<u>Definition</u> Let X be a spectral space $S \subseteq X$ is constructible if it belongs to the boolean algebra generated by qc-open subsets of X.

<u>Definition</u> Let X be a spectral space, we let X^{cons} be the topological space whose basis of open neighborhoods are constructible sets.

<u>Theorem</u> (Chevalley) Let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and let $f: X \to Y$ be finitely then f sends constructible sets to constructible sets. In particular, $f: X^{cons} \to Y^{cons}$ is an open map.

<u>Non-example</u>: $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ is not of finite presentation

the set only containing the generic point is not constructible.

Proposition If $f: X \to Y$ is finitely presented and flat then it is universally open.

Proof Since finite presentation and flatness are preserved under basechange it suffices to show $|f|: |X| \to |Y|$ is open.

WLOG $Y = \operatorname{Spec} A$. Let $U \subseteq X$ we want to show $f(U) \subseteq |Y|$ is open.

WLOG $U = \operatorname{Spec} B$. By Chevalley $f(U) \subseteq |Y|$ is constructible, by flatness it is generalizing.

But a set $T \subseteq \operatorname{Spec} A$ is open **iff** it is generalizing and open for constructible topology.

<u>Corollary</u>: Finitely presented and flat closed immersions are also open immersions.

<u>Example</u> Let k be a field

let $S \subseteq \mathbb{R}$ with $S = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ with the subspace topology.

Let $R = C^0(S, k)$ be the ring of locally constant functions in S

the map
$$\begin{aligned} R &\to k \\ f &\mapsto f(0) \end{aligned}$$

exhibits
$$\operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$$
as a flat closed immersion which is not open.

Flatness and dimension of fibers:

Proposition Let $f: X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$ and $y = f(x) \in Y$. Let X_y be the fiber over $\operatorname{Spec} k(y)$. Then

$$\dim \mathcal{O}_{X_y,x} \geq \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

If f is flat at x then equality holds.

Proof WLOG $Y = \operatorname{Spec} R$ for R a reduced Noetherian local ring and $y \in Y$ is the closed point. We prove this by induction on $\dim(Y)$.

$\dim(Y) = 0$ then $X = X_y$.

If $\dim(Y) > 0$ let $e \in A$ that is not a unit and not a zero divisor.

Let $B = \mathcal{O}_{X,x}$. let g be the image of e in B.

By Krull's principal ideal theorem $\dim(A/e) = \dim(A) - 1$ and $\dim(B/g) \geq \dim(B) - 1$. If B is flat then g is also not a zero divisor so we get equality.

Let $Y' = \operatorname{Spec} A/e$ and $X' = X \times_Y Y'$

By induction hypothesis

$$\dim \mathcal{O}_{X'_y, x} \geq \dim \mathcal{O}_{X', x} - \dim \mathcal{O}_{Y', y}$$

with equality if f is flat at x. Now $X_y = X'_y$ since y is the closed point.

$$\dim \mathcal{O}_{X_y,x} \geq \dim \mathcal{O}_{X',x} - \dim \mathcal{O}_{Y',y}$$
$$\geq (\dim \mathcal{O}_{X,x} - 1) - (\dim \mathcal{O}_{Y,y} - 1)$$
$$= \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$
with equality if f is flat at x.

<u>Definition</u> A topological space is equidimensional if every irreducible component has the same dimension.

<u>Definition</u> A locally of finite type morphism $f: X \to Y$ is of <u>relative dimension d</u> if all fibers $\{X_y\}_{y \in Y}$ are equidimensional of dimension d.

<u>Corollary</u>: Let X and Y are finite type equidimensional k-schemes and $\pi: X \to Y$ is a flat morphism then π is of relative dimension
$$d = \dim(X) - \dim(Y).$$

Proof Let $x \in X_y$ be a closed point. Since X and Y are Noetherian and π is flat we have

$$\dim \mathcal{O}_{X_y,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

On the other hand

$$\dim \mathcal{O}_{X,x} = \dim(X) - \dim \overline{\{x\}}$$
$$= \dim(X) - \text{tr.deg.}\, k(x)$$

and

$$\dim \mathcal{O}_{Y,y} = \dim(Y) - \text{tr.deg.}\, k(y)$$

Since we took $x \in X_y$ to be a closed point $[k(x) : k(y)] < \infty$ and they have same transcendence degree.

