<u>Last lecture</u> <u>Reference</u> Atiyah - Macdonald §3

·) $\mathrm{Spec}(A) = \{ \mathfrak{p} \subset A \text{ prime ideal} \}$

·) $S \subseteq A$ subset. Then there is a localization map

$$\varphi : A \longrightarrow A[S^{-1}]$$

<u>Today</u>

Study $\mathrm{Spec}(\varphi) : \mathrm{Spec}(A[S^{-1}]) \longrightarrow \mathrm{Spec}(A)$

& related topics.

§1 More on localization

A ring, $S \subseteq A$ subset, $\overline{S} :=$ multiplicative closure. Then

$$A[T_s, s \in S] / (s \cdot T_s - 1, s \in S) \cong \overline{S}^{-1} A := \left\{ \tfrac{a}{s} \;\middle|\; a \in A, s \in \overline{S} \right\} / \sim$$

provide two explicit description for the localization.

<u>Lem 1</u> $\ker(A \longrightarrow A[S^{-1}]) = \{ a \in A \mid \exists\, s \in \overline{S} \text{ s.th. } sa = 0 \}$

<u>Proof</u> RHS describes precisely those a s.th. $\frac{a}{1} \sim 0$ in the defn of $\overline{S}^{-1} \cdot A$. $\square$

Recall $s \in A$ zero-divisor $\underset{def}{=}$ $\exists\ 0 \neq a$ with $sa = 0$.

$s \in A$ regular $\underset{def}{=}$ s no zero-divisor.

Thus $A \to A[S^{-1}]$ injective $\Leftrightarrow$ S consists of regular elements.

Examples 1) A domain, $S \subseteq A \setminus \{0\}$. Then

$$\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \iff a_1 s_2 = a_2 s_1.$$ simplifies from general defn of $\sim$.

$\mathrm{Quot}(A) := A[(A \setminus \{0\})^{-1}]$ quotient field or field of fractions of A

If $S \subseteq A \setminus \{0\}$ any, then

$A[S^{-1}] \subseteq \mathrm{Quot}(A)$ subring generated by A and all s^{-1}, $s \in S$.

E.g. $\mathbb{Z}[\frac{1}{6}] = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}] \subset \mathbb{Q}$.

$k[X, \frac{1}{X(X+1)}] \subset k(X) := \mathrm{Quot}(k[X])$

2) Assume $\varepsilon \in S$ is nilpotent. Lem 1 says $A[S^{-1}] = 0$.

Can also be seen directly: If $\varepsilon^{n+1} = 0$, then

$$1 = (1 - \varepsilon T_\varepsilon)(1 + \varepsilon T_\varepsilon + \cdots + \varepsilon^n T_\varepsilon^n),$$

so $(s \cdot T_s - 1,\ s \in S) = (1)$.

More sophisticated example:

$$\mathbb{C}[X,Y]/(XY)\,[X^{-1}] = \mathbb{C}[X,Y,T]/\underbrace{(XY, XT-1)}_{\mathfrak{a}} \qquad (*)$$

Have $Y = Y\cdot(1-XT) \in \mathfrak{a}$, so

$$(*) = \mathbb{C}[X,Y,T]/(Y, XT-1) \cong \mathbb{C}[X, X^{-1}] \subset \mathbb{C}(X)$$

$$Y \longmapsto 0$$
$$T \longmapsto X^{-1}$$

§2 Localization and Ideals

Lem 2 $S \subseteq A$ any subset, $\varphi: A \to A[S^{-1}]$ the localization.

Consider $\{\text{ideals of } A\} \rightleftarrows \{\text{ideals of } A[S^{-1}]\}$

$$\mathfrak{a} \longmapsto \varphi(\mathfrak{a})\cdot A[S^{-1}]$$

$$\varphi^{-1}(\mathfrak{b}) \longleftarrow \mathfrak{b}$$

The map $\mathfrak{b} \mapsto \varphi^{-1}(\mathfrak{b})$ is injective and, more precisely,

$$\mathfrak{b} = \varphi(\varphi^{-1}(\mathfrak{b}))\cdot A[S^{-1}]. \qquad (**)$$

Proof Enough to show $(**)$: $\supseteq$ is immediate because $\mathfrak{b} \supseteq \varphi(\varphi^{-1}(\mathfrak{b}))$. Show $\subseteq$: If $\frac{a}{s} \in \mathfrak{b}$, $a \in A$, $s \in \overline{S}$, then $a = s\cdot\frac{a}{s} \in \varphi^{-1}(\mathfrak{b})$, hence $\frac{a}{s} = \frac{1}{s}\cdot a \in \text{RHS}$. □

