<u>Proposition</u> Fix maps of rings

$$R \to A \to B \quad \text{and} \quad R \to S. \quad \text{Then:}$$

1) $\Omega^1_{A \otimes_R S/S} \simeq \Omega^1_{A/R} \otimes_R S$

as $A \otimes_R S$ - modules

2) The sequence

$$\Omega^1_{A/R} \otimes B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

is exact.

3) If $A \twoheadrightarrow B$ is exact with kernel I then

$$I \otimes_A B = I/I^2 \xrightarrow{\delta} \Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} = 0$$

is exact.

4) If $A = R[x_i]_{i \in I}$ then

$$\bigoplus_i A \cdot dx_i \longrightarrow \Omega^1_{A/R}$$

is an isomorphism.

(Proof of proposition).

Let $A_S = A \otimes_R S$

1) $$\mathrm{Hom}_{A_S}(\Omega^1_{A_S/S}, M) = \mathrm{Der}_S(A_S, M)$$

$$= \mathrm{Der}_R(A, \mathrm{Forg}_A^{A_S}(M))$$

$$= \mathrm{Hom}_A(\Omega^1_{A/R}, \mathrm{Forg}_A^{A_S}(M))$$

$$= \mathrm{Hom}_A(\Omega^1_{A/R} \otimes_A A_S, M)$$

but $\Omega^1_{A/R} \otimes_A (A \otimes_R S) = \Omega^1_{A/R} \otimes_R S$

2)
$$\begin{tikzcd}
\mathrm{Rel}_{A/R} \otimes B \arrow[r] \arrow[d] & \mathrm{Rel}_{B/R} \arrow[r] \arrow[d] & \mathrm{Rel}_{B/A} \arrow[r] \arrow[d] & 0 \\
\bigoplus_{g \in A} B \cdot dg \arrow[r, hook] \arrow[d] & \bigoplus_{f \in B} B \cdot df \arrow[r] \arrow[d] & \bigoplus_{f \in B} B \cdot df \arrow[d, two heads] & \\
\Omega^1_{A/R} \otimes B \arrow[r] & \Omega^1_{B/R} \arrow[r] & \Omega^1_{B/A} \arrow[r] & 0
\end{tikzcd}$$

From here we see that it is a complex and right exact.

$\Omega^1_{B/A}$ is the quotient of $\Omega^1_{B/R}$ obtained by forcing $da = 0$ for all $a \in A$. This is precisely the image of $\Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R}$.

3) If $0 \to I \to A \twoheadrightarrow B \to 0$ then $db = 0$ for all $b \in B$.

$$B \otimes_A \Omega^1_{A/R} \longrightarrow \Omega^1_{B/R} \longrightarrow 0 = \Omega^1_{B/A}$$

Regard $I/I^2 \simeq B \otimes_A I$

then $\delta := id_B \otimes_A d$

Although $d : \underset{A}{\overset{\cup}{I}} \longrightarrow \Omega^1_{A/R}$ is only

R-linear $\delta = id_B \otimes_A \delta$ is B-linear.

Indeed, $d(ai) = a\,di + i\,da$ but $i\,da = 0$.

We also have $da = da'$ in

$\Omega^1_{B/R}$ if $a' = a + i$

$$
\begin{tikzcd}
& \bigoplus_{a\in A,\, i\in I} da - d(a+i) \arrow[d, equal] & \\
& C_1 \arrow[r, two heads] \arrow[d] & C_2 \arrow[d] \\
\mathrm{Rel}_{A/R}\otimes B \arrow[r] \arrow[d] & \bigoplus_{a\in A} da\cdot B \arrow[r] \arrow[d] & \Omega^1_{A/R}\otimes_A B \arrow[r] \arrow[d] & 0 \\
\mathrm{Rel}_{B/R} \arrow[r] & \bigoplus_{b\in B} db\cdot B \arrow[r, two heads] & \Omega^1_{B/R} \arrow[r] & 0
\end{tikzcd}
$$

C_1 and C_2 cokernels.

every element in the image

of $C_1 \to C_2$ is also

in the image of $I/I^2 \xrightarrow{\delta} C_2$

since $da - d(a+i) = di$ in $\Omega^1_{A/R}\otimes_A B$.

