§1 Definition, Examples (see §2 Atiyah-Macdonald)

<u>Defn</u> A a ring. <u>A-module</u> $\underset{\text{def}}{=}$ abelian group M together with a ring homomorphism $A \longrightarrow \mathrm{End}_{\mathrm{Ab.Grp.}}(M)$.

Equivalently, together with multiplication map $A \times M \longrightarrow M$

<u>s.th.</u>
1) $a \cdot (x+y) = a \cdot x + a \cdot y$
2) $1 \cdot x = x$
3) $(a+b) \cdot x = a \cdot x + b \cdot x$
4) $a \cdot (b \cdot x) = (ab) \cdot x$.

$\forall a, b \in A,\ x, y \in M.$

<u>Examples</u>

1) One finds from 2) & 3) that if $n \in \mathbb{Z}$, then $n \cdot x = nx$.

$\Rightarrow$ Every M has a unique structure as $\mathbb{Z}$-module, i.e.

$$\{\mathbb{Z}\text{-modules}\} = \{\text{abelian groups}\}$$

2) Axioms are identical with vector space axioms, so

$$\{k\text{-modules}\} = \{k\text{-vector spaces}\}$$

<u>Defn</u> <u>Submodule</u> of an A-module M $\underset{\text{def}}{=}$ ab. subgroup $N \subseteq M$ s.th. $a \cdot x \in N$ $\forall a \in A,\ x \in N$.

The quotient M/N then becomes A-module via

$$a \bullet (x+N) := ax + N.$$

3) A itself is an A-module via left-multiplication

$$\{\text{submodules of } A\} = \{\text{ideals of } A\}.$$

In partic., every ideal $\mathfrak{a} \subseteq A$ and every quotient $A/\mathfrak{a}$ is an A-module.

4) If $A \xrightarrow{\varphi} B$ ring map, then B becomes A-module via $a \bullet b := \varphi(a) b$.

<u>Further definitions</u> Let M, N be A-modules

•) A-module map (or homomorphism, or A-linear map) $\overline{\overline{\text{def}}}$ group homomorphism $f : M \to N$ s.th. $f(a \bullet x) = a \bullet f(x)$ $\forall a \in A$.

$\mathrm{Hom}_A(M,N) := \{A\text{-module maps } f : M \to N\}$ becomes itself A-module via $(a \bullet f)(x) := a \bullet f(x)$.

•) If $f \in \mathrm{Hom}_A(M,N)$, then

$$\ker(f), \ \mathrm{Im}(f), \ \mathrm{coker}(f) := N/\mathrm{Im}(f)$$

are all A-modules.

•) I any set, $(M_i)_{i \in I}$ tuple of A-modules. Then

$$\bigoplus_{i \in I} M_i \,, \quad \prod_{i \in I} M_i \quad \text{are again } A\text{-modules.}$$

5) An A-module M is called <u>free</u> if there exist a set I and an isomorphism (= bijective homomorphism)

$$M \cong A^{\oplus I} := \bigoplus_{i \in I} A.$$

In general, for any A-module M,

$$\mathrm{Hom}_A(A^{\oplus I}, M) = \prod_I A$$

$$f \longmapsto (f(e_i))_{i \in I}$$

Here $e_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th position}}}{1}, 0, \dots)$ is the i-th basis element of $A^{\oplus I}$.

Def tuple of elements $(x_i)_{i \in I}$, $x_i \in M$ called <u>basis</u>

$\overline{\overline{\text{def}}}$ the map $A^{\oplus I} \longrightarrow M$, $e_i \mapsto x_i$ is an isomorphism.

Equivalent: For every $x \in M$, there is a unique tuple $(a_i)_{i \in I}$ of $a_i \in A$, almost all $= 0$, s.th.

$$x = \sum_{i \in I} a_i \cdot x_i .$$

For vector spaces, the following statements hold:

1) Every vsp has a basis.
2) Every maximal set of lin. indep. elements is a basis.
3) Every minimal set of generators is a basis.

Such statements do not hold for modules in general.

E.g. consider $A = \mathbb{Z}$, i.e. A-modules $=$ abelian groups

1) $\mathbb{Z}/n$, $n \geq 1$ is not free.

2) $2 \in \mathbb{Z}$ is maximal lin. independent:

For every $0 \neq a \in \mathbb{Z}$, $\quad \underset{A}{a} \cdot \underset{M}{2} - \underset{A}{2} \cdot \underset{M}{a} = 0$

But 2 does not generate $\mathbb{Z}$.

3) $(2,3)$ is minimal set of generators because neither 2 nor 3 generates $\mathbb{Z}$, but 2,3 not lin. indep.

6) There is a matrix description of maps between free modules:

$$\mathrm{Hom}_A(A^{\oplus m}, A^{\oplus n}) \cong M_{n\times m}(A)$$

$$\left[e_j \mapsto \sum_{i=1}^{n} a_{ij} e_i\right] \longmapsto (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}.$$

Composition of maps = multiplication of matrices as usual.

