§1 The Spectrum

<u>Defn</u> An ideal $\mathfrak{p} \subseteq A$ <u>prime</u> $\underset{def}{=}$ $\mathfrak{p} \neq A$ and

$x \cdot y \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

<u>Lem 1</u> $\mathfrak{p} \subseteq A$ prime $\Leftrightarrow$ $A/\mathfrak{p}$ integral domain.

In pHic, any max. ideal is prime.

<u>Proof</u> Exercise. $\square$

<u>Lem 2</u> $\varphi: A \longrightarrow B$ ring map, $\mathfrak{q} \subset B$ prime ideal.

Then $\varphi^{-1}(\mathfrak{q}) \subset A$ is a prime ideal.

<u>Proof</u> $\mathfrak{q} \neq B \Leftrightarrow 1_B \notin \mathfrak{q} \Leftrightarrow 1_A \notin \varphi^{-1}(\mathfrak{q})$

$\Leftrightarrow \varphi^{-1}(\mathfrak{q}) \neq A$

Also, $x \cdot y \in \varphi^{-1}(\mathfrak{q}) \Leftrightarrow \varphi(xy) \in \mathfrak{q}$

$\Leftrightarrow \varphi(x) \in \mathfrak{q}$ or $\varphi(y) \in \mathfrak{q}$

$\Leftrightarrow x \in \varphi^{-1}(\mathfrak{q})$ or $y \in \varphi^{-1}(\mathfrak{q})$ $\square$

<u>Defn</u> $\mathrm{Spec}(A) := \{\mathfrak{p} \subset A \text{ prime ideal}\}$

If $\varphi: A \longrightarrow B$ ring map, put

$\mathrm{Spec}(\varphi): \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$, $\mathfrak{q} \longmapsto \varphi^{-1}(\mathfrak{q})$

Examples 1) $\pi: A \twoheadrightarrow A/\mathfrak{a}$ projection map. Then

$$\operatorname{Spec}(\pi): \operatorname{Spec}(A/\mathfrak{a}) \longrightarrow \operatorname{Spec}(A) \quad \text{is injective.}$$

$$\text{Image} = \{ \mathfrak{p} \mid \mathfrak{a} \subseteq \mathfrak{p} \}.$$

2) $\operatorname{Spec} \mathbb{Z}[i] \longrightarrow \operatorname{Spec}(\mathbb{Z})$

$$\mathfrak{p} \longmapsto \mathfrak{p} \cap \mathbb{Z}$$

is the map from last lecture. Fibers have 1 or 2 elements.

3) $\operatorname{Spec} \mathbb{Q} \longrightarrow \operatorname{Spec} \mathbb{Z}$ sends (0) to (0).

In ptic., inverse images of max ideals need not be max. again. This is why Spec is more natural than MaxSpec.

4) $\operatorname{Spec}(0) = \emptyset$. For other rings, however, Spec is non-empty:

Thm 3 Every ring $A \neq 0$ has a maximal ideal.

Proof $\Sigma = \{ \mathfrak{a} \subsetneq A \text{ ideal} \neq A \}$.

·) $\Sigma \neq \emptyset$ since $(0) \in \Sigma$.

·) Σ ordered by inclusion.

·) Let $S \subseteq \Sigma$ be a chain i.e. $\forall \mathfrak{a}, \mathfrak{b} \in S$ $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$.

Then $\mathfrak{c} = \bigcup_{\mathfrak{a} \in S} \mathfrak{a}$ is an ideal with $1 \notin \mathfrak{c}$ since $1 \notin \mathfrak{a}$ $\forall \mathfrak{a} \in S$.

Thus $\mathfrak{c} \in \Sigma$ is an upper bound for S.

·) Zorn's Lemma $\Rightarrow$ Σ contains maximal elements. $\square$

Cor 4 Assume $\mathfrak{a} \subsetneq A$ proper ideal. Then there is a max. ideal $\mathfrak{m}$ with $\mathfrak{a} \subseteq \mathfrak{m}$

Proof Apply (*) + above Thm to $A/\mathfrak{a}$. $\square$

Cor 5 Assume $0 \neq A$. Then

$x \in A$ is a unit $\Leftrightarrow$ $x \notin \mathfrak{m}$ for all max ideals $\mathfrak{m} \subset A$.

