Algebra 1
Exercise sheet 3

Solutions by: Eric Rudolph and David Čadež

11. Mai 2023

Exercise 1.

1. Height of the ideal (0) is obviously 0. Height of (f) is 1 because of irreducibility of f (easy to see). For (π, g), with $\pi \in A$ prime and $g \in A[T]$ irreducible in $(A/\pi)[T]$, we have a chain $(0) \subsetneq (g) \subsetneq (\pi, g)$. So dimension is at least 2. But it obviously cannot be more, there cannot be $(\tau, f) \subsetneq (\pi, g)$ for some prime $\tau \in A$ and $f \in A[T]$ irreducible in $(A/\tau)[T]$. We also cannot have $(f_1) \subsetneq (f_2)$ for irreducible $f_1, f_2 \in A[T]$. So we easily excluded all possible chains of length more than 2.

2. Pick any $a = \sum_{i=0}^{\infty} a_i u^i \neq 0$. If $a_0 \neq 0$, then it is invertible anyway in $k[[u]]$. Else let j be the smallest with $a_j \neq 0$. Since we treat u as invertible, we can multiply a with $(u^{-1})^j$ and get an invertible element in $k[[u]]$. Thus a is invertible in $A[u^{-1}]$. Since we can look at $A[u^{-1}] = A[T]/_{(uT-1)}$, we deduce that the ideal $(uT-1) \subseteq A[T]$ is maximal. Also, ideal $(uT-1)$ is obviously of height 1.

Exercise 2.

1. Assumption of k being algebraically closed means that the only irreducible polynomials are those of degree 1.

   Since k is a field, $k[x]$ is a PID and thus every maximal ideal in $k[x, y] = k[x][y]$ has height 2. Only maximal ideals in $k[x][y]$ are therefore (π, g) with $\pi \in k[x]$ prime and $g \in k[x][y]$ whose image in $(k[x]/\pi)[y]$ is irreducible. Because k is algebraically closed, π must be of degree 1. That means $k[x]/\pi = k$. So g must an irreducible polynomial in $k[y]$, and thus of degree 1, which is exactly what we want to show. Leading coefficients can be 1 because k is a field and we can just multiply with their inverses.

2. First write $k[x, y, T]/(xT-1)$ and $k[u, v, T]/(uT-1)$.

First we note that $\phi(xT-1)=0$ and so $\phi(T)=T$. For sure there exist more elegant ways, but for injectivity we can suppose $\phi(g)=f(uT-1)=0+(uT-1)$. Then $\phi(g)=f(\phi(xT-1))=0$. So it remains to show that ϕ is injective as a mapping $k[x,y,T]/(xT-1)\to k[u,v,T]$. That is true since it does not decrease the degrees of polynomials.

For surjectivity it is enough to show $u,v,T\in \operatorname{im}(\phi)$. Of course $\phi(x)=u$ and $\phi(T)=T$. We want $v=\phi(t)$ for some t. We multiply with u and get $uv=u\phi(t)=\phi(xt)$. Putting it all on one side we get $\phi(xt-y)=0$. We get $xt-y=0$ and so $t=Ty$. Really $\phi(Ty)=Tuv=v$. So it is also surjective.

3.

Exercise 3. Let $n=\dim A$.

Let $p_0\subsetneq p_1\subsetneq\cdots\subsetneq p_n$ prime ideals in A. Then we can increase this chain with $p_{n+1}=p_n+(T)$ and get strictly longer chain. To see that p_{n+1} is still prime, we take $ab\in p_{n+1}$. So it is of the form $ab=\gamma_0+\gamma_1T$ for $\gamma_0\in p_n$ and $\gamma_1\in A$. Write $a=\alpha_0+\alpha_1T$ and $b=\beta_0+\beta_1T$ for $\alpha_0,\beta_0\in A$, $\alpha_1,\beta_1\in A[T]$. We get that $\alpha_0\beta_0=\gamma_0\in p_n$ and thus either $\alpha_0\in p_n$ or β_0. If former, then $a\in p_{n+1}$, otherwise $b\in p_{n+1}$. This proves the lower bound.

Let $p_0\subsetneq\cdots\subsetneq p_k$ be a chain in $A[T]$. Look at the chain

$$p_0\cap A\subsetneq\cdots\subsetneq p_n\cap A. \tag{1}$$

Since every prime ideal $p_1\in A[T]$ is either $pA[T]$ or directly above $pA[T]$ (meaning there are no other prime ideal above $pA[T]$ and below p_1), where $p=p_1\cap A$. Therefore we cannot have a chain (with strict inclusions) of more than two prime ideals in $A[T]$ that would contract to the same prime ideal in A. Then we immediately see that in the chain 1 at most two consecutive elements can be the same, therefore $k\le 2n+1$.

Exercise 4.

1. Of course $S\subseteq\iota_S^{-1}((S^{-1}A)^*)$. We also easily see that for $ab\in\iota_S^{-1}((S^{-1}A)^*)$ we have $\iota_S(a)\iota_S(b)$ invertible and thus each of them must be invertible. So the set $\iota_S^{-1}((S^{-1}A)^*)$ is saturated by itself.

   Take now $a\in\iota_S^{-1}((S^{-1}A)^*)$. That means there exist $r\in A$ and $s\in S$ such that
$$\frac{r}{s}\frac{a}{1}=\frac{1}{1}.$$
   So there exists $t\in S$ such that $t(ra-s)=0$ from which we get $tra=ts\in S$ and thus $a\in\bar{S}$, which proves the other inclusion.

2. Because $S \subseteq T$, we have $\iota_T(S) \subseteq (T^{-1}A)^*$. By universal property there exists a unique map $\iota\colon S^{-1}A \to T^{-1}A$ with $\iota_T = \iota \circ \iota_S$.

3. If ι is isomorphism, then it preserves invertible elements and thus

$$\bar{S} = \iota_S^{-1}((S^{-1}A)^*) = \iota_S^{-1}(\iota^{-1}((T^{-1}A)^*)) = \iota_T^{-1}((T^{-1}A)^*) = \bar{T}.$$

   If $\bar{S} = \bar{T}$, then from uniqueness and universal property of ι it follows that $\iota = \mathrm{id}$.