ECE 707: Linear Systems (4)

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These viewgraphs are based on the text
“Linear System: Theory and Design” by Chi-Tsong Chen
Oxford University Press, 1999.

5. State-Space Solutions and Realizations

The input and output of a linear system (SISO) are related as follows

$$y(t) = \int_{\tau=t_0}^{t} g(t,\tau)u(\tau)d\tau \tag{1}$$

The equation must be digitized to compute $y(t)$ on a digital computer

$$y(k\Delta) = \sum_{m=k_0}^{k} g(k\Delta, m\Delta)u(m\Delta)\Delta \tag{2}$$

where Δ is the integration step. Problem: result is inaccurate.
For LTI system we can find the Laplace transforms $\hat{g}(s)$ and $\hat{u}(s)$, then get $\hat{y}(s) = \hat{g}(s)\hat{u}(s)$ and then transfer $\hat{y}(s)$ back to the time domain.
Problem: When there are repeated poles the computation becomes very sensitive to roundoff errors.
Better method is to get a set of state space equations from the transfer functions and then compute the solutions.

For a LTI system the state-space equation is given by

$$\begin{aligned} \underline{\dot{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \quad (3) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \quad (4) \end{aligned}$$

The problem is to find $\underline{y}(t)$ in terms of $\underline{x}(0)$ (initial state) and $\underline{u}(t), \;\; t > 0$ (input).
We can use exponential of At to get the solution. It has the properties

$$\begin{aligned} \frac{d}{dt}e^{At} &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{1}{k!}t^k A^k = \sum_{k=1}^{\infty}\frac{1}{(k-1)!}t^{k-1}A^k = A\left(\sum_{k=0}^{\infty}\frac{1}{k!}t^k A^k\right) = Ae^{At} \\ &= \left(\sum_{k=0}^{\infty}\frac{1}{k!}t^k A^k\right)A = e^{At}A \quad (5) \end{aligned}$$

also

$$e^0 = I \quad \text{Here 0 is a matrix} \quad (6)$$

Starting from (3) we have

$$
\begin{aligned}
& \underline{\dot{x}}(t) - A\underline{x}(t) = B\underline{u}(t) \\
\Rightarrow \quad & e^{-At}\underline{\dot{x}}(t) - e^{-At}A\underline{x}(t) = e^{-At}B\underline{u}(t) \quad \textcolor{red}{\text{multipling by } e^{-At}} \\
\Rightarrow \quad & \frac{d}{dt}\left(e^{-At}\underline{x}(t)\right) = e^{-At}B\underline{u}(t) \\
\Rightarrow \quad & e^{-A\tau}\underline{x}(\tau)\Big|_{\tau=0}^{t} = \int_0^t e^{-A\tau}B\underline{u}(\tau)d\tau \quad \textcolor{red}{\text{Integrating from 0 to } t} \\
\Rightarrow \quad & e^{-At}\underline{x}(t) - \underline{x}(0) = \int_0^t e^{-A\tau}B\underline{u}(\tau)d\tau \\
\Rightarrow \quad & \underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau \qquad (7)
\end{aligned}
$$

So this gives a solution to (3), to check this (1) evaluate $\underline{x}(t)$ at $t = 0$ and (2) find the derivative of $\underline{x}(t)$ and see if we get (3) back.
Now $\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t)$ gives

$$
\underline{y}(t) = Ce^{At}\underline{x}(0) + C\int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau + D\underline{u}(t) \qquad (8)
$$

Notice that Laplace transform gives

$$\hat{\underline{x}}(s) = (sI - A)^{-1}\underline{x}(0) + (sI - A)^{-1}B\hat{\underline{u}}(s) \tag{9}$$

$$\hat{\underline{y}}(s) = C(sI - A)^{-1}\underline{x}(0) + C(sI - A)^{-1}B\hat{\underline{u}}(s) + D\hat{\underline{u}}(s) \tag{10}$$

Irrespective of whether we start with

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \end{aligned}$$

or

$$\begin{aligned} \underline{x}(t) &= e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau \\ \underline{y}(t) &= Ce^{At}\underline{x}(0) + C\int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau + D\underline{u}(t) \end{aligned}$$

Discretization

If the input $\underline{u}(t)$ piecewise constant (example, input generated by digital computer), then

$$\underline{u}(t) = \underline{u}(kT) = \underline{u}[k], \quad kT \leq t < (k+1)T \tag{11}$$

