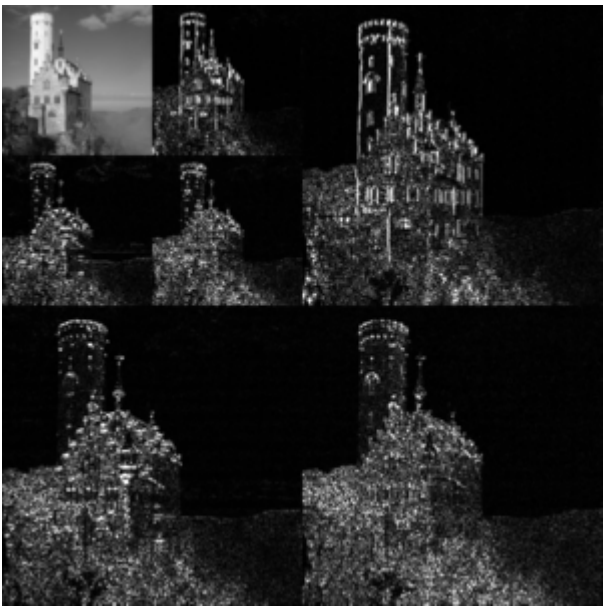


Wavelet transform

In mathematics, a **wavelet series** is a representation of a square-integrable (real- or complex-valued) function by a certain orthonormal series generated by a wavelet. This article provides a formal, mathematical definition of an **orthonormal wavelet** and of the **integral wavelet transform**.

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An example of the 2D discrete wavelet transform that is used in JPEG2000.

Definition

A function $\psi \in L^2(\mathbb{R})$ is called an **orthonormal wavelet** if it can be used to define a Hilbert basis, that is a complete orthonormal system, for the Hilbert space $L^2(\mathbb{R})$ of square integrable functions.

The Hilbert basis is constructed as the family of functions $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ by means of dyadic translations and dilations of ψ ,

$$\psi_{jk}(x) = 2^{\frac{j}{2}} \psi \left(2^j x - k \right)$$

for integers $j, k \in \mathbb{Z}$.

If under the standard inner product on $L^2(\mathbb{R})$,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$$

this family is orthonormal, it is an orthonormal system:

$$\begin{aligned}\langle \psi_{jk}, \psi_{lm} \rangle &= \int_{-\infty}^{\infty} \psi_{jk}(x) \overline{\psi_{lm}(x)} dx \\ &= \delta_{jl} \delta_{km}\end{aligned}$$

where δ_{jl} is the Kronecker delta.

Completeness is satisfied if every function $f \in L^2(\mathbb{R})$ may be expanded in the basis as

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{jk} \psi_{jk}(x)$$

with convergence of the series understood to be convergence in norm. Such a representation of f is known as a **wavelet series**. This implies that an orthonormal wavelet is self-dual.

The **integral wavelet transform** is the integral transform defined as

$$[W_{\psi} f](a, b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} \overline{\psi\left(\frac{x-b}{a}\right)} f(x) dx$$

The **wavelet coefficients** c_{jk} are then given by

$$c_{jk} = [W_{\psi} f](2^{-j}, k2^{-j})$$

Here, $a = 2^{-j}$ is called the **binary dilation** or **dyadic dilation**, and $b = k2^{-j}$ is the **binary** or **dyadic position**.

Principle

The fundamental idea of wavelet transforms is that the transformation should allow only changes in time extension, but not shape. This is affected by choosing suitable basis functions that allow for this. Changes in the time extension are expected to conform to the corresponding analysis frequency of the basis function. Based on the uncertainty principle of signal processing,

$$\Delta t \Delta \omega \geq \frac{1}{2}$$

where t represents time and ω angular frequency ($\omega = 2\pi f$, where f is temporal frequency).

The higher the required resolution in time, the lower the resolution in frequency has to be. The larger the extension of the analysis windows is chosen, the larger is the value of Δt .

