ECE 707: Control Systems Design (8)

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These viewgraphs are based on the text
"Linear System: Theory and Design" by Chi-Tsong Chen
Oxford University Press, 1999.

We have already discussed the realization problem.
A transfer matrix $\hat{G}(s)$ is called realizable if there exists a state-space equation

$$\begin{aligned} \underline{\dot{x}} &= A\underline{x} + B\underline{u} \quad (1) \\ \underline{y} &= C\underline{x} + D\underline{u} \quad (2) \end{aligned}$$

that has $\hat{G}(s)$ as its transfer matrix.
Important question is how to get a realization having smallest possible dimension.
We define that a realization of $\hat{g}(s) = N(s)/D(s)$ to be minimal if and only if it is controllable and observable.
This can be extended to show that a realization is minimal if and only if its dimension is equal to the degree of $\hat{g}(s)$.
The degree of $\hat{g}(s)$ is defined as the degree of $D(s)$ if the two polynomials $D(s)$ and $N(s)$ are coprime or have no common factors.

Consider a SISO system with proper transfer function $\hat{g}(s)$. We decompose it as

$$\hat{g}(s) = \hat{g}(\infty) + \hat{g}_{\text{sp}}(s) \tag{3}$$

where $\hat{g}_{\text{sp}}(s)$ is strictly proper and $\hat{g}(\infty)$ yields the D-matrix in every realization. So for our convenience we will only consider the strictly proper $\hat{g}(s)$. That is

$$\hat{g}(s) = \frac{N(s)}{D(s)} \quad \text{then degree of } N(s) < \text{ degree of } D(s) \tag{4}$$

Let

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \tag{5}$$

We have

$$\hat{y}(s) = \hat{g}(s)\hat{u}(s) = N(s)D(s)^{-1}\hat{u}(s) \tag{6}$$

To obtain a realization we introduce a new variable $v(t)$defined by $\hat{v}(s) = D(s)^{-1}\hat{u}(s)$. Then we have

$$D(s)\hat{v}(s) \quad = \quad \hat{u}(s) \tag{7}$$

So the first equation can give us a state update equation and the second one an output equation.
Define state variables as

$$\underline{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} v^{(3)}(t) \\ \ddot{v}(t) \\ \dot{v}(t) \\ v(t) \end{bmatrix} \tag{9}$$

or

$$\hat{\underline{x}}(s) \triangleq \begin{bmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \\ x_4(s) \end{bmatrix} = \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} \hat{v}(s) \tag{10}$$

Then we have

$$s\hat{x}_2(s) = x_1(s) \quad \Rightarrow \quad \dot{x}_2 = x_1 \tag{11}$$

$$s\hat{x}_3(s) = x_2(s) \quad \Rightarrow \quad \dot{x}_3 = x_2 \tag{12}$$

$$s\hat{x}_4(s) = x_3(s) \quad \Rightarrow \quad \dot{x}_4 = x_3 \tag{13}$$

when we consider $\underline{x}(0) = 0$.

To get the fourth state update equation we use (7)

$$
\begin{aligned}
& D(s)\hat{v}(s) = \hat{u}(s) \\
\Rightarrow \quad & (s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4)\hat{x}_4(s) = \hat{u}(s) \\
\Rightarrow \quad & s\hat{x}_1(s) = -\alpha_1\hat{x}_1(s) - \alpha_2\hat{x}_2(s) - \alpha_3\hat{x}_3(s) - \alpha_4\hat{x}_4(s) + \hat{u}(s) \\
\Rightarrow \quad & \dot{x}_1(t) = -\alpha_1 x_1(t) - \alpha_2 x_2(t) - \alpha_3 x_3(t) - \alpha_4 x_4(t) + \hat{u}(t) \qquad (14)
\end{aligned}
$$

Thus we get state update equations as

$$
\underline{\dot{x}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \qquad (15)
$$

From (8) we can get the output equation

$$
\begin{aligned}
\hat{y}(s) &= (\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4)\hat{v}(s) \\
&= \beta_1\hat{x}_1(s) + \beta_2\hat{x}_2(s) + \beta_3\hat{x}_3(s) + \beta_4\hat{x}_4(s) \\
&= \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \underline{x}(s) \qquad (16)
\end{aligned}
$$

This in time domain gives the output equation

$$y(t) = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \underline{x}(t) \tag{17}$$

The controllability matrix is given by

$$X = \begin{bmatrix} 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 & -\alpha_1^3 + 2\alpha_1\alpha_2 - \alpha_3 \\ 0 & 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

Question: Is the system controllable?

