

# Algebra 1

## Exercise sheet 9

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**Exercise 1.** First we translate the case of finding  $x, y, z \in \mathbb{Z}$  to finding rational points on  $x^2 - dy^2 = 1$ .

We do exactly like in the lectures, we consider the line  $L_q$  that goes through  $(-1, 0)$  and  $(0, q)$ . It intersects the hyperbola at exactly one point. We may assume  $x \geq 0$ ,  $y \geq 0$  and  $q \geq 0$ , since everything is symmetric.

We calculate and see that the line  $L_q$  will intersect the hyperbola at the points  $\left(\frac{1+dq^2}{1-dq^2}, \frac{2q}{1-dq^2}\right)$  and  $(-1, 0)$ . Also, if we take a rational point  $(x, y)$  on the curve, the line through it and  $(-1, 0)$  will have rational slope, namely  $q$  with  $q = \frac{y}{x+1}$ . That way we have a map from rational points on the curve to  $\mathbb{Q}$ . These two maps are each others' inverse.

For every  $q \in \mathbb{Q}$  we get the solution

$$(x, y, z) = (1 + dq^2, 2q, 1 - dq^2).$$

**Exercise 2.** Recall that prime ideals of  $k[x, y]$  are of the following forms:

1.  $(0)$
2.  $(g(y))$  for irreducible  $g(y) \in (k[x])[y]$
3.  $(x - x_0, y - y_0)$  for  $x_0, y_0 \in k$ , and these are maximal

Suppose  $f(x)$  is not a square. Then  $y^2 - f(x)$  is irreducible and  $(y^2 - f(x))$  prime. Any other prime ideal containing it must then be of the third type. Since  $k$  is algebraically closed, the only irreducible elements in  $k[x]$  are  $x - x_0$  for  $x_0 \in k$ . So prime ideal of the third type would be of the form  $(x - x_0, y \pm \sqrt{f(x_0)})$ . By taking square root we use that  $k$  is algebraically closed. So we have

$$\text{Spec}(k[x, y]/(y^2 - f(x))) = \{(x - x_0, y \pm \sqrt{f(x_0)}) \mid x_0 \in k\} \cup \{(0)\}$$

Suppose now  $f(x) = g^2(x)$  is a square. Then

$$y^2 - f(x) = y^2 - g^2(x) = (y - g(x))(y + g(x))$$

and any prime ideal would have to contain at least one of the factors. So we get prime ideals of the second type, namely  $(y - g(x))$  and  $(y + g(x))$ . For prime ideals of the third type we get the same ones as in the previous case. So we get

$$\begin{aligned} \text{Spec}(k[x, y]/(y^2 - f(x))) &= \{(y - g(x)), (y + g(x))\} \cup \\ &\cup \{(x - x_0, y \pm \sqrt{f(x_0)}) \mid x_0 \in k\} \cup \{(0)\} \end{aligned}$$