ECE 707: Control Systems Design (5)

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These viewgraphs are based on the text
"Linear System: Theory and Design" by Chi-Tsong Chen
Oxford University Press, 1999.

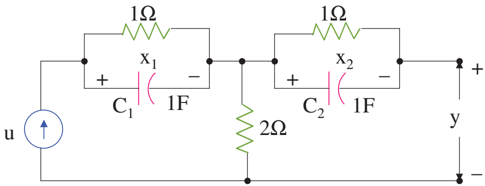


Fig. 7.1

Here we choose the state variables $x_1,\ x_2$ to be voltages across the capacitors C_1 and C_2, respectively.

The current passing through the $2-\Omega$ resistor always equal the current source u; So the state of x_1 will not appear in y. (x_1 not observable)

The open circuit across y means the input has no effect on x_2. (x_2 not controllable)

Controllability

Consider the state equation

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{1}$$

We don't need the output equation as output does not play any role in controllability.

Definition: The state equation (1) or the pair (A, B) is said to be controllable if for any initial state $\underline{x}(0)$ and any final state $\underline{x}_1$, there exists an input that can perform the transform in finite time.

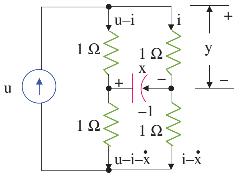


Fig. 7.2

If $x(0) = 0$, then $x(t) = 0$ for all $t \geq 0$ no matter what the input is.

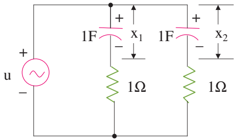


Fig. 7.3

Here due to the symmetry, if $x_1(0) = x_2(0) = 0$, then no matter what input is applied $x_1(t) = x_2(t)$ for all $t \geq 0$.

Theorem

The following statements are equivalent:

1. The pair (A, B) is controllable.
2. The matrix

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau \qquad (2)$$

is nonsingular for any $t > 0$.

Proof: Simple change of variable $\bar{\tau} = t - \tau$ can give the second form in (2) from the first form.

Proof of $W_c(t)$ is nonsingular $\Rightarrow \dot{\underline{x}} = A\underline{x} + B\underline{u}$ is controllable:
We have

$$\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau \qquad (3)$$

Now claim that for any $\underline{x}(0) = \underline{x}_0$ and $\underline{x}(t) = \underline{x}_1$, the input

$$\underline{u}(\tau) = -B'e^{A'(t-\tau)}W_c^{-1}(t)\left[e^{At}\underline{x}_0 - \underline{x}_1\right] \qquad (4)$$

will transfer $\underline{x}_0$ to $\underline{x}_1$ at time t. Substituting (4) in (3) yields

$$\begin{aligned}\underline{x}(t) &= e^{At}\underline{x}_0 - \left(\int_0^t e^{A(t-\tau)}BB'e^{A'(t-\tau)}d\tau\right)W_c^{-1}(t)[e^{At}\underline{x}_0 - \underline{x}_1] \\ &= e^{At}\underline{x}_0 - W_c(t)W_c^{-1}(t)[e^{At}\underline{x}_0 - \underline{x}_1] = \underline{x}_1\end{aligned}$$

(EOP)

Proof of $\underline{\dot{x}} = A\underline{x} + B\underline{u}$ is controllable $\Rightarrow W_c(t)$ is nonsingular:
Let (A, B) is controllable but $W_c(t)$ is singular at some t_1. Then we must have some nonzero $\underline{v}$ s.t.

$$\begin{aligned} \underline{v}'W_c(t_1)\underline{v} &= \int_0^{t_1} \underline{v}'e^{A(t_1-\tau)}BB'e^{A'(t_1-\tau)}\underline{v}d\tau \\ &= \int_0^{t_1} ||B'e^{A'(t_1-\tau)}\underline{v}||^2 d\tau = 0 \end{aligned}$$

which implies

$$B'e^{A'(t_1-\tau)}\underline{v} = \underline{0} \quad \text{or} \quad \underline{v}'e^{A(t_1-\tau)}B = \underline{0}'$$

for $0 \leq \tau \leq t_1$. Now if $\underline{\dot{x}} = A\underline{x} + B\underline{u}$ is controllable, there exists an input that can transfer the initial state $\underline{x}(0) = e^{-At_1}\underline{v}$ to $\underline{x}(t_1) = \underline{0}$ and from

$$\underline{x}(t_1) = e^{At_1}\underline{x}(0) + \int_0^{t_1} e^{A(t_1-\tau)}B\underline{u}(\tau)d\tau$$

we get

$$\underline{0} = \underline{v} + \int_0^{t_1} e^{A(t_1-\tau)} B\underline{u}(\tau)d\tau$$

