Cache Ellsworth

Linear Constrained Optimization Exercises

Exercise 8.1

See the jupyter notebook in the same folder.

Exercise 8.2

See the jupyter notebook in the same folder.

Exercise 8.3

The linear optimization problem in standard form which would maximize Kenny's profit on these two toys.

$$
\begin{aligned}
\text{maximize} \quad & 4s + 3d \\
\text{subject to} \quad & 15s + 10d \leq 1800 \\
& 2s + 2d \leq 350 \\
& d \leq 200 \\
& s, d \geq 0
\end{aligned}
$$

Exercise 8.4

The linear optimization problem.

$$
\begin{aligned}
\text{minimize} \quad & 2x_{AB} + 5x_{AD} + 2x_{BD} + 5x_{BC} + 7x_{BE} + 9x_{BF} + 4x_{DE} + 3x_{EF} + 2x_{CF} \\
\text{subject to} \quad & (x_{AB} + x_{AD}) - 0 = 10 \\
& (x_{BC} + x_{BF} + x_{BE} + x_{BE}) - (x_{AB}) = 1 \\
& (x_{CF}) - (x_{BC}) = -2 \\
& (x_{DE}) - (x_{AD}) = -3 \\
& (x_{EF}) - (x_{BE} + x_{DE}) = 4 \\
& 0 - (x_{CF} + x_{BF} + x_{EF}) = -10
\end{aligned}
$$

Exercise 8.5

i) The dictionary process is as follows

$$\begin{aligned}
\zeta = &\quad 3x_1 + x_2 \\ \hline
w_1 &= 15 - x_1 - 3x_2 \\
w_2 &= 18 - 2x_1 - 3x_2 \\
w_3 &= 4 - x_1 + x_2
\end{aligned}$$

where $x_1 = 0, x_2 = 0, w_1 = 15, w_2 = 18, w_3 = 4$

$$\begin{aligned}
\zeta = &\quad 12 - 3w_3 + 4x_2 \\ \hline
w_1 &= 11 - w_3 - 5x_2 \\
w_2 &= 10 + 2w_3 - 5x_2 \\
x_1 &= 4 - w_3 + x_2
\end{aligned}$$

where $x_1 = 4, x_2 = 0, w_1 = 11, w_2 = 10, w_3 = 0$

$$\begin{aligned}
\zeta = &\quad 20 - \frac{7}{5}w_3 - \frac{4}{5}w_2 \\ \hline
w_1 &= 1 - w_3 + w_2 \\
x_2 &= 2 + \frac{2}{5}w_3 - \frac{w_2}{5} \\
x_1 &= 6 - \frac{3}{5}w_3 - \frac{w_2}{5}
\end{aligned}$$

where $x_1 = 6, x_2 = 2, w_1 = 1, w_2 = 0, w_3 = 0$

This shows that the optimal point is $(6, 2)$ and the optimum value is 20. These agree with the same answers I got in Exercise 8.2. ii) The dictionary process is as follows

$$\begin{aligned}
\zeta = &\quad 4x + 6y \\ \hline
w_1 &= 11 + x - y \\
w_2 &= 27 - x - y \\
w_3 &= 90 - 2x - 5y
\end{aligned}$$

where $x = 0, y = 0, w_1 = 11, w_2 = 27, w_3 = 90$

$$\begin{aligned}
\zeta = &\quad 108 - 4w_2 + 2y \\ \hline
x &= 27 - w_2 - y \\
w_1 &= 38 - w_2 - 2y \\
w_2 &= 27 - x - y \\
w_3 &= 36 + 2w_2 - 3y
\end{aligned}$$

where $x = 27, y = 0, w_1 = 38, w_2 = 0, w_3 = 68$

$$\begin{aligned} \zeta = \quad & 132 - \frac{8}{3}w_2 - \frac{2}{3}w_3 \\ \hline x = & 15 - \frac{5}{3}w_2 + \frac{w_3}{3} \\ y = & 12 + \frac{2w_2}{3} - \frac{w_3}{3} \\ w_1 = & 14 - \frac{7}{3}w_2 + \frac{2}{3}w_3 \end{aligned}$$

where $x = 15, y = 12, w_1 = 14, w_2 = 0, w_3 = 0$
This shows that the optimal point is $(15, 12)$ and the optimum value is 132. These agree with the same answers I got in Exercise 8.2.

