Product DFA's

Product Construction: [Kozen, Lecture 4]

To construct **new dfa's from old** ones.

Given two **dfa's** over a (fixed) Σ,

$$M_1 = (Q_1,\ \Sigma,\ \delta_1,\ q_{01},\ F_1)$$

$$M_2 = (Q_2,\ \Sigma,\ \delta_2,\ q_{02},\ F_2)$$

we construct a **product dfa**:

$$M_3 = (Q_3,\ \Sigma,\ \delta_3,\ q_{03},\ F_3)$$

where:

$Q_3 = Q_1 \times Q_2$

$\delta_3 : Q_3 \times \Sigma \longrightarrow Q_3$

where for $(q_1, q_2) \in Q_3$, $a \in \Sigma$:

$\delta_3((q_1, q_2),\ a) \ = \ (\delta_1(q_1, a),\ \delta_2(q_2, a))$

$q_{03} \ = \ (q_{01}, q_{02})$

$F_3 \ = \ F_1 \times F_2 \ = \ \underline{\{(q_1, q_2) \in Q_3 \mid q_1 \in F_1 \wedge q_2 \in F_2\}}$

We can show: for any $u \in \Sigma^*$

$$\delta_3^*((q_1, q_2), u) \ = \ (\delta_1^*(q_1, u),\ \delta_2^*(q_2, u))$$

by **structural induction** on $u \in \Sigma^*$,

or **simple induction** on $|u|$ (cf. pp. 1-14/15, 2-4)

Q. What is $L(M_3)$?

Note in general for $M = (Q,\ \Sigma,\ \delta,\ q_0,\ F),\ u \in \Sigma^*$:

$$u \in L(M) \iff \delta^*(q_0, u) \in F.$$

So for $u \in \Sigma^*$:

$$\begin{aligned} u \in L(M_3) &\iff \delta_3^*(q_{03}, u) \in F_3 \\ &\iff (\delta_1^*(q_{01}, u),\ \delta_2^*(q_{02}, u)) \in F_1 \times F_2 \\ &\iff \delta_1^*(q_{01}, u) \in F_1 \ \wedge\ \delta_2^*(q_{02}, u) \in F_2 \\ &\iff u \in L(M_1) \ \wedge\ u \in L(M_2) \end{aligned}$$

$\therefore L(M_3) = \underline{L(M_1) \cap L(M_2)}$

We will write:

$$M_3 = M_1 \wedge M_2$$

Note (from p. 4-5):

$F_3 \ = \ \{(q_1, q_2) \in Q_1 \times Q_2 \mid q_1 \in F_1 \ \wedge\ q_2 \in F_2\}$

We construct **another product dfa**

$$M_4 \ = \ (Q_3, \ \Sigma, \ \delta_3, \ q_{03}, \ F_4)$$

where

$$F_4 \ = \ \{(q_1, q_2) \in Q_1 \times Q_2 \mid q_1 \in F_1 \ \vee \ q_2 \in F_2\}$$

Now for $u \in \Sigma^*$

$$\begin{aligned} u \in L(M_3) \ &\iff \ \delta_3^*(q_{03}, u) \in F_4 \\ &\iff \ (\delta_1^*(q_{01}, u), \ \delta_2^*(q_{02}, u)) \in F_4 \\ &\iff \ \delta_1^*(q_{01}, u) \in F_1 \ \vee \ \delta_2^*(q_{02}, u) \in F_2 \\ &\iff \ u \in L(M_1) \ \vee \ u \in L(M_2) \end{aligned}$$

