Exercises [See also Linz, Exs at end of Sec. 1.2]

Find grammars for $\Sigma = \{a, b\}$ that generate sets of all strings with

(a) **no** a

(a) **exactly** one a

(b) **at least** one a.

(c) ***exactly*** two a's

(d) $\geq$ two a's

(e) $\leq$ two a's

Notation

(1) Given a word, u, and a symbol, a:

$$n_a(u) = \textbf{\# of occurrences of } a \textbf{ in } u.$$

(2) For $u = a_1...a_n$ and $0 \leq k \leq n$:

$$u \restriction k = \textbf{prefix of } u \textbf{ up to } k = a_1...a_k.$$

(3) $\mathbf{xs}(a, b, u)$ = **excess** of a over b in u.

$$= n_a(u) - n_b(u)$$

(4) u is **balanced** $\Longleftrightarrow n_a(u) = n_b(u)$

$$\mathbf{xs}(a, b, u) = 0$$

Where $n = |u|$.

<u>Exercise:</u> [Linz, Example 1.13] Let L = the set of all **balanced** words over $\{a, b\}$.

Find a **grammar** for L.

Let G = grammar with productions:

$$S \longrightarrow a\,S\,b \mid b\,S\,a \mid S\,S \mid \lambda$$

Show $L(G) = L$ (= set of all balanced words in Σ^*).

In order to accomplish this, we must show:

(a) $L(G) \subseteq L$ and,

(b) $L \subseteq L(G)$

(a) is clear: every sentential form generated by G has an equal number of a's and b's.

(b) We will show:

If u is **balanced**, then u is generated by G.

The proof is by **induction (CVI)** on $|u|$.

Base: Suppose $|u| = 0$.
Then $u = \lambda$ which is derived in G by $S \to \lambda$.

Induction step: Suppose $|u| = n > 0$ and
$\forall v \in L$, if $|v| < n$, then v is generated by G. **(i.h.)**

There are 4 cases for u:

(1) $u = a\,v\,b$

(2) $u = b\,v\,a$

(3) $u = a\,v\,a$

(4) $u = b\,v\,b$

Case 1

$u = a\,v\,b$

Then v is balanced and $|v| < |u|$. So by the **induction hypothesis**, there is a G-derivation of v:

$$S \overset{\star}{\Longrightarrow} v$$

But then there is a G-derivation of u:

$$S \Longrightarrow a\,S\,b \overset{\star}{\Longrightarrow} a\,v\,b = u$$

Case 2

$u = b\,v\,a$

Very similar to **Case 1**.

Case 3

$u = a\,v\,a$

So, if $u = a_1 ... a_n$, then $a_1 = a$, $a_n = a$.

For $k = 0, 1, ..., n$, let

$$f(k) = \mathbf{xs}(a, b, u \upharpoonright k)$$
$$= \text{excess of } a \text{ over } b \text{ in first } k \text{ symbols of } a.$$

.

Note:

$u \restriction 0 = \lambda$, so $f(0) = \underline{0}$

$u \restriction n = u$, so $f(n) = \underline{0}$

And, for each $k < n$,

$$\begin{aligned} f(k+1) &= f(k) + 1 \quad \text{if } a_k = a \\ &= f(k) - 1 \quad \text{if } a_k = b \end{aligned}$$

So:

$$f(1) = \underline{1}$$
$$f(n-1) = \underline{-1}$$

$\therefore \exists k : \ 0 < k < n : \ f(k) = 0$

So putting

$$\begin{aligned} v_1 &= a_1 \ldots a_k \\ v_2 &= a_{k+1} \ldots a_n \end{aligned}$$

$u = v_1 \, v_2$ where v_1, v_2 are **balanced**.

So by the **induction hypothesis** there are G-derivations

$$S \overset{\star}{\Longrightarrow} v_1$$
$$S \overset{\star}{\Longrightarrow} v_2$$

But then there is a G-derivation of u:

$$\underline{S \Longrightarrow S\,S \stackrel{\star}{\Longrightarrow} v_1\,S \stackrel{\star}{\Longrightarrow} v_1\,v_2 = u}$$

Case 4

$u = b\,v\,b$

Similar to **Case 3**.