§1 Algebras

<u>Def</u> <u>A-algebra</u> $\underset{def}{=}$ ring B together with a ring map $A \to B$.

Let B, C be A-algebras. $f : B \to C$ <u>map of A-algebras</u>

$\underset{def}{=}$ f is a ring map s.th.
$$\begin{tikzcd} B \arrow[rr, "f"] & & C \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$
commutes.

<u>Motivation for this notion</u>

1) Given ring A and a (system of) polynomial equation(s) with A-coefficients $\sum_{i=0}^{n} a_i T^i = 0$, we can interpret it in any ring where we can interpret the coefficients, i.e. in any A-algebra: Say $\phi : A \to B$ is A-algebra, then
$\sum_{i=0}^{n} \phi(a_i) T^i$ is an equation over B.

<u>Example</u>

$$\mathbb{Z}[\tfrac{1}{2}] \to \mathbb{F}_3, \mathbb{Z}/27, \mathbb{F}_5, \ldots$$
$$\mathbb{Z}[\tfrac{1}{2}] \hookrightarrow \mathbb{Q}, \mathbb{R}, \mathbb{C}$$

$Y^2 - X^3/2 \mapsto$ reduce mod n (n odd)

$Y^2 - X^3/2 \mapsto$ view over $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

2) In rings like $\mathbb{C}[t]$, $\mathbb{C}[X,Y]/XY$ etc., we usually want to consider $\mathbb{C}$ as constants, i.e. a map $\mathbb{C}[t] \to \mathbb{C}[t]$ should take $a \in \mathbb{C}$ to a again. Considering $\mathbb{C}[t]$ as $\mathbb{C}$-algebra formalises this:

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[t], \mathbb{C}[t]) \xrightarrow{\sim} \mathbb{C}[t]$$
$$\phi \longmapsto \phi(t)$$

By comparison, $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{C}, \mathbb{C})$ and $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{C}[t], \mathbb{C}[t])$ are enormous, of cardinality $\geq 2^{2^{\aleph_0}}$.

<u>General fact</u> (Reminder on universal property of polynomial ring)

Let $\phi : A \to B$ be an A-algebra. Then

$$\mathrm{Hom}_{A\text{-alg}}\left(A[T_i, i \in I]/(f_j, j \in J), B \right) \cong \left\{ (b_i) \in B^I \mid \phi(f_j)(b_i) = 0 \ \forall j \in J \right\}$$
$$\phi \longmapsto (\phi(T_i))_{i \in I}.$$

<u>E.g.</u> $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[X,Y]/(XY), \mathbb{C}) \cong \{ (x,y) \in \mathbb{C}^2 \mid xy = 0 \}$.

§2 Scalar extension of modules

<u>Prop/Def</u> A ring, $A \xrightarrow{\phi} B$ an A-algebra, M an A-module

View B as A-module via $a \cdot b := \phi(a) b$.

Then $B \otimes_A M$ is a B-module via $b \cdot (x \otimes m) := (bx) \otimes m$.

It is called the <u>extension of scalars</u> from A to B of M.

<u>Proof</u> Every $b \in B$ defines an A-linear map by multiplication:

$$B \xrightarrow{[b]} B, \quad x \longmapsto bx.$$

By functoriality of the tensor product, there hence exists the A-linear

$$[b] \otimes \mathrm{id}_M : B \otimes_A M \longrightarrow B \otimes_A M, \quad x \otimes m \longmapsto bx \otimes m$$

It satisfies the B-module axioms:

$$1 \cdot (x \otimes m) = x \otimes m$$

$$c(b(x \otimes m)) = cbx \otimes m = (cb)(x \otimes m)$$

$$(b_1 + b_2)(x \otimes m) = ((b_1 + b_2)x) \otimes m = b_1(x \otimes m) + b_2(x \otimes m)$$

$$b(x_1 \otimes m_1 + x_2 \otimes m_2) = b(x_1 \otimes m_1) + b(x_2 \otimes m_2) \qquad \square$$

Note The map $\phi \otimes \mathrm{id}_M$ gives rise to the A-linear map

$$M \cong A \otimes_A M \longrightarrow B \otimes_A M$$
$$m \longmapsto 1 \otimes m \longmapsto 1 \otimes m.$$

Prop 1 $\phi : A \to B$ an A-algebra, $M = \operatorname{coker}\left(A^{\oplus J} \xrightarrow{(a_{ij})} A^{\oplus I}\right)$

Then $B \otimes_A M \cong \operatorname{coker}\left(B^{\oplus J} \xrightarrow{(\phi(a_{ij}))} B^{\oplus I}\right)$.

