ECE 707: Control Systems Design (3)

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These viewgraphs are based on the text
“Linear System: Theory and Design” by Chi-Tsong Chen
Oxford University Press, 1999.

Laplace Transform

Given the real function $f(t)$ that satisfies the condition

$$\int_0^\infty |f(t)e^{-\sigma t}|dt < \infty \quad (1)$$

for some finite real σ, the Laplace transform of $f(t)$ ($L[f(t)]$) is defined as

$$\hat{f}(s) = \int_0^\infty f(t)e^{-st}dt \quad (2)$$

The variable s is complex variable, i.e., $s = \sigma + j\omega$.
This particular definition (2) is known as one-sided Laplace Transform.

Example 1: $f(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$. Find $\hat{f}(s)$.

Solution:

$$\hat{f}(s) = \int_0^\infty f(t)e^{-st}dt \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \tag{3}$$

The last equation is valid if

$$\int_0^\infty |f(t)e^{-\sigma t}|dt = \int_0^\infty |e^{-\sigma t}|dt < \infty \tag{4}$$

which means the real part of s must be positive.
Similarly if $f(t) = e^{-\alpha t},\ t \geq 0$ then its Laplace transform is given by

$$\int_0^\infty f(t)e^{-\sigma t}dt = \int_0^\infty e^{-(\alpha+s)t}dt = -\frac{1}{s+\alpha}e^{-(s+\alpha)t}\Big|_0^\infty = \frac{1}{s+\alpha} \tag{5}$$

Important Theorems of the Laplace Transform

Theorem 1. Let k be a constant and $\hat{f}(s)$ be the Laplace transform of $f(t)$. Then

$$L\left[kf(t)\right] = k\hat{f}(s) \tag{6}$$

Theorem 2. Let $\hat{f}_1(s)$ and $\hat{f}_2(s)$ be the Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively. Then

$$L\left[f_1(t) \pm f_2(t)\right] = \hat{f}_1(s) \pm \hat{f}_2(s) \tag{7}$$

Theorem 3. Let $\hat{f}(s)$ be the Laplace transforms of $f(t)$ and $f(0)$ is the limit of f(t) as $t \to 0$. Then

$$L\left[\frac{df(t)}{dt}\right] = s\hat{f}(s) - \lim_{t\to 0} f(t) = s\hat{f}(s) - f(0) \tag{8}$$

In general for higher derivatives of $f(t)$

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n\hat{f}(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - f^{n-1}(0) \tag{9}$$

Theorem 4. Let $\hat{f}(s)$ be the Laplace transforms of $f(t)$ then

$$L\left[\int_0^t f(\tau)d\tau\right] = \frac{\hat{f}(s)}{s} \tag{10}$$

Theorem 5. Shift in time

$$L\left[f(t-T)\right] = e^{-Ts}\hat{f}(s) \tag{11}$$

Theorem 6. Initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\hat{f}(s) \tag{12}$$

Theorem 7. Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\hat{f}(s) \tag{13}$$

Theorem 8. Let $f_1(s)$ and $\hat{f}_2(s)$ be the Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively, then

$$\hat{f}_1(s)\hat{f}_2(s) = L\left[f_1(t) * f_2(t)\right] = L\left[\int_0^t f_1(\tau) f_2(t-\tau) d\tau\right] \tag{14}$$

Inverse Laplace Transform by Partial-Fraction Expansion

When Laplace transform solution is a rational function in s, it can be written as

$$\hat{f}(s) = \frac{Q(s)}{P(s)} \tag{15}$$

where $P(s)$ and $Q(s)$ are polynomials of s. It is assumed that the order of $P(s)$ in s is greater than that of $Q(s)$.

When All the Poles of $\hat{f}(s)$ Are Simple

In this case

$$\hat{f}(s) = \frac{Q(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)} \tag{16}$$

where $s_1 \neq s_2 \neq \cdots \neq s_n$.

