Definitions

$$
\begin{aligned}
&H = \sum_{j,m} \epsilon_j a^\dagger_{j,m} a_{j,m} - \frac{1}{4}\sum_{j,j'} (-1)^{j+j'-m-m'} g_{j,j'} a^\dagger_{j,m} a^\dagger_{j,m-} a_{j',-m'} a_{j',m'} \\
&S_+ = \frac{1}{2}\sum_{j,m} (-1)^{j-m} a^\dagger_{j,m} a^\dagger_{j,-m} \\
&S_- = \frac{1}{2}\sum_{j,m} (-1)^{j-m} a_{j,-m} a_{j,m} \\
&S_0 = \frac{1}{4}\sum_{j,m} \left(a^\dagger_{j,m} a_{j,m} + a^\dagger_{j,-m} a_{j,-m} - 1 \right) = \frac{1}{2}(N - \Omega) \\
&\tilde{a}_m = (-1)^{j-m} a_{-m} \\
&\Omega = \sum_j (j + \frac{1}{2})
\end{aligned}
$$

Useful Relations

$$
\begin{aligned}
[a_{j',m'}, a^\dagger_{j,m}] &= \delta_{j',j}(2a_{j',m'} a^\dagger_{j,m} - \delta_{m',m}) \\
[a_{j',m'}, a_{j,m}] &= \delta_{j',j}(2a_{j',m'} a_{j,m}) \\
[a^\dagger_{j',m'}, a^\dagger_{j,m}] &= \delta_{j',j}(2a^\dagger_{j',m'} a^\dagger_{j,m})
\end{aligned}
$$

$$
\begin{aligned}
[a^\dagger_{j',m'} a^\dagger_{j',-m'}, a^\dagger_{j,m}] &= \delta_{j',j}(2a^\dagger_{j',m'}(a^\dagger_{j',-m'} a^\dagger_{j,m} + a^\dagger_{j,m} a^\dagger_{j',-m'})) \\
&= 0 \\
[a_{j',m'} a_{j',-m'}, a_{j,m}] &= \delta_{j',j}(2a_{j',-m'}(a_{j',m'} a_{j,m} + a_{j,m} a_{j',m'})) \\
&= 0 \\
[a_{j',-m'} a_{j',m'}, a^\dagger_{j,m}] &= \delta_{j',j}(\delta_{m,m'} a_{j',-m'} - \delta_{m,-m'} a_{j',m'}) \\
[a^\dagger_{j',m'} a^\dagger_{j',-m'}, a_{j,m}] &= \delta_{j',j}(\delta_{m,-m'} a^\dagger_{j',m'} - \delta_{m,m'} a_{j',-m'}) \\
[a^\dagger_{j',m'} a_{j',m'}, a^\dagger_{j,m}] &= a^\dagger_{j',m'} \delta_{m',m} \delta_{j',j} \\
[a^\dagger_{j',m'} a_{j',m'}, \tilde{a}_{j,m}] &= -(-1)^{j-m} a_{j',m'} \delta_{m',-m} \delta_{j',j} \\
(-1)^{j+m} &= -(-1)^{j-m}
\end{aligned}
$$

Commutation Relations

Relations for $a^\dagger$

$$
\begin{aligned}
[S_+, a^\dagger_{j,m}] &= \frac{1}{2}\sum_{j',m'} (-1)^{j'-m'}[a^\dagger_{j',m'}a^\dagger_{j',-m'}, a^\dagger_{j,m}] \\
&= \boxed{0} \\
[S_-, a^\dagger_{j,m}] &= \frac{1}{2}\sum_{j',m'} (-1)^{j'-m'}[a_{j',-m'}a_{j',m'}, a^\dagger_{j,m}] \\
&= \frac{1}{2}\sum_{j',m'} (-1)^{j'-m'}\delta_{j',j}(\delta_{m,m'}a_{j,-m'} - \delta_{m,-m'}a_{j,m'}) \\
&= \frac{1}{2}\sum_{m'} (-1)^{j-m'}(\delta_{m,m'}a_{j,-m'} - \delta_{m,-m'}a_{j,m'}) \\
&= \frac{1}{2}((-1)^{j-m}a_{j,-m} - (-1)^{j+m}a_{j,-m}) \\
&= \frac{1}{2}(\tilde{a}_{j,m} - ((-1)^{2m}(-1)^{j-m}a_{j,-m})) \\
&= \frac{1}{2}(\tilde{a}_{j,m} + \tilde{a}_{j,m}) \\
&= \boxed{\tilde{a}_{j,m}} \\
[S_0, a^\dagger_{j,m}] &= \frac{1}{4}\sum_{j',m'} ([a^\dagger_{j',m'}a_{j',m'}, a^\dagger_{j,m}] + [a^\dagger_{j',-m'}a_{j',-m'}, a^\dagger_{j,m}] + [1, a^\dagger_{j,m}]) \\
&= \frac{1}{4}\sum_{j',m'} (a^\dagger_{j',m'}\delta_{m',m}\delta_{j',j} + a^\dagger_{j',-m'}\delta_{m',-m}\delta_{j',j}) \\
&= \frac{1}{4}\sum_{m'} (a^\dagger_{j,m'}\delta_{m',m} + a^\dagger_{j,-m'}\delta_{m',-m}) \\
&= \frac{1}{4}(a^\dagger_{j,m} + a^\dagger_{j,m}) \\
&= \boxed{\frac{1}{2}a^\dagger_{j,m}}
\end{aligned}
$$