It's premultiplication with $\underline{v}$ gives

$$0 = \underline{v}'\underline{v} + \int_0^{t_1} \underline{v}' e^{A(t_1-\tau)} B\underline{u}(\tau)d\tau = ||\underline{v}||^2 + 0 \neq 0$$

a contradiction. (EOP)

Theorem

The following statements are equivalent:

1. The matrix

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$$

is nonsingular for any $t > 0$.

2. The controllability matrix

$$X = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B] \tag{5}$$

has full row rank. Here A is $n \times n$.

Proof: To show 2 $\Rightarrow$ 1
Assume X does not have full row rank (2 not true) but $W_c(t)$ is nonsingular (1 is true).
We have some nonzero $\underline{v}$ s.t.

$$\underline{v}'X = [\underline{v}'B \;\; \underline{v}'AB \;\; \underline{v}'A^2B \;\; \cdots \;\; \underline{v}'A^{n-1}B] = \underline{0}'$$

i.e., $\underline{v}'A^kB = \underline{0}'$ for $k = 0, 1, \cdots, n-1$.
The matrix $e^{At}B$ can be written as linear combination of $\{B, AB, A^2B, \cdots, A^{n-1}B\}$ (since any function of $A_{n\times n}$ can be written as linear combination of $\{I, A, A^2, \cdots, A^{n-1}\}$).
So for every t we have $\underline{v}'e^{At}B = \underline{0}'$.
Now we have

$$W_c(t)\underline{v} = \int_0^t e^{A\tau}BB'e^{A'\tau}\underline{v}d\tau = \underline{0}$$

which means $W_c(t)$ is singular. Contradiction (EOP)

Proof of 1 $\Rightarrow$ 2: Assume X has full row rank (2 is true) but $W_c(t)$ is singular
(1 not true).
From $W_c(t)$ singular we have $\underline{v}'e^{At}B = \underline{0}'$ or

$$\underline{v}'B + t\underline{v}'AB + \frac{t^2}{2!}\underline{v}'A^2B + \cdots = \underline{0}' \tag{6}$$

If we evaluate (6) at $t = 0$ we get $\underline{v}'B = \underline{0}'$, next differentiate (6) w.r.t. t and evaluate at $t = 0$ we get $\underline{v}'AB = \underline{0}'$, so on
Thus we can get $\underline{v}'A^kB = \underline{0}'$ for $k = 0, 1, 2, \cdots, n-1$.
This means $\underline{v}'X = [\underline{v}'B \;\; \underline{v}'AB \;\; \underline{v}'A^2B \;\; \cdots \;\; \underline{v}'A^{n-1}B] = \underline{0}'$, or, X does not have full row rank. Contradiction (EOP).

Theorem

The following statements are equivalent:

1. The controllability matrix

$$X = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B]$$

has full row rank. Here A is $n \times n$.

2. The matrix $[A - \lambda I \quad B]$ has full row rank at every eigenvalue λ of A.

Proof: Here we will proof 1 $\rightarrow$ 2.

Let 1 $\nRightarrow$ 2 i.e., X has full row rank but there exists an eigenvalue λ_1 and left eigenvector $\underline{q}$ s.t.

$$\underline{q}'[A - \lambda_1 I \quad B] = \underline{0}'$$

this gives $\underline{q}'A = \lambda_1 \underline{q}'$ and $\underline{q}'B = \underline{0}'$. Now

$$\underline{q}'X = [\underline{q}'B \;\; \underline{q}'AB \;\; \underline{q}'A^2B \;\; \cdots \;\; \underline{q}'A^{n-1}B] = [\underline{q}'B \;\; \lambda_1\underline{q}'B \;\; \lambda_1^2\underline{q}'B \;\; \cdots \;\; \lambda_1^{n-1}\underline{q}'B] = \underline{0}'$$

Contradiction. (EOP)

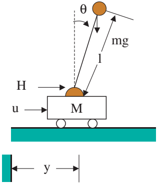


Fig. 7.4

Suppose for a given pendulum the equation becomes

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u \\ y &= [1 \;\; 0 \;\; 0 \;\; 0]\underline{x} \end{aligned}$$

we have

$$X = [B \;\; AB \;\; A^2B \;\; A^3B] = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix}$$

Example: The platform in Fig. 7.5 can be used to study suspension system of automobiles.