4) Every polynomial

$$f(x_j)_{j \in I}$$

will satisfy

$$\partial f = \sum_{i \in J}' \frac{\partial f}{\partial x_i} \, \partial x_i$$

so the $\{\partial x_i\}$ generate $\Omega^1_{R[x_i]_I / R}$

Moreover, $\partial : R[x_i]_{i \in I} \longrightarrow \bigoplus_I \partial x_i \, R[x_i]$

$$f \longmapsto \sum_{i \in J}' \frac{\partial f}{\partial x_i} \, \partial x_i$$

is a derivation

i.e. we have

$$\begin{tikzcd} \bigoplus_I R[x_i] \cdot \partial x_i \arrow[r, two heads] \arrow[rr, bend right, "\mathrm{id}"'] & \Omega^1_{R[x_i]/R} \arrow[r, two heads] & \bigoplus R[x_i] \partial x_i \end{tikzcd}$$

So it is an isomorphism.

Derivations and liftings

Lemma Consider the commutative diagram of rings

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B & \xrightarrow[\pi]{} & B/I \end{array}$$

where $I^2 = 0$.

1) Given two lifts $\phi_1, \phi_2 : A \to B$ of f then $\delta := \phi_1 - \phi_2$ is an R-linear derivation.

2) Given $\phi : A \to B$ a lift of f and $\delta : A \to I$ an R-linear derivation, then $\phi + \delta : A \to B$ is a lift of f.

3) $\mathrm{Der}_R(A, I) \circlearrowright \mathrm{Lift}(f)$ acts freely transitively, when $\mathrm{Lift}(f) \neq \emptyset$. (pseudo-torsor).

Proof

1) $\delta(ab) = \phi_1(ab) - \phi_2(ab)$

$$= \phi_1(a)\left[\phi_1(b) - \phi_2(b)\right) + \phi_2(b)\left(\phi_1(a) - \phi_2(a)\right)$$

$$= \phi_1(a)\,\delta(b) + \phi_2(b)\,\delta(a)$$

The A-action on I is through ϕ_1, ϕ_2, but this factors through B/I where $\phi_1 = \phi_2 = f$ so

$$= a \cdot \delta(b) + b \cdot \delta(a).$$

Clearly R-linear.

2) $(\phi + \delta)(ab) = \phi(a)\phi(b) + a\delta(b) + b\delta(a) + \overset{0}{(\delta(a)\cdot\delta(b))}$

$$= (\phi+\delta)(a) \cdot (\phi+\delta)(b)$$

<u>Corollary</u>: Let M be an A-module and consider the split f.o.t. $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A[M]$ then $\operatorname{Der}_R(A, M) \simeq \operatorname{Lift}(C)$

with $C =$

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, equal] \arrow[d] & \operatorname{Spec} A \arrow[d] \\
\operatorname{Spec} A[M] \arrow[r] \arrow[ur, "can"] & \operatorname{Spec} R
\end{tikzcd}$$

Corollary: A map of schemes $\operatorname{Spec} A \to \operatorname{Spec} R$ is formally unramified if and only if $\Omega^1_{A/R} = 0$

proof

$$C = \begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec}(A \oplus M) \arrow[r] \arrow[ur, dashed, "f"] & \operatorname{Spec} R \end{tikzcd}$$

with lift f. Then uniqueness of $f \Rightarrow \operatorname{Der}_R(A, M) = 0 \ \forall M$.

i.e. $\operatorname{Hom}_A(\Omega^1_{A/R}, M) = 0 \ \forall M$.

Kähler differentials as a diagonal:

Proposition $X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$ and let $\Delta_{X/S}: X \to X \times_S X$ be the diagonal map. $\mu: A \otimes_R A \to A$. Let $I = \operatorname{Ker}(\mu)$, then $\Omega^1_{A/R} = \Delta_{X/S}^* \tilde{I}(X) = I/I^2$

<u>Proof</u> Consider the derivation

$\delta : A \longrightarrow I/I^2$ given as

$\delta(a) = 1 \otimes a - a \otimes 1.$

$\delta(ab) = (1 \otimes ab - a \otimes b) + (a \otimes b - ab \otimes 1)$

$= b\,\delta(a) + a\,\delta(b)$

because on I/I^2 $a \cdot i = (a \otimes 1) i = (1 \otimes a) i$

This gives a map $\Omega^1_{A/R} \longrightarrow I/I^2$ with

$da \longmapsto 1 \otimes a - a \otimes 1.$

To construct the inverse consider

$A \oplus \Omega^1_{A/R} \varepsilon$ the split f.o.t.