Applying the partial-fraction expansion, (16) can be written as

$$\hat{f}(s) = \frac{k_{s1}}{(s - s_1)} + \frac{k_{s2}}{(s - s_2)} + \cdots + \frac{k_{sn}}{(s - s_n)} \tag{17}$$

where k_{si} are given by

$$k_{si} = \left[(s - s_i)\frac{Q(s)}{P(s)}\right]\Bigg|_{s=s_i} = \frac{Q(s_i)}{\prod_{j=1,\ j\neq i}^{n}(s_i - s_j)} \tag{18}$$

Example 2: Consider the function

$$\hat{f}(s) = \frac{5s + 3}{(s + 1)(s + 2)(s + 3)} \tag{19}$$

Find the partial-fraction expansion.
Solution: Here the partial-fraction expanded have the form

$$\hat{f}(s) = \frac{k_{-1}}{(s + 1)} + \frac{k_{-2}}{(s + 2)} + \frac{k_{-3}}{(s + 3)} \tag{20}$$

The coefficients $k_{-1},\ k_{-2}$ and k_{-3} are determined by

$$k_{-1} = \left[(s+1)\hat{f}(s)\right]\Big|_{s=-1} = \frac{5(-1)+3}{(2-1)(3-1)} = -1$$
$$k_{-2} = \left[(s+2)\hat{f}(s)\right]\Big|_{s=-2} = \frac{5(-2)+3}{(1-2)(3-2)} = \ \ 7$$
$$k_{-3} = \left[(s+3)\hat{f}(s)\right]\Big|_{s=-3} = \frac{5(-3)+3}{(1-3)(2-3)} = -6$$

Now by using the theorem 3 and example 1 we conclude that

$$f(t) = -e^{-t} + 7e^{-2t} - 6e^{-3t},\ \ t \geq 0 \tag{21}$$

When Some Poles of $\hat{f}(s)$ Are of Multiple Order

If r of the n poles of $\hat{f}(s)$ are identical, or, say, the pole at $s = s_i$ is of multiplicity r we have

$$\hat{f}(s) = \frac{Q(s)}{(s-s_1)(s-s_2)(s-s_{n-r})(s-s_i)^r} \tag{22}$$

Then $\hat{f}(s)$ can be expanded as follows

$$\hat{f}(s) = \underbrace{\frac{k_{s1}}{(s-s_1)} + \cdots + \frac{k_{s(n-r)}}{(s-s_{n-r})}}_{(n-r)\text{ simple poles}} + \underbrace{\frac{a_1}{(s-s_i)} + \cdots + \frac{a_r}{(s-s_i)^r}}_{r\text{ terms of repeated poles}} \quad (23)$$

$k_{s1}, \cdots, k_{s(n-r)}$ can be evaluated using (18). For $a_1, a_2, \cdots, a_r$ we have

$$\begin{aligned}
a_r &= \left[(s-s_i)^r \hat{f}(s)\right]\Big|_{s=s_i} \\
a_{r-1} &= \frac{d}{ds}\left[(s-s_i)^r \hat{f}(s)\right]\Big|_{s=s_i} \\
a_{r-2} &= \frac{1}{2!}\frac{d^2}{ds^2}\left[(s-s_i)^r \hat{f}(s)\right]\Big|_{s=s_i} \\
&\vdots \\
a_1 &= \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left[(s-s_i)^r \hat{f}(s)\right]\Big|_{s=s_i}
\end{aligned} \quad (24)$$

Example 3: Consider the function

$$\hat{f}(s) = \frac{1}{s(s+1)^3(s+2)} \tag{25}$$

Find the partial-fraction expansion.

