Recall

<u>Theorem</u>: Let $f: X \to Y$ be a morphism of finite type. Suppose that Y is integral and Noetherian. The following holds:

For all $\mathcal{F} \in \mathrm{Coh}(X, \mathcal{O}_X)$ there exists an open dense subset $V \subseteq Y$ such that $\mathcal{F}|_{f^{-1}(V)}$ is flat over V.

(Grothendieck freeness): Let A be a Noetherian integral domain, B a finitely generated A-algebra and M a finitely generated B-module. Then there is $a \in A$ such that $M[\frac{1}{a}]$ is a free A-module.

(Proof of Chevalley's Theorem):

Constructible sets of $S \subseteq \operatorname{Spec} B$ are the image of a map $\operatorname{Spec} C \to \operatorname{Spec} B$ with C finitely presented B-algebra.

Indeed $V(f_i)$ corresponds to $C = B/f_i$, $D(f_i)$ corresponds to $B[x]/f_i x - 1$.

Given two constructible sets S_1 and S_2 with associated algebras C_1 and C_2 then $\operatorname{Spec}(C_1 \times C_2) \to \operatorname{Spec} B$ has image $S_1 \cup S_2$ and $\operatorname{Spec}(C_1 \otimes_B C_2) \to \operatorname{Spec} B$ has image $S_1 \cap S_2$

In other words, it suffices to show $f(\mathrm{Spec}\, B) \subseteq \mathrm{Spec}\, A$ is constructible for all f finitely presented.

Now, there are finite type $\mathbb{Z}$-algebra A_0, B_0 a map $A_0 \longrightarrow A$, and a map $f_0 : \mathrm{Spec}\, B_0 \longrightarrow \mathrm{Spec}\, A_0$ making

$$\begin{tikzcd} \mathrm{Spec}\, B \arrow[r, "f"] \arrow[d] & \mathrm{Spec}\, A \arrow[d, "g"] \\ \mathrm{Spec}\, B_0 \arrow[r, "f_0"'] & \mathrm{Spec}\, A_0 \end{tikzcd}$$

cartesian.

then $f(\mathrm{Spec}\, B) = g^{-1}(f_0(\mathrm{Spec}\, B_0))$

WLOG A and B are Noetherian.

We deduce the statement by induction on $\dim(\mathrm{Spec}\, A)$.

<u>Reduction</u> $\mathrm{Spec}\, A$ is irreducible

By generic flatness there is open $U \subseteq \operatorname{Spec} A$ with either

$$U \subseteq f(\operatorname{Spec} B) \quad \text{or} \quad U \cap f(\operatorname{Spec} B) = \emptyset.$$

Since U is constructible it suffices to prove $f(\operatorname{Spec} B) \setminus U$ is constructible.

This corresponds to

$$\operatorname{Spec} B \times_{\operatorname{Spec} A} Z \longrightarrow Z \quad \text{with } Z = \operatorname{Spec} A \setminus U$$

Since $\dim(Z) < \dim(\operatorname{Spec} A)$ we conclude by induction.

Recollections on dimension theory:

Definition If X is a locally Noetherian scheme and $x \in |X|$ then

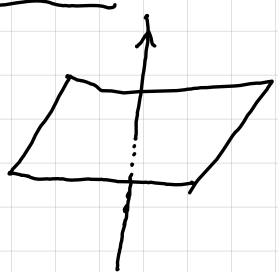
$\mathrm{codim}_X(x) = \dim \mathcal{O}_{X,x}$

We have

$$\dim \overline{\{x\}} + \mathrm{codim}_X \{x\} \leq \dim X$$

If $X = \mathrm{Spec}\, A$ and $x = \mathfrak{p}_x \subseteq A$ then $\mathrm{codim}_X(x) = \mathrm{height}_A(\mathfrak{p}_x)$

Example: $X = \mathrm{Spec}\, k[x,y,z]/(xy, xz)$



$\dim(X) = 2$

$x = \mathfrak{p}_x = (y, z, x-3)$

$\mathrm{codim}_X(x) = 1$

$\dim \overline{\{x\}} = 0$

Another example:

Non-catenary rings

$$X = \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ (\mathfrak{p}_1) \arrow[ddr] & & (\mathfrak{p}_2) \arrow[d] \\ & & (\mathfrak{p}_3) \arrow[dl] \\ & (\mathfrak{m}) & \end{tikzcd}$$

$\dim(X) = 3$

$\dim \overline{\{\mathfrak{p}_1\}} = 1$

$\operatorname{codim}_{\{\mathfrak{p}_1\}}(X) = 1$

Definition a) A ring is B is catenary if given $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ in $\operatorname{Spec} B$ then every maximal sequence $\mathfrak{p}_1 = \mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \dots \subseteq \mathfrak{q}_n \subseteq \mathfrak{p}_2$ has the same length.

b) B is universally catenary if every finite type B-algebra is catenary.

