§1 Localization of modules (Atiyah-Macdonald §3)

A ring, $S \subseteq A$ mult. subset, M an A-module.

Def $S^{-1}M \underset{def}{=} \{ m/s \mid m \in M, s \in S \} / \sim$ where

$$m_1/s_1 \sim m_2/s_2 \underset{def}{\Longleftrightarrow} \exists\, s_3 \in S \text{ s.th. } s_3 s_1 m_2 = s_3 s_2 m_1 .$$

Addition $\quad m_1/s_1 + m_2/s_2 := (s_2 m_1 + s_1 m_2)/s_1 s_2 \qquad (1)$

A-module structure $\quad a \cdot m/s := (am)/s. \qquad (2)$

It can be checked that (1) and (2) are well-defined and make $S^{-1}M$ into an A-module. Moreover, any $f: M \to N$ induces a map $S^{-1}f : S^{-1}M \to S^{-1}N$ by $m/s \mapsto f(m)/s$.

Prop 1 S^{-1} is an exact operation $\underset{def}{=}$ If $M \xrightarrow{f} N \xrightarrow{g} P$ is exact, then $S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}P$ is exact as well.

Proof $g \circ f = 0 \Longrightarrow S^{-1}g \circ S^{-1}f = 0$, so

$$\operatorname{Im}(S^{-1}f) \subseteq \ker(S^{-1}g).$$

It is left to show $\supseteq$.

So assume $(S^{-1}g)(n/s) = g(n)/s = 0$. This means there is a $t \in S$ s.th. $t \cdot g(n) = g(tn) = 0$.

Hence $t \cdot n \in \ker(g) = \mathrm{Im}(f)$

by assumption.

Say $t \cdot n = f(m)$. Then we have

$$(S^{-1}f)(m/st) = f(m)/st = tn/st = n/s$$

and the proof is complete. $\square$

<u>Prop 2</u> There is an isomorphism

$$S^{-1}A \otimes_A M \xrightarrow{\ \sim\ } S^{-1}M \qquad (*)$$

$$(a/s) \otimes m \longmapsto (am)/s.$$

<u>Proof</u> One first checks that

$$S^{-1}A \times M \longrightarrow S^{-1}M$$
$$(a/s, m) \longmapsto (am)/s$$

is well-defined and A-bilinear. Hence the map $(*)$ exists.

Since $1/s \otimes m \mapsto m/s$, it is surjective. We need to see injectivity: Assume $\displaystyle\sum_{i=1}^{n} a_i/s_i \otimes m_i \longmapsto 0.$

We can also write this element as

$$\sum_i a_i \prod_{j \neq i} s_j / s \otimes m_i = 1/s \otimes m,$$

with $s = \prod_{i=1}^{n} s_i$ and $m = \sum_i a_i \prod_{j \neq i} s_j \cdot m_i$

By assumption, $m/s = 0$, i.e. $t \cdot m = 0$ for some $t \in S$.

Then $1/s \otimes m = t/_{st} \otimes m = 1/_{st} \otimes (tm) = 0$

and we win. $\square$

In particular, $S^{-1}M$ is an $S^{-1}A$-module via

$$a/s \bullet m/t := am/st.$$

<u>Example / Cor 3</u> 1) $f: M \to N$ an A-linear map.

Then $\ker(S^{-1}f) \cong S^{-1}\ker(f)$, $\operatorname{coker}(S^{-1}f) = S^{-1}\operatorname{coker}(f)$

and $\operatorname{Im}(S^{-1}f) \cong S^{-1}\operatorname{Im}(f)$

2) $M_1, M_2 \subseteq N$ two submodules. Then

$$S^{-1}M_1 \cap S^{-1}M_2 = S^{-1}(M_1 \cap M_2) \quad \text{and}$$

$$S^{-1}M_1 + S^{-1}M_2 = S^{-1}(M_1 + M_2).$$

Some further properties (Check these!):

1) $S^{-1}(S^{-1}M) \cong S^{-1}M$.

2)
$$\begin{aligned} S^{-1}M \otimes_A N &\xrightarrow{\sim} S^{-1}M \otimes_A S^{-1}N \\ &\xrightarrow{\sim} S^{-1}M \otimes_{S^{-1}A} S^{-1}N \\ &\xrightarrow{\sim} S^{-1}(M \otimes_A N) \end{aligned}$$

uniquely characterized by $m \otimes n \mapsto m \otimes n$ for $m \in M$, $n \in N$.

3) Universal property:

Let N be an $S^{-1}A$-module ($\hat{=}$ A-module s.th. any $s \in S$ acts bijectively.) Then any A-linear $f: M \to N$ factors uniquely through $S^{-1}M$:

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "f"'] & S^{-1}M \arrow[d, dashed, "\exists!"] \\ & N \end{tikzcd}$$

Rmk This holds much more generally: Let $A \to B$ be an A-algebra, M an A-module, N a B-module.

