Smoothness vs regularity

Definition

A map of schemes $f: X \to S$ is geometrically regular if for all algebraically closed fields K and maps $\operatorname{Spec} K \to S$ the basechange

$$f_K : X \times_S \operatorname{Spec} K \to \operatorname{Spec} K$$ is a regular scheme.

Theorem Let k be a field and $f: X \to \operatorname{Spec} k$ a map of schemes: The following are equivalent:

1) f is smooth
2) f is geometrically regular.
3) There is K/k field extension with K algebraically closed s.t.

$$X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$$ is regular.

Some facts about regularity:

Let A be a Noetherian local ring.

1) A is regular iff $\hat{A} = \lim A/\mathfrak{m}^n$ is regular.

2) If $A \to B$ is local, flat and B is regular then A is regular.

3) If A is regular then $A_\mathfrak{p}$ is regular $\forall \mathfrak{p} \in \operatorname{Spec} A$.

Proof of Theorem 1) ⇒ 2)

Suppose $f: X \to \operatorname{Spec} k$ is smooth and K/k is a field extension with $\bar{K} = K$.

Locally in X_K we have a presentation

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr] & & \mathbb{A}^n_K \arrow[dl] \\ & \operatorname{Spec} K & \end{tikzcd}$$

with f étale. Fix $x \in X_K$, we want to show $\mathcal{O}_{X_K, x}$ is regular.

WLOG $x \in X(K)$. by Fact 3.

Then $\hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}}_{\mathbb{A}^n, g(x)} = K[[x_1 \dots x_n]]$ which is regular. Since $\mathcal{O}_{X,x} \to \hat{\mathcal{O}}_{X,x}$ is local flat Fact 2 implies $\mathcal{O}_{X,x}$ is regular.

2) => 3) | Easy.

3) => 1) | Fix a map of schemes $f: X \to \operatorname{Spec} k$ and a field extension K/k with $f_K: X_K \to \operatorname{Spec} K$ smooth.

WLOG $X = \operatorname{Spec} A$ and let $A_K = A \otimes_k K$.

Then $\Omega^1_{A_K/K} = K \otimes_k \Omega^1_{A/k}$,

since X_K is smooth over $\operatorname{Spec} K$ it is flat and finitely presented.

Since $\Omega^1_{A/k}$ is an A-module and $A \to A_K$ is faithfully flat $\Omega^1_{A/k}$ is flat finitely presented over A. i.e. locally free of finite rank.

Analogously, A is a finite type k-algebra since A_K is.

Let $\operatorname{Spec} A \overset{g}{\hookrightarrow} \mathbb{A}^n_k$ be a closed immersion with $A = k[x_1 \dots x_n]/I$.

Then f is smooth iff

$$0 \to I/I^2 \xrightarrow{d} \bigoplus_{i=1}^{n} A \cdot dx_i \to \Omega^1_{A/k} \to 0$$

is exact.

This is equivalent to

$$0 \to I_K/I_K^2 \xrightarrow{d_K} \bigoplus_{i=1}^{n} A_K dx_i \to \Omega^1_{A_K/K} \to 0$$

being injective.

Theorem Let k be a field, $f: X \to \operatorname{Spec} k$ locally of finite type and $x \in |X|$ a closed point. Suppose that $k(x)/k$ is a separable field extension and that $\mathcal{O}_{X,x}$ is a regular local ring. Then f is smooth at x.

Proof Since $\operatorname{spec} k(x) \to \operatorname{spec} k$ is étale we get an exact sequence

$$0 \to m_{X,x}/m_{X,x}^2 \to k(x) \otimes_{\mathcal{O}_{X,x}} \left(\Omega^1_{X/k}\right)_x \to \Omega^1_{k(x)/k}$$

coming from the triangle

$$\begin{tikzcd} \operatorname{spec} k(x) \arrow[rr, hook] \arrow[dr] & & \operatorname{spec} \mathcal{O}_{X,x} \arrow[dl] \\ & \operatorname{spec} k & \end{tikzcd}$$

