解 不妨记这些缺失的失效时间数据为 $z_1, \ldots, z_m$. 则我们有如下的联合似然函数

$$\log L(\theta|\mathbf{x}) = n(\log\theta - \theta\bar{y}) + \sum_{i=1}^{m}(\log\theta - \theta z_i)$$

取条件期望得到

$$E\left\{\log L(\boldsymbol{\theta}|\mathbf{x}) \mid \mathbf{y}, \boldsymbol{\theta}_t\right\} = (n+m)\log\theta - \theta\left[n\bar{y} + (m-r)(t+\frac{1}{\theta_t}) + r(\frac{1}{\theta_t} - th_t)\right],$$

其中

$$h_t = \frac{\exp\{-t\theta_t\}}{1-\exp\{-t\theta_t\}}.$$

则我们在第 $t+1$个M-步确定最大值 θ_{t+1}

$$\theta_{t+1} = (n+m)\left[n\bar{y} + (m-r)(t+\frac{1}{\theta_t}) + r(\frac{1}{\theta_t} - th_t)\right]^{-1}.$$

□

可观测数据：$y_1, \cdots, y_n$，r（在t时刻m个灯泡中失效的个数）

完全数据：$y_1, \cdots, y_r, z_1, \cdots, z_m$

不妨设前r个失效，后$m-r$个未失效.

$$E\left[\sum_{i=1}^{m} Z_i \mid Y=y, r, \theta^{(t)}\right]$$

$$= E\left[\sum_{i=1}^{r} Z_i \mid Z_1<t, \cdots, Z_r<t, \theta^{(t)}\right] + E\left[\sum_{i=r+1}^{m} Z_i \mid Z_{r+1} \geqslant t, \cdots, Z_m \geqslant t, \theta^{(t)}\right]$$

$$= \sum_{i=1}^{r} E\left[Z_i \mid Z_i < t, \theta^{(t)}\right] + \sum_{i=r+1}^{m} E\left[Z_i \mid Z_i \geqslant t, \theta^{(t)}\right]$$

$$= r\frac{E(Z_i I(Z_i<t))}{P(Z_i<t)} + (m-r)\frac{E(Z_i I(Z_i \geqslant t))}{P(Z_i \geqslant t)}$$

$$= r\frac{\int_0^t z\theta e^{-\theta z}dz}{F_Z(t|\theta^t)} + (m-r)\frac{\int_t^{+\infty} z\theta e^{-\theta z}dz}{1-F_Z(t|\theta^t)}$$

$$= r\frac{\frac{1}{\theta_t}(1-e^{-\theta_t t}) - te^{-\theta_t t}}{1-e^{-\theta_t t}} + (m-r)\frac{(t+\frac{1}{\theta_t})e^{-\theta_t t}}{e^{-\theta_t t}}$$

$$= r\left(\frac{1}{\theta_t} - th_t\right) + (m-r)\left(t+\frac{1}{\theta_t}\right),$$

其中 $h_t = \dfrac{e^{-\theta_t t}}{1-e^{-\theta_t t}}$.