§1 Some Terminology

An A-algebra $\phi : A \longrightarrow B$ is called

1) <u>Finite type</u> or <u>finitely generated</u> $\underset{\text{def}}{=}$ B is of form

$$B \cong A[T_1, \ldots, T_n]/\mathfrak{a} \quad \text{for some } n, \text{ some } \mathfrak{a}.$$

2) <u>Finite presentation</u> or <u>finitely presented</u> $\underset{\text{def}}{=}$

$$B \cong A[T_1, \ldots, T_n]/(f_1, \ldots, f_m) \quad \text{some } n, m, \; f_1, \ldots, f_m.$$

3) <u>Finite</u> $\underset{\text{def}}{=}$ B is finite ($=$ finitely generated) as <u>A-module</u>

4) $x \in B$ is <u>integral over A</u> $\underset{\text{def}}{=}$ $\exists$ <u>monic</u> $f \in A[T]$ s.th. $f(x) = 0$. (This is meant as $\phi(f)(x) = 0$.)

5) B <u>integral</u> $\underset{\text{def}}{=}$ every $x \in B$ is integral over A.

$$\begin{tikzcd}
 & \text{finite presentation} \arrow[d, Rightarrow] \\
\text{finite} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "(*)"] & \text{finite type} \arrow[u, dotted, bend right, Rightarrow, "\text{if } A \text{ noetherian}"'] \\
\text{integral} &
\end{tikzcd}$$

(*) see proposition below.

<u>Examples</u> 1) Finite field extensions are finite.

Algebraic field extensions are integral.

2) Every quotient $A \twoheadrightarrow A/\mathfrak{a}$ is finite.

3) If $f \in A[T]$ is <u>monic</u>, then

$$B = A[T]/(f) \cong A^{\oplus \deg(f)} \quad \text{as } A\text{-module},$$

so B is a finite A-algebra.

4) This need not be the case if f is not monic:

E.g. $A[T]/(aT-1) = A[a^{-1}]$ is often not finite,

for example $\mathbb{Z}[\frac{1}{2}]$ is not finite over $\mathbb{Z}$.

5) <u>Exercise</u> $x \in \mathbb{Q}$ is integral over $\mathbb{Z}$ $\Leftrightarrow$ $x \in \mathbb{Z}$.

For $n \in \mathbb{Z}$, $\sqrt{n} \in \mathbb{Q}(\sqrt{n})$ satisfies $(\sqrt{n})^2 - n = 0$ and is hence integral over $\mathbb{Z}$.

5) If $\mathfrak{a} \subset A$ is not fin. gen., then $A/\mathfrak{a}$ is a finite type A-algebra but not of finite presentation.

§2 Finite and integral extensions

Assume $A \subseteq B$ is a subring in the following.
(So for a general A-algebra $\phi: A \to B$, everything applies to $\phi(A) \subseteq B$.)

<u>Prop 1</u> The following are equivalent for an element $x \in B$:

1) x is integral over A
2) The A-subalgebra $A[x] \subseteq B$ generated by x is a finite A-module.
3) There exists an A-subalgebra $A \subseteq C \subseteq B$ that is finite as A-module and s.th. $x \in C$.

<u>Proof</u> 1) $\Rightarrow$ 2): Say $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$.
Then $A[x]$ is generated as A-module by $1, x, \dots, x^{n-1}$.

2) $\Rightarrow$ 3) is clear.

3) $\Rightarrow$ 1): Let $c_1, \dots, c_n \in C$ be generators as A-module
Write $x \cdot c_j = \sum_{i=1}^{n} a_{ij} c_i$ with (not necessarily unique) $a_{ij} \in A$.

In other words, we have chosen a commutative diagram

$$\begin{tikzcd} A^{\oplus n} \arrow[r, "p"] \arrow[d, "\tilde{x}"'] & C \arrow[d, "\cdot x"] \\ A^{\oplus n} \arrow[r, "p"] & C \end{tikzcd}$$

where $p(e_i) = c_i$ and where $\tilde{x} = (a_{ij})$. By Cayley-Hamilton, $\mathrm{char}_{\tilde{x}}(\tilde{x}) = 0$, meaning that $f(T) = \mathrm{char}_{\tilde{x}}(T) \in A[T]$ provides a monic polynomial s.th. $f(x) = 0$.

We conclude that x is integral as desired. $\square$

<u>Cor 2</u> We deduce that

1) B finite as A-algebra $\Rightarrow$ B integral as A-alg.

