ECE 707: Control Systems Design (6)

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These viewgraphs are based on the text
“Linear System: Theory and Design” by Chi-Tsong Chen
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Canonical Decomposition

Consider the state equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned} \quad (1)$$

Let $\bar{\underline{x}} = P\underline{x}$, then an equivalent state equation is given by

$$\begin{aligned} \dot{\bar{\underline{x}}} &= \bar{A}\bar{\underline{x}} + \bar{B}\underline{u} \\ \bar{\underline{y}} &= \bar{C}\bar{\underline{x}} + \bar{D}\underline{u} \end{aligned} \quad (2)$$

where $\bar{A} = PAP^{-1}$, $\bar{B} = PB$, $\bar{C} = CP^{-1}$, and $\bar{D} = D$.
All properties of (1), such as controllability, observability are preserved in (2).

Controllability

The new controllability matrix is given by

$$
\begin{aligned}
\bar{X} &= [\bar{B} \;\; \bar{A}\bar{B} \; \cdots \; \bar{A}^{n-1}\bar{B}] = [PB \quad PAP^{-1}PB \; \cdots \; PA^{n-1}P^{-1}PB] \\
&= P[B \;\; AB \; \cdots \; A^{n-1}B] = PX
\end{aligned} \tag{3}
$$

The new observability matrix is given by

$$
\bar{O} = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} = \begin{bmatrix} CP^{-1} \\ CP^{-1}PAP^{-1} \\ \vdots \\ CP^{-1}PA^{n-1}P^{-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} P^{-1} = OP^{-1} \tag{4}
$$

Theorem: Consider the n dimensional state equation in (1) with rank of controllability matrix

$$\rho(X) = \rho([B \;\; AB \;\; \cdots \;\; A^{n-1}B]) = n_1 < n \tag{5}$$

We form $n \times n$ matrix (for equivalence transform)
$P^{-1} \triangleq [\underline{q}_1 \;\; \underline{q}_2 \;\; \cdots \;\; \underline{q}_{n_1} \;\; \cdots \;\; \underline{q}_n]$ where the first n_1 columns are any n_1 linearly independent columns of X, and the remaining columns are arbitrarily chosen to make P nonsingular.
Then the equivalence transform $\underline{\bar{x}} = P\underline{x}$ gives

$$\begin{aligned} \begin{bmatrix} \dot{\underline{\bar{x}}}_c \\ \dot{\underline{\bar{x}}}_{\bar{c}} \end{bmatrix} &= \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} \underline{u} \\ \underline{y} &= [\bar{C}_c \;\; \bar{C}_{\bar{c}}] \begin{bmatrix} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{bmatrix} + D\underline{u} \end{aligned} \tag{6}$$

where $\bar{A}_c$ is $n_1 \times n_1$ and the n_1 dimensional subequation of (6)

$$\dot{\underline{\bar{x}}}_c = \bar{A}_c \underline{\bar{x}}_c + \bar{B}_c \underline{u} \qquad \underline{y} = \bar{C}_c \underline{\bar{x}}_c + D\underline{u} \tag{7}$$

is observable and has the same transfer function as (6).

Proof: Since $\bar{A} = PAP^{-1}$ and $P^{-1} = [\underline{q}_1 \ \ \underline{q}_2 \ \cdots \ \underline{q}_{n_1} \ \cdots \ \underline{q}_n]$, the i th column of $\bar{A}$ is the representation of $A\underline{q}_i$ w.r.t. $\{\underline{q}_1 \ \ \underline{q}_2 \ \cdots \ \underline{q}_n\}$.
To prove $A\underline{q}_i$, for $i = 1, 2, \ldots, n_1$, are linearly dependent on the set $\{\underline{q}_1 \ \ \underline{q}_2 \ \cdots \ \underline{q}_{n_1}\}$, and independent of the set $\{\underline{q}_{n_1+1} \ \ \underline{q}_{n_1+2} \ \cdots \ \underline{q}_n\}$:
For simplicity let $B = \underline{b}$ (single column). The $X = [\underline{b} \ \ A\underline{b} \ \cdots \ A^{n-1}\underline{b}]$, or, X has n columns each of which is linear combination of $\{\underline{q}_1 \ \ \underline{q}_2 \ \cdots \ \underline{q}_{n_1}\}$.
Since $A^n = c_0 I + c_1 A + \cdots + c_{n-1} A^{n-1}$ we have

$$\begin{aligned} AX &= [A\underline{b} \ \ A^2\underline{b} \ \cdots \ A^{n-1}\underline{b} \ \ A^n\underline{b}] \\ &= [A\underline{b} \ \ A^2\underline{b} \ \cdots \ A^{n-1}\underline{b} \ \ c_0\underline{b} + c_1 A\underline{b} + \cdots + c_{n-1}A^{n-1}\underline{b}] \end{aligned} \quad (8)$$

This means any column of AX is linear combination of columns of X, i.e., linear combination of $\{\underline{q}_1 \ \ \underline{q}_2 \ \cdots \ \underline{q}_{n_1}\}$.
Since $\{\underline{q}_1 \ \ \underline{q}_2 \ \cdots \ \underline{q}_{n_1}\}$ are columns of X, so $A\underline{q}_i$, for $i = 1, 2, \ldots, n_1$, are linear combination of $\{\underline{q}_1 \ \ \underline{q}_2 \ \cdots \ \underline{q}_{n_1}\}$ and independent of $\{\underline{q}_{n_1+1} \ \ \underline{q}_{n_1+2} \ \cdots \ \underline{q}_n\}$.
Same is true for any B (having multiple columns).