Exercise 8.6

The dictionary process is as follows

$$\begin{aligned} \zeta = \quad & 4s + 3d \\ \hline w_1 = & 1800 - 10d - 15s \\ w_2 = & 350 - 2d - 2s \\ w_3 = & 200 - d \end{aligned}$$

where $s = 0, d = 0, w_1 = 1800, w_2 = 350, w_3 = 200$

$$\begin{aligned} \zeta = \quad & 480 + \frac{1}{3}d - \frac{4}{15}w_1 \\ \hline s = & 120 - \frac{2}{3}d - \frac{w_1}{15} \\ w_2 = & 110 - \frac{2}{3}d + \frac{2}{15}w_1 \\ w_3 = & 200 - d \end{aligned}$$

where $s = 120, d = 0, w_1 = 0, w_2 = 110, w_3 = 200$

$$\begin{aligned} \zeta = \quad & 55 - \frac{1}{2}w_2 - \frac{1}{5}w_1 \\ \hline s = & 10 + w_2 - \frac{w_1}{5} \\ d = & 165 - \frac{3}{2}w_2 + \frac{1}{5}w_1 \\ w_3 = & 35 - \frac{3}{2}w_2 + \frac{1}{5}w_1 \end{aligned}$$

where $s = 10, d = 165, w_1 = 0, w_2 = 0, w_3 = 35$
This shows that the optimal point is $(s = 10, d = 165)$ and the optimum value is 55.

Exercise 8.7

i) The dictionary process is as follows

$$
\begin{aligned}
\zeta &= \quad -x_0 \\
\hline
x_3 &= -8 + 4x_1 + 2x_2 + x_0 \\
x_4 &= 6 + 2x_1 - 3x_2 + x_0 \\
x_5 &= 3 - x_1 + x_0
\end{aligned}
$$

where $x_0 = 0, x_1 = 0, x_2 = 0, x_3 = -8, x_4 = 6, x_5 = 3$

$$
\begin{aligned}
\zeta &= \quad -8 + 4x_1 + 2x_2 - x_3 \\
\hline
x_0 &= 8 - 4x_1 - 2x_2 + x_3 \\
x_4 &= 14 - 2x_1 - 5x_2 + x_3 \\
x_5 &= 11 - 5x_1 - 2x_2 + x_3
\end{aligned}
$$

where $x_0 = 8, x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 14, x_5 = 11$

$$
\begin{aligned}
\zeta &= \quad -x0 \\
\hline
x_1 &= 2 - \frac{1}{2}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_0 \\
x_4 &= 10 - 4x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_0 \\
x_5 &= 1 + \frac{1}{2}x_2 - \frac{1}{4}x_3 + \frac{5}{4}x_0
\end{aligned}
$$

where $x_0 = 0, x_1 = 2, x_2 = 0, x_3 = 0, x_4 = 10, x_5 = 1$
This is a feasible point.

$$
\begin{aligned}
\zeta &= \quad 2 + \frac{3}{2}x_2 + \frac{1}{4}x_3 \\
\hline
x_1 &= 2 - \frac{1}{2}x_2 + \frac{1}{4}x_3 \\
x_4 &= 10 - 4x_2 + \frac{1}{2}x_3 \\
x_5 &= 1 + \frac{1}{2}x_2 - \frac{1}{4}x_3
\end{aligned}
$$

where $x_1 = 2, x_2 = 0, x_3 = 0, x_4 = 10, x_5 = 1$

$$\begin{aligned}
\zeta = \quad & 3 + 2x_2 - x_5 \\
\hline
x_1 &= 3 - x_5 \\
x_4 &= 12 - 3x_2 - 2x_5 \\
x_3 &= 4 + 2x_2 - 4x_5
\end{aligned}$$

