Algebra 1
Exercise sheet 8

Solutions by: Eric Rudolph and David Čadež

7. Juni 2023

Exercise 3. Let $A \to B$ be a finite morphism of rings. Then B is isomorphic to a quotient polynomial ring $A[T_1, \ldots, T_n]$.

First we use that the fiber of a prime ideal $\mathfrak{p} \in A$ is equal to $\operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$ where $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

Compute

$$\begin{aligned} k(\mathfrak{p}) \otimes_A B &= k(\mathfrak{p}) \otimes_A (A[T_1, \ldots, T_n] \otimes_{A[T_1,\ldots,T_n]} B) \\ &= (k(\mathfrak{p}) \otimes_A A[T_1, \ldots, T_n]) \otimes_{A[T_1,\ldots,T_n]} B \end{aligned}$$

Observe that $k(\mathfrak{p}) \otimes_A A[T_1, \ldots, T_n] \cong k(\mathfrak{p})[T_1, \ldots, T_n]$. So we get

$$k(\mathfrak{p}) \otimes_A B = k(\mathfrak{p})[T_1, \ldots, T_n] \otimes_{A[T_1,\ldots,T_n]} B$$