Algebraic geometry 2
Exercise sheet 3

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Hey, sorry for submitting so weirdly, we usually submit to Robin, but this week I remembered a few minutes after midnight and the field for submittion was already closed in his group, and here it was still open. I also couldn't find his email immediately to send him the sheet. Could you please forward him this pdf, please :) Thank you

Exercise 1. We know that dominant morphisms between integral schemes map generic point to the generic point. So we get inclusion $k(\eta_X) \hookrightarrow k(\eta_Y)$.

Integral schemes are irreducible, so any non-empty open subset is dense. Therefore we can focus on some affine neighbourhood of $X = \operatorname{Spec}(A) \subset X$, which we also name X. Then take preimage and ...

Exercise 2.

(i) Observe that for every $x \in X$ we have

$$\dim_{k(x)} H_i(C_\bullet \otimes_A k(x)) = \dim_{k(x)} \ker(d_i \otimes k(x)) - \dim_{k(x)} \operatorname{im}(d_{i+1} \otimes k(x))$$

A map of finite free A-modules can be represented by a matrix with values in A.

Let M be an $m \times n$ matrix representing a map $A^n \to A^m$. Localizing at $x \in X$, we get a map

$$k(x)^n \cong A^n \otimes_A k(x) \to A^m \otimes_A k(x) \cong k(x)^m$$

given by this "same" matrix, denoted by M_x, whose components are images of components in M under $A \to k(x)$.

Suppose now M has rank r at some point $x \in X$. Therefore there exists an invertible minor of size $r \times r$, call it N. That means that $\det N$ does not vanish in x. Then $D(\det N)$ is an open neighbourhood of x on which M has rank $\geq r$.

This shows that $x \mapsto \dim_{k(x)} \operatorname{im}(M \otimes k(x))$ is lower semicontinuous.

Multiplying function with -1 will switch upper and lower semicontinuity.

Also note that for a given matrix we have $n = \dim\ker + \dim\operatorname{im}$ for every x where n is the dimension of the source.

Considering all that we obtain that

$$x \mapsto \dim_{k(x)} \ker(d_i \otimes k(x)) - \dim_{k(x)} \operatorname{im}(d_{i+1} \otimes k(x))$$

is a sum of upper semicontinuous function, so itself upper semicontinuous.

(ii) We have $\beta_i^{-1}(n) = \beta_i^{-1}((-\infty, n+1)) \cap \beta_i^{-1}([n, \infty))$, so intersection of an open and closed set, in particular it is constructible.

(iii) Let $k = \bar{k}$ be a field and

$$C_\bullet \colon \cdots \to 0 \to k[t] \to k[t] \to 0 \to \dots$$

be the complex of $k[t]$-modules, where the only nontrivial map is $1 \mapsto t$. We take homology at $k[t] \to k[t] \to 0$. We claim that it is not locally constant at closed point $(t) \in \mathbb{A}^1_k$.

Take $x = (t-a)$ for $a \in k$, then

$$C_\bullet \otimes_{k[t]} k(x) \colon \cdots \to 0 \to k \to k \to 0$$

where the unique nontrivial map is $1 \mapsto a$. Clearly the image of $k \to k$ will be a 1-dimensional k-vsp for $a \neq 0$ and 0-dimensional for $a = 0$.

For $x = (0)$ the generic point, we get a surjection $k(t) \to k(t), 1 \mapsto t$.

So

$$\dim_{k(x)} H(C_\bullet \otimes_{k[t]} k(x)) = \begin{cases} 1 & x = (t) \\ 0 & x = (t-a) \text{ for } a \neq 0 \text{ or } x = (0) \end{cases}$$

Exercise 3.

1. We have

$$\begin{aligned} X &= \operatorname{Spec}(A) \\ &= \operatorname{Spec}(R[T, T_1, T_2]/I_1 \cap I_2) \\ &= \operatorname{Spec}(R[T, T_1, T_2]/I_1) \cup \operatorname{Spec}(R[T, T_1, T_2]/I_2) \end{aligned}$$

where

$$\begin{aligned} X_1 = \operatorname{Spec}(R[T, T_1, T_2]/I_1) &= \operatorname{Spec}(R[\pi^{-1}, T_1, T_2]) \\ &= \operatorname{Spec}(R[\pi^{-1}][T_1, T_2]) \\ &= \operatorname{Spec}(K[T_1, T_2]) \\ &= \mathbb{A}^2_K \end{aligned}$$

and

$$\begin{aligned} X_2 = \operatorname{Spec}(R[T, T_1, T_2]/I_2) &= \operatorname{Spec}(R[T]) \\ &= \mathbb{A}^1_R. \end{aligned}$$

To show that X is equidimensional, we have to check that both irreducible components have dimension 2. Clearly they both do; X_1 is an affine plane over a field, and from Alg 1 we know $\dim \mathbb{A}^1_R = \dim R + 1 = 2$ since R is a PID that is not a field.

