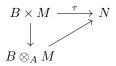
## Algebra 1 Exercise sheet 7

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## Exercise 1.

1. Lets first prove that it is well-defined. For any  $\varphi\colon M\to N$  we use the universal property of  $B\otimes_A M$ 



where  $\tau \colon B \times M \to N$  with  $\tau(b,m) = b\varphi(m)$ . It is obviously bilinear, so universal property gives us the map in the exercise. So it is well defined.

We can construct the inverse to the map given in the exercise with

$$\operatorname{Hom}_B(B \otimes_A M, N) \to \operatorname{Hom}_A(M, N)$$
  
 $\psi \mapsto (m \mapsto \psi(1 \otimes m))$ 

It is easy to check that their compositions are identities.

2. It is well defined, because it comes from a bilinear map

$$M \times N \to (M \otimes_A B) \otimes_B N$$
  
 $(m,n) \mapsto (m \otimes 1) \otimes n$ 

It is an isomorphism, because it has an inverse

$$(M \otimes_A B) \otimes_B N \to M \otimes_A N$$
$$(m \otimes b) \otimes n \mapsto m \otimes (bn)$$

It is easy to check that their compositions are identities.

3. We can define  $B=S^{-1}A$  and  $M=S^{-1}M_1, N=S^{-1}M_2$ . We just have to put this in previous part and use  $M\otimes_A S^{-1}A=M$ .

## Exercise 2.

1. We have to show

$$M_{\mathfrak{p}} = 0 \Longleftrightarrow M \otimes_A k(\mathfrak{p}) = 0. \tag{1}$$

Suppose  $M_{\mathfrak{p}} = 0$ . Then for every  $x \in M$  and  $q \in A \setminus \mathfrak{p}$  we have  $\frac{x}{q} = \frac{0}{1}$  which means there exists  $r \in A \setminus \mathfrak{p}$  such that rx = 0. Pick now any  $\sum_{i} n_{i} \otimes (b_{i} + \mathfrak{p}) \in M \otimes_{A} k(\mathfrak{p})$ . For every  $n_{i}$  we find  $r_{i} \in A \setminus \mathfrak{p}$  as above. Since  $r_{i} \notin \mathfrak{p}$ , it is non zero in  $A/\mathfrak{p}$  and thus invertible in  $\operatorname{Quot}(A/\mathfrak{p})$ . So we get

$$\sum_{i} n_{i} \otimes (b_{i} + \mathfrak{p}) = \sum_{i} n_{i} \otimes \left(\frac{r_{i} + \mathfrak{p}}{r_{i} + \mathfrak{p}}(b_{i} + \mathfrak{p})\right)$$

$$= \sum_{i} n_{i} \otimes \left(r_{i} \frac{1 + \mathfrak{p}}{r_{i} + \mathfrak{p}}(b_{i} + \mathfrak{p})\right)$$

$$= \sum_{i} r_{i} n_{i} \otimes \left(\frac{1 + \mathfrak{p}}{r_{i} + \mathfrak{p}}(b_{i} + \mathfrak{p})\right)$$

$$= \sum_{i} r_{i} n_{i} \otimes \left(\frac{1 + \mathfrak{p}}{r_{i} + \mathfrak{p}}(b_{i} + \mathfrak{p})\right)$$

$$= \sum_{i} 0 \otimes \left(\frac{1 + \mathfrak{p}}{r_{i} + \mathfrak{p}}(b_{i} + \mathfrak{p})\right)$$

$$= 0$$

Which proves one implication.

2. By associativity of the tensor product we have  $\operatorname{supp}(M \otimes_A N) \subseteq \operatorname{supp}(M) \cap \operatorname{supp}(N)$ . Suppose now  $M_p \neq 0$  and  $N_p \neq 0$ . We have to show  $(M \otimes_A N)_p \neq 0$ . Using properties of localizations we get  $M_p \otimes_{A_p} N_p \neq 0$ . We know  $A_p$  is a local ring with unique maximal ideal  $pA_p$ .

Not sure how to continue.

## Exercise 4. Let

$$\varphi \colon (f,T) \to A[T]$$
 (2)

be the inclusion. Then

$$id \otimes \varphi \colon (f,T) \otimes_{A[T]} (f,T) \to (f,T) \otimes_{A[T]} A[T]$$
 (3)

is not injective, because

$$(\mathrm{id} \otimes \varphi)(f \otimes T - T \otimes f) = f \otimes T - T \otimes f = fT \otimes 1 - fT \otimes 1 = 0 \tag{4}$$

but  $f \otimes T \neq T \otimes f \in (f, T) \otimes_{A[T]} (f, T)$ . Module (f, T) is not flat by proposition 1 in lecture notes 12.