Thus i th column of $\bar{A}$ for $i = 1, 2, \ldots, n_1$, which is the representation of $A\underline{q}_i$ w.r.t. $\{\underline{q}_1 \;\; \underline{q}_2 \;\; \cdots \;\; \underline{q}_n\}$, has $n - n_1$ zero elements, or,

$$\bar{A} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}$$

Now $\bar{B} = PB$, or the columns of $\bar{B}$ are the representation of columns of B w.r.t. $\{\underline{q}_1 \;\; \underline{q}_2 \;\; \cdots \;\; \underline{q}_n\}$.
Now columns of B are independent of $\{\underline{q}_{n_1+1} \;\; \underline{q}_{n_1+2} \;\; \cdots \;\; \underline{q}_n\}$, hence $\bar{B}$ has the following form

$$\bar{B} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

The rank of controllability matrix of $(\bar{A}, \bar{B})$ is given by

$$\begin{aligned} \rho(\bar{X}) &= \rho\left(\left[\bar{B} \;\; \bar{A}\bar{B} \;\; \cdots \;\; \bar{A}^{n-1}\bar{B}\right]\right) \\ &= \rho\left(\left[\begin{array}{cccc|ccc} \bar{B}_c & \bar{A}_c\bar{B}_c & \cdots & \bar{A}_c^{n_1-1}\bar{B}_c & \bar{A}_c^{n_1}\bar{B}_c & \cdots & \bar{A}_c^{n-1}\bar{B}_c \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right]\right) \\ &= \rho(X) = n_1 \end{aligned} \tag{9}$$

Since A_c is $n_1 \times n_1$, columns of $A_c^k B_c$ for $k \geq n_1$, are linearly dependent on the columns of $\bar{X}_c = [\bar{B}_c \;\; \bar{A}_c\bar{B}_c \;\; \cdots \;\; \bar{A}_c^{n_1-1}\bar{B}_c]$, so

$$\rho(\bar{X}_c) = \rho(\bar{X}) = n_1 \tag{10}$$

Hence the pair $(\bar{A}_c, \bar{B}_c)$ is controllable, or, the state equation (7) is controllable.
Next we show that state equations (6) and (7) have same transfer function.
We can show

$$\begin{bmatrix} sI - \bar{A}_c & -\bar{A}_{12} \\ 0 & sI - \bar{A}_{\bar{c}} \end{bmatrix}^{-1} = \begin{bmatrix} (sI - \bar{A}_c)^{-1} & M \\ 0 & (sI - \bar{A}_{\bar{c}})^{-1} \end{bmatrix} \tag{11}$$

where $M = (sI - \bar{A}_c)^{-1}\bar{A}_{12}(sI - \bar{A}_{\bar{c}})^{-1}$.
Thus the transfer matrix of (6) is

$$\begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} sI - \bar{A}_c & -\bar{A}_{12} \\ 0 & sI - \bar{A}_{\bar{c}} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D \;\; = \;\; \bar{C}_c(sI - \bar{A}_c)^{-1}\bar{B}_c + D$$

which is the transfer function of (7).

It can be seen from the equivalent state equation

$$\begin{aligned} \left[\begin{array}{c} \dot{\underline{\bar{x}}}_c \\ \dot{\underline{\bar{x}}}_{\bar{c}} \end{array}\right] &= \left[\begin{array}{cc} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{array}\right] \left[\begin{array}{c} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{array}\right] + \left[\begin{array}{c} \bar{B}_c \\ 0 \end{array}\right] \underline{u} \\ \underline{y} &= [\bar{C}_c \;\; \bar{C}_{\bar{c}}] \left[\begin{array}{c} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{array}\right] + D\underline{u} \end{aligned}$$

that the input $\underline{u}$ can't control $\underline{\bar{x}}_{\bar{c}}$, since updated $\underline{\bar{x}}_{\bar{c}}$ does not depend on $\underline{u}$ directly or indirectly through $\underline{\bar{x}}_c$.
By dropping the uncontrollable states $\underline{\bar{x}}_{\bar{c}}$, we can obtain a controllable state equation

$$\begin{aligned} \dot{\underline{\bar{x}}}_c &= \bar{A}_c \underline{\bar{x}}_c + \bar{B}_c \underline{u} \\ \underline{y} &= \bar{C}_c \underline{\bar{x}}_c + D\underline{u} \end{aligned}$$

which has same transfer matrix.

Example: Consider the three dimensional state equation

$$\underline{\dot{x}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u} \quad y = [1 \;\; 1 \;\; 1]\underline{x}$$

Is the state equation controllable? If not find a controllable state equation that has same transfer function.

Solution: Since rank of B is 2; we can use $X_2 = [B \;\; AB]$, instead of $X = [B \;\; AB \;\; A^2B]$ to check controllability. Here

$$\rho(X_2) = \rho([B \;\; AB]) = \rho\left(\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}\right) = 2 < 3 \tag{12}$$

Hence the state equation is not controllable. We can choose

$$P^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Let $\underline{\bar{x}} = P\underline{x}$. We compute

$$\begin{aligned}
\bar{A} &= PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \vdots & 0 \\ 1 & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 \end{bmatrix} \\
\bar{B} &= PB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \cdots & \cdots \\ 0 & 0 \end{bmatrix} \\
\bar{C} &= CP^{-1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & \vdots & 1 \end{bmatrix}
\end{aligned}$$

Hence the controllable state equation is given by

$$\dot{\underline{\bar{x}}}_c = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \underline{\bar{x}}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u} \qquad y = \begin{bmatrix} 1 & 2 \end{bmatrix} \underline{\bar{x}}_c$$

Theorem: Consider the n dimensional state equation in (1) with rank of observability matrix

$$\rho(O) = \rho\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n_2 < n \tag{13}$$

