§1 Exact sequences

<u>Def</u> A sequence $\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$

of maps of A-modules is called <u>exact</u> if $\mathrm{Im}(f_i) = \ker(f_{i+1})$

for all i.

<u>Example 1</u>

$0 \longrightarrow M \xrightarrow{f} N$ exact $\Leftrightarrow$ f injective

$N \xrightarrow{f} P \longrightarrow 0$ exact $\Leftrightarrow$ f surjective

$0 \longrightarrow M \longrightarrow N \xrightarrow{f} P$ exact $\Leftrightarrow$ $M \cong \ker(f)$

$M \xrightarrow{f} N \longrightarrow P \longrightarrow 0$ exact $\Leftrightarrow$ $P \cong \mathrm{coker}(f)$

<u>Short exact sequence</u> $\underset{\text{def}}{=}$ a sequence of the form

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

<u>E.g.</u> $\quad 0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$

such as $\quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$

<u>E.g.</u> $\quad 0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0$

$$x \longmapsto (x,0)$$
$$(x,y) \longmapsto y$$

Example 2 $A = k[\varepsilon]/(\varepsilon^2)$. Then following is exact:

$$\longrightarrow A \xrightarrow{\cdot\varepsilon} A \xrightarrow{\cdot\varepsilon} A \xrightarrow{\cdot\varepsilon} A \longrightarrow A/(\varepsilon) \longrightarrow 0 .$$

Example 3 Any map $f: M \to N$ can be extended to a four-term exact sequence:

$$0 \longrightarrow \ker(f) \longrightarrow M \xrightarrow{f} N \longrightarrow \operatorname{coker}(f) \longrightarrow 0$$

Prop 4 Assume $M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is exact. Let Q be any A-module. Then

$$M \otimes_A Q \xrightarrow{f \otimes \mathrm{id}_Q} N \otimes_A Q \xrightarrow{g \otimes \mathrm{id}_Q} P \otimes_A Q \longrightarrow 0$$

is exact.

Proof 1) Exactness in $P \otimes_A Q$: This means $g \otimes \mathrm{id}_Q$ surjective. Seen last lecture: For any two A-modules X and Y, $X \otimes_A Y$ is generated by the image of the universal bilinear map $X \times Y \longrightarrow X \otimes_A Y$, $(x,y) \longmapsto x \otimes y$.
(Recall Image = set of elementary tensors.)

Since $N \times Q \xrightarrow{(g,\, id_Q)} P \times Q$ is surjective by assumption, every elementary tensor of $P \otimes_A Q$ is of the form $g(n) \otimes q$, hence $g \otimes id_Q$ surjective.

2) Exactness in $N \otimes_A Q$: This means $\ker(g \otimes id_Q)$
$$= \mathrm{Im}(f \otimes id_Q)$$

Equivalent by universal propertie of image & kernel:

Given an A-linear map ϕ, the factorization $\bar{\phi}$ exists (*) if and only if

$$\begin{tikzcd} N \otimes_A Q \arrow[r, "g \otimes id_Q"] \arrow[d, "\phi"'] & P \otimes_A Q \arrow[dl, dashed, "\bar{\phi} \text{ exists ?}"] \\ X & \end{tikzcd}$$

$\phi \circ (f \otimes id_Q) = 0$. (**)

By universal property of tensor products, (*) holds if and only if

$$\begin{tikzcd} N \times Q \arrow[r, "g \otimes id_Q"] \arrow[d, "\tilde{\phi}"'] & P \times Q \arrow[dl, dashed, "\bar{\tilde{\phi}}"] \\ X & \end{tikzcd}$$

there is the factorization $\bar{\tilde{\phi}}$ for the bilinear map $\tilde{\phi}(n, q) = \phi(n \otimes q)$.

This is equivalent to $\tilde{\phi}|_{\ker(g) \times Q} \equiv 0$.

Since $\ker(g) = \mathrm{Im}(f)$ by assumption, this is equivalent to

$$\tilde{\phi}(f(m), q) = \phi(f(m) \otimes q)$$
$$= (\phi \circ (f \otimes id_Q))(m \otimes q) = 0$$

for all $m \in M$, $q \in Q$. Since the elementary tensors $\{m \otimes q, \ m \in M, q \in Q\}$ generate $M \otimes_A Q$, this is equivalent to $(**)$ as was to be shown. $\square$

§2 Presentations and tensor products

<u>Defn</u> <u>Presentation</u> of an A-module M $\underset{def}{=}$ Choice of sets I, J and maps f, p s.th. following is exact:

$$A^{\oplus J} \xrightarrow{f} A^{\oplus I} \xrightarrow{p} M \longrightarrow 0$$

(Translation: p surjective and $\ker(p) = \operatorname{Im}(f)$.)

