

# Algebraic geometry 1

## Exercise sheet 10

Solutions by: Eric Rudolph and David Čadež

19. Dezember 2023

### Exercise 1.

1. Since  $X$  is closed and irreducible, it is of the form  $X = \overline{\{p_0\}}$  for some (Eric thinks unique)  $p_0 \in \mathbb{A}_k^n$ . That means  $X \cong \text{Spec}(k[x_1, \dots, x_n]/p_0)$ . Denote  $A = k[x_1, \dots, x_n]/p_0$ .

By assumption there is a chain of specializations  $p_0 \subset \dots \subset p_d$  inside  $X$ .

Let  $Z \subseteq X \cap V(f_1, \dots, f_r)$  be a irreducible component. Thus it is the closure of a minimal prime ideal in  $A/(f_1, \dots, f_r)$ .

By Krull's principal ideal theorem we have  $\dim(A/(f_1, \dots, f_r)) \geq d - r$ .

Denote minimal prime ideals in  $A/(f_1, \dots, f_r)$  with  $q_1, \dots, q_l$ .

(Eric thinks that there is a unique generic point here again, since  $X$  is sober, so there should only be one of these prime ideals, right?)

We argue that

$$\dim(A/(f_1, \dots, f_r)/q_j) = \dim(A/(f_1, \dots, f_r)).$$

for any  $j$ .

That follows from  $A$  being catenary. If there existed a maximal chain in  $A/(f_1, \dots, f_r)$  that starts at  $q_j$  we could simply extend it below to get a maximal chain in  $A$ . Since all maximal chains in  $A$  are of the same length, we get that all maximal chains in  $A/(f_1, \dots, f_r)$  are also of the same length.

Since  $Z$  is an irreducible component, we have  $Z = \overline{\{q_i\}} \subseteq \text{Spec}(A/(f_1, \dots, f_r))$ .

Therefore any maximal chain in  $Z$  is exactly as long as the longest chain in  $A/(f_1, \dots, f_r)$ . And the longest chain in  $A/(f_1, \dots, f_r)$  is at least of length  $d - r$ .

2. The diagonal  $\Delta \subseteq \mathbb{A}_k^n \times \mathbb{A}_k^n$  can be defined as  $V(x_i \otimes x_i \mid i = 1, \dots, n) \subseteq \text{Spec}(k[x_1, \dots, x_n] \otimes_{\mathbb{Z}} k[x_1, \dots, x_n])$ .

(Should there not be a minus instead of  $\otimes$  in the above expression?)

Using exercise above we get that any irreducible component of  $X \cap Y \cong (X \times Y) \cap V(x_i \otimes x_i \mid i = 1, \dots, n)$  has dimension at least  $d + e - n$ .

3. Let  $\tilde{X} = \overline{f^{-1}(X)}$  and  $\tilde{Y} = \overline{f^{-1}(Y)}$  as in the hint.

We have  $\dim(\tilde{X}) = d + 1$  and  $\dim(\tilde{Y}) = e + 1$ . By the previous exercise we have  $\dim(\tilde{X} \cap \tilde{Y}) \geq d + 1 + e + 1 - (n + 1) = (d + e - n) + 1 \geq 1$ .

Therefore there exists  $0 \neq x \in \tilde{X} \cap \tilde{Y}$ .

Questions from Eric:

Why is  $\tilde{X}$  irreducible (to be able to use part 2) and why does the dimension increase by 1 when we go to affine space?

## Exercise 2.

1. It is enough to show that there exists a cover  $X = \cup_i \text{Spec}(A_i)$  of  $X$  by open affines such that  $f^{-1}(\text{Spec}(A_i))$  is affine for all  $i$ . Therefore it is enough to show that the hint holds, since  $X = \cup_{x \in X} U_x$ , where each  $U_x$  is an open affine with  $x \in U_x$ .

Take some  $x \in X$ . If  $x \notin f(Y)$  then by continuity and since  $f(X)$  is closed, there exists an affine  $U_x$  that is disjoint from  $f(Y)$ . In this case  $f^{-1}(U_x) = \emptyset = D(1)$  the preimage under  $f$  is of affine.

If  $x \in f(Y)$ , then  $x \in \text{Spec}(A_k)$  for some  $k$ . Now choose some principal open  $D(g) \subset Z$  with  $f^{-1}(x) \in D(g)$ .

We can now take a principal open  $D(g') \subset f(D(g))$  such that  $D(g') \subset U_k$  and  $x \in D(g')$ . Then we can show similarly to exercise 4.1 on sheet 8 that

$$f^{-1}(D(g'))$$

is a principal open again, so in particular affine.

v2

Take  $x \in |X|$ . If  $x \notin f(|Y|)$ , we can find an open  $U_x$  such that  $f^{-1}(U_x) = \emptyset$ . So assume  $x \in f(|Y|)$ . Then look at  $f^{-1}(x)$ . Take an open affine  $V_x \subseteq |Y|$  with  $f^{-1}(x) \in V_x$ . Since  $f$  is homeomorphism on its image, we have can take  $U_x = f(V_x)$  an affine neighborhood of  $x$  such that  $f^{-1}(U_x) = V_x$  is affine.