Solution: Here the partial-fraction expanded form

$$\hat{f}(s) = \frac{k_0}{s} + \frac{k_{-2}}{(s+2)} + \frac{a_1}{(s+1)} + \frac{a_2}{(s+1)^2} + \frac{a_3}{(s+1)^3} \tag{26}$$

The coefficients of the simple poles are

$$\begin{aligned} k_0 &= \left[s\hat{f}(s) \right] \Big|_{s=0} = \frac{1}{2} \\ k_{-2} &= \left[(s+2)\hat{f}(s) \right] \Big|_{s=-2} = \frac{1}{2} \end{aligned}$$

The coefficients corresponding to the third order pole are

$$a_3 = \left[(s+1)^3 \hat{f}(s)\right]\Big|_{s=-1} = -1$$

$$a_2 = \frac{d}{ds}\left[(s+2)\hat{f}(s)\right]\Big|_{s=-1} = \frac{d}{ds}\left[\frac{1}{s(s+2)}\right]\Big|_{s=-1} = 0$$

$$a_1 = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+2)\hat{f}(s)\right]\Big|_{s=-1} = \frac{1}{2}\,\frac{d^2}{ds^2}\left[\frac{1}{s(s+2)}\right]\Big|_{s=-1} = -1$$

The complete partial-fraction expansion is

$$\hat{f}(s) = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1} - \frac{1}{(s+1)^3} \tag{27}$$

Now by using the theorem 3 and example 1 we conclude that

$$f(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \frac{1}{2}t^2 e^{-t}, \;\; t \geq 0 \tag{28}$$

Since $L^{-1}\left[1/s^n\right] = t^{n-1}/(n-1)!$.

Z-Transform

Given a sequence $f[k], \;\; k = 0, 1, \cdots$ its Z-transform is given by

$$\hat{f}(z) = Z\left(f[k]\right) = \sum_{i=0}^{\infty} f[k] z^{-k} \tag{29}$$

where z is a complex quantity.

Example 4: Given $f[k] = 1, \;\; k = 0, 1, \cdots$ find $\hat{f}(z)$.

Solution:

$$\hat{f}(z) = \sum_{i=0}^{\infty} f[k] z^{-k} = \sum_{i=0}^{\infty} z^{-k} = \frac{z}{1-z}$$

for $|z| > 1$.

Important Theorems of the Laplace Transform

Theorem 1. Let $\hat{f}_1(z)$ and $\hat{f}_2(z)$ be the Laplace transforms of $f_1[k]$ and $f_2[k]$, respectively. Then

$$Z\left[f_1[k] \pm f_2[k]\right] = \hat{f}_1(z) \pm \hat{f}_2(z) \tag{30}$$

Theorem 2. If $n > 0$ then

$$Z\left(f[k-n]\right) = z^{-n}\hat{f}(z) \tag{31}$$

and

$$Z\left(f[k+n]\right) = z^{n}\left(\hat{f}(z) - \sum_{i=1}^{n-1} f[k]z^{-k}\right) \tag{32}$$

Theorem 3. Let $f_1(z)$ and $\hat{f}_2(z)$ be the Laplace transforms of $f_1[k]$ and $f_2[k]$, respectively, then

$$\hat{f}_1(z)\hat{f}_2(z) = Z\left(f_1[k] * f_2[k]\right) = Z\left(\sum_{0}^{N} f_1[k]f_2[k-N]\right) \tag{33}$$

Inverse Z-transform is obtained by first expanding $\hat{f}(z)$ by partial fraction expansion of $\hat{f}(Z)/z$ and then multiplying z both sides. This expansion gives recognizable terms, and Z-transform table is used to determine the corresponding $f[k]$.

TIME FUNCTION	LAPLACE TRANSFORM
Unit inpulse $\delta(t)$	1
Unit step $U(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$
$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$e^{-\alpha t}\sin\omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-\alpha t}\cos\omega t$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Table of Laplace transform

TIME FUNCTION	**Z-TRANSFORM**
Unit inpulse $\delta[k]$	1
Unit step $U[k]$	$\frac{z}{z-1}$
k	$\frac{z}{(z-1)^2}$
$\frac{k^2}{2}$	$\frac{z(z+1)}{2(z-1)^3}$
$\frac{k^n}{n!}$	$\lim_{\alpha\to 0} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \alpha^n} \left(\frac{z}{z-e^{-\alpha}} \right)$
$\sin \omega k$	$\frac{z \sin w}{z^2-2z\cos w+1}$
$\cos \omega k$	$\frac{z(z-\cos w)}{z^2-2z\cos w+1}$