Proposition Let $X = \operatorname{Spec} A$ for a local irreducible Noetherian ring A. If A is catenary then

$$\dim X = \dim \overline{\{x\}} + \operatorname{codim}_x(X)$$

Proof The chain $0 \subseteq \mathfrak{p}_x \subseteq \mathfrak{m}$ can be refined to a maximal chain

$$0 = \mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \cdots \mathfrak{q}_m = \mathfrak{p}_x \subseteq \mathfrak{q}_{m+1} \subseteq \cdots \mathfrak{q}_{m+d} = \mathfrak{m}$$

then
$$m = \operatorname{codim}_x(X)$$
$$d = \dim \overline{\{x\}}$$
$$\dim X = m + d.$$

Proposition If Y is irreducible finite type k-scheme and $y \in Y$ then

$$\dim Y = \dim \overline{\{y\}} + \operatorname{codim}_y(Y)$$

Definition: Let X be a topological space and let $x \in X$ be a point.

We let $\dim_x(X) := \min\{\dim(U) \mid x \in U \underset{\text{open}}{\subseteq} X\}$

Proposition If X is a finite type k-scheme then

$$\dim \overline{\{x\}} + \operatorname{codim}_X(\{x\}) = \dim_x(X)$$

Proof Let $I_1, \dots, I_n$ denote the irreducible components of X that pass through x. Then

$$\dim_x(X) = \max \dim I_i.$$

$$\operatorname{codim}_X(\{x\}) = \max\left(\dim I_i - \dim \overline{\{x\}}\right).$$

Definition Let X be a topological space a function $f: X \to \mathbb{R}$ is upper semi-continuous if for all $x \in \mathbb{R}$ $f^{-1}((-\infty, x)) \subseteq X$ is open.

Theorem If $\pi: X \to Y$ is a morphism of finite type k-schemes then the following hold:

a) $\dim_x(X_{\pi(x)}) : |X| \to \mathbb{R}$ is upper semi-continuous.

b) If π is closed then $\dim(X_y) : |Y| \to \mathbb{R}$ is upper semi-continuous.

Proof of b) We let $F_n \subseteq |Y|$ be the locus where $\dim(X_y) \geq n$.

We want to prove F_n is closed for all n

We do induction on $\dim(Y)$

<u>Reduction</u>: WLOG X and Y are integral and $\pi: X \to Y$ is dominant. Indeed $F_n \subseteq Y$ is closed in Y <u>iff</u> it's intersection with every irreducible component is closed.

Moreover if $X = \bigcup_{i=0}^{n} X_i$ with each X_i an irreducible component of X the the function

$f: y \mapsto \dim X_y$ is

$f = \max(f_0, f_1, \dots, f_n)$ where

$f_i: y \mapsto \dim X_{i,y}$

if each $f_0, \dots f_n$ is upper-semi-continuous then f is also upper semi-continuous.

If $n \leq r$ then $F_n = Y$ since:

$$\operatorname{codim}_X(x) \leq \operatorname{codim}_Y(f(x)) + \operatorname{codim}_{X_{f(x)}}(f(x)) \leq$$

$$\begin{array}{ccc} \| & & \| \\ \dim(X) - \dim\{\bar{x}\} & & \dim(Y) - \dim\{\bar{y}\} + \dim X_{f(x)} - \dim\{\bar{x}\} \end{array}$$

$$\dim(X) - \dim(Y) \leq \dim X_{f(x)}$$

If $n < r$ there is $U \subseteq Y$ with $\pi : \pi^{-1}(U) \to U$ flat. Then all fibers over U have dimension r.

Then $F_n \cap \pi^{-1}(U) = \phi$ so

$$F_n \subseteq \pi^{-1}(Y \setminus U) \longrightarrow Y \setminus U$$

we conclude by induction.

Warning: with setup as in theorem the function $|X| \to \mathbb{R}$ given by

$$x \longmapsto \dim(X_{f(x)})$$ is

not always upper semi-continuous.

Only $x \mapsto \dim_x(X_{f(x)})$ is !!!.

