Algebra 1
Exercise sheet 5

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Exercise 1.

Define $\varphi\colon A \to \bigoplus_{i=1}^n A/a_i$.

Lets show that it is injective. Pick $a \in A$ with $\varphi(a) = 0$. Then $a \in a_j$ for every $j = 1, \ldots, n$. Since intersection of these ideals is trivial, we get $a = 0$.

So A is isomorphic to the image $\varphi(A)$. The image is a submodule of the module $\bigoplus_{i=1}^n A/a_i$. The direct sum $\bigoplus_{i=1}^n A/a_i$ is noetherian because it is the sum of neotherian modules. And the submodule of a noetherian module is obviously also noetherian.

Exercise 2.

Exercise 3. In this exercise $\otimes$ will be written instead of $\otimes_A$.

1. With the condition $\Phi((\ldots, 0, n_i \otimes m, 0, \ldots)) = (\ldots, 0, n_i, 0, \ldots) \otimes m$ and additivity of Φ we can define it on each summand in $\bigoplus_{i \in I}(N_i \otimes M)$ separately. Using universal property we get unique maps

$$\begin{aligned} N_i \otimes M &\to (\bigoplus_{i \in I} N_i) \otimes M \\ n \otimes m &\mapsto (\ldots, 0, n, 0, \ldots) \otimes m. \end{aligned}$$

They define a unique map

$$\begin{aligned} \Phi\colon \bigoplus_{i \in I}(N_i \otimes M) &\to (\bigoplus_{i \in I} N_i) \otimes M \\ \sum_{i \in I}(n_i \otimes m) &\mapsto (\sum_{i \in I} n_i) \otimes m. \end{aligned}$$

Note that the definition is given on elementary tensors. It then extends linearly to all elements of the sum of tensor products. By construction it also satisfies the given condition.

Let us now construct the inverse. Using universal property of the tensor product $(\bigoplus_{i\in I} N_i)\otimes M$ on the map (it is clearly bilinear)

$$\begin{aligned}(\bigoplus_{i\in I} N_i)\times M &\to \bigoplus_{i\in I}(N_i\otimes M)\\ (\sum_{i\in I} n_i, m) &\mapsto \sum_{i\in I}(n_i\otimes m).\end{aligned}$$

We get a unique map

$$\begin{aligned}(\bigoplus_{i\in I} N_i)\otimes M &\to \bigoplus_{i\in I}(N_i\otimes M)\\ (\sum_{i\in I} n_i)\otimes m &\mapsto \sum_{i\in I}(n_i\otimes m).\end{aligned}$$

It is clearly an inverse of Φ.

2. The map

$$\begin{aligned}A/\mathfrak{a}\times M &\to M/\mathfrak{a}M\\ (a+\mathfrak{a}, m) &\mapsto ma+\mathfrak{a}\end{aligned}$$

Lets check it is well defined. If $a_1+\mathfrak{a}=a_2+\mathfrak{a}$, then $m(a_1-a_2)\in\mathfrak{a}$ and $ma_1+\mathfrak{a}=ma_2+\mathfrak{a}$.

It is clearly also bilinear.

So by universal property it gives a unique map

$$\begin{aligned}\varphi\colon A/\mathfrak{a}\otimes M &\to M/\mathfrak{a}M\\ (a+\mathfrak{a})\otimes m &\mapsto am+\mathfrak{a}M.\end{aligned}$$

Let us show the injectivity and surjectivity of φ.

Injectivity: Suppose $am\in\mathfrak{a}M$. Then $am=a_1m_1$ for some $a_1\in\mathfrak{a}$ and $m_1\in M$. Calculate

$$\begin{aligned}(a+\mathfrak{a})\otimes m &= (a(1+\mathfrak{a}))\otimes m\\ &= (1+\mathfrak{a})\otimes am\\ &= (1+\mathfrak{a})\otimes a_1m_1\\ &= a_1(1+\mathfrak{a})\otimes m_1\\ &= (a_1+\mathfrak{a})\otimes m_1\\ &= 0\otimes m_1\\ &= 0.\end{aligned}$$

Surjectivity: Take any $m+\mathfrak{a}M\in M/\mathfrak{a}M$. Then $\varphi((1+\mathfrak{a})\otimes m)=m+\mathfrak{a}M$.

We could also construct the inverse

$$\begin{aligned}\gamma\colon M/\mathfrak{a}M &\to A/\mathfrak{a}\otimes M\\ m+\mathfrak{a}M &\mapsto (1+\mathfrak{a})\otimes m.\end{aligned}$$