In this case the solution of state equation is given by

$$\underline{x}[k] \stackrel{\triangle}{=} \underline{x}(kT) = e^{AkT}\underline{x}(0) + \int_0^{kT} e^{A(kT-\tau)}B\underline{u}(\tau)d\tau \tag{12}$$

and

$$\begin{aligned}
\underline{x}[k+1] &= e^{A(k+1)T}\underline{x}(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}B\underline{u}(\tau)d\tau \\
&= e^{AT}\left[e^{AkT}\underline{x}(0) + \int_0^{kT} e^{A(kT-\tau)}B\underline{u}(\tau)d\tau\right] \\
&\quad + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}B\underline{u}(\tau)d\tau \\
&= e^{AT}\underline{x}[k] + \left(\int_0^{T} e^{A\alpha}d\alpha\right)B\underline{u}[k], \quad \alpha \stackrel{\triangle}{=} kT + T - \tau
\end{aligned} \tag{13}$$

Thus we have

$$
\begin{aligned}
\underline{x}[k+1] &= A_d \underline{x}[k] + B_d \underline{u}[k] && (14)\\
\underline{y}[k] &= C_d \underline{x}[k] + D_d \underline{u}[k] && (15)
\end{aligned}
$$

where

$$
A_d = e^{AT} \quad B_d = \int_0^T e^{A\alpha} d\alpha \quad C_d = C \quad D_d = D \tag{16}
$$

Solution of Discrete-Time Equations

Discrete-time state space equations are

$$\begin{aligned} \underline{x}[k+1] &= A\underline{x}[k] + B\underline{u}[k] && (17) \\ \underline{y}[k] &= C\underline{x}[k] + D\underline{u}[k] && (18) \end{aligned}$$

Here we can start with $\underline{x}[k]$ and get

$$\begin{aligned} \underline{x}[1] &= A\underline{x}[0] + B\underline{u}[0] && (19) \\ \underline{x}[2] &= A\underline{x}[1] + B\underline{u}[1] = A^2\underline{x}[0] + AB\underline{u}[0] + B\underline{u}[1] && (20) \end{aligned}$$

So for any $k > 0$

$$\begin{aligned} \underline{x}[k] &= A^k\underline{x}[0] + \sum_{m=0}^{k-1} A^{k-1-m}B\underline{u}[m] && (21) \\ \underline{y}[k] &= CA^k\underline{x}[0] + \sum_{m=0}^{k-1} CA^{k-1-m}B\underline{u}[m] + D\underline{u}[k] && (22) \end{aligned}$$

Easy, is not it!!

Equivalent State Equations

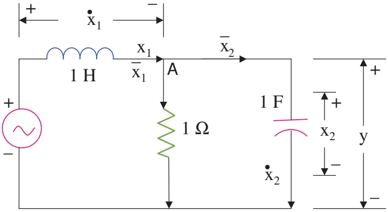


Fig. 6.1

Several sets of choices for state variables. We discuss two such sets here.

First, let inductor current x_1 and capacitor voltage x_2 are the state variables.

Voltages in the left loop gives $u = \dot{x}_1 + x_2$, currents at A gives $x_1 = x_2 + \dot{x}_2$. Hence

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= [0 \quad 1] \underline{x}(t) \end{aligned}$$

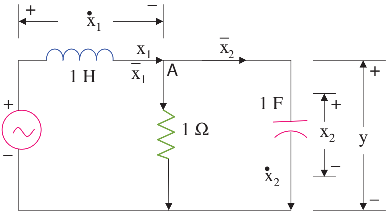


Fig. 6.1 (repeated)

Now, if $\bar{x}_1$ and $\bar{x}_2$, which are currents in the loops, are the state variables.

From left hand loop we have $u = \dot{\bar{x}}_1 + \bar{x}_1 - \bar{x}_2$.

Since voltage across capacitor is equal to voltage across resistor $(\bar{x}_1 - \bar{x}_2)$, we have $\bar{x}_2 = \dot{\bar{x}}_1 - \dot{\bar{x}}_2 = u - \bar{x}_1 - \bar{x}_2 - \dot{\bar{x}}_2$.

Thus the state equations

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\
y(t) &= [1 \quad -1]\underline{x}(t)
\end{aligned}
$$

Equivalent Transform

If p is a nonsingular matrix and $\underline{\bar{x}} = P\underline{x}$. Then the state equation

$$\begin{aligned} \dot{\underline{\bar{x}}}(t) &= \bar{A}\underline{\bar{x}}(t) + \bar{B}\underline{u}(t) \qquad (23) \\ \underline{y}(t) &= \bar{C}\underline{\bar{x}}(t) + \bar{D}\underline{u}(t) \qquad (24) \end{aligned}$$

where $\bar{A} = PAP^{-1}, \;\; \bar{B} = PB, \;\; \bar{C} = CP^{-1}, \;\; \bar{D} = D$, is said to be (algebraically) equivalent to (17)–(18) and $\underline{\bar{x}} = P\underline{x}$ is called an equivalence transform.