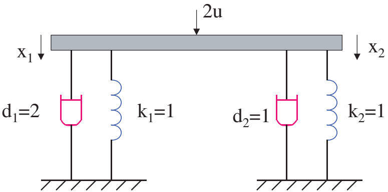


Fig. 7.5

The spring constants are 1 and the friction coefficients are 2 and 1 respectively.

Since the input is equally divided into two sides, we have $x_1 + 2\dot{x}_1 = u$ and $x_2 + \dot{x}_2 = u$, or

$$\dot{\underline{x}} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

Now we have

$$\rho([B \;\; AB]) = \rho\left(\begin{bmatrix} 0.5 & -0.25 \\ 1 & -1 \end{bmatrix}\right) = 2$$

Thus the equation is controllable.
Let us try to find the input that transfers $\underline{x}(0) = [10 \quad -1]'$ to $\underline{0}$ in 2 seconds.
For this system $W_c(t_1)$ for $t_1 = 2$ is given by

$$\begin{aligned} W_c(2) &= \int_0^2 \underbrace{\begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix}}_{e^{A\tau}} \underbrace{\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 0.5 & 1 \end{bmatrix}}_{B'} \underbrace{\begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix}}_{e^{A'\tau}} d\tau \\ &= \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix} \end{aligned} \tag{7}$$

and

$$\begin{aligned} u(t) &= -\underbrace{\begin{bmatrix} 0.5 & 1 \end{bmatrix}}_{B'} \underbrace{\begin{bmatrix} e^{-0.5(2-t)} & 0 \\ 0 & e^{-(2-t)} \end{bmatrix}}_{e^{A'(t_1-t)}} W_c^{-1}(2) \underbrace{\begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix}}_{e^{At_1}} \underbrace{\begin{bmatrix} 10 \\ -1 \end{bmatrix}}_{\underline{x}(0)} \\ &= -58.82e^{0.5t} + 27.96e^{t} \end{aligned}$$

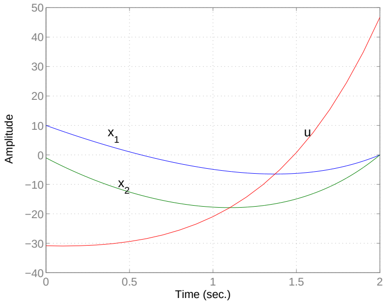


Fig. 7.6

Figure shows the states x_1, x_2 and input u as a function of time. The largest value of input is 46.7. If we had $|u(t)| < 40$, then no input exists that can transfer $[10 \quad -1]'$ to $[0 \quad 0]'$ in two seconds.

Another important property is that the input obtained the above way has the minimal energy. This means for any other input $u_1(t)$, we have

$$\int_0^{t_1} u_1(t)' u_1(t) dt \geq \int_0^{t_1} u(t)' u(t) dt \tag{8}$$

Controllability Indices

We have

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

In our study of systems we assume that $B_{n\times p}$ has rank p, i.e., B has full column rank.
If it does not then the redundant columns and the corresponding inputs are deleted.
If (A, B) is controllable, its controllability matrix
$X = [B \;\; AB \;\; A^2B \cdots A^{n-1}B]$ has n independent columns.
For example let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$X = [B \;\; AB \;\; A^2B \;\; A^3B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 & -1 & 0 \\ 1 & 0 & 0 & 2 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & -4 \\ 0 & 1 & -2 & 0 & 0 & -4 & 2 & 0 \end{bmatrix}$$