Table of Z-transform

Mathematical Descriptions of Systems

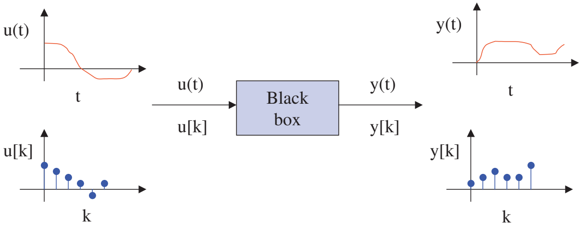


Fig. 5.1

Some Abbreviations

SISO = Single Input Single Output

MIMO = Multiple Input Multiple Output

SIMO = Single Input Multiple Output

Causality and Lumpedness

Memoryless System: If its output at t_0 ($\underline{y}(t_0)$) depends only on the inputs applied at t_0. Example: network of resistors.

Causal System: If its output at t_0 ($\underline{y}(t_0)$) depends on the inputs applied at $t \leq t_0$. Example: all physical systems.

State of a System: The state $\underline{x}(t_0)$ of a system at t_0 is the info. that, together with input $\underline{u}(t)$, for $t \geq t_0$, determines uniquely the output $\underline{y}(t)$ for all $t > t_0$.

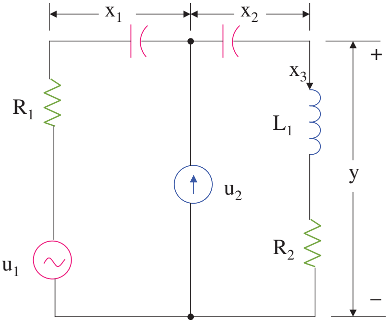


Fig. 5.2

Lumped System: If the number of state variables is finite. Otherwise the system is called distributed system.

Linear System: If for every t_0 and for any two state-input-output pairs

$$\left.\begin{array}{l} \underline{x}_i(t_0) \\ \underline{u}_i(t), \quad t \geq t_0 \end{array}\right\} \rightarrow \underline{y}_i(t), \quad t \geq t_0, \quad i = 1, 2$$

we have the following

$$\left.\begin{array}{l} \alpha_1 \underline{x}_1(t_0) + \alpha_2 \underline{x}_2(t_0) \\ \alpha_1 \underline{u}_1(t) + \alpha_2 \underline{u}_2(t), \quad t \geq t_0 \end{array}\right\} \rightarrow \alpha_1 \underline{y}_1(t) + \alpha_2 \underline{y}_2(t), \quad t \geq t_0$$

where α_1 and α_2 are real constants. This is known as superposition property.

From the above we can also conclude that response=zero-input response+ zero-state response.

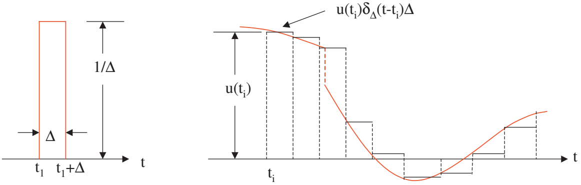
Input-Output Description

We consider a SISO linear system. We consider a pulse $\delta_\Delta(t-t_1)$ as shown in the Fig. 5.3 (a).

Then every input $u(t)$ can be approximated as Fig. 5.3 (b), i.e.,

$$u(t) \approx \sum_i u(t_i)\delta_\Delta(t-t_i)\Delta$$

Let $g_\Delta(t, t_i)$ be the output at time t excited by pulse $\delta_\Delta(t-t_i)$ applied at t_i.