In general, ideals of $A[S^{-1}]$ are simpler than those of A:

<u>Example</u> $\mathbb{Z}[\sqrt{-5}]$ is not a PID:

$$T^2+5 \equiv T^2-1 = (T+1)(T-1) \quad \text{mod } (3),$$

so there are two primes above (3):

$$\mathfrak{p}_\pm = (3, 1\pm\sqrt{-5})$$

Then $\gcd(N(3), N(1\pm\sqrt{-5})) = \gcd(9,6) = 3$,

but there are no $x+y\sqrt{-5}$ with

$N(x+y\sqrt{-5}) = x^2+5y^2 = 3$, so there is

no element dividing both 3, $1\pm\sqrt{-5}$.

$\Longrightarrow$ $\mathfrak{p}_\pm$ both not principal.

But $$\frac{3}{1\pm\sqrt{-5}} = \frac{3}{(1\pm\sqrt{-5})(1\mp\sqrt{-5})} \cdot (1\mp\sqrt{-5}) = \frac{1\mp\sqrt{-5}}{2}$$

We conclude that

$\mathfrak{p}_\pm \cdot \mathbb{Z}[\sqrt{-5}, \frac{1}{2}] = (1\mp\sqrt{-5})$ are both principal.

(In fact, $\mathbb{Z}[\sqrt{-5}, \frac{1}{2}]$ is a PID.)

Lem 3 $S \subseteq A$, φ as before, $\mathfrak{a} \subseteq A$ any ideal and $\pi : A \longrightarrow A/\mathfrak{a}$ the quotient map. Then

$$A/\mathfrak{a}\,[\pi(S)^{-1}] \xrightarrow{\cong} A[S^{-1}] \big/ \varphi(\mathfrak{a}) \cdot A[S^{-1}] .$$

Proof Let $A \xrightarrow{\alpha} \tilde{A}$ denote either of the two ring maps. Then $\alpha(\mathfrak{a}) = 0$, $\alpha(S) \subseteq \tilde{A}^\times$. Moreover, α has the universal property

$\psi : A \to B$ factors through $\alpha \iff \psi(\mathfrak{a}) = 0$, $\psi(S) \subseteq B^\times$

+ such a factorization is unique. $\square$

Alternative explicit argument:

$$LHS = A/\mathfrak{a}\,[T_{\pi(s)},\ \pi(s) \in \pi(S)] \big/ (\pi(s) T_{\pi(s)} - 1,\ \pi(s) \in \pi(S))$$

$$= A/\mathfrak{a}\,[T_s,\ s \in S] \big/ (\pi(s) T_s - 1,\ s \in S)$$ because inverses of elements are unique.

$$\cong A[T_s,\ s \in S] \big/ \mathfrak{a} + (s \cdot T_s - 1,\ s \in S)$$

$$\cong \Big(A[T_s,\ s \in S] \big/ (s \cdot T_s - 1,\ s \in S) \Big) \Big/ (\varphi(\mathfrak{a})) = RHS.$$

Prop 4 $S \subseteq A$, φ as before. Then

1) $\mathrm{Spec}(\varphi): \mathrm{Spec}(A[S^{-1}]) \longrightarrow \mathrm{Spec}(A)$ is injective.

2) Its image equals $\{\mathfrak{p} \subset A \mid S \cap \mathfrak{p} = \emptyset\}$.

3) If $\mathfrak{p} \cap S = \emptyset$, then

$$\mathrm{Spec}(\varphi)^{-1}(\mathfrak{p}) = \varphi(\mathfrak{p}) \cdot A[S^{-1}].$$

Proof 1) Clear by Lem 2.

2) Assume $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in \mathrm{Spec}(A[S^{-1}])$.

Then $\varphi(\mathfrak{p}) \cap A[S^{-1}]^{\times} \subseteq \mathfrak{q} \cap A[S^{-1}]^{\times} = \emptyset$,

hence $\mathfrak{p} \cap S = \emptyset$.

