§1 Flatness

Recall an A-module M is flat $\underset{def}{=}$ for all exact $N \to P \to Q$, also $M \otimes_A N \to M \otimes_A P \to M \otimes_A Q$ is exact.

Prop 1 Equivalent:

(1) M flat A-module

(2) $\forall$ injections $N \hookrightarrow P$, $M \otimes_A N \hookrightarrow M \otimes_A P$ is again injective.

(3) same as 2), but N and P fin. gen.

Proof (1) $\Rightarrow$ (2) is clear: Apply (1) to the exact sequence $0 \to N \to P$.

(2) $\Rightarrow$ (3) is also clear.

We show (2) $\Rightarrow$ (1): Let $N \xrightarrow{f} P \xrightarrow{g} Q$ be any exact sequence. Consider the ex. seq.

$$0 \to \ker(g) \to P \to \operatorname{im}(g) \to 0.$$

a) If (2) holds, then

$$0 \longrightarrow M \otimes_A \ker(g) \longrightarrow M \otimes_A P \longrightarrow M \otimes_A \operatorname{Im}(g) \longrightarrow 0 \qquad (*)$$

is again exact. (Recall that exactness in $M \otimes_A P$ and $M \otimes_A \operatorname{Im}(g)$ is automatic by the right-exactness of $M \otimes_A -$. We only need (2) to also have exactness in $M \otimes_A \ker(g)$.)

b) The natural map $M \otimes_A \operatorname{Im}(g) \to \operatorname{Im}(\mathrm{id}_M \otimes g)$ is surjective because $\operatorname{Im}(\mathrm{id}_M \otimes g)$ is generated by the images $m \otimes g(p)$ of elementary tensors $m \otimes p$.

c) We arrive at the situation

injective by 2^{nd} application of assumption (2)

$$\begin{tikzcd} M \otimes_A \operatorname{Im}(g) \arrow[r] \arrow[dr] & M \otimes_A Q \\ & \operatorname{Im}(\mathrm{id}_M \otimes g) \arrow[u, phantom, "\subseteq"] \end{tikzcd}$$

surjective by b)

Conclusion: $M \otimes_A \operatorname{Im}(g) \xrightarrow{\sim} \operatorname{Im}(\mathrm{id}_M \otimes g)$.

d) By exactness of (*) + c), we deduce that

$$M \otimes_A \ker(g) \xrightarrow{\sim} \ker(\mathrm{id}_M \otimes g)$$

e) We have $\ker(g) = \mathrm{Im}(f)$. The natural map

$$\mathrm{id}_M \otimes f : M \otimes_A N \longrightarrow M \otimes_A \ker(g)$$

is surjective by the argument from b).

The target in e) is $\ker(\mathrm{id}_M \otimes g)$ by d), so in summary we get $\mathrm{Im}(\mathrm{id}_M \otimes f) = \ker(\mathrm{id}_M \otimes g)$, which is the exactness of

$$M \otimes_A N - M \otimes_A P - M \otimes_A Q. \qquad \square$$

(2) $\Rightarrow$ (1).

<u>Lem 2</u> Assume $\sum_{i=1}^{r} m_i \otimes p_i = 0$ in a tensor product $M \otimes_A P$.

Then there fin. gen. A-submodules $M_0 \subseteq M$, $P_0 \subseteq P$ s.th. $m_1, \ldots, m_r \in M_0$, $p_1, \ldots, p_r \in P_0$ and s.th.

$$\sum_{i=1}^{r} m_i \otimes p_i = 0 \quad \text{in} \quad M_0 \otimes_A P_0 .$$

Proof Recall the construction of $M \otimes_A P$ as

$$\Big[\bigoplus_{(m,p) \in M \times P} A \cdot m \tilde{\otimes} p \Big] \Big/ D_{M \times P} \quad \text{with}$$

$$D_{M \times P} = \left\langle \begin{array}{l} (m_1 + m_2) \tilde{\otimes} p - m_1 \tilde{\otimes} p - m_2 \tilde{\otimes} p, \\ (am) \tilde{\otimes} p - a \cdot m \tilde{\otimes} p, \ \text{etc.} \ldots \end{array} \right\rangle$$

Here $m \tilde{\otimes} p$ is just notation for the basis vector for (m,p). Its image in $M \otimes_A P$ is $m \otimes p$.