Theorem: The controllable canonical form in (15) and (17) is observable if and only if $D(s)$ and $N(s)$ are coprime.

Proof: Part 1: (Necessary) If observable then $D(s)$ and $N(s)$ are coprime

Let $D(s)$ and $N(s)$ are not coprime, then we have a root λ_1 such that

$$N(\lambda_1) = \beta_1\lambda_1^3 + \beta_2\lambda_1^2 + \beta_3\lambda_1 + \beta_4 = 0 \tag{19}$$

$$D(\lambda_1) = \lambda_1^4 + \alpha_1\lambda_1^3 + \alpha_2\lambda_1^2 + \alpha_3\lambda_1 + \alpha_4 = 0 \tag{20}$$

Let us define a vector $\underline{v} = \left[\begin{array}{cccc} \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \end{array} \right]^T$. Then (19) can be written as $N(\lambda_1) = \underline{c}^T \underline{v} = 0$.
Also using (20) we get

$$A\underline{v} = \left[\begin{array}{cccc} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \left[\begin{array}{c} \lambda_1^3 \\ \lambda_1^2 \\ \lambda_1 \\ 1 \end{array} \right] = \left[\begin{array}{c} \lambda_1^4 \\ \lambda_1^3 \\ \lambda_1^2 \\ \lambda_1 \end{array} \right] = \lambda_1 \underline{v} \qquad (21)$$

Thus we have $A^n \underline{v} = (\lambda_1)^n \underline{v}$. We compute using $\underline{c}^T \underline{v} = 0$

$$O\underline{v} = \left[\begin{array}{c} \underline{c}^T \underline{v} \\ \underline{c}^T A \underline{v} \\ \underline{c}^T A^2 \underline{v} \\ \underline{c}^T A^3 \underline{v} \end{array} \right] = \left[\begin{array}{c} \underline{c}^T \underline{v} \\ \lambda_1 \underline{c}^T \underline{v} \\ \lambda_1^2 \underline{c}^T \underline{v} \\ \lambda_1^3 \underline{c}^T \underline{v} \end{array} \right] = \underline{0} \qquad (22)$$

which implies that the system is not observable. Thus if the system (15)-(17) is observable $D(s)$ and $N(s)$ are not coprime.

Part 2: (sufficient) If $D(s)$ and $N(s)$ are coprime then the system in (15) and (17) is observable

We start with the assumption that the system in (15) and (17) is not observable. Then observability condition 4 can be written as

$$\begin{bmatrix} A - \lambda_1 I \\ c \end{bmatrix} = 0 \tag{23}$$

or

$$A\underline{v} = \lambda_1 \underline{v} \qquad \underline{c}^T \underline{v} = 0 \tag{24}$$

where $\underline{v}$ is an eigenvector of A associated with eigenvalue λ_1.
From $\underline{c}^T \underline{v} = 0$ we get that λ_1 is a root of $N(s)$.
When $N(s)$ and $D(s)$ are coprime, $D(s)$ is a characteristic equation of A, hence λ_1 is a root of $D(s)$.
So $N(s)$ and $D(s)$ are not coprime. A contradiction. Hence, if $D(s)$ and $N(s)$ are coprime then the system (15)-(17) is observable. (EOP)

We can get another realization in the following way. We have

$$\hat{g}(s)^T = \hat{g}(s) = [\underline{c}^T(sI - A)^{-1}\underline{b}]^T = [\underline{b}^T(sI - A^T)^{-1}\underline{c}] \tag{25}$$

Thus the state equation

$$\dot{\underline{x}} = \begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\alpha_3 & 0 & 0 & 1 \\ -\alpha_4 & 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} u \tag{26}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \underline{x} \tag{27}$$

is a different realization of $\hat{g}(s)$. This is called observable canonical form. An equivalence transform (matrix P) can be used to get other controllable canonical and observable canonical form from controllable canonical and observable canonical form respectively

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

Every rational function $\hat{g}(s) = N(s)/D(s)$ can be reduced to $\hat{g}(s) = \bar{N}(s)/\bar{D}(s)$ where $\bar{N}(s)$ and $\bar{D}(s)$ are coprime and given by

$$N(s) = \bar{N}(s)R(s) \qquad D(s) = \bar{D}(s)R(s) \tag{29}$$

Such an expression is called a coprime fraction. We call $\bar{D}(s)$ a characteristic polynomial of $\hat{g}(s)$.
The degree of characteristic polynomial is defined as the degree of $\hat{g}(s)$.
Example: Let

$$\hat{g}(s) = \frac{s^2 - 1}{s^3 - 1} \tag{30}$$

Find the coprime fraction and the degree of $\hat{g}(s)$.
Solution: Here the numerator and denominator contain common factor $s - 1$. Canceling the common factor, the coprime fraction is

$$\hat{g}(s) = \frac{s + 1}{s^2 + s + 1} \tag{31}$$

Thus the rational function $\hat{g}(s)$ has degree 2.

