Algebraic geometry 1
Exercise sheet 9

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Exercise 2.

1. Suppose we have a vector bundle of rank n on $\mathbb{P}^1_k$. How do we construct a matrix $\in \mathrm{GL}_n(k[T^{\pm 1}])$?

   Take a rank n vector bundle $\mathcal{E}$. Since Picard group of U_0 and U_1 are trivial, we have isomorphisms α, β. So on $\mathrm{Spec}(k[T^{\pm 1}]) \subseteq U_0$ we have an isomorphism $\Gamma(U_0 \cap U_1, \mathcal{O}^n_{U_0}) = (k[T^{\pm 1}])^n \cong \mathcal{E}(U_0 \cap U_1)$.

   Combining this with an isomorphism $\mathcal{E}(U_0 \cap U_1) \cong (k[T^{\pm 1}])^n = \Gamma(U_0 \cap U_1, \mathcal{O}^n_{U_1})$, we get an isomorpism $(k[T^{\pm 1}])^n \cong (k[T^{\pm 1}])^n$.

   Let $\mathcal{D}$ be another rank n vector bundle on $\mathbb{P}^1_k$, and let $\varphi\colon \mathcal{E} \to \mathcal{D}$ be an isomorphism between them. On U_0 and U_1 we get induced isomorphisms

   $$(k[T])^n = \mathcal{E}(U_0) \to \mathcal{D}(U_0) = (k[T])^n$$

   and

   $$(k[T^{-1}])^n = \mathcal{E}(U_1) \to \mathcal{D}(U_1) = (k[T^{-1}])^n$$

2. Take $G = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \in \mathrm{GL}_n(k[T^{\pm 1}])$. We can writte each $p_i = \frac{g_i}{T^{k_i}}$ for some $g_i \in k[T]$. We can take $k_i = k$ to be all the same. Then

   $$\begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} = \begin{pmatrix} T^{-k} & 0 \\ 0 & T^{-k} \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \tag{1}$$

3.