A set of independent columns can be obtained in different ways.We discuss most natural way of search.
Let $\underline{b}_i$ be the i th column of B. Then columns of X can be written explicitly as

$$X = [\underline{b}_1 \;\cdots\; \underline{b}_p \vdots A\underline{b}_1 \;\cdots\; A\underline{b}_p \vdots \cdots \vdots A^{n-1}\underline{b}_1 \;\cdots\; A^{n-1}\underline{b}_p]$$

We perform our search from left to right. Consider the m th column $\underline{b}_m$ and its products with A

$$\underline{b}_m,\; A\underline{b}_m,\; A^2\underline{b}_m,\; \cdots,\; A^{\mu_m-1}\underline{b}_m,\; A^{\mu_m}\underline{b}_m,\; \cdots,\; A^{n-1}\underline{b}_m$$

If $A^{\mu_m}\underline{b}_m$ is linearly dependent on $\{\underline{b}_1, \cdots, \underline{b}_p, A\underline{b}_1, \cdots, A\underline{b}_p, \cdots, A^{\mu_m-1}\underline{b}_1, \cdots, A^{\mu_m-1}\underline{b}_p\}$ then

$$\begin{aligned} A^{\mu_m}\underline{b}_m &= c_{1,0}\underline{b}_1 + \cdots + c_{p,0}\underline{b}_p + c_{1,1}A\underline{b}_1 + \cdots + c_{p,1}A\underline{b}_p + \cdots \\ &\quad + c_{1,\mu_m-1}A^{\mu_m-1}\underline{b}_1 + \cdots + c_{p,\mu_m-1}A^{\mu_m-1}\underline{b}_p \end{aligned}$$

for some $\underline{c} = [c_{0,1} \ \cdots \ c_{p,\mu_m-1}]$. Hence

$$A^{\mu_m+1}\underline{b}_m = A\left(A^{\mu_m}\underline{b}_m\right)$$

So $A^{\mu_m+1}\underline{b}_m$ is also linearly dependent on $\{\underline{b}_1, \cdots, \underline{b}_p, A\underline{b}_1, \cdots, A\underline{b}_p, \cdots, A^{\mu_m-1}\underline{b}_1, \cdots, A^{\mu_m-1}\underline{b}_p\}$.
So for each column of $\underline{b}_i$ of B we find μ_i and arrange the columns to get the independent set.
Since rank of $X = [B \ \ AB \ \cdots \ A^{n-1}B]$ is n this means $\mu_1 + \mu_2 + \cdots + \mu_p = n$.
The set $\{\mu_1, \mu_2, \cdots, \mu_p\}$ is called controllability indices and $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_p\}$ is known as controllability index of (A, B).

If μ is the controllability index of (A, B) then it is the least integer for which

$$\rho(X_\mu) = \rho([B \;\; AB \;\; \cdots \;\; A^{\mu-1}B]) = n \tag{9}$$

Range of μ is given by

$$n/p \leq \mu \leq n - p + 1 \tag{10}$$

where the lower limit corresponds to the case when $\mu_1 = \mu_2 = \cdots = \mu_p$ and the upper limit corresponds to the case when all μ_is except one are equal to 1.

Corollary: The pair ($A_{n\times n}, B_{n\times p}$) is controllable if and only if the matrix

$$X_{n-p+1} = [B \;\; AB \;\; \cdots \;\; A^{n-p}B] \tag{11}$$

has rank n (i.e., $X_{n-p+1}X'_{n-p+1}$ is nonsingular), where $\rho(B) = p$.

Example: let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Find controllability indices.

Solution: Here $n = 4$ and $p = 2$. Hence

$$X_{n-p+1} = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 & -4 \end{bmatrix}$$

We can show that $\mu = \mu_1 = \mu_2 = 2$.

Theorem: The controllability property is invariant under any equivalence transformation.

Proof: Consider the pair (A, B) with controllability matrix

$$X = [B \;\; AB \;\; \cdots \;\; A^{n-1}B]$$

Consider its equivalent pair $(\bar{A}, \bar{B})$. By the definition of equivalence transformation we have

$$\bar{A} = PAP^{-1} \quad \text{and} \quad \bar{B} = PB$$

where P is a nonsingular matrix. The controllability matrix of $(\bar{A}, \bar{B})$ is

$$\begin{aligned} \bar{X} &= [\bar{B} \;\; \bar{A}\bar{B} \;\; \cdots \;\; \bar{A}^{n-1}\bar{B}] \\ &= [PB \;\; PAP^{-1}PB \;\; \cdots \;\; PA^{n-1}P^{-1}PB] \qquad (\text{since } \bar{A}^k = PA^kP^{-1}) \\ &= [PB \;\; PAB \;\; \cdots \;\; PA^{n-1}B] \\ &= P[B \;\; AB \;\; \cdots \;\; A^{n-1}B] = PX \end{aligned} \tag{12}$$

