Algebraic geometry 1
Exercise sheet 12

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Exercise 3.

1. Since X is smooth, all local rings are UFD's. So in this exercise we will show that locally factorial is not affine-local

2. We can write every $a \in A$ as
$$a = c_o + c_1 a_1(x) + c_2 y a_2(x),$$
so as a polynomial of degree 1 over R. Since the sum of two integral elements is integral again (StacksProject) it is enough to show that $c_2 y a(x)$ is integral over R. This is clearly true since $(c_2 y a(x))^2 \in R$.

3. We need to check that zero gets mapped to zero. This follows from
$$(-y)^2 - x^3 + x = y^2 - x^3 + x$$

4. Write an element $a(x, y) \in A$ as
$$a(x, y) = c + a_1(x) + y a_2(x).$$
We compute
$$N(a) = a(x, y)a(x, -y) = (c + a_1(x))^2 - (y a_2(x))^2 = (c + a_1(x))^2 - (x^3 - x)a_2(x)^2 \in R.$$
Multiplicativity of N follows from the properties of ring maps.

5. Take $a \in A^*$. Then $N(a) \in k[x]^* = k^*$ and since $\deg(a\sigma(a)) \leq 1$ implies $deg(a) \leq 1$ this implies $a \in k^*$.

   We can check irreducibility using the map N.

   Assume that $x = fg$ for $f, g \in A$. Then
$$x^2 = N(x) = N(fg).$$

Write $fg = a_1(x)$. Then by the calculations from above,

$$a_1(x)^2 = x^2 \in R.$$

Therefore $a_1(x) = \pm x$, but x is irreducible in R, so w.l.o.g. $f \in A^* = k^*$.

I guess a similar argument works to show that y is irreducible.

This shows that $y^2 = x(x^2 - 1)$ are two decompositions into irreducible elements, showing that A is not factorial.