2) If $x_1, \dots, x_n \in B$ are integral, then $A[x_1, \dots, x_n]$ is a finite A-algebra.

3) $\overline{A}^B := \{x \in B \text{ integral over } A\} \subseteq B$ is a subring. It is called the <u>integral closure of A in B.</u>

<u>Proof</u> 1) This is 3) $\Rightarrow$ 1) of Prop 1

2) By induction on n, $A[x_1, \dots, x_{n-1}]$ is a finite A-module. Now given that x_n is integral over A, it is

a fortiori integral over $A[x_1, \ldots, x_{n-1}]$. This implies by Prop 1 that $A[x_1, \ldots, x_n]$ is a finite module over $A[x_1, \ldots, x_{n-1}]$, hence also over A.

3) Given two integral $x, y \in B$, the ring $A[x, y]$ contains $x+y$ and $x \cdot y$. By 2), it is a finite A-module. By Prop 1, this implies that $x+y$ and $x \cdot y$ are integral again. $\square$

<u>Cor 3</u> Assume $A \subseteq B \subseteq C$ are s.th. C is integral over B and B integral over A. Then C is integral over A.

In ptic, $\overline{\overline{A}^B}^B = \overline{A}^B$.

<u>Proof</u> Given $x \in C$, let $x^n + b_{n-1} x^{n-1} + \ldots + b_0 = 0$ be an integral relation satisfied by it. Then $A[b_0, \ldots, b_{n-1}, x]$ is a finite $A[b_0, \ldots, b_{n-1}]$-module by Prop 1, hence a finite A-module. Again by Prop 1, this implies x is integral over A. For the statement about integral closures, apply this to $A \subseteq \overline{A}^B \subseteq \overline{\overline{A}^B}^B$. $\square$

<u>Prop 4</u> Assume $A \subseteq B$ is integral.

1) If $\mathfrak{b} \subseteq B$ is an ideal, then $A/_{A \cap \mathfrak{b}} \subseteq B/\mathfrak{b}$ is again integral.

2) If $S \subset A$ is a mult. subset, then $S^{-1}A \subseteq S^{-1}B$ is again integral.

3) If $A \to C$ is any A-algebra, then $C \to C \otimes_A B$ is again integral. (But not nec. injective.)

<u>Proof</u> From definitions:

1) Given $\bar{b} \in B/\mathfrak{b}$, pick relation $b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0$.

Then $\bar{b}^n + \bar{a}_{n-1} \bar{b}^{n-1} + \cdots + \bar{a}_0 = 0$ $\square$

2) Given $b/s \in S^{-1}B$, pick relation $b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0$.

Then $(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + \cdots + a_0/s^n = 0$. $\square$

3) Given an elementary tensor $c \otimes b$, pick a relation as before.

Then $(c\otimes b)^n + c\otimes a_{n-1}(c \otimes b)^{n-1} + \cdots + c^n \otimes a_0$

$= c^n \otimes (b^n + a_{n-1} b^{n-1} + \cdots + a_0) = 0$.

Elementary tensors generate $C \otimes_A B$ as C-algebra, so

Cor 2 applies. $\square$

§3 The Going-Up Theorem

<u>Situation</u> $A \subseteq B$ integral ring extension. We want to understand the map

$$\begin{aligned} \mathrm{Spec}(B) &\longrightarrow \mathrm{Spec}(A) \\ \mathfrak{q} &\longmapsto \mathfrak{q} \cap A. \end{aligned}$$

<u>Prop 5</u> Assume A and B are integral domains. Then A is a field $\Leftrightarrow$ B is a field.

<u>Proof</u> $\Rightarrow$ Assume A is a field and $0 \neq y \in B$ any. Let $y^n + a_{n-1} y^{n-1} + \dots + a_0 = 0$ is an integral dependence relation of minimal degree. Then $a_0 \neq 0$ and

$$-a_0^{-1}\left(y^{n-1} + a_{n-1} y^{n-2} + \dots + a_1\right) \cdot y = 1.$$

Hence B is a field.

