Theorem Let X be a separated scheme and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X by affine schemes.
Then for all i there is a natural isomorphism

$$\check{H}^i_{\mathcal{U}}(X, -) \simeq H^i(X, -).$$

of functors $QCoh_{(X, \mathcal{O}_X)} \longrightarrow \Gamma(X, \mathcal{O}_X)$-modules.

Lemma: If $X = \operatorname{Spec} A$ and $\mathcal{F}$ is quasicoherent, then $H^i(X, \mathcal{F}) = 0$ for all $i \geq 1$.

Proof Later!

(Proof of theorem):

Recall the Čech-to-cohomology spectral sequence

$$E_2^{p,q} = \check{H}^p_{\mathcal{U}}(X, \underline{H^q(X, \mathcal{F})}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

where $\underline{H^q(X, \mathcal{F})}$ is the presheaf

with $\underline{H^q(X,\mathcal{F})}[U] := H^q(U, \mathcal{F}_{|U})$.

By the lemma above

$$E_2^{p,q} = \begin{cases} 0 & \text{if } q \geq 1 \text{ and} \\ \check{H}^p_{\mathcal{U}}(X, \mathcal{F}) & \text{if } q = 0 \end{cases}$$

this implies $E_\infty^{p,q} = E_2^{p,q}$ and

that $H^p(X, \mathcal{F}) \simeq \check{H}^p_{\mathcal{U}}(X, \mathcal{F})$.

<u>Why does $H^n(\operatorname{Spec} A, \mathcal{F}) = 0$?</u>

we have exact equivalences of abelian categories

$$A\text{-mod} \underset{\Gamma_{Qcoh}(\operatorname{Spec} A, -)}{\overset{\widetilde{(\ )}}{\rightleftarrows}} Qcoh_{\operatorname{Spec} A}.$$

the functor $\Gamma_{Qcoh}(\operatorname{Spec} A, -) : Qcoh_{\operatorname{Spec} A} \to A\text{-mod}$ is exact.

If $\mathrm{QCoh}_{\mathrm{Spec}A}$ has enough injectives

then $R^i\Gamma_{\mathrm{QCoh}}(\mathrm{Spec}A, -) : \mathrm{QCoh}_{\mathrm{Spec}A} \to A\text{-mod}$

vanish for $i \geq 1$.

Category of A-modules has enough injectives so QCoh also does.

Did we finish?

No!

By definition, $R^i\Gamma$ are right derived functors of

$$\Gamma : (\mathrm{Spec}A, \mathcal{O}_X\text{-mod}) \to A\text{-mod}.$$

all $\mathcal{O}_X$-modules.

We have $i : \mathrm{QCoh}_{\mathrm{Spec}A} \hookrightarrow \mathcal{O}_X\text{-mod}$.

Problem: i does not preserve injectives. In particular

$R^i\Gamma_{\mathrm{QCoh}}(\mathcal{F})$ does not necessarily compute $R^i\Gamma(\mathcal{F})$.

<u>Lemma</u>: Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ a $\mathcal{O}_X$-module. Let $s \in H^n(X, \mathcal{F})$ with $n \geq 1$ the following hold:

1) There is an open cover $X = \bigcup_{i \in I} U_i$ s.t. $0 = s|_{U_i} \in H^n(U_i, \mathcal{F}|_{U_i})$.

2) Suppose there is a basis for the topology $\mathcal{U} = \{ U \subseteq_{open} X \}$ stable under finite intersections and with the property that for all $0 < i < n$ and all $U \in \mathcal{U}$ $H^i(U, \mathcal{F}) = 0$ then, there is a cover $j : \coprod U_i \to X$ s.t. $f(s) = 0$ for $f : H^n(X, \mathcal{F}) \to H^n(X, j_* j^* \mathcal{F})$.

<u>Proof</u> we do this by induction on $n \geq 1$.

1) Consider an embedding
$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{K} \to 0$$
with $\mathcal{G}$ flasque. Then $H^n(X, \mathcal{F}) \simeq H^{n-1}(X, \mathcal{K})$ for $n \geq 2$. So it suffices to show case $n = 1$. In this case
$$H^1(U, \mathcal{F}) = H^0(U, \mathcal{K}) / im(H^0(U, \mathcal{G}))$$

for all $U \subseteq X$ open.

