ECE 707: Linear Systems (7)

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These viewgraphs are based on the text
"Linear System: Theory and Design" by Chi-Tsong Chen
Oxford University Press, 1999.

Unstable system tends to burn out , disintegrate, or saturate when a signal is applied.
Stability is a basic requirement for all system.

Input-Output Stability of LTI system

Consider the SISO LTI system described by

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau \tag{1}$$

Here $g(t)$ is the impulse response of the LTI system. $y(t)$ and $u(t)$ are the input and output at time t.
An input $u(t)$ is said to be bounded if

$$|u(t)| \leq u_m < \infty \quad \text{for all } t \geq 0 \tag{2}$$

Definition: A system is said to be bounded-input bounded-output stable (BIBO) if every bounded input excites a bounded output.

This stability is defined for zero state response, i.e., the initial state is zero (relaxed system).

Theorem: A SISO system described by

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau$$

is BIBO stable if and only if

$$\int_0^\infty |g(t)|dt \leq M < \infty \qquad \text{(absolutely integrable in } [0,\infty)\text{)} \qquad (3)$$

where M is a constant.

Proof: Part 1: Absolutely integrable $g(t)$ implies BIBO stability (sufficient).

Let

$$|u(\tau)| \leq u_m < \infty \quad \text{for all } \tau \geq 0$$

Then

$$
\begin{aligned}
|y(t)| &= \left| \int_0^t g(\tau)u(t-\tau)d\tau \right| \\
&\leq \int_0^t |g(\tau)u(t-\tau)|d\tau \qquad (\text{since } \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx) \\
&\leq \int_0^t |g(\tau)||u(t-\tau)|d\tau \qquad (\text{since } |ab| \leq |a||b|) \\
&\leq u_m \int_0^t |g(\tau)|d\tau \qquad (\text{since } |u(t-\tau)| \leq u_m) \\
&\leq u_m M
\end{aligned} \tag{4}
$$

Thus the output is bounded.

Part 2: Not absolute integrable $g(t)$ implies unstable system (necessary).

If $g(t)$ is not absolutely integrable then there exists a t_1 such that

$$
\int_0^{t_1} |g(\tau)|d\tau = \infty \quad \text{and} \quad |g(\tau)| < \infty \tag{5}
$$

Let us choose

$$u(t_1 - \tau) = \begin{cases} 1 & \text{if } g(\tau) \geq 0 \\ -1 & \text{if } g(\tau) < 0 \end{cases} \tag{6}$$

Clearly input $u(t)$ is bounded as $g(t)$ is bounded. However the output at t_1

$$y(t_1) = \int_0^{t_1} g(\tau)u(t_1 - \tau)d\tau = \int_0^{t_1} |g(\tau)|d\tau = \infty \tag{7}$$

which is not bounded, so the system is not BIBO stable. (EOP)
Example: Is a system with impulse response $g(t) = 1/(t+1)$ BIBO stable? (Problem 5.3 text)
Solution: Here

$$\int_0^{\infty} |g(t)|dt = \int_0^{\infty} \frac{1}{1+t}dt = \ln(1+t)\Big|_0^{\infty} = \infty \tag{8}$$

So $g(t)$ is not absolutely integrable. System not BIBO stable.

Example: Consider the system in Fig. 9.1. Find its impulse response. Find out if the system is BIBO stable.

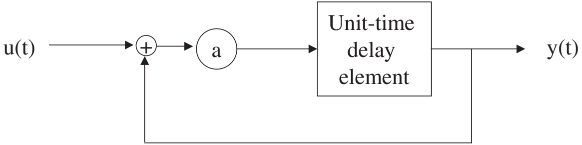

Fig. 9.1

Solution: Let $u(t) = 0$ for $t < 0$, then the output is given by

$$y(t) = \sum_{i=1}^{\infty} a^i u(t-i) = \int_0^t \left(\sum_{i=1}^{\infty} a^i \delta(t-\tau-i) \right) u(\tau) d\tau \tag{9}$$

