Algebraic geometry 2
Exercise sheet 7

Solutions by: Esteban Castillo Vargas and David Čadež

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Exercise 1.

1. Let $U \subseteq X$ and $s \in \mathrm{Isom}(\mathcal{O}_U, \mathcal{L}\mid_U)$. Then we have a map (of sets) $\mathcal{O}_X^\times(U) \to \mathrm{Isom}(\mathcal{O}_U, \mathcal{L}\mid_U)$ as $g \mapsto gs$.

   First of all, gs is an isomorphism of sheaves that on stalks looks like this:

   $$t_x \mapsto g_x\, s_x(t_x)$$

   which is still an isomorphism because $g_x \in \mathcal{O}_{X,x}^\times$. So $gs \in \mathrm{Isom}(\mathcal{O}_U, \mathcal{L}\mid_U)$.

   Suppose now $s_1, s_2 \in \mathrm{Isom}(\mathcal{O}_U, \mathcal{L}\mid_U)$. Then s_1, s_2 induce isomorphism $\mathcal{O}_V(V) \to \mathcal{L}_{|V}(V)$. Let $a \in \mathcal{O}_V(V)$ be such that $s_2^{-1}s_1(1) = a$ (Im writing s_1, s_2 also for induced group homomorphisms). The map $s_2^{-1}s_1$ is an $\mathcal{O}_V(V)$-linear automorphism of $\mathcal{O}_V(V)$. So it must be multiplication by $a \in \mathcal{O}_V(V)$. By surjectivity the section a is invertible. It is clear that $s_1 = as_2$, where multiplication with a means $\mathcal{O}_X$-multiplication with appropriate restriction of the section a. This proves the surjectivity of $\mathcal{O}_X^\times(U) \to \mathrm{Isom}(\mathcal{O}_U, \mathcal{L}\mid_U)$. Injectivity is clear by construction.

2. Injectivity: Take $\mathcal{L}, \mathcal{F}$ such that $\underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{L}) \cong \underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{F})$. Then

   $$\mathrm{Isom}(\mathcal{O}_X(U), \mathcal{L}(U)) \cong \mathrm{Isom}(\mathcal{O}_X(U), \mathcal{F}(U))$$

   for every $U \subseteq X$. So whenever $\mathrm{Isom}(\mathcal{O}_X(U), \mathcal{L}(U)) \neq \emptyset$, neither of them is, so we can find an isomorphism $\mathcal{L}(U) \cong \mathcal{F}(U)$. But we must find them in such a way that they glue to an isomorphism $\mathcal{L} \to \mathcal{F}$.

   But now I realized that we could maybe show $\mathrm{Pic}(X) \to H^1(X, \mathcal{O}_X^\times)$ is a group homomorphism. Although as far as I remember we haven't defined group structure on $H^1(X, \mathcal{O}_X^\times)$ yet.

   Anyway, under the assumption that the map is a group homomorphism, the injectivity is clear. If $\underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{L})$ has a non-zero global section, which means there exists an isomorphism $\mathcal{O}_X \to \mathcal{L}$, so $\mathcal{L}$ is trivial.

Exercise 4.

1. Since M is coherent, we can find a open affine on which all sections vanish. We do that by taking some open affine $\mathrm{Spec}(A) \subseteq X$. Then M is an A-module of finite type, let $\{x_1, \ldots, x_n\}$ be (nonzero) generators. Take $f = f_1 \ldots f_n$, where $f_i x_i = 0$ and f_i non-zero-divisors. Clearly f is neither invertible nor zero. Now we have that on $D(f)$ all elements of M vanish. So M vanishes on open subset of a curve, meaning its support is a finite set of closed points.

   With this support, M turns out to be flasque (def: restriction maps are surjections). We can see that by showing that M is isomorphic to a sheaf $M' = \oplus_i M_{x_i}$ (notation from stackexchange).

   For a flasque sheaf, taking global sections is exact (this solution is from stackexchange): Let $0 \to M \xrightarrow{f} F \xrightarrow{g} G \to 0$ be exact sequence of sheaves on X and M flasque. Let $t \in G(X)$. Because taking stalks is exact, we have a covering of $X = \cup_i U_i$ with $t_i \in F(U_i)$ such that $g(t_i) = t\mid_{U_i}$. Now take the set of all pairs (U, s), such that $g(s) = t\mid_U$. We check all conditions to use Zorn's lemma and obtain a maximal element (U^*, t^*). If $U^* \neq X$, then there exists k with $U_k \not\subseteq U^*$. Then $t^* - t_k$ restricted to $U^* \cap U_k$ is in the kernel of g. Because M is flasque, we can find a section $s* \in F(U^*)$ that gets mapped to 0 by g (i.e. comes from $M(U^*)$) and agrees with $t^* - t_k$ on $U^* \cap U_k$. So $t_* - s^*$ agrees with t_k on that intersection and gets mapped to $t \in G(X)$ at the same time. This contradicts maximality of (U^*, t^*).

   Now we use what we've shown in exercise 2 and we are done.

   (I'm worried because I haven't used anything about the space X except integrality and dimension.)

2. Because X is proper smooth curve, we can use Riemann-Roch. Ideal sheaf $\mathcal{I}$ is the line bundle $\mathcal{O}(-[x]-[y])$. Then by Riemann-Roch theorem $h^0(\mathcal{O}(-[x]-[y])) - h^1(\mathcal{O}(-[x]-[y])) = \deg(-[x]-[y]) + 1$. So dimension of $H^1(\mathcal{O}(-[x]-[y])$ has to be 1.