Introduction

Saturday, June 20, 2020 17:55

The following document is the content taught in AP Calculus BC in 2018-19, by Mr Mathew Geddes in SMUS, during the first half of the year, as recounted by me using notes and booklets that were created in that same year. This material is replicated as per my understanding; if there are any incorrect information, it is most likely due to my misunderstanding.

From what I remember, this is the first year in which the new booklets were used, and as such, there were many mistakes printed and sample questions omitted on the booklet. As a result, I am attempting to solidify my understanding by reviewing my knowledge of calculus before my journey into university.

Please bear with me as I recount my beginnings with calculus.

Boris Li
June 20, 2020

I am continuously making clarifications and additions to these notes as I trudge through the boredom known as first-year calculus. Many definitions are revised to be more mathematically specific, and instead of making a new document, I will be continuing here, since most of the material covered will be identical.

Other than that, I will also be posting additional information that might not be useful for the exam, might not be taught in first-year, but are rather extra pieces of information that I learnt over this year. They will probably be taught at some stage in university, but I just found it interesting enough to put it here, may it be a formal proof, another visualization of the same concept, or an extended concept that pertains to the calculus we learn here.

Before we get in, I would like to extend a token of gratitude to my Science One TA Rio Weil, and my fellow Science One classmates Morgan Arnold and Jocelyn Baker, for your great contributions to my relatively empty head.

Boris Li
September 23, 2020

Behaviour of Functions

Saturday, June 20, 2020 18:06

The booklet introduces us to calculus with the following statement as written by CollegeBoard:

> Big Idea 1: The idea of limits is essential for discovering and developing important ideas, definitions, formulae, and theorems in calculus.

Calculus is the study of change. To understand how things change, first you must study how they behave.

1 Characteristics of Functions

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The booklet omits the entirety of Chapter 1, but asks us to review our knowledge of functions from Pre-calculus 11 and 12. This is also reflected in the actual classes, as Mr Geddes had also skipped over it by assuming our knowledge.

2 Limits

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2.1 Determining Limits Graphically

Consider $\lim_{x\to 0}\frac{\sin x}{x}$.

If we graph the numerator and denominator separately, we would observe that the two functions intersect at the origin, which would mean that the function would, by substitution, return an indeterminate form of $\frac{0}{0}$.

As x approaches 0, $\sin x$ and x behave very similarly. Their slopes are almost identical as we get closer and closer to 0. Therefore, we can conclude that $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Now, if we graph the entire function, it would appear to pass through (0,1), but in fact, a value does not exist at x = 0. $\lim_{x\to 0}\frac{\sin x}{x} = 1$ as we plug the hole in the graph.

2.2 Exploring Limits

How to find a limit:

1. Substitute in the value; and if it does not work,
2. Factor, cancel, then substitute in the value.

A limit can only exist when both sides approach the same value; that is, when the left-hand limit equals the right-hand limit.

A floor function is a great example when demonstrating one-handed limits. It returns the largest integer to the left of the value.

$$\lim_{x\to n+}\lfloor x\rfloor = n$$
$$\lim_{x\to n-}\lfloor x\rfloor = n-1$$
$$\lim_{x\to n}\lfloor x\rfloor \quad \textit{does not exist}$$
$$n \in \mathbb{Z}$$

I will provide an almost strict definition of a limit, even though it is not AP or first-year material:
Assuming f is a function over certain set of real values called D, c is a limit point (does not have to be within the domain of f, but can be approximated by its surroundings), and $L \in \mathbb{R}$, we say:

$$\lim_{x\to c} f(x) = L$$

For every $\varepsilon > 0$, there must be a $\delta > 0$.

(If there is a Δy, there must be a Δx. That's the defintion of a function, remember?)

Under $x \in D$, if $0 < |x-c| < \delta$, then $0 < |f(x) - L| < \varepsilon$.

(The distance between x and c can always be smaller than any arbitrary number δ. Similarly, that applies to $f(x)$, L, and ε.)

The limit L is a value that can satisfy all these conditions, no matter how small ε and δ gets.

(As ε and δ get smaller, and x and c gets closer and closer, if only one value can get as close to the trend of $f(x)$, that is the limit.)

So, to put it all in one big mathematical statement:

$$\lim_{x\to c} f(x) = L \Leftrightarrow (\forall\varepsilon > 0, \exists\delta > 0, \forall x \in D, 0 < |x-c| < \delta \Rightarrow 0 < |f(x) - L| < \varepsilon)$$

2.3 Asymptotes

A **horizontal asymptote** exists when the limit as x approaches either infinity equals a constant value.

A **vertical asymptote** exists when the limit as x approaches a value from either side equals to positive or negative infinity.

2.4 End Behaviour

To find an end behaviour model g(x) for f(x), the limit as x approaches that infinity of the quotient of f(x) to g(x) must equal 1 (they must behave similarly at that end).

Find dominant terms for polynomial functions.