We form $n \times n$ matrix (for equivalence transform)

$$P^{-1} \triangleq \begin{bmatrix} \underline{p}_1^T \\ \underline{p}_2^T \\ \cdots \\ \underline{p}_{n_2}^T \\ \cdots \\ \underline{p}_n^T \end{bmatrix} \tag{14}$$

where the first n_2 rows are any n_2 linearly independent rows of O, and the remaining rows are arbitrarily chosen to make P nonsingular. Then the equivalence transform $\underline{\bar{x}} = P\underline{x}$ gives

$$\begin{aligned} \begin{bmatrix} \dot{\underline{\bar{x}}}_o \\ \dot{\underline{\bar{x}}}_{\bar{o}} \end{bmatrix} &= \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} \underline{u} \\ \underline{y} &= [\bar{C}_o \;\; 0] \begin{bmatrix} \underline{\bar{x}}_o \\ \underline{\bar{x}}_{\bar{o}} \end{bmatrix} + D\underline{u} \end{aligned} \tag{15}$$

where $\bar{A}_o$ is $n_2 \times n_2$ and the n_2 dimensional subequation of (15)

$$\dot{\underline{\bar{x}}}_o = \bar{A}_o \underline{\bar{x}}_o + \bar{B}_o \underline{u} \quad \underline{y} = \bar{C}_o \underline{\bar{x}}_o + D\underline{u} \tag{16}$$

is controllable and has the same transfer function as (15).

Theorem: Every state equation can be transformed, by an equivalence transform, into the following canonical form

$$\begin{aligned}\begin{bmatrix} \dot{\underline{x}}_{co} \\ \dot{\underline{x}}_{c\bar{o}} \\ \dot{\underline{x}}_{\bar{c}o} \\ \dot{\underline{x}}_{\bar{c}\bar{o}} \end{bmatrix} &= \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \underline{x}_{co} \\ \underline{x}_{c\bar{o}} \\ \underline{x}_{\bar{c}o} \\ \underline{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} \underline{u} \\ \underline{y} &= \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \underline{x} + D\underline{u} \end{aligned} \tag{17}$$

where the vectors

- $\underline{x}_{co}$ is controllable and observable,
- $\underline{x}_{c\bar{o}}$ is controllable but not observable,
- $\underline{x}_{\bar{c}o}$ is observable but not controllable and
- $\underline{x}_{\bar{c}\bar{o}}$ is neither controllable nor observable.

Furthermore the state equation has same transfer matrix (or zero state equivalent) to the observable and controllable state equation

$$\begin{aligned}\dot{\underline{x}}_{co} &= A_{co}\underline{x}_{co} + B_{co}\underline{u} \\ \underline{y} &= C_{co}\underline{x}_{co} + D\underline{u}\end{aligned} \tag{18}$$

The corresponding transfer function is given by

$$G(s) = C_{co}(sI - A_{co})^{-1}B_{co} + D \tag{19}$$

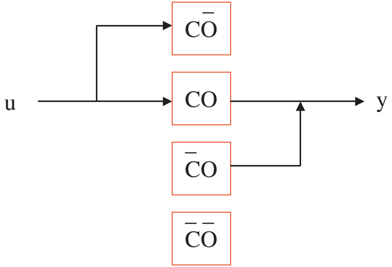

Fig. 8.1

Here we can see the reason why the transfer function description and the state-space description are not equivalent.

For example, if any portion of A matrix other than A_{co} has an eigenvalue with a positive real part, then the corresponding state variable will grow without bound and the system may burn out. This can't be detected from the transfer matrix.

Example: Consider circuit a shown in Fig. 8.2. Is it observable and controllable. If not find reduced state observable and controllable equation that has same transfer function.

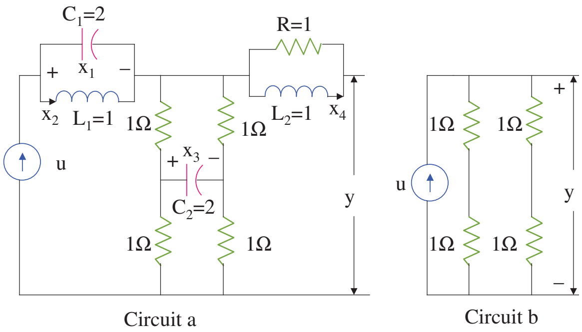

Fig. 8.2

Solution: By observation of the circuit we can say that initial state of L_1 and C_1 is not observable from the output.

Also, L_2 is not controllable by the input.

If we assign state variables as shown in circuit a of Fig. 8.2, the state equation is given by

$$
\begin{aligned}
\dot{\underline{x}} &= \begin{bmatrix} 0 & -0.5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \\
y &= [0 \;\; 0 \;\; 0 \;\; 1] \underline{x} + u
\end{aligned}
$$

Because the equation is already in the form of

$$
\begin{aligned}
\begin{bmatrix} \dot{\underline{x}}_{co} \\ \dot{\underline{x}}_{c\bar{o}} \\ \dot{\underline{x}}_{\bar{c}o} \\ \dot{\underline{x}}_{\bar{c}\bar{o}} \end{bmatrix} &= \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{c}} \end{bmatrix} \begin{bmatrix} \underline{x}_{co} \\ \underline{x}_{c\bar{o}} \\ \underline{x}_{\bar{c}o} \\ \underline{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} \underline{u} \\
\underline{y} &= \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \underline{x} + D\underline{u}
\end{aligned}
$$

it can be reduced to the following controllable state equation

$$
\dot{\underline{x}}_c = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix} \underline{x}_c + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u \qquad y = [0 \;\; 0] \underline{x}_c + u
$$

The output is independent of $\underline{x}_c$, thus it can be further reduced to $y = u$ (circuit b).

Conditions in Jordan-Form Equations

We know controllability and observability are invariant under any equivalence transform.

