MAT 137Y: Calculus!
Problem Set 7 – Solutions
Due by 3pm on Friday, February 17, 2017 online via crowdmark

1. Let h be a continuous function with domain $\mathbb{R}$. We define the function $H(x) = \displaystyle\int_0^x h(t)dt$. Calculate the following integrals. Your answers may be written in terms of values of h and H at any points (but not in terms of anything else). For example, an answer may be $h(1) + 3H(2)$.

   (a) $\displaystyle\int_1^2 h(x)dx$

   By definition of H and elementary properties of integrals,

   $$\int_1^2 h(x)dx = \int_0^2 h(x)dx - \int_0^1 h(x)dx = H(2) - H(1)$$

   (b) $\displaystyle\int_0^1 h(2x+5)dx$

   $$\begin{aligned} \int_0^1 h(2x+5)dx &= \frac{1}{2}\int_5^7 h(u)du \qquad\qquad u = 2x+5,\ du = 2dx \\ &= \frac{H(7) - H(5)}{2}. \end{aligned}$$

   (c) $\displaystyle\int_0^1 xh'(x)dx$

   Apply integration by parts with $u = x$, $du = dx$, $dv = h'(x)dx$, $v = h(x)$.

   $$\begin{aligned} \int_0^1 xh'(x)dx &= xh(x)\big|_0^1 - \int_0^1 h(x)dx \\ &= h(1) - H(1). \end{aligned}$$

   (d) $\displaystyle\int_0^1 e^{H(x)}h(x)\ dx$

   $$\begin{aligned} \int_0^1 e^{H(x)}h(x)\ dx &= \int_0^{H(1)} e^u\ du \qquad\qquad u = H(x),\ du = h(x)dx \\ &= e^{H(1)} - 1 \end{aligned}$$

   In the last equation, we have used that $H(0) = 0$.

2. For every non-negative integers n and k, we define the number

$$J(n,m) = \int_0^{2\pi} \sin^{2n} x \cos^{2m} x dx$$

In this problem you are going to find a formula for $J(n,m)$.

(a) Calculate $J(0,0)$.

$$J(0,0) = \int_0^{2\pi} dx = 2\pi. \tag{1}$$

(b)

(c) Use the same idea as in Question 2b to find a relation between $J(n,0)$ and $J(n-1,0)$.

The answer is

$$J(n,0) = \frac{2n-1}{2n} J(n-1,0). \tag{2}$$

First, observe that

$$\int_0^{2\pi} \sin^{2n}(x)dx = \int_0^{2\pi} \sin^{2n-2}(x)\left(1-\cos^2(x)\right)dx \tag{3}$$

$$= \int_0^{2\pi} \sin^{2n-2}(x)dx - \int_0^{2\pi} \sin^{2n-2}(x)\cos^2(x)dx. \tag{4}$$

In other words

$$J(n,0) = J(n-1,0) - J(n-1,1) \tag{5}$$

Next, we use integration by parts on $J(n-1,1)$. Let

$$u = \cos(x),\ du = -\sin(x)dx$$

$$dv = \sin^{2n-2}(x)\cos(x)dx,\ v = \frac{\sin^{2n-1}(x)}{2n-1}.$$

$$\int_0^{2\pi} \sin^{2n-2}(x)\cos^2(x)dx = \left.\frac{\sin^{2n-1}(x)\cos(x)}{2n-1}\right|_0^{2\pi} + \int_0^{2\pi} \frac{\sin^{2n-1}(x)}{2n-1}\sin(x)dx$$

$$= \frac{1}{2n-1}\int_0^{2\pi} \sin^{2n}(x)dx.$$

In other words

$$J(n-1,1) = \frac{1}{2n-1} J(n,0). \tag{6}$$

Finally, we put together (5) and (6) to get $J(n,0) = \dfrac{2n-1}{2n} J(n-1,0$.

(d) Find a formula for $J(n,0)$

Using formula (2) repeatedly,

$$\begin{aligned}
J(n,0) &= \frac{2n-1}{2n} J(n-1,0) \\
&= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot J(n-2,0) \\
&= \cdots \\
&= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot J(0,0) \\
&= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot 2\pi \qquad \text{by (1).}
\end{aligned}$$

If you prefer to use double-factorial notation (if you do not know what this means, ignore this part), this can also be rewritten as

$$J(n,0) = \frac{(2n-1)!!}{(2n)!!} 2\pi$$

(e) Use the same idea as in Question 2c to find a relation between $J(n,m)$ and $J(n,m-1)$.

The answer is

$$J(n,m) = \frac{2m-1}{2n+2m} J(n,m-1) \tag{7}$$

First, we use the trigonometric identity $\cos^2 x = 1 - \sin^2 x$ on $J(n,m)$:

$$\begin{aligned}
J(n,m) &= \int_0^{2\pi} \sin^{2n} x \cos^{2m} x dx = \int_0^{2\pi} \sin^{2n} x \cos^{2(m-1)} x \cos^2 x dx \\
&= \int_0^{2\pi} \sin^{2n} x \cos^{2(m-1)} x \left(1 - \sin^2 x\right) dx \\
&= \int_0^{2\pi} \sin^{2n} x \cos^{2(m-1)} x dx - \int_0^{2\pi} \sin^{2(n+1)} x \cos^{2(m-1)} x dx \\
&= J(n,m-1) - J(n+1,m-1)
\end{aligned}$$