Use the **squeeze theorem** by listing known upper and lower bounds of functions, and altering them.
If a function is between two other functions, and the two other functions share the same limit, the middle function must also share the same limit at that point.
In the case of end behaviour models, regard only limits at either infinity.

2.5 Continuity

2.5.1 Continuity requirements

A function is continuous at a point when a limit exists at that point, and is equal to the value of that point.

$$\lim_{x \to a} f(x) = f(a)$$

Then $f(x)$ is continuous at a.

A function is continuous at an endpoint when a one-sided limit exists at that endpoint.

$f(x)\ \{x | x < a, x \in \mathbb{R}\}$

$$\lim_{x \to a+} f(x) = f(a)$$

Then $f(x)$ is continuous at a.

$f(x)\ \{x | x > a, x \in \mathbb{R}\}$

$$\lim_{x \to a-} f(x) = f(a)$$

Then $f(x)$ is continuous at a.

A function is said to be continuous over an interval when all the points within that interval are continuous.

2.5.2 Types of discontinuities

A function has a **removable/point discontinuity** when the discontinuity can simply be plugged and made continuous. A limit exists, but the value does not or exists at a different value.

A function has a **jump discontinuity** when the value of the function jumps from one value to the other. Both one-sided limits exist, but they do not equal each other, and therefore a limit does not exist.

A function has an **infinite discontinuity** when a vertical asymptote exists. One-sided limits at that point equal to positive or negative infinity.

A discontinuity is said to be **oscillating** when the exact value of discontinuity cannot be determined. An example is $\sin\frac{1}{x}$ at x = 0.

2.5.3 Extended functions

When asked to find an extended function, write a function that plugs the continuity and make the resulting function continuous by writing a **piecewise function**.

2.6 Theorems

The **intermediate value theorem (IVT)** states that if a function f(x) is continuous over a closed interval [a,b], then every value between f(a) and f(b) must exist within that interval.

The **extreme value theorem (EVT)** states that if a function f(x) is continuous over a closed interval [a,b], then f(x) must have both a maximum and minimum within that interval.

Differential Calculus

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As taught usually in the first term in university, differentiation is the art of finding slopes and rates of change.

Table of Common Derivatives

Basic Properties

Constant multiple

$$\frac{d}{dx}k \cdot f(x) = k \cdot \frac{d}{dx}f(x)$$

Constant rule

$$\frac{d}{dx}k = 0$$

Binary Operations

Sum rule

$$\frac{d}{dx}f(x) + g(x) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Difference rule

$$\frac{d}{dx}f(x) - g(x) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Product rule

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x)$$

$$\therefore (uv)' = u'v + v'u$$

$$\therefore d(uv) = udv + vdu$$

Quotient rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$

$$\therefore \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$\therefore d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$

Common Rules

Chain rule

$$\frac{dy}{dx} = \frac{dy}{ds} \cdot \frac{ds}{dx}$$

Power rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

Trigonometry

Trigonometric functions

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\sec x = \tan x \sec x$$
$$\frac{d}{dx}\csc x = -\cot x \csc x$$

Inverse trigonometric functions

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

Exponents and Logarithms

Exponential functions

$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}b^x = b^x \ln b$$

Logarithmic functions

$$\frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$$

3 Derivatives

Saturday, June 20, 2020 19:33

3.1 Slope

The average rate of change over an interval is $\frac{\Delta y}{\Delta x}$.

On a graph, when connecting two points of a curve, a secant line is created, and the slope is determined by $m = \frac{f(b)-f(a)}{b-a}$.

3.2 Slope at a Point

The instantaneous rate of change over an interval is $\frac{dy}{dx}$.

The slope at point (a,f(a)) is calculated by $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. Determining this rate of change is called differentiation.

A line with slope m that passes through (a,f(a)) is called a tangent line.

3.3 Definition of the Derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

3.4 Alternate Definition of the Derivative

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

3.5 Differentiability

To find one-sided derivatives, take any of the three formulae from above and substitute in one-sided limits.

A function is only differentiable at a point when both one-sided derivates equal each other, and a limit exists at that point.

Four ways that make a function fail to be differentiable:

1. Corners ($y = |x|$ at 0);
2. Cusps ($y = x^{\frac{2}{3}}$ at 0);
3. Vertical tangents ($y = x^{\frac{1}{3}}$ at 0); and
4. Jump discontinuities ($y = \frac{|x|}{x}$ at 0).

Differentiability implies continuity; however, continuity does not ensure differentiability.

3.6 Symmetric Difference Quotient

Instead of closing in to a point from one side, closing in to a point from both sides.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}$$

Calculators use this.

3.7 Graphing Derivatives

The degree is reduced by one.

Think slope fields.

