Definition Let $f: X \to Y$ be a morphism of schemes. Let $x \in X$ we say that f is smooth at x if there is an open $x \in U \subseteq X$ s.t. $f|_U : U \to Y$ is smooth.

Theorem (Jacobian Criterion)

Let
$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{A}^n_R \arrow[d] \\ & \operatorname{Spec} R \end{tikzcd}$$
be a diagram of schemes, where i is a closed immersion of finite presentation. Let $z \in Z$, then f is smooth at z if and only if there is $U \subseteq \mathbb{A}^n_R$ open and $f_1, \dots, f_r \in \Gamma(U, \mathcal{O}_U)$ s.t. $Z \cap U = V(f_1, \dots, f_r)$ and the rank of the matrix $J(z)_{ij} = \left(\frac{\partial f_i}{\partial x_j}(z)\right)$ $i \in \{1, \dots, r\}$, $j \in \{1, \dots, n\}$ is r.

$\Rightarrow$)

<u>Proof</u> Let $z \in V \subseteq \mathbb{A}^n_R$ s.t. $f|_V : V \cap Z \to \operatorname{Spec} R$ is smooth. Let $V = \operatorname{Spec} B$ and $Z \cap V = \operatorname{Spec} B/I$. and $A = B/I$.

By shrinking V we may assume

$$0 \to I/I^2 \to A \otimes_B \Omega^1_{B/R} \to \Omega^1_{A/R} \to 0$$

is an exact sequence of free A-modules.

Let $f_1, f_2, \dots, f_r \in I$ be elements projecting to a basis of I/I^2.

Claim: there is $U \subseteq V$ open s.t. $Z \cap U = V(f_1, \dots, f_r)$.

Assuming this claim, the diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{r} e_i \arrow[r, "J"] \arrow[d, twoheadrightarrow, "{[e_i \mapsto f_i]}"'] & \bigoplus_{i=1}^{n} A \cdot dx_i \arrow[d, "\simeq"] \\
0 \to I/I^2 \arrow[r] & A \otimes_B \Omega^1_{B/R}
\end{tikzcd}$$

shows that $\operatorname{rank}(J(z)) = r$.

Proof of claim) Let $J \subseteq I$ be the ideal generated by $f_1, \dots, f_r$, and $M = I/J$

$M \otimes_B k(z) = (M \otimes_B A) \otimes_A k(z) = 0$ By Nakayama, $M \otimes_B \mathcal{O}_z = 0$

and $M|_U = 0$ for some U.

Clearly $Z \cap U = V(J) = V(f_1, \dots, f_r)$.

$\Leftarrow$) Suppose $Z \cap U = V(f_1, \dots, f_r)$

$U = \operatorname{Spec} B$ $\quad A = B/\langle f_1, \dots f_r \rangle = I$

we get a diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{r} A \cdot e_i \arrow[r, "J"] \arrow[d, "{[e_i \mapsto f_i]}"'] & \bigoplus_{i=1}^{n} A \cdot dx_i \arrow[d, "\simeq"] & \\
I/I^2 \arrow[r, "d"] & A \otimes_B \Omega^1_{B/R} \arrow[r] & \Omega^1_{A/R} \arrow[r] & 0
\end{tikzcd}$$

By hypothesis, $J(z)$ is injective.

This implies $\bigoplus_{i=1}^{r} k(z) \cdot e_i \xrightarrow{\sim} I/I^2 \otimes k(z)$

is an isomorphism. This shows

$$0 \to I/I^2 \otimes k(z) \to k(z) \otimes_B \Omega^1_{B/R} \to \Omega^1_{A/R} \otimes k(z) \to 0$$

is exact, and since $\Omega^1_{B/R}$ is finite projective, by homework

$$0 \to I/I^2 \to \Omega^1_{B/R} \to \Omega^1_{A/R} \to 0$$ is

split injective on $z \in U \underset{\text{open}}{\subseteq} \operatorname{Spec} A$.

for some U. Since $\operatorname{Spec} B$ is smooth over $\operatorname{Spec} R$, $U \cap z$ is smooth over $\operatorname{Spec} R$.

Theorem (Uniformizing Parameters):

Let $g: X \to S$ be a map of schemes, then g is smooth at $x \in X$ if and only if there is $x \in U \subseteq X$ and $f_1, \dots, f_n \in \Gamma(U, \mathcal{O}_U)$ such that

$$\begin{tikzcd} U \arrow[r, "f=(f_1,\dots,f_n)"] \arrow[dr, "g|_U"'] & \mathbb{A}^n_S \arrow[d] \\ & S \end{tikzcd}$$

commutes, the map f is étale and $\Omega^1_{X/S}|_U \xleftarrow{\sim} \bigoplus_{i=1}^{n} \mathcal{O}_U \cdot df_i$

Proof $\Leftarrow]$ Is easy since étale maps are smooth and $\mathbb{A}^n_S \to S$ is smooth.

$\Rightarrow]$ WLOG $U = \operatorname{Spec} B$ $S = \operatorname{Spec} R$ and $\Omega^1_{U/S}$ is finite free.