(a) Pulse at t_1

(b) Approximation of input signal

Fig. 5.3

Since the system is linear following input-output relations hold

$$
\begin{aligned}
&\delta_\Delta(t-t_i) \;\rightarrow\; g_\Delta(t,t_i)\\
&\delta_\Delta(t-t_i)u(t_i)\Delta \;\rightarrow\; g_\Delta(t,t_i)u(t_i)\Delta \quad \text{(homogenity)}\\
&\sum_i \delta_\Delta(t-t_i)u(t_i)\Delta \;\rightarrow\; \sum_i g_\Delta(t,t_i)u(t_i)\Delta \quad \text{(additivity)}
\end{aligned}
$$

Thus the output $y(t)$ can be approximated by

$$
y(t) \approx \sum_i g_\Delta(t,t_i)u(t_i)\Delta \tag{34}
$$

For $\Delta \rightarrow 0$ the we have

$$
y(t) = \int_{-\infty}^{+\infty} g(t,\tau)u(\tau)d\tau \tag{35}
$$

Notice that $g(t,\tau)$ is a function of two variables. Second variable denotes the time at which the impulse input was applied; the first one denotes the time at which the output is observed.

For a causal system $g(t,\tau) = 0, \;\; t < \tau$.
The system is relaxed at $t = t_0$ means its state at $t = t_0$ is zero. Thus for a causal and relaxed at t_0 system

$$y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau \tag{36}$$

If a linear system has p input terminals and q output terminals then

$$\underline{y}(t) = \int_{t_0}^{t} G(t,\tau)\underline{u}(\tau)d\tau \tag{37}$$

where

$$G(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & g_{12}(t,\tau) & \cdots & g_{1p}(t,\tau) \\ g_{21}(t,\tau) & g_{22}(t,\tau) & \cdots & g_{2p}(t,\tau) \\ \vdots & \vdots & \ldots & \vdots \\ g_{q1}(t,\tau) & g_{q2}(t,\tau) & \cdots & g_{qp}(t,\tau) \end{bmatrix}$$

State-Space Description

A linear lumped system can be described by

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t) \quad (38)$$
$$\underline{y}(t) = C(t)\underline{x}(t) + D(t)\underline{u}(t) \quad (39)$$
$$(40)$$

This is accepted as a fact!!!

Linear Time-Invariant (LTI) Systems

A system is time invariant if for every state-input-output pair

$$\left.\begin{array}{l} \underline{x}(t_0) \\ \underline{u}(t), \quad t \geq t_0 \end{array}\right\} \rightarrow \underline{y}(t), \quad t \geq t_0$$

and any T, we have

$$\left.\begin{array}{l} \underline{x}(t_0 + T) \\ \underline{u}(t - T), \quad t \geq t_0 + T \end{array}\right\} \rightarrow \underline{y}(t - T), \quad t \geq t_0 + T \quad (41)$$

Example: Computers, calculators (gives the same result whenever they are used). What about human beings?
The opposite type of systems are called time varying systems.
Example: A burning rocket, its mass decreases with time. So does its the response to the force that drives it.
For a time invariant linear system the response of the system at t to an input applied at τ is equivalent to

$$g(t,\tau) = g(t+T,\tau+T) = g(t-\tau,0) = g(t-\tau)$$

In this case function g has only one variable which is the difference of the time at which the input was applied and the output was observed. The input output relation can be summarized by

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau \tag{42}$$

Here the system is considered to be relaxed at time 0.

Using the fact that convolution in time domain gives product of functions after Laplace transform

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau \quad \Rightarrow \quad \hat{y}(s) = \hat{g}(s)\hat{u}(s) \tag{43}$$

For a MIMO LTI system with p inputs and q outputs we have

$$\begin{bmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \\ \vdots \\ \hat{y}_q(s) \end{bmatrix} = \begin{bmatrix} \hat{g}_{11}(s) & \hat{g}_{12}(s) & \cdots & \hat{g}_{1p}(s) \\ \hat{g}_{21}(s) & \hat{g}_{22}(s) & \cdots & \hat{g}_{2p}(s) \\ \vdots & \vdots & \ldots & \vdots \\ \hat{g}_{q1}(s) & \hat{g}_{q2}(s) & \cdots & \hat{g}_{qp}(s) \end{bmatrix} \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \\ \vdots \\ \hat{u}_p(s) \end{bmatrix}$$