The following discussion will show us that if a state equation is transformed to Jordan form, then the conditions for these properties become very simple.

Consider the state equation

$$\begin{aligned} \dot{\underline{x}} &= J\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned} \tag{20}$$

where J is in Jordan form. We assume J to be having two distinct eigenvalues λ_1 and λ_2 and can be written as

$$J = \text{diag}(J_1, J_2) \tag{21}$$

where J_1 and J_2 consists of Jordan blocks associated with eigenvalue λ_1 and λ_2, respectively.

Again let J_1 and J_2 can be decomposed in the following Jordan blocks

$$J_1 = \text{diag}(J_{11}, J_{12}, J_{13}) \qquad J_2 = \text{diag}(J_{21}, J_{22}) \tag{22}$$

- The row of B corresponding to the last row of J_{ij} is denoted by $\underline{b}_{lij}^T$,
- The column of C corresponding to the first column of J_{ij} is denoted by $\underline{c}_{fij}$.

Theorem: For the state equation discussed above

- The state equation is controllable if and only if the three row vectors $\{\underline{b}_{l11}^T, \underline{b}_{l12}^T, \underline{b}_{l13}^T\}$ are linearly independent and the two row vectors $\{\underline{b}_{l21}^T, \underline{b}_{l22}^T\}$ are linearly independent,
- The state equation is observable if and only if the three column vectors $\{\underline{c}_{f11}, \underline{c}_{f12}, \underline{c}_{f13}\}$ are linearly independent and the two column vectors $\{\underline{c}_{f21}, \underline{c}_{f22}\}$ are linearly independent.

Proof: For simplicity let us define

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \tag{23}$$

Controllability condition can be obtained from the requirement that $[sI - J \quad B]$ must have full row rank for the state equation to be controllable. Here

$$[sI - J\ B] = \begin{bmatrix} s-\lambda_1 & -1 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{111}^T \\ 0 & s-\lambda_1 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{l11}^T \\ 0 & 0 & s-\lambda_1 & 0 & 0 & 0 & 0 & \underline{b}_{l12}^T \\ 0 & 0 & 0 & s-\lambda_1 & 0 & 0 & 0 & \underline{b}_{l13}^T \\ 0 & 0 & 0 & 0 & s-\lambda_2 & -1 & 0 & \underline{b}_{121}^T \\ 0 & 0 & 0 & 0 & 0 & s-\lambda_2 & 0 & \underline{b}_{l21}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & s-\lambda_2 & \underline{b}_{l22}^T \end{bmatrix} \tag{24}$$

For the eigenvalue λ_1 we have

$$[\lambda_1 I - J\ B] = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{111}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{l11}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{l12}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{l13}^T \\ 0 & 0 & 0 & 0 & \lambda_1 - \lambda_2 & -1 & 0 & \underline{b}_{121}^T \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 - \lambda_2 & 0 & \underline{b}_{l21}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 - \lambda_2 & \underline{b}_{l22}^T \end{bmatrix} \tag{25}$$

By elementary column operations we can show

$$\rho([\lambda_1 I - J\ B]) = \rho\left(\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & \underline{0}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{l11}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{l12}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{l13}^T \\ 0 & 0 & 0 & 0 & \lambda_1 - \lambda_2 & 0 & 0 & \underline{0}^T \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 - \lambda_2 & 0 & \underline{0}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 - \lambda_2 & \underline{0}^T \end{bmatrix} \right) \tag{26}$$

This shows us that the rows are independent iff $\{\underline{b}_{l11}^T, \underline{b}_{l12}^T, \underline{b}_{l13}^T\}$ are linearly independent.

For the other eigenvalue λ_2 we get

$$\rho([\lambda_2 I - J \quad B]) = \rho\left(\begin{bmatrix} \lambda_2-\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{0}^T \\ 0 & \lambda_2-\lambda_1 & 0 & 0 & 0 & 0 & 0 & \underline{0}^T \\ 0 & 0 & \lambda_2-\lambda_1 & 0 & 0 & 0 & 0 & \underline{0}^T \\ 0 & 0 & 0 & \lambda_2-\lambda_1 & 0 & 0 & 0 & \underline{0}^T \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & \underline{0}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{l21}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{b}_{l22}^T \end{bmatrix}\right) \tag{27}$$

Hence for controllability we need $\{\underline{b}_{l21}^T, \underline{b}_{l22}^T\}$ to be linearly independent.

Similarly for the state equation to be observable we need $\begin{bmatrix} sI - J \\ C \end{bmatrix}$ to have full column rank for $s = \lambda_1, \lambda_2$. Here

$$\begin{bmatrix} sI - J \\ C \end{bmatrix} = \begin{bmatrix} s-\lambda_1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s-\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s-\lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s-\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s-\lambda_2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & s-\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s-\lambda_2 \\ \underline{c}_{f11} & \underline{b}_{211} & \underline{c}_{f12} & \underline{c}_{f13} & \underline{c}_{f21} & \underline{c}_{221} & \underline{c}_{f22} \end{bmatrix} \tag{28}$$

For $s = \lambda_1$ we have (after elementary row operations)

$$\rho\left(\begin{bmatrix} \lambda_1 I - J \\ C \end{bmatrix}\right) = \rho\left(\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1-\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1-\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1-\lambda_2 \\ \underline{c}_{f11} & \underline{0} & \underline{c}_{f12} & \underline{c}_{f13} & \underline{0} & \underline{0} & \underline{0} \end{bmatrix}\right) \tag{29}$$

For $s = \lambda_2$ we have (after elementary row operations)

$$\rho\left(\begin{bmatrix} \lambda_2 I - J \\ C \end{bmatrix}\right) = \rho\left(\begin{bmatrix} \lambda_2 - \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 - \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 - \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{c}_{f21} & \underline{0} & \underline{c}_{f22} \end{bmatrix}\right) \quad (3$$

From last two equations it is clear that for $\rho\left(\begin{bmatrix} sI - J \\ C \end{bmatrix}\right)$ to have full column rank we need the sets $\{\underline{c}_{f11}, \underline{c}_{f12}, \underline{c}_{f13}\}$ and $\{\underline{c}_{f21}, \underline{c}_{f22}\}$ to be linearly independent.