Hence

$$g(t) = \sum_{i=1}^{\infty} a^i \delta(t-i) \tag{10}$$

So we have

$$|g(t)| = \sum_{i=1}^{\infty} |a|^i \delta(t-i) \tag{11}$$

and

$$\begin{aligned} \int_0^{\infty} |g(t)|dt &= \int_0^{\infty} \sum_{i=1}^{\infty} |a|^i \delta(t-i)dt = \sum_{i=1}^{\infty} |a|^i \\ &= \begin{cases} \infty & \text{if } |a| \geq 1 \\ |a|/(1-|a|) < \infty & \text{if } |a| < 1 \end{cases} \end{aligned} \tag{12}$$

Hence the feedback system is stable iff $|a| < 1$.
Absolute integrable function may not be bounded. Example
Dirac-delta function $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \quad \text{but} \quad \delta(0) = \infty \tag{13}$$

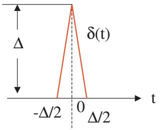

Fig. 9.2

Fig. 9.2 shows a function which becomes Dirac-delta function as $\Delta \to 0$.

If a system is BIBO stable then the output excited by $u(t) = a$, for $t \geq 0$, approaches $a\hat{g}(0)$ as $t \to \infty$

Here $\hat{g}(0)$ is the Laplace transformed $g(t)$ for $s = 0$.

Proof: Here

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau = a\int_0^t g(\tau)d\tau \tag{14}$$

which implies

$$\lim_{t\to\infty} y(t) = a\int_0^\infty g(\tau)e^{-0\tau}d\tau = a\hat{g}(0) \tag{15}$$

Theorem: A SISO system with proper rational transfer function $\hat{g}(s)$ is BIBO stable if and only if every pole of $\hat{g}(s)$ has negative real part. "Poles of $\hat{g}(s)$ has negative real part" is equivalent to "poles lie inside the left-half s-plane".

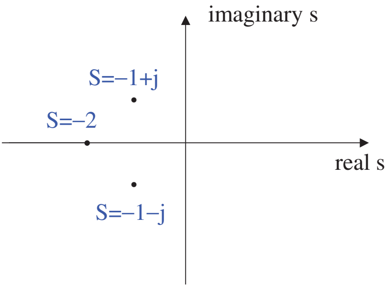

Fig. 9.3

Fig. 9.3 shows the poles of the following transfer function

$$\hat{g}(s) = \frac{1}{s+2} + \frac{0.5}{s+1+j} + \frac{0.5}{s+1-j} \tag{16}$$

Proof: If $\hat{g}(s)$ has a pole p_i with multiplicity m_i, then its partial fraction expansion contains the factors

$$\frac{1}{s-p_i},\ \frac{1}{(s-p_i)^2},\ \cdots,\ \frac{1}{(s-p_i)^{m_i}} \tag{17}$$

Thus the inverse Laplace transform of $\hat{g}(s)$ or the impulse response $g(t)$ contains the factors

$$e^{p_i t},\ te^{p_i t},\ \cdots,\ t^{m_i-1}e^{p_i t} \tag{18}$$

We have

$$\int_0^\infty t^n e^{p_i t} dt < \infty \quad \text{iff} \quad \text{real}(p_i) < 0 \tag{19}$$

Example: Is a system with impulse response $g(t) = te^{-t}$ BIBO stable? (Problem 5.3 text)

Solution: Here we have

$$\hat{g}(s) = L[te^{-t}] = \frac{1}{(s+1)^2} \tag{20}$$

So all the poles of $\hat{g}(s)$ is negative. So the system is BIBO stable.

Till now we have discussed SISO systems. Now we discuss the results for multivariate systems.

Theorem: A multivariate system with impulse response matrix $G(t) = [g_{ij}(t)]$ is BIBO stable if and only if every $g_{ij}(t)$ is absolutely integrable in $[0, \infty)$.

Theorem: A multivariate system with transfer matrix $\hat{G}(s) = [g_{ij}(s)]$ is BIBO stable if and only if every pole of $\hat{g}_{ij}(s)$ has a negative real part.

Next we discuss BIBO stability of the state equations. Consider

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \qquad (21) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \qquad (22) \end{aligned}$$

Its transfer matrix (zero state response) is given by

$$\hat{G}(s) = C(sI - A)^{-1}B + D \qquad (23)$$

The zero state response in (23) is BIBO stable if and only if every pole of $\hat{G}(s)$ has a negative real part.