4 Differentiation Techniques

Thursday, June 25, 2020 12:25

A quick summary:

Definition of the Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative at a point

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Right-hand derivative

$$f'(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$

Left-hand derivative

$$f'(a) = \lim_{h \to 0-} \frac{f(a+h) - f(a)}{h}$$

Difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

Symmetric difference quotient

$$\frac{f(a+h) - f(a-h)}{2h}$$

4.1 Polynomial Derivatives

4.1.1 Reciprocal Rule

$$\frac{d}{dx}\frac{1}{f(x)} = \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$
$$\frac{d}{dx}\frac{1}{f(x)} = \frac{f(x) - f(x+h)}{hf(x)f(x+h)}$$
$$\frac{d}{dx}\frac{1}{f(x)} = -f'(x)\frac{1}{f(x)f(x+h)}$$
$$\frac{d}{dx}\frac{1}{f(x)} = \frac{-f'(x)}{f^2(x)}$$

4.1.2 Power Rule

Definition of the derivative

$n > 0, n \in \mathbb{Z}$

$$\frac{d}{dx}x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Binomial theorem

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{\sum_{k=0}^{n}\left(\frac{n!}{k!\,(n-k)!}\right)x^{n-k}h^k - x^n}{h}$$
$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{\sum_{k=2}^{n}\left(\frac{n!}{k!\,(n-k)!}\right)x^{n-k}h^k + x^n + \left(\frac{n!}{(n-1)!}\right)x^{n-1}h - x^n}{h}$$

Factor out an h

$$\frac{d}{dx}x^n = \lim_{h\to 0}\sum_{k=2}^{n}\left(\frac{n!}{k!\,(n-k)!}\right)x^{n-k}h^{k-1} + \left(\frac{n!}{(n-1)!}\right)x^{n-1}$$
$$\frac{d}{dx}x^n = nx^{n-1}$$

$n < 0, n \in \mathbb{Z}$

$n = -m, m > 0, n \in \mathbb{Z}$

$$\frac{d}{dx}x^n = \frac{d}{dx}\frac{1}{x^m}$$
$$\frac{d}{dx}x^n = \frac{-mx^{m-1}}{x^{2m}}$$
$$\frac{d}{dx}x^n = nx^{-1-m}$$
$$\frac{d}{dx}x^n = nx^{n-1}$$

4.1.3 Sum rule

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

4.1.4 Difference rule

$$\frac{d}{dx}\big(f(x) - g(x)\big) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

4.1.5 Constant multiple

$$\frac{d}{dx}k \cdot f(x) = k \cdot \frac{d}{dx}f(x)$$

4.1.6 Constant rule

$$\frac{d}{dx}k = 0$$

4.1.7 Finding Tangent Lines

The slope equals the derivative.

$$y = m(x - x_1) + y_1$$

The point it must pass through is (x_1, y_1).

4.1.8 Finding Normal Lines

Same formula, but the slope is the **negative reciprocal** of the derivative.

4.2 Higher Order Derivatives

4.2.1 Product Rule

$$F(x) = f(x)g(x)$$
$$\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)g(x)}{h}$$
$$\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) + (f(x)\cdot g(x+h) + f(x)\cdot g(x+h)) f(x)g(x)}{h}$$
$$\frac{d}{dx}F(x) = \lim_{h\to 0}\left(\frac{g(x+h)}{h}(f(x-h) - f(x)) + \frac{f(x)}{h}(g(x-h) - g(x))\right)$$
$$\frac{d}{dx}F(x) = \lim_{h\to 0}\left(\frac{f((x-h) - f(x))}{h}g(x+h) + f(x)\frac{g((x-h) - g(x))}{h}\right)$$
$$\frac{d}{dx}F(x) = f'(x)g(x) + g'(x)f(x)$$
$$\therefore (uv)' = u'v + v'u$$
$$\therefore d(uv) = udv + vdu$$

4.2.2 Quotient Rule

$$F(x) = \frac{f(x)}{g(x)}$$
$$\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$
$$\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}\cdot g(x)\cdot g(x+h) - \frac{f(x)}{g(x)}\cdot g(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$
$$\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$
$$\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{f(x+h)g(x) + (-f(x)g(x) + f(x)g(x)) - f(x)g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$
$$\frac{d}{dx}F(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\cdot\frac{g(x)}{g(x)g(x+h)} + \frac{g(x) - g(x+h)}{h}\cdot\frac{f(x)}{g(x)g(x+h)}\right)$$
$$\frac{d}{dx}F(x) = \frac{f'(x)g(x)}{g^2(x)} + \frac{-g'(x)f(x)}{g^2(x)}$$
$$\frac{d}{dx}F(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$
$$\therefore \left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$
$$\therefore d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$

Or
$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(f(x)\frac{1}{g(x)}\right)$$
$$\frac{d}{dx}F(x) = f'(x)\frac{1}{g(x)} + \frac{-g'(x)}{g^2(x)}f(x)$$
$$\frac{d}{dx}F(x) = \frac{f'(x)}{g(x)}\frac{g(x)}{g(x)} - \frac{g'(x)f(x)}{g^2(x)}$$
$$\frac{d}{dx}F(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$

4.3 Rates of Change

Now for some actual application.

$$v = \frac{\Delta s}{\Delta t}$$
$$v = \frac{ds}{dt}$$
$$v = s'(t)$$

Notice s is a function for displacement. To calculate total distance travelled, take the absolute value of each segment where velocity switches signs.