In partic., $\sum_{i=1}^{r} m_i \otimes p_i = 0 \iff \sum_{i=1}^{r} m_i \tilde{\otimes} p_i \in D_{M \times P}$.

Since $D_{M \times P} = \bigcup_{M_0 \subseteq M,\, P_0 \subseteq P \text{ fin. gen.}} D_{M_0 \times P_0}$, there

are M_0, P_0 s.th. $\sum_{i=1}^{r} m_i \tilde{\otimes} p_i \in D_{M_0 \times P_0}$.

Enlarging M_0, P_0 if necessary, may assume $m_1, \ldots, m_r \in M_0$, $p_1, \ldots, p_r \in P_0$.

Then M_0 and P_0 have the desired properties. $\square$

Proof of (3) $\Rightarrow$ (2): Given $N \overset{f}{\hookrightarrow} P$, we want to see that $M \otimes_A N \hookrightarrow M \otimes_A P$ is again injective.

So consider any $\sum_{i=1}^{r} m_i \otimes n_i \longmapsto \sum_{i=1}^{r} m_i \otimes f(n_i) = 0$.

Our aim is to use (3) to show $\sum_{i=1}^{r} m_i \otimes n_i = 0$.

By Lem 2, there is a fin. gen. P_0 s.th.
$f(n_1), \ldots, f(n_r) \in P_0$ and $\sum_{i=1}^{r} m_i \otimes f(n_i) = 0$.
in $M \otimes_A P_0$.

Also put $N_0 = \langle n_1, \ldots, n_r \rangle$. Then f restricts to a map $f|_{N_0} : N_0 \hookrightarrow P_0$ (again injective)

s.th. $(\mathrm{id}_M \otimes f)\left(\sum_{i=1}^{r} m_i \otimes n_i \right) = 0$.

By assumption (3), this implies $\sum_{i=1}^{r} m_i \otimes n_i = 0$
in $M \otimes_A N_0$,

hence a fortiori in $M \otimes_A N$.

$\square$ Prop 1.

Prop 3 Assume M is an A-module s.th. every fin. gen. submodule $M_0 \subseteq M$ is flat. Then M is flat.

Proof Given $N \xhookrightarrow{f} P$, want to see

$M \otimes_A N \hookrightarrow M \otimes_A P$ injective again.

Consider any $\sum_{i=1}^{r} m_i \otimes n_i \longmapsto \sum_{i=1}^{r} m_i \otimes f(n_i) = 0.$

As before, we find a fin. gen. $M_0 \subseteq M$ s.th. $m_1, \dots, m_r \in M_0$ and s.th. $\sum_{i=1}^{r} m_i \otimes f(n_i) = 0$ in $M_0 \otimes_A P$.

Then $\sum_{i=1}^{r} m_i \otimes n_i = 0$ in $M_0 \otimes_A N$ by flatness of M_0. Hence also $\sum_{i=1}^{r} m_i \otimes n_i = 0$ in $M \otimes_A N$. $\square$

§2 Flatness and Torsion

Def A an integral domain, M an A-module.

$x \in M$ <u>torsion</u> $\underset{def}{\Leftrightarrow}$ $\exists\, 0 \neq f \in A$ s.th. $f \cdot x = 0$.

$M_f := \{x \in M \mid fx = 0\}$ the <u>f-torsion</u>

$M_{tors} := \{x \in M \text{ torsion}\}$ the <u>torsion submodule</u>

M <u>torsion-free</u> $\underset{def}{\Leftrightarrow}$ $M_{tors} = \{0\}$.