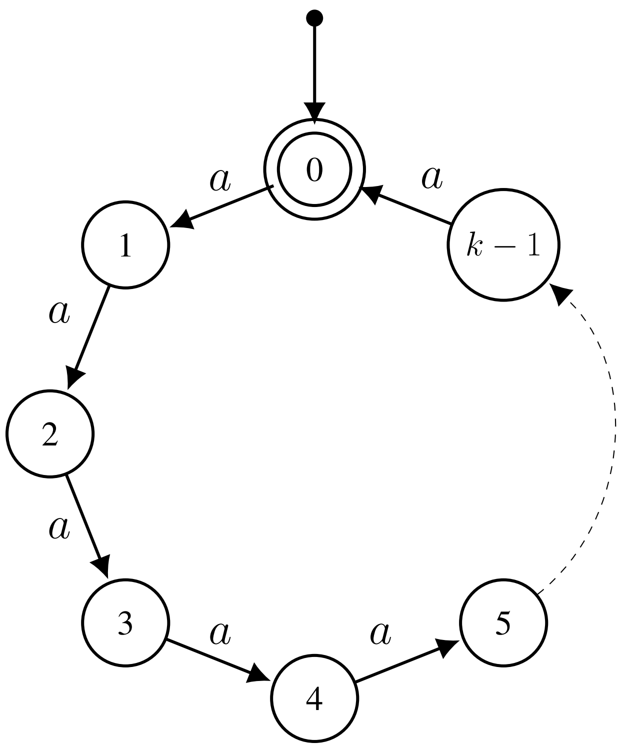
$\therefore L(M_4) \ = \ \underline{L(M_1) \cup L(M_2)}$

We will write:

$$M_4 = M_1 \vee M_2$$

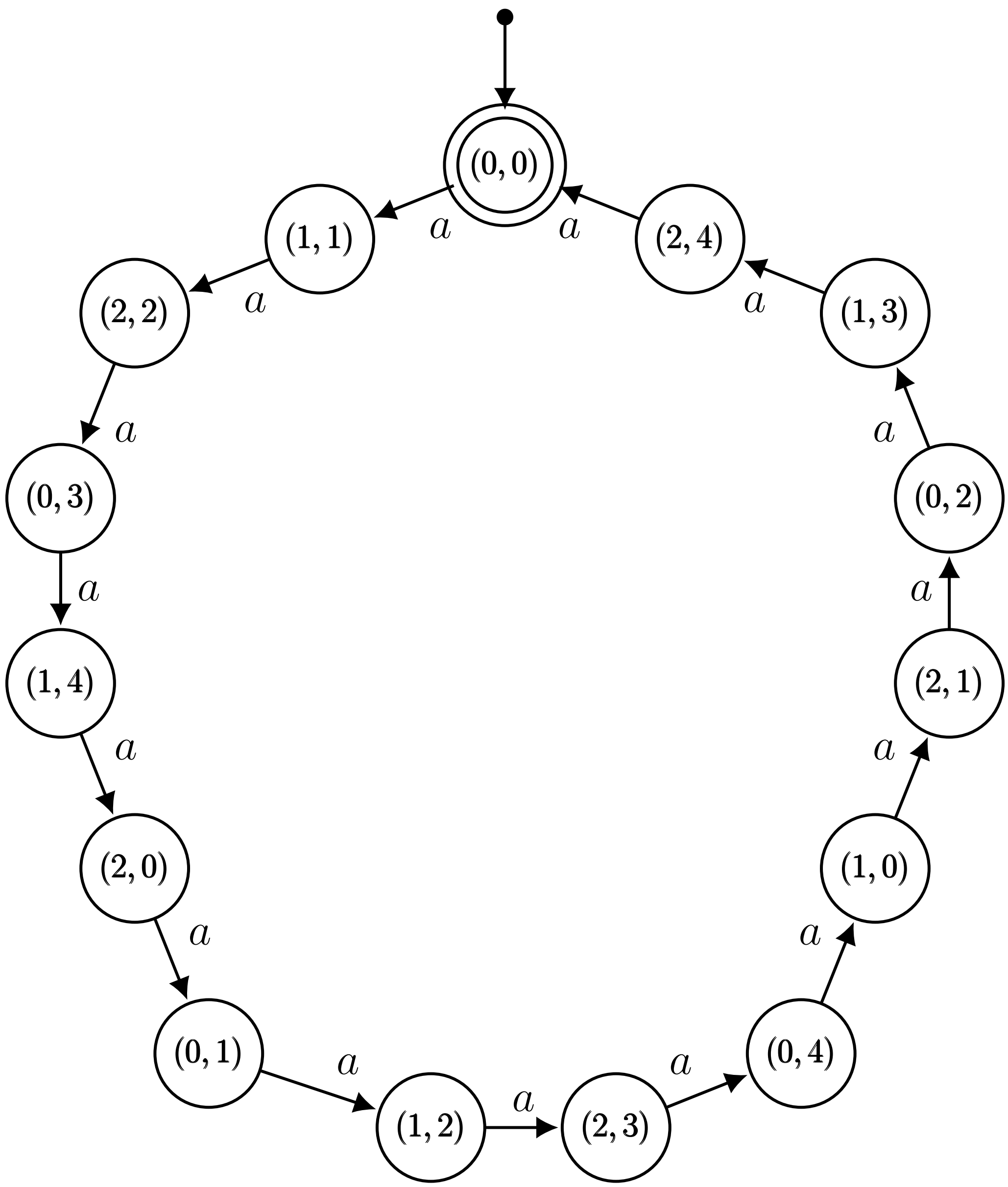
Example: Let $\Sigma = \{a\}$,

$C_k =$ the "cyclic" **dfa**:



$L(C_k) = \{u \in \Sigma^* \mid \underline{|u| \text{ is divisible by } k}\}$

What are $C_3 \wedge C_5$, $C_3 \vee C_5$?

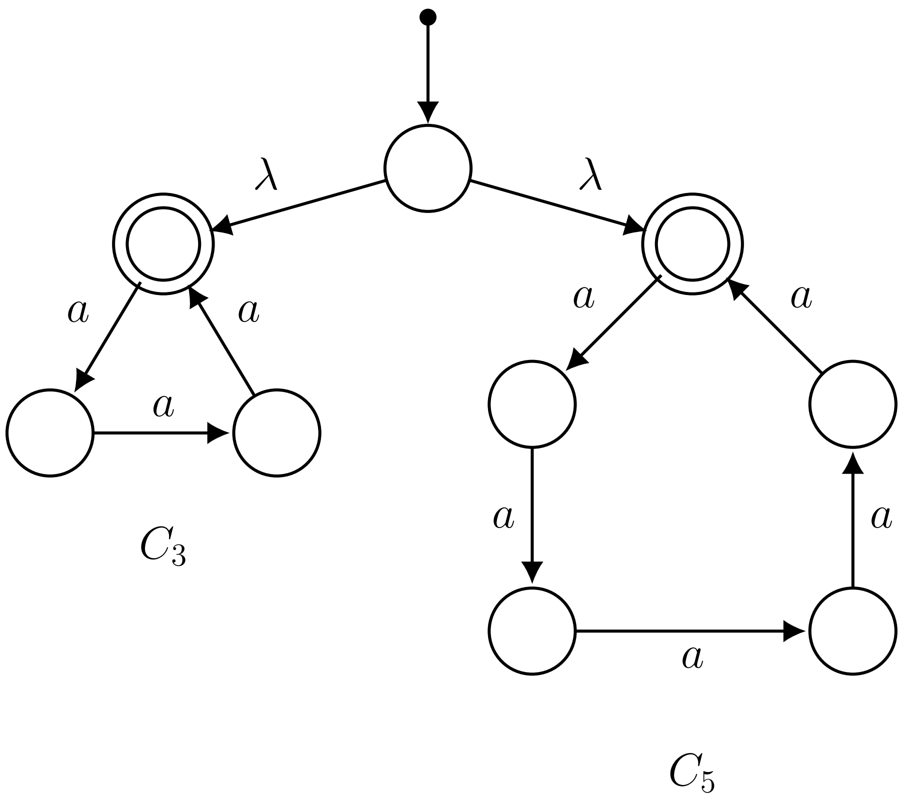


Final states of $C_3 \wedge C_5$: $(0, 0)$

Final states of $C_3 \vee C_5$: $(0, j)$ or $(i, 0)$ $(i = 0, \ldots, 2, j = 0, \ldots, 4)$

Note:

Alternative nfa for $L(C_3) \cup L(C_5)$: $C_3 \sqcup C_5$ [cf. p. 2-15]



Note:

This is **simpler** than $C_3 \vee C_5$:

$C_k \sqcup C_\ell$ has $k + \ell + 1$ nodes

$C_k \vee C_\ell$ has $lcm(k, \ell)$ nodes

Note:
$C_k \vee C_\ell$ is **deterministic**
but $C_k \sqcup C_\ell$ is **non-deterministic**

If we use **subset construction** on $C_k \sqcup C_\ell$
to get a **dfa** $(C_k \sqcup C_\ell)^D$ and then **remove unattainable states**,
we get a dfa **isomorphic** to $C_k \vee C_\ell$

(See pp. 2-12/14.)