Now since P is nonsingular we have $\rho(X) = \rho(\bar{X})$. (EOP)

Theorem: The set of controllability indices is invariant under any equivalent transformation and any reordering of columns of B.
Proof: Let us define

$$X_k = [B \;\; AB \;\; \cdots \;\; A^{k-1}B]$$

From the proof of the previous theorem we have

$$\rho(X_k) = \rho(\bar{X}_k) \quad \text{for } k = 0, 1, 2, \cdots$$

Thus the set of controllability indices are invariant under any equivalent transform.
The rearrangement of the columns of B can be achieved by $\hat{B} = BM$, where M is nonsingular matrix. Then we have

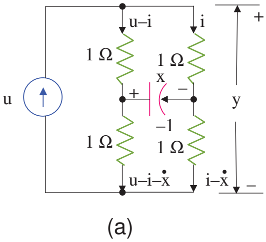
$$\hat{X}_k = [\hat{B} \;\; A\hat{B} \;\; \cdots \;\; A^{k-1}\hat{B}] = X_k \text{diag}(M, M, \cdots, M)$$

Because diag($M, M, \cdots, M$) is nonsingular, we have $\rho(X_k) = \rho(\bar{X}_k)$ for $k = 0, 1, 2, \cdots$. (EOP)

Observability Consider the state equation

$$\begin{aligned} \underline{\dot{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned} \tag{13}$$

Definition: The state equation (13) is said to be observable if there exists a finite $t_1 > 0$ s.t. the knowledge of input $\underline{u}(t)$ and output $\underline{y}(t)$ over $0 \leq t \leq t_1$ is sufficient to uniquely determine the initial state $\underline{x}(0)$.



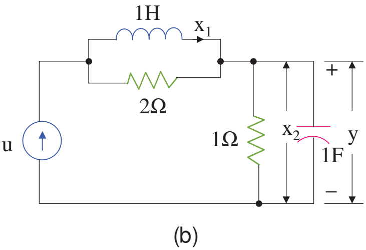


Fig. 7.7

Consider the network in Fig. 7.7 (a). If the input is zero, due to symmetry of the four resistors, the output is always zero no matter what the initial state of $x(0)$. So the network is not observable. In Fig. 7.7 (b) if the input u is zero, due to open circuit for any initial state $[a \;\; 0]'$ the output is equal to zero. Again the network is not observable.

The response generated by the initial state $\underline{x}(0)$ and the input $\underline{u}(t)$ is given by

$$\underline{y}(t) = Ce^{At}\underline{x}(0) + C\int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau + D\underline{u}(t) \tag{14}$$

When we study observability the input $\underline{u}(t)$ and the output $\underline{y}(t)$ are known; the initial state $\underline{x}(0)$ is unknown. Thus we can write (14) as

$$\underline{\bar{y}}(t) = Ce^{At}\underline{x}(0) \tag{15}$$

where

$$\underline{\bar{y}}(t) = \underline{y}(t) - C\int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau + D\underline{u}(t) \tag{16}$$

So $\underline{\bar{y}}(t)$ is a fixed vector. We need to find a solution of (15).
Note that a solution always exists here, we need to find whether the solution is unique.

Theorem: The state equation

$$\begin{aligned}\dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u}\end{aligned}$$

is observable if and only if the matrix

$$W_o(t) = \int_0^t e^{A'\tau} C' C e^{A\tau} d\tau \tag{17}$$

is nonsingular for any $t > 0$.