<u>An example</u> $A = k[X,Y]$. The ideal $\mathfrak{a} = (X,Y) \subseteq A$ is a sub-module. There is a surjection

$$A^{\oplus 2} \xrightarrow{(X\ Y)} (X,Y)$$

$$\begin{pmatrix} f \\ g \end{pmatrix} \longmapsto (X\ Y)\begin{pmatrix} f \\ g \end{pmatrix} = Xf + Yg.$$

We compute its kernel:

$Xf + Yg = 0 \Longrightarrow X|g,\ Y|f$ because X, Y are prime elements with $X \nmid Y$, $Y \nmid X$.

Write $f = Yf'$, $g = Xg'$. Then need to solve

$XY(f' + g') = 0 \iff f' = -g'$ since $XY \neq 0$ in the integral domain A.

$$\Longrightarrow A \xrightarrow[\cong]{\begin{pmatrix} -Y \\ X \end{pmatrix}} \ker((X\ Y)), \quad \text{i.e. the kernel is a free } A\text{-module of rank } 1.$$

$$f' \longmapsto f' \cdot \begin{pmatrix} -Y \\ X \end{pmatrix}$$

§2 Finiteness Properties (see §6 in A.-M.)

<u>Defn</u> •) $U \subseteq M$ a subset. <u>Submodule generated by U</u> $\overset{=}{\text{def}}$

$$(U) := \bigcap_{\substack{U \subseteq N \subseteq M \\ \text{submodule}}} N.$$

(U) agrees with $\mathrm{Im}\left(A^{\oplus U} \longrightarrow M,\ e_u \longmapsto u \right)$.

•) M called of <u>finite type</u> or <u>finitely generated</u> $\overset{=}{\text{def}}$

$M = (x_1, \ldots, x_n)$ for some $n \geq 0$ and $x_1, \ldots, x_n \in M$.

Equivalently, there exists n & a surjection

$$A^{\oplus n} \xrightarrow{\varphi} M.$$

•) Given such a φ, we may consider $\ker(\varphi) \subseteq A^{\oplus n}$ and ask if this is <u>also</u> finitely generated:

M called <u>of finite presentation</u> or <u>finitely presented</u> $\overset{=}{\text{def}}$

$\exists\, n, m$ and a map $f : A^{\oplus m} \longrightarrow A^{\oplus n}$ s.th.

$$M \cong \mathrm{coker}(f).$$

These are the modules that may be studied via finite matrices. This makes them much more accessible.

Defn A-module M noetherian $\underset{\text{def}}{=}$ Following, equivalent conditions are satisfied:

1) Every submodule $N \subseteq M$ is of finite type

2) Every ascending chain $N_1 \subseteq N_2 \subseteq \dots \subseteq M$ of submodules becomes stationary.

Proof of equivalence 1) $\Rightarrow$ 2) Given $N_1 \subseteq N_2 \subseteq \dots$, put $N := \bigcup_{i \geq 1} N_i$. By assumption, $N = (x_1, \dots, x_r)$ for suitable $x_1, \dots, x_r \in N$. If k is s.th. $x_1, \dots, x_r \in N_k$, then $N_k = N_{k+1} = \dots = N$.

2) $\Rightarrow$ 1) Given N, define $N_0 = 0$ and

$$N_i = \begin{cases} N_{i-1} + (x_i) & \text{for some } x_i \in N \setminus N_{i-1}, \text{ if exists} \\ N_{i-1} & \text{o/w}. \end{cases}$$

By assumption, there is a k s.th. $N_k = N_{k+1} = \dots = N$. Choose k minimal. Then

$$N = (x_1, \dots, x_k). \qquad \square$$

<u>Prop 1</u> Consider a submodule $K \subseteq M$ and the quotient $q: M \twoheadrightarrow Q := M/K$.

1) If M is noetherian, then K and Q are as well.

2) If K and Q are noetherian, then M is as well.

<u>Proof</u> 1) For K: Submodules of noetherian modules are noetherian by definition.

For Q: Let $N \subseteq Q$ be a submodule. Since M is noetherian, $q^{-1}(N)$ is fin. gen, say with generators $x_1, \dots, x_r$. Then $q(x_1), \dots, q(x_r)$ generate N.

2) Let $N \subseteq M$ be a submodule. By assumption, both $N \cap K$ and $q(N)$ are fin. gen., say

$$K \cap N = (x_1, \dots, x_r), \quad q(N) = (y_1, \dots, y_s).$$

Let $\tilde{y}_1, \dots, \tilde{y}_s \in N$ be lifts of the $y_1, \dots, y_s$.

<u>Claim</u> $(x_1, \dots, x_r, \tilde{y}_1, \dots, \tilde{y}_s) = N$.

Let $z \in N$ any. There exist $b_1, \dots, b_s \in A$ s.th.

$$q(z) = b_1 y_1 + \dots + b_s y_s.$$

This means $z' = z - (b_1\tilde{y}_1 + \dots + b_s\tilde{y}_s)$

$$\in \ker(q|_N) = K \cap N,$$

so there are $a_1, \dots, a_r \in A$ s.th.

$$z' = a_1 x_1 + \dots + a_r x_r$$

and we are done. $\square$

Defn A is called <u>noetherian</u> if noetherian as module over itself.