Now

$$\hat{G}(s) = C(sI - A)^{-1}B + D = \frac{1}{\det(sI - A)} C[\operatorname{Adj}(sI - A)]B + D \quad (24)$$

Every pole of A is an eigenvalue of A. Thus the system is BIBO stable if every eigenvalue of A has negative real part.
This may result from possible zero-pole cancelation.
Example: Consider the network shown in Fig. 9.4. Find if the system is BIBO stable.

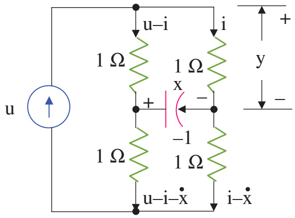

Fig. 9.4

Solution: State equation of the system is given by

$$\dot{x}(t) = x(t) + 0 \cdot u(t) \qquad y(t) = 0.5x(t) + 0.5u(t) \tag{25}$$

The A matrix is 1 and its eigenvalue is 1. It has real positive part. Now, the transfer function of the equation is given by

$$\hat{g}(s) = 0.5(s-1)^{-1} \cdot 0 + 0.5 = 0.5 \tag{26}$$

The transfer function does not have a pole.
Hence the system is BIBO stable even though A has an eigenvalue with positive real part.

Discrete-Time Case

Consider a SISO LTI system described by

$$y[k] = \sum_{m=0}^{k} g[k-m]u[m] = \sum_{m=0}^{k} g[m]u[k-m] \tag{27}$$

where $g[k]$ is the impulse response of the system. $u[k]$ and $y[k]$ are the input and output respectively, at time k.

An input sequence $u[k]$ is said to be bounded if

$$|u[k]| \leq u_m < \infty \quad \text{for} \;\; k = 0, 1, 2, \cdots \tag{28}$$

Definition: A system is said to be bounded-input bounded-output stable (BIBO) if every bounded input excites a bounded output. Same definition as the continuous case.

This stability is defined for the zero-state response and is applicable only if the system is initially relaxed.

Theorem: A discrete time SISO LTI system in (27) is BIBO stable if and only if

$$\sum_{k=0}^{\infty} |g[k]| \leq M < \infty \qquad (\text{absolutely integrable in } [0, \infty)) \tag{29}$$

for some constant M.

Example: Consider a discrete-time LTI system with impulse response sequence $g[k] = 1/k$, for $k = 1, 2, \ldots$, and $g[0] = 0$. Find if the system is BIBO stable.

Solution: We have

$$\begin{aligned}\sum_{k=0}^{\infty} |g[k]| &= \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \end{aligned} \tag{30}$$

In the first pair of parentheses, there are two terms each $\geq 1/4$; hence their sum is $> 1/2$.

In the second pair of parentheses, there are four terms each $\geq 1/8$; hence their sum is $> 1/2$, so on...

Hence

$$\sum_{k=0}^{\infty} |g[k]| > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty \tag{31}$$

The impulse response is not absolutely summable. Hence system not BIBO stable.

If a discrete-time system is BIBO stable then the output excited by $u[k] = a$, for $t \geq 0$, approaches $a\hat{g}(1)$ as $k \to \infty$.
Here $\hat{g}(z)$ is the z-transform of $g[k]$ or

$$\hat{g}(z) = \sum_{k=0}^{\infty} g[k] z^{-k} \tag{32}$$

Proof: If $u[k] = a$ for all $k \geq 0$, then

$$y[k] = \sum_{m=0}^{k} g[m] u[k-m] = a \sum_{k=0}^{k} g[m] \tag{33}$$

which implies

$$\lim_{k \to \infty} y[k] = a \sum_{m=0}^{\infty} g[m] 1^{-m} = a\hat{g}(1) \tag{34}$$

Theorem: A discrete-time SISO system with proper rational transfer function $\hat{g}(z)$ is BIBO stable if and only if every pole of $\hat{g}(z)$ has magnitude less than 1.

"Poles of $\hat{g}(z)$ has magnitude less than 1" is equivalent to "poles lie inside the unit circle".

