MAT 137Y: Calculus!
Problem Set A.

This problem set is intended to help you prepare for Test #1. It is not comprehensive: it only contains problems from some sections that were not included in past problem sets or in past tutorials. You do not need to turn in any of these problems.

1. Let f be some function for which you know only that

$$\text{if} \quad 0 < |x-3| < 1, \quad \text{then} \quad |f(x) - 5| < 0.1.$$

Which of the following statements are necessarily true?

(a) If $|x-3| < 1$ and $x \neq 3$, then $|f(x) - 5| < 0.1$.
(b) If $|x-3| < 1$, then $|f(x) - 5| < 0.1$.
(c) If $|x-2.5| < 0.3$, then $|f(x) - 5| < 0.1$
(d) $\lim_{x \to 3} f(x) = 5$.
(e) If $0 < |x-3| < 2$, then $|f(x) - 5| < 0.1$.
(f) If $0 < |x-3| < 0.5$, then $|f(x) - 5| < 0.1$.
(g) If $0 < |x-3| < \frac{1}{4}$, then $|f(x) - 5| < \frac{1}{4}(0.1)$.
(h) If $0 < |x-3| < 1$, then $|f(x) - 5| < 0.2$.
(i) If $0 < |x-3| < 2$, then $|f(x) - 4.95| < 0.05$.
(j) If $\lim_{x \to 3} f(x) = L$, then $4.9 \leq L \leq 5.1$.

Hint: The answer is "true" for exactly five of the statements.

2. Given a real number x, we defined the *floor of* x, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to x. For example, $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, and $\lfloor -0.5 \rfloor = -1$.

(a) At which points is the function $f(x) = \lfloor x \rfloor$ continuous?
(b) Consider the function $g(x) = \lfloor \sin x \rfloor$. Show that g has exactly one removable and one non-removable discontinuity inside the interval $(0, 2\pi)$.

3. Use the Intermediate Value Theorem to prove that the equation

$$\sin x = 2\cos^2 x + 0.5$$

has at least one solution.

4. Prove directly from the $\varepsilon - \delta$ definition of limit that $\lim_{x\to 1} \frac{1}{x^2+1} = \frac{1}{2}$.

5. (a) Let $a \in \mathbb{R}$. Let f be a function defined at least on an open interval around a, except possibly at a. Let $L \in \mathbb{R}$. Write the formal ε–δ definition of "$\lim_{x\to a} f(x) \neq L$" by negating the definition of "$\lim_{x\to a} f(x) = L$".

   (b) Let $a \in \mathbb{R}$. Let f be a function defined at least on an open interval around a, except possibly at a. Write the formal ε–δ definition of "$\lim_{x\to a} f(x)$ does not exist". This is different from the previous question!

   (c) Prove, formally from the definition, that $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

6. In class we have introduced the formal definition of the statement $\lim_{x\to c} f(x) = L$. (See Definition 2.2.1 on page 64 of the book.) We also introduced intuitively the notion of infinite limits (see page 58 of the book). Its formal definition is as follows:

   Let f be a function defined on at least an open interval $(c-p, c+p)$ except possibly at c itself. We say that

   $$\lim_{x\to c} f(x) = \infty$$

   when for every $M \in \mathbb{R}$, there exists $\delta > 0$ such that

   $$\text{if} \qquad 0 < |x-c| < \delta, \qquad \text{then} \qquad f(x) > M.$$

   Read the definition carefully and try to understand why this agrees with the intuitive idea for $\lim_{x\to c} f(x) = \infty$ as described on page 58 of the book. Once you are satisfied with this, answer the following questions:

   (a) Write a formal definition of the concept $\lim_{x\to c+} f(x) = \infty$.

   (b) Write a formal definition of the concept $\lim_{x\to c} f(x) = -\infty$.

   (c) Using the formal definition, prove that $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$.