ECE 707: Control Systems Design (10)

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These viewgraphs are based on the text
“Linear System: Theory and Design” by Chi-Tsong Chen
Oxford University Press, 1999.

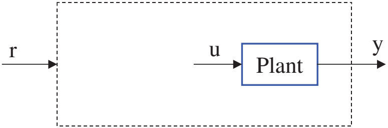

Fig. 12.1

In most of the control systems the plant (system) and the reference signal $r(t)$ are given. Our job is to find out input $u(t)$ such that $y(t)$ (the output) follows $r(t)$ as closely as possible.

If the input $u(t)$ depends only on the reference signal $r(t)$ and not on output $y(t)$ then the control is called open loop control.

Open loop control performance degrades if there are variations in the plant parameters (for example, properties of active components such as inductors and capacitors may vary with temperature) and noise disturbance (such as thermal noise).

In a control system, if input $u(t)$ depends on both $r(t)$ as well as output $y(t)$ (or state variables $\underline{x}(t)$), then the control is called closed loop control.

Closed loop control is more robust in noisy environment and under system parameter variations as it can keep track of these variations by observing the output $y(t)$ (or state variables $\underline{x}(t)$).

State Feedback

Consider the n-dimensional SISO state equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u && (1) \\ y &= \underline{c}^T\underline{x} && (2) \end{aligned}$$

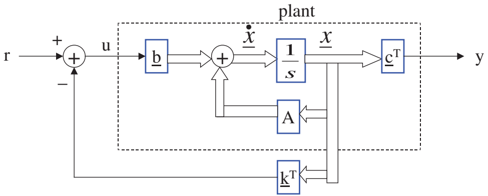

Fig. 12.2

From Fig. 12.2 it is clear that

$$u = r - \underline{k}^T \underline{x} = r - \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \underline{x} = r - \sum_{i=1}^{n} k_i x_i \tag{3}$$

Hence the state equation becomes

$$\dot{\underline{x}} = (A - \underline{b}\underline{k}^T)\underline{x} + \underline{b}r \tag{4}$$

$$y = \underline{c}^T \underline{x} \tag{5}$$

The next theorem shows that state feedback can not change the controllability of the system.

Theorem: The pair $(A - \underline{b}\underline{k}^T, \underline{b})$, for any constant vector $\underline{k}^T$, is controllable if and only if $(A, \underline{b})$ is controllable.

Proof: If $n = 4$ (number of the state variables). For the original system (plant) the controllability matrix is given by

$$X = \begin{bmatrix} \underline{b} & A\underline{b} & A^2\underline{b} & A^3\underline{b} \end{bmatrix} \tag{6}$$

For the feedback system the controllability matrix is given by

$$X_f = \left[\begin{array}{cccc} \underline{b} & (A - \underline{b}\underline{k}^T)\underline{b} & (A - \underline{b}\underline{k}^T)^2\underline{b} & (A - \underline{b}\underline{k}^T)^3\underline{b} \end{array} \right] \tag{7}$$

It can be shown that

$$X_f = X \left[\begin{array}{cccc} 1 & -\underline{k}^T\underline{b} & -\underline{k}^T(A - \underline{b}\underline{k}^T)\underline{b} & -\underline{k}^T(A - \underline{b}\underline{k}^T)^2\underline{b} \\ 0 & 1 & -\underline{k}^T\underline{b} & -\underline{k}^T(A - \underline{b}\underline{k}^T)\underline{b} \\ 0 & 0 & 1 & -\underline{k}^T\underline{b} \\ 0 & 0 & 0 & 1 \end{array} \right] \tag{8}$$

Since X_f is X multiplied by a nonsingular matrix, the rank of X_f is equal to rank of X.

Thus one of the systems is controllable if and only if the other system is controllable. (EOP)

The next example shows that the observability property can change as a result of state feedback.

Example: Consider the state equation

$$\underline{\dot{x}} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{9}$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \underline{x} \tag{10}$$

and state feedback

$$u = r - \begin{bmatrix} 3 & 1 \end{bmatrix} \underline{x} \tag{11}$$

Find out if the initial system and the feedback system are controllable and observable.
Solution: The controllability matrix and the observability matrix of the initial system

$$X = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad O = \begin{bmatrix} 1 & 2 \\ 7 & 4 \end{bmatrix} \tag{12}$$

Both of them are of full rank. So the system is controllable and observable. From the theorem we know that the feedback system is also controllable.