Moreover, $\Omega^1_{k(x)/k} = 0 \Rightarrow m_{X,x}/m_{X,x}^2 \xrightarrow{\sim} k(x) \otimes_{\mathcal{O}_X} \Omega^1_{X/k}$

Locally $x \in U \subseteq X$ we can find a closed immersion

$$\operatorname{Spec} B = U \overset{g}{\hookrightarrow} \mathbb{A}^n_k \quad \text{with}$$

$I = \operatorname{Ker}(k[x_1 \ldots x_n] \twoheadrightarrow B)$. and we get

$$I/I^2 \xrightarrow{d} g^* \Omega^1_{\mathbb{A}^n/k} \to \Omega^1_{U/k} \to 0$$

By hypothesis,

$$d = \dim_{k(x)} m_{X,x}/m_{X,x}^2 = \dim \mathcal{O}_{X,x}$$

Choose $f_1, \dots, f_{n-d} \in I$ with

$df_1, \dots, df_n$ linearly independent in

$k(x) \otimes_{k[x_1 \dots x_n]} \Omega^1_{\mathbb{A}^1/k}$

We define $U_0 \subseteq \mathbb{A}^n_k$ as $U_0 = V(f_1 \dots f_{n-d})$

By construction $U_0 \subseteq U$.

By the Jacobian criterion

U_0 is smooth at x so

$\mathcal{O}_{U_0,x}$ is regular of dimension d.

$\Rightarrow$ $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{U_0,x}$ is

a surjection of domains

with $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{U_0,x}$

$\therefore$ $\mathcal{O}_{X,x} = \mathcal{O}_{U_0,x}$ so $U = U_0$ in

a neighborhood.

Theorem Let $(R, m_R) \to (S, m_S)$ be a local map of Noetherian local rings. Let M be a finite type M module.

TFAE:

1) M is a flat R-module
2) $m \otimes_R M \to M$ is injective.
3) $\mathrm{Tor}_1^R(k, M) = 0$.

Remark: In practice $M = S$.

Theorem Let $f: X \to S$ be a map locally of finite presentation

TFAE:

1) f is smooth
2) f is flat and has smooth fibres.
3) f is flat and has smooth geometric fibres.

Proof

3) $\Leftrightarrow$ 2) By flat-descent of smoothness.

1) $\Rightarrow$ 2) clearly f has smooth fibres.

Let $x \in X$ with image $s = f(x) \in S$, we want to prove $\mathcal{O}_{X,x}$ is a $\mathcal{O}_{S,s}$-flat algebra. WLOG $S = \operatorname{Spec} R$ for R a local ring and $X = \operatorname{Spec} A$ a finitely presented R-algebra.

We let $m \subseteq R$ be the maximal ideal and $k = R/m$.

By the Jacobian criterion (possibly shrinking X) $A = R[x_1, \dots x_n]/(f_1, \dots f_m)$ and the Jacobian matrix has rank m at $x \in \operatorname{Spec} A$.

WLOG R_0 is Noetherian:

Since A is finitely presented then

$$A = A_0 \otimes_{R_0} R \quad \text{with}$$

$$A_0 = R_0[x_1, \dots, x_n] / (f_1, \dots, f_m)$$

and if x maps to x_0 under $\operatorname{Spec} A \to \operatorname{Spec} A_0$ then the Jacobian matrix at $k(x_0)$ also has rank m.

i.e. On a neighborhood of x_0

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "f"] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} A_0 \arrow[r, "f_0"'] & \operatorname{Spec} R_0 \end{tikzcd}$$

is Cartesian and f_0 is smooth.

If f_0 is flat then f is also flat.

Let $y \in \mathbb{A}^n_R$ be the image of $x \in X$ under the map

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{A}^n_S \arrow[d] \\ & \operatorname{Spec} R \end{tikzcd}$$

with $X = V(f_1, \dots, f_m)$.

Let $V_r = V(f_1, \dots, f_r)$ $0 \le r \le m$.