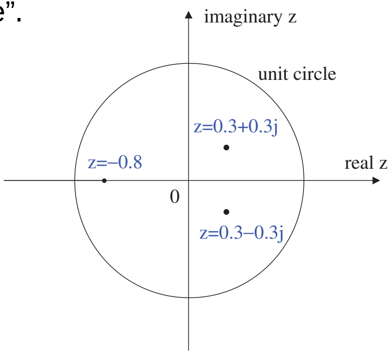

Fig. 9.5

Fig. 9.3 shows the poles of the following transfer function

$$\hat{g}(z) = \frac{1}{1 + 0.8z^{-1}} + \frac{0.5}{1 - (0.3 + 0.3j)z^{-1}} + \frac{0.5}{1 - (0.3 - 0.3j)z^{-1}} \quad (35)$$

Proof: If $\hat{g}(z)$ has a pole p_i with multiplicity m_i, then its partial fraction expansion contains the factors

$$\frac{1}{1-p_i z^{-1}},\ \frac{1}{(1-p_i z^{-1})^2},\ \cdots,\ \frac{1}{(1-p_i z^{-1})^{m_i}} \tag{36}$$

Thus the inverse Laplace transform of $\hat{g}(z)$ or the impulse response $g[k]$ contains the factors

$$p_i^k,\ kp_i^k,\ \cdots,\ k^{m_i-1}p_i^k \tag{37}$$

We have

$$\int_0^\infty k^n p_i^k dt < \infty \quad \text{iff} \quad |p_i| < 1 \tag{38}$$

In discrete-time case, if $g[k]$ is absolutely summable, then it must be bounded and approach zero as $k \to \infty$.
Next we consider multiple input multiple output discrete-time systems.
Theorem: A MIMO discrete-time system with impulse response sequence matrix $G[k] = [g_{ij}[k]]$ is BIBO stable if and only if every $g_{ij}[k]$ is absolutely summable.

Theorem: A MIMO discrete-time system with transfer matrix $\hat{G}(z) = [\hat{g}_{ij}(z)]$ is BIBO stable if and only if every pole of every $\hat{g}_{ij}(z)$ has a magnitude less than 1.

Let us consider the discrete-time state equation

$$\begin{aligned} \underline{x}[k+1] &= A\underline{x}[k] + B\underline{u}[k] && (39) \\ \underline{y}[k] &= C\underline{x}[k] + D\underline{u}[k] && (40) \end{aligned}$$

Its discrete transfer matrix (or, zero state response) is

$$\hat{G}(z) = C(zI - A)^{-1}B + D \tag{41}$$

The zero state response in (41) is BIBO stable if and only if every pole of $\hat{G}(s)$ has a negative real part.

Now

$$\hat{G}(z) = C(sI - A)^{-1}B + D = \frac{1}{\det(sI - A)} C[\text{Adj}(sI - A)]B + D \tag{42}$$

So every pole of $\hat{G}(z)$ is an eigenvalue of A.

Thus if every eigenvalue of A has a magnitude less than 1, then same can be said for the poles and the system is BIBO stable.
On the other hand, an eigenvalue of A that has magnitude ≥ 1 does not necessarily mean that the system is unstable.

Internal Stability

Now we study stability of zero input response. In continuous-time LTI zero input system is given by

$$\dot{\underline{x}} = A\underline{x}(t) \tag{43}$$

Solution of this equation for initial state $\underline{x}(0)$ is given by

$$\underline{x}(t) = e^{At}\underline{x}(0) \tag{44}$$

Definition: The zero input response or the equation $\dot{\underline{x}} = A\underline{x}(t)$ is marginally stable if every finite initial state $\underline{x}(0)$ excites a bounded response.
It is asymptotically stable if every finite initial state excites a bounded response which, in addition, approaches 0 as $t \to \infty$.

