# Algebraic geometry 1
# Exercise sheet 7

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**Exercise 1.**

1. We have the following bijection
$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_x}(\widetilde{\mathcal{N}_{|A}}, f_*\widetilde{\mathcal{N}}) &\cong \mathrm{Hom}_A(\widetilde{\mathcal{N}_{|A}}(B), f_*\widetilde{\mathcal{N}}(B)) \\ &= \mathrm{Hom}_A(N_{|A}, \widetilde{\mathcal{N}}(A)) \cong \mathrm{Hom}_{\mathcal{O}_x}(\widetilde{\mathcal{N}_{|A}}, \widetilde{\mathcal{N}_{|A}}).\end{aligned}$$
By the Yoneda lemma, this implies that $f_*\widetilde{N} \cong \tilde{N}_{|A}$.

2. For the second part of this exercise, we extend the first part as follows, using that $f_*$ is left-adjoint to $f^*$
$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_y}(f^*\tilde{\mathcal{M}}, \tilde{N}) \cong \mathrm{Hom}_{\mathcal{O}_x}(\widetilde{\mathcal{M}}, f_*\widetilde{\mathcal{N}}) \cong \mathrm{Hom}_A(\widetilde{\mathcal{M}}(B), f_*\widetilde{\mathcal{N}}(B)) \\ = \mathrm{Hom}_A(M, \widetilde{\mathcal{N}}(A)) \cong \mathrm{Hom}_B(N_{|A} \otimes_A B, \tilde{N}(A)) \cong \mathrm{Hom}_{\mathcal{O}_y}(\widetilde{\mathcal{M} \otimes_A B}, \widetilde{\mathcal{N}}).\end{aligned}$$
Now, by the Yoneda lemma we again obtain that
$$\widetilde{\mathcal{M} \otimes_A B} \cong f^*\tilde{\mathcal{M}}.$$
Next, we want to show that we can extend this exercise from affine schemes to schemes.

   Let $S_i$ with $i \in I$ be a cover of $S$ by open affines. Then for each $i \in I$ we get that $g^{-1}(S_i)$ is a subscheme of $Z_i \subset Z$ (unfortunately not necessarilary affine). Now, we cover each of these subschemes $Z_i$ by open affines $Z_{ij}$. By construction $g$ maps $Z_{ij}$ into $S_i$. Hence,
$$(g^*\mathcal{M})_{Z_{ij}} = f^*\mathcal{M}_{Z_{ij}} \cong \widetilde{M \otimes_A B},$$
showing that $g^*$ preserves quasi-coherence.

**Exercise 4.** We don't really want to do all the explicit calculations, so we only show what we think is maybe the main takeaway of this exercise.

For some polynomial $f \in \mathbb{R}[x,y]$ we have that

$$\begin{aligned}
&V(f) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \\
&\cong \mathrm{Spec}(\mathbb{R}[x,y]/(f)) \otimes_{Spec(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \\
&\cong \mathrm{Spec}(\mathbb{R}[x,y]/(f) \times_{\mathbb{R}} \mathbb{C}) \\
&\cong \mathrm{Spec}(\mathbb{C}[x,y]/(f)).
\end{aligned}$$

In the following, we take $f(x,y) := xy - 1$ and $g(x,y) := x^2 + y^2 - 1$. We know from the first sheet, that

$$\mathbb{C}[x,y]/(f) \cong \mathbb{C}[x,y]/(g),$$

but one can easily check that

$$\mathbb{R}[x,y]/(f) \ncong \mathbb{R}[x,y]/(g),$$

since the left side has strictly more units than the right side.

Therefore, this is an example showing that schemes being isomorphic is not stable under base change.