When Δt is large,

1. Bad time resolution
2. Good frequency resolution
3. Low frequency, large scaling factor

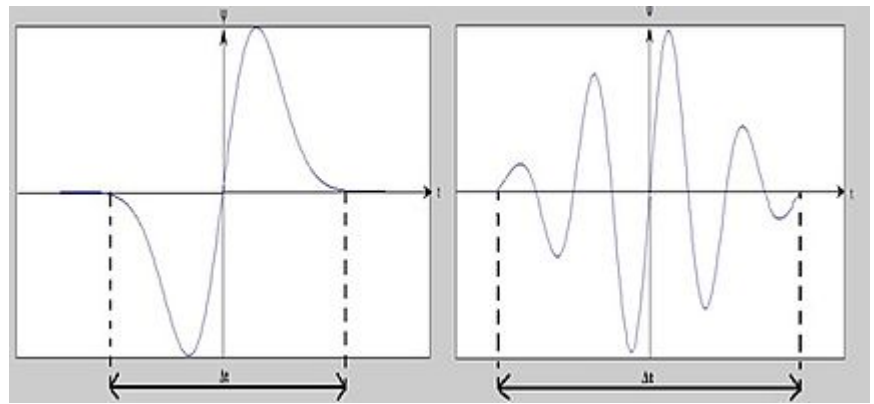
When Δt is small

1. Good time resolution

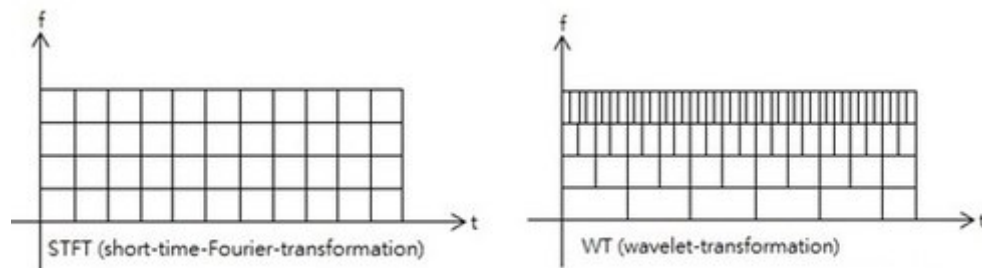
2. Bad frequency resolution

3. High frequency, small scaling factor

In other words, the basis function Ψ can be regarded as an impulse response of a system with which the function $x(t)$ has been filtered. The transformed signal provides information about the time and the frequency. Therefore, wavelet-transformation contains information similar to the short-time-Fourier-transformation,

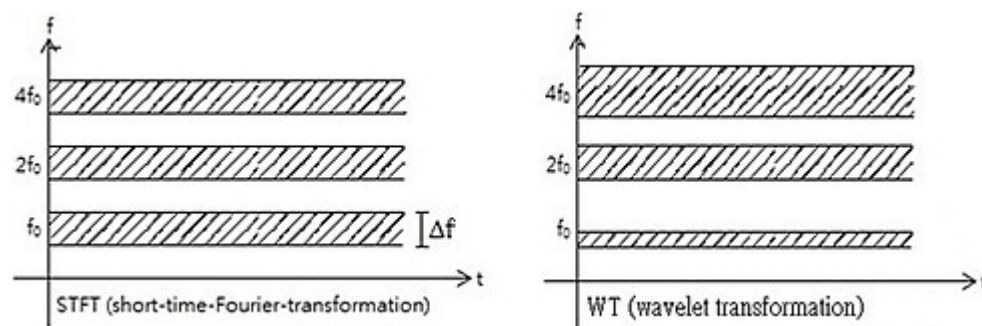


but with additional special properties of the wavelets, which show up at the resolution in time at higher analysis frequencies of the basis function. The difference in time resolution at ascending frequencies for the Fourier transform and the wavelet transform is shown below. Note however, that the frequency resolution is decreasing for increasing frequencies while the temporal resolution increases. This consequence of Heisenberg's uncertainty principle is not correctly displayed in the Figure.



This shows that wavelet transformation is good in time resolution of high frequencies, while for slowly varying functions, the frequency resolution is remarkable.

Another example: The analysis of three superposed sinusoidal signals $y(t) = \sin(2\pi f_0 t) + \sin(4\pi f_0 t) + \sin(8\pi f_0 t)$ with STFT and wavelet-transformation.



Wavelet compression

Wavelet compression is a form of data compression well suited for image compression (sometimes also video compression and audio compression). Notable implementations are JPEG 2000, DjVu and ECW for still images, CineForm, and the BBC's Dirac. The goal is to store image data in as little space as possible in a file. Wavelet compression can be either lossless or lossy.^[1] Wavelet coding is a variant of discrete cosine transform (DCT) coding that uses wavelets instead of DCT's block-based algorithm.^[2]

Using a wavelet transform, the wavelet compression methods are adequate for representing transients, such as percussion sounds in audio, or high-frequency components in two-dimensional images, for example an image of stars on a night sky. This means that the transient elements of a data signal can be represented by a smaller amount of information than would be the case if some other transform, such as the more widespread discrete cosine transform, had been used.

Discrete wavelet transform has been successfully applied for the compression of electrocardiograph (ECG) signals^[3] In this work, the high correlation between the corresponding wavelet coefficients of signals of successive cardiac cycles is utilized employing linear prediction.

Wavelet compression is not good for all kinds of data: transient signal characteristics mean good wavelet compression, while smooth, periodic signals are better compressed by other methods, particularly traditional harmonic compression (frequency domain, as by Fourier transforms and related).

