Algebraic geometry 1
Exercise sheet 6

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Exercise 1.

1. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram

$$\begin{tikzcd}
U_i \times_{S_{i,j}} V_j \arrow[dr, dotted] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & V_j \arrow[d, "\psi"] \\
& U_i \arrow[r, "\phi"] & S_{i,j}
\end{tikzcd}$$

Therefore, on the level of sets,

$$U_i \times_{S_{i,j}} V_j \subset X \times_S Y,$$

but in exercise 5.2.1, we showed that this induces an open immersion as locally ringed spaces.

Now observe that

$$\begin{tikzcd}
\bigcup_{i,j} \left(U_i \times_{S_{i,j}} V_j\right) \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

commutes, because $S = \bigcup_{i,j} S_{i,j}$. Now by uniqueness of the pullback,

$$\bigcup_{i,j} \left(U_i \times_{S_{i,j}} V_j\right) \cong U_i \times_{S_{i,j}} V_j.$$

I guess this is a good step in the direction of understanding why the pullback in the category of sheaves exists, right? If we assume X, Y, S to be sheaves and $U_i, V_j, S_{i,j}$ to be affine schemes, then by the above argument we found a cover of $X \times_S Y$ by affine schemes.

2. Surjectivity follows, because a pullback of schemes in partiular makes

$$\begin{tikzcd} \mid X \times_S Y \mid \arrow[r] \arrow[d] & \mid X \mid \arrow[d, "\psi"] \\ \mid Y \mid \arrow[r, "\phi"] & \mid S \mid \end{tikzcd}$$

commute for all ψ, ϕ from maps of schemes.

Exercise 2.

1. First let $f\colon X \to S$ be open immersion. In this case we can directly use previous exercise on the following fibred product

$$\begin{tikzcd} S \arrow[r, "\mathrm{id}"] & S \\ S \times_S S' \arrow[u, "p"] \arrow[r, "q"] & S' \arrow[u, "g"] \end{tikzcd}$$

by taking subset of $X \subseteq S$ and immediately getting open immersion $X \times_S S' \to S \times_S S'$, which we postcompose with canonical isomorphism $S \times_S S' \to S'$ and get that $X \times_S S' \to S'$ is open immersion.

Now suppose $f\colon X \to S$ is a closed immersion. So we have the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] & S \\ X \times_S S' \arrow[u, "p"] \arrow[r, "q"] & S' \arrow[u, "g"] \end{tikzcd}$$

We want to show $X \times_S S' \to S'$ is also a closed immersion. For that it satisfies to find an open covering of S' with affine subschemes such that preimages with be also affine schemes and induced maps of rings surjective.

Take $s \in S'$ and a neighborhood $g(s) \in \mathrm{Spec}(R) = U \subseteq S$. Preimage $f^{-1}(U) = \mathrm{Spec}(A)$ already is affine, since f is closed immersion, and for $g^{-1}(U)$ we have to take some smaller affine neighborhood of s. So we get $s \in \mathrm{Spec}(B) \subseteq g^{-1}(U)$. Then use previous exercise on these open sets and obtain open immersion

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_R B) \to X \times_S S'.$$

By remark at the start we have

$$\mathrm{Spec}(A \otimes_R B) = p^{-1}(\mathrm{Spec}(A)) \cap q^{-1}(\mathrm{Spec}(B)) = q^{-1}(\mathrm{Spec}(B)).$$

Only thing to argue is why the map $B \to A \otimes_R B$ is surjective. We have the following diagram

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \otimes_R B \end{tikzcd}$$

Clearly bottom arrow is also surjective, which we can check by explicit computation.

2.

Exercise 3. By definition we have to compute a fibred product of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and $\mathrm{Spec}(k(p)) \to \mathrm{Spec}(A)$ (where $k(p)$ is the residue field of $p \in \mathrm{Spec}(A)$ and $\to$ is the canonical inclusion). Since we are dealing with affine schemes, we can express it concretely as $\mathrm{Spec}(B \otimes_A k(p))$. Note that B has the structure of an A-algebra, which is induced by the starting morphism of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. So this exercise reduces to computing these tensor products.

We also observe that $k[T]$ is a PID, which means every non-zero prime ideal is a maximal ideal. This will be handy when computing residue fields, because after quotienting with a non-zero ideal we already get a field (we do not have to further take the quotient field).

1. In the first example we do now even have to calculate the tensor product, because we can rewrite $k[T,U]/(TU-1) = k[T,T^{-1}]$, so this is just a localization of $k[T]$. Morphism of spectrums, induced by inclusion into localization, is an open immersion, so fibers will be singletons if $x \in D(T)$ and empty sets otherwise. And the structure sheaf is also clear, it is just the restriction of structure sheaf $\mathcal{O}_{\mathrm{Spec}(k[T])}$.

2.

3.

4.

Exercise 4. Take $U = D(f)$ for some $f \in A$ and let $U = \cup_i D(f_i)$ be some cover. We have to check that

$$M[f^{-1}] \to \mathrm{Eq}\left[\prod_i M[f_i^{-1}] \rightrightarrows \prod_{i,j} M[(f_i f_j)^{-1}]\right]$$

is isomorphism.

This proof is exactly the same as when we proved that $\mathcal{O}_{\mathrm{Spec}(A)}$ is a sheaf, after we defined it the basis of principal opens.

Then proved that $A = \mathrm{Eq}\left[\prod_i A[f_i^{-1}] \rightrightarrows \prod_{i,j} A[(f_i f_j)^{-1}]\right]$ where $\mathrm{Spec}(A) = \cup_i D(f_i)$ is a cover.