Can be proved by replacing $\underline{x}$ by $P^{-1}\underline{\bar{x}}$ and $\dot{\underline{x}}$ by $P^{-1}\dot{\underline{\bar{x}}}$.

Notice that D is unchanged in the transform and is called the direct transform part between input and output.

Equivalent states have same characteristic equation

$$\begin{aligned} \bar{\Delta}(\lambda) &= \det(\lambda I - \bar{A}) = \det(\lambda P P^{-1} - P\bar{A}P^{-1}) = \det\left(P(\lambda I - \bar{A})P^{-1}\right) \\ &= \det(P)\det(\lambda I - \bar{A})\det(P^{-1}) = \Delta(\lambda) \qquad (25) \end{aligned}$$

Equivalent representations have same overall transfer matrix

$$\begin{aligned} \hat{\bar{G}}(s) &= \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = CP^{-1}\left(P(sI - A)P^{-1}\right)PB + D \\ &= C(sI - A)^{-1}B + D = \hat{G}(s) \end{aligned} \tag{26}$$

In our example

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \tag{27}$$

Zero-State Equivalence

Two state equations are called zero state equivalent if they have the same transfer matrix or

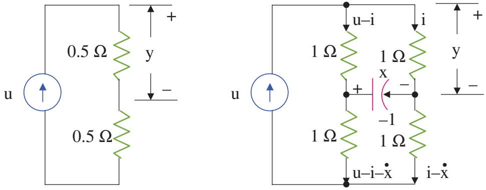
$$C(sI - A)^{-1}B + D = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} \tag{28}$$

Since $(sI - A)^{-1} = s^{-1}\sum_{k=0}^{\infty}(s^{-1}A)^k$ we have

$$\begin{aligned} & D + CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \cdots \\ & \quad = \bar{D} + \bar{C}\bar{B}s^{-1} + \bar{C}\bar{A}\bar{B}s^{-2} + \bar{C}\bar{A}^2\bar{B}s^{-3} + \cdots \end{aligned} \tag{29}$$

Theorem: Two LTI state-equations $\{A, B, C, D\}$ and $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ are zero state equivalent if and only if $D = \bar{D}$ and
$CA^mB = \bar{C}\bar{A}^m\bar{B}, \quad m = 0, 1, 2, \cdots$.
Algebraic equivalence $\Rightarrow$ zero-state equivalence. However, reverse is not true.
In order to be algebraic equivalent the state vectors need to be of same dimension. This is not true for zero-state equivalence.

Example 1: Find state-space representations and transfer matrix.



(a) Circuit 1 (b) Circuit 2

Fig. 6.2

Solution: For circuit 1 we have $y(t) = 0.5u(t)$ or $\hat{y}(s) = 0.5\hat{u}(s)$. Thus transfer function is 0.5. The circuit 1 is described by zero dimensional state equation, or, $A = B = C = 0$ and $D = 0.5$.

In circuit 2 let capacitor voltage be the state variable. Its current flows from negative to positive polarity due to negative capacitance.

Total voltage across upper right-hand loop gives $i - x - (u - i) = 0$, or, $y = i = 0.5(u + x)$.

Total voltage across lower right-hand loop gives

$i - \dot{x} - (u - i + \dot{x}) + x = 0$, or, $2\dot{x} = 2i + x - u = x + u + x - u = 2x$.

So the state-space equation are $\dot{x}(t) = x(t)$ and
$y(t) = 0.5x(t) + 0.5u(t)$.
Here $A = 1,\ B = 0,\ C = 0.5$ and $D = 0.5$.
We get that $\hat{G}(s) = C(sI - A)^{-1}B + D = D = 0.5$.

Realizations

Any LTI system can be described by the input-output description

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s)$$

if the system is lumped as well then, the system is also described by the state-space equation

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \end{aligned}$$

From the state-space equation the transfer matrix we get is unique and given by $\hat{G}(s) = C(sI - A)^{-1}B + D$.
The converse problem is to get a realization (state-space equation) from the transfer matrix.

A transfer function is called realizable if there exists a finite-dimensional state equation, or simply, $\{A, B, C, D\}$ such that

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

Theorem: A transfer matrix $\hat{G}(s)$ is realizable if and only if $\hat{G}(s)$ is a proper rational matrix.

Proof: Let

$$\hat{G}_{sp}(s) \triangleq C(sI - A)^{-1}B = \frac{1}{\text{det}(sI - A)} C[\text{Adj}(sI - A)]B \qquad (30)$$

If A is $n \times n$ then det$(sI - A)^{-1}$ has degree n.
Every entry of Adj$(sI - A)$ is the determinant of $(n-1) \times (n-1)$ submatrix of $(sI - A)$, thus it has utmost degree $(n-1)$. This is also true for their linear combinations.
Thus $C(sI - A)^{-1}B$ is a strictly proper rational matrix.
If D is nonzero matrix, then $C(sI - A)^{-1}B + D$ is proper. This proves the if part.
Note that we have $\hat{G}(\infty) = D$.