Relations for $\tilde{a}$

$$
\begin{aligned}
(1)\quad [S_-, \tilde{a}_{j,m}] &= \frac{1}{2}\sum_{j',m'}(-1)^{j'-m'}[a_{j',-m'}a_{j',m'}, \tilde{a}_{j,m}] \\
&= \boxed{0} \\
(2)\quad [S_+, \tilde{a}_{j,m}] &= \frac{1}{2}\sum_{j',m'}(-1)^{j'-m'}[a^\dagger_{j',m'}a^\dagger_{j',-m'}, \tilde{a}_{j,m}] \\
&= \frac{1}{2}\sum_{j',m'}(-1)^{j'-m'}(-1)^{j'-m}[a^\dagger_{j',m'}a^\dagger_{j',-m'}, a_{j,-m}] \\
&= \frac{1}{2}\sum_{j',m'}(-1)^{j'-m'}(-1)^{j-m}\delta_{j',j}(\delta_{-m,-m'}a^\dagger_{j',m'} - \delta_{-m,m'}a^\dagger_{j',-m'}) \\
&= \frac{1}{2}((-1)^{j-m}(-1)^{j-m}a^\dagger_{j,m} - (-1)^{j+m}(-1)^{j-m}a^\dagger_{j,m}) \\
&= \frac{1}{2}(a^\dagger_{j,m} + a^\dagger_{j,m}) \\
&= \boxed{a^\dagger_{j,m}} \\
(3)\quad [S_0, \tilde{a}_{j,m}] &= \frac{1}{4}\sum_{j',m'}([a^\dagger_{j',m'}a_{j',m'}, \tilde{a}_{j,m}] + [a^\dagger_{j',-m'}a_{j',-m'}, \tilde{a}_{j,m}] + [1, \tilde{a}_{j,m}]) \\
&= -\frac{1}{4}\sum_{j',m'}((-1)^{j-m}a_{j',m'}\delta_{m',-m}\delta_{j',j} + (-1)^{j-m}a_{j',-m'}\delta_{-m',-m}\delta_{j',j}) \\
&= -\frac{1}{4}((-1)^{j-m}a_{j,-m} + (-1)^{j-m}a_{j,-m}) \\
&= -\frac{1}{4}(\tilde{a}_{j,m} + \tilde{a}_{j,m}) \\
&= \boxed{-\frac{1}{2}\tilde{a}_{j,m}}
\end{aligned}
$$

Generator Association

$$
\begin{aligned}
(1)\quad [S_0, S_+] &= [\frac{1}{4}\sum_{j',m'}(a^\dagger_{j',m'}a_{j',m'} + a^\dagger_{j',-m'}a_{j',-m'} - 1), S_+] \\
&= \frac{1}{4}\left(\sum_{j',m'}[a^\dagger_{j',m'}a_{j',m'}, S_+] + [a^\dagger_{j',-m'}a_{j',-m'}, S_+]\right) \\
&= \frac{1}{4}\left(\sum_{j',m'} -(-1)^{j'+m'}a^\dagger_{j',m'}a^\dagger_{j',-m'} + \sum_{j',m'} -(-1)^{j'-m'}a^\dagger_{j',-m'}a^\dagger_{j',m'}\right) \\
&= \frac{1}{4}\left(\sum_{j',m'} (-1)^{j'-m'}a^\dagger_{j',m'}a^\dagger_{j',-m'} + \sum_{j',m'} (-1)^{j'+m'}a^\dagger_{j',-m'}a^\dagger_{j',m'}\right)
\end{aligned}
$$

As these two sums run from $m = -j$ to $m = j$, the two summations are equivalent, thus we have

$$
\begin{aligned}
[S_0, S_+] &= \frac{1}{2}\sum_{j',m'}(-1)^{j'-m'}a^\dagger_{j',m'}a^\dagger_{j',-m'} \\
&= \boxed{S_+}
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad [S_0, S_-] &= [\frac{1}{4}\sum_{j',m'}(a^\dagger_{j',m'}a_{j',m'} + a^\dagger_{j',-m'}a_{j',-m'} - 1), S_0] \\
&= \frac{1}{4}(\sum_{j',m'}[a^\dagger_{j',m'}a_{j',m'}, S_0] + [a^\dagger_{j',-m'}a_{j',-m'}, S_0]) \\
&= \frac{1}{4}(\sum_{j',m'} -\tilde{a}_{j',m'}a_{j',m'} + \sum_{j',m'} -\tilde{a}_{j',-m'}a_{j',-m'}) \\
&= -\frac{1}{4}(\sum_{j',m'}(-1)^{j'-m'}a_{j',-m'}a_{j',m'} + \sum_{j',m'}(-1)^{j'+m'}a_{j',m'}a_{j',-m'}) \\
&= -\frac{1}{2}\sum_{j',m'}(-1)^{j'-m'}a_{j',-m'}a_{j',m'} \\
&= \boxed{-S_-}
\end{aligned}
$$

$$
\begin{aligned}
3)\quad [S_+, S_-] &= [\frac{1}{2}\sum_{j',m'}(-1)^{j-m}a^\dagger_{j',m'}a^\dagger_{j',-m'}, S_-] \\
&= \frac{1}{2}\sum_{j',m'}(-1)^{j-m}[a^\dagger_{j',m'}a^\dagger_{j',-m'}, S_-] \\
&= \frac{1}{2}\sum_{j',m'} -(-1)^{j'-m'}(\tilde{a}_{j',m'}a^\dagger_{j',-m'} + a^\dagger_{j',m'}\tilde{a}_{j',-m'}) \\
&= \frac{1}{2}\sum_{j',m'} -(-1)^{j'-m'}((-1)^{j'-m'}a_{j',-m'}a^\dagger_{j,-m'} + (-1)^{j'+m'}a^\dagger_{j',m'}a_{j',m'}) \\
&= \frac{1}{2}\sum_{j',m'}(a^\dagger_{j',m'}a_{j',m'} + a^\dagger_{j',-m'}a_{j',-m'} - 1) \\
&= \boxed{2S_0}
\end{aligned}
$$

This tells us that S_+, S_- and S_0 are the generators of $SU(2)$, allowing us to interpret the commutation relations of $a^\dagger$ and $\tilde{a}$ with these generators as those of a spinor with $s = 1/2$

Matrix Element Calculations

As $a^\dagger_{j,m}$ and $\tilde{a}_{j,m}$ are spinors under the quasi-spin operators, we know they have a quasi-spin of 1/2, meaning we can treat them as up/down "states".

One-Body Operators

In the pairing model, one-body operators are of the form $a^\dagger_{j,m}a_{j,m} = (-1)^{j+m}a^\dagger_{j,m}\tilde{a}_{j,-m}$. Using the fact that $a^\dagger_{j,m} \cong |+\rangle$ and $\tilde{a}_{j,-m} \cong |-\rangle$, we can decompose the product $a^\dagger_{j,m}\tilde{a}_{j,-m}$ into a sum of spin 0 and spin 1 operators,

$$
a^\dagger_{j,m}\tilde{a}_{j,-m} = \frac{1}{\sqrt{2}}(\hat{O}_{1,0} + \hat{O}_{0,0}),
$$

where $\hat{O}_{i,j}$ denotes an operator with quasi-spin i and projection j. For the seniority scheme we are interested in one-body matrix elements of the form $\langle\Omega/2, S_0|a^\dagger_{j,m}\tilde{a}_{j,-m}|\Omega/2, S_0\rangle$, where $|\Omega/2, S_0\rangle$ is a state of seniority 0, and thus has quasi spin $S = \Omega/2$ and projection $S_0 = N/2 - \Omega/2$. Using our above decomposition, we have this being

$$
\langle\Omega/2, S_0|a^\dagger_{j,m}\tilde{a}_{j,-m}|\Omega/2, S_0\rangle = \frac{1}{\sqrt{2}}(\langle\Omega/2, S_0|\hat{O}_{1,0}|\Omega/2, S_0\rangle + \langle\Omega/2, S_0|\hat{O}_{0,0}|\Omega/2, S_0\rangle)),
$$

