Proposition Let $(X, \mathcal{O}_X)$ be a locally ringed space.
The following hold:

1) If $\mathcal{I} \in \mathrm{Mod}(X, \mathcal{O}_X)$ is injective then it is flasque
2) If it is flasque then it is Γ-acyclic.

Proof

1) For $j^U : U \to X$ an open subset let $j_!^U \mathcal{O}_U$ be extension by 0.

$$j_!^{U,pre} \mathcal{O}_U [V] := \begin{cases} \mathcal{O}_X(V) & \text{if } V \subseteq U \\ 0 & \text{if not} \end{cases}$$

$j_!^U \mathcal{O}_U$ is shefification of $j_!^{U,pre} \mathcal{O}_U$.

Then $\mathrm{Hom}_{\mathcal{O}_X}(j_! \mathcal{O}_U, -) \simeq \Gamma(U, -)$.

We have injection

$$0 \to j_!^V \mathcal{O}_V \to j_!^U \mathcal{O}_U$$

Since $\mathrm{Hom}(-, I)$ is exact

we see that

$$\Gamma(U, I) \to \Gamma(V, I) \to 0$$

is surjective.

2) Let $\mathcal{F}$ be flasque and set

$$0 \to \mathcal{F} \to I \xrightarrow{f} \mathcal{G} \to 0$$ with

I injective.

Claim: for all $U \underset{\text{open}}{\subseteq} X$ the map $\Gamma(U, \mathcal{I}) \to \Gamma(U, \mathcal{G})$ is surjective.

Assume claim, then $\mathcal{G}$ is flasque and

$$\begin{tikzcd} \Gamma(X, \mathcal{I}) \arrow[r] & \Gamma(X, \mathcal{G}) \arrow[r, "\delta"] \arrow[dr] & H^1(X, \mathcal{F}) \arrow[r] & H^1(X, \mathcal{I}) \arrow[d, equal] \\ & & 0 \arrow[u] & 0 \end{tikzcd}$$

So $H^1(X, \mathcal{F}) = 0$. Moreover, $H^2(X, \mathcal{F}) \simeq H^1(X, \mathcal{G})$ since $\mathcal{G}$ is also flasque both vanish.

Proof of Claim:

Let $s \in G(U)$ and consider pairs (V, t) with $V \underset{\text{open}}{\subseteq} U$ $t \in \Gamma(V, \mathcal{I})$ with $f(t) = s|_V$.

This has partial order

$$(V, t) \leq (V', t') \quad \text{if}$$

$V \subseteq V'$ and $t'|_V = t$.

By Zorn's lemma there is a maximal pair (V, t) we claim $V = U$.

pick $x \in U$ and $x \in W \subseteq U$ with and $t_W \in \Gamma(W, \mathcal{I})$ with $f(t_W) = s|_W$.

then $t - t_W \in \Gamma(V \cap W, \mathcal{F})$ so there is $r \in \Gamma(W, \mathcal{F})$ mapping to $t - t_W$

then $t_W + r \in \Gamma(W, \mathcal{I})$ glues with t.

spectral sequence:

Definition

A cohomological spectral sequence in $\mathcal{C}$ starting in page $b \geq 1$.

consists of the following data:

1) For all $p, q \in \mathbb{Z}$ and

$r \geq a$ an object $E_r^{p,q} \in \mathcal{C}$

2) For $p, q \in \mathbb{Z}$ and $r \geq a$ a

morphism $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1}$

such that

$$d_r^{p+r, q+r-1} \circ d_r^{p,q} = 0$$

* For fixed r we call the collection $(E_r^{p,q}, d_r^{p,q})$ the rth-page of the spectral sequence.

3) For all p, q, r isomorphisms

$$\alpha_r^{p,q} : H^{p,q}(E_r) \xrightarrow{\sim} E_{r+1}^{p,q}$$

Pages:

$E_1^{\bullet,\bullet}$

$\cdots$

$$\begin{tikzcd}
\cdots, & E_1^{0,2} \arrow[r] & E_1^{1,2} \arrow[r] & E^{2,2} \arrow[r] & \cdots \\
\cdots \arrow[r] & E_1^{0,1} \arrow[r] & E_1^{1,1} \arrow[r] & E_1^{2,1} \arrow[r] & \cdots \\
\cdots \arrow[r] & E_1^{0,0} \arrow[r] & E_1^{1,0} \arrow[r] & E_1^{2,0} \arrow[r] & \cdots
\end{tikzcd}$$