Second, we use integration by parts on $J(n+1, m-1)$:

$$u = \sin^{2n+1}(x) \qquad du = (2n+1)\sin^{2n}(x)\cos(x)dx$$
$$dv = \sin x \cos^{2m-2}(x)dx \qquad v = \frac{-1}{2m-1}\cos^{2m-1}(x)$$

$$\begin{aligned}
J(n+1, m-1) &= \int_0^{2\pi} \sin^{2(n+1)} x \cos^{2(m-1)} x dx \\
&= \frac{-1}{2m-1}\sin^{2n+1} x \cos^{2m-1}(x)\Big|_0^{2\pi} + \frac{2n+1}{2m-1}\int_0^{2\pi} \sin^{2n} x \cos^{2m} x dx \\
&= \frac{2n+1}{2m-1} J(n, m)
\end{aligned}$$

Putting together both equations, we get (7).

(f) Find a formula for $J(n, m)$.
We use Question 2d and Equation (7) repeteadly:

$$\begin{aligned}
J(n, m) &= \frac{2m-1}{2n+2m} J(n, m-1) \\
&= \frac{2m-1}{2n+2m} \cdot \frac{2m-3}{2n+2m-2} \cdot J(n, m-2) \\
&= \cdots \\
&= \frac{2m-1}{2n+2m} \cdot \frac{2m-3}{2n+2m-2} \cdots \frac{3}{2n+4} \cdot \frac{1}{2n+2} \cdot J(n, 0) \\
&= \frac{2m-1}{2n+2m} \cdot \frac{2m-3}{2n+2m-2} \cdots \frac{3}{2n+4} \cdot \frac{1}{2n+2} \cdot \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot 2\pi
\end{aligned}$$

If you prefer to use double-factorial notation (if you do not know what this means, ignore this part), this can also be rewritten as

$$J(n, m) = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!} 2\pi$$

3. You are going to discover integration by partial fraction decomposition *of a new type*. In this question, more than ever, it is important that you show all your process and you explain everything you are doing. You won't get any points for just the final answer. **In this problem, you are not allowed to use trigonometry substitution (the technique in Section 8.4 of the book).**

(a) **(Do not submit)**. Compute the following derivatives:

$$\frac{d}{dx}\left[\arctan x\right], \qquad \frac{d}{dx}\left[\frac{x}{1+x^2}\right]$$

We know that

$$\begin{aligned}\frac{d}{dx}\left[\arctan x\right] &= \frac{1}{1+x^2}\\ \frac{d}{dx}\left[\frac{x}{1+x^2}\right] &= \frac{(1+x^2)-2x(x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}\end{aligned}$$

(b) Calculate $\int \frac{1}{(1+x^2)^2}dx$.

Do not do a substitution! Instead, try to rewrite $\frac{1}{(1+x^2)^2}$ in terms of rational functions whose antiderivatives you know.
Hint: Use Question 3a.
Following the hint, we look for constants A, B such that

$$\frac{1}{(1+x^2)^2} = A\frac{1}{1+x^2} + B\frac{1-x^2}{(1+x^2)^2} = A\frac{1+x^2}{(1+x^2)^2} + B\frac{1-x^2}{(1+x^2)^2}$$

By inspection, $A = B = 1/2$. Thus

$$\begin{aligned}\int \frac{1}{(1+x^2)^2}dx &= \frac{1}{2}\int \frac{1}{1+x^2}dx + \frac{1}{2}\int \frac{1-x^2}{(1+x^2)^2}dx\\ &= \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{x}{1+x^2} + C.\end{aligned}$$

(c) Use a similar trick to calculate $\int \frac{x^3}{(1+x^2)^2}dx$.

Hint: It is quick to compute $\int \frac{x}{(1+x^2)^2}dx$.
First we follow the hint. Using u-substitution, we know that

$$\int \frac{x}{(1+x^2)^2}dx = \frac{1}{2}\int \frac{du}{u^2} \qquad u = 1+x^2,\ du = 2xdx \tag{8}$$

$$= \frac{-1}{2u} \tag{9}$$

$$= \frac{-1}{2}\frac{1}{1+x^2}. \tag{10}$$

Now observe that

$$\frac{x^3}{(1+x^2)^2} = \frac{x^3+x-x}{(1+x^2)^2} = \frac{x(1+x^2)}{(1+x^2)^2} - \frac{x}{(1+x^2)^2} = \frac{x}{1+x^2} - \frac{x}{(1+x^2)^2}$$

We know how to integrate both of the functions on the right, so we use this identity to calculate the integral:

$$\begin{aligned}\int \frac{x^3}{(1+x^2)^2}dx &= \int \frac{x}{1+x^2}dx - \int \frac{x}{(1+x^2)^2}dx \\ &= \frac{1}{2}\left[\ln(1+x^2) + \frac{1}{1+x^2}\right] + C\end{aligned}$$

where we have applied u-substitution to the first integral and used the computation for the hint for the second integral.

Alternative solution: We could also have used the substituation $u = x^2 + 1$ from the beginning:

$$\int \frac{x^3}{(1+x^2)^2}dx \ = \ \int \frac{x^2}{(1+x^2)^2}\frac{1}{2}(2xdx) \ = \ \frac{1}{2}\int \frac{u-1}{u^2}du$$

and then finish it from here.