Conversely, let $\mathfrak{p} \in \mathrm{Spec}(A)$. Lem 3 says that

$$A[S^{-1}] / \mathfrak{p} \cdot A[S^{-1}] = (A/\mathfrak{p})[\pi(S)^{-1}].$$

If $S \cap \mathfrak{p} = \emptyset$, then $0 \notin \pi(S)$ in the integral domain $A/\mathfrak{p}$. Then $(A/\mathfrak{p})[\pi(S)^{-1}]$ is an integral domain and hence $\mathfrak{p} \cdot A[S^{-1}]$ a prime ideal.

3) By Lem 1, $A/\mathfrak{p} \longrightarrow (A/\mathfrak{p})[\pi(S)^{-1}]$ is injective,

so $$\varphi^{-1}(\mathfrak{p}\cdot A[S^{-1}]) = \ker\left(A \longrightarrow (A/\mathfrak{p})[\pi(S)^{-1}]\right)$$
$$= \ker\left(A \longrightarrow A/\mathfrak{p}\right)$$
$$= \mathfrak{p}. \qquad \square$$

<u>Rmk</u> If $S \cap \mathfrak{p} \neq \emptyset$, then $\mathfrak{p}\cdot A[S^{-1}] = (1)$.

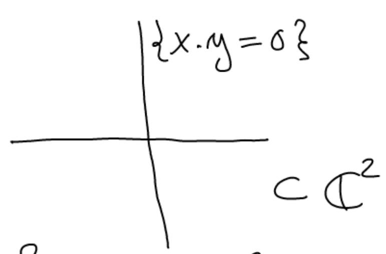
The two most common instances:

1) $f \in A$ any element.

$$\operatorname{Spec}(A[f^{-1}]) = \{\mathfrak{p} \subset A \mid f \notin \mathfrak{p}\}.$$

<u>e.g.</u> $$\operatorname{Spec} \mathbb{Z}\left[\tfrac{1}{3\cdot 5\cdot 11}\right] = \{(2),(7),(13),(17),\dots\} \cup \{(0)\}.$$

$$\operatorname{Spec} \mathbb{C}[X,Y]/(XY) = \{(X-x,Y)\} \cup \{(X,Y-y)\}$$
$$\cup \{(X),(Y)\}.$$



$$\operatorname{Spec} \mathbb{C}[X,Y]/(XY)[X^{-1}]$$
$$= \{(X-x,Y),\ x\neq 0\} \cup \{(Y)\}$$
$$\xrightarrow{1:1} \operatorname{Spec} \mathbb{C}[X,X^{-1}]$$
$$= \{(X-x),\ x \neq 0\} \cup \{(0)\}.$$

2) $\mathfrak{p} \subset A$ a prime. Then $A \setminus \mathfrak{p}$ is mult. set!

<u>Defn</u> <u>Localization of A at $\mathfrak{p}$</u> $\underset{def}{=}$ $A_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1} A$.

Then $\mathrm{Spec}(A_{\mathfrak{p}}) \xrightarrow{1:1} \{ \mathfrak{q} \in \mathrm{Spec}\, A \,,\ \mathfrak{q} \subseteq \mathfrak{p} \}$.

In ptic, $A_{\mathfrak{p}}$ has a <u>unique</u> maximal ideal, namely $\mathfrak{p} \cdot A_{\mathfrak{p}}$.

<u>Terminology</u> Rings w/ unique max ideal are <u>local rings</u>.

$\kappa(\mathfrak{p}) := A_{\mathfrak{p}} / \mathfrak{p} A_{\mathfrak{p}}$ <u>residue field</u> of $\mathfrak{p}$.

<u>By Lem 3:</u> $A_{\mathfrak{p}} / \mathfrak{p} A_{\mathfrak{p}} = \mathrm{Quot}(A/\mathfrak{p})$.

<u>Example</u> $\mathbb{Z}_{(p)} = \{ \frac{a}{b} \in \mathbb{Q} \mid p \nmid b \}$.

$$\mathbb{Z}_{(p)} / p\mathbb{Z}_{(p)} \cong \mathbb{F}_p$$

$$\frac{a}{b} \longmapsto b^{-1} a.$$

b^{-1} exists in $\mathbb{F}_p$ since $p \nmid b$.

§3 Application to Spectra

Situation $\varphi: A \longrightarrow B$ ring map, Spec A known.

How to compute Spec B ?

