

§1 Algebras

Def A-algebra $\stackrel{\text{def}}{=}$ ring B together with a ring map $A \rightarrow B$.

Let B, C be A -algebras. $f: B \rightarrow C$ map of A-algebras

$\stackrel{\text{def}}{=}$ f is a ring map s.t. $B \xrightarrow{f} C$ commutes.



Motivation for this notion

1) Given ring A and a (system of) polynomial equation(s)

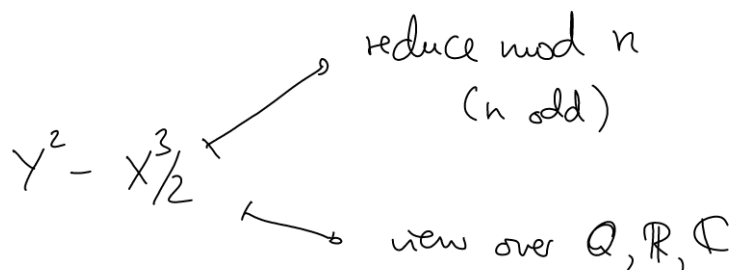
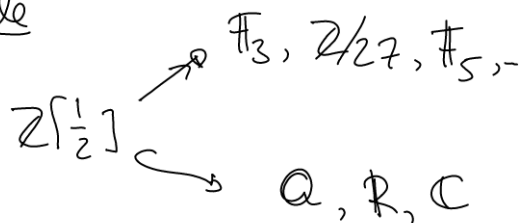
with A -coefficients $\sum_{i=0}^n a_i T^i = 0$, we can interpret it

in any ring where we can interpret the coefficients, i.e. in any

A -algebra: Say $\phi: A \rightarrow B$ is A -algebra, then

$\sum_{i=0}^n \phi(a_i) T^i$ is an equation over B .

Example



2) In rings like $\mathbb{C}[t]$, $\mathbb{C}[x, y]/(xy)$ etc., we usually want to consider \mathbb{C} as constants, i.e. a map $\mathbb{C}[t] \rightarrow \mathbb{C}[t]$ should take $a \in \mathbb{C}$ to a again. Considering $\mathbb{C}[t]$ as \mathbb{C} -algebra formalizes this:

$$\begin{aligned} \text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[t], \mathbb{C}[t]) &\xrightarrow{\sim} \mathbb{C}[t] \\ \phi &\longmapsto \phi(t) \end{aligned}$$

By comparison, $\text{Hom}_{\text{Ring}}(\mathbb{C}, \mathbb{C})$ and $\text{Hom}_{\text{Ring}}(\mathbb{C}[t], \mathbb{C}[t])$ are enormous, of cardinality $\geq 2^{2^{\aleph_0}}$.

General fact (Reminds on universal property of polynomial ring)

Let $\phi: A \rightarrow B$ be an A -algebra. Then

$$\begin{aligned} \text{Hom}_{A\text{-alg}}\left(A[t_i, i \in I] / (f_j, j \in J), B\right) &\cong \left\{ (b_i) \in B^I \mid \right. \\ &\quad \left. \phi(f_j)(b_i) = 0 \ \forall j \in J \right\} \\ \phi &\longmapsto (\phi(t_i))_{i \in I}. \end{aligned}$$

E.g. $\text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x, y]/(xy), \mathbb{C}) \cong \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}.$

§2 Scalar extension of modules

Prop/Def A ring, $A \xrightarrow{\phi} B$ an A -algebra, M an A -module

View B as A -module via $a \cdot b := \phi(a)b$.

Then $B \otimes_A M$ is a B -module via $b \cdot (x \otimes m) := (bx) \otimes m$.

It is called the extension of scalars from A to B of M .