Counterexample:

$$Y = \mathbb{A}^1 \qquad X = \mathbb{A}^2 \setminus \mathbb{A}^1 \times \{0\} \sqcup Y$$

$\pi : X \to Y$ is given

by first projection on $\mathbb{A}^2 \setminus \mathbb{A}^1 \times \{0\}$

and by inclusion on Y

then $F_1 = \{x \in X \mid \dim(X_{f(x)}) \geq 1\}$

is $F_1 = \mathbb{A}^2 \setminus \mathbb{A}^1 \times \{0\} \sqcup Y \setminus \{0\}$

which is not closed, whereas

$F_1' = \{x \in X \mid \dim_x(X_{f(x)}) \geq 1\} = \mathbb{A}^2 \setminus \mathbb{A}^1 \times \{0\}$

Important facts for Noetherian local rings:

<u>Krull's principal ideal theorem:</u>

If A is a Noetherian ring and $f \in A$, then every minimal prime $\mathfrak{p} \subseteq A$ containing f has height at most 1.

If f is not a zero-divisor then every such prime has height 1.

Corollary: If A is a catenary Noetherian local ring and $a \in m \subseteq A$ is not a zero divisor, then

$$\dim(\operatorname{Spec} A/a) = \dim(\operatorname{Spec} A) - 1$$

Proof Let $\operatorname{Spec} A = \bigcup_{i=1}^{n} I_i$ where each I_i is an irreducible component, then $\operatorname{Spec} A/a = \bigcup_{i=1}^{n} (I_i \cap \operatorname{Spec} A/a)$ and $\dim(\operatorname{Spec} A/a) = \max_{i=1}^{n} \dim(I_i \cap \operatorname{Spec} A/a)$

By KPIT all generic points of $I_i \cap \operatorname{Spec} A/a$ are codimension 1 in I_i

Since I_i is catenary

$$\dim I_i = \dim I_i \cap \operatorname{Spec} A/a + 1$$

Potential worry:

$$\begin{tikzcd}
 & 0 \arrow[dl] \arrow[dr] & \\
(\mathfrak{p}_1) \arrow[ddr] & & (\mathfrak{p}_2) \arrow[d] \\
 & & (\mathfrak{p}_3) \arrow[dl] \\
 & (\mathfrak{m}) &
\end{tikzcd}$$

what if there is $f \in A$ s.t. $\overline{\{\mathfrak{p}_1\}} = V(f)$?

Krull's height theorem:

Suppose $X = \operatorname{Spec} A$ where A is Noetherian and Z is an irreducible component of $V(f_1, \dots, f_\ell)$ then the codimension of Z is at most ℓ.

Proposition Let A be a Noetherian local ring with maximal ideal $\mathfrak{m} \subseteq A$. We let $\delta(A)$ be the minimum number such that $\{\mathfrak{m}\} = V(f_1, \dots, f_{\delta(A)})$ for some elements in A. Then $\dim(A) = \delta(A)$.

Proof

$\dim(A) \leq \delta(A)$: Suppose

$\{m\} = V(f_1 \dots f_d)$ then by

Krull's height theorem

height$(m) = \dim(A) \leq \delta(A)$

$\delta(A) \leq \dim(A)$: Let $d = \dim(A)$ we show

by induction on d that there are

d-elements with $\{m\} = V(f_1 \dots f_d)$.

Pick $f_1 \in A$ not a zero-divisor

then $\dim(\mathrm{Spec}(A/f_1)) \leq d-1$

by induction we can $(d-1)$-elements

cutting $m/(f_1)$.

proof of Krull's height theorem

WLOG $X = \operatorname{Spec} A$ for a Noetherian local ring and $Z = \{m\} = V(f_1, \dots, f_\ell)$

Let $\mathfrak{q} \subseteq m$ be any prime with no other prime in between. $\operatorname{height}(m) = \max_{\mathfrak{q}} (\operatorname{height}(\mathfrak{q}) + 1)$.

We may assume $f_1 \notin \mathfrak{q}$. Then $V(\mathfrak{q}, f_1) = m$.

We have $f_j^N \in (\mathfrak{q}, f_1)$ $\forall j \in \{2, \dots \ell\}$.

and some $N \gg 0$.

$$f_j^N = q_j + a_j f_1 \quad \text{for some } q_j \in \mathfrak{q}$$

and $a_j \in A$.

Then

$$V(f_1, q_2, \dots, q_n) = V(f_1, f_2^N, \dots f_\ell^N) = \{m\}$$

The ring $A/(q_2 \dots q_n)$ has unique maximal ideal of codimension 1 by KPIT so $\mathfrak{q}$ is minimal in $\operatorname{Spec} A/(q_2, \dots q_\ell)$. By induction,

$\operatorname{height}(\mathfrak{q}) \leq \ell - 1$