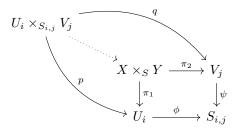
Algebraic geometry 1 Exercise sheet 6

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Exercise 1.

1. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram



Therefore, on the level of sets,

$$U_i \times_{S_{i,j}} V_j \subset X \times_S Y$$
,

but in exercise 5.2.1, we showed that this induces an open immersion as locally ringed spaces.

Now observe that

$$\bigcup_{i,j} (U_i \times_{S_{i,j}} V_j) \longrightarrow Y$$

$$\downarrow \qquad \qquad \downarrow$$

$$X \longrightarrow S$$

commutes, because $S = \bigcup_{i,j} S_{i,j}$. Now by uniqueness of the pullback,

$$\bigcup_{i,j} (U_i \times_{S_{i,j}} V_j) \cong U_i \times_{S_{i,j}} V_j.$$

I guess this is a good step in the direction of understanding why the pullback in the category of sheaves exists, right? If we assume X, Y, S to be

sheaves and $U_i, V_j, S_{i,j}$ to be affine schemes, then by the above argument we found a cover of $X \times_S Y$ by affine schemes.

2. Surjectivity follows, because a pullback of schemes in partial makes

$$\begin{array}{c|c} \mid X \times_S Y \mid & \longrightarrow \mid X \mid \\ & \downarrow^{\psi} \\ \mid Y \mid & \stackrel{\phi}{\longrightarrow} \mid S \mid \end{array}$$

commute for all ψ, ϕ .

Exercise 3. By definition we have to compute a fibred product of $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ and $\operatorname{Spec}(k(p)) \to \operatorname{Spec}(A)$ (where k(p) is the residue field of $p \in \operatorname{Spec}(A)$ and \to is the canonical inclusion). Since we are dealing with affine schemes, we can express it concretely as $\operatorname{Spec}(B \otimes_A k(p))$. Note that B has the structure of an A-algebra, which is induced by the starting morphism of schemes $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$. So this exercise reduces to computing these tensor products.

We also observe that k[T] is a PID, which means every non-zero prime ideal is a maximal ideal. This will be handy when computing residue fields, because after quotienting with a non-zero ideal we already get a field (we do not have to further take the quotient field).

- 1. In the first example we do now even have to calculate the tensor product, because we can rewrite $k[T,U]/(TU-1)=k[T,T^{-1}]$, so this is just a localization of k[T]. Morphism of spectrums, induced by inclusion into localization, is an open immersion, so fibers will be singletons if $x \in D(T)$ and empty sets otherwise. And the structure sheaf is also clear, it is just the restriction of structure sheaf $\mathcal{O}_{\text{Spec}(k[T])}$.
- 2.
- 3.
- 4.

Exercise 4. Take U = D(f) for some $f \in A$ and let $U = \bigcup_i D(f_i)$ be some cover. We have to check that

$$M[f^{-1}] \to \operatorname{Eq}\left[\prod_{i} M[f_i^{-1}] \Longrightarrow \prod_{i,j} M[(f_i f_j)^{-1}]\right]$$

is isomorphism.

This proof is exactly the same as when we proved that $\mathcal{O}_{\mathrm{Spec}(A)}$ is a sheaf,

after we defined it the basis of principal opens.

Then proved that $A = \text{Eq}\left[\prod_i A[f_i^{-1}] \rightrightarrows \prod_{i,j} A[(f_if_j)^{-1}]\right]$ where $\text{Spec}(A) = \mathbb{E}[A]$ $\cup_i D(f_i)$ is a cover.

We can simply tensor the whole diagram and, since tensor product commute with direct limits, we have that

$$M = \operatorname{Eq}\left[\prod_{i} M \otimes_{A} A[f_{i}^{-1}] \Rightarrow \prod_{i,j} M \otimes_{A} A[(f_{i}f_{j})^{-1}]\right].$$