Back to lifting criteria:

Proposition Formal smoothness is local on the source.

Theorem (Gruson–Raynaud)
Let A be a ring and M be an A-module. Let $A \to B$ be a faithfully flat map. If $M \otimes_A B$ is projective B-module, then M is a projective A-module.

Rmk - projectivity satisfies faithfully flat descent.
- we showed being finite projective satisfies faithfully flat descent.

Let $f: X \to S$ be a morphism of schemes, let $X = \bigcup_{i \in I} U_i$ open cover with $U_i = \operatorname{Spec} A_i$ such that $f|_{U_i}: U_i \to S$ is formally smooth.

Fix a diagram

$$\begin{tikzcd} T_0 \arrow[r, "u_0"] \arrow[d, hook, "i"'] & X \arrow[d] \\ T \arrow[r, "u"'] & S \end{tikzcd}$$

with $T = \operatorname{Spec} B$ and $T_0 = \operatorname{Spec} B/I$ s.t. $I^2 = 0$.

By hypothesis

$$\begin{tikzcd} T_0 \cap U_i \arrow[r, "u_0"] \arrow[d, hook, "i"'] & U_i \arrow[d] \\ T \cap U_i \arrow[r, "u"'] \arrow[ur, dotted, "l_i"] & S \end{tikzcd}$$

has a lift l_i

Question Do the l_i glue to a map $T \to X$?

Definition

Let G be a group.

1) A G-pseudo-torsor is a set P with action $G \times P \to P$ s.t. $\forall\, p_1, p_2 \in P$ $\exists!\, g \in G$ with $g \cdot p_1 = p_2$.

2) A G-pseudo-torsor P is a G-torsor if $P \neq \emptyset$.

Let T be a topological space and G a sheaf of groups.

1) A G-pseudo-torsor is a sheaf P over T with an action

$$G \times P \to P$$

s.t. $\forall\, U \subseteq T$ open and $\forall\, p \in P(U)$ the map

$$\begin{aligned} G(U) &\to P(U) \\ g &\mapsto g \cdot p \end{aligned}$$

is an isomorphism.

2) A pseudo-torsor P is a torsor if for all $x \in T$ $P_x \neq \emptyset$.

Example: Consider $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$,

let $\pi : S^1 \to S^1$ $\quad \pi(e^{i\theta}) = e^{2i\theta}$ $\quad \forall\, \theta \in \mathbb{R}$.

For all $U \underset{\text{open}}{\subseteq} S^1$ let

$$P_U = \{ \sigma : U \to S^1 \mid \pi \circ \sigma = \mathrm{id}_U \}.$$

Consider the constant group sheaf

$G = \mathbb{Z}/2\mathbb{Z}$.

we have a action:

$$G \times P \longrightarrow P$$

$$(e, \sigma) \longmapsto \sigma$$

$$(1, \sigma) \longmapsto \sigma + \pi i$$

Then P is a G-torsor.

With setup as before

let

$$P : \mathrm{Ouv}_T \longrightarrow \mathrm{Sets}$$

$$\{ U \underset{\text{open}}{\subseteq} T \} \longmapsto \left\{ \begin{array}{l} l_U \\ \text{Lifts of} \end{array} \begin{tikzcd} U_0 \arrow[r] \arrow[d, "p"] & T_0 \arrow[r] & X \arrow[d] \\ U \arrow[rru, dashed, "l_U"] \arrow[r, phantom, "\subseteq"] & T \arrow[r] & S \end{tikzcd} \right\}$$

Claim: P is a
$\mathcal{H}om_{T_0}(u_0^* \Omega^1_{X/S}, I)$ - torsor.

Remark:
when $X = \operatorname{Spec} A$, $S = \operatorname{Spec} C$
this says
$\operatorname{Der}_C(A, I)$ acts simply transitively on the set of lifts (See Lecture 6)

Key point: This construction glues.