Put differently, we write M as

$$M \cong \bigoplus_{i \in I} A \cdot e_i \Big/ \langle u_j, \ j \in J \rangle$$

Each u_j can be written as

$$u_j = \sum_{i \in I} a_{ij} \cdot e_i \qquad \text{(for fixed } j\text{, almost all } a_{ij} \text{ zero)}$$

Lemma 5 Every module M admits a presentation

Proof Let $m_i \in M$, $i \in I$, be generators.
(Example : $I = M$ and $m_m := m$.) Then

$$A^{\oplus I} \xrightarrow{p} M, \quad p(e_i) = m_i \quad \text{is surjective.}$$

Let $u_j \in \ker(p)$, $j \in J$, be generators. Then

$$\begin{aligned} A^{\oplus J} &\xrightarrow{\quad f \quad} A^{\oplus I} \\ e_j &\longmapsto u_j \end{aligned}$$

is a surjection onto $\ker(p)$

and hence (f, p) give a presentation. $\square$

Example 6 $A = k[X, Y]$. $M = (X, Y) \subset A$.
We have seen in lecture 6 that

$$A \xrightarrow{f = \begin{pmatrix} -Y \\ X \end{pmatrix}} A^{\oplus 2} \xrightarrow{(X \ Y)} M \longrightarrow 0$$

is a presentation. Bonus feature here: f injective. This is not required by the definition.

Example 7 $\mathfrak{a} \subseteq A$ ideal generated by $(a_1, \dots, a_n)$. Then

$$\begin{aligned} A^{\oplus n} &\longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0 \\ e_i &\longmapsto a_i \end{aligned}$$

is a presentation.

Prop 7 Properties of the tensor product:

1) $A \otimes_A M \cong M$ via $a \otimes x \longmapsto ax$

2) $M \otimes_A N \cong N \otimes_A M$ via $x \otimes y \longmapsto y \otimes x$

3) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$ via $(x \otimes y) \otimes z \longmapsto x \otimes (y \otimes z)$

4) $\left(\bigoplus_{i \in I} M_i\right) \otimes N \cong \bigoplus_{i \in I} M_i \otimes N$ via $\left(\sum_i x_i\right) \otimes y \longmapsto \sum_i x_i \otimes y$.

First recall the tensor calculus rules from our proof of existence of the tensor product:

$$(ax) \otimes y = a \cdot (x \otimes y) = x \otimes (ay)$$

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y \qquad @$$

$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$

The proof strategy for Prop 7 is to use these + universal property of tensor product to construct all maps in question. I prove 1) & 2) as examples and leave 3) & 4) as exercise.

Proof 1) $A \times M \longrightarrow M$, $(a,m) \longmapsto a \cdot m$ is bilinear, hence factors through $A \otimes_A M \xrightarrow{\phi} M$

$$a \otimes m \longmapsto am.$$

It is clearly surjective. For injectivity: Let $x \in A \otimes_A M$ be any, write it as linear combination of elementary tensors, say $x = \sum_{i=1}^{n} a_i \otimes x_i$, assume

$$\phi(x) = \sum_{i=1}^{n} a_i x_i = 0.$$

By @, $x = \sum_{i=1}^{n} 1 \otimes a_i x_i = 1 \otimes \sum_{i=1}^{n} a_i x_i$

$$= 1 \otimes 0 = 0. \quad \square \ 1)$$

Rmk 1) reflects

$$\mathrm{BiHom}_A(A, M; P) \cong \mathrm{Hom}_A(A, \mathrm{Hom}_A(M,P)) \cong \mathrm{Hom}_A(M,P).$$

$$f \longmapsto [a \mapsto f(a,-)] \longmapsto f(1,-)$$

2) By @, $M \times N \longrightarrow N \otimes_A M$, $N \times M \longrightarrow M \otimes_A N$

$$(m,n) \longmapsto n \otimes m, \quad (n,m) \longmapsto m \otimes n$$

are A-bilinear, hence factor through two maps

$$M \otimes_A N \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} N \otimes_A M, \qquad m \otimes n \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} n \otimes m.$$

Hence $\Psi \circ \Phi(m \otimes n) = m \otimes n$, $\Phi \circ \Psi(n \otimes m) = n \otimes m$.

