Algebra 1
Exercise sheet 6

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Exercise 1. We note that the map is defined on elementary tensors.

Well defined: Suppose $\sum_{i=1}^n a_i \otimes b_i = 0$. Then $(\sum_{i=1}^n a_i \cdot b_i)(1 \otimes 1) = 0$.

Injectivity: Take $\sum_{i=1}^n a_i \otimes b_i \in \mathfrak{a} \otimes_A \mathfrak{b}$. Suppose $\sum_{i=1}^n a_i \cdot b_i = 0$. Then we can simply observe that since $a \otimes b = 1 \otimes (ab)$ we get

$$\sum_{i=1}^n a_i \otimes b_i = \sum_{i=1}^n 1 \otimes (a_i \cdot b_i) = 1 \otimes (\sum_{i=1}^n a_i \cdot b_i) = 1 \otimes 0 = 0$$

which proves injectivity.

Surjectivity: An element $\sum_{i=1}^n a_i \cdot b_i \in \mathfrak{a} \cdot \mathfrak{b}$ is the image of an element $\sum_{i=1}^n a_i \otimes b_i \in \mathfrak{a} \otimes_A \mathfrak{b}$.

Exercise 2.

1. Let $M = (b_1, \ldots, b_k)$ be the generators. Pick any

   $$\prod_{i \in I} m_i \otimes n_i \in \prod_{i \in I} M \otimes_A N_i.$$

   It is enough to only look at these kinds of elements, because they generate $\prod_{i \in I} M \otimes_A N_i$.

   Then write $m_i = \sum_{j=1}^k a_{ij} b_j$ for every $i \in I$. Then just observe

   $$\prod_{i \in I} (\sum_{j=1}^k a_{ij} b_j) \otimes n_i = \sum_{j=1}^k \prod_{i \in I} a_{ij} b_j \otimes n_i = \sum_{j=1}^k \left(b_j \otimes (\prod_{i \in I} a_{ij} n_i) \right)$$

   which is the image of $\sum_{j=1}^k \left(b_j \otimes (\prod_{i \in I} a_{ij} n_i) \right) \in M \otimes_A \prod_{i \in I} N_i$.

Exercise 3.

Exercise 4. Let J be the "inverse" of I, so $I \otimes_A J \cong A$. During the lectures we've shown that tensoring preserves exact sequences and in the proof we've shown that tensoring preserves surjectivity.