We can determine these matrix elements by using the Wigner-Eckart theorem, which yields

$$\langle \Omega/2, S_0 | a^\dagger_{j,m} \tilde{a}_{j,-m} | \Omega/2, S_0 \rangle = \frac{1}{\sqrt{2}} (\alpha(1) \langle \Omega/2 | \, |\hat{O}_1| \, |\Omega/2\rangle + \alpha(0) \langle \Omega/2 | \, |\hat{O}_0| \, |\Omega/2\rangle),$$

where $\alpha(i) = \langle \Omega/2, S_0; i, 0 | \Omega/2, S_0 \rangle$ are the Clebsch–Gordan coefficients, with $\alpha(0) = 1$. To find the reduced matrix elements $\langle \Omega/2 | \, |\hat{O}_i| \, |\Omega/2\rangle$, we can exploit their invariance with respect to S_0 to choose the vacuum case of $N = 0 \implies S_0 = -\Omega/2$. Doing so for the case of $\hat{O}_1 = \hat{O}_{1,0}$, where we denote $|\Omega/2, -\Omega/2\rangle = |\emptyset\rangle$ as the vacuum state, yields

$$\begin{aligned}
\langle \emptyset | \hat{O}_{1,0} | \emptyset \rangle &= \frac{1}{\sqrt{2}} \langle \emptyset | (a^\dagger_{j,m} \tilde{a}_{j',m'} + \tilde{a}_{j,m} a^\dagger_{j',m'}) | \emptyset \rangle \\
&= \frac{1}{\sqrt{2}} \langle \emptyset | \tilde{a}_{j,m} a^\dagger_{j',m'} | \emptyset \rangle \\
&= \frac{(-1)^{j-m}}{\sqrt{2}} \langle j, -m | j', m' \rangle \\
&= \frac{(-1)^{j-m}}{\sqrt{2}} \delta_{j,j'} \delta_{m',-m}
\end{aligned}$$

Thus, $\langle \Omega/2, -\Omega/2; 1, 0 | \Omega/2, -\Omega/2 \rangle \langle \Omega/2 | \, |\hat{O}_1| \, |\Omega/2\rangle = \frac{(-1)^{j-m}}{\sqrt{2}} \delta_{j,j'} \delta_{m',-m}$. An analogous calculation gives

$$\begin{aligned}
\langle \emptyset | \hat{O}_{0,0} | \emptyset \rangle &= \frac{1}{\sqrt{2}} \langle \emptyset | (a^\dagger_{j,m} \tilde{a}_{j',m'} - \tilde{a}_{j,m} a^\dagger_{j',m'}) | \emptyset \rangle \\
&= -\frac{1}{\sqrt{2}} \langle \emptyset | \tilde{a}_{j,m} a^\dagger_{j',m'} | \emptyset \rangle \\
&= -\frac{(-1)^{j-m}}{\sqrt{2}} \langle j', -m | j', m' \rangle \\
&= -\frac{(-1)^{j-m}}{\sqrt{2}} \delta_{j,j'} \delta_{-m,m'}
\end{aligned}$$

meaning $\langle \Omega/2, -\Omega/2; 0, 0 | \Omega/2, -\Omega/2 \rangle \langle \Omega/2 | \, |\hat{O}_0| \, |\Omega/2\rangle = -\frac{(-1)^{j-m}}{\sqrt{2}} \delta_{j,j'} \delta_{-m,m'}$. Of course, $\langle \Omega/2, -\Omega/2; 0, 0 | \Omega/2, -\Omega/2 \rangle = 1$.

Combining this with our previous result thus tells us that

$$\langle \Omega/2, S_0 | a^\dagger_{j,m} \tilde{a}_{j',m'} | \Omega/2, S_0 \rangle = \frac{(-1)^{j-m}}{2} \delta_{j,j'} \delta_{-m,m'} \left(\frac{\langle \Omega/2, S_0; 1, 0 | \Omega/2, S_0 \rangle}{\langle \Omega/2, -\Omega/2; 1, 0 | \Omega/2, -\Omega/2 \rangle} - 1 \right)$$

For the case of the pairing model, where $m' = -m$ and $j' = j$, we have the one-body matrix elements being

$$\begin{aligned}
\langle \Omega/2, S_0 | a^\dagger_{j,m} a_{j,m} | \Omega/2, S_0 \rangle &= (-1)^{j+m} \langle \Omega/2, S_0 | a^\dagger_{j,m} \tilde{a}_{j',-m} | \Omega/2, S_0 \rangle \\
&= -\frac{1}{2} \left(\frac{\langle \Omega/2, S_0; 1, 0 | \Omega/2, S_0 \rangle}{\langle \Omega/2, -\Omega/2; 1, 0 | \Omega/2, -\Omega/2 \rangle} - 1 \right)
\end{aligned}$$

We can simplify this result by considering the number operator

$$\hat{N} = \sum_{j,m} a^\dagger_{j,m} a_{j,m}$$

and noting that

$$\begin{aligned}
\langle \Omega/2, S_0 | \hat{N} | \Omega/2, S_0 \rangle &= \sum_{j,m} \langle \Omega/2, S_0 | a^\dagger_{j,m} a_{j,m} | \Omega/2, S_0 \rangle \\
&= -\frac{1}{2} \sum_{j,m} \left(\frac{\langle \Omega/2, S_0; 1, 0 | \Omega/2, S_0 \rangle}{\langle \Omega/2, -\Omega/2; 1, 0 | \Omega/2, -\Omega/2 \rangle} - 1 \right) \\
&= -\left(\frac{\langle \Omega/2, S_0; 1, 0 | \Omega/2, S_0 \rangle}{\langle \Omega/2, -\Omega/2; 1, 0 | \Omega/2, -\Omega/2 \rangle} - 1 \right) \left(\frac{1}{2} \sum_{j,m} 1 \right) \\
&= -\left(\frac{\langle \Omega/2, S_0; 1, 0 | \Omega/2, S_0 \rangle}{\langle \Omega/2, -\Omega/2; 1, 0 | \Omega/2, -\Omega/2 \rangle} - 1 \right) \Omega,
\end{aligned}$$

where $\frac{1}{2}\sum_{j,m} 1 = \sum_j (j+1/2) = \Omega$. As this summation is simply N (due to $\hat{N}$ being the number-operator acting on an N-particle state), we have that

$$\begin{aligned} -\left(\frac{\langle\Omega/2, S_0; 1, 0|\Omega/2, S_0\rangle}{\langle\Omega/2, -\Omega/2; 1, 0|\Omega/2, -\Omega/2\rangle} - 1\right)\Omega &= N \\ \implies -\left(\frac{\langle\Omega/2, S_0; 1, 0|\Omega/2, S_0\rangle}{\langle\Omega/2, -\Omega/2; 1, 0|\Omega/2, -\Omega/2\rangle} - 1\right) &= \frac{N}{\Omega}. \end{aligned}$$