Then
$$\begin{aligned} \mathrm{Hom}_A(M, N) &\cong \mathrm{Hom}_B(B \otimes_A M, N) \\ \phi &\mapsto [b \otimes m \mapsto b \cdot \phi(m)]. \end{aligned}$$

Example: Consider $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n \to 0$,

Apply $\mathbb{Q} \otimes_{\mathbb{Z}} -$:

$$\begin{array}{ccccccccc} & & & & & & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n & & \\ & & & & & & \| & & \\ 0 & \to & \mathbb{Q} & \xrightarrow{\cdot n} & \mathbb{Q} & \to & 0 & \to & 0 \end{array}$$

is again exact.

Example: If $M \cong A^{\oplus I}$ is free, then $S^{-1}M \cong (S^{-1}A)^{\oplus I}$ because $S^{-1}A \otimes_A -$ and $(\ )^{\oplus I}$ commute.

E.g. $\{p^{\mathbb{Z}_{\geq 0}}\}^{-1} \mathbb{Z}[T] \cong \mathbb{Z}[p^{-1}][T]$

(consider as $\mathbb{Z}$-module)

But S^{-1} does not in general commute w/ infinite products:

$$\mathbb{Z}[\![T]\!][p^{-1}] \subsetneq (\mathbb{Z}[p^{-1}])[\![T]\!].$$

because e.g. $1 + T/p + T^2/p^2 + T^3/p^3 + \cdots \notin$ LHS.

§2 Passing to local rings

<u>Notation</u> $\mathfrak{p} \in \operatorname{Spec} A$. Then $M_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1} M$ called localization of M in $\mathfrak{p}$. It is an $A_{\mathfrak{p}}$-module.

<u>Prop 4</u> Let M be an A-module. Equivalent:

(1) $M = 0$

(2) $M_{\mathfrak{p}} = 0 \quad \forall\, \mathfrak{p} \in \operatorname{Spec} A$

(3) $M_{\mathfrak{m}} = 0 \quad \forall\, \mathfrak{m} \in \operatorname{Max\,Spec} A$.

<u>Proof</u> (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear. Assume (3), let $x \in M$. We want to see $x = 0$. Consider the <u>annihilator ideal</u> of x: $\operatorname{Ann}(x) := \{ a \in A \mid a \cdot x = 0 \}$.

Now $M_{\mathfrak{m}} = 0 \;\Rightarrow\; x/1 = 0$ in $M_{\mathfrak{m}}$

$\Leftrightarrow\; \exists\, s \in (A \setminus \mathfrak{m})$ s.th. $s \cdot m = 0$

Def of $S^{-1}M$

$\Rightarrow\; \operatorname{Ann}(x) \not\subseteq \mathfrak{m}$.

We have this for all $\mathfrak{m}$, so $\operatorname{Ann}(x) = A$, in ptic $x = 1 \cdot x = 0$. $\square$

<u>Prop 5</u> (Converse to Prop 1) Let $M \xrightarrow{f} N \xrightarrow{g} P$ be s.th. $g \circ f = 0$. Assume for all $\mathfrak{m} \in \mathrm{MaxSpec}(A)$,

$$M_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} N_{\mathfrak{m}} \xrightarrow{g_{\mathfrak{m}}} P_{\mathfrak{m}} \quad \text{is exact.}$$

Then $M \xrightarrow{f} N \xrightarrow{g} P$ is exact.

<u>Proof</u> By Prop 1, localization is an exact operation, so

$$\ker(g_{\mathfrak{m}}) / \mathrm{Im}(f_{\mathfrak{m}}) = \left(\ker(g) / \mathrm{Im}(f)\right)_{\mathfrak{m}} \quad @.$$

By assumption, $@ = 0$ $\forall \mathfrak{m}$. Thus Prop 4 implies $\ker(g)/\mathrm{Im}(f) = 0$, i.e. that $M \to N \to P$ is exact. $\square$

<u>Cor 6</u> Let $f : A^{\oplus J} \longrightarrow A^n$ be a map s.th.

$$I_n(f) := \left(\det(f_Q) \;\middle|\; Q \subset J,\ |Q| = n \right)$$

quadratic minor w/ columns J.

equals the unit ideal. Then f is surjective.

(<u>Rmk</u> This answers an open question from Lecture 7.)

Proof By Prop 5, it suffices to show that $f_{\mathfrak{m}}$ is surjective for all max ideals $\mathfrak{m}$. But $A_{\mathfrak{m}}$ is a local ring, so $A_{\mathfrak{m}}^\times = A_{\mathfrak{m}} \setminus \mathfrak{m}A_{\mathfrak{m}}$. Hence

$I_n(f) = (1) \implies \exists\, Q \subseteq J$ (depending on $\mathfrak{m}$) s.th. $\det(f_Q) \in A_{\mathfrak{m}}^\times$. Hence

$f_{Q,\mathfrak{m}} = f_{\mathfrak{m},Q} = f_{\mathfrak{m}} \circ incl_Q$ is an isomorphism, so $f_{\mathfrak{m}}$ surjective.

