4 Closure Properties of Regular Languages

[Linz § 4.1]

We show **closure** under some **simple set operations**

(1) $\boldsymbol{L_1,\ L_2}$ **regular** $\boldsymbol{\Longrightarrow L_1 \cup L_2}$ **regular**

Proof : Suppose for some nfa's, M_1, M_2:

$$\boldsymbol{L_1 = L(M_1),\ L_2 = L(M_2)}$$

Let $\boldsymbol{M_1 \sqcup M_2}$ $=$ the "**union**" of $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$
shown on p. 3-8 (bottom).

Then $L_1 \cup L_2 = L(M_1 \sqcup M_2)$ is regular.

(2) $\boldsymbol{L_1,\ L_2}$ **regular** $\boldsymbol{\Longrightarrow}$ $\boldsymbol{L_1 L_2}$ **regular**

Proof : Suppose $L_1 = L(M_1),\ L_2 = L(M_2)$ for nfa's $M_1,\ M_2$

Let $M_1 \cdot M_2 =$ the "**concatenation**" of M_1 and M_2
Shown on p. 3-9 (top).

Then $L_1 L_2 = L(M_1 \cdot M_2)$ is regular.

(3) $\boldsymbol{L}$ **regular** $\Longrightarrow$ $\boldsymbol{L^*}$ **regular**

Proof : Suppose $L = L(M)$ for some nfa M.

Let $M^* =$ the "**star**" of M
shown on p. 3-9 (bottom).

Then $L^* = L(M^*)$ is regular.

Given $L \subseteq \Sigma^*$,
$\overline{L} = \Sigma^* \backslash L =$ the **complement** of L in Σ.

Then:
(4) $\boldsymbol{L}$ **regular** $\Longrightarrow$ $\boldsymbol{\overline{L}}$ **regular**

To prove this, we must first define the "**complement**" of a Σ-nfa.
How?

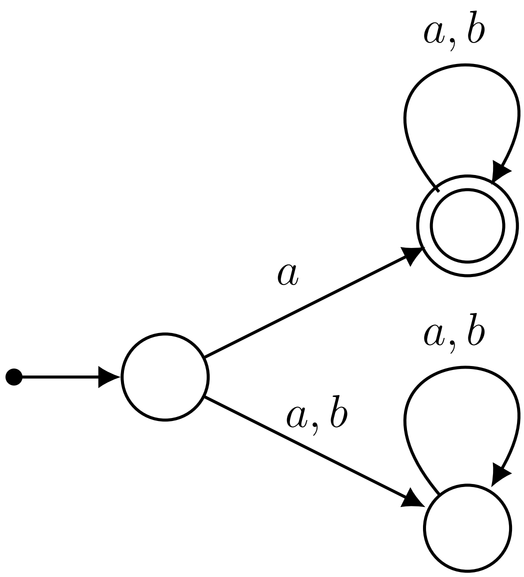
Proposal: Given $\boldsymbol{M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)}$

Define $$\boldsymbol{\overline{M} = (Q,\ \Sigma,\ \delta,\ q_0,\ \overline{F})}$$

where $$\boldsymbol{\overline{F} = Q \backslash F}$$

Problem Consider e.g. $\boldsymbol{M} : \Sigma = \{\boldsymbol{a}, \boldsymbol{b}\}$

$\overline{M}$:



$$\boldsymbol{L(M) = \{au \mid u \in \Sigma^*\}}$$

$$\boldsymbol{L(\overline{M}) = \Sigma^*}$$

Solution Replace M by **equivalent dfa $\boldsymbol{M^D}$** (p. 2-13)

Then if $$\boldsymbol{L = L(M)},$$

then $$\boldsymbol{\overline{L} = L(\overline{M^D})}$$

From now on we write, for any **nfa** M,

$$\overline{M} \quad \textbf{means} \quad \overline{M^D}$$

so that if
$$L \;=\; L(M)$$
then
$$\overline{L} \;=\; L(\overline{M})$$
is regular.

(5) $L_1,\ L_2$ **regular** $\Longrightarrow$ $L_1 \cap L_2$ **regular**

Why?
Suppose $L_1 = L(M_1),\ \ L_2 = L(M_2)$ (**nfa's**).

Define the nfa
$$M_1 \sqcap M_2 \;=\; \underline{\overline{\overline{M_1} \sqcup \overline{M_2}}}$$

Then

$$L_1 \cap L_2 \;=\; \underline{\overline{\overline{L_1} \;\cup\; \overline{L_2}}}$$

$$=\; \underline{L(M_1 \sqcap M_2)}$$