To prove the only if part let $\hat{G}(s)$ is a $p \times q$ proper rational matrix.
We can decompose $\hat{G}(s)$ as $\hat{G}(s) = \hat{G}(\infty) + \hat{G}_{sp}(s)$. Here $\hat{G}_{sp}(s)$ is the strictly rational part.
Let

$$d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1} s + \alpha_r \quad (31)$$

be the least common denominator of the elements of $\hat{G}_{sp}(s)$.
Then strictly proper rational matrix $\hat{G}_{sp}(s)$ is given by

$$\hat{G}_{sp}(s) = \frac{1}{d(s)}[N(s)] = \frac{1}{d(s)}\left[N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_{r-1} s + \alpha_r\right] \quad (32)$$

Now we claim that set of equations

$$\underline{\dot{x}} = \begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & \cdots & -\alpha_{r-1} I_p & -\alpha_r I_p \\ I_p & 0 & \cdots & 0 & 0 \\ 0 & I_p & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} I_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \underline{u} \quad (33)$$

$$\underline{y} = [N_1 \;\; N_2 \cdots N_{r-1} \;\; N_r]\, \underline{x} + \hat{G}(\infty)\underline{u} \quad (34)$$

is a realization of $\hat{G}(s)$.
Here length of the state vector is of length rp. Let

$$Z \triangleq \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_r \end{bmatrix} \triangleq (sI - A)^{-1} B \quad (35)$$

Here Z_i are $p \times p$. We have $(sI - A)Z = B$ or $sZ = AZ + B$. If we look at the form of A and B we get

$$sZ_i = Z_{i-1}, \quad \text{for} \quad i = 2, \cdots, r \quad \text{equating second to last } p \times p \text{ blocks} \quad (36)$$

This with the first $p \times p$ block gives

$$\begin{aligned} sZ_1 &= -\alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - \alpha_r Z_r + I_p \\ &= -\left(\alpha_1 + \frac{\alpha_2}{s} + \cdots + \frac{\alpha_r}{s^{r-1}}\right) Z_1 + I_p \end{aligned}$$

or

$$\left(s + \alpha_1 + \frac{\alpha_2}{s} + \cdots + \frac{\alpha_r}{s^{r-1}}\right) Z_1 = \frac{d(s)}{s^{r-1}} Z_1 = I_p \qquad (37)$$

Thus we have

$$Z_1 = \frac{s^{r-1}}{d(s)} I_p, \quad Z_2 = \frac{s^{r-2}}{d(s)} I_p, \cdots, Z_r = \frac{1}{d(s)} I_p \qquad (38)$$

Since $C(sI - A)^{-1}B = N_1 Z_1 + N_2 Z_2 + \cdots N_r Z_r$. We get

$$C(sI - A)^{-1}B = \frac{1}{d(s)} \left[N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_r\right] \qquad (39)$$

This shows $\hat{G}(s)$ has a realization if it is proper rational matrix. (EOP)

Let $\hat{G}(s)$ be given by

$$\hat{G}(s) = \left[\begin{array}{c} d_1 \\ d_2 \end{array} \right] + \frac{1}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \left[\begin{array}{c} \beta_{11} s^3 + \beta_{12} s^2 + \beta_{13} s + \beta_{14} \\ \beta_{21} s^3 + \beta_{22} s^2 + \beta_{23} s + \beta_{24} \end{array} \right] \tag{40}$$

Then its realization can be directly obtained as

$$\underline{\dot{x}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \underline{u} \tag{41}$$

$$\underline{y} = \left[\begin{array}{cccc} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{array} \right] \underline{x} + \left[\begin{array}{c} d_1 \\ d_2 \end{array} \right] \underline{u} \tag{42}$$

Example: Consider the proper rational matrix

$$\hat{G}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-12}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix} \quad (43)$$

Find a realization for it.

Solution: The strictly proper rational matrix part can be arranged as

$$\begin{aligned} \hat{G}_{sp}(s) &= \frac{1}{s^3 + 4.5s^2 + 6s + 2} \begin{bmatrix} -6(s+2)^2 & 3(s+2)(s+0.5) \\ 0.5(s+2) & (s+1)(s+0.5) \end{bmatrix} \\ &= \frac{1}{d(s)} \left(\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} -24 & 7.5 \\ 0.5 & 1.5 \end{bmatrix} s + \begin{bmatrix} -24 & 3 \\ 1 & 0.5 \end{bmatrix} \right) \end{aligned}$$

and a realization

$$\dot{\underline{x}} = \begin{bmatrix} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{u} \quad (44)$$

$$\underline{y} = \begin{bmatrix} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{bmatrix} \underline{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \underline{u} \tag{45}$$

For a proper transfer matrix there are many realizations. Here we discuss realizations that implements the rows or columns of the transfer matrix and the collects the results.