<u>Equivalent:</u> Every ideal of A is finitely generated.

<u>Cor 2</u> A noetherian ring, M a fin gen A-module. Then M is noetherian.

<u>Proof</u> •) We can write $A^{\oplus n-1} \cong A^{\oplus n} / A \oplus 0^{n-1}$. By induction and Prop 1 (2), we find that $A^{\oplus n}$ is a noetherian A-module for all $n \geq 0$.

•) M fin. gen. means there exists an n and a surjection

$$A^{\oplus n} \twoheadrightarrow M.$$

Then, Prop 1 (1) states M is noetherian. $\square$

<u>Cor 3</u> Let A be noetherian. Every finite type A-module M is even of finite presentation.

<u>Proof</u> Pick any surjection $f: A^{\oplus n} \twoheadrightarrow M$. As $A^{\oplus n}$ is noetherian (see before), $\ker(f)$ is finitely generated. $\square$.

Some examples of non-noetherian rings:

1) $k[T_1, T_2, \dots]$

The ideal $(T_1, T_2, \dots)$ is not finitely generated.

2) $C^\infty(\mathbb{R}, \mathbb{R})$

The ideals $\mathfrak{a}_n = \{ f \mid f|_{[-\frac{1}{n}, \frac{1}{n}]} \equiv 0 \}$

form an ascending, non-stabilizing chain.

3) $\mathbb{Z}[p^{1/2}, p^{1/4}, p^{1/8}, p^{1/16}, \dots] \subset \mathbb{R}$.

The ideal $(p^{1/2}, p^{1/4}, p^{1/8}, \dots)$ is not finitely generated.

§3 Noetherian Rings (see §7 in A.–M.)

<u>Prop 4</u> Let A be a noetherian ring.

1) Any quotient ring $A/\mathfrak{a}$ is noetherian.

2) Any localization $S^{-1}A$ is noetherian.

<u>Proof</u> Ideals in $A/\mathfrak{a}$ and $S^{-1}A$ are of the form $\mathfrak{b}/\mathfrak{a}$ (for $\mathfrak{a} \subseteq \mathfrak{b}$) resp. $\underbrace{\mathfrak{b} \cdot S^{-1}A \text{ (for } \mathfrak{b} \subseteq A \text{ any)}}_{\text{see last lecture}}$.

By assumption on A, all $\mathfrak{b}$ in question are fin. gen., hence all ideals in $A/\mathfrak{a}$ and $S^{-1}A$ are fin. gen. $\square$

Thm 5 (Hilbert Basis Theorem) If A is noetherian, then $A[T]$ is noetherian.

Proof Let $\mathfrak{a} \subseteq A[T]$ an ideal. Consider the ideal

$$\mathfrak{b} := \left(a \mid \exists f = a \cdot T^n + a_{n-1} T^{n-1} + \dots + a_0 \in \mathfrak{a} \right) \subseteq A.$$

A noetherian $\Rightarrow$ $\mathfrak{b} = (a_1, \dots, a_r)$ for suitable $a_i \in \mathfrak{b}$.

Pick $f_i \in \mathfrak{a}$ s.th. $f_i = a_i T^{n_i} +$ (lower terms), $i = 1, \dots, r$.

Put $n := \max_{i=1}^{r} \{ \deg f_i \}$.

Claim Any $g \in \mathfrak{a}$ can be written as $g = h + \sum_{i=1}^{r} h_i f_i$

with $h, h_1, \dots, h_r \in A[T]$, $\deg(h) < n$.

Proof Write $g = b_m T^m + b_{m-1} T^{m-1} + \dots$

If $n < m$, then we can choose $g = h$, $h_1 = \dots = h_r = 0$

and are done.

If $m \geq n$, we proceed as follows: By defn, $b_m \in \mathfrak{b}$, so

can be written as $b_m = x_1 a_1 + \dots + x_r a_r \quad x_i \in A.$

Put $g' = g - \sum_{i=1}^{r} x_i T^{m - \deg(f_i)} \cdot f_i.$

Then $g' \in \mathfrak{a}$ and $\deg(g') < \deg(g)$, so we can conclude by induction. $\square$ Claim.

<u>End of proof</u>: Consider the A-submodule $M = \bigoplus_{i=0}^{n-1} A \cdot T^i$ of $A[T]$. It is finitely generated (even $\cong A^{\oplus n}$).

A noetherian $\implies$ $\mathfrak{a} \cap M$ is a fin gen A-module.

Using the claim,

$$\mathfrak{a} = (\mathfrak{a} \cap M) + (f_1, \dots, f_r)$$

is hence finitely generated. $\square$

<u>Cor 6</u> A noetherian. Any ring of the form $B = A[T_1, \dots, T_n]/\mathfrak{a}$ is noetherian. Moreover, in this situation, $\mathfrak{a} = (f_1, \dots, f_m)$ is finitely generated.

<u>Proof</u> Induction with Thm 5 + Prop 4. $\square$

This in particular applies if A is a PID.