2. Clearly only prime ideal that contains $I_1 \cup I_2$ is $(\pi T - 1, T_1, T_2)$ which is consequently also a closed point. Denote $x = (\pi T - 1, T_1, T_2)$.

   To calculate $\dim \mathcal{O}_{X_1,x}$ we have to find ideals that are between I_1 and x. Those are exactly primes of $K[T_1, T_2]_{(T_1,T_2)}$, so $\dim \mathcal{O}_{X_1,x} = 2$.

   And for $\dim \mathcal{O}_{X_2,x}$ we have to find ideals that are between I_2 and x. Those are prime ideals of $R[T]_{(\pi T-1)}$. This is same as asking what is the height of $(\pi T - 1)$, which is 1, so the localization has dimension 1.

Ex.4) (i) $\cdots \to C(f)_{n+1} \longrightarrow C(f)_n \longrightarrow C(f)_{n-1} \to \cdots$

$$D_{n+1} \oplus C_n \longrightarrow D_n \oplus C_{n-1} \longrightarrow D_{n-1} \oplus C_{n-2}$$

$$\begin{pmatrix} d_{n+1} \\ c_n \end{pmatrix} \longmapsto \begin{pmatrix} \partial^D_{n+1}(d_{n+1}) + f_n(c_n) \\ -\partial^C_n(c_n) \end{pmatrix} \longmapsto \begin{pmatrix} T \\ -\partial^C_{n-1}(-\partial_n(c_n)) \end{pmatrix}$$

$$\begin{pmatrix} T \\ -\partial^C_{n-1}(-\partial_n(c_n)) \end{pmatrix} = \begin{pmatrix} T \\ 0 \end{pmatrix}$$

where $T = \partial^D_n\left(\partial^D_{n+1}(d_{n+1}) + f_n(c_n)\right) + f_{n-1}\left(-\partial^C_n(c_n)\right)$

$$= \cancel{\partial^D_n \circ \partial^D_{n+1}(d_{n+1})} + \cancel{\partial^D_n \circ f_n(c_n) - f_{n-1} \circ \partial^C_n(c_n)}$$

$= 0$ because $D_\bullet$ is a complex and f a complex morphism.

(ii) Let us state snake lemma applied to the short exact seq. of the hint:

$$
\begin{tikzcd}
 & \vdots & \vdots & C_{n-1} \arrow[d] & \\
0 \arrow[r] & D_{n+1} \arrow[r, "\iota_{n+1}"] \arrow[d, "\partial^D_{n+1}"] & D_{n+1} \oplus C_n \arrow[r, "\pi_n"] \arrow[d, "\partial^D_{n+1} \oplus \partial^C_n"] & C_n \arrow[r] \arrow[d, "\partial^C_n"] & 0 \\
0 \arrow[r] & D_n \arrow[r] \arrow[d, "\partial^D_n"] & D_n \oplus C_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \\
 & D_{n-1} & \vdots & \vdots &
\end{tikzcd}
$$

which gives : $0 \to \ker(\partial^D_{n+1}) \xrightarrow{\iota^*_{n+1}} \ker(\partial^D_{n+1} \oplus \partial^C_n) \xrightarrow{\pi^*_n} \ker(\partial^C_n) \xrightarrow{\mu} \operatorname{coker}(\partial^D_n) \xrightarrow{\iota^*_n} \operatorname{coker}(-) \to \operatorname{coker}(\partial^C_n) \to 0$

this gives on homology the following :

$$H_{n+1}(D_\bullet) \longrightarrow H_{n+1}(C(f)_\bullet) \longrightarrow H_n(C_\bullet)$$

$$\longrightarrow H_n(D_\bullet) := \frac{\ker(\partial^D_n)}{\operatorname{im}(\partial^D_{n+1})} \supseteq \operatorname{im}(\mu) \longrightarrow H_n(C(f)_\bullet) \longrightarrow \cdots$$

Exactness at every point of the seq. is clear but at $H_n(D_\bullet)$.
Since $\mu(m) \xmapsto{t_n^*} \in \operatorname{im}(\partial^D_{n+1} \oplus \partial^C_n) \equiv 0$ in $H_n(C(f)_\bullet)$
we have exactness there too. Repeating this process creates the desired long exact seq. from the short one.