The input and the output are related as $\hat{y}(s) = \hat{G}(s)\hat{u}(s)$. It can also be expressed as

$$\hat{y}(s) = \hat{G}_{c1}(s)\hat{u}(s) + \hat{G}_{c2}(s)\hat{u}(s) + \cdots = \hat{y}_{c1}(s) + \hat{y}_{c2}(s) + \cdots \tag{46}$$

where $\hat{G}_{ci}$ are the columns of $\hat{G}$. This implementation is shown in Fig. 6.3(a).

Another way is that each output term can be implemented as

$$\hat{y}_i(s) = \hat{G}_{ri}(s)\hat{u}(s) \tag{47}$$

where $\hat{G}_{ri}$ are the columns of $\hat{G}$. This implementation is shown in Fig. 6.3(b).

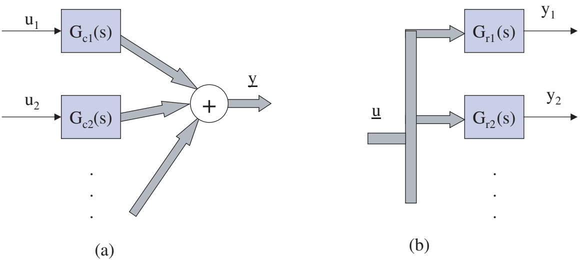


Fig. 6.3

Example: Consider the proper rational matrix

$$\hat{G}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix} \quad (48)$$

Find the realization of its columns and then of the whole matrix.

Solution: First column is

$$\hat{G}_{c1}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} \\ \frac{1}{(2s+1)(s+2)} \end{bmatrix} \quad (49)$$

We get the following realization

$$\begin{aligned} \underline{\dot{x}}_1 &= A_1\underline{x}_1 + \underline{b}_1 u_1 = \begin{bmatrix} -2.5 & -1 \\ 1 & 0 \end{bmatrix} \underline{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 \qquad (50) \\ \underline{y}_{c1} &= C_1\underline{x}_1 + \underline{d}_1 u_1 = \begin{bmatrix} -6 & -12 \\ 0 & 0.5 \end{bmatrix} \underline{x}_1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1 \qquad (51) \end{aligned}$$

The second column is given by

$$\hat{G}_{c2}(s) = \begin{bmatrix} \frac{3}{s+2} \\ \frac{s+1}{(s+2)^2} \end{bmatrix}$$

and the corresponding realization

$$\begin{aligned} \underline{\dot{x}}_2 &= A_2\underline{x}_2 + \underline{b}_2 u_2 = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix} \underline{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2 \qquad (52) \\ \underline{y}_{c2} &= C_2\underline{x}_2 + \underline{d}_2 u_2 = \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix} \underline{x}_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_2 \qquad (53) \end{aligned}$$

The overall state-space equation

$$\begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \begin{bmatrix} \underline{b}_1 & 0 \\ 0 & \underline{b}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{54}$$

$$\underline{y} = \underline{y}_{c1} + \underline{y}_{c2} = [C_1 \;\; C_2]\underline{x} + [\underline{d}_1 \;\; \underline{d}_2]\underline{u} \tag{55}$$

Solution of Linear Time-Varying Equation

Consider the linear time-varying state equation

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t) \tag{56}$$

$$\underline{y}(t) = C(t)\underline{x}(t) + D(t)\underline{u}(t) \tag{57}$$

For simplicity let's find a solution for $\dot{\underline{x}}(t) = A(t)\underline{x}(t)$. Scalar time-varying equation $\dot{x} = a(t)x$ due to initial state $x(0)$ is

$$x(t) = e^{\int_0^t a(\tau)d\tau} x(0)$$

Here

$$\dot{x} = \frac{d}{dt} e^{\int_0^t a(\tau)d\tau} x(0) = a(t) e^{\int_0^t a(\tau)d\tau} x(0) = a(t)x$$

An extension of the scalar case

$$\underline{x} = e^{\int_0^t A(\tau)d\tau}\underline{x}(0) \tag{58}$$

In this case

$$\underline{\dot{x}} = \frac{d}{dt}e^{\int_0^t A(\tau)d\tau}\underline{x}(0) \neq A(t)\underline{x}$$