Proof Last lecture we showed that

$$B \otimes_A M \cong \operatorname{coker}\left(B^{\oplus J} \xrightarrow{(a_{ij})} B^{\oplus I}\right)$$

the A-linear map $b_j \longmapsto \sum_i a_{ij} \cdot b_i$

But $a_{ij} \cdot b_i = \phi(a_{ij}) b_i$ by defn of A-module str. on B. $\square$

Special cases

1) $A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$ (can view as $A/\mathfrak{a}$-module)

2) $A[S^{-1}] \otimes_A M \cong M[S^{-1}]$ (studied next week)

3) $A[T] \otimes_A M \cong \bigoplus_{i=0}^{\infty} M \cdot T^i$

Ad 3) $A[T] = \bigoplus_{i=0}^{\infty} A \cdot T^i$ is a free A-module with generators $1, T, T^2, \dots$. Then we use $A^{\oplus I} \otimes_A M \cong M^{\oplus I}$.

§3 Tensor product of A-algebras

<u>Prop 2</u> Let $A \xrightarrow{\phi} B$, $A \xrightarrow{\psi} C$ be two A-algebras. Then $B \otimes_A C$ becomes a ring with respect to the multiplication

$$(b_1 \otimes c_1)(b_2 \otimes c_2) := b_1 b_2 \otimes c_1 c_2 .$$

The two maps $\left\{ \begin{array}{l} B \to B \otimes_A C, \ b \mapsto b \otimes 1 \\ C \to B \otimes_A C, \ c \mapsto 1 \otimes c \end{array} \right\}$ are ring maps

and the diagram
$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"'] & C \arrow[d] \\ B \arrow[r] & B \otimes_A C \end{tikzcd}$$
commutes.

Moreover, it has the following universal property:

For every pair of A-algebra maps $B \xrightarrow{f} D$, $C \xrightarrow{g} D$ to an A-algebra D, there is a <u>unique</u> A-algebra map $B \otimes_A C \xrightarrow{h} D$ s.th. $f = h \circ \phi$, $g = h \circ \psi$:

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"'] & C \arrow[d] \arrow[ddr, bend left, "g"] & \\ B \arrow[r] \arrow[drr, bend right, "f"'] & B \otimes_A C \arrow[dr, dotted, "h"] & \\ & & D \end{tikzcd}$$

<u>Proof</u> 1) Multiplication is well-defined: Consider

$$B \times C \times B \times C \longrightarrow B \otimes_A C$$
$$(b_1, c_1, b_2, c_2) \longmapsto b_1 b_2 \otimes c_1 c_2 .$$

It is A-multilinear. By small extension of tensor calculus (see Atiyah-Macdonald, Prop 2.12*), it factors through an A-linear map

$$B \otimes_A C \otimes_A B \otimes_A C \longrightarrow B \otimes_A C$$
$$b_1 \otimes c_1 \otimes b_2 \otimes c_2 \longmapsto b_1 b_2 \otimes c_1 c_2 .$$

Composing with the universal bilinear map, we obtain the A-bilinear

$$B \otimes_A C \times B \otimes_A C \longrightarrow B \otimes_A C$$
$$(b_1 \otimes c_1 , b_2 \otimes c_2) \longmapsto b_1 b_2 \otimes c_1 c_2 .$$

2) Makes $B \otimes_A C$ into an A-algebra + $B, C \longrightarrow B \otimes_A C$ are A-algebra maps: Can be checked directly (Exercise)

The A-algebra structure here is given by

$$\phi \otimes \psi : A = A \otimes_A A \longrightarrow B \otimes_A C$$
$$a \longmapsto a \otimes 1 = 1 \otimes a \longmapsto \phi(a) \otimes 1 = 1 \otimes \psi(a).$$

The unit element is $1 \otimes 1$.

3) Universal property: Assume $f: B \to D$, $g: C \to D$ are A-algebra maps. Then

$$B \times C \longrightarrow D, \quad (b,c) \longmapsto f(b)g(c)$$

is an A-bilinear map, hence factors uniquely through

$$B \otimes_A C \longrightarrow D, \quad b \otimes c \longmapsto f(b)g(c).$$

This is a ring map:

$$\left(\sum_{i=1}^{n} b_i \otimes c_i \right) \cdot \left(\sum_{j=1}^{m} b_j' \otimes c_j' \right)$$

$$= \sum_{i,j} b_i b_j' \otimes c_i c_j' \longmapsto \sum_{i,j} f(b_i b_j')\, g(c_i c_j')$$

$$= \left(\sum_i f(b_i) g(c_i) \right) \cdot \left(\sum_j f(b_j') g(c_j') \right)$$

Even an A-algebra map: Let $\chi: A \to D$ be the A-algebra structure on D. Then

$$\begin{tikzcd} B \otimes_A C \ni \phi(a) \otimes 1 \arrow[rr, mapsto] & & f(\phi(a)) = \chi(a) \in D \\ & a \arrow[ul, mapsto] \arrow[ur, mapsto] & \end{tikzcd}$$

because f is an A-algebra map by assumption. $\square$

§4 Generators and Relations

<u>Defn</u> Presentation of an A-alg. C $\underset{def}{=}$ choice of set I, ideal $\mathfrak{a} \subseteq A[T_i, i \in I]$ and an isomorphism

$$A[T_i, i\in I]/\mathfrak{a} \xrightarrow{\sim} C.$$

<u>Lem 3</u> Every A-alg. C admits a presentation.

