Regular languages and nfa's

Theorem : [Linz, Theorem 3.1]

Every **regular language**, L, is **accepted** by some **nfa**.

I.e., For every **Σ-regular expression**, r, there is a Σ-**nfa** M, s.t.

$$L(r) = L(M).$$

Proof. By **structural induction** on $r \in \textbf{\textit{RegExp}}(\Sigma)$,

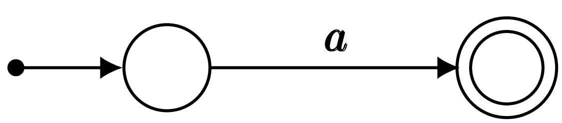
i.e. according to the **inductive definition** of ***RegExp***(Σ) (p. 3-1). □

(***Note***: By note (2) on p. 3-6, we can assume that every **nfa** we construct in the course of the proof has exactly one **final state**.)

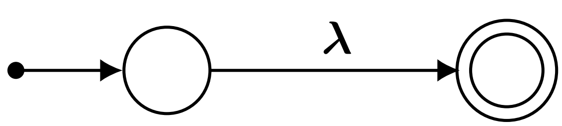
We proceed with the **inductive proof**:

Base cases:

- $r = a$ ($\in \Sigma$). Then $L(r) = \{a\}$ is accepted by:



- $r = \lambda$. Then $L(r) = \{\lambda\}$ is accepted by:

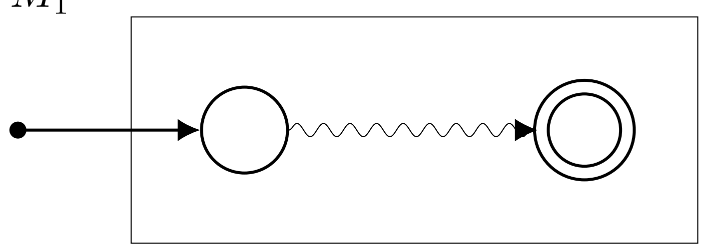


- $r = \emptyset$. Then $L(r) = \emptyset$ is accepted by:

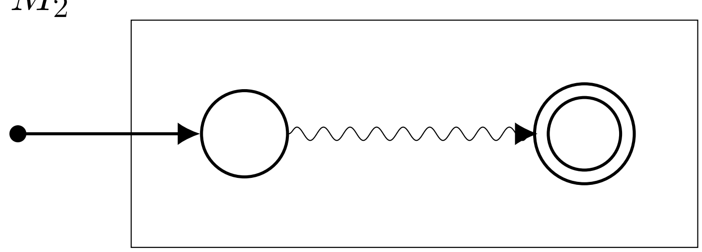
Recursive steps:

- $r = r_1 + r_2$. Suppose (**induction hypothesis**) we have **nfa's** M_1 and M_2 which accept $L(r_1)$, $L(r_2)$ respectively. Then $L(r) = L(r_1) \cup L(r_2)$ is accepted by:

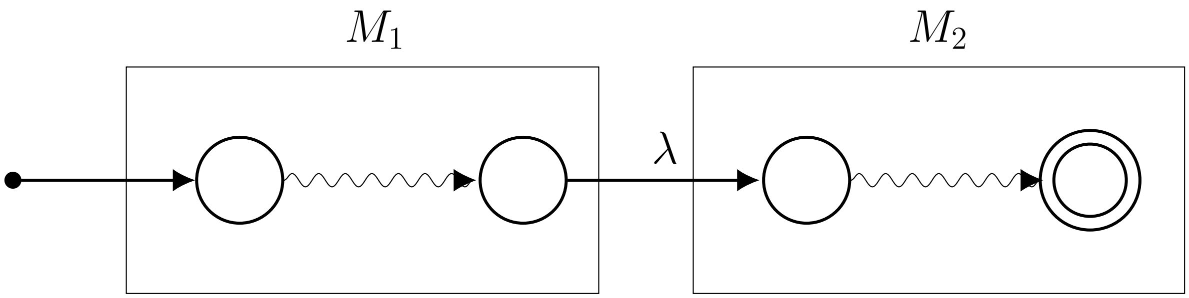
M_1



M_2

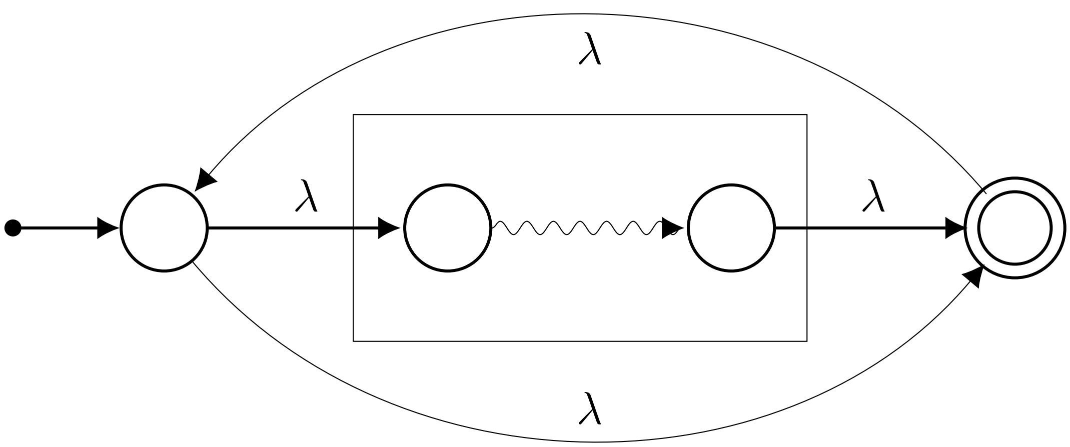


- $\boldsymbol{r = r_1 \cdot r_2}$. Suppose (**induction hypothesis**) we have **nfa's** M_1 and M_2 which accept $\boldsymbol{L(r_1)}$, $\boldsymbol{L(r_2)}$ respectively. Then $\boldsymbol{L(r) = L(r_1) \cdot L(r_2)}$ is accepted by:



- $\boldsymbol{r = r_1^*}$. Suppose (**induction hypothesis**) we have an **nfa** M_1 which accepts $L(r_1)$. Then, $L(r) = L(r_1)^*$ is accepted by:

M



$Q.$ Is the following true?

$\forall \boldsymbol{r} \in \boldsymbol{RegExp}(\Sigma),\ \exists\ \Sigma$-**dfa** which accepts $\boldsymbol{L}(\boldsymbol{r})$

YES

$Q.$ Is the **converse** true?

$\forall\ \Sigma$-**nfa** $\boldsymbol{M},\ \exists\, \boldsymbol{r} \in \boldsymbol{RegExp}(\Sigma):\ \boldsymbol{L}(\boldsymbol{r}) = \boldsymbol{L}(\boldsymbol{M})$

YES

Theorem : [Linz, Theorem 3.2]

For every nfa, M, $L(M)$ is **regular**.

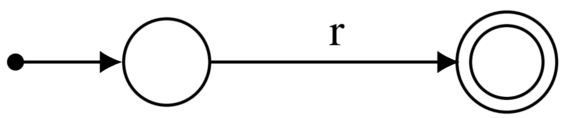
I.e. $\forall\ \Sigma$-**nfa** $\boldsymbol{M},\ \exists\, \boldsymbol{r} \in \boldsymbol{RegExp}(\Sigma):\ \boldsymbol{L}(\boldsymbol{r}) = \boldsymbol{L}(\boldsymbol{M})$

Proof : (**outline**)

The proof uses the concept of **generalized transition graphs**, i.e. graphs where the edges are labeled not just by **elements** of Σ, but by Σ-**regular expressions**.

The graph for M is **transformed** into an equivalent graph with **fewer nodes**, and edges with **more complex** Σ-expressions.

This is repeated until we arrive at a graph of the form:



Then $L(M) = L(r)$, i.e. $L(M)$ is **regular**.

Combining Theorems 3.1 and 3.2, we have proven:

Theorem : For any Σ-language $L \subseteq \Sigma^*$,
L is **regular** $\iff$ L is accepted by some **nfa**.

Note:
We have already seen:
For every Σ-**nfa**, N, we can find an **equivalent** Σ-dfa, M.
i.e. with $L(M) = L(N)$
So we can **rewrite** the above Theorem:

Corollary : For any Σ-language $L \subseteq \Sigma^*$,
L is **regular** $\iff$ L is accepted by some **dfa**

Note: Linz **defines** a language to be "regular" if it is accepted by some **dfa** [Linz, Def. 2.3]. We have not done that here – but it comes to the same thing by the theorems above.

Example: "**Reversing**" an **nfa**

Show if L is **regular**, so is L^{R}.

Use Corollary on p. 3-11.

Suppose L is regular.

Then, L, is accepted by some **nfa**:

$M = (Q, \Sigma, \delta, q_0, F)$

Define $M^{\mathsf{R}} = (Q, \Sigma, \delta^{\mathsf{R}}, q_0^{\mathsf{R}}, F^{\mathsf{R}})$ as follows:

1) **Reverse all arrows**

i.e. $q_1 \in \delta^{\mathsf{R}}(q_2, a) \Longleftrightarrow q_2 \in \delta^{\mathsf{R}}(q_1, a)$

2) **Interchange starting and final states**

i.e. $F^{\mathsf{R}} = \{q_0\}$

Let q_0^{R} = **new start state**, joined to **all old final states** by λ- transitions. (see p. 2-17)

Then, $L(M^{\mathsf{R}}) = (L(M))^{\mathsf{R}}$.