Definition: A map of torsors P_1, P_2 is a G-equivariant map of sheaves.

Rmk: Any map of G-torsors is automatically an isomorphism

Definition: Given a topological space T and a sheaf of groups G we let $H^1(T, G) = \{G\text{-torsors}\}/\simeq$

Remark $H^1(T, G)$ is a pointed set with distinguished element being the isomorphism class of the trivial torsor.

Remark If $s \in P(T)$ then the map
$$\begin{aligned} G &\longrightarrow P \\ g &\longmapsto g.s \end{aligned}$$
induces an isomorphism of G-torsors.

In other words $P = *$ in $H^1(T, G)$
$$\iff P(T) \neq \emptyset.$$

Functoriality of torsors:

Proposition

If $G_1 \longrightarrow G_2$ is a map of groups, and P_1 is a G_1-torsor, then $P_2 := G_2 \overset{G_1}{\times} P_1 = G_2 \times P_1 / G_1$ is a G_2-torsor.

Proof For $x \in T$ we can compute the stalk $(P_2)_x = G_{2x} \times_{G_{1x}} P_{1x} \cong G_{2,x}$.

If $s \in P_2(U)$ we want to show

$$G_2(U) \to [P_2(U)]$$
$$g \longmapsto g \cdot s$$

is an isomorphism. We can check this map is an isomorphism on stalks.

Rmk

In general $P_2(U) \neq [G_2(U) \overset{P_1(U)}{\times} G_1(U)]$ since we have to sheafify.

Proposition $H^1(T, G_1 \times G_2) = H^1(T, G_1) \times H^1(T, G_2)$.

proof There are evident maps

$$P_{1,2} \longmapsto (P_{1,2} \overset{G_1 \times G_2}{\times} G_1, P_{1,2} \overset{G_1 \times G_2}{\times} G_2)$$

$$P_1 \times P_2 \longleftarrow (P_1, P_2)$$

since stalks commute with finite

products one shows the constructions are inverse to each other.

<u>Proposition</u> $H^1(T, \prod_{j \in J} G_j) \subseteq \prod_{j \in J} H^1(T, G_j)$

<u>Proof</u> If P is a $\prod_{j \in J} G_j$-torsor then $\left(P_j = P \overset{\prod G_j}{\times} G_j\right)_{j \in J}$ is a family of G_j-torsors indexed by $j \in J$.

Given a family of torsors $(P_j)_{j \in J}$, we let $P = \prod P_j$, this is only a pseudo-torsor since some stalks might be empty. Key point: stalks do not commute with infinite product.

Given a diagram

$$D = \begin{tikzcd} T_0 \arrow[r, "\mu_0"] \arrow[d, hook] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

we set a torsor

$$P_D \in H^1(T, \mathcal{H}om(\mu_0^* \Omega^1_{X/S}, I)).$$

and we want to show

$P_D = *$ in this H^1.

Claim: $u_0^* \Omega^1_{X/S}$ is projective.

proof $u_0^* \Omega^1_{X/S}|_{T_0 \cap U_i}$ is projective since $U_i \to S$ is formally smooth. Since $\bigcup T_0 \cap U_i \to T_0$ is a faithfully flat cover the result follows from (Gruson - Raynaud).

Claim: $H^1(T, \mathcal{H}om(u_0^* \Omega^1_{X/S}, I)) = *$

Let $M = u_0^* \Omega^1_{X/S}$, then M is a direct summand of $\bigoplus_J A$ since it is projective.

Then $\mathcal{H}om(M, I)$ is a direct summand of $\mathcal{H}om(\bigoplus_J A, I) = \prod_J I$

This gives us an injection
$H^1(T, \mathcal{H}om(M, I)) \subseteq H^1(T, \prod_J I) \subseteq \prod_J H^1(T, I)$

We have reduced this to showing

$H^1(T, I) = *$ for all quasi-coherent sheaves I and affine $T = \operatorname{Spec} B$.