Algebra 1
Exercise sheet 3

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Exercise 1.

1. So $\{f_n\}_n \in A[[T]]$ is a sequence of elements such that $f_n \in (T)^n$. Then we can define $f = \sum_{n=0}^{\infty} f_n$, because every coefficient will have only finitely many summands. Then we obviously have $f - \sum_{k=0}^{n} f_k \in (T)^{n+1}$ by the definition of f. Also, if there would be $g \in A[[T]]$ which is not equal to f at the coefficient at degree m, then $g - \sum_{k=0}^{m} f_k \notin (T)^{m+1}$.

2. Suppose A is noetherian and $I \subseteq A[[T]]$ ideal. At the lectures we have shown that in this case $A[T]$ is noetherian, which we will use. Define the set
$$B = \{aT^n \mid \exists f \in I \colon f = aT^n + \text{higher terms}\}$$
and the ideal it generates
$$J = (B) \subseteq A[T].$$

Claim. For any generating set B of an ideal J in a noetherian ring A, we can find a finite subset $C \subseteq B$ that generates this ideal J.

Beweis. Set $C = \{c\}$ for any $c \in B$. If $(C) = J$, we are done. Otherwise pick any $a \in J \setminus (C)$ and write $a = \sum_{i=1}^{n} a_i c_i$. Since $a \notin (C)$, there is $c_i \notin C$ and we can add all these c_i to C. In such a step we increase (C) by at least a. We also add finitely many elements in every step. Since the ring is noetherian, this process terminates and we get a finite set $C \subseteq B$ that generates J. $\square$

Since $A[T]$ is neotherian and B generates J, we can find $\{a_1 T^{n_1}, \dots, a_k T^{n_k}\} \subseteq B$ such that $J = (a_1 T^{n_1}, \dots, a_k T^{n_k})$.

Denote with $f_i = a_i T^{n_i} + \text{higher terms} \in I$ suitable $f_i \in I$. Pick any $g = bT^m + \text{higher terms} \in I$. Then we have $c_m^1, \dots, c_m^k \in A[T]$ such that

$\sum_{i=1}^k c_m^i a_i T^{n_i} = bT^m$. Observe that c_m^i can be taken to be monomials (powers $\neq m$ cancel anyway). So

$$g - \sum_{i=1}^{k} c_m^i f_i \in I \cap (T)^{m+1}.$$

Continuing so forth we get power series $g_1, \ldots, g_k \in A[[T]]$ defined by

$$g_i = c_m^i T^m + c_{m+1}^i T^{m+1} + c_{m+2}^i T^{m+2} \cdots$$

Note that $c_m^i \in A[T]$, so this does not present the coefficients exactly. But such g_i exists and is unique, which we can show using first part of the exercise. By construction we have

$$g = \sum_{i=1}^{k} g_i f_i,$$

which shows $I = (f_1, \ldots, f_k)$.

Exercise 2.

1. First observe that every element $f \in \mathbb{C}[[z]]$ with non-zero constant term is invertible in the ring of power series $\mathbb{C}[[z]]$ with positive radius of convergence. We know it is invertible as a formal power series from the lectures. The radius of convergence is positive, since the inverse $\frac{1}{f}$ is bounded on some small neighbourhood around 0 (follows simply from continuity of f). So there is a ball around 0 where $\frac{1}{f}$ does not have singularities and because the radius of convergence is the distance to the nearest singularity, it is positive.

   Let $I \subseteq \mathbb{C}[[z]]$ be an ideal. Pick $f \in I$. If constant term of f is non-zero, then $I = (1)$. Otherwise there exists $k \in \mathbb{N}$ such that $f = z^k g$, where g is a unit. This k is just the position of the first non-zero coefficient. So $z^k \in I$.

   So I is clearly defined by the minimum position of non-zero coefficient over all elements $f \in I$. Let $l \in \mathbb{N}$ be such. Then for every $h \in I$ either $h = z^l g$ for some $g \in \mathbb{C}[[z]]$ or $h = z^m g$ for some unit g. In first case we have $h \in (z^l)$ and in the other contradiction with the minimality of l. So $I = (z^l)$. So $\mathbb{C}[[z]]$ seems to even be a PID.