$\Leftarrow$ Assume B is a field and $0 \neq x \in A$. The inverse $x^{-1} \in B$ is integral over A, so there is a relation

$$x^{-n} + a_{n-1} x^{-n+1} + \dots + a_1 x^{-1} + a_0 = 0.$$

Then $x^{-1} = -\left(a_{n-1} + a_{n-2} x + \dots + a_0 x^{n-1}\right) \in A.$ $\square$

Cor 6 Let $\mathfrak{q} \subseteq B$ be a prime ideal. Then

$$\mathfrak{q} \text{ maximal} \iff \mathfrak{p} = \mathfrak{q} \cap A \text{ maximal.}$$

Proof Apply Prop 5 to $A/\mathfrak{p} \subseteq B/\mathfrak{q}$. $\square$

Cor 7 Assume $\mathfrak{q}, \mathfrak{q}' \subseteq B$ are primes with $\mathfrak{q} \subseteq \mathfrak{q}'$ and $\mathfrak{p} := \mathfrak{q} \cap A = \mathfrak{q}' \cap A$. Then $\mathfrak{q} = \mathfrak{q}'$.

Recall two results from prev. lectures:

(1) Localization of A-modules is exact, in phic preserves injections and finite intersections

(2) If $S \subseteq A$ is a mult. subset, $\phi : A \to S^{-1}A$ the localization map, then

$$\begin{aligned} \operatorname{Spec}(S^{-1}A) &\cong \{ \mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \cap S = \emptyset \} \\ \mathfrak{p} &\longmapsto \phi^{-1}(\mathfrak{p}). \\ \mathfrak{p} \cdot S^{-1}A. &\longleftarrow\!\shortmid \mathfrak{p} \end{aligned}$$

Proof of Cor 7 Being integral is stable under localization (see Prop 4 (2)), so $A_\mathfrak{p} \subseteq B_\mathfrak{p}$ is integral.

Then $\mathfrak{q}B_{\mathfrak{p}} \subseteq \mathfrak{q}'B_{\mathfrak{p}}$ are two prime ideals of $B_{\mathfrak{p}}$ (see (2) above) s.th. (e.g. by (1) above)

$$\mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{q}'B_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}.$$

But this ideal is maximal: $A_{\mathfrak{p}}$ is local w/ max ideal $\mathfrak{p}A_{\mathfrak{p}}$.

By Cor 6, $\mathfrak{q}B_{\mathfrak{p}}$ and $\mathfrak{q}'B_{\mathfrak{p}}$ are both maximal. But one is contained in the other by assumption, so they are equal. Hence $\mathfrak{q} = \phi^{-1}(\mathfrak{q}B_{\mathfrak{p}}) = \phi^{-1}(\mathfrak{q}'B_{\mathfrak{p}}) = \mathfrak{q}'$, where $\phi : B \longrightarrow B_{\mathfrak{p}}$ is the localization map, which is what we wanted to show. $\square$

<u>Cor 8</u> The map $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is surjective.

<u>Proof</u> Let $\mathfrak{p} \in \mathrm{Spec}(A)$ be any. The map $A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}}$ is injective, so $B_{\mathfrak{p}}$ is not the zero-ring and hence has a maximal ideal, say $\mathfrak{q}B_{\mathfrak{p}}$ with $\mathfrak{q} \subset B$. Then $\mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}}$ is maximal again (Cor 6) and hence equal to $\mathfrak{p}A_{\mathfrak{p}}$. Conclusion: $\mathfrak{q} \cap A = \mathfrak{p}$. $\square$

Thm 9 (Going-Up) Assume we are given prime ideals:

$$\begin{array}{ccccccccccc} \mathfrak{q}_1 & \subseteq & \cdots & \subseteq & \mathfrak{q}_m & & & & & & \\ \cup\!\mid & & & & \cup\!\mid & & & & & & \\ \mathfrak{p}_1 & \subseteq & \cdots & \subseteq & \mathfrak{p}_m & \subseteq & \mathfrak{p}_{m+1} & \subseteq & \cdots & \subseteq & \mathfrak{p}_n \end{array}$$

where $\mathfrak{p}_i \subseteq A$, $\mathfrak{q}_i \subseteq B$ and $\mathfrak{p}_i = \mathfrak{q}_i \cap A$.

Then there are $\mathfrak{q}_{m+1} \subseteq \cdots \subseteq \mathfrak{q}_n \subseteq B$ s.th.

$\mathfrak{p}_i = \mathfrak{q}_i \cap A$ also for $i > m$.

