Algebraic geometry 1
Exercise sheet 3

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Exercise 1.

1. Define

$$\begin{aligned}\pi^{-1} : U &\longrightarrow \pi^{-1}(U)\\ (x_1, ..., x_n) &\mapsto (x_1, ..., x_n)[x_1 : ... : x_n].\end{aligned}$$

   This is well-defined, because by definition of U, not all x_i can be zero at the same time, so $[x_1 : ... : x_n]$ is actually a point in projective space. We also have $(x_1, ..., x_n)[x_1 : ... : x_n] \in Z$ for $(x_1, ..., x_n) \in U$, because $x_i x_j = x_j x_i$ for all $1 \leq i, j \leq n$. To see injectivity of π^{-1}, let $(x_1, ..., x_n) \in U$ with $x_j \neq 0$. Then we have $y_j \neq 0$, because if we assume $x_j \neq 0$ and $y_j = 0$, then for some $y_i \neq 0$ (which exists since $[y_1 : ... : y_n]$ is a point in projective space) we have $0 \neq x_j y_i = x_i y_j = 0$. Therefore, we can just set $y_j = 1$. Then

$$x_i y_j = x_j y_i \implies y_1 = \frac{x_1 y_j}{x_j} = \frac{x_1}{x_j},$$

   showing that all the y_i are fixed up to a scalar after fixing all the x_i.

2. Define

$$\begin{aligned}\phi : V_i &\to \mathbb{A}_n^k\\ (x, y) &\mapsto (\frac{x_1}{y_i}, \dots, x_i, \dots, \frac{x_n}{y_i}),\end{aligned}$$

   where the inverse map is given by

$$\begin{aligned}\phi^{-1} : \mathbb{A}_n^k &\to V_i\\ (x_1, \dots, x_n) &\mapsto (x_1 x_i, \dots, x_i, \dots, x_n x_i)[x_1 : \dots : x_{i-1} : 1 : \dots : x_n].\end{aligned}$$

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