Thus, we have the simplifed result

$$\langle\Omega/2, S_0|a^\dagger_{j,m} a_{j,m}|\Omega/2, S_0\rangle = \frac{N}{2\Omega}$$

The more general form of this simplifed result is

$$\langle\Omega/2, S_0|a^\dagger_{j,m} a_{j',m'}|\Omega/2, S_0\rangle = \frac{N}{2\Omega}\delta_{j,j'}\delta_{m,m'}$$

Simpler Calculation

Exploiting commutation relations and properties of S_+ and S_- allows us to compute the above matrix elements much more efficiently. Using $|N_i\rangle = |\frac{1}{2}\Omega, \frac{1}{2}(N_i - \Omega)\rangle$, we have

$$\begin{aligned} \langle N|a^\dagger_{j,m} a_{j',m'}|N\rangle &\propto \langle N|a^\dagger_{j,m} a_{j',m'} S_+^{N/2}|0\rangle \\ &= \langle N|[a^\dagger_{j,m} a_{j',m'}, S_+^{N/2}]|0\rangle \\ &= \langle N|a^\dagger_{j,m}[a_{j',m'}, S_+^{N/2}]|0\rangle \\ &= \langle N|a^\dagger_{j,m}(-(-1)^{j'+m'}\frac{N}{2})a^\dagger_{j',-m'} S_+^{N/2-1}|0\rangle \\ &= -(-1)^{j'+m'}\frac{N}{2}\langle N|a^\dagger_{j,m} a^\dagger_{j',-m'} S_+^{N/2-1}|0\rangle \\ &= -(-1)^{j'+m'}\frac{N}{2}\langle N|S_+^{N/2-1} a^\dagger_{j,m} a^\dagger_{j',-m'}|0\rangle \\ &\propto -(-1)^{j'+m'}\frac{N}{2}\langle 2|a^\dagger_{j,m} a^\dagger_{j',-m'}|0\rangle \\ &\propto -(-1)^{j'+m'}\frac{N}{2}\langle 0|S_- a^\dagger_{j,m} a^\dagger_{j',-m'}|0\rangle \\ &= -(-1)^{j'+m'}\frac{N}{2}\langle 0|[S_- a^\dagger_{j,m}] a^\dagger_{j',-m'}|0\rangle \\ &= -(-1)^{j'+m'}\frac{N}{2}\langle 0|\tilde{a}_{j,m} a^\dagger_{j',-m'}|0\rangle \\ &= -(-1)^{j'+m'}(-1)^{j-m}\frac{N}{2}\langle j, -m|j', -m'\rangle \\ &= -(-1)^{j'+m'}(-1)^{j-m}\frac{N}{2}\delta_{j,j'}\delta_{m,m'} \\ &= \frac{N}{2}\delta_{j,j'}\delta_{m,m'} \end{aligned}$$

Thus, we have $\langle N|a^\dagger_{j,m} a_{j',m'}|N\rangle = \alpha\frac{N}{2}\delta_{j,j'}\delta_{m,m'}$. While we could determine the constant of proportionality α by tracking the coefficients induced by repeated applications of S_+ and S_-, a more straightforward way is to utilize the number operator $\hat{N} = \sum_{j,m} a^\dagger_{j,m} a_{j,m}$, which has an expectation value given by

$$\begin{aligned} \langle N|\hat{N}|N\rangle &= \sum_{j,m}\langle N|a^\dagger_{j,m} a_{j,m}|N\rangle \\ &= \alpha\frac{N}{2}\sum_{j,m} 1 \\ &= \alpha\frac{N}{2}(2\Omega) \\ &= \alpha N\Omega \end{aligned}$$

As $\langle N|\hat{N}|N\rangle = N$, we thus have $\alpha N\Omega = N \implies \alpha = 1/\Omega$. Thus, our final result is

$$\langle N|a^\dagger_{j,m}a_{j',m'}|N\rangle = \frac{N}{2\Omega}\delta_{j,j'}\delta_{m,m'}$$

Two-Body Operators

Now let's consider the two-body matrix elements. In the pairing Hamiltonian, the two-body operators are of the form $a^\dagger_{j,m}a^\dagger_{j',m'}a_{j'',m''}a_{j''',m'''}$. Using associativity of tensor products, we get this being decomposed as

$$\begin{aligned}
a^\dagger_{j,m}a^\dagger_{j',m'}a_{j'',m''}a_{j''',m'''} &= (-1)^{j''+j'''}(-1)^{m''+m'''}a^\dagger_{j,m}a^\dagger_{j',m'}\tilde{a}_{j'',-m''}\tilde{a}_{j''',-m'''} \\
&= (-1)^{j''+j'''}(-1)^{m''+m'''}\hat{O}_{1,1}\hat{O}_{1,-1} \\
&= (-1)^{j''+j'''}(-1)^{m''+m'''}\left(\frac{1}{\sqrt{6}}\hat{O}_{2,0} + \frac{1}{\sqrt{2}}\hat{O}_{1,0} + \frac{1}{\sqrt{3}}\hat{O}_{0,0}\right)
\end{aligned}$$

By linearity of the inner product we thus know that

$$\begin{aligned}
\langle \Omega/2, S_0|a^\dagger_{j,m}a^\dagger_{j',m'}a_{j'',m''}a_{j''',m'''}|\Omega/2, S_0\rangle = (-1)^{j''+j'''}(-1)^{m''+m'''}\Bigg(\frac{1}{\sqrt{6}}\langle \Omega/2, S_0|\hat{O}_{2,0}|\Omega/2, S_0\rangle + \\
\frac{1}{\sqrt{2}}\langle \Omega/2, S_0\hat{O}_{1,0}|\Omega/2, S_0\rangle + \langle \Omega/2, S_0\frac{1}{\sqrt{3}}\hat{O}_{0,0}|\Omega/2, S_0\rangle\Bigg)
\end{aligned}$$

meaning our goal is to find $\langle \Omega/2, S_0|\hat{O}_{2,0}|\Omega/2, S_0\rangle$, $\langle \Omega/2, S_0\hat{O}_{1,0}|\Omega/2, S_0\rangle$ and $\langle \Omega/2, S_0\frac{1}{\sqrt{3}}\hat{O}_{0,0}|\Omega/2, S_0\rangle$.