Exercise M any. Then M/M_{tors} is torsion-free.

Example A a PID, M fin. gen. Then

M torsion-free $\Leftrightarrow$ $M \cong A^{\oplus n}$ free.

(Proof: Apply Structure Thm for fin. gen. modules over PIDs.)

Thm 4 A a PID. Then M torsion-free $\Leftrightarrow$ M flat.

Proof M torsion-free $\Rightarrow$ any fin. gen. $M_0 \subseteq M$ is torsion-free.

Structure Thm $\Rightarrow$ any such M_0 is free, hence flat.

Then by Prop 3, M is flat.

Conversely, assume M flat. $\forall\, 0 \neq f \in A$, consider the injection $A \xrightarrow{f} A$. (Injective because A is an integral domain.)

Since $M \otimes_A -$ exact, $M \xrightarrow{f} M$ is again injective which means $M_f = 0$. Having this for all f says $M_{tors} = 0$. $\square$

<u>Examples</u> We already know that $\mathbb{Q}$, $\mathbb{Z}[\frac{1}{n}]$ etc. are flat because localizations are flat.

New examples:

- $\prod_{i \in \mathbb{Z}} \mathbb{Z}$ is flat. One can prove it is not free.
- $M = \prod_{p \text{ prime}} \mathbb{F}_p$ contains the non-torsion element $(1, 1, \dots)$

  Hence M/M_{tors} is a non-zero, flat $\mathbb{Z}$-module.
- $M = \langle \frac{1}{p},\ p \text{ prime} \rangle \subseteq \mathbb{Q}$ is a subgroup that is not finitely generated and not free, but flat.

The following two results (stated without proof) generalize / improve Prop 1 and Thm 4.

Prop 5 A any ring. Then equivalent

Reference: Stacks Project Tag 00HD

1) M is a flat A-module

2) $\forall$ fin. gen. ideals $\mathfrak{a} \subseteq A$,

$$\begin{aligned} \mathfrak{a} \otimes_A M &\longrightarrow M \\ a \otimes m &\longmapsto am \end{aligned}$$

is again injective.

Remarks

a) This improves Prop 1 in that it further restricts the class of injections against which one has to check flatness.

b) The implication 1) $\Rightarrow$ 2) is immediate: Apply $M \otimes_A -$ to $\mathfrak{a} \hookrightarrow A$. The proof of 2) $\Rightarrow$ 1) is not difficult and only omitted because it is a bit lengthy.

c) The image of $\mathfrak{a} \otimes_A M \longrightarrow M$ is $\mathfrak{a} \cdot M$.

So M flat $\iff$ $\mathfrak{a} \otimes_A M \xrightarrow{\sim} \mathfrak{a}M \quad \forall \mathfrak{a}$.

d) If $\mathfrak{a} = (f)$ and if f is regular ($=$ no zero-divisor)

then $\ker\left((f) \otimes_A M \longrightarrow M \right)$ is M_f.

In this sense, Prop 5 should be understood as

"Flat $\iff$ $\mathfrak{a}$-torsion free $\forall$ ideals $\mathfrak{a}$."

<u>Prop 6</u> Let M be a <u>fin. gen.</u> A-module. Then

M flat $\iff$ $\forall \mathfrak{p} \in \mathrm{Spec}(A)$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module.

<u>Remark</u> If A is a PID and M fin. gen.

Then

M flat $\iff$ M free.

Prop 6 shows that parts of this remain true over any ring A.