For the feedback system the state equation is given by

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \qquad (13) \\ y &= \begin{bmatrix} 1 & 2 \end{bmatrix} \underline{x} \qquad (14) \end{aligned}$$

The controllability matrix and the observability matrix of the feedback system

$$X = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad O = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \qquad (15)$$

The observability matrix does not have full row rank. So the system which was initially observable becomes unobservable when we use feedback.

Consider a SISO system with transfer function

$$\hat{g}(s) = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \qquad (16)$$

Consider a controllable realization

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u && (17)\\ y &= \underline{c}^T \underline{x} && (18) \end{aligned}$$

It has controllability matrix

$$X = \begin{bmatrix} \underline{b} & A\underline{b} & A^2\underline{b} & A^3\underline{b} \end{bmatrix} \quad (19)$$

also consider controllable canonical form

$$\begin{aligned} \dot{\bar{\underline{x}}} &= \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\underline{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u && (20)\\ y &= \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \bar{\underline{x}} && (21) \end{aligned}$$

Inverse of the controllability matrix for this form

$$\bar{X}^{-1} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

Thus the equivalence transform $\underline{\bar{x}} = P\underline{x}$ in this case is given by

$$P^{-1} = X\bar{X}^{-1} = \begin{bmatrix} \underline{b} & A\underline{b} & A^2\underline{b} & A^3\underline{b} \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

This transform is important as we will see in the next theorem.
Theorem: If a n-dimensional state equation is controllable, then by state feedback $u = r - \underline{k}^T\underline{x}$, where $\underline{k}$ is real vector, the eigenvalues of $A - \underline{bk}^T$ can arbitrarily assigned provided that complex conjugate eigenvalues are assigned in pairs.

Proof: Again we prove the theorem for $n = 4$.
Let us consider a controllable system equation

$$\begin{aligned} \underline{\dot{x}} &= A\underline{x} + \underline{b}u && (24) \\ y &= \underline{c}^T \underline{x} && (25) \end{aligned}$$

Let us assume that the transfer function is given by

$$\hat{g}(s) = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \qquad (26)$$

Then as we have already shown, the equivalence transform $\underline{\bar{x}} = P\underline{x}$ will form controllable canonical form of the state equation

$$\begin{aligned} \underline{\dot{\bar{x}}} &= \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{\bar{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u && (27) \\ y &= \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \underline{\bar{x}} && (28) \end{aligned}$$

where the transform matrix P is given by

$$P^{-1} = \left[\begin{array}{cccc} \underline{b} & A\underline{b} & A^2\underline{b} & A^3\underline{b} \end{array} \right] \left[\begin{array}{cccc} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{array} \right] \tag{29}$$

Now consider that a state feedback $u = r - \underline{k}^T \underline{x}$ is used in the original system. The feedback for the transformed system (that gives the same effect) is given by

$$u = r - \underline{k}^T \underline{x} = r - \underline{k}^T P^{-1} \underline{\bar{x}} \stackrel{\triangle}{=} r - \underline{\bar{k}}^T \underline{\bar{x}} \tag{30}$$

where $\underline{\bar{k}}^T = \underline{k}^T P^{-1}$. Since

$$\bar{A} - \underline{\bar{b}}\underline{\bar{k}}^T = P(A - \underline{b}\underline{k}^T)P^{-1} \tag{31}$$

$A - \underline{b}\underline{k}^T$ and $\bar{A} - \underline{\bar{b}}\underline{\bar{k}}^T$ have the same eigenvalues.
If $\underline{\bar{k}}^T$ is chosen as

$$\underline{\bar{k}}^T = \left[\begin{array}{cccc} \bar{\alpha}_1 - \alpha_1 & \bar{\alpha}_2 - \alpha_2 & \bar{\alpha}_3 - \alpha_3 & \bar{\alpha}_4 - \alpha_4 \end{array} \right] \tag{32}$$

then the feedback equation becomes

$$\dot{\underline{\bar{x}}} = \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{\bar{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r \tag{33}$$

$$y = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \underline{\bar{x}} \tag{34}$$

because of the companion form the characteristic equation of $\bar{A} - \underline{\bar{b}\bar{k}}^T$ and, consequently, of $A - \underline{bk}^T$ becomes equal to

$$\Delta_f(s) = s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4 \tag{35}$$