To do so, let us work backwards and decompose our spin 2,1,0 tensors in terms of our original spin-1/2 states (ie, $a^\dagger_{j,m}$, $a^\dagger_{j',m'}$, $\tilde{a}_{j'',-m''}$ and $\tilde{a}_{j''',-m'''}$). For $\hat{O}_{2,0}$ we have that

$$\begin{aligned}
\hat{O}_{2,0} &= \frac{1}{\sqrt{6}}\hat{O}_{1,1}\hat{O}_{1,-1} + \sqrt{\frac{2}{3}}\hat{O}_{1,0}\hat{O}_{1,0} + \frac{1}{\sqrt{6}}\hat{O}_{1,-1}\hat{O}_{1,1} \\
&= \frac{1}{\sqrt{6}}a^\dagger_{j,m}a^\dagger_{j',m'}\tilde{a}_{j'',-m''}\tilde{a}_{j''',-m'''} + \sqrt{\frac{2}{3}}\hat{O}_{1,0}\hat{O}_{1,0} + \frac{1}{\sqrt{6}}\tilde{a}_{j,m}\tilde{a}_{j',m'}a^\dagger_{j'',-m''}a^\dagger_{j''',-m'''}
\end{aligned}$$

For the middle term we know that

$$\begin{aligned}
\hat{O}_{1,0}\hat{O}_{1,0} = \frac{1}{2}(&a^\dagger_{j,m}\tilde{a}_{j',m'}\tilde{a}_{j'',-m''}a^\dagger_{j''',-m'''} + a^\dagger_{j,m}\tilde{a}_{j',m'}a^\dagger_{j'',-m''}\tilde{a}_{j''',-m'''} \\
&+ \tilde{a}_{j,m}a^\dagger_{j',m'}a^\dagger_{j'',-m''}\tilde{a}_{j''',-m'''} + \tilde{a}_{j,m}a^\dagger_{j',m'}\tilde{a}_{j'',-m''}a^\dagger_{j''',-m'''})
\end{aligned}$$

The vacuum-state matrix element then is

$$\begin{aligned}
\langle\emptyset|\hat{O}_{2,0}|\emptyset\rangle = &\frac{1}{\sqrt{6}}\langle\emptyset|a^\dagger_{j,m}a^\dagger_{j,-m}\tilde{a}_{j',-m'}\tilde{a}_{j',m'}|\emptyset\rangle + \sqrt{\frac{2}{3}}\frac{1}{2}\langle\emptyset|\tilde{a}_{j,m}a^\dagger_{j',m'}\tilde{a}_{j'',-m''}a^\dagger_{j''',-m'''}|\emptyset\rangle \\
&+ \frac{1}{\sqrt{6}}\langle\emptyset|\tilde{a}_{j,m}\tilde{a}_{j',m'}a^\dagger_{j'',-m''}a^\dagger_{j''',-m'''}|\emptyset\rangle \\
&= 0 + \sqrt{\frac{1}{6}}(-1)^{j-m}(-1)^{j''+m''}\langle j,-m|a^\dagger_{j',m'}a_{j'',m''}|j''',-m'''\rangle \\
&+ (-1)^{j-m}(-1)^{j'-m'}\frac{1}{\sqrt{6}}\langle j,-m|a_{j',-m'}a^\dagger_{j'',-m''}|j''',-m'''\rangle
\end{aligned}$$

Using $a_{j',-m'}a^\dagger_{j'',-m''} = \delta_{j',j''}\delta_{-m'',-m'} - a^\dagger_{j'',-m''}a_{j',-m'}$ gives

$$\begin{aligned}
\langle j,-m|a_{j',-m'}a^\dagger_{-m''}|j''',-m'''\rangle &= \langle j,-m|\delta_{j',j''}\delta_{-m'',-m'}|j''',-m'''\rangle - \langle j,-m|a^\dagger_{j'',-m''}a_{j',-m'}|j''',-m'''\rangle \\
&= \delta_{j,j'''}\delta_{j',j''}\delta_{-m,-m'''}\delta_{-m',-m''} - \delta_{j,j''}\delta_{j',j'''}\delta_{-m',-m'''}\delta_{-m,-m''}
\end{aligned}$$

Now using $\langle j,-m|a^\dagger_{j',m'}a_{j'',m''}|j''',-m'''\rangle = \delta_{j,j'}\delta_{j'',j'''}\delta_{-m,m'}\delta_{m'',-m'''}$, we have

$$\begin{aligned}
\langle \Omega/2, S_0|\hat{O}_{2,0}|\Omega/2, S_0\rangle &= \frac{\langle \Omega/2, S_0; 2,0|\Omega/2,S_0\rangle}{\langle \Omega/2,-\Omega/2;2,0|\Omega/2,-\Omega/2\rangle}\langle \Omega/2,-\Omega/2|\hat{O}_{2,0}|\Omega/2,-\Omega/2\rangle \\
&= \sqrt{\frac{1}{6}}\frac{\langle \Omega/2, S_0; 2,0|\Omega/2,S_0\rangle}{\langle \Omega/2,-\Omega/2;2,0|\Omega/2,-\Omega/2\rangle}((-1)^{j-m}(-1)^{j''+m''}\delta_{j,j'}\delta_{j'',j'''}\delta_{-m,m'}\delta_{m'',-m'''} \\
&+ (-1)^{j-m}(-1)^{j'-m'}(\delta_{j,j'''}\delta_{j',j''}\delta_{-m,-m'''}\delta_{-m',-m''} \\
&- \delta_{j,j''}\delta_{j',j'''}\delta_{-m',-m'''}\delta_{-m,-m''}))
\end{aligned}$$

Next, let's consider $\hat{O}_{1,0} = \frac{1}{\sqrt{2}}\left(\hat{O}_{1,1}\hat{O}_{1,-1} - \hat{O}_{1,-1}\hat{O}_{1,1}\right)$. Using our previous work we can easily evaluate this, giving

$$\begin{aligned}
\langle \Omega/2, S_0|\hat{O}_{1,0}|\Omega/2, S_0\rangle = -(-1)^{j-m}(-1)^{j'-m'}\frac{1}{\sqrt{2}}\frac{\langle \Omega/2, S_0; 2,0|\Omega/2,S_0\rangle}{\langle \Omega/2,-\Omega/2;2,0|\Omega/2,-\Omega/2\rangle}\times \\
(\delta_{j,j'''}\delta_{j',j''}\delta_{-m,-m'''}\delta_{-m',-m''} - \delta_{j,j''}\delta_{j',j'''}\delta_{-m',-m'''}\delta_{-m,-m''})
\end{aligned}$$

