Recall:

Defined (F^i, δ^i) universal cohomological δ-functors.

__Proposition__ If (F^i, δ^i) is erasable, then it is universal.

__Definition__: Let C be an abelian category

1) An object $I \in C$ is injective if $\mathrm{Hom}_C(-, I)$ is exact

2) C has enough injectives if for each $A \in C$ there is I injective and a monomorphism $A \longrightarrow I$.

Lemma Let $F: \mathcal{C} \to \mathcal{D}$ be an additive functor.

1) If F is erasable and I is injective then $F(I) = 0$

2) If $\mathcal{C}$ has enough injectives then F is erasable $\Leftrightarrow$ $F(I) = 0$ for all injectives.

Proof 1) Let $I \hookrightarrow B$ s.t. $F(B) = 0$

since $B = I \oplus B/I$ and F is additive

$F(I) = 0$.

2) Easy.

Theorem / **Definition**.

Let $F : \mathcal{C} \to D$ be a left-exact functor and assume that $\mathcal{C}$ has enough injectives. Then there exists a universal cohomological δ-functor extending F.

Lemma: If $\mathcal{C}$ has enough injectives then for all $A \in \mathcal{C}$ there is an injective resolution

$$0 \to A \to I^0 \xrightarrow{d_A^1} I^1 \xrightarrow{d_A^2} \cdots \to I^n \xrightarrow{d_A^{n+1}} \cdots$$

(i.e. each I^i is injective and
$$H^i(I^\bullet) = \begin{cases} A & \text{if } i=0 \\ 0 & \text{if } i \geq 1 \end{cases}$$
).

Def On objects

$$R^iF(A) = H^i(F(I^\bullet)) = \frac{\ker(F(d_A^{i+1}))}{\operatorname{im}(F(d_A^i))}.$$

For morphisms:

<u>Theorem</u> (Fundamental lemma of homological algebra)
Let $\mathcal{C}$ be an abelian category.
Given

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & I^0 \arrow[r, "d_A^1"] & I^1 \arrow[r, "d_A^2"] & I^2 \arrow[r] & \cdots \\
0 \arrow[r] & B \arrow[r, "d_B"'] & J^0 \arrow[r, "d_B^1"] & J^1 \arrow[r, "d_B^2"] & J^2 \arrow[r] & \cdots
\end{tikzcd}$$

injective resolutions there is a lift

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & I^0 \arrow[r, "d_A^1"] \arrow[d, dotted, "f^0"] & I^1 \arrow[r, "d_A^2"] \arrow[d, dotted, "f^1"] & I^2 \arrow[r] \arrow[d, dotted, "f^2"] & {} \\
0 \arrow[r] & B \arrow[r, "d_B"'] & J^0 \arrow[r, "d_B^1"] & J^1 \arrow[r, "d_B^2"] & J^2 \arrow[r] & \cdots
\end{tikzcd}$$

Moreover, every pair of lifts $(f^i)_{i=0}^{\infty}$ and $(f'^i)_{i=0}^{\infty}$ are chain homotopic.

$$\left[\ f^i - f'^i = d_B^i \circ h^i + h^{i+1} d_A^{i+1}\ \right]$$

Lemma (Horseshoe lemma) $\mathcal{C}$ abelian category with enough injectives

given

$$\begin{tikzcd} & 0 \arrow[d] \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \\ & B \arrow[d] \\ 0 \arrow[r] & C \arrow[r] \arrow[d] & K^0 \arrow[r] & K^1 \arrow[r] & \cdots \\ & 0 \end{tikzcd}$$

with I^i and K^i injective resolution one can construct commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d, dashed] & I^1 \arrow[r] \arrow[d, dashed] & \cdots \\ 0 \arrow[r] & B \arrow[r, dashed] \arrow[d] & J^0 \arrow[r, dashed] \arrow[d, dashed] & J^1 \arrow[r, dashed] \arrow[d, dashed] & \cdots \\ 0 \arrow[r] & C \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & {} \end{tikzcd}$$

with J^i injective resolution of B. and exact columns.

To make it δ-functor:

<u>Lemma</u> (Snake Lemma) Let $\mathcal{C}$ be an abelian category given

$$\begin{array}{ccccccc} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to 0 \\ & \downarrow a & & \downarrow b & & \downarrow c & \\ 0 \to & A' & \to & B' & \to & C' & \end{array}$$

with exact rows we get exact sequence

$$\ker a \to \ker b \to \ker c \xrightarrow{\partial} \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c.$$

<u>Theorem</u> Let $F : \mathcal{C} \to \mathcal{D}$ be a left-exact functor of abelian categories.

1) $$R^i F(A) := H^i(F(I^\bullet_A))$$
for an injective resolution $A \to I^\bullet_A$ can be made into an additive functor.