Theorem: A state equation $(A, \underline{b}, \underline{c}^T, d)$ is a minimal realization of a proper rational function $\hat{g}(s)$ if and only if $(A, \underline{b})$ is controllable and $(A, \underline{c}^T)$ is observable.
Proof: If $(A, \underline{b})$ is not controllable or if $(A, \underline{c}^T)$ is not observable then the state equation can be reduced to a lesser dimensional state equation.
Thus $(A, \underline{b}, \underline{c}^T, d)$ is not a minimal realization.
Thus the condition is necessary.
To show sufficiency we consider the n dimensional state equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{bu} && (32) \\ \underline{y} &= \underline{c}^T\underline{x} + d\underline{u} && (33) \end{aligned}$$

Its controllability matrix

$$X_{n\times n} = \begin{bmatrix} \underline{b} & A\underline{b} & \cdots & A^{n-1}\underline{b} \end{bmatrix} \quad (34)$$

Its observability matrix

$$O_{n\times n} = \begin{bmatrix} \underline{c}^T \\ \underline{c}^T A \\ \vdots \\ \underline{c}^T A^{n-1} \end{bmatrix} \tag{35}$$

both have rank n.

Let us consider there exists a $\bar{n}$ dimensional minimal realization of $\hat{g}(s)$, with $\bar{n} < n$.

$$\dot{\bar{\underline{x}}} = \bar{A}\bar{\underline{x}} + \bar{\underline{b}}\bar{\underline{u}} \tag{36}$$

$$\bar{\underline{y}} = \bar{\underline{c}}^T\bar{\underline{x}} + \bar{d}\bar{\underline{u}} \tag{37}$$

Thus we have

$$\hat{g}(s) = \underline{c}^T(sI - A)^{-1}\underline{b} + d = \bar{\underline{c}}^T(sI - \bar{A})^{-1}\bar{\underline{b}} + \bar{d} \tag{38}$$

Expanding we have

$$d + \underline{c}^T\underline{b}s^{-1} + \underline{c}^T A\underline{b}s^{-2} + \underline{c}^T A^2\underline{b}s^{-3} + \cdots = \bar{d} + \bar{\underline{c}}^T\bar{\underline{b}}s^{-1} + \bar{\underline{c}}^T\bar{A}\bar{\underline{b}}s^{-2} + \bar{\underline{c}}^T\bar{A}^2\bar{\underline{b}}s^{-3} + \cdots \tag{39}$$

The equality of both sides means

$$
\begin{aligned}
d &= \bar{d} && (40)\\
\underline{c}^T A^m \underline{b} &= \bar{\underline{c}}^T \bar{A}^m \bar{\underline{b}} \qquad m = 0, 1, 2, \ldots && (41)
\end{aligned}
$$

Now we can consider the product

$$
\begin{aligned}
OX &= \begin{bmatrix} \underline{c}^T \\ \underline{c}^T A \\ \vdots \\ \underline{c}^T A^{n-1} \end{bmatrix} \begin{bmatrix} \underline{b} & A\underline{b} & \cdots & A^{n-1}\underline{b} \end{bmatrix} \\
&= \begin{bmatrix} \underline{c}^T \underline{b} & \underline{c}^T A\underline{b} & \cdots & \underline{c}^T A^{n-1}\underline{b} \\ \underline{c}^T A\underline{b} & \underline{c}^T A^2\underline{b} & \cdots & \underline{c}^T A^n\underline{b} \\ \vdots & \vdots & \ldots & \vdots \\ \underline{c}^T A^{n-1}\underline{b} & \underline{c}^T A^n\underline{b} & \cdots & \underline{c}^T A^{2(n-1)}\underline{b} \end{bmatrix} && (42)
\end{aligned}
$$

Using (41) we can replace every $\underline{c}^T A^m \underline{b}$ by $\bar{\underline{c}}^T \bar{A}^m \bar{\underline{b}}$. Thus we have

$$
OX = \bar{O}_n \bar{X}_n \qquad (43)
$$

where $\bar{X}_n$ and $\bar{O}_n$ are the from (34)-(35) where $(A, \underline{b}, \underline{c}^T)$ replaced by $(\bar{A}, \bar{\underline{b}}, \bar{\underline{c}}^T)$.

