ECE 707: Control Systems Design (11)

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These viewgraphs are based on the text
“Linear System: Theory and Design” by Chi-Tsong Chen
Oxford University Press, 1999.

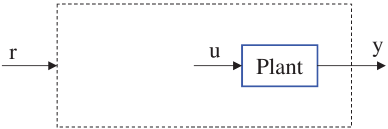

Fig. 13.1

In most of the control systems the plant (system) and the reference signal $r(t)$ are given. Our job is to find out input $u(t)$ such that $y(t)$ (the output) follows $r(t)$ as closely as possible.

If the input $u(t)$ depends only on the reference signal $r(t)$ and not on output $y(t)$ then the control is called open loop control.

Fig. 13.2(a) shows an open-loop control.

In a control system, if input $u(t)$ depends on both $r(t)$ as well as output $y(t)$ (or state variables $\underline{x}(t)$), then the control is called closed loop control.

There may be different kind of closed loop configurations.

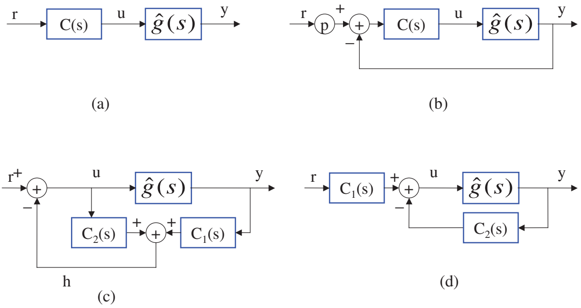

Fig. 13.2

The simplest is the unity feedback configuration shown in Fig. 13.2(b). Here the design aspects are gain p and the compensator $C(s)$. Thus we have

$$\hat{u}(s) = C(s)[p\hat{r}(s) - \hat{y}(s)] \tag{1}$$

The reference signal $r(t)$ and the plant output $y(t)$ drive the same compensator C to generate $u(t)$. In this case the designer is said to have one degree of freedom.

The estimator-state-feedback type system is shown in Fig. 13.2(c). Here the signal $\hat{h}(s)$ is given by

$$\hat{h}(s) = C_2(s)\hat{y}(s) + C_1(s)\hat{u}(s) \tag{2}$$

Hence Laplace transform of input $\hat{u}(s)$ is given by

$$\begin{aligned} \hat{u}(s) &= \hat{r}(s) - \hat{h}(s) \\ &= \hat{r}(s) - C_2(s)\hat{y}(s) - C_1(s)\hat{u}(s) \end{aligned} \tag{3}$$

or,

$$\hat{u}(s) = \frac{1}{1 + C_1(s)}\hat{r}(s) - \frac{C_2(s)}{1 + C_1(s)}\hat{h}(s) \tag{4}$$

Here $r(t)$ and $y(t)$ drive two independent compensators to generate $u(t)$. Here the designer has two degrees of freedom.
A more natural two-degree-of-freedom configuration is shown in Fig. 13.2(d). Here

$$\hat{u}(s) = C_1(s)\hat{r}(s) - C_2(s)\hat{y}(s) \tag{5}$$

One design objectives is to form an overall stable system i.e., no poles in the right s-plane. An additional design goal can be to have all zeros in the negative s-plane. Such a design is called minimum phase design and has less frequency distortion.

Pole Placement for Unity-Feedback Configuration

The system shown in Fig. 9.1(b) will be designed here. The plant transfer function $\hat{g}(s)$ is considered to be strictly proper. The degree of $\hat{g}(s)$ is considered to be equal to n.

The problem here is to design a proper compensator $C(s)$ of least possible degree m so that the overall system can have any set of $n+m$ poles.

The transfer function from r to y can be calculated as follows:

We have

$$\hat{u}(s) = C(s)[p\hat{r}(s) - \hat{y}(s)] \tag{6}$$

Since $\hat{y}(s) = \hat{u}(s)\hat{g}(s)$, we have

$$\frac{\hat{y}(s)}{\hat{g}(s)} = C(s)[p\hat{r}(s) - \hat{y}(s)] \tag{7}$$

$$\Rightarrow \quad \hat{y}(s)[1 + C(s)\hat{g}(s)] = pC(s)\hat{g}(s)\hat{r}(s) \tag{8}$$

So, the overall transfer function is given by

$$\hat{g}_0(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = \frac{pC(s)\hat{g}(s)}{1 + C(s)\hat{g}(s)} \tag{9}$$

Let $\hat{g}(s) = N(s)/D(s)$ and $C(s) = B(s)/A(s)$. Then the overall transfer function from r to y becomes

$$\hat{g}_0(s) = \frac{p\frac{B(s)N(s)}{A(s)D(s)}}{1 + \frac{B(s)N(s)}{A(s)D(s)}} = \frac{pB(s)N(s)}{A(s)D(s) + B(s)N(s)} \tag{10}$$

One important observation is that the zeros of original system (solution of $N(s)$) remains unchanged. Total set of zeros is zeros of $N(s)$ and zeros of $B(s)$.