Since $G \to K$ is surjective for every $s \in H^0(X, K)$ there is open cover $\mathcal{U}$ s.t. $s|_{U_i}$ $\forall U_i \in \mathcal{U}$ comes from $H^0(U_i, G)$.

2) How does $H^n(X, j_*j^*\mathcal{F})$ look like?

$$H^n(X, j_*j^*\mathcal{F}) = H^n(X, R^0 j_* j^* \mathcal{F})$$

if $n=1$ the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q j_* j^* \mathcal{F}) \Rightarrow H^{p+q}(X, j^*\mathcal{F})$$

$$\begin{array}{lllll} & \cdots & & \cdots & \\ 0 & H^0(X, R^1 j_* j^* \mathcal{F}) & & H^1(X, R^1 j_* j^* \mathcal{F}) & \cdots \\ 0 & H^0(X, R^0 j_* j^* \mathcal{F}) & & H^1(X, R^0 j_* j^* \mathcal{F}) & \cdots \\ 0 & 0 & & 0 & 0 \quad 0 \end{array}$$

So $H^1(X, R^0 j_* j^* \mathcal{F})$ is $E_\infty^{1,0}$-term

So $0 \to H^1(X, R^0 j_* j^* \mathcal{F}) \to H^1(\coprod U_i, j^* \mathcal{F})$

Aside:

Mnemonic: "If on every page you are a kernel in the end you are a cokernel".

The $(0,p)$ entry of spectral sequence all pass page by taking kernel, so $E_\infty^{(0,p)}$ is cokernel of $R^p(G\circ F)$.

Similarly, $E^{(p,0)}$ pass page by taking cokernel, so $E_\infty^{(p,0)} \subseteq R^p(G\circ F)$.

If $U \in \mathcal{U}$ with induced map $j^U : U \to X$ then for $0 < i < n$

$R^i j^U_* j^{U*} \mathcal{F}$ is sheafification

of rule

$$V \mapsto H^i(U \cap V, \mathcal{F}) = H^i(j^{U,-1}(V), j_U^* \mathcal{F})$$

since $V \cap U \in \mathcal{U}$ for all $V \in \mathcal{U}$.

this shows $R^i j^U_* \mathcal{F} = 0 \;\forall i\; 0 < i < n$.

$E_2^{p,q} = 0 \;\forall\; 0 < q < n$ so that

$$E_2^{n,0} = E_\infty^{n,0} = H^n(X, j_* j^* \mathcal{F})$$

Then $H^n(X, j_* j^* \mathcal{F}) \hookrightarrow H^n(\coprod U_i, \mathcal{F})$ and we can choose it to vanish.

Proposition If $X = \operatorname{Spec} A$ and $\mathcal{F}$ is quasicoherent sheaf, then $H^n(X, \mathcal{F}) = 0$ $\forall n \geq 1$

Proof. Assume this holds for all $0 < i < n$, for all affine schemes and all quasicoherent sheaves on them.

Let $\mathcal{F} \in \mathrm{Qcoh}_X$, and fix $s \in H^n(X, \mathcal{F})$.

By previous lemma (since $H^i(D(f), \mathcal{F}) = 0$ $\forall$ $0 < i < n$ and all $f \in A$) there is a cover $X = \bigcup_{i=1}^{n} D(f_i)$ s.t.

s maps to 0 on $H^n(X, j_* j^* \mathcal{F})$

Consider the SES in Qcoh_X

$$0 \to \mathcal{F} \to \bigoplus_{i=1}^{n} j_*^{D(f_i)} \mathcal{F}_{|D(f_i)} \to \mathcal{G} \to 0$$

taking LES

$$H^{n-1}(X, \mathcal{G}) \xrightarrow{\delta} H^n(X, \mathcal{F}) \xrightarrow{g} H^n\left(X, j_*^{D(f_i)} \mathcal{F}_{|D(f_i)}\right)$$

by construction $g(s) = 0$ so $s = \delta(t)$,

but $H^{n-1}(X, \mathcal{G}) = 0$. when $n \geq 1$

and when $n=1$

the map $H^0(X,G) \to H^1(X,F)$

is the 0 map since $H^0(X,-)$
is exact on quasicoherent sheaves
on affine schemes.

