Queen's University
Faculty of Engineering and Applied Science
Department of Electrical and Computer Engineering

ELEC 326: Probability and Random Processes

Design Problem I, Fall 2018

Post date: October 17- Due date October 28th at 11 pm EDT

Instructor: Prof. Shahram Yousefi, PhD, P.Eng., SM'IEEE

Last Name: Angus **First Name:** Cache

ID Number: 20000629 **Program:** Computer Eng

There are 4 pages in this document

Instructions: [**Please read carefully**]

1. "**PLEASE NOTE:** The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer."
2. This is an open-book assignment. All non-human aids are allowed. You are to do this problem alone and without consultation with any other student or others. Help from humanoids, robots, bots, and other artificial intelligence tools is also allowed. Cheating will **NOT** be tolerated and will be dealt with according to **MOST** stringent regulations.
3. This question booklet has 2 pages including this cover. Make sure you carefully read both pages.
4. There is 1 question with a total mark of **50**.
5. Clearly show all the steps used in your solutions. No mark will be given for final answers without shown method and details and/or nontrivial assumptions made. Carry 2 decimal digits in your numerical calculations.
6. Give units for answers. **Highlight or box** the final answers clearly. Write **as neat as possible**.
7. **IMPORTANT:** Please submit your answers (and not this question 2-pager) in a single PDF (only) file that is legible and not larger than 10mb. On page 1, add your full name, your student ID, your program (like above) and total number of pages in the PDF file. This can be a typed document or hand-written and scanned document.

   GOOD LUCK!

Design problem [50 marks] Telecommunications (telecom) is the art of transmitting information from one place, say point A, to another, say point B. In *digital* telecom, every T seconds, a finite waveform (signal) from a finite set of signals $\mathcal{S}$ is picked and sent over a channel connecting point A to point B.

A channel is a mathematical model for a physical medium connecting the points like a coaxial cable or a wifi radio frequency (RF) link among many others. In binary telecom, the signal set $\mathcal{S}$ consists of only two waveforms $s_1(t)$ and $s_2(t)$ which are both varying voltages. Using standard transformations, in our problem, we assume that we can simply represent the two signals as two scalars of $s_1 = -1$ and $s_2 = +1$ volts. See the system block diagram below. When one of the two scalar signals is transmitted through the channel, the net effect in an additive-noise channel model is the addition of a noise voltage n. That is, the output of the channel, r, being fed to the receiver is $(s_i + n)$ when s_i is sent.

As you can see, the receiver's job sitting after the channel is to look at the corrupted voltage (signal plus noise) and decide what signal was really sent by the transmitter. This is called the estimate $\hat{s}$ of the transmitted signal which will be delivered to the end point B.

In our case, noise is a continuous random variable whose PDF, $P_N(n)$, is shown below. Here c is a constant.

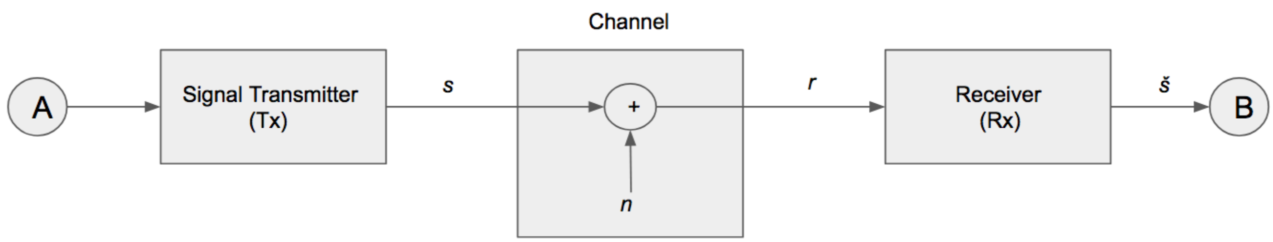
We use a biased coin every T seconds to pick and transmit $s_1 = -1$ if we see a tail and to pick and transmit $s_2 = +1$ if we see a head. Assume that $P(\text{Head})$=0.25.

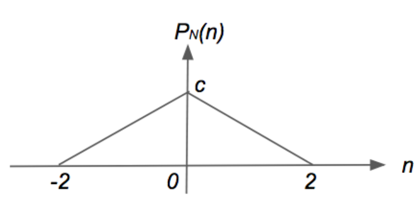
a. Design the receiver (Rx) for this system to minimize the probability of error. Draw the *optimal* receiver block diagram and carefully show all the important pieces and parameters on it. Derive and justify all of these.

Remark: Error is the event of sending s_1 and deciding in favour of s_2, or vice versa, at the receiver. In an optimal receiver, to minimize the probability of error, one can show that we need to maximize the so-called *a posteriori* probability $P(s_i|r)$.

b. If we use the above coin a very very large number of times (repeat the independent flips) and send many many signals, what will the average probability of error we will observe at the point B of this system be assuming we use the optimal receiver in part **a**? Calculate this probability.

Remark: in both **a** and **b** above, you are not to provide responses in terms of c. They have to be in known numbers.





