Cache Ellsworth

Spectral Theory Exercises

Exercise 2

The only distinct eigenvalue is 0. The eigenspace is $[1,0,0]^T$. The Geometric and algebraic multiplicities are then one and three.

Exercise 4

(i) Let A be a Hermitian 2×2 matrix with entries of a, b, c, d $\in \mathbb{C}$ in the obvious entries. So,

$$\lambda\pm = \frac{1}{2}(\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\mathrm{det}(A)}).$$

We know that $A = A^H$ and so b must equal $\bar{c}$. Thus the discriminant is:

$$(a+d)^2 - 4(ad - bc) = (a+d)^2 - 4(ad - b\bar{b}) = a^2 + 2ad + d^2 - 4ad + 4b\bar{b} = a^2 - 2ad + d^2 + 4b\bar{b} = (a-d)^2 + 4b\bar{b}.$$

We know that a complex number times it's conjugate is always a positive real number. Thus the discriminant is a positive number and so the eigenvalue is real.

(ii) Let A be a skew-Hermitian 2×2 matrix with entries of a, b, c, d $\in \mathbb{C}$ in the obvious entries. So,

$$\lambda\pm = \frac{1}{2}(\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\mathrm{det}(A)}).$$

We know that $A^H = -A$ and so $a = -\bar{a}$ and $d = -\bar{d}$. So the real parts of these numbers are equal to 0 , the imaginary parts are still there though which are represented by $a_z i, d_z i$. Thus the discriminant is:

$$\begin{aligned}
(a+d)^2 - 4(ad - bc) &= (a+d)^2 - 4(ad - bc) \\
&= a^2 + 2ad + d^2 - 4ad + 4bc \\
&= a^2 - 2ad + d^2 + 4bc \\
&= (a-d)^2 - 4c\bar{c} \\
&= (a_z i - d_z i)^2 - 4c\bar{c} \\
&= (i)^2(a_z - d_z)^2 - 4c\bar{c} \\
&= -(a_z - d_z)^2 - 4c\bar{c}
\end{aligned}$$

We know that a complex number times it's conjugate is always a positive real number. So, then the discriminant is a negative number and so the eigenvalue has an imaginary part. Furthermore, $\mathrm{Tr}(A) =$ a negative complex number and so the eigenvalue is fully imaginary.

Exercise 6

Proof. Let A be an $n \times n$ upper-triangular matrix with a_{ii} in the diagonal . Then,

$$\det(\lambda I - A) = \prod_{i=1}^{n} (\lambda - a_{ii})$$

by definition of the determinate of an upper-triangular matrix. Setting this equal to 0 to find our eigenvalues, we see that $a_i i$ is equal to λ and nothing else will be equal to λ. Therefore, the diagonal entries of A are the eigenvalues. $\square$

Exercise 8

Let V be the span of the set $S = \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ in the vector space $C^{\infty}(\mathbb{R}; \mathbb{R})$. (i) Because V is the span of S, we only need to check if S is linearly independent. In the last problem set we saw that the elements in S are equal to 0 if we multiply them to a different element in the set and take the integral. This shows that they are are orthonormal under the inner product. Therefore, they are independent and are a basis of V. (ii)

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

(iii) a) $\text{span}\{\sin(x), \cos(x)\}$. b) $\text{span}\{\sin(2x), \cos(2x)\}$.

Exercise 13

$$P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}. \text{ So, } P^{-1}AP = D = \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Exercise 15

Proof. Let $(\lambda)_{i=1}^{n}$ be the eigenvalues of a semi-simple matrix $A \in M_n(\mathbb{F})$ and $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial. Then,

$$f(A) = a_0 + a_1 A + \cdots + a_n A^n = a_0 I + a_1 P^{-1} D P + \cdots + a_n P^{-1} D^n P$$

by proposition 4.3.10 and $A = P^{-1}DP$ where $D = \text{diag}(\lambda_i)_{i=1}^{n}$ because A is semi-simple by Theorem 4.3.7. We know that a_i is a constant and so the equation becomes,

$$f(A) = P^{-1} a_0 P + P^{-1} * a_1 * DP + \cdots + P^{-1} * a_n * D^n P = P^{-1} f(D) P$$