Since elementary tensors generate, this implies

$$\Psi \circ \Phi = \mathrm{id}, \quad \Phi \circ \Psi = \mathrm{id}. \qquad \square \text{ 2)}.$$

<u>Cor 8</u> 1) $A^{\oplus I} \otimes_A N \cong N^{\oplus I}$

2) Assume $M \cong \operatorname{coker}\left(A^{\oplus J} \xrightarrow{f = (a_{ij})_{i,j}} A^{\oplus I}\right)$ is a presented module.

Then $M \otimes_A N \cong \operatorname{coker}\left(N^{\oplus J} \xrightarrow{(a_{ij})_{i,j}} N^{\oplus I}\right)$

<u>Proof</u> 1) Combine Prop 7, 1) and 4).

2) By Prop 4, $M \otimes_A N \cong \operatorname{coker}(f \otimes \mathrm{id}_N)$. We compute this map using the isomorphism in 1):

$$\begin{tikzcd}
A^{\oplus J} \otimes_A N \arrow[r, "f \otimes \mathrm{id}_N"] \arrow[d, "\cong"] & A^{\oplus I} \otimes_A N \arrow[d, "\cong"] \\
N^{\oplus J} \arrow[r, "???"] & N^{\oplus I}
\end{tikzcd}
\qquad
\begin{array}{ccc}
e_j \otimes n & \longmapsto & \left(\sum a_{ij} e_i\right) \otimes n \\
\uparrow & & = \sum_i a_{ij} \cdot e_i \otimes n \\
& & \downarrow \\
n \cdot e_j & \longmapsto & \sum_i a_{ij} n \cdot e_i
\end{array}$$

Here I have written $n \cdot e_j = (0, \ldots, 0, \underset{\substack{\uparrow \\ j\text{-th component}}}{n}, 0, \ldots)$

That is, $f \otimes \mathrm{id}_N$ is given by the same matrix $(a_{ij})_{i,j}$ under the identifications from part 1). $\square$

§3 Examples

Example 9 Let $\mathfrak{a} = (X, Y) \subseteq A = k[X, Y]$. Then

$$\mathfrak{a} \otimes_A A/\mathfrak{a} \cong \operatorname{coker}\left(A \xrightarrow{\binom{-Y}{X}} A^{\oplus 2}\right) \otimes_A A/\mathfrak{a}$$

$$\cong \operatorname{coker}\left(A/\mathfrak{a} \xrightarrow{\binom{-Y}{X}} (A/\mathfrak{a})^{\oplus 2}\right)$$

but $X, Y \in \mathfrak{a}$, so $X \cdot A/\mathfrak{a} = Y \cdot A/\mathfrak{a} = 0$. Hence

$$\cong \operatorname{coker}\left(A/\mathfrak{a} \xrightarrow{0} (A/\mathfrak{a})^{\oplus 2}\right)$$

$$\cong (A/\mathfrak{a})^{\oplus 2} \cong k^{\oplus 2}$$

is a two-dimensional k-vsp.

Example 10 A any ring, $\mathfrak{a}, \mathfrak{b} \subseteq A$ two ideals. Then

$$\mathfrak{a} \times \mathfrak{b} \longrightarrow \mathfrak{a} \cdot \mathfrak{b} \qquad (a, b) \longmapsto a \cdot b \quad \text{is } A\text{-bilinear,}$$

hence factors through an A-linear map

$$\mathfrak{a} \otimes_A \mathfrak{b} \longrightarrow \mathfrak{a} \cdot \mathfrak{b}$$
$$a \otimes b \longmapsto a \cdot b$$

This map is always surjective since $\mathfrak{a}\mathfrak{b} = (f \cdot g \mid f \in \mathfrak{a}, g \in \mathfrak{b})$.

When is it injective? This is a subtle question:

<u>Eg.</u> $A = k[\varepsilon]/(\varepsilon^2)$, $\mathfrak{a} = \mathfrak{b} = (\varepsilon)$.

