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Algebraic Geometry II

10. Exercise sheet

Exercise 1 (4 points):
Let S be a scheme. Prove that there exists an exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n_S/S} \xrightarrow{\varphi} \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n_S/S}(-1) \cdot e_i \xrightarrow{(x_0,\ldots,x_n)} \mathcal{O}_{\mathbb{P}^n_S/S} \to 0$$

where $x_0, \ldots, x_n \in H^0(X, \mathcal{O}_{\mathbb{P}^n_S/S}(1)) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n_S/S}(-1), \mathcal{O}_{\mathbb{P}^n_S/S})$ are the canonical sections.
Hint: One can reduce to $S = \mathrm{Spec}(\mathbb{Z})$. Argue locally and use $\varphi(d(x_j/x_i)) = 1/x_i^2(x_ie_j - x_je_i)$.

Exercise 2 (4 points):
Let k be a field and $X := \mathbb{P}^n_k$ with $n \geq 2$. Show that $\Omega^1_{X/k}$ is not an iterated extension of line bundles, i.e., there does not exist a flag $0 \subsetneq \mathcal{F}_n \subsetneq \ldots \subsetneq \mathcal{F}_1 = \Omega^1_{X/k}$ of $\mathcal{O}_X$-modules with $\mathcal{F}_i/\mathcal{F}_{i+1}$ a line bundle for $i = 1, \ldots, n-1$.

Exercise 3 (4 points):
Let k be a field and let X be a proper scheme over k. For a coherent sheaf $\mathcal{F}$ on X we define the Euler characteristic $\chi(X, \mathcal{F}) := \sum_{i=0}^{\infty}(-1)^i \dim_k H^i(X, \mathcal{F})$.
i) Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. Prove that $\chi(X, \mathcal{F}') = \chi(X, \mathcal{F}) + \chi(X, \mathcal{F}'')$.
ii) Prove that for $d \in \mathbb{Z}$ we have $\chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = \binom{n+d}{n} := \prod_{i=1}^{n} \frac{d+i}{i}$
iii) Assume that X is geometrically integral and $X = V(f) \subseteq \mathbb{P}^2_k$ for some non-zero $f \in H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d))$, $d > 0$. Prove that $\dim_k H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}$.
Remark: You may freely use that here $H^i(X, \mathcal{F})$, $i \geq 0$, is finite dimensional and zero for $i \gg 0$. This will be proven, for X projective, in the lecture.

Exercise 4 (4 points):
Let X be a spectral space, let I be a filtered category and let $\mathcal{F}_i, i \in I$, be a direct system of abelian sheaves on X. For $U \subseteq X$ open let $\Psi_U \colon \varinjlim_I \mathcal{F}_i(U) \to (\varinjlim_I \mathcal{F}_i)(U)$ be the canonical morphism.
i) Assume $U \subseteq X$ is open and quasi-compact. Prove that Ψ_U is injective.
ii) Assume that $U \subseteq X$ is open and qcqs. Prove that Ψ_U is bijective.
iii) Prove that for any $n \geq 0$

$$\varinjlim_I H^n(X, \mathcal{F}_i) \cong H^n(X, \varinjlim_I \mathcal{F}_i).$$

Hint: You may assume, or prove, that there exists functorial injective resolutions for abelian sheaves on X. Using this there exists a direct system $\mathcal{G}_i, i \in I$, of complexes of injective abelian sheaves and a quasi-isomorphism $\{\mathcal{F}_i\} \to \{G_i\}$ of direct systems, i.e., each $\mathcal{F}_i \to \mathcal{G}_i$ is a quasi-isomorphism. Prove $H^n(X, \varinjlim_I \mathcal{G}_i) = 0$ for any $n > 0$ using Cech cohomology and conclude by induction on n.

To be handed in on: Thursday, 27.06.2024 (during the lecture or via eCampus).