MAT 137
Tutorial #19– Taylor series
March 27-28, 2017

There are four main power series we know. If you have not learned them in lecture yet, you will soon. For now, just accept them. They are:

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots && \text{for all } x \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots && \text{for all } x \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots && \text{for all } x \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots && \text{for } |x| < 1
\end{aligned}
$$

Today you will practice how to manipulate these four functions to write many more functions as power series. In Tutorial 20 you will use this to actually calculate the sum of a lot of series.

1. Compute the Maclaurin series of the following functions by performing algebraic manipulations of the main four Taylor series. Indicate the domain where the expansion is valid.

   Hint: As an example, for Question 1a, use the substitution $u = -x$ and use the Maclaurin expansion for $\frac{1}{1-u}$. Similar tricks work for many of the others.

   (a) $f(x) = \frac{1}{1+x}$

   (b) $f(x) = \frac{1}{1+x^2}$

   (c) $f(x) = x^2 e^x$

   (d) $f(x) = \cos x^4$

   (e) $f(x) = \sin(2x)$

   (f) $f(x) = e^{-x}$

   (g) $f(x) = \frac{e^x - e^{-x}}{2}$

   (h) $f(x) = \frac{\sin x}{x}$

   Note: Question 1h actually means $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

2. Compute the Maclaurin series of the following functions using differentiation or integration of other known series. Indicate the domain where the expansion is valid.

   (a) $\ln(1+x)$

   (b) $\arctan x$

   (c) $f(x) = \displaystyle\int_0^x e^{-t^2}\,dt$

   Hint: As an example, for Question 2a, notice that you already know the Maclaurin expansion for $\dfrac{d}{dx}\ln(1+x) = \dfrac{1}{1+x}$. Then integrate that series term by term. Pay attention to the constant of integration.

3. Compute the Taylor series of the following functions about $x = a$ for the given value of a. Indicate the domain where the expansion is valid.

   (a) $f(x) = e^x$ about $a = 1$.

   (b) $f(x) = \dfrac{1}{1-x}$ about $a = 3$.

   (c) $f(x) = \sin x$ about $a = \pi/4$.

   Hint: You can always use the substitution $u = x - a$ and try to turn the resulting function into something you recognize.