Equivalent to $\underline{\hat{y}}(s) = \hat{G}(s)\underline{\hat{u}}(s)$.
Using Laplace transform the state-space equations can be rewritten as

$$\left.\begin{array}{l} \underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \end{array}\right\} \Rightarrow \left\{\begin{array}{r} s\underline{\hat{x}}(s) - \underline{x}(0) = A\underline{\hat{x}}(s) + B\underline{\hat{u}}(s) \\ \underline{\hat{y}}(s) = C\underline{\hat{x}}(s) + D\underline{\hat{u}}(s) \end{array}\right. \tag{44}$$

The previous equation implies

$$\underline{\hat{x}}(s) = (sI - A)^{-1}\underline{x}(0) + (sI - A)^{-1}B\underline{\hat{u}}(s)$$
$$\underline{\hat{y}}(s) = C(sI - A)^{-1}\underline{x}(0) + C(sI - A)^{-1}B\underline{\hat{u}}(s) + D\underline{\hat{u}}(s) \quad (45)$$

Example 5:

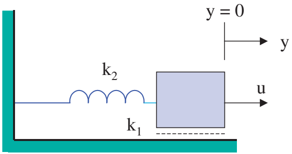


Fig. 5.4

Find out state-space relations and input output relation considering zero initial conditions.

Solution: Given k_1 to be viscous friction coefficient, k_2 to be the spring constant and mass of the block to be m we have

$$m\ddot{y} = u - k_1\dot{y} - k_2 y \quad (46)$$

Applying Laplace transform we get

$$ms^2\hat{y}(s) = \hat{u}(s) - k_1 s\hat{y}(s) - k_2\hat{y}(s)$$

which implies

$$\hat{y}(s) = \frac{1}{ms^2 + k_1 s + k_2}\hat{u}(s)$$

If $m = 1$, $k_1 = 3$ and $k_2 = 2$ then the impulse response

$$g(t) = L^{-1}\left[\frac{1}{s^2 + 3s + 2}\right] = L^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] = e^{-t} - e^{-2t}$$

Let us select the displacement and velocity of the block as state variables, i.e., $x_1 = y$, $x_2 = \dot{y}$. We have

$$\dot{x}_1 = x_2 \quad \text{and} \quad m\dot{x}_2 = u - k_1 x_2 - k_2 x_1$$

They can be expressed as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2/m & -k_1/m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

The output is related with the state variables as follows

$$y(t) = [1 \;\; 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Example 6:

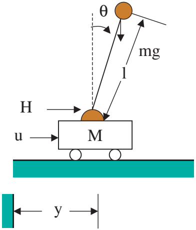


Fig. 5.5

Find out state-space relations and input output relation considering zero initial conditions.

Solution: From Newton's law of linear movement we have

$$M\ddot{y} = u - H = u - m\frac{d^2}{dt^2}(y + l\cos\theta) = m\ddot{y} + ml\ddot{\theta}\cos\theta - ml(\dot{\theta})^2\sin\theta$$

The application of Newton's law to the rotational movement gives

$$mgl\sin\theta = ml^2\ddot{\theta} + m\ddot{y}l\cos\theta$$

For small θ and $\dot{\theta}$ we have $\sin\theta = \theta$, $\cos\theta = 1$ and terms having θ^2, $(\dot{\theta})^2$, $\theta\dot{\theta}$ and $\theta\ddot{\theta}$ can be neglected. We have

$$\begin{aligned} M\ddot{y} &= u - mg\theta \\ Ml\ddot{\theta} &= (M+m)g\theta - u \end{aligned}$$

For zero initial condition Laplace transform gives

$$\begin{aligned} Ms^2\hat{y}(s) &= \hat{u}(s) - mg\hat{\theta}(s) \\ Mls^2\hat{\theta}(s) &= (M+m)g\hat{\theta}(s) - \hat{u}(s) \end{aligned}$$

The transfer functions $\hat{g}_{yu}(s)$ and $\hat{g}_{\theta u}(s)$ can be obtained from the above relations.