On the other hand

$$\dim X_y = \max_{\substack{x \in |X_y| \\ \text{closed}}} \dim \mathcal{O}_{X_y,x} = \dim X - \dim Y$$

Generic flatness:

Theorem: Let $f: X \to Y$ be a morphism of finite type. Suppose that Y is integral and Noetherian. The following holds:

For all $\mathcal{F} \in Coh(X, \mathcal{O}_X)$ there exists an open dense subset $V \subseteq Y$ such that $\mathcal{F}|_{f^{-1}(V)}$ is flat over V.

Example: 1) We often apply it to $\mathcal{F} = \mathcal{O}_X$.

2) If $Y = \mathbb{A}^1_k = \operatorname{Spec} k[T]$ and $f: \operatorname{Spec} k \to \mathbb{A}^1_k$ is the inclusion of a point then f becomes flat on $Y \setminus f(*)$.

3) $\mathbb{A}^1_k \longrightarrow \alpha$ becomes an isomorphism away from the node.

Proof WLOG $Y = \operatorname{Spec} A$, then X is qc. Let $X = \bigcup_{i=1}^{n} \operatorname{Spec} B_i$ if $\mathcal{F}|_{\operatorname{Spec} B_i \cap f^{-1}(V_i)}$ is flat over V_i we can let $V = \bigcap_{i=1}^{n} V_i$ so that $\mathcal{F}|_{f^{-1}(V) \subseteq X}$ is flat over V. In other words, WLOG $X = \operatorname{Spec} B$.

(Grothendieck freeness): Let A be a Noetherian integral domain, B a finitely generated A-algebra and M a finitely generated B-module. Then there is $a \in A$ such that $M[\frac{1}{a}]$ is a free A-module.

We argue by induction:

Base case: Suppose that $B = A$.

$M|_{\operatorname{Frac} A}$ is free and since M is finitely presented (A being Noetherian) a basis spreads to an open neighborhood.

<u>Lemma</u>: If B is a finite type A-algebra such that every finite type B-module is generically free over A, then $B[T]$ also satisfies this.

<u>proof</u>: Suppose $\bigoplus_{i=1}^{n} B[T]\cdot e_i \twoheadrightarrow M$

let $M_0 = 0$, $M_1 = \bigoplus_{i=1}^{n} B\cdot e_i \subseteq M$

and $M_{n+1} = M_n + TM_n \subseteq M$ as B-submodules.

Multiplication by T induces a surjective map of B-modules

$$\psi_n : M_n/M_{n-1} \longrightarrow M_{n+1}/M_n$$

the sequence $\psi_1, \psi_2 \circ \psi_1, \ldots \psi_n \circ \psi_{n-1} \circ \ldots \psi_1, \ldots$

induce submodules

$$\ker(\psi_1) \subseteq \ker(\psi_2 \circ \psi_1) \subseteq \ldots \subseteq M_1$$

since M_1 is a f.s. B-module this sequence stabilizes i.e.

$$\psi_n \text{ is an isomorphism } n \gg 0.$$

Pick $a_0, a_1, \ldots, a_n \in A$ s.t.

$(M_0)[\frac{1}{a_0}]$, $M_i/M_{i-1}[\frac{1}{a_i}]$ are free $A[\frac{1}{a_i}]$-modules

Let $a = a_0 \cdot a_1 \cdot \ldots \cdot a_n$ then all of the $[M_n/M_{n-1}][\frac{1}{a}]$ are free $A[\frac{1}{a}]$-modules.

Lemma If $M = \bigcup_{i=0}^{\infty} M_i$ A-modules with $M_0 = 0$ and each M_{i+1}/M_i is free, then M is free.

Proof $M_1 = M_1/M_0$ is free

$$0 \to \underbrace{M_1}_{\text{free}} \to M_2 \to \underbrace{M_2/M_1}_{\text{free}} \to 0$$

then $M_2 = M_1 \oplus M_2/M_1$ is free.

Inductively: $M_n = M_1 \oplus M_2/M_1 \oplus \dots \oplus M_n/M_{n-1}$

So that $M = M_1 \oplus M_2/M_1 \oplus \dots \oplus M_n/M_{n-1} \oplus \dots$

End of proof:

Any finite type A-algebra has the form $B \cong A[x_1, \dots x_n]/I$. Moreover, any finite B-module can be regarded as a finite $A[x_1 \dots x_n]$-module.