$\square$

Example Assume A has a unique max ideal, e.g. $A = k[[T_1, \ldots, T_n]]$. Then $A = A^\times \sqcup \mathfrak{m}$.

Recall Nilradical $\mathrm{nil}(A) = \{\text{nilpotent } x \in A\}$.

Prop 6 $\mathrm{nil}(A) = \bigcap_{\mathfrak{p} \subset A \text{ PI}} \mathfrak{p}$.

Proof $\subseteq$ Let $f \in \mathrm{nil}(A)$ and $\mathfrak{p}$ a PI. Then $f^n = 0 \in \mathfrak{p}$ for $n \gg 0$, hence $f \in \mathfrak{p}$ by defn. of prime ideal.

$\supseteq$ Let $f \notin \mathrm{nil}(A)$. Put $\Sigma = \{\text{ideals } \mathfrak{a} \text{ s.th. } f^n \notin \mathfrak{a} \ \forall n\}$.

·) $(0) \in \Sigma$ by assumption, so $\Sigma \neq \emptyset$

·) Σ ordered by inclusion + every chain has upper bound

Zorn's Lemma $\Rightarrow$ $\exists$ maximal elements in Σ. Pick one, say $\mathfrak{p}$.

Claim $\mathfrak{p}$ is a prime ideal.

Proof Let $x, y \notin \mathfrak{p}$. Then $\mathfrak{p} + (x)$, $\mathfrak{p} + (y) \notin \Sigma$ by maximality, so $f^n \in \mathfrak{p} + (x)$, $\mathfrak{p} + (y)$ for $n \gg 0$, say

$f^n = p_1 + ax = p_2 + by$. Then

$$f^{2n} = p_1 p_2 + axp_2 + byp_1 + ab\,xy \in \mathfrak{p} + (abxy)$$

Thus $\mathfrak{p} + (abxy) \notin \Sigma$, hence $xy \notin \mathfrak{p}$, thus $\mathfrak{p}$ prime.

By defn of Σ, $f \notin \bigcap_{\mathfrak{p} \subset A} \mathfrak{p}$ as was to be shown. $\square$

Cor 7 Consider $\pi : A \longrightarrow A/\mathrm{nil}(A)$.

The following, induced map is a bijection:

$$\mathrm{Spec}(\pi) : \mathrm{Spec}\,(A/\mathrm{nil}(A)) \longrightarrow \mathrm{Spec}\,(A).$$

This reduces the study of spectra to reduced rings.

Aim for next few lectures: Develop techniques for the study of the spectrum. This will generalize and conceptualize our arguments for $\mathrm{Spec}\ \mathbb{Z}[T]/(g) \longrightarrow \mathrm{Spec}\ \mathbb{Z}$ from last lecture.

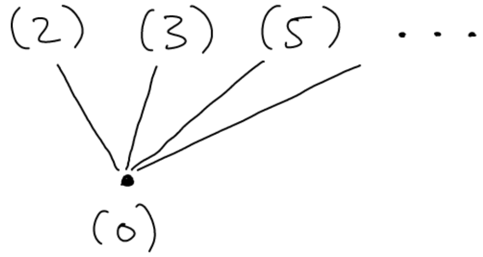
As motivation, we next consider a non-trivial example:

§2 Spec $\mathbb{Z}[T]$

First some terminology + a recap. from Introduction to algebra:

$\operatorname{Spec}(A)$ forms a partially ordered set w.r.t. inclusion.

Spec $\mathbb{Z}$:



(similar picture for every PID that is not a field.)

<u>Defn</u> 1) <u>Height</u> $\operatorname{ht}(\mathfrak{p})$ of $\mathfrak{p} \in \operatorname{Spec}(A)$ = supremum over n s.th. there are $\mathfrak{p}_0, \ldots, \mathfrak{p}_n \in \operatorname{Spec}(A)$ with

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}.$$

2) <u>Krull dimension</u> of A: $\dim(A) := \sup_{\mathfrak{p} \in \operatorname{Spec}(A)} \operatorname{ht}(\mathfrak{p})$

Prop 8 $\dim(\mathbb{Z}[T]) = 2$. $\operatorname{Spec} \mathbb{Z}[T]$ consists of the following elements:

(0) of height 0

(f), $f \in \mathbb{Z}[T]$ irreducible of height 1

(p, f), $p \in \mathbb{Z}$ prime, $f \in \mathbb{Z}[T]$ of height 2.

s.th. $f \bmod (p)$ irreducible

Moreover, every ht 1 prime ideal is contained in height 2 prime ideals, i.e. not maximal.