$$a = \frac{\Delta v}{\Delta t}$$
$$a = \frac{dv}{dt}$$
$$a = v'(t) = s''(t)$$

4.4 Trigonometric Derivatives

4.4.1 Sine

$$\frac{d}{dx}\sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$
$$\frac{d}{dx}\sin x = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$\because \cos 0 = 1$

$$\frac{d}{dx}\sin x = \lim_{h \to 0} \frac{\sin x + \cos x \sin h - \sin x}{h}$$
$$\frac{d}{dx}\sin x = \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$
$$\frac{d}{dx}\sin x = \cos x$$

4.4.2 Cosine

$$\frac{d}{dx}\cos x = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h}$$
$$\frac{d}{dx}\cos x = \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$\because \cos 0 = 1$

$$\frac{d}{dx}\cos x = \lim_{h \to 0} \frac{\cos x - \sin x \sin h - \cos x}{h}$$
$$\frac{d}{dx}\cos x = -\sin x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$
$$\frac{d}{dx}\cos x = -\sin x$$

4.4.3 Tangent

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$
$$\frac{d}{dx}\tan x = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x}$$
$$\frac{d}{dx}\tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$$
$$\frac{d}{dx}\tan x = \sec^2 x$$

4.4.4 Cotangent

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x}$$
$$\frac{d}{dx}\cot x = \frac{(\cos x)'\sin x - (\sin x)'\cos x}{\sin^2 x}$$
$$\frac{d}{dx}\cot x = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$
$$\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x}$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$

4.4.5 Secant

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x}$$
$$\frac{d}{dx}\sec x = \frac{-(\cos x)'}{\cos^2 x}$$
$$\frac{d}{dx}\sec x = \frac{\sin x}{\cos^2 x}$$
$$\frac{d}{dx}\sec x = \tan x \sec x$$

4.4.6 Cosecant

$$\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x}$$
$$\frac{d}{dx}\csc x = \frac{-(\sin x)'}{\sin^2 x}$$
$$\frac{d}{dx}\csc x = -\frac{\cos x}{\sin^2 x}$$
$$\frac{d}{dx}\csc x = -\cot x \csc x$$

4.5 Chain Rule

4.5.1 Leibniz notation

Let f and g be differentiable everywhere.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}$$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta u}\cdot\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}$$

From step 2, as Δx approach 0, Δu approach 0.

$$\frac{dy}{dx} = \lim_{\Delta u \to 0}\frac{\Delta y}{\Delta u}\cdot\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}$$
$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

4.5.2 Lagrange notation

$$\frac{d}{dx} f(g(x)) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$
$$\frac{d}{dx} f(g(x)) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \cdot \frac{g(x+h) - g(x)}{g(x+h) - g(x)}$$
$$\frac{d}{dx} f(g(x)) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$
$$\frac{d}{dx} f(g(x)) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot g'(x)$$
Let $k = g(x+h) - g(x)$, as $h \to 0$, $k \to 0$.
$$\frac{d}{dx} f(g(x)) = \lim_{k\to 0} \frac{f(g(x+h)) - f(g(x))}{k} \cdot g'(x)$$
$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

5 Advanced Differentiation Techniques

Friday, July 3, 2020 12:07

Disclaimer: I seem to have lost the booklet for this, and therefore I will be trying my best to remember what was taught from memory.

5.1 Composite Functions

$(f \circ g)(x) = f(g(x))$

5.2 Implicit Differentiation

Differentiating the left side and the right side is allowed.

5.3 Inverse Derivatives

$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$

$(f^{-1})'(a)$ is the slope of the inverse graph.

The point $(a, f^{-1}(a))$ is on the inverse graph.

The point $(f^{-1}(a), a)$ therefore must be on the original graph, and the slope at that point, $f'(f^{-1}(a))$ must be the reciprocal of the slope of the inverse graph.

5.4 Exponential and Logarithmic Derivatives

5.4.1 Natural Exponent

$\frac{d}{dx}e^x = e^x$

By definition.

5.4.2 Natural Logarithm

$y = \ln x$

$e^y = x$

$\frac{d}{dx}e^y = \frac{dx}{dx}$

$e^y \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y}$

$\frac{d}{dx}\ln x = \frac{1}{x}$

5.4.3 General Exponents

$y = b^x$

$\ln y = x \ln b$

$\frac{d}{dx}\ln y = \frac{d}{dx}x \ln b$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln b$

$\frac{dy}{dx} = y \ln b$

$$\frac{d}{dx}b^x = b^x \ln b$$

5.4.4 General Logarithms

$$y = \log_b x$$
$$y = \frac{\ln x}{\ln b}$$
$$\frac{dy}{dx} = \frac{1}{x \ln b}$$
$$\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$$

5.5 Logarithmic Differentiation

When in doubt, try taking the natural logarithm of both sides. This should eliminate nested exponents.