where $x_1 = 3, x_2 = 0, x_3 = 4, x_4 = 12, x_5 = 0$

$$\begin{aligned}
\zeta = \quad & 11 - \frac{2}{3}x_4 - \frac{7}{3}x_5 \\
\hline
x_1 &= 3 - x_5 \\
x_2 &= 4 - \frac{1}{3}x_4 - \frac{2}{3}x_5 \\
x_3 &= 4 - \frac{8}{3}x_4 - \frac{16}{3}x_5
\end{aligned}$$

where $x_1 = 3, x_2 = 4, x_3 = 4, x_4 = 0, x_5 = 0$
So the optimal point is $(x_1 = 3, x_2 = 4)$ where the optimal point is 11.
ii) The dictionary process is as follows changing it to the auxiliary problem

$$\begin{aligned}
\zeta = \quad & -x_0 \\
\hline
x_3 &= 15 - 5x_1 - 3x_2 + x_0 \\
x_4 &= 15 - 3x_1 - 5x_2 + x_0 \\
x_5 &= -12 - 4x_1 + 3x_2 + x_0
\end{aligned}$$

where $x_0 = 0, x_1 = 15, x_2 = 0, x_3 = 15, x_4 = 15, x_5 = -12$

$$\begin{aligned}
\zeta = \quad & -12 - 4x_1 + 3x_2 - x_5 \\
\hline
x_3 &= 27 - x_1 - 6x_2 + x_5 \\
x_4 &= 27 + x_1 - 8x_2 + x_5 \\
x_0 &= 12 + 4x_1 - 3x_2 + x_5
\end{aligned}$$

where $x_0 = 12, x_1 = 0, x_2 = 0, x_3 = 27, x_4 = 27, x_5 = 0$

$$
\begin{aligned}
\zeta = \quad & -\frac{15}{8} - \frac{29}{8}x_1 - \frac{3}{8}x_4 - \frac{5}{8}x_5 \\
\hline
x_3 = & \frac{27}{4} - \frac{7}{4}x_1 + \frac{3}{4}x_4 + \frac{1}{4}x_5 \\
x_2 = & \frac{27}{8} + \frac{1}{8}x_1 - \frac{1}{8}x_4 + \frac{1}{8}x_5 \\
x_0 = & \frac{15}{8} + \frac{29}{8}x_1 + \frac{3}{8}x_4 + \frac{5}{8}x_5
\end{aligned}
$$

where $x_0 = \frac{15}{8}, x_1 = 0, x_2 = \frac{27}{8}, x_3 = \frac{27}{4}, x_4 = 0, x_5 = 0$

This problem has no feasible solutions because the objective function is negative.

iii) The dictionary process is as follows

$$
\begin{aligned}
\zeta = \quad & -3x_1 + x_2 \\
\hline
w_1 = & \; 4 - x_2 \\
w_2 = & \; 6 + 2x_1 - 3x_2
\end{aligned}
$$

where $x_1 = 0, x_2 = 0, w_1 = 4, w_2 = 6$

$$
\begin{aligned}
\zeta = \quad & 2 - \frac{7}{3}x_1 - \frac{1}{3}w_2 \\
\hline
w_1 = & \; 2 - x_1 + \frac{1}{3}w_2 \\
x_2 = & \; 2 + x_1 - \frac{1}{3}w_2
\end{aligned}
$$

Thus the optimal point is $(x_1 = 0, x_2 = 2)$ where the optimal value is 2.

Exercise 8.8

$$
\begin{aligned}
\text{maximize } & 5z + x + y \\
& z \leq 4 \\
& z - x \leq 3 \\
& x, y, z \geq 0
\end{aligned}
$$

In this example, y is clearly unbounded. So the region is closed and unbounded. Also there is a unique feasible maximizer at $[1, 0, 4]^T$.