So (58) is not a solution of $\underline{\dot{x}}(t) = A(t)\underline{x}(t)$.
Let $A(t)$ is $n \times n$. For every initial state $\underline{x}_i(t)$, there exists an unique solution $\underline{x}_i(t)$.
We can arrange n of the solutions as $X = [\underline{x}_1 \;\; \underline{x}_2 \cdots \underline{x}_n]$. Since each $\underline{x}_i$ satisfies $\underline{\dot{x}}(t) = A(t)\underline{x}(t)$ we have

$$\dot{X}(t) = A(t)X(t)$$

If $X(t_0)$ is nonsingular, then $X(t)$ is called a fundamental matrix of $\underline{\dot{x}}(t) = A(t)\underline{x}(t)$. (not unique)

Example: Consider the equation (homogeneous)

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \underline{x}(t)$$

or $\dot{x}_1(t) = 0$ and $\dot{x}_2(t) = tx_1(t)$.
Solution of $\dot{x}_1(t)$ is $x_1(t) = x(0)$. The solution of $\dot{x}_2(t) = tx_1(t)$ is given by

$$x_2(t) = \int_0^t \tau x_1(0) d\tau + x_2(0) = 0.5t^2 x_1(0) + x_2(0)$$

Thus we have

$$\underline{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\Rightarrow\; \underline{x}(t) = \begin{bmatrix} 1 \\ 0.5t^2 \end{bmatrix}$$

and

$$\underline{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;\Rightarrow\; \underline{x}(t) = \begin{bmatrix} 1 \\ 0.5t^2 + 2 \end{bmatrix}$$

Two initial states are independent thus

$$X(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix} \tag{59}$$

is a fundamental matrix.

Claim: The fundamental matrix is nonzero for all t.

Proof: Let us consider that the fundamental matrix $X(t)$ singular at $t = t_1$.

Then we have a solution $\underline{x}(t_1) = X(t_1)\underline{v} = 0$.

This means that $\dot{\underline{x}} = 0$, or, $\underline{x}(t)$ is zero for all t, a contradiction...

State Transitional Matrix: Let $X(t)$ be any fundamental matrix of $\dot{\underline{x}}(t) = A(t)\underline{x}(t)$. Then

$$\Phi(t, t_0) \stackrel{\triangle}{=} X(t)X^{-1}(t_0) \tag{60}$$

is called the state transition matrix of $\dot{\underline{x}}(t) = A(t)\underline{x}(t)$.

It is also unique solution of

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0) \tag{61}$$

with the initial condition $\Phi(t_0, t_0) = I$.
Some properties of transition matrix

$$\Phi(t, t) = I \tag{62}$$

$$\Phi^{-1}(t, t_0) = \left[X(t)X^{-1}(t_0)\right]^{-1} = X(t_0)X^{-1}(t) = \Phi(t_0, t) \tag{63}$$

$$\Phi(t, t_0) = X(t)X^{-1}(t_0) = X(t)X^{-1}(t_1)X(t_1) = \Phi(t, t_1)\Phi(t_1, t_0) \tag{64}$$

Now consider

$$\begin{aligned} \dot{\underline{x}}(t) &= A(t)\underline{x}(t) + B(t)\underline{u}(t) \\ \underline{y}(t) &= C(t)\underline{x}(t) + D(t)\underline{u}(t) \end{aligned}$$

We claim that solution of time-varying state space equations excited by initial state $\underline{x}(t_0)$ and input $\underline{u}(t)$ is given by

$$\begin{aligned} \underline{x}(t) &= \Phi(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau \\ &= \Phi(t,t_0)\left[\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t_0,\tau)B(\tau)\underline{u}(\tau)d\tau\right] \end{aligned} \tag{65}$$

At $t = t_0$

$$\underline{x}(t_0) = \Phi(t_0,t_0)\underline{x}(t_0) + \int_{t_0}^{t_0} \Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau = I\underline{x}(t_0) + 0 = \underline{x}(t_0) \tag{66}$$

and

$$\begin{aligned} \frac{d}{dt}\underline{x}(t) &= \frac{\partial}{\partial t}\Phi(t,t_0)\underline{x}(t_0) + \frac{\partial}{\partial t}\int_{t_0}^{t} \Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau \\ &= A(t)\Phi(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t} \left(\frac{\partial}{\partial t}\Phi(t,\tau)\right) B(\tau)\underline{u}(\tau)d\tau + \Phi(t,t)B(t)\underline{u}(t) \\ &= A(t)\Phi(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t} A(t)\Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau + B(t)\underline{u}(t) \end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\underline{x}(t) &= A(t)\left(\Phi(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau\right) + B(t)\underline{u}(t) \\
&= A(t)\underline{x}(t) + B(t)\underline{u}(t)
\end{aligned}
$$

Now $\underline{y}(t)$ is given by

$$
\underline{y}(t) = C(t)\Phi(t,t_0)\underline{x}(t_0) + C(t)\int_{t_0}^{t} \Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau + D(t)\underline{u}(t) \tag{67}
$$

If the input is identically 0, then we have

$$
\begin{aligned}
\underline{x}(t) &= \Phi(t,t_0)\underline{x}(t_0) \\
\underline{y}(t) &= C(t)\Phi(t,t_0)\underline{x}(t_0)
\end{aligned} \tag{68}
$$

This is zero state response.