Proof: We have

$$\bar{\underline{y}}(t) = Ce^{At}\underline{x}(0)$$

Premultiplying by $e^{A't}C'$ and then integrating over 0 to t_1 gives.

$$\left(\int_0^{t_1} e^{A't} C' C e^{At} dt\right) \underline{x}(0) = \int_0^{t_1} e^{A't} C' \bar{\underline{y}}(t) dt \tag{18}$$

If $W_0(t)$ is nonsingular, then

$$\underline{x}(0) = W_0^{-1}(t_1) \int_0^{t_1} e^{A't} C' \bar{\underline{y}}(t) dt \tag{19}$$

This $\underline{x}(0)$ is unique. So if $W_0(t)$ is nonsingular for any $t > 0$ (here t_1) then the state equation is observable.
Now let $W_0(t_1)$ is singular, then there exists nonzero $\underline{v}$ s.t.

$$\begin{aligned} \underline{v}' W_o(t_1) \underline{v} &= \int_0^{t_1} \underline{v}' e^{A'\tau} C' C e^{A\tau} \underline{v} d\tau \\ &= \int_0^{t_1} ||C e^{A\tau} \underline{v}||^2 d\tau = 0 \end{aligned}$$

which implies $C e^{A\tau} \underline{v} = \underline{0}$ for all $0 \leq \tau \leq t_1$.
If $\underline{u} = \underline{0}$, then $\underline{x}(0) = \underline{v} \neq \underline{0}$ and $\underline{x}(0) = \underline{0}$ both yield the same

$$\underline{y}(t) = C e^{At} \underline{x}(0) = \underline{0}$$

This means we can't uniquely determine $\underline{x}(0)$. Thus the state equation is not observable.

Note that if the state equation is

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned}$$

Then controllability depends only on (A, B) and observability depends only on (A, C).

Theorem: The pair (A, B) is controllable if and only if the pair (A', B') is observable.

Proof: The pair (A, B) is controllable if and only if

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau$$

is nonsingular at any $t > 0$. The pair (A', B') is observable if and only if

$$W_o(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau \quad (\text{by replacing } A \text{ by } A' \text{ and } C \text{ by } B') \tag{20}$$

is nonsingular for any $t > 0$.

The two conditions are identical.

Theorem: The following statements are equivalent

1. The pair (A, C) is observable.
2. The matrix

$$W_o(t) = \int_0^t e^{A'\tau} C' C e^{A\tau} d\tau$$

is nonsingular for any $t > 0$.

3. The observability matrix

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{21}$$

has rank n (full column rank).

4. The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank at every eigenvalue, λ, of A.

Observability Indices

Consider the observability pair $(A_{n\times n}, C_{q\times n})$. We assume that C has full row rank (i.e., $\rho(C) = q$).

If (A, C) is observable then rank of the observability matrix

$$\rho(O) = \rho\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$$

In this case we search for independent rows from top to bottom.

If C has rows c_m^T for $m = 1, 2, \cdots, q$, then similar to the controllability matrix case, we can define observability indices ν_m.

ν_m is the number of independent rows associated with c_m^T.

If rank of O is n then

$$\nu_1 + \nu_2 + \cdots + \nu_q = n \tag{22}$$

The observability index is given by $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_q\}$.

Observability index have the following range

$$n/q \leq \nu \leq n - q + 1 \tag{23}$$

Corollary: The pair (A, C) is observable if and only if the matrix

$$O_{n-q+1} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-q} \end{bmatrix} \tag{24}$$

has rank n or the matrix $O'_{n-q+1}O_{n-q+1}$ is nonsingular.
Theorem: The observability property is invariant under any equivalence transformation.
Theorem: The set of observability indices of (A, C) is invariant under any equivalence transformation and any reordering of the rows of C.

We have

$$\underline{\bar{y}}(t) = Ce^{At}\underline{x}(0)$$

By repeatedly differentiating and evaluating at $t = 0$ we get

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\nu-1} \end{bmatrix} \underline{x}(0) = \begin{bmatrix} \underline{\bar{y}}(0) \\ \underline{\bar{y}}^{1}(0) \\ \vdots \\ \underline{\bar{y}}^{\nu-1}(0) \end{bmatrix} \tag{25}$$

or

$$O_\nu \underline{x}(0) = \tilde{Y}(0) \tag{26}$$

If (A, C) is observable then $O_\nu' O_\nu$ is nonsingular. Hence

$$\underline{x}(0) = (O_\nu' O_\nu)^{-1} O_\nu' \tilde{Y}(0) \tag{27}$$

This gives another way to solve determine $\underline{x}(0)$.