This means that the dimension of $\bar{X}_n$ is $\bar{n} \times n$ and that of $\bar{O}_n$ is $n \times \bar{n}$.
Here $\rho(OX) = n$, since the original system is observable and controllable (here this means O and X are nonsingular).
But the rank of $\bar{O}_n \bar{X}_n$ can not be more than $\bar{n}$. This shows that there is a contradiction in (43).
This shows that if we obtain a controllable and observable state equation the dimension of the state equation can't be reduced further, i.e., the realization is minimal.

Theorem: All minimal realizations of $\hat{g}(s)$ are equivalent.

Proof: Let $(A, \underline{b}, \underline{c}^T, d)$ and $(\bar{A}, \underline{\bar{b}}, \underline{\bar{c}}^T, \bar{d})$ be minimal realizations of $\hat{g}(s)$.

Then we have

$$
\begin{aligned}
OX &= \begin{bmatrix} \underline{c}^T \\ \underline{c}^T A \\ \vdots \\ \underline{c}^T A^{n-1} \end{bmatrix} \begin{bmatrix} \underline{b} & A\underline{b} & \cdots & A^{n-1}\underline{b} \end{bmatrix} \\
&= \begin{bmatrix} \underline{c}^T \underline{b} & \underline{c}^T A\underline{b} & \cdots & \underline{c}^T A^{n-1}\underline{b} \\ \underline{c}^T A\underline{b} & \underline{c}^T A^2\underline{b} & \cdots & \underline{c}^T A^{n}\underline{b} \\ \vdots & \vdots & \ldots & \vdots \\ \underline{c}^T A^{n-1}\underline{b} & \underline{c}^T A^{n}\underline{b} & \cdots & \underline{c}^T A^{2(n-1)}\underline{b} \end{bmatrix} \\
&= \begin{bmatrix} \bar{\underline{c}}^T \bar{\underline{b}} & \bar{\underline{c}}^T \bar{A}\bar{\underline{b}} & \cdots & \bar{\underline{c}}^T \bar{A}^{n-1}\bar{\underline{b}} \\ \bar{\underline{c}}^T \bar{A}\bar{\underline{b}} & \bar{\underline{c}}^T \bar{A}^2\bar{\underline{b}} & \cdots & \bar{\underline{c}}^T \bar{A}^{n}\bar{\underline{b}} \\ \vdots & \vdots & \ldots & \vdots \\ \bar{\underline{c}}^T \bar{A}^{n-1}\bar{\underline{b}} & \bar{\underline{c}}^T \bar{A}^{n}\bar{\underline{b}} & \cdots & \bar{\underline{c}}^T \bar{A}^{2(n-1)}\bar{\underline{b}} \end{bmatrix} \\
&= \bar{O}\bar{X}
\end{aligned}
\tag{44}
$$

Also we have

$$
\begin{aligned}
OAX &= \begin{bmatrix} \underline{c}^T A \\ \underline{c}^T A^2 \\ \vdots \\ \underline{c}^T A^n \end{bmatrix} \begin{bmatrix} \underline{b} & A\underline{b} & \cdots & A^{n-1}\underline{b} \end{bmatrix} \\
&= \begin{bmatrix} \underline{c}^T A\underline{b} & \underline{c}^T A^2\underline{b} & \cdots & \underline{c}^T A^n\underline{b} \\ \underline{c}^T A^2\underline{b} & \underline{c}^T A^3\underline{b} & \cdots & \underline{c}^T A^{n+1}\underline{b} \\ \vdots & \vdots & \ldots & \vdots \\ \underline{c}^T A^n\underline{b} & \underline{c}^T A^{n+1}\underline{b} & \cdots & \underline{c}^T A^{2n-1}\underline{b} \end{bmatrix} \\
&= \bar{O}A\bar{X}
\end{aligned} \tag{45}
$$

Now since the systems are minimal realizations so they controllable and observable. Here this means that all controllability and observability matrices are nonsingular.
We define a matrix P

$$
P \triangleq \bar{O}^{-1}O = \bar{X}X^{-1} \tag{46}
$$

from (44).