Finally, we have for $\hat{O}_{0,0} = \frac{1}{\sqrt{3}}\left(\hat{O}_{1,1}\hat{O}_{1,-1} - \hat{O}_{1,0}\hat{O}_{1,0} + \hat{O}_{1,-1}\hat{O}_{1,1}\right)$ that

$$\begin{aligned}
\langle \Omega/2, S_0|\hat{O}_{0,0}|\Omega/2, S_0\rangle &= \frac{1}{\sqrt{3}}(-\frac{1}{2}(-1)^{j-m}(-1)^{j''+m''}\delta_{j,j'}\delta_{j'',j'''}\delta_{-m,m'}\delta_{m'',-m'''} \\
&+ (-1)^{j-m}(-1)^{j'-m'}(\delta_{j,j'''}\delta_{j',j''}\delta_{-m,-m'''}\delta_{-m',-m''} \\
&- \delta_{j,j''}\delta_{j',j'''}\delta_{-m',-m'''}\delta_{-m,-m''}))
\end{aligned}$$

After performing simplifications, we can express our final result as

$$\begin{aligned}
\langle \Omega/2, S_0|a^\dagger_{j,m}a^\dagger_{j',m'}a_{j'',m''}a_{j''',m'''}|\Omega/2, S_0\rangle &= \frac{(-1)^{j''+j'''-m''-m'''}}{4\Omega(\Omega-1)}[N(N-2)(-1)^{j+j'-m-m'}(\delta_{j,j'''}\delta_{j',j''}\delta_{-m',-m''}\delta_{-m,-m'''} \\
&- \delta_{j,j''}\delta_{j',j'''}\delta_{-m',-m'''}\delta_{-m,-m''}) + N(N-2\Omega)(-1)^{j+j''-m+m''} \\
&(\delta_{j,j'}\delta_{j'',j'''}\delta_{-m,m'}\delta_{m'',-m'''})]
\end{aligned}$$

Defining $a,b,c,d = (j,m),(j',m'),(j'',m''),(j''',m''')$, $\overline{q} = (j_q,-m_q)$ and $p_{q\sigma} = (-1)^{j_q+j_\sigma-m_q-m_\sigma}$, we can simplify this further as

$$\langle \Omega/2, S_0|a^\dagger_a a^\dagger_b a_c a_d|\Omega/2, S_0\rangle = \frac{p_{cd}}{4\Omega(\Omega-1)}[N(N-2)p_{ab}(\delta_{\overline{a},\overline{d}}\delta_{\overline{b},\overline{c}} - \delta_{\overline{a},\overline{c}}\delta_{\overline{b},\overline{d}}) + N(N-2\Omega)p_{a\overline{c}}\delta_{\overline{a},b}\delta_{\overline{c},d}]$$

Utilizing the Kronecker δ's allow us to eliminate some of the phases, yielding

$$\langle \Omega/2, S_0|a^\dagger_a a^\dagger_b a_c a_d|\Omega/2, S_0\rangle = \frac{1}{4\Omega(\Omega-1)}[N(N-2)(\delta_{\overline{a},\overline{d}}\delta_{\overline{b},\overline{c}} - \delta_{\overline{a},\overline{c}}\delta_{\overline{b},\overline{d}}) - N(N-2\Omega)p_{ad}\delta_{\overline{a},b}\delta_{\overline{c},d}]$$

Three-Body Operators

The three-body matrix elements are

$$\langle N|a^\dagger_a a^\dagger_b a^\dagger_c a_d a_e a_f|N\rangle\,,$$

where $|N\rangle$ and $a,b,c,...$ are defined as above. Exploiting properties of S_+ and S_- allows for us to compute this element as

$$\langle N|a^\dagger_a a^\dagger_b a^\dagger_c a_d a_e a_f|N\rangle = p_a\frac{N}{2}\left(\frac{N}{2}-1\right)\left(\frac{1}{2\Omega\left(\Omega-1\right)}\left(\delta_{\overline{a},b}p_c A + p_b B\right) + \frac{1}{6\Omega\left(\Omega-1\right)\left(\Omega-2\right)}\left(\frac{N}{2}-2\right)p_{bc}C\right)$$

where

$$A = 2p_f\left(p_{\overline{d}}\delta_{\overline{f},e}\delta_{\overline{d},\overline{c}} + p_{\overline{e}}\left(\delta_{\overline{e},d}\delta_{\overline{f},\overline{c}} - \delta_{\overline{f},d}\delta_{\overline{e},\overline{c}}\right)\right)$$

$$B = 2p_{\overline{f}}\left(p_{\overline{d}}\delta_{\overline{f},e}\left(\delta_{\overline{b},c}\delta_{\overline{a},\overline{d}} - \delta_{\overline{a},c}\delta_{\overline{b},\overline{d}}\right) + p_{\overline{e}}\left(\delta_{\overline{b},c}\left(\delta_{\overline{e},d}\delta_{\overline{a},\overline{f}} - \delta_{\overline{f},d}\delta_{\overline{a},\overline{e}}\right) - \delta_{\overline{a},c}\left(\delta_{\overline{e},d}\delta_{\overline{b},\overline{f}} - \delta_{\overline{f},d}\delta_{\overline{b},\overline{e}}\right)\right)\right)$$

$$C =$$

Flow Equations

If we assume our Hamiltonian is (in the j-scheme)

$$H = \sum_{j,m} \epsilon_m a^\dagger_{j,m} a_{j,m} - \frac{1}{4} \sum_{j,j',m,m'} g_{m,m'} (-1)^{j+j'-m-m'} a^\dagger_{j,m} a^\dagger_{j,-m} a_{j',-m'} a_{j',m'}$$

where ϵ is the single particle energy and g represents the two-particle coupling, then converting to m scheme by letting $j = j' = J$ for a single J large enough to encompass all N-particles gives (where we are taking $a_m \equiv a_{J,m}$ and so on)

$$H = \sum_m \epsilon_m a^\dagger_m a_m + \frac{1}{4} \sum_{m,m'} g_{m,m'} (-1)^{-m-m'} a^\dagger_m a^\dagger_{-m} a_{-m'} a_{m'}$$

