8 Closure Properties of Context Free Language

[Linz Ch. 8]

Recall (from Sec. 4):

If $L_1, L_2 \subseteq \Sigma^*$ are **regular**, then so are $L_1 \cup L_2$, $L_1 \cap L_2$.

For $\cup$: 3 proofs:

(1) Suppose $L_1 = L(M_1)$, $L_2 = L(M_2)$: M_1, M_2 are NFA's.

Then $L_1 \cup L_2 = L(M_1 \sqcup M_2)$ [See p. 2-15, 4-1, 4-10]

(2) **Product Construction**:

Let $L_1 = L(M_1)$, $L_2 = L(M_2)$: M_1, M_2 are DFA's

Define **product DFA**'s: [pp. 4-5 to 4-7]

$$M_3 = M_1 \vee M_2, \quad L(M_3) = L(M_1) \cup L(M_2)$$

$$M_4 = M_1 \wedge M_2, \quad L(M_4) = L(M_1) \cap L(M_2)$$

(3) $\exists$ regular expressions r_1, r_2 s.t. $L_1 = L(r_1)$, $L_2 = L(r_2)$.

Then $L_1 \cup L_2 = \underline{L(r_1 + r_2)}$

Note also: If L is regular, then so is $\overline{L} = \Sigma^* \backslash L$. [See p. 4-4.]

Now for **CF** languages:

(1) **Union**:

Suppose L_1, L_2 are CF. Then $L_1 \cup L_2$ is CF.

Proof : Suppose $L_1 = L(G_1,\ L_2 = L(G_2)$ for CFG's G_1, G_2.

Say
$$G_1 = (N_1,\ \Sigma,\ S_1,\ P_1)$$
$$G_2 = (N_2,\ \Sigma,\ S_2,\ P_2)$$

We can assume w.l.o.g. $N_1 \cap N_2 = \emptyset$. (Why?)

Let
$$G = (N,\ \Sigma,\ S,\ P)$$

where S is a **new start symbol** and

$$N = N_1 \cup N_2 \cup \{S\}$$

and $$P = \underline{\{S \to S_1 \mid S_2\} \cup P_1 \cup P_2}$$

.

Then $L(G) = \underline{L_1 \cup L_2}$

(2) **Concatenation**:

Let $L_1 = L(G_1)$, $L_2 = L(G_2)$ as above,

with $N_1 \cap N_2 = \emptyset$.

Define
$$G = (N,\ \Sigma,\ S,\ P)$$

where S is a **new start symbol** and

$$N = N_1 \cup N_2 \cup \{S\}$$

and
$$P = \underline{\{S \to S_1 S_2\} \cup P_1 \cup P_2}$$

Then $L(G) = \underline{L_1 L_2}$

(3) **Star Closure**:

Let $L_0 = L(G_0)$, $G_0 = (N_0,\ \Sigma,\ S_0,\ P_0)$

Define
$$G = (N,\ \Sigma,\ S,\ P)$$

where S is a **new start symbol** and

$$N = N_0 \cup \{S\}$$

and
$$P = P_0 \cup \underline{\{S \to S_0 S \mid \lambda\}.}$$

Then $L(G) = \underline{L(G_0)^*}$

- **Intersection**?

NO! (See Homework 10.)

What is the **problem**?

Recall: to show **regular languages** closed under intersection uses **product construction** on DFA's:

if $$L_1 = L(M_1), \quad L_2 = L(M_2)$$

then $$L_1 \cap L_2 \ = \ L(M_1 \wedge M_2).$$ [pp.4-5/6]

Suppose now: $L_1 = L(M_1),\ \ L_2 = L(M_2)$

where M_1, M_2 are PDA's.

Q. Why can't we form a "product" of M_1 and M_2?

A. <u>How can we work with 2 **stacks**?</u>

But, we **can** form a product of a PDA and a regular DFA!

Hence we can show:

Theorem: If $L_1, L_2 \subseteq \Sigma^*$, L_1 is a **CFL**, L_2 is **regular**,

then $L_1 \cap L_2$ is a **CFL**.

Proof : Say $L_1 = L(M_1)$, M_1 **is a PDA**

$L_2 = L(M_2)$, M_2 **is a DFA**

Let $$M_1 = (Q_1,\ \Sigma,\ \Gamma,\ \delta_1,\ s_1,\ \perp,\ F_1)$$
$$M_1 = (Q_2,\ \Sigma,\ \delta_2,\ s_2,\ F_2)$$

We construct a **product** PDA,

$$M = M_1 \wedge M_2 = (Q,\ \Sigma,\ \Gamma,\ \delta,\ s,\ \perp,\ F)$$

where

$$Q = \underline{Q_1 \times Q_2}$$
$$\Gamma$$
$$s = \underline{(s_1, s_2)} \qquad \text{(start state of } M\text{)}$$
$$\perp$$
$$F = \underline{F_1 \times F_2} \qquad \text{(set of final states of } M\text{)}$$

and δ is defined by:

For $p_1, p_2, q_1, q_2 \in Q, \quad a \in \Sigma, \quad A \in \Gamma, \quad \alpha \in \Gamma^*$:

if

$$\delta_1 : \ (p_1,\ a,\ A) \ \rightarrow \ (q_1,\ \alpha)$$
$$\delta_2 : \ (p_2,\ a) \qquad \rightarrow \ q_2$$

then

$$\delta : \ ((p_1, p_2), a, A) \ \rightarrow \ \underline{((q_1, q_2), \alpha)}$$

Then

$$L(M) \;=\; L(M_1) \cap L(M_2) \;=\; L_1 \cap L_2.$$

<u>Exercise:</u> Let $L = \{w \in \{a, b, c\}^* \mid n_a(w) = n_b(w) = n_c(w).\}$

Show that L is **not** a CFL.