Algebraic geometry 1
Exercise sheet 2

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Exercise 1. Let $I = (f_1, \dots, f_r) \subseteq k[x_1, \dots, x_n]$ be an ideal and $X = V(I) \subseteq \mathbb{A}^n(k)$ be its vanishing locus.

1. We have to show

$$\overline{X} = \bigcap \{V^+(h) \mid h \text{ homogenous}, h(X) = 0\} = V(\{\tilde{g} \mid g \in I\}).$$

   Pick any homogenous h that vanishes on X. By letting $x_{n+1} = 1$ we see that $h(x_1, \dots, x_n, 1) \in \sqrt{I}$, since it vanishes on X. Therefore we have $l \in \mathbb{N}$ such that $h(x_1, \dots, x_n, 1)^l = \sum_{i=1}^r \alpha_i f_i$ for some $\alpha_i \in k[x_1, \dots, x_n]$. There also exists $m \in \mathbb{N}$ such that $x_{n+1}^m h(x_1, \dots, x_n, 1) = h$. Adding these two together we get $x_{n+1}^{ml} \sum_{i=1}^r \alpha_i f_i = h^l$.