Algebraic geometry 1
Exercise sheet 2

Solutions by: Eric Rudolph and David Čadež

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Exercise 1. Let $I = (f_1, \ldots, f_r) \subseteq k[x_1, \ldots, x_n]$ be an ideal and $X = V(I) \subseteq \mathbb{A}^n(k)$ be its vanishing locus.

1. We have to show
$$\overline{X} = \bigcap \{V^+(h) \mid h \text{ homogenous}, h(X) = 0\} = V(\{\tilde{g} \mid g \in I\}).$$
Pick any homogenous h that vanishes on $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$. By letting $x_{n+1} = 1$ we see that $h(x_1, \ldots, x_n, 1) \in \sqrt{I}$, since it vanishes on X. Therefore we have $l \in \mathbb{N}$ such that $h(x_1, \ldots, x_n, 1)^l = \sum_{i=1}^r \alpha_i f_i$ for some $\alpha_i \in k[x_1, \ldots, x_n]$.
Homogenization is almost a bijection between homogenous polynomials in $n+1$ variables and all polynomials in n variables. It is bijective between homogenous polynomials in $n+1$ variables that are not divisible by x_{n+1} and polynomials in n variables. The "inverse" to homogenization would be the letting $x_{n+1} = 1$.
Adding these two together we get $x_{n+1}^{ml} \widetilde{\sum_{i=1}^r \alpha_i f_i} = h^l$. Therefore
$$h = \sum_{i=1}^r \beta_i \widetilde{f_i}$$
and thus $\bigcap \{V^+(h) \mid h \text{ homogenous}, h(X) = 0\} \supseteq V(\{\tilde{g} \mid g \in I\})$.

2. We calculate $\overline{X}$ using the first part of the exercise and take the numbering of the varieties from the first exercise sheet:
$$\begin{aligned}
\overline{X_1} \cap \mathbb{P}_k^2 \backslash \mathbb{A}_k^2 &= V(x^2 - xz) \cap V(z) = (0:1:0), \\
\overline{X_2} \cap \mathbb{P}_k^2 \backslash \mathbb{A}_k^2 &= V(z^2 - xy) \cap V(z) = (1:0:0) \cup (0:1:0), \\
\overline{X_3} \cap \mathbb{P}_k^2 \backslash \mathbb{A}_k^2 &= V(xy) \cap V(z) = (1:0:0) \cup (0:1:0), \\
\overline{X_4} \cap \mathbb{P}_k^2 \backslash \mathbb{A}_k^2 &= V(x^2 - xy) \cap V(z) = (0:1:0), \\
\overline{X_5} \cap \mathbb{P}_k^2 \backslash \mathbb{A}_k^2 &= V(x) \cap V(z) = (0:1:0).
\end{aligned}$$

3.

Define $f(x,y) := xy - 1$ and $g(x,y) := xy$. Then we have

$$X = V(f,g) = \emptyset,$$

and since $\emptyset$ is closed in projective space as the vanishing set of a non-zero constant, we have $\overline{X} = \emptyset$. However,

$$(0:1:0) \in V^+(xy - z, xy) = V^+(\tilde{f}, \tilde{g}),$$

proving that in this case $\overline{X} \neq V(\tilde{f}, \tilde{g})$.

Exercise 3.

1. We define a closed subset $V(f-g) \subseteq X$, which contains the open set $\mathcal{U} \subseteq V(f-g)$. By definition the completent $\mathcal{U}^C$ is closed and $V(f-g) \cup \mathcal{U}^C = X$. Since X is irreducible and $\mathcal{U}$ is non-empty, we have $f = g$.

2. Lets show first that $\chi\colon A \mapsto \chi_A(A)$ vanishes on diagonalizable matrices with pairwise different eigenvalues: For $A = TDT^{-1}$ we have

$$\chi_A(A) = \sum_{i=0}^{n} \alpha_i A^i = \sum_{i=0}^{n} \alpha_i (TDT^{-1})^i = T \left(\sum_{i=0}^{n} \alpha_i D^i \right) T^{-1}.$$

Denote $D = \operatorname{diag}(d_1, \ldots, d_n)$ and notice that

$$\sum_{i=0}^{n} \alpha_i D^i = \sum_{i=0}^{n} \operatorname{diag}(\alpha_i d_1^i, \ldots, \alpha_i d_n^i) = \operatorname{diag}(\chi_A(d_1), \ldots, \chi_A(d_n)) = 0$$

because characteristic polynomial vanishes on eigenvalues d_i.

Since $\mathbb{A}^{n\times n}(L)$ is irreducible and χ vanishes on open subset of it, namely on diagonalizable matrices with pairwise different eigenvalues, χ must be 0 on whole $\mathbb{A}^{n\times n}(L)$.

Exercise 4.

1. First, we note that points in projective space are closed. To see this, take some point $a = (a_0 : ... : a_n) \in \mathbb{P}_k^n$. Then $a = V(\langle f \in k[x_0, ..., x_n] : f = x_i a_j - x_j a_i \rangle)$. Now, we let $X \subset \mathbb{P}_k^n$ be some quasi-projective variety and $x, y \in X$ with $x \neq y$. Define $U^c := \{x\}^c$, which is open as the complement of a closed set and by definition fulfills the properties $x \notin U$ and $y \in U$.