If the initial state is zero, then

$$
\begin{aligned}
\underline{y}(t) &= C(t)\int_{t_0}^{t}\Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau + D(t)\underline{u}(t) \\
&= \int_{t_0}^{t}\left[C(t)\Phi(t,\tau)B(\tau) + D\delta(t-\tau)\right]\underline{u}(\tau)d\tau \qquad (69)
\end{aligned}
$$

This is zero-input response. Here the impulse response matrix is given by

$$
G(t,\tau) = C(t)\Phi(t,\tau)B(\tau) + D\delta(t-\tau) \qquad (70)
$$

For the simple cases where $A(t)$ is diagonal or constant, i.e.,

$$
A(t)\left(\int_{t}^{t_0} A(\tau)d\tau\right) = \left(\int_{t}^{t_0} A(\tau)d\tau\right)A(t)
$$

We have

$$
\Phi(t,t_0) = e^{\int_{t_0}^{t} A(\tau)d\tau} \qquad (71)
$$

For constant A we have

$$\Phi(t, t_0) = e^{A(t-t_0)} = \Phi(t - t_0) \tag{72}$$

Equivalent Time-Varying Equations

We have

$$\begin{aligned} \dot{\underline{x}}(t) &= A(t)\underline{x}(t) + B(t)\underline{u}(t) \\ \underline{y}(t) &= C(t)\underline{x}(t) + D(t)\underline{u}(t) \end{aligned}$$

Let $P(t)$ be nonsingular and both $P(t)$ and $\dot{P}(t)$ are continuous for all t. Let $\bar{\underline{x}}(t) = P(t)\underline{x}(t)$. Then the state equation

$$\begin{aligned} \dot{\bar{\underline{x}}}(t) &= \bar{A}(t)\bar{\underline{x}}(t) + \bar{B}(t)\underline{u}(t) \\ \bar{\underline{y}}(t) &= \bar{C}(t)\bar{\underline{x}}(t) + \bar{D}(t)\underline{u}(t) \end{aligned} \tag{73}$$

where

$$\begin{aligned} \bar{A}(t) &= [P(t)A(t) + \dot{P}(t)]P^{-1}(t) \\ \bar{B}(t) &= P(t)B(t) \end{aligned}$$

$$
\begin{aligned}
\bar{C}(t) &= C(t)P^{-1}(t) \\
\bar{D}(t) &= D(t)
\end{aligned}
$$

We claim that $\bar{X}(t) = P(t)X(t)$ is a fundamental matrix for $\dot{\underline{\bar{x}}}(t) = \bar{A}(t)\underline{\bar{x}}(t)$.
Proof: We have

$$
\begin{aligned}
\frac{d}{dt}[P(t)X(t)] &= \dot{P}(t)X(t) + P(t)\dot{X}(t) = \dot{P}(t)X(t) + P(t)A(t)X(t) \\
&= [\dot{P}(t) + P(t)A(t)][P^{-1}(t)P(t)]X(t) = \bar{A}(t)[P(t)X(t)]
\end{aligned}
$$

Theorem: Let A_0 be an arbitrary constant matrix. Then there exists an equivalent transformation from (73) to state equation that has $\bar{A}(t) = A_0$.
Proof: Let $X(t)$ be a fundamental matrix of $\dot{\underline{x}}(t) = A(t)\underline{x}(t)$.
By differentiating $X^{-1}(t)X(t) = I$ gives

$$
\dot{X}^{-1}(t)X(t) + X^{-1}(t)\dot{X}(t) = 0 \tag{74}
$$

or

$$\dot{X}^{-1}(t) = -X^{-1}(t)A(t)X(t)X^{-1}(t) = -X^{-1}(t)A(t) \tag{75}$$

Since $\bar{A}(t) = A_0$, $\bar{X}(t) = e^{A_0 t}$ is a fundamental matrix of $\dot{\underline{\bar{x}}}(t) = \bar{A}(t)\underline{\bar{x}}(t)$.