Theorem: The equation $\dot{\underline{x}} = A\underline{x}(t)$ is marginally stable if and only if all eigenvalues of A have zero or negative real parts and those with zero real parts have Jordan block of order 1 associated with it.
The equation $\dot{\underline{x}} = A\underline{x}(t)$ is asymptotically stable if and only if all eigenvalues of A have negative real parts.
Equivalence transform does not alter stability of a state equation.
Consider the equivalence transform $\underline{z} = P\underline{x}$.
Studying stability of

$$\dot{\underline{x}} = A\underline{x}(t)$$

is equivalent to studding stability of

$$\dot{\underline{z}} = \bar{A}\underline{z}(t) \tag{45}$$

where $\bar{A} = PAP^{-1}$.
If we choose columns of P^{-1} to be the eigenvectors of A, then $\bar{A}$ is in Jordan form.

Solution of this state equation is given by

$$\underline{z}(t) = e^{\bar{A}t}\underline{z}(0) \tag{46}$$

Now consider the following $\bar{A}$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{47}$$

Hence $e^{\bar{A}t}$ is given by

$$e^{\bar{A}t} = \begin{bmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & e^{2t} \end{bmatrix} \tag{48}$$

Solution of the state equation is given by

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \\ z_5(t) \end{bmatrix} = \begin{bmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \\ z_5(0) \end{bmatrix} \tag{49}$$

We get the following equations

$$z_1(t) = z_1(0) + tz_2(0) \tag{50}$$

$$z_2(t) = z_2(0) \tag{51}$$

$$z_3(t) = z_3(0) \tag{52}$$

$$z_4(t) = e^{-t} z_4(0) \tag{53}$$

$$z_5(t) = e^{2t} z_5(0) \tag{54}$$

If we need finite solution for all t we can't have (50) and (54).
So no eigenvalue with positive (real part) for marginal stability. Also we can't have a Jordan block of order ≥ 2 associated with zero eigenvalue (real part).
If we further need the solution to be zero at $t \to \infty$ we can't have (51) or (52).

So no nonnegative (real part) eigenvalue of $\bar{A}$ or A for asymptotic stability.
As we know that $e^{jx} = \cos(x) + j\sin(x)$. The imaginary part of eigenvalue of $\bar{A}$ or A can't change the stability of $\underline{z}(t) = e^{\bar{A}t}\underline{z}(0)$.
As discussed earlier, every pole of the transfer function

$$\hat{G}(s) = C(sI - A)^{-1}B + D \tag{55}$$

is an eigenvalue of A. A sufficient condition for BIBO stability is that these poles have negative real part.
Hence Asymptotic stability $\Rightarrow$ BIBO stability.
The converse is not true. Consider the example

$$\dot{x}(t) = x(t) + 0.u(t) \quad y(t) = 0.5x(t) + 0.5u(t) \tag{56}$$

Here the A matrix has eigenvalue 1, hence not asymptotically stable.
However we have already seen that this system is BIBO stable.

Discrete-Time Case

Consider the zero input state equation

$$\underline{x}[k+1] = A\underline{x}[k] \tag{57}$$

Again we have same definitions of marginal stability and asymptotic stability.

Theorem: The system is marginally stable iff all eigenvalue of A has magnitude ≤ 1 and those equal to 1 have Jordan block of order 1 associated with them.

Theorem: The system is asymptotically stable iff all eigenvalue of A has magnitude less than 1.

Stability of Linear Time Variant System

A SISO linear time-varying system is described by

$$y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau \tag{58}$$

The condition for BIBO stability is

$$\int_{t_0}^{t} |g(t,\tau)| d\tau < \infty \tag{59}$$

for all t_0 and $t \geq t_0$.
Similarly for a multivariate case the condition becomes

$$\int_{t_0}^{t} ||G(t,\tau)|| d\tau < \infty \tag{60}$$

Next we study the stability of zero input response. Consider the equation

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) \tag{61}$$

The solution of this equation is given by

$$\underline{x}(t) = \Phi(t,t_0)\underline{x}(0) \tag{62}$$

So the response is marginally stable if and only if

$$||\Phi(t,t_0)|| < \infty \tag{63}$$

for all t_0 and $t \geq t_0$.
The equation $\dot{\underline{x}}(t) = A(t)\underline{x}(t)$ is asymptotically stable if and only if

$$||\Phi(t,t_0)|| < \infty \quad \text{and} \quad \lim_{t \to \infty} ||\Phi(t,t_0)|| = 0 \tag{64}$$