$$\begin{tikzcd}
A_{\mathfrak{m}}^n \arrow[r, hook, "incl_Q"] \arrow[dr, "f_{\mathfrak{m},Q}"', "\cong"] & A_{\mathfrak{m}}^{\oplus J} \arrow[d] \\
& A_{\mathfrak{m}}^n .
\end{tikzcd}$$

$\square$

Nakayama's Lemma is an equivalent way of stating this by applying it to $\mathrm{coker}(f)$:

Cor 7 (Nakayama's Lemma) Let A be a local ring with max ideal $\mathfrak{m}$ and M a fin. gen. A-module. Then

$$M/\mathfrak{m}M = 0 \implies M = 0.$$

Proof Choose a presentation (J need not be finite)

$$A^{\oplus J} \xrightarrow{f} A^{\oplus n} \longrightarrow M \longrightarrow 0.$$

Recall that $M/\mathfrak{m}M = \kappa(\mathfrak{m}) \otimes_A M$ and that $\kappa(\mathfrak{m}) \otimes_A -$ is right exact. So

$$\kappa(\mathfrak{m})^{\oplus J} \xrightarrow{f \bmod \mathfrak{m}} \kappa(\mathfrak{m})^n \longrightarrow \underbrace{M/\mathfrak{m}M}_{=0} \longrightarrow 0$$

is again exact, meaning ($f \bmod \mathfrak{m}$) is surjective.

Thus there is $Q \subseteq J$, $|Q| = n$, s.th. $\det(f_Q) \notin \mathfrak{m}$

Since A local, this means $\det(f_Q) \in A^\times$ and we conclude from Cor 7 that f is surjective. This means $M = \operatorname{coker}(f) = 0$ as cl · d. $\square$

Cor 8 (Variant) A any ring, N a fin gen A-module, $M \xrightarrow{f} N$ an A-linear map s.th.

$$f \bmod \mathfrak{m} : M/\mathfrak{m}M \longrightarrow N/\mathfrak{m}N$$

surjective for all max ideals $\mathfrak{m} \subseteq A$. Then f surjective. $\square$

Example Consider $M = \mathbb{Z} \hookrightarrow N = \mathbb{Q}$ which is not surjective. But for all primes $p \in \mathbb{Z}$,

$$\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Q}/p\mathbb{Q} = 0$$

is surjective. This shows that the assumption for N being fin. gen. in Cor. 7 and 8 is necessary.

§3 Flatness

Defn A-module M is flat $\underset{\text{def}}{=}$ $\forall$ exact $N \to P \to Q$,

$$M \otimes_A N \to M \otimes_A P \to M \otimes_A Q \quad \text{is again exact.}$$

Examples 1) A is flat as A-module.

2) If $(M_i)_{i \in I}$ are all flat, then $\bigoplus_{i \in I} M_i$ is flat.

In particular, any free module $M \cong A^{\oplus I}$ is flat.

3) By Prop. 1 and 2, $S^{-1}A$ is flat. More generally, if M is flat, then $S^{-1}M$ is flat (see the properties of $\otimes$ and S^{-1} on page 4.)

Example Let $f \in A$ be regular (i.e. $A \xrightarrow{\cdot f} A$ injective).
Then $A/(f)$ is not flat as A-module:

$$A/(f) \otimes_A \left(A \xrightarrow{\cdot f} A\right) \cong A/(f) \xrightarrow{\cdot 0} A/(f)$$

is not injective anymore. (Of course, $A/(f)$ is flat as $A/(f)$-module.)

Prop 9 Let M be an A-module. Equivalent:

(1) M is flat as A-module

(2) $M_{\mathfrak{p}}$ is flat as $A_{\mathfrak{p}}$-module $\forall \mathfrak{p} \in \operatorname{Spec} A$

(3) $M_{\mathfrak{m}}$ is flat as $A_{\mathfrak{m}}$-module $\forall \mathfrak{m} \in \operatorname{MaxSpec} A$.

Proof (1) $\Rightarrow$ (2): Let $N \xrightarrow{f} P \xrightarrow{g} Q$, be an exact sequence of $A_{\mathfrak{p}}$-modules. It holds that

$$\begin{aligned} M \otimes_A N &\xrightarrow{\sim} M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N \\ m \otimes n &\longmapsto m/1 \otimes n \end{aligned}$$

(see page 4)

and similarly for P, Q. So (2) follows from (1).

$(2) \Rightarrow (3)$ is clear.

$(3) \Rightarrow (1)$: Let $N \xrightarrow{f} P \xrightarrow{g} Q$ be ex seq of A-modules. By assumption,

$$M_m \otimes_{A_m} N_m \longrightarrow M_m \otimes_{A_m} P_m \longrightarrow M_m \otimes_{A_m} Q_m$$

is exact. It holds that

$$M_m \otimes_{A_m} N_m = (M \otimes_A N)_m \qquad \text{(see page 4)}$$

and similarly for P, Q, so we can apply Prop 5 to conclude that

$$M \otimes_A N \longrightarrow M \otimes_A P \longrightarrow M \otimes_A Q$$

is exact. $\square$