5.6 Inverse Trigonometric Derivatives

A note on the notation: I am a proponent of using the prefix 'arc-' instead of the -1 exponent; however, Microsoft provides me with no such easy tools. This will be fixed when I transfer my notes over to LaTeX.

$$\sin^{-1} x = \arcsin x$$
$$\sin^n x = (\sin x)^n$$

5.6.1 Arcsine

$$y = \sin^{-1} x$$
$$x = \sin y$$

By the definition of sine, a right-angle triangle can be constructed with an angle measuring y, its opposite side measuring x, and the hypotenuse measuring 1.
The adjacent side to the referenced angle, c, can be calculated using the Pythagorean theorem:

$$x^2 + c^2 = 1$$
$$c = \sqrt{1 - x^2}$$

$$\frac{dx}{dx} = \frac{d \sin y}{dx}$$
$$1 = \frac{dy}{dx}\cos y$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

And by definition of cosine, being the ratio of adjacent to hypotenuse in terms of the referenced angle:

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

5.6.2 Arccosine

$$y = \cos^{-1} x$$
$$x = \cos y$$

$$x^2 + c^2 = 1$$
$$c = \sqrt{1 - x^2}$$

Where c is the opposite, x is the adjacent, and the hypotenuse measures 1, all in reference to the angle that measures y.

$$\frac{dx}{dx} = \frac{d\cos y}{dx}$$
$$1 = -\frac{dy}{dx}\sin y$$
$$\frac{dy}{dx} = -\frac{1}{\sin y}$$
$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

5.6.3 Arctangent

$$y = \tan^{-1} x$$
$$x = \tan y$$

$$x^2 + 1 = c^2$$
$$c = \sqrt{1+x^2}$$
Where x is the opposite, the adjacent measures 1, c is the hypotenuse, all in reference to the angle that measures y.

$$\frac{dx}{dx} = \frac{d\tan y}{dx}$$
$$1 = \frac{dy}{dx}\sec^2 y$$
$$\frac{dy}{dx} = \cos^2 y$$
$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1+x^2}}\right)^2$$
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

5.6.4 Arccotangent

$$y = \cot^{-1} x$$
$$x = \cot y$$

$$x^2 + 1 = c^2$$
$$c = \sqrt{1+x^2}$$
Where the opposite measures 1, x is the adjacent, c is the hypotenuse, all in reference to the angle that measures y.

$$\frac{dx}{dx} = \frac{d\cot y}{dx}$$
$$1 = -\frac{dy}{dx}\csc^2 y$$
$$\frac{dy}{dx} = -\sin^2 y$$
$$\frac{dy}{dx} = -\left(\frac{1}{\sqrt{1+x^2}}\right)^2$$
$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

5.6.5 Arcsecant

$$y = \sec^{-1} x$$
$$x = \sec y$$

$$c^2 + 1 = x^2$$
$$c = \sqrt{x^2 - 1}$$
Where c is the opposite, the adjacent measures 1, x is the hypotenuse, all in reference to the angle that measures y.

$$\frac{dx}{dx} = \frac{d \sec y}{dx}$$
$$1 = \frac{dy}{dx} \tan y \sec y$$
$$\frac{dy}{dx} = \frac{\cos y}{\tan y}$$

Secant is positive in quadrants 1 and 4, and negative in 2 and 3. When we take the inverse of this function, information is lost.
If we enter in a positive value into arcsecant, the answer from quadrant 1 is returned; alternatively, if a negative value is entered, the answer from quadrant 2 is returned.
Since tangent and secant are both positive in quadrant 1 and both negative in quadrant 2, the product of the two must therefore always be positive given the domain of arcsecant.
$$\frac{dy}{dx} = \left|\frac{1}{x} \cdot \frac{1}{\sqrt{x^2 - 1}}\right|$$
$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}$$

5.6.6 Arccosecant

$$y = \csc^{-1} x$$
$$x = \csc y$$

$$c^2 + 1 = x^2$$
$$c = \sqrt{x^2 - 1}$$
Where the opposite measures 1, c is the adjacent, x is the hypotenuse, all in reference to the angle that measures y.

$$\frac{dx}{dx} = \frac{d \csc y}{dx}$$
$$1 = -\frac{dy}{dx} \cot y \csc y$$
$$\frac{dy}{dx} = -\sin y \tan y$$

Cosecant is positive in quadrants 1 and 2, and negative in quadrants 3 and 4.
Arccosecant returns the answer from quadrants 1 and 4.
Since sine and tangent are both positive in quadrant 1 and both negative in quadrant 4, the product must be positive within the domain of arccosecant.
$$\frac{dy}{dx} = -\left|\frac{1}{x} \cdot \frac{1}{\sqrt{x^2 - 1}}\right|$$
$$\frac{d}{dx} \csc^{-1} x = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

5.7 L'Hôpital's Rule

When the limit equals $\frac{0}{0}$, take the derivative of both top and bottom.

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$$
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Applications of Derivatives

Friday, July 3, 2020 16:35

Now we know how to find derivatives, let us figure out when to find derivatives.

6 Analyzing Functions

Friday, July 3, 2020 17:28

6.1 First Derivative Test

Critical points: $f'(c) = 0$ or undefined.