Thus, we can use state feedback to achieve any characteristic equation of $A - \underline{bk}^T$ or, we can place eigenvalues (zeros of (35)) place in the complex plane we want.
If we have complex eigenvalues they will appear in pairs as the coefficients of $\Delta_f(s)$ are real. (EOP)

The Procedure of Placing Eigenvalues

Let us consider a n-dimensional system equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u && (36) \\ y &= \underline{c}^T \underline{x} && (37) \end{aligned}$$

Let us consider that we want to place the eigenvalues (at $\bar{\lambda}_i$ with multiplicity m_i) such that the characteristic equation of $A - \underline{bk}^T$ becomes

$$\Delta_f(s) = s^n + \bar{\alpha}_1 s^{n-1} + \cdots + \bar{\alpha}_{n-1} s + \bar{\alpha}_n = \prod_{i=1}^{p} (s - \bar{\lambda}_i)^{m_i} \quad (38)$$

To do this we have to find out characteristic equation of A

$$\Delta(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n \quad (39)$$

The feedback gain vector is given by

$$\underline{k}^T = [\bar{\alpha}_1 - \alpha_1 \;\; \bar{\alpha}_2 - \alpha_2 \;\; \cdots \;\; \bar{\alpha}_n - \alpha_n] P \quad (40)$$

where P can be obtain from

$$P^{-1} = \left[\begin{array}{cccc} \underline{b} & A\underline{b} & \cdots & A^{n-1}\underline{b} \end{array} \right] \begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_{n-1} \\ 0 & 1 & \cdots & \alpha_{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{41}$$

Notice that if the initial system has a transfer function

$$\hat{g}(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_n}{s^4 + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n} \tag{42}$$

Then after state feedback the transfer function (from r to s) becomes

$$\hat{g}_f(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_n}{s^4 + \bar{\alpha}_1 s^{n-1} + \cdots + \bar{\alpha}_{n-1} s + \alpha_n} \tag{43}$$

So the poles of the transfer function changes however there is no change in the zeros.
Thus if any of the replaced poles coincide with zeros, then numerator and denominator of $\hat{g}_f(s)$ are not coprime. Hence the feedback system becomes unobservable.

Example: Consider the state equation

$$\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u \qquad (44) \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \underline{x} \qquad (45) \end{aligned}$$

Place the eigenvalues at $-1.5 \pm 0.5j$ and $-1 \pm j$ by using state feedback.

Solution: We can show that the characteristic equation of A-matrix (How?) is given by

$$\Delta(s) = s^2(s^2 - 5) = s^4 + 0 \cdot s^3 - 5 \cdot s^2 + 0 \cdot s + 0 \qquad (46)$$

The desired characteristic function of the feedback system

$$\begin{aligned} \Delta_f(s) &= (s + 1.5 - 0.5j)(s + 1.5 + 0.5j)(s + 1 - j)(s + 1 + j) \\ &= s^4 + 5s^3 + 10.5s^2 + 11s + 5 \qquad (47) \end{aligned}$$

Thus we have

$$\underline{k}^T = [5 - 0 \;\; 10.5 + 5 \;\; 11 - 0 \;\; 5 - 0] P \tag{48}$$

where P is given by

$$\begin{aligned} P^{-1} &= X\bar{X}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} && (49) \\ &= \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 0 & -3 & 0 \\ 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} && (50) \end{aligned}$$

Hence

$$P = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{6} \\ -\frac{1}{3} & 0 & -\frac{1}{6} & 0 \end{bmatrix} \tag{51}$$

So

$$\underline{k}^T = [-1.6667 \quad -3.6667 \quad -8.5833 \quad -4.3333] \tag{52}$$

Regulation and Tracking

The problem to find a state feedback gain so that the response due to nonzero initial conditions die out at a desired rate is called a regulation problem.