Then $\mathfrak{a} \cong \operatorname{coker}(A \xrightarrow{\cdot \varepsilon} A)$, so

$$\mathfrak{a} \otimes_A \mathfrak{a} \cong \operatorname{coker}(\mathfrak{a} \xrightarrow{\cdot \varepsilon} \mathfrak{a}) \cong \operatorname{coker}(\mathfrak{a} \xrightarrow{0} \mathfrak{a}) \cong \mathfrak{a}.$$

but $\mathfrak{a}^2 = (\varepsilon)^2 = 0$.

<u>E.g.</u> A PID, $\mathfrak{a} = (f)$, $\mathfrak{b} = (g)$ with $fg \neq 0$. Then

$$\begin{aligned} A &\xrightarrow{\sim} \mathfrak{a} \\ 1 &\longmapsto f \end{aligned} \quad , \quad \begin{aligned} A &\xrightarrow{\sim} \mathfrak{b} \\ 1 &\longmapsto g \end{aligned} \quad , \quad \text{hence}$$

$$\begin{aligned} A &\xrightarrow{\sim} \mathfrak{a} \otimes_A \mathfrak{b} \\ 1 &\longmapsto f \otimes g \end{aligned} \quad \text{and} \quad \begin{aligned} \mathfrak{a} \otimes_A \mathfrak{b} &\xrightarrow{\sim} \mathfrak{a}\mathfrak{b} = (fg) \\ f \otimes g &\longmapsto fg. \end{aligned}$$

<u>E.g.</u> $A = k[X, Y]$, $\mathfrak{a} = \mathfrak{b} = (X, Y)$. Then $\mathfrak{a}^2 = (X^2, XY, Y^2)$.

By sheet 5, Ex. 3, $\mathfrak{a}^2 \otimes_A A/\mathfrak{a} \cong \mathfrak{a}^2/\mathfrak{a}^3 \cong (A/\mathfrak{a})^{\oplus 3}$

Recall that this quotient was computed on sheet 2.

On the other hand, by Prop 5,

$$
\begin{aligned}
(\mathfrak{a} \otimes_A \mathfrak{a}) \otimes_A A/\mathfrak{a} &\cong \mathfrak{a} \otimes_A (\mathfrak{a} \otimes_A A/\mathfrak{a}) \\
&\cong \mathfrak{a} \otimes_A (A/\mathfrak{a})^{\oplus 2} \\
\text{Example 9} \qquad &\cong (\mathfrak{a} \otimes_A A/\mathfrak{a})^{\oplus 2} \\
&\cong (A/\mathfrak{a})^{\oplus 4}.
\end{aligned}
$$

Thus $\mathfrak{a} \otimes_A \mathfrak{a} \twoheadrightarrow \mathfrak{a}^2$ surjective, but no isomorphism.

Concretely: $z = X \otimes Y - Y \otimes X$ lies in kernel, is $\neq 0$. (Check this!)

Moreover: $X \cdot z = X^2 \otimes Y - XY \otimes X$

$$= X \otimes XY - X \otimes XY = 0.$$

($\otimes$ allows to juggle X and Y around as long as multiplication happens within the A-module $\mathfrak{a}$.)

Similarly, $Y \cdot z = 0$. So even though $\mathfrak{a}$ is a torsion-free module over an integral domain, there is an inclusion

$$k \cong A/\mathfrak{a} \cdot z \hookrightarrow \mathfrak{a} \otimes_A \mathfrak{a}$$

§ Appendix After the lecture, I realized that the reference to Sheet 5, Ex 3 on page 10 can be easily circumvented:

Claim $\mathfrak{a} \subseteq A$ any. Then

$$\mathfrak{a}^n \otimes_A A/\mathfrak{a} \cong \mathfrak{a}^n/\mathfrak{a}^{n+1}.$$

Proof Let $a_i \in \mathfrak{a}$, $i \in I$, be generators of $\mathfrak{a}$.

Then $A^{\oplus I} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$ is a presentation of $A/\mathfrak{a}$.

$e_i \longmapsto a_i$

Hence $$\mathfrak{a}^n \otimes_A A/\mathfrak{a} \cong \mathfrak{a}^n \otimes_A \operatorname{coker}\left(A^{\oplus I} \xrightarrow{(a_i)} A\right)$$

$$\cong \operatorname{coker}\left((\mathfrak{a}^n)^{\oplus I} \xrightarrow{(a_i)} \mathfrak{a}^n\right)$$

$$\cong \mathfrak{a}^n/\mathfrak{a}^{n+1}. \qquad \square$$