2) For any two injective resolutions
$$H^i(F(I^\bullet_A)) \xrightarrow{\sim} H^i(F(J^\bullet_A))$$
canonically

3) Each R^iF is erasable.

4) There exists natural maps

$$\delta^i : R^iF(C) \to R^{i+1}F(A)$$

when $0 \to A \to B \to C \to 0$ is an exact sequence making $(R^iF, \delta)_{i\geq 0}^{\infty}$ into a cohomological δ-functor extending F.

5) The data (R^iF, δ^i) is a universal cohomological δ-functor extending F.

<u>Proof</u> 1) Given $A \xrightarrow{f} B$ we get

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & I_A^0 \arrow[r] \arrow[d, "f^0"] & I_A^1 \arrow[d, "f^1"] & \cdots \\
0 \arrow[r] & B \arrow[r] & I_B^0 \arrow[r] & I_B^1 &
\end{tikzcd}$$

this yields a morphism

$$\begin{tikzcd}
R^iF(A) \arrow[r, "R^iF(f^i)"] \arrow[d, equal] & R^iF(B) \arrow[d, equal] \\
H^i(F(I_A^i)) \arrow[r, "H^i(F(f^i))"'] & H^i(F(I_B^i))
\end{tikzcd}$$

the arrow $\mathcal{F}(f^i)$ depends on f^i but $H^i(\mathcal{F}(f^i))$ doesn't. Since $f^i \underset{h}{\sim} f'^i$ are homotopic so $\mathcal{F}(f^i) \overset{\mathcal{F}(h)}{\sim} \mathcal{F}(f'^i)$ are homotopic.

One can check $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ and that $\mathcal{F}$ is additive.

2)

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I_A^0 \arrow[r] \arrow[d, "f^0"] & \cdots \\
& A \arrow[r] \arrow[d, equal] & J_A^0 \arrow[r] \arrow[d, "g^0"] & {} \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I_A^0 \arrow[r] \arrow[d, "f^0"] & \cdots \\
0 \arrow[r] & A \arrow[r] & J_A^0 \arrow[r] & \cdots
\end{tikzcd}
$$

$g^i \circ f^i \sim id_{I^i}$ $\qquad$ $f^i \circ g^i \sim id_{J^i}$

so $H^i(\mathcal{F}(f^i))$ $\qquad$ $H^i(\mathcal{F}(g^i))$

are inverses to each other.

3) suffices to show $i \geq 1$ $R^i\mathcal{F}(I) = 0$ for I injective.

But $0 \to I \xrightarrow{id} I \to 0$ is an injective resolution.

4) Given $0 \to A \to B \to C \to 0$ we make

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & I_A^0 \arrow[r] \arrow[d] & I_A^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B \arrow[r] \arrow[d] & I_B^0 \arrow[r] \arrow[d] & I_B^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & C \arrow[r] \arrow[d] & I_C^0 \arrow[r] \arrow[d] & I_C^1 \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The columns are split exact except possibly the first one.

Applying F also gives exact sequence of complexes

$$0 \to F(I_A^\bullet) \to F(I_B^\bullet) \to F(I_C^\bullet) \to 0$$

Snake lemma gives

$$0 \to H^0(F(I_A^\bullet)) \to H^0(F(I_B^\bullet)) \to H^0(F(I_C^\bullet)) \xrightarrow{\delta} H^1(F(I_A^\bullet))$$

$\cdots$

Example:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(I^0_A) \arrow[r] \arrow[d, "\mathcal{F}(d^1)"] & \mathcal{F}(I^0_B) \arrow[r] \arrow[d, "\mathcal{F}(d^1)"] & \mathcal{F}(I^0_C) \arrow[r] \arrow[d, "\mathcal{F}(d^1)"] & 0 \\
0 \arrow[r] & \mathcal{F}(I^1_A) \arrow[r] & \mathcal{F}(I^1_B) \arrow[r] & \mathcal{F}(I^1_C) \arrow[r] & 0
\end{tikzcd}
$$

use snake lemma

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(I^0_A) \arrow[r] \arrow[d] & \mathcal{F}(I^0_B) \arrow[r] \arrow[d] & \mathcal{F}(I^0_C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \mathcal{F}(d^2_A) \arrow[r] & \ker \mathcal{F}(d^2_B) \arrow[r] & \ker \mathcal{F}(d^2_C) \arrow[r, "\delta"'] & \operatorname{coker} \mathcal{F} d^2_A
\end{tikzcd}
$$

use snake lemma

$$
0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \xrightarrow{\delta} R^1\mathcal{F}(A) \to R^1\mathcal{F}(B) \to R^1\mathcal{F}(C)
$$