2. Assume that  $f$  is universally closed. We want to show that  $f$  is integral, surjective and universally injective. By the first part of this exercise we get that  $f$  is affine. It says on The Stacks project that maps that are affine and universally closed are also integral. The other two properties follow immediately from the assumptions.

On the other hand, assume that  $f$  is integral, surjective and universally injective. We now from algebra 1 that integral maps are closed and we

learned in this course that the property of a morphism being integral is stable under base change. If you put these two facts together you get that integral maps are universally closed.

We also know from The Stacks project that the property of a map of schemes being surjective is stable under base change, so  $f$  surjective already implies  $f$  univversally surjective. All in all, we get that  $f$  is univversally bijective.

### Exercise 3.

1. We have a map  $k \rightarrow \Gamma(X, \mathcal{O}_X)$ . For any  $f \in \Gamma(X, \mathcal{O}_X)$  we can define  $k[x] \rightarrow \Gamma(X, \mathcal{O}_X)$  by  $x \mapsto f$ .

Showing that  $g(X)$  does not contain the generic point of  $\mathbb{A}_k^1$  is equivalent to showing that  $k[x] \rightarrow \Gamma(X, \mathcal{O}_X)$  is not injective.

We have a composition  $k \rightarrow k[x] \rightarrow \Gamma(X, \mathcal{O}_X)$ . So also  $X \rightarrow \mathbb{A}_k^1 \rightarrow \text{Spec}(k)$ .

Map  $X \rightarrow \text{Spec}(k)$  is proper.

Map  $\mathbb{A}_k^1 \rightarrow \text{Spec}(k)$  is separated, since it is a map of affine schemes. (Follows from the fact that  $k[x] \otimes_k k[x] \rightarrow k[x]$  is surjective, and thus  $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1 \times_k \mathbb{A}_k^1$  a closed immersion.)

Thus by the proposition from the lectures, the map  $g: X \rightarrow \mathbb{A}_k^1$  is proper. In particular it is closed. Since  $X$  is connected, the image  $g(X)$  must be connected as well.

Using the hint, we can postcompose to obtain  $X \rightarrow \mathbb{A}_k^1 \rightarrow \mathbb{P}_k^1$ . Now the conclusion should be that the image of  $X$  in  $\mathbb{P}_k^1$  is also closed. Since  $\mathbb{A}_k^1 \subseteq \mathbb{P}_k^1$  is not closed, the image of  $X$  in  $\mathbb{A}_k^1$  can also not be closed. Therefore it must be a single point.

Since we did not exactly understand why should  $X \rightarrow \mathbb{P}_k^1$  be closed, we decided to rather show that  $X \rightarrow \mathbb{A}_k^1$  cannot be surjective, as that would imply  $\mathbb{A}_k^1$  being universally closed over  $\text{Spec}(k)$  (which we've shown during the lectures to be false).

Instead of doing it abstractly, we can show that  $X \rightarrow \mathbb{A}_k^1$  being surjective would imply  $\mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1$  being closed.

By the universal property of  $\mathbb{A}_k^2$  we get a map  $X \rightarrow \mathbb{A}_k^2$ , induced by  $X \rightarrow \mathbb{A}_k^1$ . So we have a map  $X \rightarrow \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1$ . Denote  $\alpha: X \rightarrow \mathbb{A}_k^2$  and  $\beta: \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1$ . If  $\alpha$  would be surjective, then for any  $U \subseteq \mathbb{A}_k^2$  we would have  $(\beta \circ \alpha)(\alpha^{-1}(U)) = \beta(U)$ . Since  $\beta \circ \alpha$  is closed by assumption, this would prove that  $\beta$  is closed. That is not true, so  $\beta \circ \alpha$  is not surjective.

We've shown that the image of  $X \rightarrow \mathbb{A}_k^1$  is a single point. Since this point is closed, it is not the generic point. This shows that  $k[x] \rightarrow \Gamma(X, \mathcal{O}_X)$  induced by  $f \in \Gamma(X, \mathcal{O}_X)$  is not injective.

2. We have a map  $k \rightarrow \Gamma(X, \mathcal{O}_X)$ . It cannot be 0, since  $X$  is locally finite type over  $\text{Spec}(k)$ . So it is injective.

It is also surjective, since for any  $f \in \Gamma(X, \mathcal{O}_X)$  the map  $k[x] \rightarrow \Gamma(X, \mathcal{O}_X)$  defined by  $x \mapsto f$  is not injective. Therefore  $k \cong \Gamma(X, \mathcal{O}_X)$ .

**Exercise 4.**

1. Steps:

- We use equalizers to show that the scheme  $\text{im}(f)$  exists.
- Using composition  $X \rightarrow Z \rightarrow S$  we get that ideal sheaf of  $\text{im}(f)$  is indeed contained in the kernel of  $f^\# : \mathcal{O}_S \rightarrow f_*(\mathcal{O}_X)$ .
- Show that ideal sheaf of a closed immersion is indeed a quasi-coherent sheaf.
- Show that every quasi-coherent ideal sheaf  $I \subseteq \ker(f^\#)$  induces a closed immersion  $Z_I \hookrightarrow S$ .
- Show that if  $I \subseteq J \subseteq \ker(f^\#)$ , then  $Z_I \hookrightarrow Z_J \hookrightarrow S$ , where  $Z_I$  and  $Z_J$  are closed immersion that they induce.

2. Steps:

- 

- 3.