We also have

$$\bar{X} = \bar{O}^{-1} O X = P X \tag{47}$$

or

$$\left[\begin{array}{cccc} \underline{\bar{b}} & \bar{A}\underline{\bar{b}} & \cdots & \bar{A}^{n-1}\underline{\bar{b}} \end{array} \right] = P \left[\begin{array}{cccc} \underline{b} & A\underline{b} & \cdots & A^{n-1}\underline{b} \end{array} \right] \tag{48}$$

So we have $\underline{\bar{b}} = P\underline{b}$.
Again

$$\bar{O} = O \bar{X}^{-1} X = O P^{-1} \tag{49}$$

or

$$\left[\begin{array}{c} \underline{\bar{c}}^T \\ \underline{\bar{c}}^T \bar{A} \\ \vdots \\ \underline{\bar{c}}^T \bar{A}^{n-1} \end{array} \right] = \left[\begin{array}{c} \underline{c}^T \\ \underline{c}^T A \\ \vdots \\ \underline{c}^T A^{n-1} \end{array} \right] P^{-1} \tag{50}$$

So we have $\underline{\bar{c}}^T = \underline{c}^T P^{-1}$.

From $OAX = \bar{O}\bar{A}\bar{X}$ we get

$$\bar{A} = \bar{O}^{-1}OAX\bar{X}^{-1} = PAP^{-1} \tag{51}$$

Thus $(A, \underline{b}, \underline{c}^T, d)$ and $(\bar{A}, \underline{\bar{b}}, \underline{\bar{c}}^T, \bar{d})$ meet the conditions of algebraic equivalence.
Consider a rational function $\hat{g}(s) = N(s)/D(s)$.
If $N(s)$ and $D(s)$ are coprime then every root of $D(s)$ is a pole of $\hat{g}(s)$ and vice versa. Consider the example

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{s-1}{s^2+3s+2} = \frac{s-1}{(s+1)(s+2)} \tag{52}$$

Here every root of $D(s)$ is a pole of $\hat{g}(s)$. This is not true if $D(s)$ and $N(s)$ are not coprime. For example

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{s^2-1}{s^3-1} = \frac{(s+1)(s-1)}{(s^2+s+1)(s-1)} = \frac{s+1}{s^2+s+1} \tag{53}$$

We can see that although $s-1$ is a factor of $D(s)$, 1 is not a pole of $\hat{g}(s)$.
Let us consider that $(A, \underline{b}, \underline{c}^T, d)$ is a minimal realization of $\hat{g}(s) = N(s)/D(s)$.

Then we have

$$\frac{N(s)}{D(s)} = \underline{c}^T(sI-A)^{-1}\underline{b} + d = \frac{1}{\det(sI-A)}\underline{c}^T[\text{Adj}(sI-A)]\underline{b} + d \quad (54)$$

If $N(s)$ and $D(s)$ are coprime, then

$$\text{degree of } D(s) = \text{degree of } \hat{g}(s) = \text{degree of } \det(sI-A) = \text{dimension of } A \quad (55)$$

This shows that if a state equation is controllable and observable, then every eigenvalue of A is a pole of $\hat{g}(s)$ and vice versa.

Computing Coprime Factors

Consider a proper transfer matrix

$$\hat{g}(s) = \frac{N(s)}{D(s)} \quad (56)$$

where $N(s)$ and $D(s)$ are polynomials. For simplicity let us consider

$$\text{degree of } N(s) \leq \text{degree of } D(s) = n = 3 \quad (57)$$

Let us consider that polynomials $\bar{N}(s)$ and $\bar{D}(s)$ exist such that

$$\text{degree of } \bar{N}(s) \leq \text{degree of } \bar{D}(s) < n = 3 \tag{58}$$

and

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{\bar{N}(s)}{\bar{D}(s)} \tag{59}$$

which implies

$$D(s)(-\bar{N}(s)) + N(s)\bar{D}(s) = 0 \tag{60}$$

Here the condition that $\bar{D}(s) < 3$ is important as otherwise there are infinitely many solutions.

