Algebraic geometry 1
Exercise sheet 3

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Exercise 1.

1. Define

$$\pi^{-1} : U \longrightarrow \pi^{-1}(U)$$
$$(x_1, ..., x_n) \mapsto (x_1, ..., x_n)[x_1 : ... : x_n].$$

This is well-defined, because by definition of U, not all x_i can be zero at the same time, so $[x_1 : ... : x_n]$ is actually a point in projective space. We also have $(x_1, ..., x_n)[x_1 : ... : x_n] \in Z$ for $(x_1, ..., x_n) \in U$, because $x_i x_j = x_j x_i$ for all $1 \leq i, j \leq n$. To see injectivity of π^{-1}, let $(x_1, ..., x_n) \in U$ with $x_j \neq 0$. Then we have $y_j \neq 0$, because if we assume $x_j \neq 0$ and $y_j = 0$, then for some $y_i \neq 0$ (which exists since $[y_1 : ... : y_n]$ is a point in projective space) we have $0 \neq x_j y_i = x_i y_j = 0$. Therefore, we can just set $y_j = 1$. Then

$$x_i y_j = x_j y_i \implies y_1 = \frac{x_1 y_j}{x_j} = \frac{x_1}{x_j},$$

showing that all the y_i are fixed up to a scalar after fixing all the x_i.

2. Define

$$\phi : V_i \to \mathbb{A}_n^k$$
$$(x, y) \mapsto (\frac{x_1}{y_i}, \dots, x_i, \dots, \frac{x_n}{y_i}),$$

where the inverse map is given by

$$\phi^{-1} : \mathbb{A}_n^k \to V_i$$
$$(x_1, \dots, x_n) \mapsto (x_1 x_i, \dots, x_i, \dots, x_n x_i)[x_1 : \cdots : x_{i-1} : 1 : \cdots : x_n].$$

Exercise 2. The part which to me seemed the hardest was to calculate the closure of $\pi^{-1}(Y \setminus \{t\})$ in Z. So to for that we look at $\pi^{-1}(Y)$ and decompose it into irreducible components. Or actually we first cover it V_i and then look at them inside each V_i. In these two cases it decomposed into nice irreducible components, one of which is the blow-up and the other whole $\pi^{-1}(t)$.

1. Let $Y = V(x_1^2 - x_2^3) \subseteq \mathbb{A}^2(k)$. We look at

$$\pi^{-1}(Y) = \{((x_1, x_2), [y_1 : y_2]) \mid x_1 y_2 = x_2 y_1, x_1^2 - x_2^3 = 0\}. \tag{1}$$

   We can cover it with V_i $(i = 1, 2)$. Lets first look inside $V_1 \cong \mathbb{A}^2(k) \times \mathbb{A}^1(k)$, where $y_1 = 1$. Equations then become $x_1 y_2 = x_2$ and $x_1^2(1 - x_1 y_2^3) = 0$. The latter equation can be decomposed, so we get two closed subsets:

   - $\{x_1 = x_2 = 0\} \subseteq V_1 \subseteq \mathbb{A}^2(k) \times \mathbb{P}^1(k)$
   - $\{x_1 y_2^3 - 1 = 0, x_2 y_2^2 - 1 = 0\} \subseteq V_1 \subseteq \mathbb{A}^2(k) \times \mathbb{P}^1(k)$

   First one lies in $\pi^{-1}(t)$, so $\{x_1 = x_2 = 0\} \cap \pi^{-1}(Y \setminus \{t\}) = \emptyset$. Therefore

$$\pi^{-1}(Y \setminus \{t\}) \cap V_1 = \{x_1 y_2^3 - 1 = 0, x_2 y_2^2 - 1 = 0, x_1 \neq 0, x_2 \neq 0\} \cap V_1 \tag{2}$$

   Therefore taking the closure inside V_1:

$$\mathrm{BL}_t(Y) \cap V_1 = \{x_1 y_2^3 - 1 = 0, x_2 y_2^2 - 1 = 0\} \cap V_1. \tag{3}$$

   Before definition a morphism $\mathrm{BL}_t(Y) \to \mathbb{A}^1(k)$, lets look at $\mathrm{BL}_t(Y) \cap V_2$. Similar as before we set $y_2 = 1$ and get that $\pi^{-1}(Y) \cap V_2$ is made up of two closed subsets

   - $\{x_1 = x_2 = 0\} \subseteq V_2 \subseteq \mathbb{A}^2(k) \times \mathbb{P}^1(k)$
   - $\{x_1 = y_1^3, x_2 = y_1^2\} \subseteq V_2 \subseteq \mathbb{A}^2(k) \times \mathbb{P}^1(k)$

   This intersection contains more information, since it also contains $\mathrm{BL}_t(Y) \cap \pi^{-1}(t)$.

