ECE 707: Control Systems Design (9)

T. Kirubarajan

Department of Electrical and Computer Engineering
McMaster University
Hamilton, Ontario, Canada

These viewgraphs are based on the text
“Linear System: Theory and Design” by Chi-Tsong Chen
Oxford University Press, 1999.

McMaster University

Minimal Realizations: Matrix Case

Characteristic Polynomial: The characteristic polynomial of a matrix $\hat{G}(s)$ is defined as the least common denominator of all minors of $\hat{G}(s)$.

Degree of $\hat{G}(s)$: The degree of $\hat{G}(s)$ is equal to the degree of the characteristic polynomial.

Consider the rational matrix

$$\hat{G}_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{bmatrix} \tag{1}$$

Its minors of order 1 are $1/(s+1)$, $1/(s+2)$, $1/(s+1)$, $1/(s+2)$.
Its minor of order 2 is det$(\hat{G}_1(s)) = 0$.
So the least common denominator of all minors is $(s+1)(s+2)$.
Hence characteristic equation of $\hat{G}_1(s)$ is $s^2 + 3s + 2$ and the degree of $\hat{G}_1(s)$ is 2.

Example: Find the degree of the following transfer function

$$\hat{G}(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \\ \frac{-1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s} \end{bmatrix} \quad (2)$$

Solution: The entries of $\hat{G}(s)$ are its minors of order 1.
Three minors of order 2

$$\frac{s}{(s+1)^2(s+2)} + \frac{1}{(s+1)^2(s+2)} = \frac{s+1}{(s+1)^2(s+2)} = \frac{1}{(s+1)(s+2)} \quad (3)$$

$$\frac{s}{(s+1)}\frac{1}{s} + \frac{1}{(s+1)(s+3)} = \frac{s+4}{(s+1)(s+3)} \quad (4)$$

$$\frac{1}{(s+1)(s+2)s} - \frac{1}{(s+1)(s+2)(s+3)} = \frac{3}{s(s+1)(s+2)(s+3)} \quad (5)$$

The least common denominator of these minors is $s(s+1)(s+2)(s+3)$. Thus the degree of $\hat{G}(s)$ is 4.
Note: To calculate degree of $\hat{G}(s)$ its every must be reduced to coprime fraction.

Characteristic Polynomial of A: If λ_i for $i = 1, 2, \ldots, n$ are the eigenvalues of A with multiplicity m_i, then characteristic polynomial of A is given by

$$\text{Characteristic polynomial of } A = \det(sI - A) = \prod_{i=1}^{n} (s - \lambda_i)^{m_i} \tag{6}$$

Minimal Polynomial of A: If λ_i for $i = 1, 2, \ldots, n$ are the eigenvalues of A with multiplicity m_i and $\bar{m}_i$ are the size of largest Jordan block associated with λ_i, then minimal polynomial of A is given by

$$\text{Minimal polynomial of } A = \prod_{i=1}^{n} (s - \lambda_i)^{\bar{m}_i} \tag{7}$$

Let (A, B, C, D) be a controllable and observable realization of $\hat{G}(s)$. Then without proof we can mention the following properties

- Least common denominator of all minors of $\hat{G}(s)$ = characteristic polynomial of A.
- Least common denominator of all entries of $\hat{G}(s)$ = minimal polynomial of A.

Following two theorems are same as SISO case:
Theorem: A state equation (A, B, C, D) is a minimal realization of a proper rational matrix $\hat{G}(s)$ if and only if (A, B) is controllable and (A, C) is observable or if and only if

$$\text{dim } A = \text{degree of } \hat{G}(s) \tag{8}$$

Theorem: All minimal realizations of $\hat{G}(s)$ are equivalent.
Matrix Polynomial Fractions Every $p \times q$ proper matrix can be expressed as

$$\hat{G}(s) = N(s)D^{-1}(s) \tag{9}$$

where $N(s)$ and $D(s)$ are $q \times p$ and $p \times p$ polynomial matrices.
For example consider the following transfer matrix

$$\hat{G}(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \\ \frac{-1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s} \end{bmatrix} \tag{10}$$

It can be expressed as

$$\hat{G}(s) = \begin{bmatrix} s & 1 & s \\ -1 & 1 & s+3 \end{bmatrix} \begin{bmatrix} s+1 & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & s(s+3) \end{bmatrix}^{-1} \tag{11}$$