Instead of solving (60) directly, we will try to solve a set of linear algebraic equations. Here

$$\begin{aligned}
D(s) &= D_0 + D_1 s + D_2 s^2 + D_3 s^3 && (61)\\
N(s) &= N_0 + N_1 s + N_2 s^2 + N_3 s^3 && (62)\\
\bar{D}(s) &= \bar{D}_0 + \bar{D}_1 s + \bar{D}_2 s^2 && (63)\\
\bar{N}(s) &= \bar{N}_0 + \bar{N}_1 s + \bar{N}_2 s^2 && (64)
\end{aligned}$$

Hence (60) becomes

$$
\begin{aligned}
0 = & -\bar{N}_0 D_0 + N_0 \bar{D}_0 + (-\bar{N}_0 D_1 - \bar{N}_1 D_0 + N_0 \bar{D}_1 + N_1 \bar{D}_0)s \\
& +(-\bar{N}_2 D_0 - \bar{N}_1 D_1 - \bar{N}_0 D_2 + N_2 \bar{D}_0 + N_1 \bar{D}_1 + N_0 \bar{D}_2)s^2 \\
& +(-\bar{N}_2 D_1 - \bar{N}_1 D_2 - \bar{N}_0 D_3 + N_3 \bar{D}_0 + N_2 \bar{D}_1 + N_1 \bar{D}_2)s^3 \\
& +(-\bar{N}_2 D_2 - \bar{N}_1 D_3 + N_3 \bar{D}_1 + N_2 \bar{D}_2)s^4 + (-\bar{N}_2 D_3 + N_3 \bar{D}_2)s^5
\end{aligned}
\tag{65}
$$

This can be written in the form of following equation

$$
S\underline{m} \triangleq \begin{bmatrix} D_0 & N_0 & 0 & 0 & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 & 0 & 0 \\ D_2 & N_2 & D_1 & N_1 & D_0 & N_0 \\ D_3 & N_3 & D_2 & N_2 & D_1 & N_1 \\ 0 & 0 & D_3 & N_3 & D_2 & N_2 \\ 0 & 0 & 0 & 0 & D_3 & N_3 \end{bmatrix} \begin{bmatrix} -\bar{N}_0 \\ \bar{D}_0 \\ -\bar{N}_1 \\ \bar{D}_1 \\ -\bar{N}_2 \\ \bar{D}_2 \end{bmatrix} = \underline{0} \tag{66}
$$

Clearly for a nonzero solution of the equation S should be singular. In other words $D(s)$ and $N(s)$ are coprime if and only if S is nonsingular.

How to find coprime $\bar{N}(s)$ and $\bar{D}(s)$

Since $D_3 \neq 0$, the D columns in the S matrix are independent of the left columns.

However, N columns may not be independent. It can be shown that if a N column is not independent of its left columns then the columns to the right of this are all dependent.

We form a matrix S_1 with first dependent N column and all columns to the left of it. The vector in the null space of this matrix gives the solution for coprime factors.

Example: Let

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{s^2 - 1}{s^3 - 1} \tag{67}$$

Find if $N(s)$ and $D(s)$ are coprime. If not, find the coprime polynomials for $\hat{g}(s)$.

Solution: Here S matrix is given by

$$S = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{68}$$

In this case the matrix S has rank 5. Hence, the third N column is first dependent column. So $S_1 = S$.
If $[-\bar{N}_0 \quad \bar{D}_0 \quad -\bar{N}_1 \quad \bar{D}_1 \quad -\bar{N}_2 \quad \bar{D}_2]$ is a vector in the null space, then

$$\begin{aligned} 0 &= \bar{N}_0 - \bar{D}_0 && (69) \\ 0 &= \bar{N}_1 - \bar{D}_1 && (70) \\ 0 &= \bar{D}_0 + \bar{N}_2 - \bar{D}_2 && (71) \\ 0 &= -\bar{N}_0 + \bar{D}_1 && (72) \\ 0 &= -\bar{N}_1 + \bar{D}_2 && (73) \\ 0 &= \bar{N}_2 && (74) \end{aligned}$$

We consider $D_2 = 1$, then we have

$$D(s) = 1 + s + s^2 \quad (75)$$
$$N(s) = 1 + s \quad (76)$$

So given $N(s)$ and $D(s)$ form the $S_{n\times n}$ matrix and find out its rank.
If rank $r < n$ then the corresponding $N(s)$ and $D(s)$ are not coprime.
If rank is r then number of independent N columns is $\mu = r - n/2$.
Form a matrix S_1 using the first $2\mu + 2$ columns of S matrix.
Find a vector in the null space of S_1 and get the solution for coprime $\bar{N}(s)$ and $\bar{D}(s)$.