

# Algebra 1

## Exercise sheet 9

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**Exercise 1.** First we translate the case of finding  $x, y, z \in \mathbb{Z}$  to finding rational points on  $x^2 - dy^2 = 1$ .

We do exactly like in the lectures, we consider the line  $L_q$  that goes through  $(-1, 0)$  and  $(0, q)$ . It intersects the hyperbola at exactly one point. We may assume  $x \geq 0$ ,  $y \geq 0$  and  $q \geq 0$ , since everything is symmetric.

We calculate and see that the line  $L_q$  will intersect the hyperbola at the points  $\left(\frac{1+dq^2}{1-dq^2}, \frac{2q}{1-dq^2}\right)$  and  $(-1, 0)$ . Also, if we take a rational point  $(x, y)$  on the curve, the line through it and  $(-1, 0)$  will have rational slope, namely  $q$  with  $q = \frac{y}{x+1}$ . That way we have a map from rational points on the curve to  $\mathbb{Q}$ . These two maps are each others' inverse.

For every  $q \in \mathbb{Q}$  we get the solution

$$(x, y, z) = (1 + dq^2, 2q, 1 - dq^2).$$