MAT 137Y: Calculus!
Problem Set 8 – Solutions
Due by 3pm on Friday, March 3, 2017 online via crowdmark

For all these questions, let $\{a_n\}_{n=1}^{\infty}$ be a sequence. I define two new sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ from it via the equations

- $\forall n \geq 1,\ b_n = a_{2n}$
- $\forall n \geq 1,\ c_n = a_{2n-1}$

The following six questions are six theorems about the relation between these three sequences. Are they true or false? If a theorem is true, give a formal proof (from the definitions, of course). If a theorem is false, show it with a counterexample.

1. **Theorem 1:**

   - IF the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing,
   - THEN the sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are increasing.

   This theorem is true.

   Proof: Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence.

   - First we prove that $\{b_n\}_{n=1}^{\infty}$ is increasing. Fix $n \in \mathbb{N}$ arbitrary. Since the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing, we know that

     $$b_{n+1} = a_{2n+2} > a_{2n+1} > a_{2n} = b_n.$$

     Since n was arbitrary, we have shown that the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing.
   - Next we prove that $\{c_n\}_{n=1}^{\infty}$ is increasing. Fix $n \in \mathbb{N}$ arbitrary. Since the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing, we know that

     $$c_{n+1} = a_{2n+1} > a_{2n} > a_{2n-1} = c_n.$$

     Since n was arbitrary, we have shown that the sequence $\{c_n\}_{n=1}^{\infty}$ is increasing.

   $\square$

2. **Theorem 2:**

   - IF the sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are increasing,
   - THEN the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing.

   This theorem is false.

   Example: Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the formula $a_n = n + (-1)^n$.

   - The terms of the sequence $\{b_n\}_{n=1}^{\infty}$ are $b_n = 2n + (-1)^{2n} = 2n + 1$. This is an increasing sequence.
   - The terms of the sequence $\{c_n\}_{n=1}^{\infty}$ are $c_n = 2n - 1 + (-1)^{2n-1} = 2n - 2$. This is an increasing sequence.
   - However, $a_2 = 2 + 1 = 3$ and $a_3 = 3 - 1 = 2$. Thus $a_3 < a_2$, so the sequence $\{a_n\}_{n=1}^{\infty}$ is not increasing.

   Thus $\{a_n\}_{n=1}^{\infty}$ is a sequence such that both $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are increasing AND $\{a_n\}_{n=1}^{\infty}$ is NOT increasing. $\square$

3. **Theorem 3:**

   - IF the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded,
   - THEN the sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are bounded.

   This theorem is true.

   Proof: Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence. This means that there exists $M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, |a_n| \leq M$.

   - For all $n \in \mathbb{N}$, it follows that
     $$|b_n| = |a_{2n}| \leq M.$$
     So $\{b_n\}_{n=1}^{\infty}$ is bounded.
   - For all $n \in \mathbb{N}$, it also follows that
     $$|c_n| = |a_{2n-1}| \leq M.$$
     So $\{c_n\}_{n=1}^{\infty}$ is bounded.

   $\square$

4. **Theorem 4:**

    - IF the sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are bounded,
    - THEN the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.

    This theorem is true.

    Proof: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are bounded. We prove that $\{a_n\}_{n=1}^{\infty}$ is also bounded.

    - Since $\{b_n\}_{n=1}^{\infty}$ is bounded, there exists $M_1 \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, |b_n| \leq M_1$.
    - Since $\{c_n\}_{n=1}^{\infty}$ is bounded, there exists $M_2 \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, |c_n| \leq M_2$.

    Let $M = \max\{M_1, M_2\}$.

    If n is even, then
    $$|a_n| = |b_{n/2}| \leq M_1 \leq M$$

    If n is odd, then
    $$|a_n| = |c_{(n+1)/2}| \leq M_2 \leq M$$

    Thus, $\forall n \in \mathbb{N}, |a_n| \leq M$. In other words, $\{a_n\}_{n=1}^{\infty}$ is bounded. $\square$

5. **Theorem 5:** Let $L \in \mathbb{R}$.

- IF the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to L,
- THEN the sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are convergent to L.

This theorem is true.

Proof: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to $L \in \mathbb{R}$.

- First we show that $\{b_n\}_{n=1}^{\infty}$ converges to L. i.e. we want to prove the statement

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \colon n \geq N \Rightarrow |b_n - L| < \varepsilon.$$

  Fix $\varepsilon > 0$ arbitrary. Since $\{a_n\}_{n=1}^{\infty}$ converges to L, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - L| < \varepsilon$. Thus

$$n \geq N \Rightarrow |b_n - L| = |a_{2n} - L| < \varepsilon$$

  since $2n > n \geq N$. Since ε was arbitrary, we have proven that $\{b_n\}_{n=1}^{\infty}$ converges to L.

- Next we show that $\{c_n\}_{n=1}^{\infty}$ converges to L. i.e. we want to prove the statement

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \colon n \geq N \Rightarrow |c_n - L| < \varepsilon.$$

  Fix $\varepsilon > 0$ arbitrary. Since $\{a_n\}_{n=1}^{\infty}$ converges to L, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - L| < \varepsilon$. Thus

$$n \geq N \Rightarrow |c_n - L| = |a_{2n-1} - L| < \varepsilon$$

  since $2n - 1 \geq n \geq N$ for all $n \in \mathbb{N}$, $n \geq 1$.

  Since ε was arbitrary, we have proven that $\{b_n\}_{n=1}^{\infty}$ converges to L.

$\square$

6. **Theorem 6:** Let $L \in \mathbb{R}$.

- IF the sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are convergent to L,
- THEN the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to L.

This theorem is true.

Proof: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are convergent to $L \in \mathbb{R}$. We must prove that $\{a_n\}_{n=1}^{\infty}$ is convergent to L. This means

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \colon n \geq N \Rightarrow |a_n - L| < \varepsilon.$$

Fix $\varepsilon > 0$ arbitrary.

- Since $\{b_n\}_{n=1}^{\infty}$ is convergent to L, we know that there exists $N_1 \in \mathbb{N}$ such that
$$n \geq N_1 \Rightarrow |b_n - L| < \varepsilon.$$
- Since $\{c_n\}_{n=1}^{\infty}$ is convergent to L, we know that there exists $N_2 \in \mathbb{N}$ such that
$$n \geq N_2 \Rightarrow |c_n - L| < \varepsilon.$$

Let $N = \max\{2N_1, 2N_2 - 1\}$.

If $n \geq N$ is even, then $n/2 \geq N_1$ so

$$|a_n - L| = |b_{n/2} - L| < \varepsilon.$$

If $n \geq N$ is odd, then $(n+1)/2 \geq N_2$ so

$$|a_n - L| = |c_{(n+1)/2} - L| < \varepsilon.$$

Thus $n \geq N \Rightarrow |a_n - L| < \varepsilon$. Since ε was arbitrary, we are done. $\square$