<u>Theorem</u> (Serre's Criterion) Let X
be a quasicompact scheme.
The following are equivalent:

1) X is affine
2) $H^n(X,F)=0$ for all $F \in QCoh(X)$.
3) $H^1(X,I)=0$ for all quasicoherent
ideal sheaves.

<u>Proof</u> <u>1) => 2)</u> | Proposition above

<u>2) => 3)</u> | Clear

<u>3) => 1) :</u> | Let $A = \Gamma(X, \mathcal{O}_X)$, we show
that $f: X \to \operatorname{Spec} A$ is an
affine morphism.

Being an affine morphism is LOCT

So it suffices to find $\{a_1, \dots a_n\} \in A$ with $1 \in \langle a_1, \dots a_n \rangle$. s.t. $f^{-1}(D(a_i)) \subseteq X$ is affine.

Fix $x \in X$ closed and $x \in U \subseteq X$ with U affine.

Let $Z = X \setminus U$ with reduced scheme structure and ideal sheaf $\mathcal{I}_Z \subseteq \mathcal{O}_X$.

We have SES

$$0 \to \mathcal{I}_{Z \cup \{x\}} \to \mathcal{I}_Z \to k(x) \to 0$$

with $k(x)$ the skyscraper sheaf over x and $k(x)$ residue field of x.

We get exact sequence

$$0 \to H^0(X, \mathcal{I}_{Z \cup \{x\}}) \to H^0(X, \mathcal{I}_Z) \to H^0(X, k(x)) \to 0$$

$$// \quad H^1(X, \mathcal{I}_{Z \cup \{x\}})$$

there is $a \in H^0(X, \mathcal{I}_Z)$ with $a_x = 1$ in $k(x)$.

This shows $x \in f^{-1}(D(a))$

Moreover, $f^{-1}(D(a)) \subseteq \mathcal{U}$ since $a \in \mathcal{I}_Z$

Since $\mathcal{U}$ is affine, $f^{-1}(D(a)) = f^{-1}(D(a)) \cap \mathcal{U}$ wich is a distinguished open set of $\mathcal{U}$. hence affine.

For all closed point $x \in X$ we constructed $a \in A$ s.t. $f^{-1}(D(a))$ is affine and $x \in f^{-1}(D(a))$.

Let $W = \bigcup f^{-1}(D(a))$, $X \setminus W$ is closed, but quasicompact schemes have a closed point.

Let us show the $D(a)$ constructed this way cover $\operatorname{Spec} A$.

Choose a_i s.t $X = \bigcup_{i=1}^{m} f^{-1}(D(a_i))$ and define

$$\alpha : \mathcal{O}_X^m \longrightarrow \mathcal{O}_X$$
$$(b_1, \dots b_m) \longmapsto \sum_{i=1}^{m} a_i b_i$$

Since $(a_i)_x \neq 0$ for some i $\forall x \in X$

α is surjective.

Let $\mathcal{F} = \ker(\alpha)$ we have

a filtration

$$0 \subseteq \mathcal{O}_X \cap \mathcal{F} \subseteq \mathcal{O}_X^2 \cap \mathcal{F} \subseteq \cdots \mathcal{O}_X^m \cap \mathcal{F} = \mathcal{F}$$

Let $K^i = \operatorname{coker}(\mathcal{O}_X^i \cap \mathcal{F} \to \mathcal{O}^{i+1} \cap \mathcal{F})$

then

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & S \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X^i \cap \mathcal{F} \arrow[r] \arrow[d] & \mathcal{O}_X^{i+1} \cap \mathcal{F} \arrow[r] \arrow[d] & K^i \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & \mathcal{O}^i \arrow[r] \arrow[d, "\alpha"] & \mathcal{O} \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & \alpha(\mathcal{O}^i) \arrow[r] & \alpha(\mathcal{O}^{i+1}) & &
\end{tikzcd}$$

Snake lemma $S = 0$ so $K^m \hookrightarrow \mathcal{O}_X$

this gives $H^1(X, K^i) = 0$ and inductively

$$H^1(X, \mathcal{O}_X^i \cap \mathcal{F}) = 0$$

This gives $\alpha : H^0(X, \mathcal{O}_X^m) \to H^0(X, \mathcal{O}_X)$

i.e. $1 = a_1 b_1 + \cdots + a_m b_m$.