Proof Every $b \in B$ defines an A -linear map by multiplication:

$$B \xrightarrow{[b]} B, x \mapsto bx.$$

By functoriality of the tensor product, there hence exists the A -linear

$$[b] \otimes \text{id}_M : B \otimes_A M \longrightarrow B \otimes_A M, x \otimes m \mapsto bx \otimes m$$

It satisfies the B -module axioms:

$$1 \cdot (x \otimes m) = x \otimes m$$

$$c(b(x \otimes m)) = cbx \otimes m = (cb)(x \otimes m)$$

$$(b_1 + b_2)(x \otimes m) = ((b_1 + b_2)x) \otimes m = b_1(x \otimes m) + b_2(x \otimes m)$$

$$b(x_1 \otimes m_1 + x_2 \otimes m_2) = b(x_1 \otimes m_1) + b(x_2 \otimes m_2) \quad \square$$

Note The map $\phi \otimes \text{id}_M$ gives rise to the A -linear map

$$\begin{aligned} M &\cong A \otimes_A M \longrightarrow B \otimes_A M \\ m &\longmapsto 1 \otimes m \longmapsto 1 \otimes m. \end{aligned}$$

Prop 1 $\phi: A \rightarrow B$ an A -algebra, $M = \text{coker}(A^{\oplus J} \xrightarrow{(a_{ij})} A^{\oplus I})$

$$\text{Then } B \otimes_A M \cong \text{coker}(B^{\oplus J} \xrightarrow{(\phi(a_{ij}))} B^{\oplus I}).$$

Proof Last lecture we showed that

$$B \otimes_A M \cong \text{coker}(B^{\oplus J} \xrightarrow{(a_{ij})} B^{\oplus I})$$

the A -linear map $b_j \mapsto \sum a_{ij} \cdot b_i$

But $a_{ij} \cdot b_i = \phi(a_{ij})b_i$ by defn of A -module str. on B . \square

Special cases

$$1) A/\mathfrak{o} \otimes_A M \cong M/\mathfrak{o}M \quad (\text{can view as } A/\mathfrak{o}\text{-module})$$

$$2) A[S^{-1}] \otimes_A M \cong M[S^{-1}] \quad (\text{studied next week})$$

$$3) A[T] \otimes_A M \cong \bigoplus_{i=0}^{\infty} M \cdot T^i$$

Ad 3) $A[T] = \bigoplus_{i=0}^{\infty} A \cdot T^i$ is a free A -module with

generators $1, T, T^2, \dots$. Then we use $A^{\oplus I} \otimes_A M \cong M^{\oplus I}$.

§3 Tensor product of A -algebras

Prop 2 Let $A \xrightarrow{\phi} B$, $A \xrightarrow{\psi} C$ be two A -algebras. Then

$B \otimes_A C$ becomes a ring with respect to the multiplication

$$(b_1 \otimes c_1)(b_2 \otimes c_2) := b_1 b_2 \otimes c_1 c_2.$$

The two maps $\left\{ \begin{array}{l} B \rightarrow B \otimes_A C, b \mapsto b \otimes 1 \\ C \rightarrow B \otimes_A C, c \mapsto 1 \otimes c \end{array} \right\}$ are ring maps

and the diagram

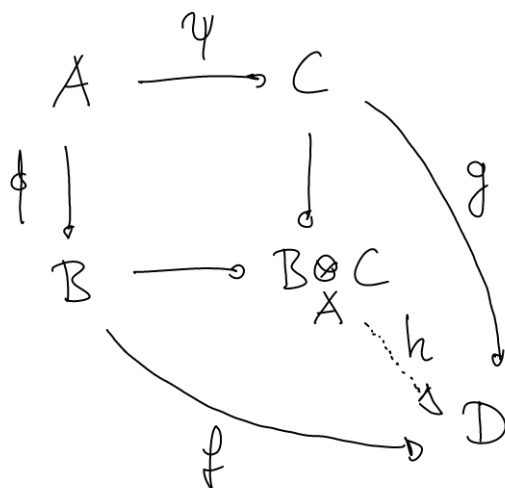
$$\begin{array}{ccc} A & \xrightarrow{\psi} & C \\ \phi \downarrow & & \downarrow b \\ B & \longrightarrow & B \otimes_A C \end{array}$$

commutes.