Let the state variables be $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \theta$, $x_4 = \dot{\theta}$. Then from this relations we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (M+m)g/Ml & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/Ml \end{bmatrix} u(t)$$

and $y = [1\ 0\ 0\ 0]\underline{x}$.

Example 7:

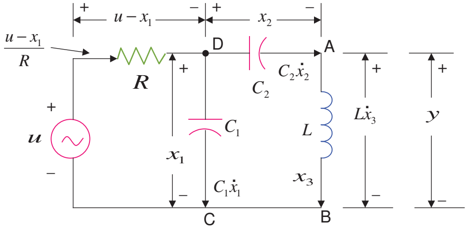


Fig. 5.6

Find out state-space relations.

Solution: We consider voltages across the capacitors and currents through the inductors as state variables.
Equating currents at A and D and voltage in the loop ABCD we get

$$\frac{u - x_1}{R} = C_1\dot{x}_1 + C_2\dot{x}_2 = C_1\dot{x}_1 + x_3$$
$$C_2\dot{x}_2 = x_3$$
$$L\dot{x}_3 = x_1 - x_2$$

And the output is given by

$$y = L\dot{x}_3 = x_1 - x_2 \tag{47}$$

Thus

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1/RC_1 & 0 & -1/C_1 \\ 0 & 0 & 1/C_2 \\ 1/L & -1/L & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1/RC_1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

and $y = [1 \;\; -1 \;\; 0]\underline{x} + 0.u$.

Example 7:

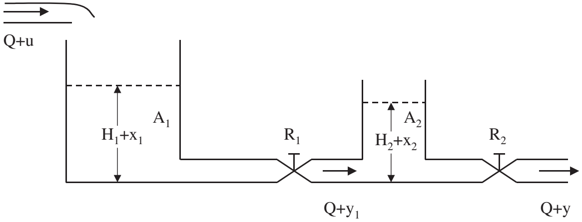


Fig. 5.7

It is assumed that under normal operation, inflows and outflows of both tanks are equal to Q and levels are H_1 and H_2.
Let u be the inflow perturbation of the first tank which causes a change of liquid level x_1 and outflow y_1 for this tank, and level variation x_2 and flow change y_2 for the second tank.
It is assumed that $y_1 = \frac{x_1 - x_2}{R_1}$ and $y = x_2 R_2$.
Find out state-space relations.

Solution: Changes in the liquid levels are governed by

$$A_1 dx_1 = (u - y_1)dt \quad \text{and} \quad A_2 dx_2 = (y_1 - y)dt$$

Thus

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1/A_2R_2 & 1/A_1R_1 \\ 1/A_2R_1 & -(1/A_2R_1 + 1/A_2R_2) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/A_1 \\ 0 \end{bmatrix} u(t)$$

and $y = [0\ 1/R_2]\underline{x} + 0.u$.

Discrete Time Systems

For a linear discrete time system the output sequence $y[k]$ is input sequence $u[k]$ and the system response $g[k,m]$ are related as

$$y[k] = \sum_{m=-\infty}^{+\infty} g[k,m]u[m] \tag{48}$$

For a causal system $g[k,m] = 0, \quad k < m$. In addition if the system is relaxed at k_0, we have

$$y[k] = \sum_{m=k_0}^{k} g[k,m]u[m] \tag{49}$$

For a time invariant system we can write

$$y[k] = \sum_{m=0}^{k} g[k-m]u[m] \tag{50}$$

Z-transform of the above equation gives

$$\hat{y}(z) = \hat{g}(z)\hat{u}(z) \tag{51}$$

For every lumped discrete time system we have

$$\begin{aligned} \underline{x}[k+1] &= A[k]\underline{x}[k] + B[k]\underline{u}[k] \\ y[k] &= C[k]\underline{x}[k] + D[k]\underline{u}[k] \end{aligned} \tag{52}$$