<u>Reference</u>: Matsumura "Comm. Alg." Prop. (3.G)

§3 The Snake Lemma

Lem 7 Assume we are given a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0
\end{tikzcd}$$

Then there is a natural connecting map $\delta : \ker(f_3) \to \operatorname{coker}(f_1)$ s.th. the following sequence is exact:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f_1) \arrow[r] & \ker(f_2) \arrow[r] & \ker(f_3) \arrow[dll, rounded corners, "\delta"] & \\
 & \operatorname{coker}(f_1) \arrow[r] & \operatorname{coker}(f_2) \arrow[r] & \operatorname{coker}(f_3) \arrow[r] & 0
\end{tikzcd}$$

Proof Define δ in the following way:

$$\begin{tikzcd}
0 \arrow[r, dash] & M_1 \arrow[r, dash] \arrow[d] & M_2 \arrow[rr, dash] \arrow[dd] & & M_3 \arrow[rr] \arrow[dd] & & 0 \\
 & & & 2 \arrow[rr, mapsto] \arrow[dd, mapsto] & & 1 \arrow[dd, mapsto] & \\
0 \arrow[r] & N_1 \arrow[r, dash] & N_2 \arrow[rr] & & N_3 \arrow[rr] & & 0 \\
 & 3 \arrow[rr, equal] & & 3 \arrow[rr, mapsto] & & 0 &
\end{tikzcd}$$

Given $①\in \ker(f_3)$, pick any preimage $②\in M_2$.
(Such a $②$ exists by exactness of the top row.)

Then $③ := f_2(②)$ lies in N_1 by commutativity of the square on the right and exactness of the bottom row.

<u>Define</u> $\delta(①) := ③ \mod \operatorname{Im}(f_1)$.

This is <u>well-defined</u>: If $②'$ is another choice of preimage, then $② - ②' \longmapsto ① - ① = 0$,
so $② - ②' \in \ker(M_2 \longrightarrow M_3) \underset{\uparrow}{=} M_1$

by exactness of top row.

It follows that $f_2(②') - ③ \in \operatorname{Im}(f_1)$, so both define the same class in $\operatorname{coker}(f_1)$.

<u>Exactness</u> There are six places where one has to check exactness. I leave this mostly as an exercise, but do exactness in $\ker(f_3)$ as example.

Even though this is one of the most "difficult" of the six checks, the proof is simple and mechanical:

<u>$\operatorname{Im}(\ker(f_2) \to \ker(f_3)) \subseteq \ker(\delta)$:</u>

Assume $1 \in \operatorname{Im}(\ker(f_2) \to \ker(f_3))$.

Then we may pick $2 \in \ker(f_2)$.

Then $3 = 0$ and hence $\delta(1) = 0$.

<u>$\ker(\delta) \subseteq \operatorname{Im}(\ker(f_2) \to \ker(f_3))$:</u>

Given 1, let 2 and 3 be as in defn of δ.

Assume $\delta(1) = 0$, meaning $3 \in \operatorname{Im}(f_1)$.

Pick $4 \in M_1$ s.th. $f_1(4) = 3$.

By exactness of top row, $2 - 4 \mapsto 1 - 0$,

so $2' = 2 - 4 \in M_2$ is another possible preimage of 1 in M_2. But $f_2(2 - 4) = 3 - 3 = 0$,

so $2' \in \ker(f_2)$. Hence $1 \in \operatorname{Im}(\ker(f_2) \to \ker(f_3))$

as desired. $\square$

Example Consider multiplication by n on the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & n^{-1}\mathbb{Z}/\mathbb{Z} \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot n"] & \mathbb{Q} \arrow[r] \arrow[d, "\cdot n"] & \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\cdot n"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[r] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & \mathbb{Z}/n & 0 & 0 &
\end{tikzcd}$$

The Snake Lemma states that there is a natural isomorphism $\delta : n^{-1}\mathbb{Z}/\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/n$.

Explicitly For $① = \frac{a}{n} + \mathbb{Z}$, can choose $② = \frac{a}{n}$.

Then $③ = f_2(②) = n \cdot \frac{a}{n} = a$.

Thus $\delta\left(\frac{a}{n} + \mathbb{Z}\right) = a \mod (n)$..

