Algebra 1
Exercise sheet 5

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8. Mai 2023

Exercise 1. Pick an ascending chain of ideals in A

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots .$$

For every $j = 1, \dots, n$ define $p_j \colon A \to A/a_j$ and then $p_j(I_i) \subseteq A/a_j$ and observe the chains

$$p_j(I_0) \subseteq p_j(I_1) \subseteq p_j(I_2) \subseteq \cdots .$$

Since they all terminate, pick M to be the index when the longest one terminates. Suppose $0 \neq a \in I_{M+1} \setminus I_M$. Because the intersection of a_j is $\{0\}$, we have $a \notin a_j$ for some j. Fix that j. That means $p_j(a) \neq 0$ for that j. That means $p_j(a) \in p_j(I_{M+1}) \setminus p_j(I_M)$, which is contradiction.

Since $p_j(I_{M+1}) \setminus p_j(I_M) = \emptyset$, we have $a \in p_j(I_M)$ for every j. That means $a = b_j + c_j$ for $b_j \in I_M$ and $c_j \in a_j$. Look at

$$a^n = (b_1 + c_1) \cdots (b_n + c_n) \in I_M$$