
6		6
7		4
8		3
9		10
10		6
11		4
12		8
13		9
14		4
15		3
16		5
Total		**100**

1. *[15 points - 3 points each]*

Calculate the following limits. If a limit does not exist, indicate whether it is ∞, $-\infty$, or neither.

(a) $\lim_{x \to 1} \left(e^{x^{137}} - 1 \right)$

Your answer:

(b) $\lim_{x \to \infty} x^2 \sin\left(\frac{1}{4x^2}\right)$

Your answer:

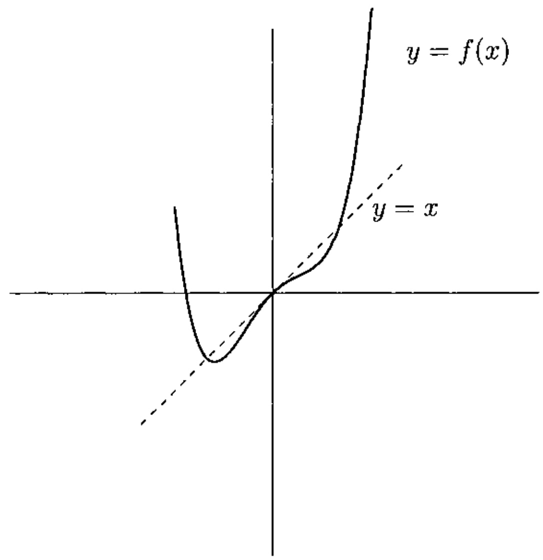
(c) $\lim_{x\to 0}(\cos x)^{1/x}$

Your answer:

(d) $\lim_{x\to 0} \frac{\int_0^{2\sin x} \cos(t^2)\, dt}{2x}$

Your answer:

(e) The graph of a function f and its tangent line at 0 are shown. What is the value of $\lim_{x\to 0} \frac{f(x)}{e^x - 1}$?



Your answer:

2. *[9 points - 3 points each]* Evaluate the following integrals.

(a) $\displaystyle\int \frac{\sin x + 2}{\sin^2 x - 1} \cos x dx$

Your answer:

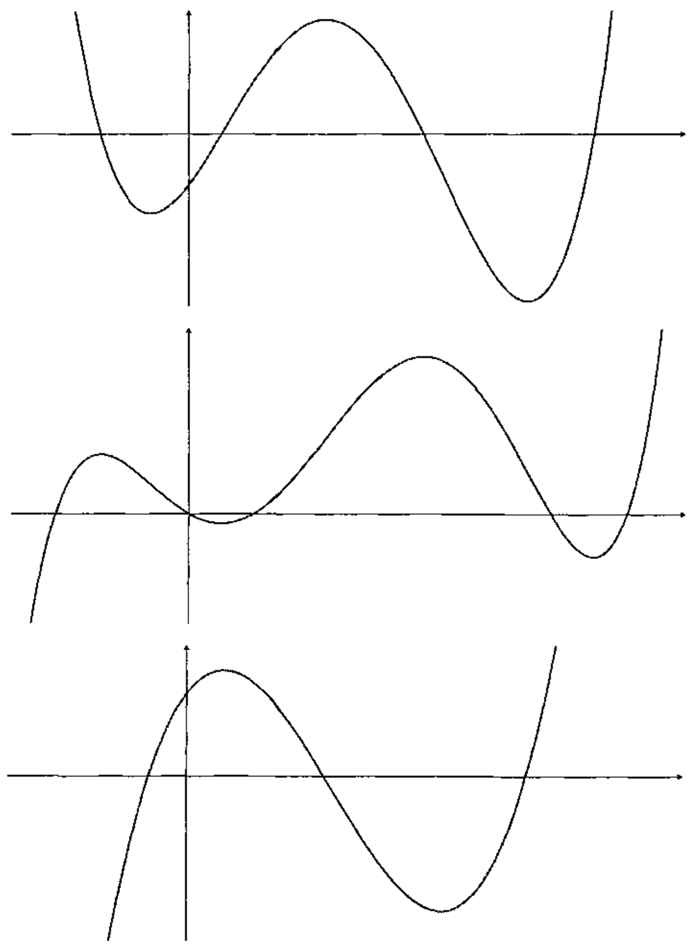
(b) $\displaystyle\int x \tan^2 x dx$

Your answer:

(c) $\displaystyle\int_{-12}^{e-13} \frac{x-2}{x+13}dx$

Your answer:

3. *[3 points]* The figure below gives the graph of a function f, the graph of its first derivative f', and the graph of its second derivative f'', but not in the correct order. Which curve is the graph of which function?



(a)

(b)

(c)

Your answer: f: f': f'':

4. *[6 points]*

(a) *[2 points]* Let f be a function with domain $(-\infty, \infty)$. Give the formal definition ($\epsilon - \delta$ definition) of the statement "$\lim_{x\to 0} f(x) = 5$".