In this problem our design goal is to have any $F(s)$ of degree $m+n$ (at most) such that

$$A(s)D(s) + B(s)N(s) = F(s) \tag{11}$$

where $N(s)$ and $D(s)$ are known and $F(s)$ is the system to be designed. We have to be find out $A(s)$ and $B(s)$.
Since we are considering $\hat{g}(s)$ and $C(s)$ to be proper, the degree of $N(s) \leq$ degree of $D(s) = n$ and degree of $B(s) \leq$ degree of $A(s) = m$. We can write

$$\begin{aligned} D(s) &= D_0 + D_1 s + D_2 s^2 + \cdots + D_n s^n && (12)\\ N(s) &= N_0 + N_1 s + N_2 s^2 + \cdots + N_n s^n && (13)\\ A(s) &= A_0 + A_1 s + A_2 s^2 + \cdots + A_m s^m && (14)\\ B(s) &= B_0 + B_1 s + B_2 s^2 + \cdots + B_m s^m && (15)\\ F(s) &= F_0 + F_1 s + F_2 s^2 + \cdots + F_{n+m} s^{n+m} && (16) \end{aligned}$$

all the coefficients are real constants.

The resulting equations can be arranged in the matrix form as

$$[A_0 \;\; B_0 \;\; A_1 \;\; B_1 \;\; \cdots \;\; A_m \;\; B_m]S_m = [F_0 \;\; F_1 \;\; F_2 \;\; \cdots \;\; F_{n+m}] \tag{17}$$

where

$$S_m = \begin{bmatrix} D_0 & D_1 & \cdots & D_n & 0 & \cdots & 0 \\ N_0 & N_1 & \cdots & N_n & \cdots & 0 & \\ 0 & D_0 & \cdots & D_{n-1} & D_n & \cdots & 0 \\ 0 & N_0 & \cdots & N_{n-1} & N_n & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & D_0 & \cdots & D_n \\ 0 & 0 & \cdots & 0 & N_0 & \cdots & N_n \end{bmatrix} \tag{18}$$

Taking transpose we can get the standard problem of system of equations

$$A\underline{x} = \underline{y} \tag{19}$$

where A, $\underline{y}$ are known and $\underline{x}$ is the unknown.
We know that for a solution to exist the number of equations should be less than or equal to the number of unknowns.

Here that means S_m should be square or have more rows than columns, i.e.,

$$2(m+1) \geq n+m+1 \quad \text{or} \quad m \geq n-1 \tag{20}$$

If $m = n-1$, S_{n-1} is a square matrix of order $2n$. This is nonsingular if $D(s)$ and $N(s)$ are coprime. In that case (17) has unique solution.

Application in Regulation and Tracking

For the regulation problem, the reference signal $r(t) = 0$. Here the problem is to bring the system to zero state at a desired rate.
For this problem all poles of overall transfer function $g_0(s)$ should have negative real part and $p = 1$ (any value of p will do).
Intracking problem $r(t) = a$, hence $\hat{r}(s) = a/s$ and the output $\hat{y}(s)$ is given by

$$\hat{y}(s) = \hat{g}_0(s)\frac{a}{s} \tag{21}$$

From the final-value theorem of the Laplace transform we have

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s) = \hat{g}_0(0)a \tag{22}$$

Since our aim is to have $y(t)$ to follow $r(t)$, which is a, we need $\hat{g}_0(0) = 1$. We have

$$\hat{g}_0(s) = \frac{pN(s)B(s)}{F(s)} \tag{23}$$

hence for $\hat{g}_0(0) = 1$ we need

$$1 = \frac{N(0)B(0)p}{F(0)} \tag{24}$$

or

$$p = \frac{F(0)}{N(0)B(0)} \tag{25}$$

Hence this value of p along with the poles of the overall system will achieve the tracker. To design a tracking system we need $N(0) \neq 0$.

Example: Consider a plant with transfer function

$$\hat{g}(s) = \frac{s-2}{s^2-1} \tag{26}$$

Find a compensator $C(s)$ so that the output $y(t)$ tracks any step-input r.

Solution: Let the selected poles be -2 and $-1 \pm j$. Hence $F(s)$ (denominator of overall system) is given by

$$F(s) = (s+2)(s+1+j)(s+1-j) = s^3 + 4s^2 + 6s + 4 \tag{27}$$

From $\hat{g}(s)$ we have

$$\begin{aligned} D(s) &= -1 + 0 \cdot s + 1 \cdot s^2 && (28) \\ N(s) &= -2 + 1 \cdot s + 0 \cdot s^2 && (29) \end{aligned}$$

Hence the corresponding system of equations

$$[A_0 \;\; B_0 \;\; A_1 \;\; B_1] \begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 1 & 1 \end{bmatrix} = [4 \;\; 6 \;\; 4 \;\; 1] \tag{30}$$

The solution is

$$[A_0 \;\; B_0 \;\; A_1 \;\; B_1] = [34/3 \;\; -23/3 \;\; 1 \;\; -22/3] \tag{31}$$

Hence

$$A(s) = s + 34/3 \quad \text{and} \quad B(s) = -22/3s - 23/3 \tag{32}$$

From these we get $C(s) = B(s)/A(s)$. Again

$$p = \frac{F(0)}{N(0)B(0)} = \frac{4}{(-2)\cdot(-23/3)} = \frac{6}{23} \tag{33}$$

The model matching problem is to obtain a transfer function $\hat{g}_0(s)$ for a given system, i.e., we need to shift poles as well as zeros of the system.
In this case we need two degrees of freedom (Fig. 13.2(d)).