   Define a morphism $\phi\colon \mathrm{BL}_t(Y) \to \mathbb{A}^2(k)$

2.

Exercise 4.

1. Lets first prove that V_U are stable under intersections:

   Claim. Take $U, W \subseteq X$ open subsets. Then $V_{U \cap W} = V_U \cap V_W$.

   Proof of claim. Inclusion $V_{U \cap W} \subseteq V_U \cap V_W$ is clear.

   For the other inclusion take $Z \in V_U \cap V_W$. By definition $Z \cap U \neq \emptyset$ and $Z \cap V \neq \emptyset$. Suppose $Z \cap (U \cap V) = \emptyset$. Then $(Z \cap U)^c \cup (Z \cap V)^c = X$. But since Z is irreducible, and is covered by $U^c \cup V^c$, we must have (WLOG) $Z \subseteq U^c$. That is in contradiction with $Z \cap U \neq \emptyset$. $\square$(of claim)

It also behaves well under unions:

$$\begin{aligned}
V_{U\cup W} &= \{Z \text{ cl. irred.} \mid Z \cap (U \cup W) \neq \emptyset\} \\
&= \{Z \text{ cl. irred.} \mid (Z \cap U) \neq \emptyset \text{ or } (Z \cap W) \neq \emptyset\} \\
&= \{Z \text{ cl. irred.} \mid (Z \cap U) \neq \emptyset\} \cup \{Z \text{ cl. irred.} \mid (Z \cap W) \neq \emptyset\} \\
&= V_U \cup V_W
\end{aligned}$$

and practically same argument applies to infinite unions.

Therefore every open subset of X^{sob} can be written as V_U for some open $U \subseteq X$ (in general it could've been just a base of topology, but this shows it is the whole topology).

Claim. If $V_{U_1} = V_{U_2}$ then $U_1 = U_2$.

Proof of claim. Suppose $x \in U_1 \setminus U_2$, then $\overline{\{x\}} \in V_{U_1} \setminus V_{U_2}$. This proves the claim. $\square$(of claim) Next claim is a direct consequence of one above.

Claim. Closed irreducible subsets of X^{sob} are exactly V_U^c for open $U \subseteq X$ such that (closed) subset $U^c \subseteq X$ is irreducible.

Proof of claim. Let V_U^c be irreducible and $U^c = U_1^c \cup U_2^c \subseteq X$. Then $V_U = V_{U_1 \cap U_2} = V_{U_1} \cap V_{U_2}$ and thus $V_U^c = V_{U_1}^c \cup V_{U_2}$. Since V_U^c is irreducible, we must have $V_U^c = V_{U_1}^c$ and thus $U = U_1$, which proves irreducibility of U^c.

For the other implication, let U^c be irreducible and $V_U^c = V_{U_1}^c \cup V_{U_2}^c$. Then $U_1 \cap U_2 = U$. Since U^c is irreducible, we must have (WLOG) $U_1 = U$ and therefore $V_U = V_{U_1}$. $\square$(of claim)

Let us show X^{sob} is sober. Let V_U^c be closed irreducible. Then by last claim U^c is closed and irreducible. The set U^c is the generic point with $\overline{\{U^c\}} = V_U^c$. The inclusion $\overline{\{U^c\}} \subseteq V_U^c$ is obvious, because V_U^c contains the point U^c and is a closed set. For the other inclusion take a closed set that V_W^c that contains U^c. That means $U^c \cap W = \emptyset$ and thus $W \subseteq U$. Then we have $V_W \subseteq V_U$ and $V_U^c \subseteq V_W^c$. This proves that V_U^c is the closure of the point U^c.

2. Define

$$\begin{aligned}
h \colon X^{\text{sob}} &\to Z \\
W &\mapsto \text{unique generic point of } \overline{g(W)}.
\end{aligned}$$

Note that: a continuous image of an irreducible set is irreducible and the closure of an irreducible set is irreducible. So $\overline{g(W)} \subseteq Z$ is a closed irreducible subset and thus has a unique generic point in Z.

Let's now prove $g = h \circ f$. Take $x \in X$. We have to prove $g(x)$ is the unique generic point of $\overline{g\left(\overline{\{x\}}\right)}$. Clearly $g(x) \in \overline{g\left(\overline{\{x\}}\right)}$. Take any closed $W \subseteq Z$ with $g(x) \in W$. Then, by definition, $x \in g^{-1}(W)$. Because $g^{-1}(W)$

is closed, also $\overline{\{x\}} \subseteq g^{-1}(W)$. So $g\left(\overline{\{x\}}\right) \subseteq W$. But since W is closed we have $\overline{g\left(\overline{\{x\}}\right)} \subseteq W$. This proves that $g(x)$ is indeed a generic point of $\overline{g\left(\overline{\{x\}}\right)}$. So we have $g = h \circ f$.