For example, bringing an aircraft to zero deviation from its desired altitude is a regulation problem.

In this case the reference signal r is zero.

Consider another problem of designing an overall system so that $y(t)$ approaches $r(t) = a$ at $t \to \infty$. This is called asymptotic tracking of a step reference input.

Consider a state equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \quad (53) \\ y &= \underline{c}^T\underline{x} \quad (54) \end{aligned}$$

The regulation problem is has a simple solution, use state feedback to shift the eigenvalues of $A - \underline{bk}^T$ so that the system becomes stable. Then the state feedback equation becomes

$$\begin{aligned} \dot{\underline{x}} &= (A - \underline{bk}^T)\underline{x} + \underline{b}r && (55) \\ y &= \underline{c}^T \underline{x} && (56) \end{aligned}$$

Since $r(t)$ is zero the response caused by $\underline{x}_0$ is

$$y(t) = \underline{c}^T e^{(A - \underline{bk}^T)t} \underline{x}_0 \tag{57}$$

If all eigenvalues of $(A - \underline{bk}^T)$ are in the left s plane then the output will decay rapidly to zero.
For the tracking problem, in addition to the state feedback we need a feedforword gain p

$$u(t) = pr(t) - \underline{k}^T \underline{x}(t) \tag{58}$$

Then we have (for $n = 4$)

$$\hat{g}(s) = p\frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4} \tag{59}$$

As $r(t) = a$ then the output approaches $\hat{g}_f(0) \cdot a$ as $t \to \infty$.
Thus $y(t)$ will track the input asymptotically if

$$1 = \hat{g}_f(0) = p\frac{\beta_4}{\bar{\alpha}_4} \quad \text{or} \quad p = \frac{\bar{\alpha}_4}{\beta_4} \tag{60}$$

We can see that tracking is not possible if $\beta_4 = 0$, i.e., if the transfer function has a zero at $s = 0$.

Stabilization

Again, consider a system equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u && (61) \\ y &= \underline{c}^T \underline{x} && (62) \end{aligned}$$

If $(A, \underline{b})$ is controllable, all eigenvalues can be assigned arbitrarily.

The system can be stabilized by shifting all eigenvalues of A to the left s-plane.
However this is not true for a uncontrollable system as we see in the following.
Any uncontrollable state equation can be transformed into

$$\begin{bmatrix} \dot{\underline{x}}_c \\ \dot{\underline{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} \underline{x}_c \\ \underline{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \underline{b}_c \\ 0 \end{bmatrix} u \tag{63}$$

If we introduce the state feedback

$$u = r - \underline{k}^T \underline{x} = r - [\underline{k}_1^T \;\; \underline{k}_2^T] \begin{bmatrix} \underline{x}_c \\ \underline{x}_{\bar{c}} \end{bmatrix} \tag{64}$$

Then the state update equation becomes

$$\begin{bmatrix} \dot{\underline{x}}_c \\ \dot{\underline{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} A_c - \underline{b}_c \underline{k}_1^T & A_{12} - \underline{b}_c \underline{k}_2^T \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} \underline{x}_c \\ \underline{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \underline{b}_c \\ 0 \end{bmatrix} r \tag{65}$$

We can see that $A_{\bar{c}}$ and consequently its eigenvalues are not effected by the state feedback. Thus the state equation is still uncontrollable.

This also shows that the system can be stabilized by state feedback only if $A_{\bar{c}}$ is stable.

State Estimator

In order to apply state feedback, we need the state variables of a system.

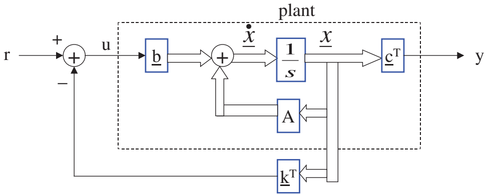

Fig. 12.2

The state variables may not be accessible for direct connection as sensing devices may be either not available or very expansive.

We need to design a state estimator for our feedback system to work.

Consider the state equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u && (66) \\ y &= \underline{c}^T \underline{x} && (67) \end{aligned}$$

where A,$\underline{b}$ and $\underline{c}^T$ are known and the input and output are also available.

