Smooth proper curves:

Let k be a field and

$X \to \operatorname{Spec} k$ a smooth proper curve.

Definition

1) Given a coherent sheaf $\mathcal{F} \in \operatorname{Coh}(X)$ we define the Euler characteristic

$$\chi(X, \mathcal{F}) := \underbrace{\sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F})}_{\text{well-defined by finiteness in cohomology and Grothendieck vanishing.}}$$

2) Given $\mathcal{L} \in \operatorname{Pic}(X)$ we let

$$\deg(\mathcal{L}) := \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X)$$

3) We let $g(X) = \dim_k H^0(X, \Omega^1_{X/k})$ the genus of X.

Last Semester:

Theorem There is an exact sequence:

$$0 \to k^\times \to K(X)^\times \xrightarrow[\mathrm{div}]{} \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0$$

Theorem (conditional) The degree map

$$\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}$$

$$\sum_{\substack{x \in |C| \\ x \text{ closed}}} n_x [x] \longmapsto \sum n_x \deg\left(k(x)/k\right)$$

factors as

$$\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$$

$$D \longmapsto \mathcal{O}(D)$$

$$\mathcal{L} \longmapsto \deg(\mathcal{L})$$

$\underline{\text{Sketch:}}$ Given $i_x : \operatorname{Spec} k(x) \hookrightarrow X$

a closed point we set

SES:

$$0 \to \mathcal{L} \to \mathcal{L}(x) \to i_{x,*} k(x) \to 0$$

Giving LES:

$$0 \to H^0(X, \mathcal{O}(x)) \to H^0(X, \mathcal{O}) \to \dots$$

all finite dim k-vector spaces.

passing to $\chi(X, -)$ gives

$$\chi(X, \mathcal{L}(x)) = \chi(X, \mathcal{L}) + \dim_k k(x)$$

Since $\operatorname{Spec} k(x) \to \operatorname{Spec} k$ is affine.

Then

$$\deg(\mathcal{L}(x)) = \deg(\mathcal{L}) + \dim_k(k(x)).$$

inductively $\deg(\mathcal{O}(D)) = \deg(D)$.

NOT conditional anymore ∇.

Serre duality:

For all smooth proper schemes X over $\mathrm{Spec}\, k$ of dimension n and vector bundles $\mathcal{E}$ over X we have an isomorphism

$$H^i(X, \mathcal{E}) \simeq \mathrm{Hom}_k\left(H^{n-i}(X, \Omega^n_{X/k} \otimes \mathcal{E}^\vee), k\right)$$

In particular,

$$\dim_k H^i(X, \mathcal{E}) = \dim_k H^{n-i}(X, \Omega^n_{X/k} \otimes \mathcal{E}^\vee)$$

(we prove this soon!)

Theorem (Riemann–Roch, conditional).

If $\mathcal{L} \in \operatorname{Pic}(X)$ then

$$\dim_k H^0(X,\mathcal{L}) - \dim_k H^0(X, \Omega^1_{X/k} \oplus \mathcal{L}^\vee) = \deg(\mathcal{L}) + 1 - g(X)$$

Sketch:

$$\deg(\mathcal{L}) = \chi(X,\mathcal{L}) - \chi(X,\mathcal{O}_X)$$

$$= \dim_k H^0(X,\mathcal{L}) - \dim_k H^1(X,\mathcal{L})$$

$$- \left(\dim_k H^0(X,\mathcal{O}_X) - \dim_k H^1(X,\mathcal{O}_X)\right)]$$

Smooth proper $\Rightarrow$ $\dim_k H^0(X,\mathcal{O}_X) = 1$

$$\deg(\mathcal{L}) + 1 = \dim_k H^0(X,\mathcal{L}) - \dim_k H^1(X,\mathcal{L}) + \dim_k H^1(X,\mathcal{O}_X)$$

$$g(X) := \dim_k H^0(X, \Omega^1_{X/k})$$

By Serre duality $\quad g(X) = \dim_k H^1(X,\mathcal{O}_X)$

and $\quad \dim_k H^1(X,\mathcal{L}) = \dim_k H^0(X, \mathcal{L}^\vee \otimes \Omega^1_{X/k})$

$$\deg(\mathcal{L}) + 1 - g(X) = \dim_k H^0(X,\mathcal{L}) - \dim_k H^0(X, \Omega^1_{X/k} \oplus \mathcal{L}^\vee)$$

[Conditional on Serre duality].