$\Omega^1_{U/S}$ is generated by df_i $f_i \in B$ and we pick a basis $\bigoplus_{i=1}^{n} \mathcal{O}_U \cdot df_i \xrightarrow{\sim} \Omega^1_{B/R}$ after shrinking U around x

We get a map

$$\begin{tikzcd} U \arrow[r, "f=(f_1 \ldots f_n)"] \arrow[dr] & \operatorname{Spec} A \arrow[d] \\ & \operatorname{Spec} R \end{tikzcd}$$

$A = R[x_1, \ldots x_n]$

$x_i \mapsto f_i \in B$.

We get an exact sequence

$$0 \to \Omega^1_{A/R} \otimes B \xrightarrow{\sim} \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0 \qquad (*)$$

with $dx_i \mapsto df_i$ which is an isomorphism by construction.

$$\boxed{\begin{array}{l} \underline{\text{Recall}} \quad \begin{tikzcd} X \arrow[r, "p"] \arrow[dr] & Y \arrow[d, "t"] \\ & S \end{tikzcd} \\ \text{when top is smooth and} \\ (*) \text{ is exact then } p \text{ is smooth} \end{array}}$$

So f is formally smooth. Since $\Omega^1_{B/A} = 0$ f is formally unramified i.e. formally étale. But f is finitely presented.

Étale maps and "local homeomorphisms":

Proposition Let
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$
be maps of schemes. If f is formally étale, then $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism.

Proof
$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \underset{\overset{\parallel}{0}}{\Omega^1_{X/Y}} \to 0$$
and since f is formally smooth the above sequence is split exact. (i.e. an isomorphism).

Proposition Suppose $f: X \to S$ is formally étale, $x \in X$, $s \in S$, $s = f(x)$ and $k(s) \xrightarrow{\sim} k(x)$.

Then
$$\hat{\mathcal{O}}_{S,s} = \varprojlim \mathcal{O}_{S,s}/m^n_{S,s} \to \hat{\mathcal{O}}_{X,x} = \varprojlim \mathcal{O}_{X,x}/m^n_x$$
is an isomorphism.

Proof

Claim: For all $n \geq 0$ the map

$$\mathcal{O}_{S,s}/\mathfrak{m}_s^n \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^n$$ is an

isomorphism. (Here $\mathfrak{m}_s \subseteq \mathcal{O}_{S,s}$ is the maximal ideal).

Base-case is $k(s) \simeq k(x)$.

Let $\mathcal{C}_{\leq n}$ be the category of local rings $(A, \mathfrak{m}_A, i)$ with $\mathfrak{m}_A^n = 0$ and

$i: k(s) \xrightarrow{\sim} A/\mathfrak{m}_A$.

Let
$$F_n : \mathcal{C}_{\leq n} \longrightarrow \text{Sets}$$
$$G_n$$

$F_n = \mathrm{Hom}\left(\left(\mathcal{O}_{X,x}/\mathfrak{m}_x^n, i_x\right), -\right)$ $G_n = \mathrm{Hom}\left(\left(\mathcal{O}_{S,s}/\mathfrak{m}_s^n, i_s\right), -\right)$

The map $\mathcal{O}_{S,s}/\mathfrak{m}_s^n \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ induces a morphism $F_n \xrightarrow{\Phi_n} G_n$.

Claim Φ_n is isomorphism and by Yoneda

$$\mathcal{O}_{X,x}/\mathfrak{m}_x^n \xleftarrow{\sim} \mathcal{O}_{S,s}/\mathfrak{m}_s^n$$

Indeed, let $A \in \mathcal{C}_{\leq n}$ and $A' = A/m_A^{n-1} \in \mathcal{C}_{\leq n-1}$.

Then $F_n(A') = F_{n-1}(A') = G_{n-1}(A') = G_n(A')$

Moreover, $F_n(A) = F_n(A') \times_{G_n(A')} G_n(A)$

in other words

$$\begin{tikzcd}
\operatorname{Spec} A' \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O}_{X,x}/m_x^n \arrow[r, hook] \arrow[d] & \operatorname{Spec} \mathcal{O}_X \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists!"] & \operatorname{Spec} \mathcal{O}_{S,s}/m_s^n \arrow[r] & \operatorname{Spec} \mathcal{O}_S
\end{tikzcd}$$

Smoothness vs regularity

<u>Definition</u> 1) Let (A, m) be a Noetherian local ring, let $k = A/m$.
Then A is <u>regular</u> whenever

$$\dim_k(m/m^2) = \dim A.$$

2) A scheme X is called regular if it is locally Noetherian and for all $x \in X$ $\mathcal{O}_{X,x}$ is a regular local ring.

3) A map of schemes $f: X \to S$ is geometrically regular if for all algebraically closed fields K and maps $\operatorname{Spec} K \to S$ the basechange

$$f_K : X \times_S \operatorname{Spec} K \to \operatorname{Spec} K \text{ is a}$$

regular scheme.

