§1 Tensor products (§2 in Atiyah-Macdonald)

M, N, P A-modules.

Defn $f : M \times N \to P$ called *A-bilinear* $\underset{def}{\Leftrightarrow}$ $\forall x \in M, y \in N$,

$f(x,-) : N \to P$, $f(-,y) : M \to P$ are A-linear.

Like for vector spaces or abelian groups,

$$\mathrm{BiHom}_A(M,N;P) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P))$$
$$f \longmapsto [x \mapsto f(x,-)]$$

Prop 1 Given M, N, there exist a pair (T, g) where

·) T an A-module

·) $g : M \times N \to T$ A-bilinear

s.th. for all P and all A-bilinear $f : M \times N \to P$,

there is a *unique* A-linear $h : T \to P$ s.th.

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "f"'] & T \arrow[d, "h"] \\ & P \end{tikzcd}$$

commutes.

The pair (T, g) is unique up to unique isomorphism. It is called a tensor product of M and N over A.

Example 2 Say $M = A^m$ and $N = A^n$. We have

$$\mathrm{BiHom}_A(A^m, A^n; P) \cong M_{m\times n}(P) \qquad (*)$$

$$f \longmapsto (f(e_i, e_j))_{i=1,\dots,m,\ j=1,\dots,n}$$

(This is the matrix representation of bilinear maps.)

Put $T = A^{m\times n}$, call its standard basis (e_{ij}).

Let $g : A^m \times A^n \longrightarrow A^{m\times n}$ be the unique bilinear map that satisfies $g(e_i, e_j) = e_{ij}$.

Its matrix representation is precisely $(e_{ij})_{ij} \in M_{m\times n}(A^{m\times n})$. Then $(A^{m\times n}, g)$ is a tensor product of A^m and A^n:

Given f as in $(*)$, the A-linear map

$$h : A^{m\times n} \longrightarrow P, \quad e_{ij} \longmapsto f(e_i, e_j)$$

is the unique one that satisfies $f = h \circ g$.

Observation g is not surjective, but $\mathrm{Im}(g)$ generates T as A-module.

(The latter has to be the case by the universal property.)

Proof of Prop 1 1) Consider the (huge!) module

$$\tilde{T} := \bigoplus_{(x,y) \in M \times N} A \cdot e_{(x,y)} \quad \text{and the map}$$

$$\tilde{g} : M \times N \to \tilde{T}, \quad (x,y) \mapsto e_{(x,y)}$$

Here, $\tilde{g}$ is just a map of sets, no further properties.

Note that if $f : M \times N \to P$ is any map, then there is a unique A-linear map $\tilde{h} : \tilde{T} \to P$ s.th. $f = \tilde{h} \circ \tilde{g}$, namely $\tilde{h}(e_{(x,y)}) = f(x,y)$.

2) Let $U \subseteq \tilde{T}$ be the smallest A-submodule s.th. the composition $g = [\pi : \tilde{T} \to \tilde{T}/U] \circ \tilde{g}$ is an A-bilinear map.

Concretely, U is the submodule generated by

$$e_{(x, y_1+y_2)} - e_{(x,y_1)} - e_{(x,y_2)}, \quad e_{(x, ay)} - a e_{(x,y)}$$

$$e_{(x_1+x_2, y)} - e_{(x_1,y)} - e_{(x_2,y)}, \quad e_{(ax,y)} - a e_{(x,y)}$$

for $a \in A$, $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$. @

Put $T = \tilde{T}/U$.

3) Claim (T, g) is a tensor product.

Proof Given $f : M \times N \to P$, let $\tilde{h} : \tilde{T} \to P$ be the unique A-linear map s.th. $f = \tilde{h} \circ \tilde{g}$:

$$\begin{tikzcd}
M \times N \arrow[r, "\tilde{g}"] \arrow[dr, "f"'] & \tilde{T} \arrow[r, "\pi"] \arrow[d, "\tilde{h}"] & T \arrow[dl, dashed, "h"] \\
& P &
\end{tikzcd}$$

We need to show that the factorization h exists. Equivalently, the claim is that $\tilde{h}(u) = 0 \ \forall\, u \in U$.

But $\tilde{h}(z) = 0$, where z is any of the generators from @, <u>because f is bilinear</u>. The claim follows. $\square$

<u>Definition/Notation</u> We write

$$(M \otimes_A N, \ (x, y) \mapsto x \otimes y)$$

for the tensor product of M and N over A.