Theorem (conditional) A line bundle $\mathcal{L}$ is ample if and only if $\deg(\mathcal{L}) > 0$

Sketch (1st semester)

Ingredient:

a) If $\mathcal{L}$ is a line bundle s.t. for all quasicoherent ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_X$ there is $N \gg 0$ with $\mathcal{L}^{\otimes N} \otimes \mathcal{I}$ globally generated then $\mathcal{L}$ is ample.

b) Riemann-Roch $\Rightarrow$ If $\mathcal{L}' \in \mathrm{Pic}(X)$ and $\deg(\mathcal{L}') \geq 2g(X)$ then $\mathcal{L}'$ is globally generated

[Serre duality $\Rightarrow$ Riemann-Roch $\Rightarrow$ Theorem].

Definition Let $f: X \to Y$ be a non-constant map of smooth proper curves over $\operatorname{Spec} k$.

Suppose that the field extension $[K(X):K(Y)]$ is separable.

1) We let $\deg(f) = \dim_{K(Y)} K(X)$.

2) We let $R_f \subseteq X$ be the **ramification locus**. This is the schematic support of $\Omega'_{X/Y}$.

3) We let $Br_f \subseteq Y$ be the branching locus defined as $Br_f = f(R_f)$.

Theorem (Riemann – Hurwitz formula)
With notation as above,

$$2g(X)-2 = \deg(f)(2g(Y)-2) + \dim_k H^0(X, \Omega^1_{X/Y})$$

Proof From the triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

we have a right exact sequence:

$$f^*\Omega^1_{Y/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

Moreover, both $\Omega^1_{X/k}$ and $\Omega^1_{Y/k}$ are line bundles since X and Y are smooth.

Since $k(X)/k(Y)$ is separable and

$$\begin{tikzcd} \operatorname{Spec} k(X) \arrow[r] \arrow[d] & \operatorname{Spec} k(Y) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is Cartesian.

we have $\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} K(X) = \Omega^1_{\mathrm{Spec}\, K(X)/K(Y)} = 0$

This shows $\Omega^1_{X/Y}|_U = 0$ for some $U \subseteq X$ open.

In particular $R_f \subseteq X$ is a 0-dimensional scheme. and

$$f^*\Omega^1_{Y/k}|_U \twoheadrightarrow \Omega^1_{X/k}|_U$$

Now, maps of line bundles $\neq 0$ on the generic point are injective so

$$0 \to f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$$

passing to Euler characteristic we have

$$\chi(X, \Omega^1_{X/k}) = \chi(X, f^*\Omega^1_{Y/k}) + \dim_k H^0(X, \Omega_{X/Y})$$

Substracting $\chi(X, \mathcal{O}_X)$ on both sides gives

$$\deg(\Omega^1_{X/k}) = \deg(f^*\Omega^1_{Y/k}) + \dim_k H^0(X, \Omega^1_{X/Y})$$

Since

$\deg(\Omega^1_{X/k}) = 2g(X) - 2$ it suffices to show the following claim:

Claim $\deg(f^*\Omega^1_{Y/k}) = \deg(f) \cdot \deg(\Omega^1_{Y/k})$

Actually:

$$\begin{tikzcd}
\mathrm{Pic}(Y) \arrow[r, "f^*"] \arrow[d, "\deg"'] & \mathrm{Pic}(X) \arrow[d, "\deg"] \\
\mathbb{Z} \arrow[r, "(-)\cdot\deg(f)"] & \mathbb{Z}
\end{tikzcd}$$

Given $y \in |Y|$ a closed point we have

$$0 \to \mathcal{O}(-y) \to \mathcal{O} \to i_* k(y) \to 0$$

Since $f: X \to Y$ is flat

$f^*: Qcoh(Y) \to Qcoh(X)$ is exact and we have

$$0 \to f^*\mathcal{O}(-y) \to \mathcal{O}_X \to f^* i_* k(y) \to 0$$

$$\begin{tikzcd} X_y \arrow[r, "i'"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r, "i"'] & Y \end{tikzcd}$$