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(I^1_A) \arrow[r] \arrow[d, "\mathcal{F}(d^2)"] & \mathcal{F}(I^1_B) \arrow[r] \arrow[d] & \mathcal{F}(I^1_C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}(I^2_A) \arrow[r] & \mathcal{F}(I^2_B) \arrow[r] & \mathcal{F}(I^2_C) \arrow[r] & 0
\end{tikzcd}
$$

snake lemma twice.

$$
\begin{tikzcd}
& \operatorname{coker} \mathcal{F}(d^1_A) \arrow[r] \arrow[d] & \operatorname{coker} \mathcal{F}(d^1_B) \arrow[r] \arrow[d] & \operatorname{coker}(\mathcal{F}(d^1_c)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \mathcal{F}(d^3_A) \arrow[r] & \ker \mathcal{F}(d^3_B) \arrow[r] & \ker(\mathcal{F} d^3_C) &
\end{tikzcd}
$$

5) Since $(R^i F, \delta^i)$ is erasable then it is universal.

<u>Theorem</u> Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category of $\mathcal{O}_X$-modules has enough injectives.

<u>Example</u> a) $\mathcal{O}_X = \underline{\mathbb{Z}}$

then $\mathrm{Mod}(X, \mathcal{O}_X) = \mathrm{Shv}_{Ab}$.

b) $X = *$

then

$\mathrm{Mod}(X, \mathcal{O}_X) = \mathrm{Mod}_R$

<u>Sketch</u> - Mod_R exercise 4 hwk 1

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and $x \in X$ a point.

we have a map

$$j_x : (\{x\}, \mathcal{O}_{X,x}) \longrightarrow (X, \mathcal{O}_X)$$

of ringed spaces

Choose an injection
$$\mathcal{F}_x \overset{\alpha_x}{\hookrightarrow} I_x \quad \text{with} \quad I_x$$
injective $\mathcal{O}_{X,x}$-module.

Set $\quad \mathcal{I} = \prod_{x \in X} (j_x)_* I_x$

then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) \simeq \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, (j_x)_* \mathcal{I}_x) \simeq \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{I}_x)$$

the family $\{\alpha_x\}_{x \in X}$ determines an injective map $\mathcal{F} \hookrightarrow \mathcal{I}$

Moreover, $\mathcal{I}$ is injective.

since j_x^* is exact and each $\mathcal{I}_x$ is injective.

<u>Definition</u> Let $(X, \mathcal{O}_X)$ be a ringed space and $R = \Gamma(X, \mathcal{O}_X)$.

The i-th cohomology functor
$$H^i(X, -) : \mathrm{Mod}_{(X, \mathcal{O}_X)} \to R\text{-modules}$$
is the right derived functor of $\Gamma(X, -)$.

__Definition__ Let $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ bee a left exact functor. Suppose that $\mathcal{C}$ has enough injectives. An object $B \in \mathcal{C}$ is called __$\mathcal{F}$-acyclic__ if

$$R^i\mathcal{F}(B) = 0 \quad \forall\, i \geq 1.$$

__Lemma__ Let $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ bee left exact and suppose that

$$0 \to A \to B^0 \to B^1 \to \cdots$$

is a resolution by $\mathcal{F}$-acyclic objects

then $R^i\mathcal{F}(A) \cong H^i(\mathcal{F}(B^\bullet)) \quad \forall\, i.$

__Proof__ Write

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & A \arrow[r] & B^0 \arrow[rr] \arrow[dr] & & B^1 \arrow[rr] \arrow[dr] & & B^2 \\
& & 0 \arrow[r] & A^0 \arrow[ur] \arrow[r] & 0 \quad 0 \arrow[r] & A^1 \arrow[ur] \arrow[r] & 0
\end{tikzcd}$$

with $0 \to A^{i-1} \to B^i \to A^i \to 0$ exact etc.

then

$$R^1\mathcal{F}(A) \cong \operatorname{Coker}\left(\mathcal{F}(B^0) \to \mathcal{F}(A^0)\right)$$

since $R^1\mathcal{F}(B) = 0$

but $\mathcal{F}(A^0) = \operatorname{Ker}\left(\mathcal{F}(B^1) \to \mathcal{F}(B^2)\right)$

so $R^1\mathcal{F}(A) \cong H^1\left(\mathcal{F}(B^\bullet)\right)$

for $i \geq 2$

$$R^i\mathcal{F}(A) \cong R^{i-1}\mathcal{F}(A^0) \cong R^{i-2}\mathcal{F}(A^1) \dots \cong R^1\mathcal{F}(A^{i-2})$$

and $R^1\mathcal{F}(A^{i-2}) \cong H^i\left(\mathcal{F}(B^\bullet)\right)$

<u>Definition</u> A sheaf in $\mathrm{Mod}(X, \mathcal{O}_X)$ is Flasque if the transition maps are surjective.