<u>Theorem</u> Let k be a field and $f: X \to \operatorname{Spec} k$ a map of schemes:

The following are equivalent:

1) f is smooth

2) f is geometrically regular.

3) There is K/k field extension with K algebraically closed s.t.

$$X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K \text{ is regular.}$$

<u>Example:</u> If $k = \mathbb{F}_p(t) = \operatorname{Frac}(\mathbb{F}_p[t])$.

and $X = \operatorname{Spec} \mathbb{F}_p(t^{1/p}) = \operatorname{Spec} k[T]/_{T^p - t}$

then X is regular but

$X_{\overline{\mathbb{F}_p}} = \operatorname{Spec} \overline{\mathbb{F}_p}[T]/_{T^p}$ is not regular.

<u>Lemma</u>: Let $A \to B$ be a faithfully flat map of rings and M an A-module, then:

1) M is of finite type (resp. finite presentation) <u>iff</u> $M \otimes_A B$ is of finite type (resp. finite presentation).

2) M is flat over A if and only if $M \otimes_A B$ is flat over B.

<u>Proof</u> For both $\Rightarrow]$ direction is easy.

1) $\underline{\Leftarrow]}$ write $M = \operatorname{colim}_{i \in I} N_i$ as N_i ranges over f.g. submodules.

Then $M \otimes_A B = \operatorname{colim}(N_i \otimes_A B)$ and there is i s.t. $M \otimes B = N_i \otimes B$.

Now, $M/N_i \otimes_A B = 0 \Rightarrow M = N_i$.

Applying this to $K = \ker(\bigoplus_{i=1}^n A \to M)$ we also get f.p. case.

2) $\Leftarrow$) Let $N_1 \to N_2$ be injective then

$$M \underset{A}{\otimes} N_1 \to M \underset{A}{\otimes} N_2 \quad \text{is injective}$$

iff $$B \underset{A}{\otimes} M \underset{A}{\otimes} N_1 \to B \underset{A}{\otimes} M \underset{A}{\otimes} N_2 \quad \text{is injective}$$

$$\begin{array}{ccc} \| & & \| \end{array}$$

$$(B \underset{A}{\otimes} M) \underset{B}{\otimes} (B \underset{A}{\otimes} N_1) \to (B \underset{A}{\otimes} M) \underset{B}{\otimes} (B \underset{A}{\otimes} N_2).$$

which is injective since $(B \otimes_A M)$ is a flat B-module.

Some facts about regularity:

Let A be a Noetherian local ring.

1) A is regular iff $\hat{A} = \lim A/\mathfrak{m}^n$ is regular.

2) If $A \to B$ is local, flat and B is regular then A is regular.

3) If A is regular then $A_\mathfrak{p}$ is regular $\forall \mathfrak{p} \in \operatorname{Spec} A$.

Proof of Theorem 1) => 2)

Suppose $f: X \to \operatorname{Spec} k$ is smooth and K/k is a field extension with $\bar{K} = K$.

Locally in X_K we have a presentation

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr] & \mathbb{A}^n_K \arrow[d] \\ & \operatorname{Spec} K \end{tikzcd}$$

with g étale. Fix $x \in X_K$, we want to show $\mathcal{O}_{X_K, x}$ is regular.

WLOG $x \in X(K)$ by Fact 3.

Then $\hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}}_{\mathbb{A}^n, g(x)} = K[[x_1 \dots x_n]]$ which is regular. Since $\mathcal{O}_{X,x} \to \hat{\mathcal{O}}_{X,x}$ is local flat Fact 2 implies $\mathcal{O}_{X,x}$ is regular.

2) => 3) | Easy.

3) => 1) | Fix a map of schemes $f: X \longrightarrow \operatorname{Spec} k$ and a field extension K/k with $f_K : X_K \to \operatorname{Spec} K$ smooth.

WLOG $X = \operatorname{Spec} A$ and let $A_K = A \otimes_k K$.

Then $\Omega^1_{A_K/K} = K \otimes_k \Omega^1_{A/k}$,

since X_K is smooth over $\operatorname{Spec} K$ it is flat and finitely presented.

Since $\Omega^1_{A/k}$ is an A-module and $A \longrightarrow A_K$ is faithfully flat $\Omega^1_{A/k}$ is flat finitely presented over A. i.e. locally free of finite rank.

Analogously, A is a finite type k-algebra since A_K is.

Let $\operatorname{Spec} A \xhookrightarrow{g} \mathbb{A}^n_k$ be a closed immersion with $A = k[x_1 \ldots x_n]/I$.

Then f is smooth iff

$$0 \to I/I^2 \xrightarrow{d} \bigoplus_{i=1}^{n} A \cdot dx_i \to \Omega^1_{A/k} \to 0$$

is exact.

This is equivalent to

$$0 \to I_K/I_K^2 \xrightarrow{d_K} \bigoplus_{i=1}^{n} A_K\, dx_i \to \Omega^1_{A_K/K} \to 0$$

being injective.