We can duplicate the original system as

$$\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + \underline{b}u \qquad (68)$$

where $\hat{\underline{x}}$ be an estimate of $\underline{x}$.

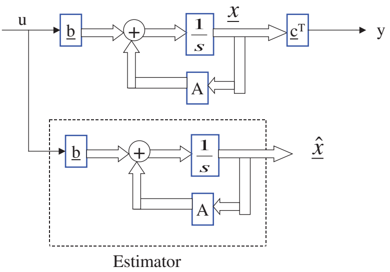

Fig. 12.3

Note that the original system and the duplicated system can be made of different types of components.

For example, original system can be electromechanical system and the duplicate system can be an op-amp circuit that copies the behavior of the original system.

Fig. 12.3 shows such an estimator. This kind of duplication is called open-loop estimator.

Here if the initial state same in the original and the duplicate circuit, then estimation is perfect.

The problem here is that we need perfect knowledge of the initial state of the original system for the estimator to work.

Next we will discuss an estimator which uses input as well as the output of the original system.

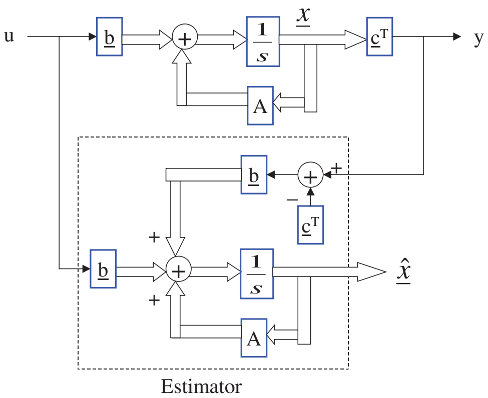

Fig. 12.4

Fig. 12.4 shows a closed-loop estimator. Here we have gain vector $\underline{l}$ that amplifies the difference of the output of the original system and the estimated output.

The vector obtained by this operation is added to the contribution of input to get state update.

Duplicate system equation of the closed-loop estimator

$$\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + \underline{b}u + \underline{l}(y - \underline{c}^T\hat{\underline{x}}) \tag{69}$$

which can be written as

$$\dot{\hat{\underline{x}}} = (A - \underline{l}\underline{c}^T)\hat{\underline{x}} + \underline{b}u + \underline{l}y \tag{70}$$

Let us define

$$\underline{e}(t) \stackrel{\triangle}{=} \underline{x}(t) - \hat{\underline{x}}(t) \tag{71}$$

Hence

$$\begin{aligned}
\dot{\underline{e}}(t) &= \dot{\underline{x}}(t) - \dot{\hat{\underline{x}}}(t) = A\underline{x} + \underline{b}u - (A - \underline{l}\underline{c}^T)\hat{\underline{x}} - \underline{b}u - \underline{l}y \\
&= (A - \underline{l}\underline{c}^T)\underline{x} - (A - \underline{l}\underline{c}^T)\hat{\underline{x}} \\
&= (A - \underline{l}\underline{c}^T)\underline{e}(t)
\end{aligned} \tag{72}$$

If all eigenvalues of $(A - \underline{l}\underline{c}^T)$ can be placed arbitrarily then the rate at which $\underline{e}(t)$ approaches zero can be controlled.
The following theorem tells the condition under which this can be done.

Theorem: All eigenvalues of $(A - \underline{l}\underline{c}^T)$ can be assigned arbitrarily by selecting real constant vector $\underline{l}$ if and only if $(A, \underline{c}^T)$ is observable.
Proof: We know the pair $(A, \underline{c}^T)$ is observable if and only if the pair $(A^T, \underline{c})$ is controllable.
Now if $(A^T, \underline{c})$ is controllable, the eigenvalues of $A^T - \underline{c}\underline{l}^T$ can be placed arbitrarily by the choice of $\underline{l}$.
Which means if $(A, \underline{c}^T)$ is observable, the eigenvalues of $A - \underline{l}\underline{c}^T$ can be placed arbitrarily by the choice of $\underline{l}$. (EOP)
The procedure for computing state feedback gain can be used to compute the gain $\underline{l}$.