Proof Idea Compute fibers of $\operatorname{Spec} \mathbb{Z}[T] \longrightarrow \operatorname{Spec} \mathbb{Z}$.

Let $0 \neq \mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[T]$ in the following:

1st Case: $\mathfrak{p} \cap \mathbb{Z} = (p)$. Then $\bar{\mathfrak{p}} = \mathfrak{p}/p\mathbb{Z}[T]$ is prime ideal of $\mathbb{F}_p[T]$ and

$$\mathfrak{p} = \begin{cases} (p) & \text{if } \bar{\mathfrak{p}} = 0 \\ (p, f) & \text{if } \bar{\mathfrak{p}} = (\bar{f}) \text{ where } f \in \mathbb{Z}[T] \text{ any lift.} \end{cases}$$

(see last lecture)

<u>2nd Case</u> $\mathfrak{p} \cap \mathbb{Z} = (0)$. This requires a new technique, namely <u>localization</u>.

Consider $\mathfrak{q} := \mathfrak{p} \cdot \mathbb{Q}[T]$ (Ideal generated by elements from $\mathfrak{p}$ in $\mathbb{Q}[T]$.)

<u>Claim</u> $\mathfrak{q}$ is a prime ideal of $\mathbb{Q}[T]$.

<u>First</u> we show $\mathfrak{q} \neq \mathbb{Q}[T]$. Namely $1 \in \mathfrak{q}$ means there are $f_i \in \mathbb{Q}[T]$, $a_i \in \mathfrak{p}$ s.th. $1 = \sum f_i \cdot a_i$.

Let $0 \neq m \in \mathbb{Z}$ be s.th. $m \cdot f_i \in \mathbb{Z}[T]$ $\forall i$.

Then $\quad m = \sum (mf_i) a_i \in \mathfrak{p}$, contrary to assumptions.

<u>Next</u> assume $g \cdot h \in \mathbb{Q}[T]$, say $g \cdot h = \sum f_i \cdot a_i$ with $f_i \in \mathbb{Q}[T]$, $a_i \in \mathfrak{p}$. Pick m s.th. mg, mh and the $m \cdot f_i$ are all in $\mathbb{Z}[T]$. Then

$$mf \cdot mg \in \mathfrak{p} \xRightarrow{\mathfrak{p} \text{ prime}} mf \in \mathfrak{p} \text{ or } mg \in \mathfrak{p}.$$

This implies $f \in \mathfrak{q}$ or $g \in \mathfrak{q}$ because $m^{-1} \in \mathbb{Q}$. $\square$ Claim.

Write $\mathfrak{q} = (h)$ with $h \in \mathbb{Z}[T]$ <u>primitive</u> meaning $\gcd(\text{coeff. of } h) = 1$.

Lemma of Gauss $h \in \mathbb{Z}[T]$ primitive, $f \in \mathbb{Z}[T]$ any.

Then $h \mid f$ in $\mathbb{Q}[T]$ $\Longleftrightarrow$ $h \mid f$ in $\mathbb{Z}[T]$.

(Proof: Schröer's lecture.)

Consequence: $\mathfrak{q} \cap \mathbb{Z}[T] = (h)$.

Left to show: $\mathfrak{q} \cap \mathbb{Z}[T] = \mathfrak{p}$ and claim about heights. $\longrightarrow$ Exercise. $\square$ Prop 8.

Recall A is a unique factorization domain (UFD)

$:\Longleftrightarrow$ A is an integral domain + every $f \in A \setminus \{0\}$ has a factorization into prime elements.

(Such a factorization is necessarily unique up to ordering and units.)

Prop 9 A UFD $\Longleftrightarrow$ $A[T]$ UFD

(Proof: Schröer's lecture)

Prop 10 Let A be a UFD.

