# Elliptic curves and their moduli spaces
# Exercise sheet 10

Solutions by: Esteban Castillo Vargas and David Čadež

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**Problem 1.**

1. 

2. If $\phi$ is divisible in $L$, then it is clearly divisible in $\mathrm{Hom}(T_l(E_1), T_l(E_2))$. Suppose $\phi$ is divisible by $l$ in $\mathrm{Hom}(T_l(E_1), T_l(E_2))$. Then $\phi$ vanishes on $E_1[l]$, so by proposition 10.1 from the lectures we have that $\phi$ is already divisible by $l$.

3. If $\mathrm{End}^0(E)$ is a quaternion algebra, then $\mathrm{End}(E)$ is of rank 4 (its always free of rank $\leq 4$). The map $\mathbb{Z}_l \otimes_{\mathbb{Z}} \mathrm{End}(E) \cong (\mathbb{Z}_l)^4 \to \mathrm{End}(T_l(E)) \cong M_2(\mathbb{Z}_l)$ (we used $T_l(E) \cong \mathbb{Z}_l$) is injective. By previous part invertible elements in $\mathbb{Z}_l \otimes_{\mathbb{Z}} \mathrm{End}(E)$ get mapped to invertible elements in $\mathrm{End}(T_l(E))$, so the map is also surjective.

**Problem 2.** By assumption both $E_1$ and $E_2$ have some nontrivial endomorphism, lets denote them by $\tau_1$ and $\tau_2$. By assumption there is an isomorphism $\phi\colon \mathbb{Q}(\tau_1) \to \mathbb{Q}(\tau_2)$. It follows that $\phi(\tau_1) = a\tau_2 + b$. Since $K$ is an imaginary-quadratic field, $\tau_1$ and $\tau_2$ satisfy equations $\tau_1^2 + d_1 = 0$ and $\tau_2^2 + d_2 = 0$ for some integers $d_1, d_2 > 0$.

$$0 = \phi(\tau_1)^2 + d_1 = a^2\tau_2^2 + 2ab\tau_2 + b^2 + d_1 = -a^2 d_2 + 2ab\tau_2 + b^2 + d_1$$

Therefore $ab = 0$.

If $a = 0$, then $b^2 + d_1 = 0$ which doesn't have a solution for $b \in \mathbb{Q}$. So $b = 0$. Then $a^2 d_2 = d_1$. Write $a = \frac{a_1}{a_2} \in \mathbb{Q}$. Then $a_1^2 d_2 = a_2^2 d_1$ and $a_2\tau_1 = \pm a_1\tau_2$. So there exists a nonconstant morphism from $E_1$ to $E_2$, namely multiplication with $a_2 \in \mathbb{Z}$.