Example: Consider the Jordan form state equation

$$
\begin{aligned}
\underline{\dot{x}} &= \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \underline{u} \\
\underline{y} &= \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 & 0 & 2 & 0 \end{bmatrix} \underline{x}
\end{aligned}
\tag{31}
$$

Find if the state equation is controllable and observable.

Solution: The matrix J has two distinct eigenvalues λ_1 and λ_2. There are three Jordan blocks, with order 2,1, and 1 associated with λ_1. The rows of B corresponding to the last row of the Jordan blocks associated with λ_1 are $\{[1\ \ 0\ \ 0], [0\ \ 1\ \ 0], [1\ \ 1\ \ 1]\}$. These three rows are linearly independent.

There is only one Jordan block of order 3 associated with λ_2. The row corresponding to the last row of Jordan block is $[1 \;\; 1 \;\; 1]$, which is nonzero, so linearly independent.
Hence the state equation is controllable.
The columns of C corresponding to first columns of Jordan blocks for λ_1 are $\{[1 \;\; 1 \;\; 1]', [2 \;\; 1 \;\; 2]', [0 \;\; 2 \;\; 3]'\}$ are linearly independent.
But the columns of C corresponding the first column of the Jordan block corresponding to λ_2 is $[0 \;\; 0 \;\; 0]$, not linearly independent.
Hence the state equation is not observable.

Now consider the Jordan form state equation

$$
\begin{aligned}
\dot{\underline{x}} &= \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \underline{x} + \begin{bmatrix} \underline{b}_{111}^T \\ \underline{b}_{l11}^T \\ \underline{b}_{l12}^T \\ \underline{b}_{l13}^T \\ \underline{b}_{121}^T \\ \underline{b}_{221}^T \\ \underline{b}_{l21}^T \end{bmatrix} \underline{u} \\
\underline{y} &= \begin{bmatrix} \underline{c}_{f11} & \underline{c}_{211} & \underline{c}_{f12} & \underline{c}_{f13} & \underline{c}_{f21} & \underline{c}_{221} & \underline{c}_{321} \end{bmatrix} \underline{x}
\end{aligned} \tag{32}
$$

Since $(sI-J)^{-1}$ is given by

$$
(sI-J)^{-1} = \begin{bmatrix} \frac{1}{1-\lambda_1} & \frac{1}{(1-\lambda_1)^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1-\lambda_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\lambda_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-\lambda_2} & \frac{1}{(1-\lambda_2)^2} & \frac{1}{(1-\lambda_2)^3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\lambda_2} & \frac{1}{(1-\lambda_2)^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\lambda_2} \end{bmatrix} \tag{33}
$$

From the relation between $\underline{u}$ and $\underline{y}$, i.e., $\underline{y}(s) = C(sI - J)^{-1}B\underline{u}(s)$. Now the output can be written as

$$\underline{y}(s) = \underline{y}_1(s) + \underline{y}_2(s) + \underline{y}_3(s) + \underline{y}_4(s) + \underline{y}_5(s) + \underline{y}_6(s) + \underline{y}_7(s) \tag{34}$$

$$\underline{y}_1(s) = \underline{c}_{f11}[\frac{1}{(s-\lambda_1)^2}\underline{b}_{l11}^T\underline{u}(s) + \frac{1}{s-\lambda_1}\underline{b}_{111}^T\underline{u}(s)] \tag{35}$$

$$\underline{y}_2(s) = \underline{c}_{211}[\frac{1}{s-\lambda_1}\underline{b}_{l11}^T\underline{u}(s)] \tag{36}$$

$$\underline{y}_3(s) = \underline{c}_{f12}[\frac{1}{s-\lambda_1}\underline{b}_{l12}^T\underline{u}(s)] \tag{37}$$

$$\underline{y}_4(s) = \underline{c}_{f13}[\frac{1}{s-\lambda_1}\underline{b}_{l13}^T\underline{u}(s)] \tag{38}$$

$$\underline{y}_5(s) = \underline{c}_{f21}[\frac{1}{(s-\lambda_2)^3}\underline{b}_{l21}^T\underline{u}(s) + \frac{1}{(s-\lambda_2)^2}\underline{b}_{221}^T\underline{u}(s) + \frac{1}{s-\lambda_2}\underline{b}_{121}^T\underline{u}(s)] \tag{39}$$

$$\underline{y}_6(s) = \underline{c}_{221}[\frac{1}{(s-\lambda_2)^2}\underline{b}_{l21}^T\underline{u}(s) + \frac{1}{s-\lambda_2}\underline{b}_{221}^T\underline{u}(s)] \tag{40}$$

$$\underline{y}_7(s) = \underline{c}_{321}[\frac{1}{s-\lambda_2}\underline{b}_{l21}^T\underline{u}(s)] \tag{41}$$