See Diary Of An x264 Developer: The problems with wavelets (<https://web.archive.org/web/20100228145846/http://x264dev.multimedia.cx/?p=317>) (2010) for discussion of practical issues of current methods using wavelets for video compression.

Method

First a wavelet transform is applied. This produces as many coefficients as there are pixels in the image (i.e., there is no compression yet since it is only a transform). These coefficients can then be compressed more easily because the information is statistically concentrated in just a few coefficients. This principle is called transform coding. After that, the coefficients are quantized and the quantized values are entropy encoded and/or run length encoded.

A few 1D and 2D applications of wavelet compression use a technique called "wavelet footprints".^{[4][5]}

Comparison with Fourier transform and time-frequency analysis

Transform	Representation	Input
Fourier transform	$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} dx$	ξ , frequency
Time-frequency analysis	$X(t, f)$	t , time; f , frequency
Wavelet transform	$X(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \overline{\Psi\left(\frac{t-b}{a}\right)} x(t) dt$	a , scaling; b , time

Wavelets have some slight benefits over Fourier transforms in reducing computations when examining specific frequencies. However, they are rarely more sensitive, and indeed, the common Morlet wavelet is mathematically identical to a short-time Fourier transform using a Gaussian window function.^[6] The exception is when searching for signals of a known, non-sinusoidal shape (e.g., heartbeats); in that case, using matched wavelets can outperform standard STFT/Morlet analyses.^[7]

Other practical applications

The wavelet transform can provide us with the frequency of the signals and the time associated to those frequencies, making it very convenient for its application in numerous fields. For instance, signal processing of accelerations for gait analysis,^[8] for fault detection,^[9] for design of low power pacemakers and also in ultra-wideband (UWB) wireless communications.^{[10][11][12]}

1. Discretizing of the c-τ-axis

Applied the following discretization of frequency and time:

$$\begin{aligned}c_n &= c_0^n \\ \tau_m &= m \cdot T \cdot c_0^n\end{aligned}$$

Leading to wavelets of the form, the discrete formula for the basis wavelet:

$$\Psi(k, n, m) = \frac{1}{\sqrt{c_0^n}} \cdot \Psi \left[\frac{k - mc_0^n}{c_0^n} T \right] = \frac{1}{\sqrt{c_0^n}} \cdot \Psi \left[\left(\frac{k}{c_0^n} - m \right) T \right]$$

Such discrete wavelets can be used for the transformation:

$$Y_{DW}(n, m) = \frac{1}{\sqrt{c_0^n}} \cdot \sum_{k=0}^{K-1} y(k) \cdot \Psi \left[\left(\frac{k}{c_0^n} - m \right) T \right]$$

2. Implementation via the FFT (fast Fourier transform)

As apparent from wavelet-transformation representation (shown below)

$$Y_W(c, \tau) = \frac{1}{\sqrt{c}} \cdot \int_{-\infty}^{\infty} y(t) \cdot \Psi \left(\frac{t - \tau}{c} \right) dt$$

where c is scaling factor, τ represents time shift factor

and as already mentioned in this context, the wavelet-transformation corresponds to a convolution of a function y(t) and a wavelet-function. A convolution can be implemented as a multiplication in the frequency domain. With this the following approach of implementation results into:

- Fourier-transformation of signal y(k) with the FFT
- Selection of a discrete scaling factor c_n
- Scaling of the wavelet-basis-function by this factor c_n and subsequent FFT of this function
- Multiplication with the transformed signal YFFT of the first step
- Inverse transformation of the product into the time domain results in $Y_W(c, \tau)$ for different discrete values of τ and a discrete value of c_n
- Back to the second step, until all discrete scaling values for c_n are processed

There are many different types of wavelet transforms for specific purposes. See also a full list of wavelet-related transforms but the common ones are listed below: Mexican hat wavelet, Haar Wavelet, Daubechies wavelet, triangular wavelet.

See also

- Continuous wavelet transform

- Discrete wavelet transform
- Complex wavelet transform
- Constant-Q transform
- Stationary wavelet transform
- Dual wavelet
- Multiresolution analysis
- MrSID, the image format developed from original wavelet compression research at Los Alamos National Laboratory (LANL).
- ECW, a wavelet-based geospatial image format designed for speed and processing efficiency
- JPEG 2000, a wavelet-based image compression standard
- DjVu format uses wavelet-based IW44 algorithm for image compression
- scaleograms, a type of spectrogram generated using wavelets instead of a short-time Fourier transform.
- Wavelet
- Haar wavelet
- Daubechies wavelet
- Morlet wavelet
- Gabor wavelet
- Chirplet transform
- Time-frequency representation
- S transform
- Set partitioning in hierarchical trees
- Short-time Fourier transform

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External links

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