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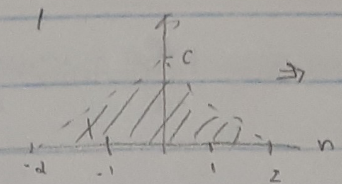
Design Problem Part I

Cache Angus

1) $S_1 = -1$ V

$S_2 = +1$ V

→ pdf



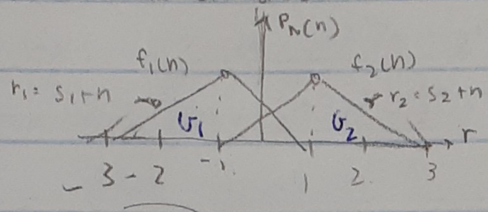
$\Rightarrow P_N(n) = \begin{cases} \frac{C}{2}x + C & -2 < x \le 0 \\ -\frac{C}{2} + C & \text{if } 0 < x \le 2 \\ \emptyset & \text{otherwise} \end{cases}$

→ $C = \int_{-2}^{0} \frac{C}{2}x + C + \int_{0}^{2} -\frac{C}{2}x - C = \{1 \Rightarrow \left[\frac{(C/2)x + C(x+2)}{2}\right]_{-2}^{0} + \left[\frac{2C}{2} + \frac{(-C/2x + 2C)x}{2}\right]_{0}^{2} = 1 = C$

$= \left[\frac{(C/2)(0) + C + (0+2)}{2}\right] - \emptyset + \left[C + \left(\frac{-\frac{C}{2}\cdot 2 + 2C)2}{2}\right)\right] - \emptyset = 1$

$= 3C - C = 1 = \boxed{C = 1/2}$

→ adjust pdf of n with S_1 and S_2 inputs → P



For $S_1 = -1$, fro R receive S1 → $P(S_1)$ 0.25

$P_R(r_1) = \begin{cases} 1/4\, r + 3/4 & \text{if } -3 < r \le -1 \\ -1/4\, r + 1/4 & \text{if } -1 < r \le 1 \\ 0 & \text{otherwise} \end{cases}$

Note: to choose signal 1 or 2

↳ r has to have a greater a posteriori

prob → $P_i p\{\bar{U} | S_i\}$

So we say H_1 vs H_0, where we choose

$\hat{H}_1$ as the hypothesis S_1 was sent

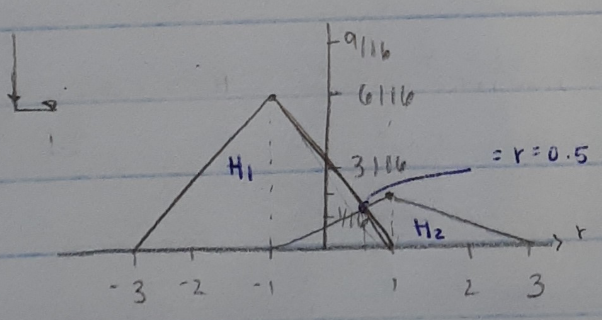
$\Rightarrow P_1 p\{r | S_1\} > P_2 p\{r | S_1\}$

For $S_2 = 1$ pr R receive S_2 → $P(S_2) = 0.75$

$P_R(r_2) = \begin{cases} 1/4\, r + 1/4 & \text{if } -1 < r \le 1 \\ -1/4\, r + 3/4 & 1 < r \le 3 \\ 0 & \text{otherwise} \end{cases}$

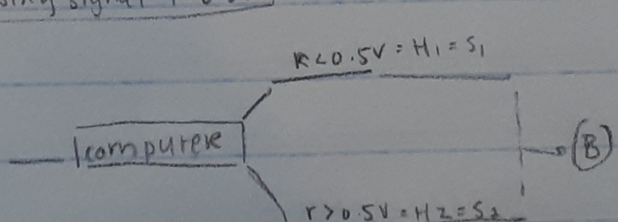
H_1 (chosen) $\cdot p(r|S_1) = 0.75 P_R(r_1) = \begin{cases} 3/16\, r + 9/16 & \text{if } -3 < r \le -1 \\ -3/16\, r + 3/16 & \text{if } -1 < r \le 1 \\ 0 & \text{otherwise} \end{cases}$

$H_2 = p(r|S_2) = 0.25 P_R(r_2) = \begin{cases} 1/16\, r + 1/16 & \text{if } -1 < r \le 1 \\ -1/16\, r + 3/16 & \text{if } 1 < r \le 3 \\ 0 & \text{otherwise} \end{cases}$



When $r < 0.5$, then H_1 should be chosen, otherwise $r > 0.5$ → H_2

Threshold voltage for choosing signal 1 = 0.5V



b) Threshold = 0.5 V

False alarm → $P[S_1 \text{ sel} | S_2 \text{ sent}] = P[(r|S_1) > 0.5] = \int_{0.5}^{\infty} f_{r|S_1}(x)\,dx = \int_{0.5}^{1} \left(\frac{1}{4} - \frac{1}{4}x\right) dx = \frac{1}{4}\int_{0.5}^{1} (1-x)\,dx = \frac{1}{4}\left[x - \frac{x^2}{2}\right]_{0.5}^{1}$

↳ S_1 selected when S_2 sent

$= \frac{1}{8} - \frac{3}{32} = \boxed{\frac{1}{32}}$

False dismissal → $P[S_2 \text{ sel} | S_1 \text{ sent}] = P[(r|S_2) < 0.5] = \int_{-\infty}^{0.5} f_{r|S_2}(x)\,dx = \int_{-1}^{0.5} \left(\frac{1}{4} + \frac{1}{4}x\right) dx = \frac{1}{4}\int_{-1}^{0.5} (1-x)\,dx = \frac{1}{4}\left[x + \frac{x^2}{2}\right]_{-1}^{0.5}$

↳ S_2 selected when S_1 sent

$= \frac{6}{32} + \frac{1}{8} = \boxed{\frac{9}{32}}$

Error signal sent isn't signal recieved $= P[S_2 \text{ sel} | S_1 \text{ sent}] \cdot P(S_1) + P[S_1 \text{ sel} | S_2 \text{ sent}] \cdot P(S_2)$

$= \frac{9}{32} \cdot 0.25 + \frac{1}{32} \cdot 0.75 = \frac{3}{32} = 0.09375$

$\boxed{\therefore \text{ error is } 9.395\%}$