• To show connection between PDA's and CFL's, we use **another normal form** for CFG's:

Greibach Normal Form (GNF)

Definition: A CFG is in GNF if its productions are all of the form:

$$A \longrightarrow a\,B_1\,B_2\,\ldots\,B_k \quad (k \geq 0)$$

where $A, B_1, \ldots, B_k$ are nonterminals and $a \in \Sigma$.

Note:

Can have $k = 0$ giving production

$$A \longrightarrow a$$

$Theorem \quad (Greibach)$

For any CFG G, there is a CFG G^G in **GNF** s.t.

$$L(G^G) \;=\; L(G)\backslash\{\lambda\}.$$

[see Linz Sec. 6.2 or Kozen Lecture 21]

Notes:

- GNF gives "leftmost derivation" of a string
- GNF is used to prove **equivalence** of **CFGs** and **PDAs**.

Equivalence of CFG's and PDA

[Linz § 7.2, Kozen L. 25]

Theorem 1 : Given a CFG $G = (N, \Sigma, S, P)$, we can construct a PDA M s.t.

$$L(M) = L(G).$$

Proof (outline):

By **Greibach's Theorem**, we can assume all productions of G are of the form:

$$A \longrightarrow aB_1B_2...B_k \quad (k \geq 0)$$

where $a \in \Sigma \cup \{\lambda\}$

Define a PDA

$$M = (\{q\},\ \Sigma,\ \Gamma,\ \delta,\ q,\ \overset{\perp}{S},\ \varnothing)$$

which accepts by empty stack, where

- q is the only state
- Σ is the same as for G

- $\Gamma = N$ (i.e. stack alphabet = G's set of non-terminals)
- $S\ (= \bot)$ is the initial stack symbol
- δ is defined by:

  for each production of G

$$A \longrightarrow aB_1...B_k$$

  δ contains the transition

$$(q, a, A) \longrightarrow (q, B_1...B_k).$$

Then we can show: $L(G) = L(M)$. □

To prove the **converse**: First note:

A PDA can always be simulated by a PDA which

- accepts by empty stack, and
- has only 1 state.

Theorem 2 (converse):
Given a PDA M, we can construct a CFG G s.t.

$$L(G) = L(M).$$

Proof (outline): *Step 1*: Replace M by a PDA which accepts by empty stack and has only one state.
Step 2: **Reverse** the construction in the proof of Thm 1. □

Example 4: Give a PDA for $\{uu^{\mathsf{R}} \mid u \in \{a,b\}^*\}$
i.e. palindromes of even length over $\{a,b\}$.

Define M :

$$
\begin{aligned}
Q &= \{s,\, q,\, f\} \\
\Sigma &= \{a,\, b\} \\
\Gamma &= \{\perp,\, A,\, B\} \\
F &= \{f\}
\end{aligned}
$$

δ :

(1) $\left(s,\, a,\, \begin{matrix}\perp \\ A \\ B\end{matrix}\right) \longrightarrow \left(s,\, \begin{matrix}A\perp \\ AA \\ AB\end{matrix}\right)$ **Build up** stack to **copy** 1st half of string

(2) $\left(s,\, b,\, \begin{matrix}\perp \\ A \\ B\end{matrix}\right) \longrightarrow \left(s,\, \begin{matrix}B\perp \\ BA \\ BB\end{matrix}\right)$ **Build up** stack to **copy** 1st half of string

(3) $(s,\, a,\, A) \longrightarrow (q,\, \lambda)$ **Nondeterministic**: **guess** '***aa***' in middle and change state.

(4) $(s,\, b,\, B) \longrightarrow (q,\, \lambda)$ **Nondeterministic**: **guess** '***bb***' in middle and change state.

(5) $(q,\, a,\, A) \longrightarrow (q,\, \lambda)$ 2nd half **matches** so far!

(6) $(q,\, b,\, B) \longrightarrow (q,\, \lambda)$ 2nd half **matches** so far!

(7) $(q,\, \lambda,\, \perp) \longrightarrow (f,\, \lambda)$ Success! Word is a palindrome.

What about the empty string?

Must add: $(8) \quad (s,\ \lambda,\ \perp) \longrightarrow (f,\ \lambda)$

Note:

No transitions

$$
\begin{array}{ll}
(q,\ a,\ B) & \longrightarrow \ldots \\
(q,\ b,\ A) & \longrightarrow \ldots \\
\left(q,\ \begin{matrix} a \\ b \end{matrix},\ \perp\right) & \longrightarrow \ldots
\end{array}
$$

These are not palindromes!