Exercise 8.9

$$
\begin{aligned}
\text{maximize } & x_1 + x_2 + x_3 \\
& x_1 \leq 1 \\
& x_2 \leq 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Exercise 8.10

$$
\begin{aligned}
\text{maximize } & x_1 + x_2 + x_3 \\
& -x_1 \leq -2 \\
& x_1 \leq 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Exercise 8.11

$$
\begin{aligned}
\text{maximize } & x_1 + x_2 + x_3 \\
& x_3 - x_2 - x_1 \leq -1 \\
& x_1 \leq 1 \\
& x_2 \leq 1 \\
& x_3 \leq 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Exercise 8.12

The dictionary process is as follows:

$$
\begin{aligned}
\zeta = &\quad 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
\hline
w_1 = &-0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
w_2 = &-0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
w_3 = &\;1 - x_1
\end{aligned}
$$

where $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, w_1 = 0, w_2 = 0, w_3 = 1$.

$$
\begin{aligned}
\zeta = &\quad -27x_2 + x_3 - 44x_4 - 20w_1 \\
\hline
x_1 = &\;3x_2 + x_3 - 2x_4 - 2w_1 \\
w_2 = &\;4x_2 + 2x_3 - 8x_4 + x_5 \\
w_3 = &\;1 - 3x_2 - x_3 + 2x_4 + 2w_1
\end{aligned}
$$

where $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, w_1 = 0, w_2 = 0, w_3 = 1$.

$$\begin{aligned} \zeta = \quad 1 - 30x_2 - 42x_4 - 18w_1 - w_3 \\ \hline x_1 = 1 - w_3 \\ w_2 = 2 - 2x_2 - 4x_4 + 5w_1 - 2w_3 \\ x_3 = 1 - 3x_2 + 2x_4 + 2w_1 - w_3 \end{aligned}$$

where $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0, w_1 = 0, w_2 = 2, w_3 = 0$.
So the optimal point is $(1, 0, 1, 0)$ with an optimal value of 1.

Exercise 8.15

Proof. Let $x \in \mathrm{R}^n$ be a primal feasible point and $y \in mathrmR^m$ is a feasible point for the dual problem. We know that $Ax \leq b$ and $A^T y \leq c$. Applying the transpose of the first equation.

$$x^T A^T \leq b^T. \implies x^T A^T y \leq x^T c \leq y^T b \implies c^T x \leq b^T y.$$

□

Exercise 8.17

Proof. We change the dual form into the standard form:

$$\begin{aligned} \text{maximize } & (-b^T)y \\ \text{subject to } & (-A)^T y \preceq -c \\ & y \succeq 0 \end{aligned}$$

Then the dual of this version (The dual of the dual) is:

$$\begin{aligned} \text{minimize } & (-c^T)x \\ \text{subject to } & ((-A)^T)^T x \succeq -b \\ & x \succeq 0 \end{aligned}$$

Changing this into the standard form.

$$\begin{aligned} \text{maximize } & c^T x \\ \text{subject to } & Ax \succeq b \\ & x \succeq 0 \end{aligned}$$

This is the original primal problem

□

Exercise 8.18

The dictionary process is as follows

$$\begin{array}{rl} \zeta = & x_1 + x_2 \\ \hline w_1 = & 3 - 2x_1 - x_2 \\ w_2 = & 5 - x_1 - 3x_2 \\ w_3 = & 4 - 2x_1 - 3x_2 \end{array}$$

where $x_1 = 0, x_2 = 0, w_1 = 3, w_2 = 5, w_3 = 4$

$$\begin{array}{rl} \zeta = & \frac{3}{2} + \frac{1}{2}x_2 - \frac{1}{2}w_1 \\ \hline x_1 = & \frac{3}{2} - \frac{1}{2}x_2 - \frac{1}{2}w_1 \\ w_2 = & \frac{7}{2} - \frac{5}{2}x_2 + \frac{1}{2}w_1 \\ w_3 = & 1 - 2x_2 + w_1 \end{array}$$