Consider
$$\begin{array}{ccc} \mathrm{Spec}(B) & \supseteq & \mathrm{Spec}(\varphi)^{-1}(\mathfrak{p}) \\ \big\downarrow \mathrm{Spec}(\varphi) & & \big\downarrow \\ \mathrm{Spec}(A) & \ni & \mathfrak{p} \end{array}$$

Then $\mathrm{Spec}(\varphi)^{-1}(\mathfrak{p}) = \left\{ \mathfrak{q} \subset B \;\middle|\; \begin{array}{l} \varphi(\mathfrak{p}) \subset \mathfrak{q} \text{ and} \\ s \notin \mathfrak{p} \Rightarrow \varphi(s) \notin \mathfrak{q} \end{array} \right\}$

$$= \mathrm{Spec}\; B/\varphi(\mathfrak{p}) \left[\varphi(S)^{-1}\right] \quad S = A \setminus \mathfrak{p}.$$

via $\quad \pi^{-1}(\overline{\mathfrak{q}}) \longleftarrow\!\shortmid \overline{\mathfrak{q}}$

where $\quad \pi: B \longrightarrow B/\varphi(\mathfrak{p}) \left[\varphi(S)^{-1}\right]$ is the natural map.

If, for example, $B = A[T_1, \dots, T_n]/(f_1, \dots, f_m)$,

then $B/\varphi(\mathfrak{p}) \left[\varphi(S)^{-1}\right] = \kappa(\mathfrak{p})[T_1, \dots, T_n]/(f_1, \dots, f_m)$

and $\quad \mathrm{Spec}(\varphi)^{-1}(\mathfrak{p}) = \mathrm{Spec}\; \kappa(\mathfrak{p})[T_1, \dots, T_n]/(f_1, \dots, f_m).$

$$\pi^{-1}(\overline{\mathfrak{q}}) \longleftarrow\!\shortmid \overline{\mathfrak{q}}$$

<u>Example</u> $\varphi: \mathbb{Z} \to \mathbb{Z}[T]$ from last time. Then, for $(0) \subset \mathbb{Z}$, residue field is $\mathbb{Q}$ and

$$\mathrm{Spec}(\varphi)^{-1}((0)) = \mathrm{Spec}\, \mathbb{Q}[T]$$

$$\mathbb{Z}[T] \cap \mathfrak{q} \longleftarrow\!\!\shortmid\, \mathfrak{q}$$

In fact, for any ring A, we obtain the description

$$\mathrm{Spec}(A) = \coprod_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{Spec}\, \kappa(\mathfrak{p})[T].$$

§4 Appendix: Geometric Intuition

$X := \mathbb{R}^n$, $A := C^\infty(X, \mathbb{R})$ smooth fcts. on X

$X \to \operatorname{MaxSpec}(A)$, $x \mapsto \mathfrak{m}_x := \{ f \mid f(x) = 0 \}$

Residue field $\kappa(\mathfrak{m}_x) = A/\mathfrak{m}_x \xrightarrow{\sim} \mathbb{R}$

$f \longmapsto f(x)$.

Given $f_1, \ldots, f_n \in A$, have two operations:

1) $V = V(f_1, \ldots, f_n) = \{ x \in X \mid f_1(x) = \cdots = f_n(x) = 0 \}$

Then V is closed. For $\bar{g} \in A/(f_1, \ldots, f_n)$ and $x \in V$, the value $\bar{g}(x) \in \mathbb{R}$ is well-defined, so $A/(f_1, \ldots, f_n)$ is a ring of functions on V.

2) $D = D(f_1, \ldots, f_n) = \{ x \in X \mid (f_1 \cdots f_n)(x) \neq 0 \}$

This set is open and $A[f_1^{-1}, \ldots, f_n^{-1}]$ is a ring of functions on D.

<u>Rmk</u> V is for "vanishing set",

D is for "does not vanish set".

For general ring A and $X = \mathrm{Spec}(A)$, the intuition is exactly the same. Two differences:

1) f viewed as "function" on X takes values

$$f(\mathfrak{p}) := \text{image of } f \text{ in } \kappa(\mathfrak{p})$$

in the varying fields $\kappa(\mathfrak{p})$.

2) X, $V(f_1, \ldots, f_n)$, $D(f_1, \ldots, f_n)$ are completely described by their rings as

$$X = \mathrm{Spec}\, A$$

$$V(f_1, \ldots, f_n) = \mathrm{Spec}\, A/(f_1, \ldots, f_n)$$

$$D(f_1, \ldots, f_n) = \mathrm{Spec}\, A[f_1^{-1}, \ldots, f_n^{-1}].$$