$\cdots$

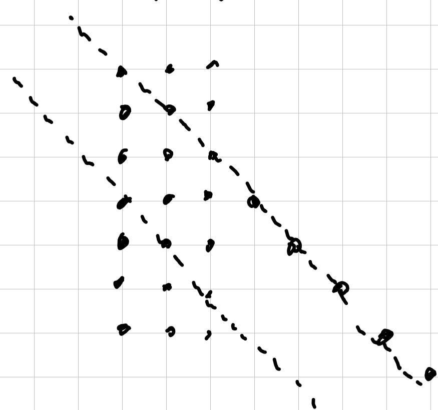
$E_2^{\bullet,\bullet}$

$$\begin{tikzcd}
\cdots, & E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} & \cdots \\
\cdots & E_2^{0,1} \arrow[drr] & E_2^{1,1} & E_2^{2,1} & \cdots \\
\cdots & E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & \cdots
\end{tikzcd}$$

Definition Let $\mathcal{C}$ be an abelian category. Let $E = (E_r^{p,q}, d_r^{p,q}, \alpha_r^{p,q})$ be a spectral sequence in $\mathcal{C}$ starting in page b

1) E is called bounded if for all n there is only finitely many terms $E_b^{p,q}$ that are non-zero and such that $p+q=n$

Visually



2) when $E_\cdot^{\cdot,\cdot}$ is bounded for fixed (p,q) there is $N^{p,q} \geq 0$ s.t. $\forall r \geq N^{p,q}$

$$\underbrace{d_r^{p,q}}_{\substack{\text{from}\\ E_r^{p,q}}} = 0 = \underbrace{d^{p-r,q+r-1}}_{\substack{\text{to}\\ E_r^{p,q}}}$$

in particular $E_r^{p,q} \simeq E_{r+1}^{p,q}$

we let $E_\infty^{p,q} := E_{N^{p,q}}^{p,q}$ and call it <u>limit term of E at (p,q)</u>.

3) If there is r_0 s.t. $d_r^{p,q} = 0$ for all p, q and $r \geq r_0$ we say that E degenerates on the E_{r_0}-page.

4) Given a collection of objects $H^n \in \mathcal{C}$ we say that E converges to $H^\bullet$ if for each H^n there is a finite filtration

$$0 = F^t H^n \subseteq \dots \subseteq F^{p+1} H^n \subseteq \dots F^s H^n = H^n$$

such that $E_\infty^{p,q} \simeq F^p H^{p+q} / F^{p+1} H^{p+q}$.

We write $E_a^{p,q} \Rightarrow H^{p+q}$

Definition A filtration F on a chain complex $C^\bullet$ is an ordered family of chain complexes.

$$F_{p-1}C \subseteq F_p C \subseteq \dots \subseteq \dots C$$

We say it is bounded if for all n there are integers $s \leq t$ s.t. $F_s C^n = 0$ and $F_t C^n = C^n$.

Theorem (Convergence Theorem)

Suppose that the filtration on C is bounded. Then there is a bounded spectral sequence

$$E^1_{p,q} = H^{p+q}(F_pC / F_{p-1}C) \Rightarrow H^{p+q}(C)$$

converging to $H_*(C)$.

Motto: You can approximate the cohomology of a complex by filtering it.

Main Example

If $C^{\bullet,\bullet}$ is a double complex

$$\begin{tikzcd}
\cdots & & \\
C^{0,2} \arrow[r] & C^{1,2} \arrow[r] & C^{2,2} \\
C^{0,1} \arrow[r] \arrow[u] & C^{1,1} \arrow[r] \arrow[u] & C^{2,1} \arrow[u] & \vdots \\
C^{0,0} \arrow[r] \arrow[u] & C^{1,0} \arrow[r] \arrow[u] & C^{2,0} \arrow[u]
\end{tikzcd}$$

it has a natural filtration

$$F_1^l C^{p,q} = \begin{cases} 0 & \text{if } l \leq p \\ C^{p,q} & \text{otherwise} \end{cases}$$

then $F_1^l \mathrm{Tot}(C^{\bullet,\bullet}) / F^{l-1} \mathrm{Tot}(C^{\bullet,\bullet}) = C^{l,\bullet}$

$$\begin{tikzcd}
\cdots & & \\
0 \arrow[r] & C^{l,q+1} \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & C^{l,q} \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & C^{l,q-1} \arrow[r] \arrow[u] & 0 \arrow[u] \\
\cdots & &
\end{tikzcd}$$

So there is a E_1-spectral sequence with $E_1^{p,q}$ the cohomologies of columns $C^{p,\bullet}_{\cdot} \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))$

We can also filter by rows. (equivalently take transpose of $C^{\bullet,\bullet}$ and filter by columns)

we get E_1-spectral sequence with $E_1^{p,q} \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))$ and $E^{p,q}$ the cohomology of columns.

<u>Example:</u> Re proving snake lemma:

take diagram

$$\begin{tikzcd} & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D \arrow[r] & E \arrow[r] & F & \end{tikzcd}$$

Rewrite it as:

$$S = \begin{tikzcd} C \arrow[r] & F \\ B \arrow[u] \arrow[r] & E \arrow[u] \\ A \arrow[u, "f"] \arrow[r] & D \arrow[u, "g"'] \end{tikzcd}$$

double complex

$\rightsquigarrow$ pass to E_1-page

$$\begin{matrix} 0 & \to & \operatorname{coker}(g) \\ 0 & \to & 0 \\ \ker(f) & \to & 0 \end{matrix}$$

already degenerates.