Moreover, it has the following universal property:

For every pair of A -algebra maps $B \xrightarrow{f} D$, $C \xrightarrow{g} D$ to an A -algebra D , there is a unique A -algebra map

$B \otimes_A C \xrightarrow{h} D$ s.t. $f = h \circ \phi$, $g = h \circ \psi$:



Proof 1) Multiplication is well-defined: Consider

$$B \times C \times B \times C \longrightarrow B \otimes_A C$$

$$(b_1, c_1, b_2, c_2) \longmapsto b_1 b_2 \otimes c_1 c_2.$$

It is A -multilinear. By small extension of tensor calculus

(see Atiyah-Macdonald, Prop 2.12*), it factors through an

A -linear map $B \otimes_A C \otimes_A B \otimes_A C \longrightarrow B \otimes_A C$

$$b_1 \otimes c_1 \otimes b_2 \otimes c_2 \longmapsto b_1 b_2 \otimes c_1 c_2.$$

Composing with the universal bilinear map, we obtain the A -bilinear

$$B \otimes_A C \times B \otimes_A C \longrightarrow B \otimes_A C$$

$$(b_1 \otimes c_1, b_2 \otimes c_2) \longmapsto b_1 b_2 \otimes c_1 c_2.$$

2) Makes $B \otimes_A C$ into an A -algebra + $B, C \longrightarrow B \otimes_A C$ are

A -algebra maps: Can be checked directly (Exercise)

The A -algebra structure here is given by

$$\phi \otimes \psi : A = A \otimes_A A \longrightarrow B \otimes_A C$$

$$a \longmapsto a \otimes 1 = 1 \otimes a \longmapsto \phi(a) \otimes 1 = 1 \otimes \psi(a).$$

The unit element is $1 \otimes 1$.

3) Universal property: Assume $f: B \rightarrow D$, $g: C \rightarrow D$ are A -algebra maps. Then

$$B \times C \longrightarrow D, (b, c) \longmapsto f(b)g(c)$$

is an A -bilinear map, hence factors uniquely through

$$B \otimes_A C \longrightarrow D, b \otimes c \longmapsto f(b)g(c).$$

This is a ring map:

$$\left(\sum_{i=1}^n b_i \otimes c_i \right) \cdot \left(\sum_{j=1}^m b'_j \otimes c'_j \right)$$

$$= \sum_{i,j} b_i b'_j \otimes c_i c'_j \longmapsto \sum_{i,j} f(b_i b'_j) g(c_i c'_j)$$

$$= \left(\sum_i f(b_i) g(c_i) \right) \cdot \left(\sum_j f(b'_j) g(c'_j) \right)$$

Even an A -algebra map: Let $\chi: A \rightarrow D$ be the A -algebra structure on D . Then

$$B \otimes_A C \ni f(a) \otimes 1 \longmapsto f(f(a)) = \chi(a) \in D$$

$$\nwarrow \quad \nearrow$$

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because f is an A -algebra map by assumption. \square

§ 4 Generators and Relations

Defn Presentation of an A -alg. $C \stackrel{\text{def}}{=} \text{choice of}$
set I , ideal $\alpha \subseteq A[T_i, i \in I]$ and an isomorphism
 $A[T_i, i \in I] / \alpha \xrightarrow{\sim} C$.

Lemma 3 Every A -alg. C admits a presentation.

Proof Take $I = C$. Then $A[T_c, c \in C] \rightarrow C$ is surjective.
 $T_c \mapsto c$

If α is its kernel, then $A[T_c, c \in C] / \alpha \cong C \quad \square$.

Prop 4 Let $\phi: A \rightarrow B$ be an A -algebra. There is an isomorphism of B -algebras

$$B \otimes_A (A[T_i, i \in I] / \alpha) \xrightarrow{\sim} B[T_i, i \in I] / (\phi(\alpha)).$$

Special cases:

- 1) $A/\alpha \otimes_A A/\beta \cong A/(\alpha + \beta)$
- 2) $B \otimes_A A[s^{-1}] \cong B[\phi(s)^{-1}]$.
- 3) $\mathfrak{p} \subset A$ prime ideal. Then $\kappa(\mathfrak{p}) \otimes_A B \xrightarrow{\sim} (B/\mathfrak{p})[\phi(A \setminus \mathfrak{p})^{-1}]$.