Now, $f^* i_* k(y) = i'_* \mathcal{O}_{X_y} = k(y) \otimes_{\mathcal{O}_Y} f_* \mathcal{O}_X$

$$\chi(X, f^*\mathcal{O}(-y)) - \chi(X, \mathcal{O}_X) = \dim_k H^0(X_y, \mathcal{O}_{X_y})$$

Since $f: X \to Y$ is finite flat $f_*\mathcal{O}_X$ is a vector bundle of rank $\deg(f)$.

So $H^0(X_y, \mathcal{O}_{X_y})$ is a $k(y)$-algebra of $\dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y}) = \deg(f)$

So $\dim_k(H^0(X_y, \mathcal{O}_{X_y})) = \deg(f) \cdot \dim_k(k(y))$

This shows

$$\deg(f^*\mathcal{O}(-y)) = \deg(f)\cdot\deg(\mathcal{O}(-y))$$

as we wated to show. $\blacksquare$

Analysing $\Omega^1_{X/Y}$.

let $x \in \mathrm{Supp}(\Omega^1_{X/Y})$, $y = f(x)$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\, \mathcal{O}_{X,x} & \longrightarrow & \mathrm{Spec}\, \mathcal{O}_{Y,y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Moreover, $(\Omega^1_{X/Y})_x = \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}}$

From the triangle

$$\begin{array}{ccc} \mathrm{Spec}\, \mathcal{O}_{X,x} & \longrightarrow & \mathrm{Spec}\, \mathcal{O}_{Y,x} \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}\, \mathbb{R} & \end{array}$$

We set

$$\Omega^1_{\mathcal{O}_{Y,x}/k} \otimes \mathcal{O}_{X,x} \longrightarrow \Omega^1_{\mathcal{O}_{X,x}/k} \longrightarrow \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}} \longrightarrow 0$$

$$\begin{array}{ccc} \shortparallel & & \shortparallel \\ \mathcal{O}_{X,x} \cdot dt_y & \longrightarrow & \mathcal{O}_{X,x} \cdot dt_x \\ dt_y & \longmapsto & d(f(t_y)) \end{array}$$

Here $t_y \in \mathcal{O}_{Y,y}$ and $t_x \in \mathcal{O}_{X,x}$ are uniformizers.

Now $f(t_y) = u\, t_x^e$ for some $e \geq 0$. Then

$$d f(t_y) = t_x^e\, du + e\, t_x^{e-1} u\, dt_x$$

We can write $du = p\, dt_x$ for some $p \in \mathcal{O}_{X,x}$.

Then $d f(t_y) = (eu + t_x p)\left[t_x^{e-1}\, dt_x \right]$.

Case 1: $e \in k^\times$ tame ramification

$(eu + t_x p)$ is a unit so

$$(\Omega^1_{X/Y})_x \simeq k(x)[t]/t^{e-1} \text{ as } k\text{-modules.}$$

Case 2: $e = 0$ in k

i.e. $p | e$ for $p = \mathrm{char}(k)$

wild ramification ∇.

Harder to describe, depends on the unit.

Example: $u = 1 + t_x^n$ then

$$du = n t^{n-1} dt_x$$

If $k = \mathbb{C}$ all points have tame ramification.

$$X_y \simeq \coprod_{x \in f^{-1}(y)} \mathrm{Spec}\, \mathbb{C}[t]/t^{e_x}$$

$$\deg(f) - \# X_y(\mathbb{C}) = \sum_{\substack{x \in f^{-1}(y) \\ x \in X(\mathbb{C})}} e_x - 1 = \sum_{\substack{x \in f^{-1}(y) \\ x \in X(\mathbb{C})}} \dim_{\mathbb{C}} (\Omega^1_{X/Y})_x .$$

$$\text{"Total branching number"} = \sum_{y \in Y(\mathbb{C})} \left(\deg(f) - \# X_y(\mathbb{C}) \right) = \dim_{\mathbb{C}} H^0(X, \Omega^1_{X/Y}).$$

Left to do?

Prove Serre duality.