<u>Proof</u> Take $I = C$. Then $A[T_c, c \in C] \longrightarrow C$ is surjective.

$$T_c \longmapsto c$$

If $\mathfrak{a}$ is its kernel, then $A[T_c, c\in C]/\mathfrak{a} \cong C$ $\square$.

<u>Prop 4</u> Let $\phi : A \longrightarrow B$ be an A-algebra. There is an isomorphism of B-algebras

$$B \otimes_A \left(A[T_i, i\in I]/\mathfrak{a}\right) \xrightarrow{\sim} B[T_i, i\in I]/(\phi(\mathfrak{a})).$$

Special cases:

1) $A/\mathfrak{a} \otimes_A A/\mathfrak{b} \cong A/\mathfrak{a}+\mathfrak{b}$

2) $B \otimes_A A[S^{-1}] \cong B[\phi(S)^{-1}]$.

3) $\mathfrak{p} \subset A$ prime ideal. Then $\kappa(\mathfrak{p}) \otimes_A B \xrightarrow{\sim} (B/\mathfrak{p})[\phi(A\setminus\mathfrak{p})^{-1}]$.

Proof Consider

$$\begin{tikzcd} & A[T_i, i\in I]/\sigma \arrow[d] \\ B \arrow[r] & B[T_i, i\in I]/\sigma \\ b \arrow[r, mapsto] & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[d, mapsto] & T_i \arrow[d, mapsto] \\ \phi(a) & T_i . \end{tikzcd}$$

By universal property (Prop 2), factor through unique map

$$\begin{aligned} B\otimes_A \left(A[T_i, i\in I]/\sigma\right) &\longrightarrow B[T_i, i\in I]/\sigma \\ b\otimes a &\longmapsto \phi(a)b \\ b\otimes T_i &\longmapsto bT_i \end{aligned}$$

This is clearly surjective. For injectivity, we note that both have same universal property:

$$\mathrm{Hom}_{A\text{-alg}}(-, D) = \{ B\text{-algebra structure map } B\to D + \text{tuple } (d_i)\in D^I \text{ s.th. } \chi(f)(d_i)=0 \ \forall f\in\sigma. \}$$

Here $\chi: A\to D$ is the A-algebra structure.

Special cases:

1) Set $B = A/\mathfrak{b}$, take $I=\emptyset$, σ any.

2) B any, $I=S$, $\sigma = (s\cdot T_s - 1, s\in S)$

3) Combine 1) & 2). $\square$

§5 Example: Tensoring field extensions

① L/K finite extension. Primitive Element Thm states there exists a K-algebra isomorphism $L \cong K[T]/(f(T))$.
Assume L/K separable $\iff$ f only simple zeroes in $\overline{K}$.

Let M be splitting field of L, e.g. $\overline{K}$ or a normal closure of L in $\overline{K}$. Then

$$M \otimes_K L \cong M \otimes_K K[T]/(f(T)) \cong \prod_{i=1}^{[L:K]} M.$$

$\uparrow$ factors into linear polynomials.

In ptic, if L Galois, then $L \otimes_K L \cong \prod_{i=1}^{[L:K]} L$.

② Now consider $K = \mathbb{F}_p(x)$, $L = \mathbb{F}_p(x^{1/p})$ with x transcendental. This is an inseparable extension and

$$L \otimes_K L \cong \mathbb{F}_p(x^{1/p})[T]/_{T^p - x}$$

$$\cong L[T]/_{(T - x^{1/p})^p} \quad \text{is non-reduced.}$$

3) Now consider two transcendental extensions:

K any, $\quad L = K(X) = \mathrm{Quot}(K[X])$

$\quad\quad\quad\quad M = K(Y) = \mathrm{Quot}(K[Y])$.

Prop 4 states that

$$L \otimes_K M \cong K[X,Y]\left[(K[X]\setminus\{0\})^{-1}, (K[Y]\setminus\{0\})^{-1} \right].$$

<u>Recall</u>: a) $\mathrm{Spec}\, A[S^{-1}] = \{ \mathfrak{p} \in \mathrm{Spec}\, A \mid \mathfrak{p} \cap S = \emptyset \}$

b) $\mathrm{Spec}\, K[X,Y] = \{(0)\} \sqcup \{ (f) \mid f \text{ irreducible} \}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad \sqcup \{ \text{maximal ideals} \}$

Moreover, each maximal ideal intersects $K[X]$ and $K[Y]$ non-trivially, so

$$\mathrm{Spec}(L \otimes_K M) = \{(0)\} \cup \left\{ (f) \;\middle|\; \begin{array}{l} f \in k[X,Y] \text{ irred.} \\ f \notin k[X] \cup k[Y] \end{array} \right\}$$

<u>Eg.</u> $X + aY + b$ with $a, b \in K$, $a \neq 0$ all define mutually different prime ideals of $L \otimes_K M$.

$Y^2 + X$, $Y^5 - X^3 + X$ etc. yield further primes.