If $f'(x)$ goes from positive to negative at c, $f(c)$ is a local maximum.
If $f'(x)$ goes from negative to positive at c, $f(c)$ is a local minimum.

In a closed interval $[a, b]$:
If $\lim_{x\to a+} f'(x) < 0$, then $f(a)$ is a local maximum.
If $\lim_{x\to a+} f'(x) > 0$, then $f(a)$ is a local minimum.
If $\lim_{x\to b-} f'(x) > 0$, then $f(b)$ is a local maximum.
If $\lim_{x\to b-} f'(x) < 0$, then $f(b)$ is a local minimum.

6.2 Modelling and Optimization

These problems usually require a solution of a maximum or a minimum. Use the first derivative test to find such maxima and minima.

6.3 Second Derivative Test

Concavity: Whether the function is curving upward or downwards; concave up ('smile') or concave down ('frown'). The concavity switches at a point of inflection.

If $f''(c) > 0$, f' is increasing, and f is concave up.
If $f''(c) < 0$, f' is decreasing, and f is concave down.
If $f''(c) = 0$, f' has a critical point, and f may be a point of inflection.

> If the two one-sided limits for $f''(c)$ are both positive or both negative, f' does not change direction and f therefore does not change concavity, resulting in a lack of point of inflection.

6.4 Curve Sketching

Horizontal inflection point: f'' changes signs, $f''(c) = 0$ and $f'(c) = 0$
Vertical inflection point: Vertical tangent, $f(c)$ exists, but $f'(c)$ and $f''(c)$ do not.

6.5 Mean Value Theorem

6.5.1 Mean Value Theorem

If f is **continuous** $[a, b]$ and **differentiable** (a, b), there must be a value $c \in (a, b)$ that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
There must be a point on the graph that has the same slope as the average slope.

6.5.2 Rolle's Theorem

Given the conditions of the MVT, and $f(a) = f(b)$, there must be a value $c \in (a, b)$ that $f'(c) = 0$.

7 Solving Problems

Tuesday, July 7, 2020 11:18

7.1 Related Rates: Area & Volume

7.1.1 Circles

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

7.1.2 Cylinders

$V = \pi r^2 h$

$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$

(constant radius, draining straight down)

7.1.2 Spheres

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

7.1.3 Cones

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{1}{3}\pi \left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$

(simplify r & h down to one variable)

7.2 Related Rates: Motion

7.2.1 Right Triangle

$a^2 + b^2 = c^2$

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

$a\frac{da}{dt} + b\frac{db}{dt} = c\frac{dc}{dt}$

7.3 Related Rates: Periodic

7.3.1 Position versus Angle

$y = r\sin\theta$

$\frac{dy}{dt} = r\cos\theta\frac{d\theta}{dt}$

7.3.2 Chords & Clocks

Assuming a & b, the two 'radii', are constants

$c^2 = a^2 + b^2 + 2ab\cos\theta$

$2c\frac{dc}{dt} = -2ab\sin\theta\frac{d\theta}{dt}$

$$\frac{dc}{dt} = -\frac{ab}{c}\sin\theta\frac{d\theta}{dt}$$

7.4 Linearization & Differentials

7.4.1 Differential

$dy = f'(x)dx$

7.4.2 Linearization

$L(a + \Delta x) = f(a) + f'(a)(\Delta x)$

Integral Calculus

Tuesday, July 7, 2020 11:53

Integrals, antiderivatives, and how to define them. As the name suggests, the antiderivative is to reverse the process of differentiation, finding the original function given the derivative.

The integral is same, but also different. The integral exists to find the area under a curve, and although is similar to the antiderivative in evaluation and calculation, is fundamentally different in concept.

The fundamental theorem of calculus ties this all together, linking the idea of a derivative to its respective antiderivative, differentiation and its relationship to integration, and allows us to solve differential equations.

Table of Common Antiderivatives

Basic Properties

Constant multiple

$$\int kf(x)dx = k\int f(x)dx + C$$

Constant rule

$$\int \quad = C$$

$$\int dx = x + C$$

Binary Operations

Sum rule

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx + C$$

Difference rule

$$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx + C$$

Common Rules

Integration by Substitution

$$\int f(u) \cdot u'(x)dx = \int f(u) \cdot \frac{du}{dx}dx = \int f(u)du$$

Reverse Power rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Reciprocal rule

$$\int \frac{dx}{x} = \ln|x| + C$$

Trigonometry

Trigonometric functions

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C = -\ln|\csc x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

Trigonometric derivatives

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \tan x \sec x \, dx = \sec x + C$$

$$\int \cot x \csc x \, dx = -\csc x + C$$

Inverse trigonometric functions

These require integration by parts, and will be proven in Appendix 2 of AP Calculus BC.

