Algebra 1
Exercise sheet 9

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Exercise 1. First we translate the case of finding $x, y, z \in \mathbb{Z}$ to finding rational points on $x^2 - dy^2 = 1$.

We do exactly like in the lectures, we consider the line L_q that goes through $(-1, 0)$ and $(0, q)$. It intersects the hyperbola at exactly one point. We may assume $x \geq 0$, $y \geq 0$ and $q \geq 0$, since everything is symmetric.

We calculate and see that the line L_q will intersect the hyperbola at the points $\left(\frac{1+dq^2}{1-dq^2}, \frac{2q}{1-dq^2}\right)$ and $(-1, 0)$. Also, if we take a rational point (x, y) on the curve, the line through it and $(-1, 0)$ will have rational slope, namely q with $q = \frac{y}{x+1}$. That way we have a map from rational points on the curve to $\mathbb{Q}$. These two maps are each others' inverse.

For every $q \in \mathbb{Q}$ we get the solution

$$(x, y, z) = (1 + dq^2, 2q, 1 - dq^2).$$

Exercise 2. Recall that prime ideals of $k[x, y]$ are of the following forms:

1. (0)
2. $(g(y))$ for irreducible $g(y) \in (k[x])[y]$
3. $(x - x_0, y - y_0)$ for $x_0, y_0 \in k$, and these are maximal

Suppose $f(x)$ is not a square. Then $y^2 - f(x)$ is irreducible and $(y^2 - f(x))$ prime. Any other prime ideal containing it must then be of the third type. Since k is algebraically closed, the only irreducible elements in $k[x]$ are $x - x_0$ for $x_0 \in k$. So prime ideal of the third type would be of the form $(x - x_0, y \pm \sqrt{f(x_0)})$. By taking square root we use that k is algebraically closed. So we have

$$\operatorname{Spec}\left(k[x, y]/(y^2 - f(x))\right) = \{(x - x_0, y \pm \sqrt{f(x_0)}) \mid x_0 \in k\} \cup \{(0)\}$$

Suppose now $f(x) = g^2(x)$ is a square. Then

$$y^2 - f(x) = y^2 - g^2(x) = (y - g(x))(y + g(x))$$

and any prime ideal would have to contain at least one of the factors. So we get prime ideals of the second type, namely $(y - g(x))$ and $(y + g(x))$. For prime ideals of the third type we get the same ones as in the previous case. So we get

$$\begin{aligned} \operatorname{Spec}\big(k[x,y]/(y^2 - f(x))\big) = \{\, (y - g(x)), (y + g(x)) \,\} \cup \\ \cup \{\, (x - x_0, y \pm \sqrt{f(x_0)}) \mid x_0 \in k \,\} \cup \{\, (0) \,\} \end{aligned}$$

Lets now compute the cardinality of the fibers of the given map. First note that $\operatorname{Spec}(k[x]) = \{\, (x - x_0) \mid x_0 \in k \,\} \cup \{(0)\}$. Take some $(x - x_0)$. We have to find all $p \in \operatorname{Spec}(k[x,y]/(y^2 - f(x)))$ such that $p \cap k[x] = (x - x_0)$. Looking at the sets we see that there are two such prime ideals in $k[x,y]/(y^2 - f(x))$, namely $(x - x_0, y \pm \sqrt{f(x_0)})$. So the cardinality of fibers is 2.