The three diagonal elements of $D(s)$ are the least common denominators of the three columns of $\hat{G}(s)$.
The fraction in (9) and (11) are called right polynomial fraction.
Similarly any $p \times q$ proper matrix can be expressed as

$$\hat{G}(s) = \bar{D}^{-1}(s)\bar{N}(s) \tag{12}$$

where $\bar{N}(s)$ and $\bar{D}(s)$ are $q \times p$ and $q \times q$ polynomial matrices.
For the same example we have

$$\hat{G}(s) = \begin{bmatrix} (s+1)(s+2)(s+3) & 0 \\ 0 & s(s+1)(s+2) \end{bmatrix}^{-1} \begin{bmatrix} s(s+2)(s+3) & (s+3) & (s+1)(s+2) \\ -s(s+2) & s & (s+1)(s+2) \end{bmatrix}$$

The two diagonal elements of $\bar{D}(s)$ are the least common denominators of the two rows of $\hat{G}(s)$.

The fraction in (12) and (13) are called left polynomial fraction.
Let $R(s)$ be any $p \times p$ nonsingular polynomial matrix. Then we have

$$\hat{G}(s) = [N(s)R(s)][D(s)R(s)]^{-1} = N(s)D^{-1}(s) \tag{14}$$

Thus right fractions are not unique. Same holds for left fractions.
Unimodal Matrix: A square matrix is called a unimodal matrix if its determinant is nonzero and independent of s.
The following matrices are unimodal

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} -2 & s^{10}+s+1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} s & s+1 \\ s-1 & s \end{bmatrix} \tag{15}$$

If $A(s)$ and $B(s)$ (square matrices of the same order) are two unimodal matrices then $A(s)B(s)$ is a unimodal matrix since

$$\text{det}(A(s)B(s)) = \text{det}(A(s))\text{det}(B(s)) \tag{16}$$

Inverse of an unimodal matrix is unimodal, since

$$\det(M^{-1}(s)) = 1/\det(M(s)) \tag{17}$$

Consider $A(s) = B(s)C(s)$ where $A(s)$, $B(s)$ and $C(s)$ are polynomial matrices of compatible order.
We call $C(s)$ a right divisor of $A(s)$ and $B(s)$ a left divisor of $A(s)$.
Similarly we call $A(s)$ a right multiple of $B(s)$ and a left multiple of $C(s)$.
Greatest Common Right Divisor: A square polynomial matrix $R(s)$ is called greatest common right divisor of $D(s)$ and $N(s)$ if

- $R(s)$ is a common right divisor of $D(s)$ and $N(s)$,
- $R(s)$ is a left multiple of every common right divisor of $D(s)$ and $N(s)$

i.e.,

$$D(s) = \hat{D}(s)R(s) \quad \text{and} \quad N(s) = \hat{N}(s)R(s) \tag{18}$$

and for any polynomial matrix $R_1(s)$ such that

$$D(s) = \hat{D}(s)R_1(s) \quad \text{and} \quad N(s) = \hat{N}(s)R_1(s) \tag{19}$$

$$R(s) = P(s)R_1(s) \tag{20}$$

where all of $D(s), \hat{D}(s), N(s), \hat{N}(s), R(s), P(s)$ are polynomial matrices.

If greatest common right divisor is a unimodal matrix, then $D(s)$ and $N(s)$ are said to be right coprime.

Similarly we can define Greatest common left divisor and left coprime.

Consider a proper rational matrix $\hat{G}(s)$ factored as

$$\hat{G}(s) = N(s)D^{-1}(s) = \bar{D}^{-1}(s)\bar{N}(s) \tag{21}$$

where $N(s)$ and $D(s)$ are right coprime, and $\bar{N}(s)$ and $\bar{D}(s)$ are left coprime.

Then the characteristic polynomial of $\hat{G}(s)$ is defined as det($D(s)$) or det($\bar{D}(s)$).
The degree of $\hat{G}(s)$ is defined as

$$\text{degree of } \hat{G}(s) = \text{degree of } \det(D(s)) = \text{degree of } \det(\bar{D}(s)) \tag{22}$$

Let us consider the polynomial matrix

$$M(s) = \begin{bmatrix} 3s^2 + 2s & 2s + 1 \\ s^2 + s - 3 & s \end{bmatrix} \tag{23}$$

We can write $M(s)$ as

$$M(s) = \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}}_{M_{hc}} \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}}_{H_c(s)} + \underbrace{\begin{bmatrix} 2s & 1 \\ s - 3 & 0 \end{bmatrix}}_{M_{lc}(s)} \tag{24}$$

The constant matrix M_{hc} is called column-degree coefficient matrix.
The diagonal elements of $H_c(s)$ has diagonal elements as s^k, where k corresponds to the maximum power in that column.

