Elliptic curves and their moduli spaces
Exercise sheet 4

Solutions by: Esteban Castillo Vargas and David Čadež

8. Mai 2024

Problem 1. As suggested in the hint, we know that y and $3x^2 + a$ do not both vanish at no point on the curve (they are coefficients of the Jacobi matrix, which is invertible on E).

The differential form $\mathrm{d}x/2y$ is defined on $D(y)$ and $\mathrm{d}y/(3x^2 + a)$ is defined on $D(3x^2 + a)$.

We have to check they match on $D(y(3x^2 + a))$. On $E \cap D_+(z)$ we have $y^2 - x^3 - ax - b = 0$, so also $2y \; \mathrm{d}y - (3x^2 + a) \; \mathrm{d}x = 0$ and therefore $\mathrm{d}x/2y = \mathrm{d}y/(3x^2 + a)$ whenever both y and $3x^2 + a$ are invertible.

Now we show that this form can be extended to a global section in $\Omega^1_{E/k}(E)$. Observe that only point k-rational point on the curve that is not contained in $D(z)$ is $[0 : 1 : 0]$. That means we can cover the curve, as usually, with $D(z) \cup D(y)$. But clearly the section $\mathrm{d}x/2y$ can be extended (or, is defined) on the whole $D(y)$, so indeed on whole $D(y) \cap E$.