So $H^0(S) = \ker(f)$

$H^1(S) = 0$

$H^2(S) = 0$

$H^3(S) = \operatorname{coker}(g)$

Second filtration:

$$\begin{tikzcd} D \arrow[r, "s"] & E \arrow[r] & F \\ A \arrow[u, "a"] \arrow[r, "f"'] & B \arrow[u, "b"] \arrow[r] & C \arrow[u, "c"'] \end{tikzcd} \qquad S^{\text{trans}}$$

$\rightsquigarrow$ E_1-page

$$\begin{tikzcd} \operatorname{coker}(a) \arrow[r, "h"] & \operatorname{coker}(b) \arrow[r] & \operatorname{coker}(c) \\ \ker(a) \arrow[r] & \ker(b) \arrow[r] & \ker(c) \end{tikzcd}$$

$\rightsquigarrow$ E_2-page

$$\begin{tikzcd} \ker h \arrow[drr] & 0 & \operatorname{coker}(g) \\ \ker(f) & 0 & \dfrac{\ker(c)}{\ker(b)} \end{tikzcd}$$

has to be isomorphism.

E_3-page

$$\begin{matrix} 0 & 0 & \operatorname{coker} g \\ \ker f & 0 & 0 \end{matrix}$$

this gives

$\ker(a) \to \ker(b)$

<u>Theorem</u> (Grothendieck spectral sequence)

Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that F sends injective objects to G-acyclic objects. Then for each $A \in \mathcal{A}$ there is a convergent spectral sequence

$$E_2^{p,q} = R^pG \circ R^qF \Rightarrow R^{p+q}(G \circ F)(A).$$

<u>Example</u> Low dimensional case

$$\begin{tikzcd}
R^0GR^1F \arrow[drr] & R^1GR^1F \arrow[drr] & & \\
R^0GR^0F & R^1GR^0F & R^2GR^0F &
\end{tikzcd}$$

$$F^2(R^1(F\circ G)) = 0 \qquad F^{1+0}(R^1(F\circ G)) = E^{1,0} = R^1G R^0F$$

$$F^0(R^{0+1}(F\circ G)) \big/ F^1(R^{0+1}(F\circ G)) = E_\infty^{0,1} = \ker(R^0GR^1F \to R^2GR^0F)$$

We set

$$0 \to R^1 G \circ R^0 F \to R^1(G \circ F) \to R^0 G \circ R^1 F \to R^2 G R^0 F$$

is exact.

<u>Sketch of proof of Thm</u>:

Let $A \in \mathcal{A}$ with injective resolution $A \to I^\bullet$.

Consider $F(I^\bullet)$, this is a complex of G-acyclic objects.

<u>Definition</u> A Cartan-Eilenberg resolution of a chain complex $C_\bullet$ is a bicomplex $C^\bullet \to I^{\bullet,\bullet}$

$$\begin{tikzcd}
\cdots \arrow[r] & I^{0,1} \arrow[r] & I^{1,1} \arrow[r] & \cdots \\
I^{-1,0} \arrow[r] \arrow[u] & I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[u] \\
C^{-1} \arrow[r] \arrow[u] & C^0 \arrow[r] \arrow[u] & C^1 \arrow[r] \arrow[u] & C^2 \arrow[u]
\end{tikzcd}$$

s.t. for each p

$$C^p \to I^{p,\bullet}$$ is an injective resolution and taking kernels, images and cohomology horizontally in $I^{\bullet,\bullet}$ give injective resolutions of the kernels, images and cohomology of $C^\bullet$.

<u>Proposition</u> If direct sums are exact in $\mathcal{C}$ (AB4) every complex has a Cartan-Eilenberg resolution.

We write a double complex

$$\mathcal{F}(I^\bullet) \longrightarrow D^{\bullet,\bullet}$$ a Cartan-Eilenberg resolution st $\mathcal{F}(I^\bullet)$.

and we let $C^{\bullet,\bullet} = G(D^{\bullet,\bullet})$

we have two spectral sequences converging to $\mathrm{Tot}(C^{\bullet,\bullet})$.

vertical cohomology gives a E_1-page $E_1^{p,q} = RG^q(\mathcal{F}(I^p))$

Since $\mathcal{F}(I^q)$ is G-acyclic this vanishes unless $q=0$

E_2-page degenerates and

$$E_\infty^{p,q} = \begin{cases} R^p(G\circ\mathcal{F})(A) & q=0 \\ 0 & \text{otherwise.} \end{cases}$$

For the other spectral sequence.

horizontal cohomology, by definition of Cartan-Eilenberg resolution.

Gives a E_1-page with G applied to an injective resolution of $R^p\mathcal{F}$ passing to the E_2-page gives $R^qG\circ R^p\mathcal{F}$.