Proof Pass to $A/\mathfrak{p}_m \subseteq B/\mathfrak{q}_m$ and apply Cor 8 to find $\overline{\mathfrak{q}_{m+1}} \subseteq B/\mathfrak{q}_m$ above $\mathfrak{p}_{m+1}/\mathfrak{p}_m$. Taking its preimage in B constructs $\mathfrak{q}_{m+1}$. Now induct. $\square$

Cor 10 For their Krull dimensions,

$$\dim(A) = \dim(B).$$

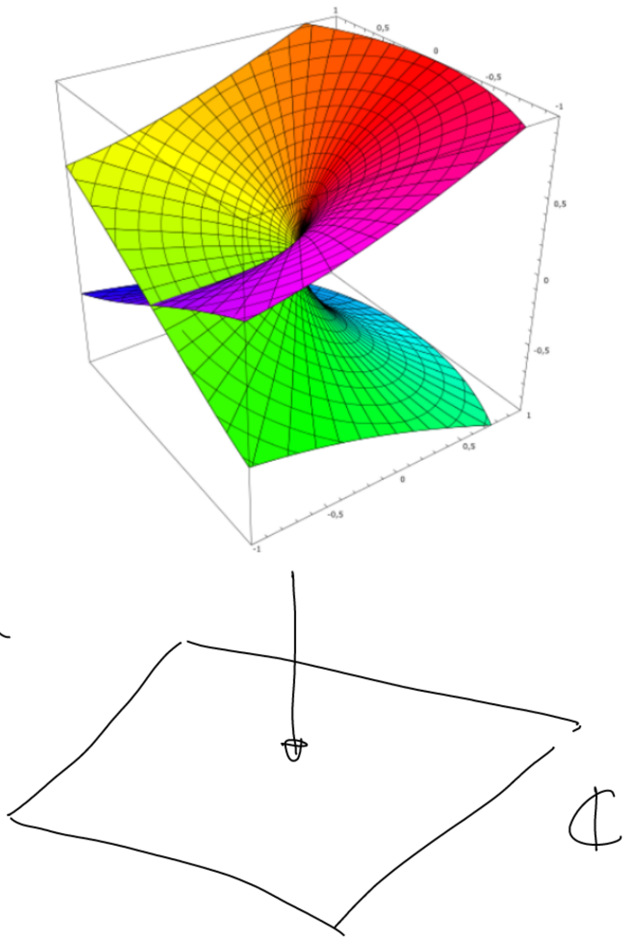
Proof Thm 9 implies $\leq$, Cor 7 $\geq$. $\square$

<u>Example</u> $\mathbb{C}[x] \subseteq \mathbb{C}[y]$, $x \longmapsto y^n$.

$$\begin{array}{lccccc}
\operatorname{Spec} \mathbb{C}[y] & (0), & (y-a) & & \mathbb{C} & a \\
@ \downarrow & \downarrow & \downarrow & \longleftrightarrow & \downarrow & \downarrow \\
\operatorname{Spec} \mathbb{C}[x] & (0), & (x-a^n) & & \mathbb{C} & a^n.
\end{array}$$

This is the algebraic incarnation of the finite map of Riemann surfaces $\mathbb{C} \longrightarrow \mathbb{C}$, $a \longmapsto a^n$.

Here is a picture from the Wikipedia article called "Ramification (mathematics)" for $n=2$. In general, fibers of @ have $\leq n$ elements.



Note that $\mathbb{C}[y] \cong \mathbb{C}[x][T]/_{T^n - x}$
$y \longmapsto T$
is indeed a finite extension.

Consider now $\quad \mathbb{C}[x_1, x_2] \subseteq \mathbb{C}[y_1, y_2]$

$$x_i \longmapsto y_i^2 .$$

Recall that we have a full description of primes in polynomial rings in 2 variables.

The following describes the primes above the chain

$$\mathfrak{p}_0 = (0) \subseteq \mathfrak{p}_1 = (x_1 - x_2) \subseteq \mathfrak{p}_2 = (x_1 - a, x_2 - a)$$

in the sense of the Going-Up Theorem. $\quad 0 \neq a \in \mathbb{C}$.

$$\begin{tikzcd}
 & & (y_1 - \sqrt{a}, y_2 - \sqrt{a}) \\
 & (y_1 - y_2) \arrow[ur, dash] \arrow[r, dash] & (y_1 + \sqrt{a}, y_2 + \sqrt{a}) \\
(0) \arrow[ur, dash] \arrow[dr, dash] & & (y_1 - \sqrt{a}, y_2 + \sqrt{a}) \\
 & (y_1 + y_2) \arrow[ur, dash] \arrow[r, dash] & (y_1 + \sqrt{a}, y_2 - \sqrt{a})
\end{tikzcd}$$

$$(0) \subseteq (x_1 - x_2) \subseteq (x_1 - a, x_2 - a)$$