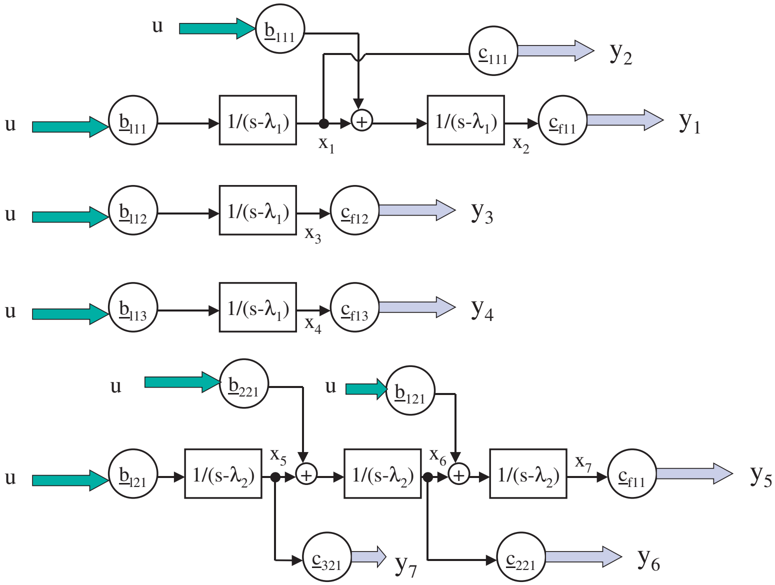
u
b111
c111
y2
u
b111
1/(s-λ1)
x1
+
1/(s-λ1)
x2
cf11
y1
u
b112
1/(s-λ1)
x3
cf12
y3
u
b113
1/(s-λ1)
x4
cf13
y4
u
b221
u
b121
u
b121
1/(s-λ2)
x5
+
1/(s-λ2)
x6
+
1/(s-λ2)
x7
cf11
y5
c321
y7
c221
y6

Fig. 8.3

Corollary: A single-input Jordan form state equation is controllable iff there is only one Jordan block associated with each distinct eigenvalue and every entry of B (which is a column) corresponding to the last row of each Jordan block is nonzero.
In this case $B = [b_1 \;\; b_2 \;\; \cdots \;\; b_n]^T$.
Corollary: A single-output Jordan form state equation is observable iff there is only one Jordan block associated with each distinct eigenvalue and every entry of C (which is a row) corresponding to the first column of each Jordan block is nonzero.
In this case $C = [c_1 \;\; c_2 \;\; \cdots \;\; c_n]$.

Consider the single-input single-output state equation

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 10 \\ 9 \\ 0 \\ 1 \end{bmatrix} u \\ y &= [1 \;\; 0 \;\; 0 \;\; 2]\underline{x} + u \end{aligned}$$

Here we have two Jordan blocks, one with order 3 associated with eigenvalue 0, other one of order 1 and associated with eigenvalue -2. Hence one Jordan block for each distinct eigenvalue.
But the entry in B corresponding to the last row of the first Jordan block is zero. Hence not controllable.
The entries in C corresponding to the first column of both Jordan block is nonzero. Hence observable.

Discrete-Time State Equations

Consider the discrete time state equation (time invariant)

$$\begin{aligned} \underline{x}[k+1] &= A\underline{x}[k] + B\underline{u}[k] \\ \underline{y}[k] &= C\underline{x}[k] + D\underline{u}[k] \end{aligned} \tag{42}$$

Theorem: The following statements are equivalent:

1. The state equation (42) or the pair (A, B) is controllable.
2. The matrix

$$W_{dc}[n-1] = \sum_{m=0}^{n-1} A^m BB'(A')^m \tag{43}$$

is nonsingular

3. The controllability matrix $X_d = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B]$ has full row rank.
4. The matrix $[A - \lambda I \;\; B]$ has full row rank for every eigenvalue of A.

Solution of (42) at $k = n$ is given by

$$\underline{x}[n] = A^n \underline{x}[0] + \sum_{m=0}^{n-1} A^{n-1-m} B \underline{u}[m] \tag{44}$$

which can be written as

$$\underline{x}[n] - A^n \underline{x}[0] = [B \;\; AB \;\; \cdots \;\; A^{n-1}B] \begin{bmatrix} \underline{u}[n-1] \\ \underline{u}[n-2] \\ \vdots \\ \underline{u}[0] \end{bmatrix} \tag{45}$$

It can be shown that for any $\underline{x}[0]$ and $\underline{x}[n]$ a solution of (45) exists iff X_d has full row rank. (equivalence of 1 and 3 proved)
Now the matrix $W_{dc}[n-1]$ can be written as

$$W_{dc}[n-1] = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B] \begin{bmatrix} B' \\ B'A' \\ \vdots \\ B'(A')^{n-1} \end{bmatrix} = X_d X_d' \tag{46}$$

Thus X_d is of full row rank iff $W_{dc}[n-1]$ is nonsingular. (equivalence of 2 and 3 proved)

Theorem: The following statements are equivalent:

1. The state equation (42) or the pair (A, C) is observable.
2. The matrix

$$W_{do}[n-1] = \sum_{m=0}^{n-1} (A')^m C' C A^m \tag{47}$$

is nonsingular

3. The controllability matrix

$$O_d = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{48}$$

has full column rank.

4. The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for every eigenvalue of A.

Controllability to the Origin and Reachability

We have three different type of controllability definition:

1. Transfer any state to any other state. (we have discussed this controllability)
2. Transfer any state to zero state, called controllability to origin.
3. Transfer zero state to any state, called reachability.

In continuous time case the three definitions are equivalent, as e^{At} is nonsingular.

In discrete-time case, if A is nonsingular, the three definitions are equivalent. If A is singular, still (1) and (3) are equivalent, not (2) and (3).

Let us consider the following state equation

$$\underline{x}[k+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underline{x}[k] + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u[k]$$

The controllability matrix is 0. Hence not controllable. But since $A^k = 0$ for $k \geq 3$. Thus we have $\underline{x}[3] = A^3 \underline{x}[0] = \underline{0}$ for any initial $\underline{x}[0]$. Thus the equation is controllable to the origin.