Now defining the generalized normal-ordered operators

$$\begin{aligned} : a^\dagger_m a_{m'} : &= a^\dagger_m a_{m'} - \langle \Omega/2, S_0 | a^\dagger_m a_{m'} | \Omega/2, S_0 \rangle \\ &= a^\dagger_m a_{m'} - \frac{N}{2\Omega} \delta_{m,m'} \end{aligned}$$

and

$$\begin{aligned} : a^\dagger_m a^\dagger_{m'} a_{m''} a_{m'''} : &= a^\dagger_m a^\dagger_{m'} a_{m''} a_{m'''} - \langle \Omega/2, S_0 | a^\dagger_m a^\dagger_{m'} a_{m''} a_{m'''} | \Omega/2, S_0 \rangle + \text{singles} + \text{doubles} \\ &= a^\dagger_m a^\dagger_{m'} a_{m''} a_{m'''} - \frac{1}{4\Omega(\Omega-1)} [N(N-2)(\delta_{-m,-m'''} \delta_{-m',-m''} - \delta_{-m,-m''} \delta_{-m',-m'''}) \\ &+ N(N-2\Omega)(-1)^{m+m'''} \delta_{m,-m'} \delta_{m'',-m'''}] + \frac{N}{2\Omega} (-\delta_{m,m'''} : a^\dagger_{m'} a_{m''} : + \delta_{m,m''} : a^\dagger_{m'} a_{m'''} : \\ &- \delta_{m',m''} : a^\dagger_m a_{m'''} + \delta_{m',m'''} : a^\dagger_m a_{m''}) \end{aligned}$$

gives us

$$H = E + \sum_m f_m : a^\dagger_m a_m : + \frac{1}{4} \sum_{m,m'} \Gamma_{m,m'} : a^\dagger_m a^\dagger_{-m} a_{-m'} a_{m'} :$$

where

$$\begin{aligned} \Gamma_{m,m'} &= g_{m,m'} (-1)^{-m-m'} \\ f_m &= \epsilon_m - \frac{N}{4\Omega} (g_{-m,-m} + g_{m,-m}) \\ E &= \frac{N}{2\Omega} \sum_m \epsilon_m - \frac{1}{4} \left[\frac{N(N-2)}{4\Omega(\Omega-1)} \left(\sum_m g_{m,m} + \sum_m g_{m,-m} \right) - \frac{N(N-2\Omega)}{4\Omega(\Omega-1)} \sum_{m,m'} g_{m,m'} \right] \end{aligned}$$

Having established our Hamiltonian, we can utilize the flow equations for a multi-reference state, as given in [**?**], to obtain

$$\frac{dE}{ds} = \frac{1}{4}\sum_{m,m'}\left[\left(\eta_{m',m}\Gamma_{m,m'} - \Gamma_{m',m}\eta_{m,m'}\right)\left(\frac{N}{2\Omega}\left(1-\frac{N}{2\Omega}\right)\right)^2 + \left(\frac{d}{ds}\Gamma_{m,m'}\right)\lambda^{m,-m}_{m',-m'}\right]$$
$$+\frac{1}{4}\sum_{a,b,c,d,e}\left(\eta^{b,c}_{d,e}\Gamma_{a,b}\lambda^{c,a,-a}_{d,e,-b} - \Gamma_{a,b}\eta^{c,d}_{a,e}\lambda^{-a,c,d}_{b,-b,e}\right),$$

$$\frac{d}{ds}f_m = \left(\eta^m_m f_m - f_m\eta^m_m\right) + \left(1-\frac{N}{2\Omega}\right)\frac{N}{4\Omega}\sum_a\left(\eta^{m,-m}_{a,-a}\Gamma_{m,a} - \Gamma_{a,m}\eta^{a,-a}_{m,-m}\right) + \frac{1}{4}\sum_{a,b,c}\left(\eta^{m,-m}_{b,c}\Gamma_{a,m}\lambda^{a,-a}_{b,c} - \Gamma_{m,a}\eta^{b,-a}_{m,-m}\lambda^{b,c}_{a,-a}\right)$$
$$+\sum_{a,b,c}\left(\eta^{m,b}_{a,c}\Gamma_{a,m}\lambda^{b,-a}_{c,-m} - \Gamma_{m,a}\eta^{a,b}_{m,c}\lambda^{-m,b}_{-a,c}\right) - \frac{1}{2}\sum_{a,b,c}\left(\eta^{m,a}_{m,b}\Gamma_{c,a}\lambda^{c,-c}_{b,a} - \Gamma_{m,m}\eta^{b,c}_{-m,a}\lambda^{b,c}_{-m,a}\right)$$
$$+\frac{1}{2}\sum_{a,b,c}\left(\eta^{m,c}_{m,a}\Gamma_{a,b}\lambda^{c,-a}_{b,-b} - \Gamma_{m,m}\eta^{-m,a}_{b,c}\lambda^{-m,a}_{b,c}\right)$$

$$\frac{d}{ds}\Gamma_{m,m'} = \left(\eta^m_m + \eta^{-m}_{-m} - \eta^{m'}_{m'} - \eta^{-m'}_{-m'}\right)\Gamma_{m,m'} - \left(f_m + f_{-m} - f_{m'} - f_{-m'}\right)\eta^{m,-m}_{m',-m'}$$
$$+\frac{1}{2}\left(1-\frac{N}{\Omega}\right)\sum_a\left(\eta^{m,-m}_{a,-a}\Gamma_{a,m'} - \Gamma_{m,a}\eta^{a,-a}_{m',-m'}\right)$$

where η is our generator and λ are the irreducible matrix elements. These irreducible matrix elements are given by

$$\lambda^{a,b}_{c,d} = \langle N|a^\dagger_a a^\dagger_b a_d a_c|N\rangle - \langle N|a^\dagger_a a_c|N\rangle\langle N|a^\dagger_b a_d|N\rangle + \langle N|a^\dagger_a a_d|N\rangle\langle N|a^\dagger_b a_c|N\rangle$$
$$= \left(\frac{1}{4\Omega(\Omega-1)}\left[N(N-2)(\delta_{-a,-c}\delta_{-b,-d} - \delta_{-a,-d}\delta_{-b,-c}) + N(N-2\Omega)(-1)^{a+c}\delta_{-a,b}\delta_{-d,c}\right]\right) + \left(\frac{N}{2\Omega}\right)^2(\delta_{a,d}\delta_{b,c} - \delta_{a,c}\delta_{b,d})$$

and

$$\lambda^{a,b,c}_{d,e,f} = \langle N|a^\dagger_a a^\dagger_b a^\dagger_c a_f a_e a_d|N\rangle - \mathcal{A}\{\lambda^a_d\lambda^{b,c}_{e,f}\} - \mathcal{A}\{\lambda^a_d\lambda^b_e\lambda^c_f\}$$

where $\mathcal{A}$ is the antisymmetrization operator.

