Algebraic geometry 1
Exercise sheet 5

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Exercise 1.

We make a pushout of the diagram $U_1 \leftarrow V_1 \rightarrow U_2$, where $V_1 \rightarrow U_1$ is the inclusion and $V_1 \rightarrow U_2$ and composition of φ and inclusion.

Let X be the pushout in terms of topological spaces and let $\alpha_1 \colon U_1 \rightarrow X$ and $\alpha_2 \colon U_2 \rightarrow X$ be the associated morphisms.

We define a sheaf $\mathcal{O}_X$ in the following way. Take an open subset $Z \subseteq X$. Then $Z \cap \alpha_1(U_1) = Z_1$ and $Z \cap \alpha_2(U_2) = Z_2$ are an open cover of Z in X. Then let

1. Define

$$X := U_1 \coprod U_2 /\sim,$$

where $x \sim y$ if $x = \varphi(y)$ and for $i \in \{1, 2\}$

$$\begin{aligned} \pi_i \colon U_i &\rightarrow X \\ x &\mapsto \bar{x}. \end{aligned}$$

We can now give X the topology by defining a subset $U \subset X$ to be open if $\pi_i^{-1}(U) \in$ are open in U_i.

We basically take X to be the pushout of $U_1 \leftarrow V_1 \rightarrow U_2$, where $V_1 \rightarrow U_1$ is the inclusion and $V_1 \rightarrow U_2$ and composition of φ and inclusion.

Notice, that π_i are homeomorphic onto open subsets of X. This will become important later. Next we want to define a structure sheaf on X that behaves well with restricting to U_i.

For $U \subset X$ open, let

$$\begin{aligned} \mathcal{O}_X(U) := \ker(\mathcal{O}_{U_1}(\pi^{-1}(U)) \oplus \mathcal{O}_{U_2}(\pi^{-1}(U)) &\rightarrow \mathcal{O}_{U_1}(\pi^{-1}(U) \cap U_1) \\ (x, y) &\mapsto x_{|\pi^{-1}(U) \cap V_1} - \varphi^\sharp(\pi_2^{-1}(U) \cap V_2)(y_{|\pi_2^{-1}(U) \cap V_2})), \end{aligned}$$

where the substraction in the above term comes from the group structure of $\mathcal{O}_{U_1}(\pi^{-1}(U) \cap V_1)$. This is of course a group again, as the kernel of a ring map.

We conclude, that $(X, \mathcal{O}_X)$ is a scheme, because $X = \pi_1(U_1) \cup \pi_2(U_2)$ can be covered by affine schemes using the cover from U_1 and U_2 and since by construction of the structure sheaf $\mathcal{O}_{X|U_1} = \mathcal{O}_{x_i}$. Here we finally used, as promised, that π_i are homeomorphisms onto open subsets of X.

Exercise 2.

1. Take two isomorphic open immersions $(Z, \mathcal{O}_Z)$ and $(W, \mathcal{O}_W)$ as schemes over $(Y, \mathcal{O}_Y)$. So we have a commutative diagram

$$\begin{tikzcd} (Z,\mathcal{O}_Z) \arrow[r, hook] \arrow[d, "\cong"] & (Y,\mathcal{O}_Y) \\ (W,\mathcal{O}_W) \arrow[ur, hook] & \end{tikzcd}$$

   from which we get a diagram of topological spaces

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\cong"] & Y \\ W \arrow[ur, hook] & \end{tikzcd}$$

   from which it clearly follows that Z and W must be equal as sets.

   For the other way, we want to show that for every open $Z \subseteq Y$ there is a unique sheaf $\mathcal{O}_Z$ for which $(\varphi, \varphi^{\#})\colon (Z, \mathcal{O}_Z) \hookrightarrow (Y, \mathcal{O}_Y)$ is an open embedding. Take any two sheaves $\mathcal{O}_Z$ and $\mathcal{O}'_Z$ on Z for which $(\mu, \mu^{\#})\colon (Z, \mathcal{O}'_Z) \hookrightarrow (Y, \mathcal{O}_Y)$ is also open embedding. Then by definition of an open embedding we have isomorphisms $\mu^{-1}\mathcal{O}_Y \to \mathcal{O}_Z$ and $\varphi^{-1}\mathcal{O}_Y \to \mathcal{O}'_Z$. But $\varphi^{-1}\mathcal{O}_Y$ and $\mu^{-1}\mathcal{O}_Y$ are the same, since $\varphi = \mu$, so $\mathcal{O}'_Z \cong \mathcal{O}_Z$. As for the existence: there clearly exists such a sheaf $\mathcal{O}_Z$ simply by taking a restriction $\mathcal{O}_Y \mid_Z$. But (as it says in Davies/Scholze notes) it is not obvious. We have to show that we can cover Z with open subsets, where each of them is isomorphic to an affine scheme. Let $Y = \cup_i Y_i$, where $Y_i \cong \operatorname{Spec} B_i$. Then for every point $x \in Z$ we choose i such that $x \in Y_i \cap Z$. That means there exists some $f \in B_i$ such that $x \in D_{Y_i}(f) \subseteq V_i \cap U$. Since $D_{Y_i}(f) \cong B_[f^{-1}]$, we found a neighborhood of $x \in Z$ that is isomorphic to an affince scheme. We can do that for every $x \in Z$ and thus cover it. So Z is itself a scheme.

Exercise 3.

1. Remember, that the contravariant functor $A \mapsto (Spec(A), \mathcal{O}_{Spec(A)})$ is an equivalence of categories, meaning that $\mathcal{O}^{op}$ is equivalent to $Rings$. Then the affine case follows from the Yoneda embedding.
   How to show general case?

   We want to be in the situation of the second part of this exercise. We choose $\mathcal{C} = Psh, \mathcal{D} = \mathcal{E} = Sh.$ and $\mathcal{F}$ is sheafification, $\tilde{\mathcal{F}}$ is pullback along f, with $\mathcal{G}$ just being inclusion and $\tilde{\mathcal{G}}$ being the pushforeward along f. So we can conclude that the concatenation of $\mathcal{F} \circ \tilde{\mathcal{F}} \dashv \mathcal{G} \circ \tilde{\mathcal{G}}$. Now the claim follows using the first part of the exercise.

Exercise 4.

1. Let $F\colon C \to D$ be a functor with adjoints $G, G'\colon D \to C$. By the definition of adjointness, for every arrow $f\colon Fc \to d$ we have unique arrows $\phi f\colon c \to Gd$ and $\mu f\colon c \to G'd$, such that ϕ and μ are natural. In this case take some $d \in D$ and $c = Gd$. Then we have a unique arrow $Gd \to G'd$.

   We just have to show this is natural in d, so pick some other $e \in D$ and $FGe \to e$. Same as before we get an arrow $Gb \to G'b$. Using adjointness we have a commutative diagram

$$\begin{array}{ccc} FGa & \longrightarrow & a \\ \downarrow & & \downarrow \\ FGb & \longrightarrow & b \end{array}$$

   Then, using the naturality of μ gives that

$$\mu(FGa \to a \to b) = Ga \to G'a \to G'b$$

   and

$$\mu(FGa \to FGb \to b) = Ga \to Gb \to G'b$$

   Which proves that $a \mapsto (Ga \to G'a)$ is natural. We could easily construct an inverse $a \mapsto (G'a \to Ga)$ which would compose to identity.

2.