The polynomial matrix $M_{lc}(s)$ contains the remaining terms. Similarly we can express $M(s)$ as

$$M(s) = \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix}}_{M_r(s)} \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}}_{H_{hr}(s)} + \underbrace{\begin{bmatrix} 2s & 1 \\ s-3 & 0 \end{bmatrix}}_{M_{lr}(s)} \quad (25)$$

Computing Matrix Coprime Factors

Consider a $q \times p$ proper rational matrix $\hat{G}(s)$ expressed as

$$\hat{G}(s) = \underbrace{\bar{D}^{-1}(s)\bar{N}(s)}_{\text{left fractions}} = \underbrace{N(s)D^{-1}(s)}_{\text{right fractions}} \quad (26)$$

Given any left fraction $\bar{D}^{-1}(s)\bar{N}(s)$ (not necessarily left coprime) we can obtain a right coprime fraction $N(s)D^{-1}(s)$ by solving the polynomial matrix equation

$$\bar{D}(s)(-N(s)) + \bar{N}(s)D(s) = 0 \quad (27)$$

Let us consider an example to discuss the process. Let

$$\hat{\hat{G}}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix} \tag{28}$$

First we find $\hat{G}(s)$ that is strictly proper fraction (i.e., numerator power is less than the denominator power). Here

$$\hat{\hat{G}}(s) = \underbrace{\begin{bmatrix} \frac{-12}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}}_{\hat{G}(s)} + \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \tag{29}$$

Last part is constant and it gives D matrix for all realizations. In the following we consider $\hat{G}(s)$ only.

Next we find a left fraction using least common denominator for each row.

$$\hat{G}(s) = \underbrace{\begin{bmatrix} (2s+1)(s+2) & 0 \\ 0 & (2s+1)(s+2)^2 \end{bmatrix}^{-1}}_{\bar{D}^{-1}(s)} \underbrace{\begin{bmatrix} -12(s+2) & 3(2s+1) \\ s+2 & (s+1)(2s+1) \end{bmatrix}}_{} \bar{N}(s) \quad (30)$$

Thus we have

$$\bar{D}(s) = \begin{bmatrix} 2s^2+5s+2 & 0 \\ 0 & 2s^3+9s^2+12s+4 \end{bmatrix} \quad (31)$$

$$= \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}}_{\bar{D}_0} + \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & 12 \end{bmatrix}}_{\bar{D}_1} s + \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix}}_{\bar{D}_2} s^2 + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}_{\bar{D}_3} s^3 \quad (32)$$

and

$$\bar{N}(s) = \begin{bmatrix} -12s-24 & 6s+3 \\ s+2 & 2s^2+3s+1 \end{bmatrix} \quad (33)$$

$$= \underbrace{\begin{bmatrix} -24 & 3 \\ 2 & 1 \end{bmatrix}}_{\bar{N}_0} + \underbrace{\begin{bmatrix} -12 & 6 \\ 1 & 3 \end{bmatrix}}_{\bar{N}_1} s + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}_{\bar{N}_2} s^2 + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\bar{N}_3} s^3 \quad (34)$$

Now (27) can be written as

$$\begin{bmatrix} \bar{D}_0 & \bar{N}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{D}_1 & \bar{N}_1 & \bar{D}_0 & \bar{N}_0 & 0 & 0 & 0 & 0 \\ \bar{D}_2 & \bar{N}_2 & \bar{D}_1 & \bar{N}_1 & \bar{D}_0 & \bar{N}_0 & 0 & 0 \\ \bar{D}_3 & \bar{N}_3 & \bar{D}_2 & \bar{N}_2 & \bar{D}_1 & \bar{N}_1 & \bar{D}_0 & \bar{N}_0 \\ 0 & 0 & \bar{D}_3 & \bar{N}_3 & \bar{D}_2 & \bar{N}_2 & \bar{D}_1 & \bar{N}_1 \\ 0 & 0 & 0 & 0 & \bar{D}_3 & \bar{N}_3 & \bar{D}_2 & \bar{N}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{D}_3 & \bar{N}_3 \end{bmatrix} \begin{bmatrix} -N_0 \\ D_0 \\ -N_1 \\ D_1 \\ -N_2 \\ D_2 \\ -N_3 \\ D_3 \end{bmatrix} = 0 \qquad (35)$$

