Elliptic curves and their moduli spaces
Exercise sheet 5

Solutions by: Esteban Castillo Vargas and David Čadež

16. Mai 2024

Problem 1.

(i) Pick $f \in k(\eta) \setminus k$ (η is the generic point of the curve C). We will show that f can be viewed as a non-constant function $C \to \mathbb{P}^1_k$.

First we will define $U \to \mathbb{A}^1_k$ on some open cofinite subscheme $U \in C$. Then we will postcompose this morphism with $\mathbb{A}^1_k \to \mathbb{P}^1_k$. At we will extend map $U \to \mathbb{P}^1_k$ to $\varphi \colon C \to \mathbb{P}^1_k$. By construction it will clear that the pullback of t is f.

Take some open affine $\operatorname{Spec}(A) \subseteq C$. By integrality of C, $\operatorname{Spec}(A)$ contains the generic point, so we have $\operatorname{Quot}(A) = k(\eta)$. Write $f = \frac{f_1}{f_2} \in \operatorname{Quot}(A)$. Then f is a section on $D(f_2)$. Because A is 1-dimensional, the complement $V(f_2)$ is finite. We use the bijection $\Gamma(\operatorname{Spec}(A), \mathcal{O}_X) \cong \operatorname{Mor}_k(\operatorname{Spec}(A), \mathbb{A}^1_k)$ to realize f as a morphism $D(f_2) \to \mathbb{A}^1_k$. We do this on an finite open affine cover to obtain $f \colon U \to \mathbb{A}^1_k$, where $U \subseteq C$ is open, cofinite and dense. We can pick finite cover by quasicompactness and density of U is by integrality. Note here that this definition on affine cover clearly agrees on intersections because we always had the same $f \in k(\eta)$.

Now note that local rings at closed points on the curves are valuation rings. At this point we could recall a proposition from alg geo 1 that said that every dominant rational map between proper normal curves is represented by a morphism of schemes. By our definition curves are proper and smooth, so that would work. (And any morphism representing $U \to \mathbb{A}^1_k$ would pullback t to f.)

Postcompose with $\mathbb{A}^1_k \to \mathbb{P}^1_k$ to obtain $f \colon U \to \mathbb{P}^1_k$.

Now all we need to do is extend this f to the whole C.

We use valuative criteria to do it.

$$\begin{tikzcd}
\operatorname{Spec}(k(\eta)) \arrow[r] \arrow[d] & \operatorname{Spec}(\mathcal{O}_{C,x}) \arrow[d, dashed] \arrow[dr] & \\
U \arrow[r] & \mathbb{P}^1_k \arrow[r] & \operatorname{Spec}(k)
\end{tikzcd}$$

We obtain dashed arrow by valuative criteria for properness of $\mathbb{P}^1_k$ over k. Since $\mathbb{P}^1_k$ is of finite presentation over k, we can use the spreading argument to obtain an open neighbourhood $x \in V \subset C$ and a morphism $V \to \mathbb{P}^1_k$ that extends $\operatorname{Spec}(\mathcal{O}_{C,x}) \to \mathbb{P}^1_k$. Morphisms $V \to \mathbb{P}^1_k$ and $U \to \mathbb{P}^1_k$ match on generic point and, since C is separated, the equalizer is closed subscheme, so they must match on $U \cup V$ (here we use integrality of C).

There are only finitely many points in $C \setminus U$, so we can do this process finitely many times and obtain $C \to \mathbb{P}^1_k$.

Let us now argue why the pullback of t is f. When defining $U \to \mathbb{A}^1_k$ we defined it using identification

$$\operatorname{Hom}_k(k[t], \Gamma(\operatorname{Spec}(A), \mathcal{O}_X)) \cong \operatorname{Mor}_k(\operatorname{Spec}(A), \mathbb{A}^1_k).$$

So we defined the morphism by saying that the pullback of t should be f. Also notice that if f is not in k, then $k[t] \to \Gamma(\operatorname{Spec}(A), \mathcal{O}_X))$ is injective, so the induced morphism maps generic point to generic point and is thus not constant.

(ii) We simply follow the definitions

$$\operatorname{div}(\varphi^*(f)) = \sum_{x \in C_1} \operatorname{ord}_x(\varphi^*(f))[x]$$

and

$$\begin{aligned}
\varphi^* \operatorname{div}(f) &= \varphi^* \left(\sum_{y \in C_2} \operatorname{ord}_y(f)[y] \right) \\
&= \sum_{y \in C_2} \operatorname{ord}_y(f) \left(\sum_{x \in \varphi^{-1}(y)} e_x[x] \right) \\
&= \sum_{x \in C_1} \operatorname{ord}_{\varphi(x)}(f) e_x[x].
\end{aligned}$$

So we have to show that for any $x \in C_1$ we have $\operatorname{ord}_x(\varphi^* f) = \operatorname{ord}_{\varphi(x)}(f) e_x$.

Denote $n = \operatorname{ord}_{\varphi(x)}(f)$.

By normality we can pick a uniformizers

$$m_{C_2,\varphi(x)} = (t_{C_2,\varphi(x)}) \quad \text{and} \quad m_{C_1,x} = (t_{C_1,x}).$$

The map $\varphi_x^\#\colon \mathcal{O}_{C_2,\varphi(x)} \to \mathcal{O}_{C_1,x}$ is a restriction of $\varphi^*\colon k(\eta_2) \to k(\eta_1)$, so we can use φ^* for both.

By the definition of ramification index $\varphi^*(m_{C_2,\varphi(x)})\mathcal{O}_{C_1,x} = m_{C_1,x}^{e_x}$. So $\varphi^*(t_{C_2,\varphi(x)}) = \beta t_{C_1,x}^{e_x}$ for some $\beta \in \mathcal{O}_{C_1,x}^\times$.

Write $f = \alpha t_{C_2,\varphi(x)}^n$ with $\alpha \in \mathcal{O}_{C_2,\varphi(x)}^\times$. Then

$$\varphi^*(f) = \varphi^*(\alpha)\varphi^*(t_{C_2,\varphi(x)})^n = \varphi^*(\alpha)\beta^n t_{C_1,x}^{ne_x} \in \mathcal{O}_{C_1,x}^\times t_{C_1,x}^{ne_x}.$$

So $\operatorname{ord}_x(\varphi^*(f)) = ne_x$ which is what we needed to show.