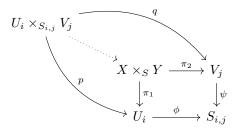# Algebraic geometry 1
Exercise sheet 6

Solutions by: Eric Rudolph and David Čadež

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Exercise 1.

1. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram

$$
\begin{array}{c}
U_i \times_{S_{i,j}} V_j \\
\end{array}
\quad
\begin{array}{ccc}
 & \xrightarrow{\;\;q\;\;} & \\
X \times_S Y & \xrightarrow{\pi_2} & V_j \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \psi} \\
U_i & \xrightarrow{\;\phi\;} & S_{i,j}
\end{array}
$$

Therefore, on the level of sets,

$$ U_i \times_{S_{i,j}} V_j \subset X \times_S Y, $$

but in exercise 5.2.1, we showed that this induces an open immersion as locally ringed spaces.

Now observe that

$$
\begin{array}{ccc}
\bigcup_{i,j} \left(U_i \times_{S_{i,j}} V_j \right) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & S
\end{array}
$$

commutes, because $S = \bigcup_{i,j} S_{i,j}$. Now by uniqueness of the pullback,

$$ \bigcup_{i,j} \left(U_i \times_{S_{i,j}} V_j \right) \cong U_i \times_{S_{i,j}} V_j. $$

I guess this is a good step in the direction of understanding why the pullback in the category of sheaves exists, right? If we assume X, Y, S to be

sheaves and $U_i, V_j, S_{i,j}$ to be affine schemes, then by the above argument we found a cover of $X \times_S Y$ by affine schemes.

2. Surjectivity follows, because a pullback of schemes in partiular makes

$$
\begin{array}{ccc}
\mid X \times_S Y \mid & \longrightarrow & \mid X \mid \\
\downarrow & & \downarrow \psi \\
\mid Y \mid & \xrightarrow{\ \phi\ } & \mid S \mid
\end{array}
$$

commute for all ψ, ϕ from maps of schemes.

Exercise 3. By definition we have to compute a fibred product of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and $\mathrm{Spec}(k(p)) \to \mathrm{Spec}(A)$ (where $k(p)$ is the residue field of $p \in \mathrm{Spec}(A)$ and $\to$ is the canonical inclusion). Since we are dealing with affine schemes, we can express it concretely as $\mathrm{Spec}(B \otimes_A k(p))$. Note that B has the structure of an A-algebra, which is induced by the starting morphism of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. So this exercise reduces to computing these tensor products.

We also observe that $k[T]$ is a PID, which means every non-zero prime ideal is a maximal ideal. This will be handy when computing residue fields, because after quotienting with a non-zero ideal we already get a field (we do not have to further take the quotient field).

1. In the first example we do now even have to calculate the tensor product, because we can rewrite $k[T,U]/(TU-1) = k[T,T^{-1}]$, so this is just a localization of $k[T]$. Morphism of spectrums, induced by inclusion into localization, is an open immersion, so fibers will be singletons if $x \in D(T)$ and empty sets otherwise. And the structure sheaf is also clear, it is just the restriction of structure sheaf $\mathcal{O}_{\mathrm{Spec}(k[T])}$.

2.

3.

4.

Exercise 4. Take $U = D(f)$ for some $f \in A$ and let $U = \cup_i D(f_i)$ be some cover. We have to check that

$$
M[f^{-1}] \to \mathrm{Eq}\left[\prod_i M[f_i^{-1}] \rightrightarrows \prod_{i,j} M[(f_i f_j)^{-1}]\right]
$$

is isomorphism.

This proof is exactly the same as when we proved that $\mathcal{O}_{\mathrm{Spec}(A)}$ is a sheaf, after we defined it the basis of principal opens.

Then proved that $A = \mathrm{Eq}\left[\prod_i A[f_i^{-1}] \rightrightarrows \prod_{i,j} A[(f_i f_j)^{-1}]\right]$ where $\mathrm{Spec}(A) = \cup_i D(f_i)$ is a cover.

We can simply tensor the whole diagram and, since tensor product commute with direct limits, we have that

$$M = \mathrm{Eq}\left[\prod_i M \otimes_A A[f_i^{-1}] \rightrightarrows \prod_{i,j} M \otimes_A A[(f_i f_j)^{-1}]\right].$$