or

$$
\underbrace{\begin{bmatrix}
2 & 0 & -24 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & -12 & 6 & 2 & 0 & -24 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & 1 & 3 & 0 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 5 & 0 & -12 & 6 & 2 & 0 & -24 & 3 & 0 & 0 & 0 & 0 \\
0 & 9 & 0 & 2 & 0 & 12 & 1 & 3 & 0 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 5 & 0 & -12 & 6 & 2 & 0 & -24 & 3 \\
0 & 2 & 0 & 0 & 0 & 9 & 0 & 2 & 0 & 12 & 1 & 3 & 0 & 4 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 5 & 0 & -12 & 6 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 9 & 0 & 2 & 0 & 12 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 9 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0
\end{bmatrix}}_{S}
\begin{bmatrix}
-N_0 \\ D_0 \\ -N_1 \\ D_1 \\ -N_2 \\ D_2 \\ -N_3 \\ D_3
\end{bmatrix} = 0
$$

If we use matlab function 'qr' (QR decomposition) we will get R as upper diagonal form of S, where Q is an unitary matrix.

The rows of R shows the dependence of columns of S. Here

$$R = \begin{array}{c}\begin{bmatrix}
x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x \\
0 & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x \\
0 & 0 & x & x & x & x & x & x & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & x & x & x & x & x & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0
\end{bmatrix} \\
\begin{matrix} d_1 & d_2 & n_1 & n_2 & d_1 & d_2 & n_1 & n_2 & d_1 & d_2 & n_1 & n_2 & d_1 & d_2 & n_1 & n_2 \end{matrix}
\end{array} \quad (37)$$

We can see here three first N columns of R (S) are independent of the previous (left) columns. Also two second N columns are independent of left columns.

Next for each N column we get a matrix that include all columns upto the first dependent one. For first N column we get

$$\begin{bmatrix} 2 & 0 & -24 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & -12 & 6 & 2 & 0 & -24 & 0 & 0 & 0 \\ 0 & 12 & 1 & 3 & 0 & 4 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 5 & 0 & -12 & 2 & 0 & -24 \\ 0 & 9 & 0 & 2 & 0 & 12 & 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 5 & 0 & -12 \\ 0 & 2 & 0 & 0 & 0 & 9 & 0 & 0 & 12 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -n_0^{11} \\ -n_0^{21} \\ d_0^{11} \\ d_0^{21} \\ -n_1^{11} \\ -n_1^{21} \\ d_1^{11} \\ -n_2^{11} \\ -n_2^{21} \\ d_2^{11} \end{bmatrix} = 0 \qquad (38)$$

Null vector with last element equal to 1 is

$$\begin{bmatrix} 12 & -1/2 & 1 & 0 & 6 & 0 & 5/2 & 0 & 0 & 1 \end{bmatrix} \qquad (39)$$

For second N column we get

$$\begin{bmatrix} 2 & 0 & -24 & 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & -12 & 6 & 2 & 0 & -24 & 3 \\ 0 & 12 & 1 & 3 & 0 & 4 & 2 & 1 \\ 2 & 0 & 0 & 0 & 5 & 0 & -12 & 6 \\ 0 & 9 & 0 & 2 & 0 & 12 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 9 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -n_0^{12} \\ -n_0^{22} \\ d_0^{12} \\ d_0^{22} \\ -n_1^{12} \\ -n_1^{22} \\ d_1^{12} \\ d_1^{22} \end{bmatrix} = 0 \tag{40}$$

Null vector with last element equal to 1 is

$$\begin{bmatrix} 9 & -1 & 1 & 2 & 0 & 0 & 2 & 1 \end{bmatrix} \tag{41}$$

Hence the corresponding $D(s)$ and $N(s)$ can be obtained from

$$\begin{bmatrix} -n_0^{11} & -n_0^{12} \\ -n_0^{21} & -n_0^{22} \\ d_0^{11} & d_0^{12} \\ d_0^{21} & d_0^{22} \\ -n_1^{11} & -n_1^{12} \\ -n_1^{21} & -n_1^{22} \\ d_1^{11} & d_1^{12} \\ d_1^{21} & d_1^{22} \\ -n_2^{11} & -n_2^{12} \\ -n_2^{21} & -n_2^{22} \\ d_2^{11} & d_2^{12} \\ d_2^{21} & d_2^{22} \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ -1/2 & -1 \\ 1 & 1 \\ 0 & 2 \\ 6 & 0 \\ 0 & 0 \\ 5/2 & 2 \\ & 1 \\ 0 & \\ 0 & \\ 1 & \\ & \end{bmatrix} \tag{42}$$