To prove h is continuous we take an open set $U \subseteq Z$. we want to see that $h^{-1}(U)$ is open. Since $g^{-1}(U) = f^{-1}(h^{-1}(U))$ is open and f^{-1} induces a bijection of open sets, the set $h^{-1}(U)$ is open as well. So h is continuous.

We should also argue why h is unique. Take $h, h' \colon X^{\text{sob}} \to Z$ both continuous and satisfying $g = h \circ f = h' \circ f$. Pick any closed irreducible $W \subseteq X$. Suppose $h(W) \neq h'(W)$. WLOG there exists open $U \subseteq Z$ such that $h(W) \in U$ and $h'(W) \notin U$ (because requiring *unique* generic point implies T_0 property). Open sets $h^{-1}(U)$ and $h'^{-1}(U)$ therefore differ. Using one of the claims above, they are of the form $V_{U_1} = h^{-1}(U)$ and $V_{U_2} = h'^{-1}(U)$. So we have $W \in V_{U_1}$ and $W \notin V_{U_2}$. Then there exists $w \in W \cap U_1$, for which $\overline{\{w\}} \in V_{U_1}$ and $\overline{\{w\}} \notin V_{U_2}$. By definition $\overline{\{w\}} \in h^{-1}(U)$ and $\overline{\{w\}} \notin h'^{-1}(U)$ which means that $h(\overline{\{w\}}) \in U$ and $h'(\overline{\{w\}}) \notin U$. But that is a contradiction with assumption $g = h \circ f = h' \circ f$.

3. We can define $h \colon V \to \operatorname{MaxSpec}(A)$ with $h(x_1, \dots, x_n) = (X_1 - x_1, \dots, X_n - x_n)$. Due to Hilberts Nullstellensatz this is a bijection (because k alg. closed). Take a closed subset $C \subseteq V$. Again by Hilberts Nullstellensatz we get a radical ideal I such that $V = V(I)$. So $h(C) = \{m \in \operatorname{MaxSpec} \mid I \subseteq m\}$, which is a closed set. And if we take a closed set $V(I) \subseteq \operatorname{MaxSpec}(A)$ for some $I \subseteq A$ we have $h^{-1}(V(I)) = \{x \in V \mid \forall f \in I \colon f(x) = 0\}$. So h is a homeomorphism.

   Observe that if we take $X = \operatorname{MaxSpec}(A)$, then $X^{\text{sob}} = \operatorname{Spec}(A)$. As sets that is clear, because all irreducible closed sets of V are exactly vanishing sets of prime ideals in A. And topology on $V \cong \operatorname{MaxSpec}(A)$ is defined by the basis of closed sets being the vanishing sets of prime ideals, so $X^{\text{sob}} = \operatorname{Spec}(A)$.

   Then we have a diagram

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "i"] & \operatorname{MaxSpec}(A) \arrow[d, "j"] \\ V^{\text{sob}} \arrow[r] & \operatorname{Spec}(A) \end{tikzcd}$$

   where $i \colon x \mapsto \overline{\{x\}}$ and $j \colon m \mapsto \{m\}$ (since maximal ideals are already closed points).

   By 2. part we know there exists $g \colon V^{\text{sob}} \to \operatorname{Spec}(A)$ with $j \circ h = g \circ i$. And similar there exists $f \colon \operatorname{Spec}(A) \to V^{\text{sob}}$ with $i \circ h^{-1} = f \circ j$. Combining these two equations gives us $j = g \circ f \circ j$ and $i = f \circ g \circ i$.

   Observe that the map $g \circ f$ satisfies the "universal condition" from 2. part for the map $i \colon V \to V^{\text{sob}}$. So it is the unique map $\pi \colon V \to V^{\text{sob}}$ that makes

$$\begin{tikzcd}
V \arrow[r, "i"] \arrow[d, "i"] & V^{\text{sob}} \\
V^{\text{sob}} \arrow[ur] &
\end{tikzcd}$$

And identity would also satisfy that condition, so $g \circ f$ must be the identity on V^{sob}.

We argue exactly the same that $f \circ g$ is the identity on $\operatorname{Spec}(A)$.

This proves that $V^{\text{sob}} \cong \operatorname{Spec}(A)$.