Then by definition of diagonalizable the eigenvalues of $f(A)$ are $(f(\lambda_i))_{i=1}^{n}$. $\square$

Exercise 16

(i). If we separate A as in exercise 13(. So, $P^{-1}AP = D = \begin{bmatrix} 0.4^k & 0 \\ 0 & 1 \end{bmatrix}$.) and take the limit we see that the D matrix is: . So, $P^{-1}AP = D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus,

$$B = \text{inv}(\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix})$$

Taking the difference gives us a number raised to the k power which will converge to 0. (ii) The ∞-norm: $||A^k - B||_\infty = 0.4^k$. This converges to 0.
The Frobenius norm: $\sqrt{\text{tr}((A^k - B)^T(A^k - B))}$ = some number that converges to 0. Verified by Python.
(iii) The eigenvalues are $f(1) = 9$ and $f(0.4) = 5.064$.

Exercise 18

Proof. We see that the transpose of A has the same eigenvalues as A. Considering the characteristic polynomial of $A : \det(\lambda I - A) = \det((\lambda I - A)^T)$ which is the characteristic polynomial of A^T. Therefore we know that if λ is a an eigenvalue of A then it is an eigenvalue of A^T. So then $A^T\mathbf{x} = \lambda\mathbf{x}$. Taking the transpose of this equation, we have: $\mathbf{x^T A} = \lambda\mathbf{x^T}$ □

Exercise 20

Proof. Let A be Hermitian and orthonormally similar to B. Therefore $A^H = H$ and $\exists$ an orthonormal matrix U such that $B = U^H AU$. Computing the hermitian of both sides we get:

$$B^H = (U^H AU)^H = U^H A^H U^{H^H} = U^H A^H U = U^H AU = B$$

Thus B is also a Hermitian matrix. □

Exercise 24

We let A be a hermitian matrix and so $A^H = A$. We notice:

$$\bar{x^H Ax} = (x^H Ax)^H = x^h A^H x = x^H Ax$$

Notice we move from to the first inequality because the transpose of a scalar. And also if the conjugate is equal to the real value then there is no imaginary part. Thus, the Rayleigh quotient can only take on real values for Hermitian matrices.
Now, we let A be a skew-Hermitian matrix. Then:

$$\bar{x^H Ax} = (x^H Ax)^H = x^h A^H x = -xAx$$

The conjugate is equal to the negated value. Which, means that there is no real part to the number; only imaginary. Thus, the Rayleigh quotient can only take on imaginary values for skew-Hermitian matrix.

Exercise 25

Exercise 27

Proof. Let $A \in M_n(\mathbb{F})$ is positive definite. Therefore, A is hermitian. So $A = A^H$. Thus the diagonal elements are real numbers. We also know that $x^T Ax > 0 \implies \langle x, Ax\rangle > 0$. This shows that the diagonals are positive. $\square$

Exercise 28

This is Albi's proof, but we worked on it together.*

By proposition 4.5.7, There exist matrices S_A and S_B such that $A = S_A^H A_A$ and $B = S_B^H S_B$. Then

$$\mathrm{Tr}(AB) = \mathrm{Tr}(S_A^H S_A S_B^H S_B) = \mathrm{Tr}(S_B S_A^H S_A S_B^H) = \mathrm{Tr}((S_A S_B^H)^H S_A S_B^H) = ||S_A S_B^H||_F^2 \geq 0.$$

By Proposition 4.5.6 $A = Q_A D_A Q_A^H$ and $B = Q_B D_B Q_D^H$, where Q_A and Q_B are orthonormal and D_A, D_B are diagonal matrices containing the eigenvalues of A and B respectively. Since the trace is invariant under orthonormal transformations we have

$$\mathrm{Tr}(AB) = \mathrm{Tr}(D_A D_B) = \sum_i \lambda_i^A \lambda_i^B \leq \left(\sum_i \lambda_i^A\right)\left(\sum_i \lambda_i^B\right) = \mathrm{Tr}(A)\mathrm{Tr}(B),$$

which concludes the proof.

Exercise 31

Exercise 32

Exercise 33

Exercise 36

Exercise 38