This example applies much more generally:

A any ring, $f \in A$ any, $0 \to M_1 \to M_2 \to M_3 \to 0$ any ex. seq.

Then we may consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow \cdot f & & \downarrow \cdot f & & \downarrow \cdot f & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

The Snake Lemma states that there is an exact seq.

$$0 \to M_{1,f} \to M_{2,f} \to M_{3,f} \xrightarrow{\delta} M_1/fM_1$$
$$\to M_2/fM_2 \to M_3/fM_3 \to 0$$

Note that taking f-torsion is a left-exact functor, i.e. $\forall$ exact $0 \to M_1 \to M_2 \to M_3 \to 0$,

$0 \to M_{1,f} \to M_{2,f} \to M_{3,f}$ is exact.

The above explains how to connect this situation with the right-exact functor $M \mapsto M/fM \cong A(f) \otimes_A M$.

§4 Application The above results in ptic show the following:

Cor 8 Assume $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact and that M_3 is f-torsion-free $M_{3,f}=0$.

Then $A/(f) \otimes_A (0 \to M_1 \to M_2 \to M_3 \to 0)$ is again exact.

In the remainder, I want to hint at how this can be applied in general:

Situation: M an A-module, $0 \to M_1 \to M_2 \to M_3 \to 0$ exact.

1) Pick a presentation $A^{\oplus J} \xrightarrow{z} A^{\oplus I} \to M \to 0$

2) Since $A^{\oplus J} \otimes_A -$ and $A^{\oplus I} \otimes_A -$ are exact, the following commutative diagram has exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M_1^{\oplus J} \arrow[r] \arrow[d, "X_1"] & M_2^{\oplus J} \arrow[r] \arrow[d, "X_2"] & M_3^{\oplus J} \arrow[r] \arrow[d, "X_3"] & 0 \\
0 \arrow[r] & M_1^{\oplus I} \arrow[r] & M_2^{\oplus I} \arrow[r] & M_3^{\oplus I} \arrow[r] & 0
\end{tikzcd}$$

3) The Snake Lemma gives an exact sequence

$$0 \to \ker(X_1) \to \ker(X_2) \to \ker(X_3) \xrightarrow{\delta} M \otimes_A M_1 \to M \otimes_A M_2 \to M \otimes_A M_3 \to 0$$

<u>Conclusion</u> $M \otimes_A (0 \to M_1 \to M_2 \to M_3 \to 0)$

exact on the left $\iff$ $\ker(X_2) \to \ker(X_3)$ surjective.

<u>Cor 9</u> Assume $\mathfrak{a} = (f, g) \subseteq A$ is an ideal that is generated by two elements f, g.

Assume $A \xrightarrow{\binom{-g}{f}} A^{\oplus 2} \xrightarrow{(f\ g)} \mathfrak{a} \to 0$ is exact.

(Examples: $(X, Y) \subseteq R[X, Y]$ R any ring, more generally $(f, Y) \subseteq R[Y]$ with $f \in R$ regular (= not zero-divisor))

If $M_{3,f} \cap M_{3,g} = \{0\}$, then

$$\mathfrak{a} \otimes_A (0 \to M_1 \to M_2 \to M_3 \to 0)$$

is again exact.

<u>Proof</u> Apply the Snake Lemma to

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\begin{pmatrix}-g\\ f\end{pmatrix}"'] & M_2 \arrow[r] \arrow[d, "\begin{pmatrix}-g\\ f\end{pmatrix}"'] & M_3 \arrow[r] \arrow[d, "\begin{pmatrix}-g\\ f\end{pmatrix}"'] & 0 \\ 0 \arrow[r] & M_1^{\oplus 2} \arrow[r] & M_2^{\oplus 2} \arrow[r] & M_3^{\oplus 2} \arrow[r] & 0 \end{tikzcd}$$

$\square$