Algebraic geometry 1
Exercise sheet 11

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Exercise 1. We claim that $\mathcal{O}_{X,x}$ is a normal, local noetherian domain of dimension at most one. Normality is by definition of normality of X. Stalks are of course local rings by definition of locally ringed space. Noetherian comes from the assumption that X is of finite type over k. Also,

$$\dim(X) = \sup_{x \in X} \dim(\mathcal{O}_{X,x}).$$

Hence, $\dim(\mathcal{O}_{X,x}) \leq 1$.

If $\dim(\mathcal{O}_{X,x}) = 0$, then $\mathcal{O}_{X,x}$ is of course regular as the only prime ideal is (0).

If $\dim(\mathcal{O}_{X,x}) = 1$, then we know from the lecture that the maximal ideal $m \subset \mathcal{O}_{X,x}$ is principal (and not zero) and hence $\mathcal{O}_{X,x}$ is regular.

Exercise 3.

1. Since k is algebraically closed, the only irreducible polynomials $f \in k[x, y]$ are of degree 1.

   Hence, we can write

   $$f_r = l_1 \dots l_r,$$

   where $l_i \in k[x, y]$ is of degree 1. From the assumption that f_r is homogenous it follows that the l_i are homogenous.

   Therefore, we can write

   $$Z = V(f_r) = V(l_1 \dots l_r) = \cup_i V(l_i)$$

   and since $V(l_i)$ is a line through the origin, Z can be written as the finite union of lines through the origin.

2. We first want to prove that $\dim(\mathcal{O}_{X,(x,y)}) = 1$ for all r. The prime ideals p in this ring fulfil $(f) \subset p \subset (x, y)$. Remember that we can write down these prime ideals explicitely as in "What do primes of k[x,y] look like". From this the claim follows.

   We know that $\dim_k(m_{\mathcal{O}_{X,(x,y)}}/m^2_{\mathcal{O}_{X,(x,y)}})$ is the number of generators of $m_{\mathcal{O}_{X,(x,y)}}$.

   Now if $r = 1$, then we can write $f = g(x, y)x + h(x, y)y$ and w.l.o.g. we have $g(0, 0) = 1$, meaning that it is invertible (after localizing). Therefore $f = x + h(x, y)y$, so $y \mid x$ meaning $(x, y) = (y)$ On the other hand, if $r > 1$, then $x \nmid y$ and $y \nmid x$ meaning that m is no principal ideal showing that X is singular at zero in this case. (This can be seen by writing f as $f = x^2h_1(x, y) + xyh_2(x, y) + y^2h_3(x, y)$).

3. By part two of this exercise, all the schemes have a singular point at the origin. I don't know why they do not have singular points anywhere else.

Exercise 4. Intuitively, since

$$\Gamma(X, \mathcal{O}_X) \subset \Gamma(U, \mathcal{O}_X)$$

one can always restrict sections on X to sections on U. In this exercise we basically show that under these special conditions we can also uniquely extend a section on U to one on the whole X.

We will show that the restriction map on global sections is an isomorphism, i.e. that

$$\Gamma(X, \mathcal{O}_X) \cong \Gamma(U, \mathcal{O}_X).$$

This immediately implies the claim of the exercise by definition of vector bundle (if rings are isomorphic then so is their finite sum).

By definition, $\mathrm{codim}(Z) \geq 2$. (where $\mathrm{codim}(Z) = \inf_{z\in Z}(\mathcal{O}_{X,z})$) as defined in Görz Wedhorn.

Take $Y \subset X$ an irreducible component of codimension 1. By construction, Y and U intersect nontrivially (either $Z \cap Y = \emptyset$ or $Y \subsetneq Z$). In particular, U contains the generic point μ of Y. This means that $\Gamma(U, \mathcal{O}_X) \subset \mathcal{O}_{X,\mu}$.

The hint tells us that A is the intersection of all localizations of A at prime ideals p of height 1. Those prime ideals of height 1 correspond to irreducible closed subsets of codimension 1. Hence, we have just shown that

$$\Gamma(U, \mathcal{O}_X) \subset A = \Gamma(X, \mathcal{O}_X).$$

The other inclusion is immediate.