Ex. 4) i) Context:

$$\begin{tikzcd} & C \arrow[d, "\psi"] \\ B \arrow[ur, dashed, "\exists f"] \arrow[r, "\varphi"'] & A \end{tikzcd}$$

with B and C polynomial R-algebras. Assume $B = R[T_i \mid i \in I]$ and the axiom of choice. Then we have a choice function:

$$F = \{\psi^{-1}(\varphi(T_i))\}_{i\in I} \xrightarrow{\alpha} \bigcup F$$

and we can define f as follows:

$$f(T_i) := \alpha\left(\psi^{-1}(\varphi(T_i))\right)$$

this gives the following map:

$$\begin{tikzcd} \cdots 0 \arrow[r] & J_B/J_B^2 \arrow[r] \arrow[d, "f_0"] & A \otimes_B \Omega^1_{B/R} \arrow[r] \arrow[d, "f_1"] & 0 \cdots \\ \cdots 0 \arrow[r] & J_C/J_C^2 \arrow[r] & A \otimes_C \Omega^1_{C/R} \arrow[r] & 0 \cdots \end{tikzcd}$$

$$\begin{tikzcd}
b \arrow[r, mapsto] \arrow[d, mapsto] & 1 \otimes db \arrow[dr, mapsto] & & \\
f(b) \arrow[r, mapsto] & 1 \otimes df(b) = 1 \otimes df(b) & =: f_1(1 \otimes db)
\end{tikzcd}$$

Hence a map/morphism of complexes.

ii) We write down the homotopy:

$$\begin{tikzcd}
\cdots 0 \arrow[r] \arrow[d] & I_B/I_B^2 \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "f'"] & A \otimes_B \Omega^1_{B/R} \arrow[r] \arrow[dl, "H"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "f'"] & 0 \cdots \arrow[dl] \\
\cdots 0 \arrow[r] & I_C/I_C^2 \arrow[r] & A \otimes_C \Omega^1_{C/R} \arrow[r] & 0 \cdots
\end{tikzcd}$$

Defining $H(1 \otimes db) := f(b) - f'(b)$ gives the result. We argue that H exists by universal property of Kähler differentials:

$$\begin{tikzcd}
B \arrow[r, "d"] \arrow[dr, "f-f'"'] & \Omega^1_{B/R} \arrow[d, dashed, "\exists H"] \\
& I_C/I_C^2
\end{tikzcd}$$

Indeed $f-f'$ is an R-derivation of B into the A-module I_C / I_C^2:

$$R\text{-}\underline{\text{linearity}}:\ (f-f')(rb) := f(rb) - f'(rb)$$
$$= r f(b) - r f'(b)$$
$$= r\,(f-f')(b)$$

Leibniz rule:

(iii) We repeat previous arguments and obtain maps

$$\begin{tikzcd} B \arrow[r, "f\,g"] \arrow[drr] \arrow[rrr, bend left, "id"] & C \arrow[r, "g"] \arrow[dr] \arrow[r, bend left] & B \arrow[r, "f"] \arrow[d] & C \arrow[dl] \\ & & A & \end{tikzcd}$$

where $f \circ g \neq id$, but we will show that

$h_{f\circ g} \underset{ht}{\simeq} h_{id}$. Indeed the (ii) part proves it.