Hence $D(s)$ and $N(s)$ are given by

$$\begin{aligned} D(s) &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 5/2 & 2 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2 \\ &= \begin{bmatrix} s^2 + \frac{5}{2}s + 1 & 2s+1 \\ 0 & s+2 \end{bmatrix} \end{aligned} \tag{43}$$

$$\begin{aligned} N(s) &= \begin{bmatrix} -12 & -9 \\ 1/2 & 1 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s^2 \\ &= \begin{bmatrix} -6s - 12 & -9 \\ \frac{1}{2} & 1 \end{bmatrix} \end{aligned} \tag{44}$$

Hence we have got the right coprime factors for $\hat{G}(s)$, or

$$\hat{G}(s) = \underbrace{N(s)D^{-1}(s)}_{\text{right coprime}} \tag{45}$$

Now we can write $D(s)$ as

$$D(s) = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{D_{hc}} \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}}_{H(s)} + \underbrace{\begin{bmatrix} 5/2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{D_{lc}} \underbrace{\begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{L(s)} \tag{46}$$

We have

$$\underline{\hat{y}}(s) = \hat{G}(s)\underline{\hat{u}}(s) = N(s)D^{-1}(s)\underline{\hat{u}}(s) \tag{47}$$

Now we define $\hat{v}(s)$ as

$$\underline{\hat{v}}(s) = D^{-1}(s)\underline{\hat{u}}(s) \tag{48}$$

Then we have

$$D(s)\underline{\hat{v}}(s) = \underline{\hat{u}}(s) \tag{49}$$

$$\underline{\hat{y}}(s) = N(s)\underline{\hat{v}}(s) \tag{50}$$

We also define $\hat{x}(s)$ as

$$\underline{\hat{x}}(s) = L(s)\underline{\hat{v}}(s) = \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{v}_1(s) \\ \hat{v}_2(s) \end{bmatrix} \tag{51}$$

or, in time domain

$$\begin{aligned} x_1(t) &= \dot{v}_1(t) && (52) \\ x_2(t) &= v_1(t) && (53) \\ x_3(t) &= v_2(t) && (54) \end{aligned}$$

Hence we have

$$\dot{x}_2(t) = x_1(t) \tag{55}$$

We need to find equations for $\dot{x}_1(t)$ and $\dot{x}_3(t)$.

From

$$D(s) = D_{hc}H(s) + D_{lc}L(s) \tag{56}$$

we get

$$[D_{hc}H(s) + D_{lc}L(s)]\underline{\hat{v}}(s) = \underline{\hat{u}}(s) \tag{57}$$

$$\Rightarrow \quad H(s)\underline{\hat{v}}(s) = -D_{hc}^{-1}D_{lc}L(s)\underline{\hat{v}}(s) = D_{hc}^{-1}\underline{\hat{u}}(s) \tag{58}$$

$$\Rightarrow \quad H(s)\underline{\hat{v}}(s) = -D_{hc}^{-1}D_{lc}\underline{\hat{x}}(s) + D_{hc}^{-1}\underline{\hat{u}}(s) \tag{59}$$

$$\Rightarrow \quad \begin{bmatrix} s^2\hat{v}_1(s) \\ s\hat{v}_2(s) \end{bmatrix} = \begin{bmatrix} -5/2 & -1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \\ \hat{x}_3(s) \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \underline{\hat{u}}(s)$$

Thus we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -5/2 & -1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \underline{u}(t) \tag{60}$$

Hence the total update equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -5/2 & -1 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}(t) \tag{61}$$

Now we need to get the output equation. We can write

$$N(s) = \begin{bmatrix} -6 & -12 & -9 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -12 & -9 \\ 0 & 1/2 & 1 \end{bmatrix} L(s) \tag{62}$$

Since

$$\underline{\hat{y}}(s) = N(s)\underline{\hat{v}}(s) \quad (63)$$

$$= \begin{bmatrix} -6 & -12 & -9 \\ 0 & 1/2 & 1 \end{bmatrix} L(s)\underline{\hat{v}}(s) \quad (64)$$

$$= \begin{bmatrix} -6 & -12 & -9 \\ 0 & 1/2 & 1 \end{bmatrix} \underline{\hat{x}}(s) \quad (65)$$

Hence considering the total transfer matrix $\hat{\underline{\hat{G}}}(s)$ we have

$$\underline{y}(t) = \begin{bmatrix} -6 & -12 & -9 \\ 0 & 1/2 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}(t) \quad (66)$$

This gives the output equation.