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C$$

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \sqrt{1 - x^2} + C$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2}\ln(1 + x^2) + C$$

$$\int \cot^{-1} x \, dx = x \cot^{-1} x + \frac{1}{2}\ln(1 + x^2) + C$$

$$\int \sec^{-1} x \, dx = x \sec^{-1} x - \ln\left(|x| + \sqrt{x^2 - 1}\right) + C$$

$$\int \csc^{-1} x \, dx = x \csc^{-1} x - \ln\left(|x| - \sqrt{x^2 - 1}\right) + C$$

Inverse trigonometric derivatives

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C_1 = -\cos^{-1}\frac{x}{a} + C_2$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C_1 = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C_2$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C_1 = -\frac{1}{a}\csc^{-1}\left|\frac{x}{a}\right| + C_2$$

Exponents and Logarithms

Exponential functions

$$\int e^x dx = e^x + C$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Logarithmic functions

These require integration by parts, and will be proven in Appendix 2 of AP Calculus BC.

$$\int \ln x \, dx = x \ln x - x + C$$

$$\int \ln x \, dx = x \ln ax - x + C$$

$$\int \log_b x \, dx = x \log_b x - \frac{x}{\ln b} + C$$

8 Antidifferentiation

Tuesday, July 7, 2020 12:07

8.1 Antiderivative

8.1.1 Reverse Power Rule

The power rule states that:

$f(x) = x^n$

$f'(x) = nx^{n-1}$

Then by reversing the process:

$f'(x) = x^n$

$$f(x) = \frac{x^{n+1}}{n+1} + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

8.1.2 Notation

Derivative notation

$$\frac{dy}{dx} = f'(x)$$

Differential notation (multiply both sides by dx)

$dy = f'(x)dx$

Integral notation (integrate both sides)

$$y = \int f'(x)dx$$

8.2 Trigonometric Antiderivative Results

$$\frac{d}{dx}\sin x = \cos x$$

$$\int \cos x \, dx = \sin x + C$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\int \sin x \, dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\frac{d}{dx}\sec x = \tan x \sec x$$
$$\int \tan x \sec x \, dx = \sec x + C$$

$$\frac{d}{dx}\csc x = -\cot x \csc x$$
$$\int \cot x \csc x \, dx = -\csc x + C$$

8.3 Antidifferentiation via Substitution

$$\int f(u) \cdot \frac{du}{dx} \cdot dx = \int f(u) du$$

8.4 Advanced Antidifferentiation Techniques

Sometimes the method above is unavoidable, since x cannot be completely cancelled out when du replaces dx, and the remaining x's have to be substituted as u's.

8.5 Exponential and Logarithmic Antiderivative Results

8.5.1 Reciprocals

$$\frac{d}{dx}\ln x = \frac{1}{x}, x > 0$$
$$\frac{d}{dx}\ln(-x) = \frac{1}{x}, x < 0$$
$$\int \frac{dx}{x} = \begin{cases} \ln x + C, x > 0 \\ \text{does not exist } x = 0 \\ \ln(-x) + C, x < 0 \end{cases}$$
$$\int \frac{dx}{x} = \ln|x| + C$$

8.5.2 Exponents

$$\frac{d}{dx}e^x = e^x$$
$$\int e^x dx = e^x + C$$

8.6 Inverse Trigonometric Antiderivative Results

$$\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}$$
$$\frac{d}{dx}\cos^{-1}\frac{x}{a} = -\frac{1}{\sqrt{a^2 - x^2}}$$
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C_1 = -\cos^{-1}\frac{x}{a} + C_2$$

$$\frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2 + x^2}$$
$$\frac{d}{dx}\cot^{-1}\frac{x}{a} = -\frac{a}{a^2 + x^2}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C_1 = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C_2$$

$$\frac{d}{dx}\sec^{-1}\frac{x}{a} = \frac{a}{|x|\sqrt{x^2 - a^2}}$$

$$\frac{d}{dx}\csc^{-1}\frac{x}{a} = -\frac{a}{|x|\sqrt{x^2 - a^2}}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C_1 = -\frac{1}{a}\csc^{-1}\left|\frac{x}{a}\right| + C_2$$

9 Differential Equations

Tuesday, July 7, 2020 13:20

9.1 Analytical Approach

A differential equation is one that involves differentials. To solve one:

1. Separate the variables, y & dy on one side, x and dx on the other; then
2. Integrate.

Remember your constants of integration!

9.2 Graphical Approach

Sketching slope fields. This is useful for visualizing the possible solutions for the differential equation. With an initial value, a particular solution can be extracted from the graph.

9.3 Riemann Sums

A definite integral has limits of integration. It evaluates the area under the curve, bounded by the lower and upper limit. Riemann sums cuts the curve into pieces, and then uses geometric shapes to approximate the area.

In the following methods, $\Delta x = \frac{b-a}{n}$, where n is a natural number of segments cut into.

9.3.1 Left Rectangular Approximation Method (LRAM)

Taking the left value of each segment.

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} f(x_{k-1})\Delta x\,, x_m = a + m\Delta x$$

$$\int_a^b f(x)dx \approx \Delta x\big(f(a) + f(a+\Delta x) + f(a+2\Delta x) + \cdots + f(b-2\Delta x) + f(b-\Delta x)\big)$$

9.3.2 Right Rectangular Approximation Method (RRAM)

Taking the right value of each segment.