Controllability After Sampling

Consider the state equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \tag{49}$$

If the input is piecewise constant, or,

$$u[k] \overset{\triangle}{=} u(kT) = u(t) \quad \text{for} \;\; kT \le t < (k+1)T \tag{50}$$

Then the equation can be described by

$$\underline{\bar{x}}[n] = \bar{A}\underline{\bar{x}}[0] + \bar{B}\underline{u}[k] \tag{51}$$

with

$$\bar{A} = e^{AT} \qquad \bar{B} = \left(\int_0^T e^{At} dt\right) B \tag{52}$$

It can be shown that if (49) is uncontrollable then (51) also uncontrollable, but if (49) is controllable then (51) may not be controllable for all values of T.

Linear Time Variant Equations

Consider the state equation

$$\begin{aligned} \dot{\underline{x}}(t) &= A(t)\underline{x}(t) + B(t)\underline{u}(t) \\ \underline{y}(t) &= C(t)\underline{x}(t) \end{aligned} \tag{53}$$

The state equation is said to be controllable at t_0, if there exists a finite $t_1 > t_0$ such that for any initial state $\underline{x}(t_0)$ and final state $\underline{x}(t_1)$ there exists an input that can perform the transfer.

Difference with LTI: for LTI if the system is controllable it is controllable at each t_0, and the transfer can be performed in any interval $\tau = t_1 - t_0$.

Theorem: The state equation (53) or the pair $(A(t), B(t))$ is controllable at time t_0 iff there exists $t_1 > t_0$ such that the matrix

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B'(\tau)\Phi'(t_1, \tau)d\tau \tag{54}$$

is nonsingular, where $\Phi(t, \tau)$ is the state transition matrix of $\dot{\underline{x}}(t) = A(t)\underline{x}(t)$.

Proof: The response of the system at t_1 is given by

$$\underline{x}(t_1) = \Phi(t_1, t_0)\underline{x}(0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)\underline{u}(\tau)d\tau \tag{55}$$

We claim that the input

$$\underline{u}(t) = -B'(t)\Phi'(t_1, t)W_c^{-1}(t_0, t_1)[\Phi(t_1, t_0)\underline{x}_0 - \underline{x}_1] \tag{56}$$

can transfer $\underline{x}$ from $\underline{x}_0$ to $\underline{x}_1$. Indeed substituting (56) in (55) we get

$$\begin{aligned}
\underline{x}(t_1) &= \Phi(t_1, t_0)\underline{x}_0 \\
&\quad - \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B'(\tau)\Phi'(t_1, \tau)W_c^{-1}(t_0, t_1)[\Phi(t_1, t_0)\underline{x}_0 - \underline{x}_1]d\tau \\
&= \Phi(t_1, t_0)\underline{x}_0 \\
&\quad - \left(\int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B'(\tau)\Phi'(t_1, \tau)d\tau\right) W_c^{-1}(t_0, t_1)[\Phi(t_1, t_0)\underline{x}_0 - \underline{x}_1] \\
&= \underline{x}_1
\end{aligned} \tag{57}$$

Thus the equation is controllable if $W_c(t_0, t_1)$ is invertible, i.e., nonsingular.

Now let us consider that $W_c(t_0, t_1)$ is singular but the equation is controllable. Then there exists one vector $\underline{v}$ such that

$$\begin{aligned} \underline{v}' W_c(t_0, t_1)\underline{v} &= \int_{t_0}^{t_1} \underline{v}'\Phi(t_1, \tau)B(\tau)B'(\tau)\Phi'(t_1, \tau)\underline{v}d\tau \\ &= \int_{t_0}^{t_1} ||B'(\tau)\Phi'(t_1, \tau)\underline{v}||^2 d\tau = 0 \end{aligned} \tag{58}$$

which implies

$$B'(\tau)\Phi'(t_1, \tau)\underline{v} = \underline{0} \quad \text{or} \quad \underline{v}'\Phi(t_1, \tau)B(\tau) = \underline{0}' \tag{59}$$

for all τ such that $t_0 \le \tau \le t_1$. Since the state equation is controllable there exists an input that can transfer the state $\underline{x}_0 = \Phi(t_0, t_1)\underline{v}$ at t_0 to $\underline{x}(t_1) = \underline{0}$.

Then (55) becomes

$$\underline{0} = \underbrace{\Phi(t_1, t_0)\Phi(t_0, t_1)}_{I}\underline{v} + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)\underline{u}(\tau)d\tau \tag{60}$$

Its premultiplication with $\underline{v}'$ gives

$$\begin{aligned} 0 &= \underline{v}'\underline{v} + \int_{t_0}^{t_1} \underline{v}'\Phi(t_1,\tau)B(\tau)\underline{u}(\tau)d\tau \\ &= \underline{v}'\underline{v} + 0 \end{aligned} \tag{61}$$

This means the construct is true only if $\underline{v} = \underline{0}$. That means if the state equation is controllable $W_c(t_0, t_1)$ is nonsingular. (EOP)
Since the transition matrix may not be available we need controllability condition that does not involve it.
Let us define matrices

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t) \tag{62}$$

where $M_0(t) = B(t)$.
It can be shown that

$$\frac{\partial^m}{\partial t^m}\Phi(t_2, t)B(t) = \Phi(t_2, t)M_m(t) \tag{63}$$

for $m = 0, 1, 2, \ldots$.