We prove by induction that $\mathcal{O}_{V_r, y}$ is a flat R-module.

the base case is $\mathcal{O}_{\mathbb{A}^n_R, y} = \mathcal{O}_{V_0, y}$.

we also let $V_r^s = V_r \times_S \operatorname{Spec} k(s)$.
which are smooth $k(s)$-varieties.
By local criterion of flatness

it suffices to show that

$$\operatorname{Tor}_1^R(\mathcal{O}_{V_{r+1}}, k) = 0$$

We have an exact sequence

$$0 \to \underset{\substack{\parallel \\ K}}{\mathcal{O}_{V_r,Y}\cdot f_r} \longrightarrow \mathcal{O}_{V_r,Y} \longrightarrow \mathcal{O}_{V_{r+1},Y} \to 0$$

$$0 \to \mathrm{Tor}_1^R(\mathcal{O}_{V_{r+1}},k) \to K\otimes_R k \longrightarrow \mathcal{O}_{V_r,Y}\otimes_R k \longrightarrow \mathcal{O}_{V_{r+1},Y}\otimes_R k \to 0$$

since $\mathcal{O}_{V_r}$ is R-flat by induction.

It suffices to show that

$$K\otimes_R k \xrightarrow{\alpha} \mathcal{O}_{V_r,Y}\otimes_R k \quad \text{is injective.}$$

$$\begin{tikzcd}
 & K/mK \arrow[dr, "\alpha"] & \\
\mathcal{O}_{V_r,Y}/m\mathcal{O}_{V_r,Y} \arrow[ur, "1 \mapsto \bar{f}_r"] \arrow[rr, "1 \longmapsto \bar{f}_r"'] \arrow[d, equal] & & \mathcal{O}_{V_r,Y}/m\mathcal{O}_{V_r,Y} \arrow[d, equal] \\
\mathcal{O}_{V_r^s,x} & & \mathcal{O}_{V_r^s,x}
\end{tikzcd}$$

but $\mathcal{O}_{V_r^s,x}$ is an integral domain.

$\boxed{2) \Rightarrow 1)}$

$S = \operatorname{Spec} R \qquad X = \operatorname{Spec} A, \qquad A = B/I$

$B = R[x_1 \dots x_n].$

we consider the sequence

$$I/I^2 \to A \otimes \Omega^1_{B/R} \to \Omega^1_{A/R} \to 0 \qquad (*)$$

$\operatorname{Spec} A$ is smooth $/\operatorname{Spec} R$ iff $(*)$ is exact and locally split.

Equivalently, for all $x \in X$

$$0 \to I/I^2 \otimes_A k(x) \to k(x) \otimes_B \Omega^1_{B/R} \to k(x) \otimes_A \Omega^1_{A/R} \to 0$$

is exact.

Fix $s \in S$ $\quad X_s = \operatorname{Spec} A \otimes_R k(s)$ $\quad B_s = k(s)[x_1 \ldots x_n]$

we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & J/J^2 \arrow[r] & A_s \otimes \Omega^1_{B_s/k(s)} \arrow[r] & \Omega^1_{A_s/k(s)} \arrow[r] & 0 \quad (*_s) \\
& I/I^2 \otimes_R k(s) \arrow[u] \arrow[r] & (A \otimes \Omega^1_{B/R}) \otimes_R k(s) \arrow[u, "S"] \arrow[r] & \Omega^1_{A/R} \otimes_R k(s) \arrow[u, "S"] \arrow[r] & 0
\end{tikzcd}$$

Now, $0 \to I \to B \to A \to 0$ and A flat $\Rightarrow$

$$0 \to I \otimes_R k(s) \to B \otimes_R k(s) \to A \otimes_R k(s) \to 0$$

So $I/I^2 \otimes k(s) \cong J/J^2$

and $(*)$ is exact locally after any basechange $s \in S$.

This shows $(*)$ is exact after basechange to A_s, but over A_s it is exact locally split so $(*_s) \otimes_{A_s} k(x)$ is also exact.