Algebraic geometry 1
Exercise sheet 3

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Exercise 1.

1. Let X be a finite set that is irreducible with respect to some topology $\mathcal{F}$ on X. Then we get $\mid \mathcal{F} \mid < \infty$ and since finite unions of closed sets are closed again, we get that

$$X' := \bigcup_{U \subsetneq X \text{closed}} U$$

is closed in X. Since X is by assumption irreducible, $X \neq X'$, so we can pick $x_0 \in X \backslash X'$, which is by construction generic. For the second part of the exercise we use part 2 of Hochster's Theorem. As a finite set, X is quasicompact and as a basis $\mathcal{B}$ consisting of quasicompact open sets stable under finite intersections take all of the open sets.
It remains to show that X is sober. We need to check that every irreducible subset of X has a unique generic point. The existence of a generic point comes from part of of this exercise. Uniqueness of this point is due to the fact that generic points in T_0 spaces are unique if they exist, which follows directly from the definition of T_0.

1. Let X be a finite irreducible topological space. Since

$$X = \bigcup_{x \in X} \overline{\{x\}}$$

is a finite decomposition into closed sets, we must have $\overline{\{x\}} = X$ for some $x \in X$. This x is a generic point of X.

If we additionally assumed X is T_0, then this point x would be unique, since in a T_0 space we have $\overline{\{x\}} \neq \overline{\{y\}}$ for $x \neq y$. Also in a finite space the conditions of quasicompactness and the basis being stable under finite intersections are clearly fulfilled. So finite T_0 spaces are spectral.

2. Let us first describe what $\mathrm{Spec}(\mathbb{Z})$ looks like. The ring $\mathbb{Z}$ a PID with prime ideals being those (a), for which $a \in \mathbb{Z}$ is either a prime number or $a = 0$. So

$$\mathrm{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime}\} \cup \{(0)\}.$$

In this space (0) is a unique generic point.

Closed sets in $\mathrm{Spec}(\mathbb{Z})$ are by definiton

$$V((a)) = \{(p) \in \mathrm{Spec}(\mathbb{Z}) \mid (a) \subseteq (p)\} = \{(p) \in \mathrm{Spec}(\mathbb{Z}) \mid p \text{ divides } a\}.$$

So if $a = \Pi_i p_i^{k_i}$, then $V((a)) = \{(p_i)\}_i \subseteq \mathrm{Spec}(\mathbb{Z})$. Since any $a \in \mathbb{Z}$ is only divisible by finitely many prime numbers, we get the cofinite topology on $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$.

Adding generic point (0) to $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$ is the construction $X \to X^{\mathrm{sob}}$ which we did last week.

So closed sets in $\mathrm{Spec}(\mathbb{Z})$ are exactly all finite subsets of $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$, the whole space and the empty set.

Define the diagram with objects

$$F_i := \{(p_1), \dots, (p_i)\} \cup \{(0)\}$$

with subspace topology, as $F_i \subseteq \mathrm{Spec}(\mathbb{Z})$, and morphisms

$$f_i \colon F_{i+1} \to F_i$$

mapping $f_i((p_j)) = (p_j)$ for $j \le i$ and $f_i((p_{i+1})) = f_i((0)) = (0)$.

We view F_i as a subspace of F_j for $i < j$.

First we have to check that f_i are continuous. Take a closed subset $U \subseteq F_i$. If $(0) \in U$, then $U = F_i$ and $f_i^{-1}(U) = F_{i+1}$. And if $U \subseteq \{(p_1), \dots, (p_i)\}$, then $f_i^{-1}(U) = \{(p_1), \dots, (p_i)\} \subseteq F_{i+1}$.

We claim $\mathrm{Spec}(\mathbb{Z})$ is the limit of this diagram. For that we define $\alpha_i \colon \mathrm{Spec}(\mathbb{Z}) \to F_i$ by $\alpha_i((p_j)) = (p_j)$ for $j \le i$ and $\alpha_i((p_j)) = \alpha_i((0)) = (0)$ for $j > i$. Clearly we have $\alpha_i = f_i \circ \alpha_{i+1}$.

Suppose now there exists an object B with $\beta_i \colon B \to F_i$. Define $\alpha \colon B \to \mathrm{Spec}(\mathbb{Z})$ in the following way. Take $b \in B$ and look at $\beta_i(b)$ for different $i = 1, 2, \dots$. If $\beta_i(b) = (0)$ for all i, then define $\alpha(b) = (0) \in \mathrm{Spec}(\mathbb{Z})$. If $\beta_{i_0}(b) \neq (0)$ for some i_0, then $\beta_j(b) = \beta_{i_0}(b)$ for all $j > i_0$ using commutativity of the diagram.

Lets show that α is continuous. Take a closed subset $A \subseteq \mathrm{Spec}(\mathbb{Z})$. If $(0) \in A$, then $\alpha^{-1}(A) = B$. Else A is finite subset of $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$. Then A is contained in some F_i (pick i to be largest such that $(p_i) \in A$) and $(\alpha_i^{-1} \circ \alpha_i)(A) = A$ holds. Therefore

$$\alpha^{-1}(A) = (\alpha^{-1} \circ \alpha_i^{-1} \circ \alpha_i)(A) = (\beta_i^{-1} \circ \alpha_i)(A) = \beta_i^{-1}(\alpha_i(A)).$$

The set $\alpha_i(A)$ is a finite subset of F_i that does not contain (0), so it is closed. Since β_i is continuous, the set $\alpha^{-1}(A)$ is then closed, which proves continuity of α.

The map α was defined using condition that the diagram must commute. It is clear that any other definition of α would not satisfy commutativity conditions $\beta_i = \alpha_i \circ \alpha$ for all i. So α was unique and thus $\operatorname{Spec}(\mathbb{Z})$ is the limit of the constructed diagram.

Exercise 2. Denote $A = \lim A_i$, $B = \lim B_i$ and $C = \lim C_i$. Also denote maps $A_i \to A$ with f_i, $B_i \to B$ with g_i and $C_i \to C$ with h_i.

By composing α_i and g_i we get $A_i \to B$ defined as $g_i \circ \alpha_i$. Then by the definition of a colimit we have a unique map $A \to B$, denoted by α, such that $g_i \circ \alpha_i = \alpha \circ f_i$. In the same way we obtain $\beta\colon B \to C$. With these definitions the whole diagram commutes.