Theorem: Let $A(t)_{n\times n}$ and $B(t)_{n\times p}$ be $n-1$ times continuously differentiable. Then the n dimensional pair $(A(t), B(t))$ is controllable at t_0 if there exists a finite $t_1 > t_0$ such that

$$\rho([M_0(t_1) \;\; M_1(t_1) \;\; \cdots \;\; M_{n-1}(t_1)]) = n \tag{64}$$

Note that this is a sufficient condition, not a necessary condition.
Proof: Let us consider that (64) holds, but $W_c(t_0, t)$ is singular for some $t_2 \geq t_1$. Then there exists one vector $\underline{v}$ such that

$$\begin{aligned} \underline{v}' W_c(t_0, t_2)\underline{v} &= \int_{t_0}^{t_2} \underline{v}'\Phi(t_1,\tau)B(\tau)B'(\tau)\Phi'(t_1,\tau)\underline{v}d\tau \\ &= \int_{t_0}^{t_1} ||B'(\tau)\Phi'(t_1,\tau)\underline{v}||^2 d\tau = 0 \end{aligned} \tag{65}$$

which implies

$$B'(\tau)\Phi'(t_1,\tau)\underline{v} = \underline{0} \quad \text{or} \quad \underline{v}'\Phi(t_1,\tau)B(\tau) = \underline{0}' \tag{66}$$

for all τ such that $t_0 \leq \tau \leq t_2$. Differentiations w.r.t. τ yields

$$\underline{v}'\Phi(t_1, \tau)M_m(\tau) = \underline{0}' \tag{67}$$

for $m = 0, 1, 2, \ldots, n-1$, and all $t_0 \leq \tau \leq t_2$. Since $t_2 \geq t_1$ we have

$$\underline{v}'\Phi(t_2, t_1)[M_0(t_1) \;\; M_1(t_1) \;\; \cdots \;\; M_{n-1}(t_1)] = \underline{0}' \tag{68}$$

Since $\Phi(t_2, t_1)$ is nonsingular $\underline{v}'\Phi(t_2, t_1) \neq \underline{0}'$. Hence (64) is not valid. A contradiction. Hence (64) valid implies $W_c(t_0, t_1)$ nonsingular, or, the state equation is controllable at t_0.

Example: Consider

$$\underline{\dot{x}} = \begin{bmatrix} t & -1 & 0 \\ 0 & -t & t \\ 0 & 0 & t \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

Prove that the matrix is controllable for all t.

Solution: We have $M_0 = B = [0 \;\; 1 \;\; 0]'$ and we can get

$$\begin{aligned} M_1 &= -A(t)M_0 + \frac{d}{dt}M_0 = \begin{bmatrix} 1 \\ 0 \\ -t \end{bmatrix} \\ M_2 &= -A(t)M_1 + \frac{d}{dt}M_1 = \begin{bmatrix} -t \\ t^2 \\ t^2 - 1 \end{bmatrix} \end{aligned}$$

The determinant of the matrix

$$[M_0 \;\; M_1 \;\; M_2] = \begin{bmatrix} 0 & 1 & -t \\ 1 & 0 & t^2 \\ 1 & -t & t^2 - 1 \end{bmatrix}$$

is $t^2 + 1$, which is nonzero for all t. Thus the state equation is controllable for all t.

Example: Consider

$$\underline{\dot{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \underline{x} + \begin{bmatrix} e^{t} \\ e^{2t} \end{bmatrix} u$$

Find if the matrix is controllable.

Solution: Here since A does not vary with time

$$\Phi(t,\tau) = \exp\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} (t-\tau) \right) = \begin{bmatrix} e^{t-\tau} & 0 \\ 0 & e^{2(t-\tau)} \end{bmatrix} \quad (69)$$

and

$$\Phi(t,\tau)B(\tau) = \begin{bmatrix} e^{t-\tau} & 0 \\ 0 & e^{2(t-\tau)} \end{bmatrix} \begin{bmatrix} e^{\tau} \\ e^{2\tau} \end{bmatrix} = \begin{bmatrix} e^{t} \\ e^{2t} \end{bmatrix} \quad (70)$$

We can find

$$
\begin{aligned}
W_c(t_0, t) &= \int_{t_0}^{t} \Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)d\tau = \int_{t_0}^{t} \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix} \begin{bmatrix} e^t & e^{2t} \end{bmatrix} d\tau \\
&= \begin{bmatrix} \int_{t_0}^{t} e^{2t} d\tau & \int_{t_0}^{t} e^{3t} d\tau \\ \int_{t_0}^{t} e^{3t} d\tau & \int_{t_0}^{t} e^{4t} d\tau \end{bmatrix} = \begin{bmatrix} e^{2t}(t-t_0) & e^{3t}(t-t_0) \\ e^{3t}(t-t_0) & e^{4t}(t-t_0) \end{bmatrix} \qquad (71)
\end{aligned}
$$

Its determinant is identically zero for all t_0 and t. Thus the state equation is not controllable at any t_0.

Theorem: The state equation (53) or the pair $(A(t), C(t))$ is observable at time t_0 iff there exists $t_1 > t_0$ such that the matrix

$$
W_o(t_0, t_1) = \int_{t_0}^{t_1} \Phi'(\tau, t_0)C'(\tau)C(\tau)\Phi(\tau, t_0)d\tau \qquad (72)
$$

is nonsingular, where $\Phi(t,\tau)$ is the state transition matrix of $\underline{\dot{x}}(t) = A(t)\underline{x}(t)$.

Theorem: Let $A(t)_{n\times n}$ and $C(t)_{q\times n}$ be $n-1$ times continuously differentiable. Then the n dimensional pair $(A(t), C(t))$ is controllable at t_0 if there exists a finite $t_1 > t_0$ such that

$$\rho\left(\begin{bmatrix} N_0(t_1) \\ N_1(t_1) \\ \vdots \\ N_{n-1}(t_1) \end{bmatrix}\right) = n \tag{73}$$

where

$$N_{m+1}(t) = N_m(t)A(t) + \frac{d}{dt}N_m(t) \quad \text{for} \;\; m = 0, 1, 2, \ldots, n-1 \tag{74}$$

with $N_0 = C(t)$.