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} f(x_k)\Delta x\,, x_m = a + m\Delta x$$

$$\int_a^b f(x)dx \approx \Delta x\big(f(a+\Delta x) + f(a+2\Delta x) + f(a+3\Delta x) + \cdots + f(b-\Delta x) + f(b)\big)$$

9.3.3 Midpoint Rectangular Approximation Method (MRAM)

Taking the middle value of each segment.

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} f\left(\frac{x_k - x_{k-1}}{2}\right)\Delta x\,, x_m = a + m\Delta x$$

$$\int_a^b f(x)dx$$
$$\approx \Delta x\left(f\left(a + \frac{1}{2}\Delta x\right) + f\left(a + \frac{3}{2}\Delta x\right) + f\left(a + \frac{5}{2}\Delta x\right) + \cdots + f\left(b - \frac{3}{2}\Delta x\right) + f\left(b - \frac{1}{2}\Delta x\right)\right)$$

9.3.4 Trapezoidal Approximation

Taking both endpoints, draw a line, area under that line.

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1})\right)\frac{\Delta x}{2}, x_m = a + m\Delta x$$

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left(f(a) + 2f(a + \Delta x) + 2f(a + 2\Delta x) + \cdots + 2f(b - \Delta x) + f(b)\right)$$

9.4 Graphically Calculating Area

Simply use geometric shapes. And your brain.

10 Fundamental Theorem of Calculus

Tuesday, July 7, 2020 14:57

10.1 Integrals

10.1.1 Definite Integral

A definite integral is when we cut a curve into infinitely many slices over an interval, and add them together.

Let f be a continuous function over $[a, b]$.
The curve is subdivided into intervals of length $\Delta x = \frac{b-a}{n}$.
As $n \to \infty$, $\Delta x \to 0$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx$$

10.1.2 Fundamental Theorem of Calculus

Let $f(x)$ be a continuous function over $[a, b]$, and $F'(x) = f(x)$.

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

10.1.3 Connections

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$
$$\int_a^a f(x)dx = 0$$
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

10.2 Derivative of Definite Integrals

10.2.1 First Fundamental Theorem of Calculus

$$\int_a^b f'(x)dx = f(x)\Big|_a^b = f(b) - f(a)$$

10.2.2 Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

If $g(x) = \int_a^x f(t)dt$ then $g'(x) = f(x)$.

10.2.3 Extension to Second Fundamental Theorem

The 2nd FTC also applies to functions.

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = f(u)\frac{du}{dx} - f(v)\frac{dv}{dx}$$

10.3 Average Value of a Function

10.3.1 Average Value

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

10.3.2 Mean Value Theorem

If f is continuous over $[a, b]$, then there exists a c such that:

$$\int_a^b f(x)dx = f(c)(b-a)$$

There must be a value that is equal to the average value.
Then the average value of the first derivative can also be shown as:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{1}{b-a} \cdot \big(f(b) - f(a)\big)$$

$$f'(c) = \frac{1}{b-a} \cdot \int_a^b f'(x)dx$$

10.4 Integral as Net Change

Just as before, since $v(t) = x'(t)$,

$$\int x'(t)dt = \int v(t)dt = x(t) + C$$

$$\int_a^b v(t)dt = x(t)\Big|_a^b = x(b) - x(a)$$

Then,

$$x(b) = x(a) = \int_a^b v(t)dt$$

Applications of Integration

Tuesday, July 7, 2020 16:31

I mean, if we have applications of derivatives, we kind of need applications of integration?

11 Area and Volume

Tuesday, July 7, 2020 16:33

11.1 Area Between Two Curves

$$A = \int_a^b \left(f(x) - g(x)\right) dx$$

11.2 Horizontal Slices

$$A = \int_a^b \left(f^{-1}(y) - g^{-1}(y)\right) dy$$

11.3 Volumes

11.3.1 Revolutions around x-axis

$$V = \pi \int_a^b f^2(x) dx$$

11.3.2 Revolutions around y-axis

$$V = \pi \int_c^d f^2(y) dy$$

11.4 Washer Method

$$V = \pi \int_a^b \left(f^2(x) - g^2(x)\right) dx$$

11.5 Cross-Sectional Area

$$V = \int_a^b A(x) dx$$

Conclusion

Tuesday, July 7, 2020 15:53

AP Calculus as taught in high school covers most of the main concepts, such as a general idea of how derivatives and antiderivatives work. However, it is taught in such a way that it requires intuition and favours simplicity over rigourous proofs. I have attempted to condense most of the concepts into a single line of mathematical statements, and have sometimes provided my thought process and logic through intermediate steps, but this is not enough.

Looking forward, as I enter into university, I will once again relearn all the concepts, albeit in a different manner, this time formally using all the strict methods of manipulation and analysis. I hope this document can serve as an intermediate, a comfortable medium between the familiarity of words and the elegance of mathematics.

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July 7, 2020