Algebra 1
Exercise sheet 7

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Exercise 1.

1. Lets first prove that it is well-defined. For any $\varphi\colon M \to N$ we use the universal property of $B \otimes_A M$

$$\begin{tikzcd} B \times M \arrow[r, "\tau"] \arrow[d] & N \\ B \otimes_A M \arrow[ur] & \end{tikzcd}$$

   where $\tau\colon B \times M \to N$ with $\tau(b, m) = b\varphi(m)$. It is obviously bilinear, so universal property gives us the map in the exercise. So it is well defined.

   We can construct the inverse to the map given in the exercise with

$$\begin{aligned} \mathrm{Hom}_B(B \otimes_A M, N) &\to \mathrm{Hom}_A(M, N) \\ \psi &\mapsto (m \mapsto \psi(1 \otimes m)) \end{aligned}$$

   It is easy to check that their compositions are identities.

2. It is well defined, because it comes from a bilinear map

$$\begin{aligned} M \times N &\to (M \otimes_A B) \otimes_B N \\ (m, n) &\mapsto (m \otimes 1) \otimes n \end{aligned}$$

   It is an isomorphism, because it has an inverse

$$\begin{aligned} (M \otimes_A B) \otimes_B N &\to M \otimes_A N \\ (m \otimes b) \otimes n &\mapsto m \otimes (bn) \end{aligned}$$

   It is easy to check that their compositions are identities.

3. We can define $B = S^{-1}A$ and $M = S^{-1}M_1, N = S^{-1}M_2$. We just have to put this in previous part and use $M \otimes_A S^{-1}A = M$.

Exercise 2.

1. We have to show
$$M_{\mathfrak{p}} = 0 \iff M \otimes_A k(\mathfrak{p}) = 0. \tag{1}$$
Suppose $M_{\mathfrak{p}} = 0$. Then for every $x \in M$ and $q \in A \setminus \mathfrak{p}$ we have $\frac{x}{q} = \frac{0}{1}$ which means there exists $r \in A \setminus \mathfrak{p}$ such that $rx = 0$. Pick now any $\sum_i n_i \otimes (b_i + \mathfrak{p}) \in M \otimes_A k(\mathfrak{p})$. For every n_i we find $r_i \in A \setminus \mathfrak{p}$ as above. Since $r_i \notin \mathfrak{p}$, it is non zero in $A/\mathfrak{p}$ and thus invertible in $\mathrm{Quot}(A/\mathfrak{p})$. So we get
$$\begin{aligned}\sum_i n_i \otimes (b_i + \mathfrak{p}) &= \sum_i n_i \otimes \left(\frac{r_i + \mathfrak{p}}{r_i + \mathfrak{p}}(b_i + \mathfrak{p})\right) \\ &= \sum_i n_i \otimes \left(r_i \frac{1 + \mathfrak{p}}{r_i + \mathfrak{p}}(b_i + \mathfrak{p})\right) \\ &= \sum_i r_i n_i \otimes \left(\frac{1 + \mathfrak{p}}{r_i + \mathfrak{p}}(b_i + \mathfrak{p})\right) \\ &= \sum_i r_i n_i \otimes \left(\frac{1 + \mathfrak{p}}{r_i + \mathfrak{p}}(b_i + \mathfrak{p})\right) \\ &= \sum_i 0 \otimes \left(\frac{1 + \mathfrak{p}}{r_i + \mathfrak{p}}(b_i + \mathfrak{p})\right) \\ &= 0\end{aligned}$$
Which proves one implication.

2. By associativity of the tensor product we have $\mathrm{supp}(M \otimes_A N) \subseteq \mathrm{supp}(M) \cap \mathrm{supp}(N)$. Suppose now $M_p \neq 0$ and $N_p \neq 0$. We have to show $(M \otimes_A N)_p \neq 0$. Using properties of localizations we get $M_p \otimes_{A_p} N_p \neq 0$. We know A_p is a local ring with unique maximal ideal pA_p.

   Not sure how to continue.

Exercise 4. Let
$$\varphi\colon (f, T) \to A[T] \tag{2}$$
be the inclusion. Then
$$\mathrm{id} \otimes \varphi\colon (f, T) \otimes_{A[T]} (f, T) \to (f, T) \otimes_{A[T]} A[T] \tag{3}$$
is not injective, because
$$(\mathrm{id} \otimes \varphi)(f \otimes T - T \otimes f) = f \otimes T - T \otimes f = fT \otimes 1 - fT \otimes 1 = 0 \tag{4}$$
but $f \otimes T \neq T \otimes f \in (f, T) \otimes_{A[T]} (f, T)$. Module (f, T) is not flat by proposition 1 in lecture notes 12.