Algebraic geometry 2
Exercise sheet 8

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Exercise 3.

1.

2. We can show that $j_!$ is exact. We know it preserves epimorphisms because it has a right adjoint. But monomorphisms are preserved because they can be checked on stalks.

   Let $0 \to A \to B \to C \to 0$ be exact. We want to show $0 \to j_!A \to j_!B \to j_!C \to 0$ is exact. For $x \in V$ we clearly have $(j_!A)_x = A_x$ and for $x \not\in V$ we have $(j_!A)_x = 0$ (for this we use explicit definition of sheafification from alggeo1). So $j_!A \to j_!B$ is a monomorphism.

   Now we show that j^* preserves injectives.

   Let F be injective $\mathcal{O}_X$-module. Let $0 \to A \to B \to C \to 0$ be exact sequence of $\mathcal{O}_V$-modules. Then $0 \to j_!A \to j_!B \to j_!C \to 0$ is exact. By injectiveness of F we have exact sequence $0 \to \mathrm{Hom}(j_!A, F) \to \mathrm{Hom}(j_!B, F) \to \mathrm{Hom}(j_!C, F) \to 0$ is exact. Then we use that $j_!$ and j^* are adjoint pair and we get that $0 \to \mathrm{Hom}(A, j^*F) \to \mathrm{Hom}(B, j^*F) \to \mathrm{Hom}(C, j^*F) \to 0$ is exact. So $\mathrm{Hom}(, j^*F)$ preserves exact sequences, which is what we wanted to show.

3. We can take $X = \mathrm{Spec}(\mathbb{Z})$ and $V = X \setminus \{p\}$ for prime p. We will show that $j_!\mathcal{O}_V$ has no global sections.

   Denote the presheaf defined in the exercise by F.

   At alggeo1 we constructed sheafification of a presheaf F explicitly as the sheaf

   $$\tilde{F} \colon U \mapsto \{(s_x)_{x \in U} \in \Pi_{x \in U} F_x \mid (s_x)_{x \in U} \text{ satisfies condition below}\}$$

   for all $x \in U$ there exists a neighbourhood V_x and $t \in F(V_x)$ such that for all $y \in V_x$ we have $s_y = t_y$.

   So global sections of $j_!\mathcal{O}_V$ are $(s_x)_{x \in X}$ (that satisfy the condition). But for any neighbourhood V_p of p we have $F(V_p) = 0$, so there has to be an open set V_p such that $s_y = 0$ for all $y \in V_p$. But then $s_y = 0$ for all $y \in X$, since any non-zero section of $(j_! \mathcal{O}_V) \mid_V$ is zero only on finitely many points.

The map $0 = (j_! \mathcal{O}_V)(X)[p^{-1}] \to \mathbb{Z}[p^{-1}]$ is is therefore not an isomorphism, and $j_! \mathcal{O}_V$ not a quasi-coherent sheaf.