Of course, due to commutativity of matrix elements, we can simplify our flow equations to

$$\frac{dE}{ds} = \frac{1}{4}\sum_{m,m'}\left(\frac{d}{ds}\Gamma_{m,m'}\right)\lambda^{m,-m}_{m',-m'} + \frac{1}{4}\sum_{a,b,c,d,e}\left(\eta^{b,c}_{d,e}\Gamma_{a,b}\lambda^{c,a,-a}_{d,e,-b} - \Gamma_{a,b}\eta^{c,d}_{a,e}\lambda^{-a,c,d}_{b,-b,e}\right),$$

$$\frac{d}{ds}f_m = \left(1-\frac{N}{2\Omega}\right)\frac{N}{4\Omega}\sum_a\left(\eta^{m,-m}_{a,-a}\Gamma_{m,a} - \Gamma_{a,m}\eta^{a,-a}_{m,-m}\right) + \frac{1}{4}\sum_{a,b,c}\left(\eta^{m,-m}_{b,c}\Gamma_{a,m}\lambda^{a,-a}_{b,c} - \Gamma_{m,a}\eta^{b,-a}_{m,-m}\lambda^{b,c}_{a,-a}\right)$$
$$+\sum_{a,b,c}\left(\eta^{m,b}_{a,c}\Gamma_{a,m}\lambda^{b,-a}_{c,-m} - \Gamma_{m,a}\eta^{a,b}_{m,c}\lambda^{-m,b}_{-a,c}\right) - \frac{1}{2}\sum_{a,b,c}\left(\eta^{m,a}_{m,b}\Gamma_{c,a}\lambda^{c,-c}_{b,a} - \Gamma_{m,m}\eta^{b,c}_{-m,a}\lambda^{b,c}_{-m,a}\right)$$
$$+\frac{1}{2}\sum_{a,b,c}\left(\eta^{m,c}_{m,a}\Gamma_{a,b}\lambda^{c,-a}_{b,-b} - \Gamma_{m,m}\eta^{-m,a}_{b,c}\lambda^{-m,a}_{b,c}\right)$$

$$\frac{d}{ds}\Gamma_{m,m'} = \left(\eta^m_m + \eta^{-m}_{-m} - \eta^{m'}_{m'} - \eta^{-m'}_{-m'}\right)\Gamma_{m,m'} - \left(f_m + f_{-m} - f_{m'} - f_{-m'}\right)\eta^{m,-m}_{m',-m'}$$
$$+\frac{1}{2}\left(1-\frac{N}{\Omega}\right)\sum_a\left(\eta^{m,-m}_{a,-a}\Gamma_{a,m'} - \Gamma_{m,a}\eta^{a,-a}_{m',-m'}\right)$$

Generator

If we take our generator to be the Brillouin generator [?], then

$$\eta_j^i = -1/2 \sum_m (\Gamma_{j,m} \lambda_{m,m}^{i,-j} - \Gamma_{i,-i}^{m,-m} \lambda_{j,-i}^{m,-m})$$

$$\begin{aligned}\eta_{k,l}^{i,j} &= f_i \lambda_{k,l}^{i,j} - f_j \lambda_{k,l}^{j,i} - f_k \lambda_{k,l}^{i,j} + f_l \lambda_{l,k}^{i,j} \\ &+ \frac{1}{2} \sum_m \left(\Gamma_{m,-m}^{k,-k} \lambda_{m,-m,l}^{-k,i,j} - \Gamma_{m,-m}^{l,-l} \lambda_{m,-m,k}^{-l,i,j} - \Gamma_{i,-i}^{m,-m} \lambda_{-i,k,l}^{m,-m,j} + \Gamma_{j,-j}^{m,-m} \lambda_{-j,k,l}^{m,-m,i} \right)\end{aligned}$$

Rediagonalization

$$\begin{aligned}\langle 0 | a_{j,m}^\dagger a_{j',m'} | 0 \rangle &= \frac{N}{2\Omega} \delta_{j,j'} \delta_{m,m'} \\ \langle 2_{JM} | a_{j,m}^\dagger a_{j',m'} | 0 \rangle &= \frac{N}{\sqrt{2\Omega}} (-1)^{j'-m'} \langle jm; j' - m' | JM \rangle \left(1 - (-1)^{j+j'-J}\right) \\ \langle 2_{JM} | a_{j,m}^\dagger a_{j',m'} | 2_{J'M'} \rangle &= \frac{(-1)^{j'+m'}}{\sqrt{2}} \left[\frac{\langle \frac{\Omega}{2} - 1, \frac{N}{2} - \frac{\Omega}{2}; 1, 0 | \frac{\Omega}{2} - 1, \frac{N}{2} - \frac{\Omega}{2} \rangle}{\langle \frac{\Omega}{2} - 1, 1 - \frac{\Omega}{2}; 1, 0 | \frac{\Omega}{2} - 1, 1 - \frac{\Omega}{2} \rangle} \gamma_1(JJ'jj'|MM'mm') + \gamma_2(JJ'jj'|MM'mm') \right]\end{aligned}$$

where

$$\begin{aligned}\gamma_1(JJ'jj'|MM'mm') &= (-1)^{j'+m'} \sum_{m_1} \langle j', m'; j_1, m_1 | J'M' \rangle \langle j, m; j_1, m_1 | JM \rangle \left(1 - (-1)^{j+j_1-J} - (-1)^{j'+j_1-J} + (-1)^{j+j'+2j_1-2J}\right) \\ &+ p(a) \Big[\delta_{mm'} \delta_{jj'} \delta_{JJ'} \delta_{MM'} \left(1 + (-1)^J\right) + \sum_{m_1} \langle j', m'; j_1, m_1 | J'M' \rangle \langle j, m; j_1, m_1 | JM \rangle \Big(1 - (-1)^{j+j_1-J} \\ &- (-1)^{j'+j_1-J} + (-1)^{j+j'+2j_1-2J}\Big) \Big]\end{aligned}$$

and

$$\begin{aligned}\gamma_2(JJ'jj'|MM'mm') &= (-1)^{j'+m'} \sum_{m_1} \langle j', m'; j_1, m_1 | J'M' \rangle \langle j, m; j_1, m_1 | JM \rangle \left(1 - (-1)^{j+j_1-J} - (-1)^{j'+j_1-J} + (-1)^{j+j'+2j_1-2J}\right) \\ &- p(a) \Big[\delta_{mm'} \delta_{jj'} \delta_{JJ'} \delta_{MM'} \left(1 + (-1)^J\right) + \sum_{m_1} \langle j', m'; j_1, m_1 | J'M' \rangle \langle j, m; j_1, m_1 | JM \rangle \Big(1 - (-1)^{j+j_1-J} \\ &- (-1)^{j'+j_1-J} + (-1)^{j+j'+2j_1-2J}\Big) \Big]\end{aligned}$$

for $m_1 = -j_1, ... j_1$, where j_1 is the angular momentum shell our nucleons are contained in.

References