Algebraic geometry 1
Exercise sheet 3

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7. November 2023

Exercise 1.

1. Let X be a finite set that is irreducible with respect to some topology $\mathcal{F}$ on X. Then we get $\mid \mathcal{F} \mid < \infty$ and since finite unions of closed sets are closed again, we get that

$$X' := \bigcup_{U \subsetneq X \text{closed}} U$$

is closed in X. Since X is by assumption irreducible, $X \neq X'$, so we can pick $x_0 \in X \backslash X'$, which is by construction generic. For the second part of the exercise we use part 2 of Hochster's Theorem. As a finite set, X is quasicompact and as a basis $\mathcal{B}$ consisting of quasicompact open sets stable under finite intersections take all of the open sets.
It remains to show that X is sober. We need to check that every irreducible subset of X has a unique generic point. The existence of a generic point comes from part of of this exercise.
Uniqueness of this point is due to the fact that generic points in T_0 spaces are unique if they exist, which follows directly from the definition of T_0.

1. Let X be a finite irreducible topological space. Since

$$X = \bigcup_{x \in X} \overline{\{x\}}$$

is a finite decomposition in closed sets, we must have $\overline{\{x\}} = X$ for some $x \in X$. This x is a generic point of X.

If we additionally assumed X is T_0, then this point x would be unique, since in a T_0 space we have $\overline{\{x\}} \neq \overline{\{y\}}$ for $x \neq y$. Also in a finite space the conditions of quasicompactness and the basis being stable under finite intersections are clearly fulfilled. So finite T_0 spaces are spectral.

2. Let us first describe what Spec($\mathbb{Z}$) looks like. It is a PID with prime ideals being those (a) for which $a \in \mathbb{Z}$ is a prime number or $a = 0$. So

$$\mathrm{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime}\} \cup \{(0)\}.$$

In this space (0) is a unique generic point.

Closed sets in Spec($\mathbb{Z}$) are by definiton

$$V((a)) = \{(p) \in \mathrm{Spec}(\mathbb{Z}) \mid (a) \subseteq (p)\} = \{(p) \in \mathrm{Spec}(\mathbb{Z}) \mid p \text{ divides } a\}.$$

So if $a = \Pi_i p_i^{k_i}$, then $V((a)) = \{(p_i)\}_i \subseteq \mathrm{Spec}(\mathbb{Z})$. Since any $a \in \mathbb{Z}$ is only divisible by finitely many prime numbers, we get the finite completement topology on $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$.

Adding generic point (0) to $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$ is actually the construction $X \to X^{\mathrm{sob}}$ which we did last week.

Exercise 2. Denote $A = \lim A_i$, $B = \lim B_i$ and $C = \lim C_i$. Also denote maps $A_i \to A$ with f_i, $B_i \to B$ with g_i and $C_i \to C$ with h_i.

By composing α_i and g_i we get $A_i \to B$ defined as $g_i \circ \alpha_i$. Then by the definition of a colimit we have a unique map $\alpha : A \to B$, such that $g_i \circ \alpha_i = \alpha \circ f_i$. In the same way we obtain $\beta \colon B \to C$. With these definitions the whole diagram commutes.

Exercise 3.

1. Let $\mathcal{F}, \mathcal{G}$ be presheaves on X and $(\phi_U)_{U \in Ouv(X)}, (\psi_U)_{U \in Ouv(X)}$ morphism of presheaves from $\mathcal{F}$ to $\mathcal{G}$.
We define for $U \in Ouv(X)$

$$(\phi + \psi)(U) := \phi(U) + \psi(U).$$

This is indeed a map of presheaves again, because we can restrict ϕ and ψ independently.
The zero object in this category is the presheaf that sends every open set $U \in Ouv(X)$ to the trivial group $(0, +)$. It is inital and terminal, because group maps send 0 to 0. Define the presheaf kernel as

$$ker(\phi)(U) := ker(\phi(U)).$$

We check that this is indeed a sheaf!!!???