Elements of the form $x \otimes y$ are called <u>elementary tensors</u>.

They generate $M \otimes_A N$ as A-module.

<u>Example</u> $e_1 \otimes e_1 + e_1 \otimes e_2 = e_1 \otimes (e_1 + e_2) \in A^2 \otimes_A A^2$ is elementary, while $e_1 \otimes e_1 + e_2 \otimes e_2$ is not (if $A \neq 0$).

§2 Properties

Functoriality Let $\phi : M_1 \to M_2$, $\psi : N_1 \to N_2$ be A-linear maps. Consider the commutative diagram

$$\begin{tikzcd}
M_1 \times N_1 \arrow[r] \arrow[d, "{(\phi,\psi)}"'] \arrow[dr] & M_1 \otimes_A N_1 \arrow[d, dotted, "\phi\otimes\psi"] \\
M_2 \times N_2 \arrow[r] & M_2 \otimes_A N_2
\end{tikzcd}$$

The diagonal map is A-bilinear, so factors through a unique A-linear $\phi\otimes\psi : M_1 \otimes_A N_1 \to M_2 \otimes_A N_2$.

It is given on elementary tensors by

$$(\phi\otimes\psi)(x\otimes y) = \phi(x)\otimes\psi(y).$$

Quotients Special case: Let $U \subseteq M$ be an A-submodule. Then $M\otimes_A N \twoheadrightarrow M/U \otimes_A N$ is surjective because it is surjective on elementary tensors and these are generators.

How can we describe the kernel? Answer:

$$\begin{aligned}
\mathrm{BiHom}(M/U, N; P) &= \{ f \in \mathrm{BiHom}(M,N;P) \mid f|_{U\times N} \equiv 0 \} \\
&= \{ h \in \mathrm{Hom}(M\otimes N, P) \mid h\circ(U\otimes N \to M\otimes N) = 0 \} \\
&= \mathrm{Hom}(M\otimes N / \mathrm{Im}(U\otimes N \to M\otimes N), P).
\end{aligned}$$

Thus $M/U \otimes_A N \xleftarrow{\sim} M \otimes_A N / \mathrm{Im}(U \otimes_A N \to M \otimes_A N)$

$(x+U) \otimes y \longmapsfrom x \otimes y$

<u>Example 3</u> Together with Example 2, this gives a <u>completely general description</u> for $M \otimes_A N$:

Choose presentations $M \cong A^{\oplus I}/U$, $N \cong A^{\oplus J}/V$. Then

$$M \otimes_A N \cong A^{\oplus I} \otimes_A A^{\oplus J} \Big/ \mathrm{Im}\left(U \otimes_A A^{\oplus J}\right) + \mathrm{Im}\left(A^{\oplus I} \otimes_A V\right)$$

<u>Concrete example</u>:

$A = \mathbb{Z}$, $M = \mathbb{Z}/u\mathbb{Z}$, $N = \mathbb{Z}/v \cdot \mathbb{Z}$, $u, v \in \mathbb{Z}$.

Let e, f be generators of M, N.

<u>Example 2</u>: $\mathbb{Z} \cdot e \otimes_{\mathbb{Z}} \mathbb{Z} \cdot f \xrightarrow{\sim} \mathbb{Z} \cdot e \otimes f$.

$$\mathrm{Im}\left((u\mathbb{Z} \cdot e) \otimes_{\mathbb{Z}} \mathbb{Z}f \longrightarrow \mathbb{Z}e \otimes f\right) = u \cdot \mathbb{Z}\, e \otimes f.$$

$$(ue) \otimes f \longmapsto u \cdot e \otimes f$$

$$\mathrm{Im}\left(\mathbb{Z}e \otimes (v \cdot \mathbb{Z}f) \longrightarrow \mathbb{Z} \cdot e \otimes f\right)$$

$$e \otimes vf \longmapsto v \cdot e \otimes f$$

Conclusion: $\mathbb{Z}e/u\mathbb{Z}e \otimes \mathbb{Z}f/v\mathbb{Z}f$

$$\cong \mathbb{Z}\, e\otimes f / (u;v)\, e\otimes f \cong \mathbb{Z}/\gcd(u,v)\mathbb{Z}.$$