(b) *[4 points]* Assume that

$$\lim_{x\to 1} f(x) = 5$$

Prove, directly from the $\epsilon - \delta$ definition of limit, that

$$\lim_{x\to 0} f(2x+1) = 5.$$

5. *[5 points]* The top of a ladder slides down a vertical wall at a rate of 1.5 m/h. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 2 m/h. How long is the ladder?

Your answer:

6. *[6 points]* Lord Sandwich's chef is running out of flour. He only has enough flour to make a rectangular slice of bread with an area of 128 cm^2. Lord Sandwich loves jam, but to prevent it from spilling out of his sandwich, the chef must leave a 2 cm margin at the top and bottom and a 1 cm margin at each side. What should the dimensions of the slice of bread be to maximize the area of jam on the toast?

Your answer:

7. *[4 points]* Show that the following equation has exactly one real root.

$$2x + \cos x = 0.$$

8. *[3 points]* Use logarithmic differentiation to show the following.

If $f(x) = x^m(x-1)^n$, where $m, n \in \mathbb{N}$, then

$$f'(x) = \left(\frac{m}{x} + \frac{n}{x-1}\right) f(x).$$

9. *[10 points]* Consider the function $f(x) = \frac{1}{x^2}$ in the interval $[1, \infty)$.

(a) *[3 points]* Approximate $\int_1^\infty f(x)dx$ with a Riemann sum using the infinite partition

$$P = \{1, 2, 3, \dots\}$$

and $x_i^* = x_i$, i.e. the right point from the interval. Your final answer should be a series using the sigma notation. You don't have to evaluate it.

Your answer:

(b) *[1 point]* Does this series converge?

Your answer:

(c) *[5 points]* The Riemann sum in part (a) represents the sum of the area of rectangles. Rotate each of those rectangles around the y-axis. For each interval $[x_{i-1}, x_i]$ you get a solid of revolution (hint: each of those solids is a cylindrical shell with thickness 1). Using the same partition P, write a Riemann sum that sums the volume of all of those solids of revolution. Your final answer should be a series using the sigma notation. You don't have to evaluate it.

... Or for *[2 points]* (Note you cannot do both for 7 points!)

Use the shell method to express the volume of the rotation of $f(x)$ between 1 and ∞ around the y-axis.

Your answer:

(d) *[1 point]* Does the series of the integral you get in part (c) converge?

Your answer:

10. *[6 points - 3 points each]* A nonnegative function f defined on $(-\infty, \infty)$ is called a *probability density function* if

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

The *mean* of a probability density function is defined as

$$\mu = \int_{-\infty}^{\infty} xf(x)dx.$$

Let $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

(a) Show that $f(x)$ is a probability density function.

(b) Calculate the mean of $f(x)$.

Your answer:

11. *[4 points]* Rotate the area between the curve $y = x$ and the curve $y = x^2$ around the x-axis. What is the volume of the solid you obtained?

Your answer:

12. *[8 points - 2 point each]* Let a_n be a decreasing sequence, with least upper bound M and greatest lower bound $m < M$. Are the following statements true or false? No need to justify your answers.

(a) If $m < 0 < M$ then $b_n = 1/a_n$ is increasing.

Your answer:	**True** ☐	**False** ☐

(b) If $0 < m < M$ then $b_n = 1/a_n$ is increasing.

Your answer:	**True** ☐	**False** ☐

(c) If $0 < m < M$ then $b_n = 1/a_n$ is bounded.

Your answer:	**True** ☐	**False** ☐

(d) $b_n = \sin(a_n)$ has to be increasing.

Your answer:	**True** ☐	**False** ☐

13. *[9points - 3 point each]* For each of the following sums, prove whether they converge or diverge. Make sure to state the name of any convergence test you use.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{\sqrt{n}}$

Your answer:	**Converges** ☐	**Diverges** ☐

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{2^k3^k}{k^k}$

Your answer:	**Converges** ☐	**Diverges** ☐

(c) $\displaystyle\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$

Your answer:	**Converges** ☐	**Diverges** ☐

14. *[4 points]* Recall that the sum of the geometric series is

$$\frac{1}{1-x} = \sum_{k=1}^{\infty} x^k \qquad \text{for } |x| < 1.$$

(a) *[3 points]* Starting from the geometric series, find the Taylor series of $f(x) = \dfrac{1}{(1+x)^2}$ around 0. $\Big[$Do not calculate the derivatives of $f(x) = \frac{1}{(1+x)^2}$, but you will need to calculate another derivative.$\Big]$

Your answer:

(b) *[1 point]* What is the radius of convergence?

Your answer:

15. *[3 points]* Find the Taylor series of $x^2 + 3$ around 0.

Your answer:

16. *[5 points]* Find the interval of convergence of

$$\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k x^k.$$

Your answer:

This page is intended for extra work in case you run out of space. If you use it for any problem, clearly indicate so on the corresponding problem page.