Proof Consider

$$\begin{array}{ccc}
 & A[T_i, i \in I] / \sigma & a \quad T_2 \\
 & \downarrow & \downarrow \\
 B \longrightarrow & B[T_i, i \in I] / \sigma & \phi(a) \quad T_i \\
 b \longmapsto & b &
 \end{array}$$

By universal property (Prop 2), factor through unique map

$$\begin{array}{ccc}
 B \otimes_A (A[T_i, i \in I] / \sigma) & \longrightarrow & B[T_i, i \in I] / \sigma \\
 b \otimes a & \longmapsto & \phi(a)b \\
 b \otimes T_i & \longmapsto & b T_i
 \end{array}$$

This is clearly surjective. For injectivity, we note that both have same universal property:

$$\text{Hom}_{A\text{-alg}}(-, D) = \left\{ \begin{array}{l} B\text{-algebra structure map } B \rightarrow D \\ + \text{ tuple } (d_i) \in D^I \text{ s.t. } \chi(f)(d_i) = 0 \\ \forall f \in \sigma. \end{array} \right\}$$

Here $\chi: A \rightarrow D$ is the A -algebra structure.

Special cases:

1) Set $B = A/\mathfrak{b}$, take $I = \emptyset$, or any.

2) B any, $I = S$, $\sigma = (s \cdot T_s - 1, s \in S)$

3) Combine 1) & 2).

□

§5 Example: Tensoring field extensions

① L/K finite extension. Primitive Element Thm states

there exists a K -algebra isomorphism $L \cong K[T]/(f(T))$.

Assume L/K separable $\Leftrightarrow f$ only simple zeros in \bar{K} .

Let M be splitting field of L , e.g. \bar{K} or a normal closure of L in \bar{K} . Then

$$M \otimes_K L \cong M \otimes_K K[T]/(f(T)) \cong \prod_{i=1}^{[L:K]} M.$$

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factors into linear polynomials.

In partic, if L Galois, then $L \otimes_K L \cong \prod_{i=1}^{[L:K]} L$.

② Now consider $K = \mathbb{F}_p(x)$, $L = \mathbb{F}_p(x^{1/p})$ with

x transcendental. This is an inseparable extension and

$$L \otimes_K L \cong \mathbb{F}_p(x^{1/p})[T] /_{T^p - x}$$

$$\cong \mathbb{F}_p(x^{1/p})[T] / (T - x^{1/p})^p \quad \text{is non-reduced.}$$

3) Now consider two transcendental extensions:

$$K \text{ any, } L = K(x) = \text{Quot}(K[x])$$

$$M = K(y) = \text{Quot}(K[y]).$$

Prop 4 states that

$$L \otimes_K M \cong K[x, y] [(K[x] - \{0\})^{-1}, (K[y] - \{0\})^{-1}].$$

Recall: a) $\text{Spec } A[S^{-1}] = \{ \mathfrak{p} \in \text{Spec } A \mid \mathfrak{p} \cap S = \emptyset \}$

b) $\text{Spec } K[x, y] = \{ (0) \} \cup \{ (\mathfrak{p}) \mid \mathfrak{p} \text{ irreducible} \}$
 $\cup \{ \text{maximal ideals} \}$

Moreover, each maximal ideal intersects $K[x]$ and $K[y]$ non-trivially, so

$$\text{Spec}(L \otimes_K M) = \{ (0) \} \cup \left\{ (\mathfrak{p}) \mid \begin{array}{l} \mathfrak{p} \in K[x, y] \text{ irred.} \\ \mathfrak{p} \not\subset K[x] \cup K[y] \end{array} \right\}$$

E.g. $X + aY + b$ with $a, b \in K$, $a \neq 0$ all

define mutually different prime ideals of $L \otimes_K M$.

$Y^2 + X$, $Y^5 - X^3 + X$ etc. yield further primes.