Consider maps $\phi_1, \phi_2 : A \to A \oplus \Omega^1_{A/R} \varepsilon$

$\phi_1 : a \longmapsto a + 0 \cdot \varepsilon$

$\phi_2 : b \longmapsto b + db \cdot \varepsilon$

$\phi_1 \otimes \phi_2 : A \otimes_R A \to A \oplus \Omega^1_{A/R}$

$a \otimes b \longmapsto ab + a\,db\,\varepsilon$

$$\Phi_1 \otimes \Phi_2 (1 \otimes a - a \otimes 1) = da$$

providing the inverse.

<u>Proposition</u> Let $f: R \to A$ be a map of rings. $S \subseteq A$ a multiplicative subset, then the map

$$\Omega^1_{A/R} \otimes_R A[S^{-1}] \to \Omega^1_{A[S^{-1}]/R}$$ is

an isomorphism.

<u>Proof</u> Let $I = \ker(\mu: A \otimes_R A \to A)$ and $J = \ker(\mu: A[S^{-1}] \otimes_R A[S^{-1}] \to A[S^{-1}])$, then

$$\begin{tikzcd}
\Omega^1_{A/R} \otimes_R A[S^{-1}] \arrow[r] \arrow[d, "\simeq"'] & \Omega^1_{A[S^{-1}]/R} \arrow[d, "\simeq"] \\
I/I^2 \otimes_R A[S^{-1}] \arrow[r, "\simeq"'] & J/J^2
\end{tikzcd}$$

Definition Let $X \to S$ be a map of schemes, recall that $\Delta_{X/S} : X \to X \times_S X$ is a locally closed immersion, so there is $X \underset{\substack{\Delta^u \\ \text{closed}}}{\subseteq} U \underset{\text{open}}{\subseteq} X \times_S X$.

We define $\Omega^1_{X/S} := (\Delta^u_{X/S})^*(\mathcal{I})$

where $\mathcal{I} = \ker(\mathcal{O}_U \to \Delta^u_* \mathcal{O}_X)$.

Proposition If we have a commutative square

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, phantom, "\text{open } \cap"] & \operatorname{Spec} R \arrow[d, phantom, "\cap \text{ open}"] \\ X \arrow[r] & S \end{tikzcd}$$

then $\Omega^1_{X/S}(\operatorname{Spec} A) \cong \Omega^1_{A/R}$

Proof we have a Cartesian diagram.

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec}(A \otimes_R A) \arrow[d, phantom, "\cap \text{ open}"] \\ X \arrow[r] & X \times_S X \end{tikzcd}$$

Proposition Let $f: X \to Y$, $g: Y \to S$ and $h: T \to S$ be map of schemes. The followings hold:

1) $h^* \Omega^1_{Y/S} \simeq \Omega^1_{X_T/T}$ (basechange)

2) $$f^* \Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$
is right exact.

3) If f is a closed immersion with $I = \ker(\mathcal{O}_Y \to f_* \mathcal{O}_X)$ then
$$f^* I \xrightarrow{d} f^* \Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0$$ is exact.

Proof All can be checked on affines.

Proposition Suppose that $f: X \to Y$ is locally of finite type, then it is unramified **iff** $\Delta_{X/Y}: X \to X \times_Y X$ is an open immersion

Proof $\Leftarrow$] Since $\Omega^1_{X/Y} = \Delta^*_{X/Y} I$ and $\Delta_{X/Y}$ is an open immersion then $I = 0$ and $\Omega^1_{X/Y} = 0$.

$\Rightarrow$] Since f is loc. finite type then $\Delta_{X/Y}$ is loc finitely presented.

If $A = R[a_1 \ldots a_n]/I$ then $I = \ker(\mu: A \otimes_R A \to A)$ is generated by $(a_i \otimes 1 - 1 \otimes a_i)$.

WLOG $X = \operatorname{Spec} A$ $Y = \operatorname{Spec} B$ and let $C = A \otimes_B A$, we have an exact sequence

$$0 \to I \to C \to A \to 0$$

with I f.g. C-module.

Since $\left(I/I^2\right)_x = 0$ $\forall$ $x \in \operatorname{Spec} A$

then by Nakayama $I_x = 0$,

so A_x is flat over

C_x. But flat finitely presented

closed immersions are open.