2. We claim $(\sin(x)) \subsetneq (\sin(\frac{x}{2})) \subsetneq (\sin(\frac{x}{4})) \subsetneq (\sin(\frac{x}{8})) \subsetneq \cdots$ is an infinite chain that does not terminate. Inclusions follow from equation

   $$\sin(\frac{x}{2^{n+1}}) \cos(\frac{x}{2^{n+1}}) = \sin(\frac{x}{2^n}).$$

   And they are strict, because

   $$\frac{\sin(\frac{x}{2^{n+1}})}{\sin(\frac{x}{2^n})}$$

is not a holomorphic function, it has a pole at $2^n\pi$.

Exercise 3.

1.

2.

Exercise 4. Let A be a PID.

1. Let $a \in A \setminus \{0\}$ and $\pi \in A$ prime.

   Lets first suppose that $\pi^{n+1} \nmid a$ and show $\dim_{A/\pi} \pi^n B/\pi^{n+1}B = 0$.

   Pick $\pi^n b + (a) \in \pi^n B$. Since $\gcd(\pi^{n+1}, a) = \pi^n$, we have $\alpha, \beta \in A$ such that $\alpha\pi^{n+1} + \beta a = \pi^n$. So $\alpha b\pi^{n+1} + \beta ba = b\pi^n$ and since $\beta ba \in (a)$ we have $b\pi^n + (a) = \alpha b\pi^{n+1} + (a) \in \pi^{n+1}B$. This proves $\pi^{n+1}B \subseteq \pi^n B$, which we had to show.

   Lets suppose now $\pi^{n+1} \mid a$. Write $a = u\pi^{n+1}$. We claim $\pi^n + (a) \in \pi^n B \setminus \pi^{n+1}B$. That is true, because $\pi^n + xa = \pi^n + xu\pi^{n+1}$ is not divisible by π^{n+1} for any $x \in A$. So we have found a non-trivial element in the vector space $\pi^n B/\pi^{n+1}B$ and the dimension must be at least 1. To show it is exactly 1 we can show that every two elements are linearly dependent. Pick $\pi^n b + (a), \pi^n c + (a) \in \pi^n B$. If $\pi \mid b$ or $\pi \mid c$, then one of the vector is 0 and they are linearly dependent. Otherwise pick $\alpha + (\pi), \beta + (\pi) \in A/\pi$ such that $\alpha b + (\pi) + \beta c + (\pi) = 0 + (\pi)$ (here we use gcd and the fact that A is a PID again). Then $\alpha\pi^n b + \beta\pi^n c + (a) = \pi^n(\alpha b + \beta c) + (a) \in \pi^{n+1}B$ and thus a zero vector.

2. Suppose $M = A^r \oplus A/a_1 \oplus \cdots \oplus A/a_k$, $N = A^s \oplus A/b_1 \oplus \cdots \oplus A/b_l$ with $a_1, \dots, a_k, b_1, \dots, b_l \in A$ non-zero and $a_1 \mid a_2 \mid \dots \mid a_k, b_1 \mid b_2 \mid \dots \mid b_l$. Suppose also $M \cong N$ as A-modules.

   If we have an isomorvarphism $\varphi\colon M \to N$, we can easily show $\mathrm{Tor}(M) \cong \mathrm{Tor}(N)$. Then we can quotient with these torsion parts and get isomorvarphism between free modules, which are isomorvarphic exactly when their ranks are the same. So we have $r = s$. Also, since φ maps torsion elements to torsion elements, we can remove free parts of both M and N.

   For every $x \in A/a_1 \oplus \cdots \oplus A/a_k$ we have $a_1 x = 0$ and thus $a_1\varphi(x) = 0$. Because φ is surjective, we get $a_1 \mid b_1$. By the same argument, only using φ^{-1} we get $b_1 \mid a_1$. So $a_1 = u_1 b_1$ for some unit $u_1 \in A$. Suppose now $a_2 \neq u_2 b_2$ for any unit $u_2 \in A$. WLOG $b_2 \nmid a_2$. Then $b_2\varphi(1) \in A/b_1 \subseteq N$ where $1 \in A/a_2 \subseteq M$. Since $0 \neq b_2 1 \in A/a_2$, we have $0 \neq \varphi(b_2 1) \in A/b_1$ and so ... We somehow continue this process and maybe show what we need to.