Feedback from Estimated States

Consider a system equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \quad (73) \\ y &= \underline{c}^T \underline{x} \quad (74) \end{aligned}$$

Consider that $\underline{x}$ is not available. We construct a close-loop state estimator as

$$\dot{\hat{\underline{x}}} = (A - \underline{l}\underline{c}^T)\hat{\underline{x}} + \underline{b}u + \underline{l}y \tag{75}$$

we apply the feedback gain to the estimated state to get

$$u = r - \underline{k}^T\hat{\underline{x}} \tag{76}$$

Substituting this into (73) and (75) we get

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} - \underline{b}\underline{k}^T\hat{\underline{x}} + \underline{b}r & (77) \\ \dot{\hat{\underline{x}}} &= (A - \underline{l}\underline{c}^T)\hat{\underline{x}} + \underline{b}(r - \underline{k}^T\hat{\underline{x}}) + \underline{l}y & (78) \end{aligned}$$

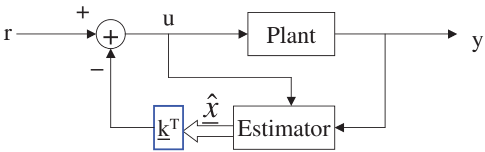

Fig. 12.5

Fig. 12.5 shows the corresponding system.

The combined state equation of the controller-estimator is given by

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\hat{\underline{x}}} \end{bmatrix} = \begin{bmatrix} A & -\underline{b}\underline{k}^T \\ \underline{l}\underline{c}^T & A - \underline{l}\underline{c}^T - \underline{b}\underline{k}^T \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} + \begin{bmatrix} \underline{b} \\ \underline{b} \end{bmatrix} r \tag{79}$$

$$= y = [\underline{c}^T \quad \underline{0}^T] \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} \tag{80}$$

Let us consider the following equivalence transform

$$\begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} = \begin{bmatrix} \underline{x} \\ \underline{x} - \hat{\underline{x}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} \triangleq P \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} \tag{81}$$

Computing P^{-1}, same as P, we can get the corresponding $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ matrices. This gives

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{e}} \end{bmatrix} = \begin{bmatrix} A - \underline{b}\underline{k}^T & \underline{b}\underline{k}^T \\ 0 & A - \underline{l}\underline{c}^T \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} + \begin{bmatrix} \underline{b} \\ \underline{0} \end{bmatrix} r \tag{82}$$

and

$$y = [\underline{c}^T \quad \underline{0}^T] \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} \tag{83}$$

The A-matrix is block triangular. Thus the eigenvalues of the main system (corresponding to states $\underline{x}$) remains unchanged.
So inserting the state estimator does not effect the eigenvalues of main system and vice versa.
Thus the design of state feedback and state estimator can be done independently (separation property).

State Estimator–Multivariate Case

Consider the n-dimensional p-input and q-output system equation

$$\begin{aligned} \underline{\dot{x}} &= A\underline{x} + B\underline{u} && (84) \\ \underline{y} &= C\underline{x} && (85) \end{aligned}$$

In state feedback, the input $\underline{u}$ is given by

$$\underline{u} = \underline{r} - K\underline{x} \tag{86}$$

where K is $p \times n$ constant matrix and $\underline{r}$ is a reference signal. The state equation of the feedback system

$$\begin{aligned} \dot{\underline{x}} &= (A - BK)\underline{x} + B\underline{r} && (87) \\ \underline{y} &= C\underline{x} && (88) \end{aligned}$$

Theorem: The pair $(A - BK, B)$ is controllable if and only if (A, B) is controllable.
Theorem: All eigenvalues of $(A - BK)$ can be assigned arbitrarily (complex conjugate eigenvalues in pairs) if and only if (A, B) is controllable.

Cyclic Design

A square matrix A is called cyclic if its characteristic polynomial and minimal polynomial are the same i.e., if only one Jordan block is associated with each distinct eigenvalue.

Theorem: If the pair (A, B) is controllable then for almost any vector $\underline{v}$ the pair $(A, B\underline{v})$ is controllable.

We argue the validity of this theorem by an example.