where $x_1 = \frac{3}{2}, x_2 = 0, w_1 = 0, w_2 = \frac{7}{2}, w_3 = 1$

$$\begin{array}{rl} \zeta = & \frac{7}{4} - \frac{1}{4}w_1 - \frac{1}{4}w_3 \\ \hline x_1 = & \frac{3}{2} - \frac{1}{2}x_2 - \frac{1}{2}w_1 \\ w_2 = & \frac{7}{2} - \frac{5}{2}x_2 + \frac{1}{2}w_1 \\ x_2 = & \frac{1}{2} + \frac{1}{2}w_1 - \frac{1}{2}w_3 \end{array}$$

where $x_1 = \frac{3}{2}, x_2 = \frac{1}{2}, w_1 = 0, w_2 = \frac{7}{2}, w_3 = 0$ This shows that the optimal point is $(x_1 = \frac{5}{4}, x_2 = \frac{1}{2})$ and the optimum value is $\frac{7}{4}$.

The Dual problem's dictionary process.

$$\begin{array}{rl} \text{minimize} & 3y_1 + 5y_2 + 4y_3 \\ \text{subject to} & 2y_1 + y_2 + 2y_3 \geq 1 \\ & y_1 + 3y_2 + 3y_3 \geq 1 \\ & y_1 + 3y_2 + 3y_3 \geq 1 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Dictionary process

$$\begin{array}{rl}\zeta = & -v_0 \\ \hline v_1 = & -1 + 2y_1 + y_2 + 2y_3 + v_0 \\ v_2 = & -1 + y_1 + 3y_2 + 3y_3 + v_0\end{array}$$

where $v_0 = 0, v_1 = -1, v_2 = -1, y_1 = 0, y_2 = 0, y_3 = 0$

$$\begin{array}{rl}\zeta = & -1 + 2y_1 + y_2 + 2y_3 - v_1 \\ \hline v_0 = & 1 - 2y_1 - y_2 - 2y_3 + v_1 \\ v_2 = & 2 - 5y_1 - 3y_3 + 3v_1 - 2v_0\end{array}$$

where $v_0 = 1, v_1 = 0, v_2 = 2, y_1 = 0, y_2 = 0, y_3 = 0$

$$\begin{array}{rl}\zeta = & -v_0 \\ \hline y_2 = & 1 - 2y_1 - 2y_3 + v_1 - v_0 \\ v_2 = & 2 - 5y_1 - 3y_3 + 3v_1 - 2v_0\end{array}$$

where $v_0 = 0, v_1 = 0, v_2 = 2, y_1 = 0, y_2 = 1, y_3 = 0$

$$\begin{array}{rl}\zeta = & -2 + y_1 - 3y_2 - 2v_1 \\ \hline y_3 = & \frac{1}{2} - y_1 - \frac{1}{2}y_2 + \frac{1}{2}v_1 \\ v_2 = & \frac{1}{2} - 2y_1 + \frac{3}{2}y_2 + \frac{3}{2}v_1\end{array}$$

where $v_0 = 0, v_1 = 0, v_2 = \frac{1}{2}, y_1 = 0, y_2 = 0, y_3 = \frac{1}{2}$

$$\begin{array}{rl}\zeta = & -\frac{7}{4} - \frac{3}{2}y_2 - \frac{5}{4}v_1 - \frac{1}{2}v_2 \\ \hline y_3 = & \frac{1}{4} - y_1 - \frac{7}{2}y_2 - \frac{1}{4}v_1 + \frac{1}{2}v_2 \\ y_1 = & \frac{1}{4} + \frac{3}{2}y_2 + \frac{3}{4}v_1 - \frac{1}{2}v_2\end{array}$$

where $v_0 = 0, v_1 = 0, v_2 = 0, y_1 = \frac{1}{4}, y_2 = 0, y_3 = \frac{1}{4}$
This shows that the optimal point is $(y_1 = \frac{1}{4}, y_2 = 0, y_3 = \frac{1}{4})$ and the optimum value is $-\frac{7}{4}$.

Changing this to the primal form we get the same optimum value.