1) $\mathfrak{p} \in \operatorname{Spec}(A)$ has height 1 $\Longleftrightarrow$ $\exists$ prime element π s.th. $\mathfrak{p} = (\pi)$.

2) $\dim(A) \leq 1$ $\Longrightarrow$ A is a PID.

§3 Localization

Defn A ring, $S \subseteq A$ subset. Localization of A in S $\overset{=}{\text{def}}$ ring map $A \xrightarrow{\varphi} A[S^{-1}]$ with following properties:

1) $\varphi(S) \subseteq A[S^{-1}]$

2) Given $\psi: A \to B$ s.th. $\psi(S) \subseteq B^\times$, there is a unique factorization

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & A[S^{-1}] \arrow[d, dashed, "\exists!"] \\ & B \end{tikzcd}$$

It follows from the uniqueness that a localization is unique up to unique isomorphism.

Prop 11 Localizations exist.

Proof For $s \in S$, let T_s be a variable. Consider

$$A \xrightarrow{\varphi} \tilde{A} := A[T_s,\ s \in S] \Big/ (s \cdot T_s - 1,\ s \in S)$$

1) $\varphi(S) \subseteq \tilde{A}^\times$ because $s \cdot T_s = 1$ in $\tilde{A}$.

2) Let $A \xrightarrow{\psi} B$ be a ring map with $\psi(S) \subseteq B^\times$.

There is at most one factorization

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & \tilde{A} \arrow[d, "\alpha"] \\ & B \end{tikzcd}$$

because $\alpha(s \cdot T_s - 1) = 0 \iff \varphi(s) \cdot \alpha(T_s) = 1$

$\iff \alpha(T_s) = \varphi(s)^{-1}$.

For existence, we consider $\beta : A[T_s, s \in S] \longrightarrow B$

$T_s \longmapsto \varphi(s)^{-1}$.

Then $\beta(sT_s - 1) = 0$, so β factors through $\tilde{A}$. $\square$

We next give a more explicit description (see Schröer's lecture for proofs.)

Def 1) $S \subseteq A$ <u>multiplicative</u> $\underset{\text{def}}{=}$ $1 \in S$ and $a \in S, b \in S \Rightarrow ab \in S$.

2) S <u>saturated</u> $\underset{\text{def}}{=}$ S multiplicative and $ab \in S \Rightarrow a \in S, b \in S$.

Every subset $S \subseteq A$ has a multiplicative closure S^{mult} and a saturated hull S^{sat}. Then

$$A[S^{-1}] \cong A[S^{\text{mult},-1}] \cong A[S^{\text{sat},-1}]$$

because all three have the same univ. property:

$$xy \in B^\times \iff x \in B^\times \text{ and } y \in B^\times.$$

<u>Example</u> $S = \{8\} \subset \mathbb{Z}$. $S^{\text{mult}} = \{8^n, n \geq 1\}$

$S^{\text{sat}} = \{\pm 2^n, n \geq 0\}$.

Let $S \subseteq A$ multiplicative. Then

$$\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \quad :\Longleftrightarrow \exists s_3 \in S : \; s_3(a_1 s_2 - a_2 s_1) = 0$$

defines an equivalence relation on $R \times S$ and

$$S^{-1}A := \left\{ \frac{a}{s}, \; a \in A, \; s \in S \right\} / \sim$$

is a ring with usual multiplication rules for fractions.
Via the map $A \longrightarrow S^{-1}A$, $a \longmapsto \frac{a}{1}$, $S^{-1}A$ is a localization of A in S.

<u>Example</u> A integral domain, $S = A \setminus \{0\}$.

Then $\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \iff a_1 s_2 = a_2 s_1$ and

$$A[S^{-1}] = \mathrm{Quot}(A) \text{ is the quotient field of } A$$

$\mathrm{Quot}(\mathbb{Z}) = \mathbb{Q}$, $\mathrm{Quot}(\mathbb{Z}[i]) = \mathbb{Q}(i)$,

$\mathrm{Quot}(k[T]) = k(T) = \left\{ \frac{f}{g} \; f, g \in k[T], \; g \neq 0 \right\}$

$\mathrm{Quot}(k[T_1, \dots, T_n]) = k(T_1, \dots, T_n)$ similar.