Controllability is invariant under equivalence transform, so without loss of generality we can assume A to be in Jordan form. Consider the example

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 2 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} \tag{89}$$

A is cyclic and (A, B) is controllable.

Let us consider an arbitrary $\underline{v}$

$$B\underline{v} = B \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} x \\ x \\ \alpha \\ x \\ \beta \end{bmatrix} \tag{90}$$

It is clear that if α and β are nonzero then $(A, B\underline{v})$ is controllable. We have

$$\alpha = v_1 + 2v_2 \text{ and } \beta = v_1 \tag{91}$$

Thus v_1 and v_2 can't have the following values

$$v_1 = 0 \quad \text{or} \quad v_1 = -2v_2 \tag{92}$$

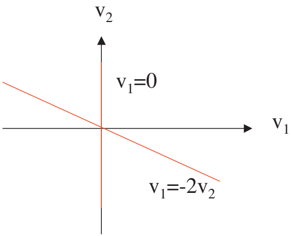

Fig. 12.6

Fig. 12.6 shows that the probability of an arbitrary $\underline{v}$ to give a controllable $(A, B\underline{v})$ is 1.

Theorem: If the pair (A, B) is controllable, then for almost any real constant matrix K, the matrix $(A - BK)$ has only distinct eigenvalues (cyclic).

Again the theorem is shown intuitively for $n = 4$. Let the characteristic polynomial of $(A - BK)$ be

$$\Delta_f(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 \tag{93}$$

where a_i are functions of entries of K. Differentiation of $\Delta_f(s)$ w.r.t. s gives

$$\Delta'_f(s) = 4s^3 + 3a_1 s^2 + 2a_2 s + a_3 \tag{94}$$

If $\Delta_f(s)$ has repeated roots, then $\Delta_f(s)$ and $\Delta'_f(s)$ are not coprime, in that case

$$\begin{vmatrix} a_4 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 2a_2 & a_4 & a_3 & 0 & 0 & 0 & 0 \\ a_2 & 3a_1 & a_3 & 2a_2 & a_4 & a_3 & 0 & 0 \\ a_1 & 4 & a_2 & 3a_1 & a_3 & 2a_2 & a_4 & a_3 \\ 1 & 0 & a_1 & 4 & a_2 & 3a_1 & a_3 & 2a_2 \\ 0 & 0 & 1 & 0 & a_1 & 4 & a_2 & 3a_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & a_1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} = 0 \tag{95}$$

It is obvious that the above equation is satisfied by a small subset of all possible K. Hence arbitrary K has probability 1 of not satisfying this or eigenvalues of $(A - BK)$ are distinct.
Next we discuss the cyclic design.

Let us consider that A is not cyclic, but (A, B) is controllable. We introduce $\underline{u} = \underline{w} - K_1\underline{x}$ as shown in Fig. 12.7.

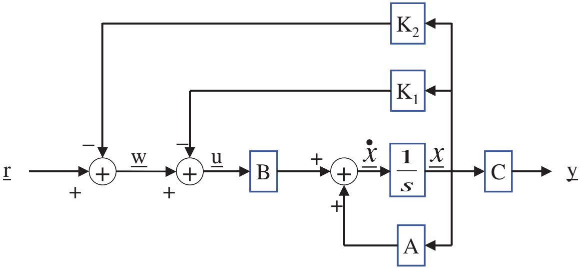

Fig. 12.7

Then the update equation becomes

$$\dot{\underline{x}} = (A - BK_1)\underline{x} + B\underline{w} = \bar{A}\underline{x} + B\underline{w} \tag{96}$$

For arbitrary choice of K_1 $\bar{A}$ is cyclic. Now because (A, B) is controllable so is $(\bar{A}, B)$.

Hence there exists vector $\underline{v}$ such that $(\bar{A}, B\underline{v})$ is controllable.

Next we introduce another feedback $\underline{w} = \underline{r} - K_2\underline{x}$, where $K_2 = \underline{vk}^T$. Then the update equation becomes

$$\underline{\dot{x}} = (\bar{A} - B\underline{vk}^T)\underline{x} + B\underline{w} \tag{97}$$

Now the eigenvalues can be assigned arbitrarily by selecting $\underline{k}^T$ (SISO case).