Let $P(t) \triangleq \bar{X}(t)X^{-1}(t) = e^{A_0 t}X^{-1}(t)$ and have

$$\begin{aligned}
\bar{A}(t) &= [P(t)A(t) + \dot{P}(t)]P^{-1}(t) \\
&= [e^{A_0 t}X^{-1}(t)A(t) + A_0 e^{A_0 t}X^{-1}(t) + e^{A_0 t}\dot{X}^{-1}(t)]X(t)e^{-A_0 t} \\
&= [e^{A_0 t}X^{-1}(t)A(t) + A_0 e^{A_0 t}X^{-1}(t) + e^{A_0 t}(-X^{-1}(t)A(t))]X(t)e^{-A_0 t} \\
&= A_0 e^{A_0 t}X^{-1}(t)X(t)e^{-A_0 t} = A_0
\end{aligned} \tag{76}$$

$\underline{\bar{x}}(t) = P(t)\underline{x}(t)$ gives the corresponding transformation.

Lyapunov Transformation: A matrix $P(t)$ is called a Lyapunov transformation if $P(t)$ is nonsingular, $P(t)$ and $\dot{P}(t)$ are continuous, and $P(t)$ and $P^{-1}(t)$ are bounded for all t.

The state equations corresponding to $\underline{x}(t)$ and $\underline{\bar{x}}(t)$ are called Lyapunov equivalent if $\underline{\bar{x}}(t) = P(t)\underline{x}(t)$.

Periodic State Equations

Consider

$$\begin{aligned}\underline{\dot{x}}(t) &= A(t)\underline{x}(t) + B(t)\underline{u}(t) \\ \underline{y}(t) &= C(t)\underline{x}(t) + D(t)\underline{u}(t)\end{aligned}$$

It is assumed that $A(t+T) = A(t)$ for all t, or, $A(t)$ is periodic with period T.

Let $X(t)$ be a fundamental matrix of $\underline{\dot{x}} = A(t)\underline{x}$. Then we have

$$\dot{X}(t+T) = A(t+T)X(t+T) = A(t)X(t+T) \tag{77}$$

So $X(t+T)$ is also a fundamental matrix. Now we claim that $X(t+T)$ can be expressed as

$$X(t+T) = X(t)X^{-1}(0)X(T) \tag{78}$$

To verify this let us consider a matrix $\bar{A}$ such that $e^{\bar{A}T} = X^{-1}(0)X(T)$. Now $X(t+T) = X(t)e^{\bar{A}T}$.

The corresponding transform is given by $P(t) \overset{\triangle}{=} e^{-\bar{A}t}X^{-1}(t)$.

Here

$$P(t+T) = e^{\bar{A}(t+T)}X^{-1}(t+T) = e^{\bar{A}t}e^{\bar{A}T}\left[e^{-\bar{A}T}X^{-1}(t)\right] = e^{-\bar{A}t}X^{-1}(t) = P(t)$$

Thus $P(t)$ is periodic with period T.

Discrete Time Case

Consider discrete-time state equations

$$\begin{aligned} \underline{x}[k+1] &= A[k]\underline{x}[k] + B[k]\underline{u}[k] \quad (79) \\ \underline{y}[k] &= C[k]\underline{x}[k] + D[k]\underline{u}[k] \quad (80) \end{aligned}$$

The solution can be computed recursively once the initial state $x[k_0]$ and the input $\underline{u}[k]$ $k \geq k_0$ are known.
We can define discrete state transition matrix as the solution of $\Phi[K+1, k_0] = A[k]\Phi[k, k_0]$ with $\Phi[k_0, k_0] = I$.
Its solution can be obtained directly as

$$\Phi[k, k_0] = A[k-1]A[k-2]\cdots A[k_0], \quad k > k_0 \quad (81)$$

In continuous case the state transition matrix $\Phi(t, t_0)$ is defined for $t \geq t_0$ as well as $t < t_0$.
However, in discrete case since $A[k]$ may be singular, inverse of $\Phi[k, k_0]$ may not exist.
Thus $\Phi[k, k_0]$ is defined only for $k \geq k_0$. Here we have

$$\Phi[k, k_0] = \Phi[k, k_1]\Phi[k_1, k_0], \quad \text{for} \ \ k \geq k_1 \geq k_0 \tag{82}$$

Using the discrete state transition matrix we can express the solution of (79)–(80) as

$$\underline{x}[k] = \Phi[k, k_0]\underline{x}[k_0] + \sum_{m=k_0}^{k-1} \Phi[k, m+1]B[m]\underline{u}[m] \tag{83}$$

$$\underline{y}[k] = C[k]\Phi[k, k_0]\underline{x}[k_0] + C[k] \sum_{m=k_0}^{k-1} \Phi[k, m+1]B[m]\underline{u}[m] + D[k]\underline{u}[k] \tag{84}$$