

# Algebraic geometry 1

## Exercise sheet 5

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### Exercise 1.

1. Define

$$X := U_1 \cup U_2 / \sim,$$

where  $x \sim y$  if  $x = \varphi(x_2)$  and for  $i \in \{1, 2\}$

$$\begin{aligned} \pi_i : U_i &\rightarrow x \\ x &\mapsto \bar{x}. \end{aligned}$$

We can now give  $X$  the structure of a topological space by defining a subset  $U \subset X$  to be open if  $\pi^{-1}(U) \in \mathcal{O}_{U_i}$  are open in  $U_i$ .

Notice, that  $\pi_i$  are homeomorphic onto open subsets of  $X$ . This will become important later. Next we want to define a structure sheaf on  $X$  that behaves well with restricting to  $U_i$ .

For  $U \subset X$  open, let

$$\begin{aligned} \mathcal{O}_X(U) &:= \ker(\mathcal{O}_{U_1}(\pi^{-1}(U)) \oplus \mathcal{O}_{U_2}(\pi^{-1}(U)) \rightarrow \mathcal{O}_{U_1}(\pi^{-1}(U) \cap U_1) \\ &\quad (x, y) \mapsto x|_{\pi^{-1}(U) \cap U_1} - \varphi^\#(\pi_2^{-1}(U) \cap U_2)(y|_{\pi_2^{-1}(U) \cap U_2})), \end{aligned}$$

where the subtraction in the above term comes from the group structure of  $\mathcal{O}_{U_1}(\pi^{-1}(U) \cap U_1)$ . This is of course a group again, as the kernel of a ring map.

We conclude, that  $(X, \mathcal{O}_X)$  is a scheme, because  $X = \pi_1(U_1) \cup \pi_2(U_2)$  can be covered by affine schemes using the cover from  $U_1$  and  $U_2$  and by construction of the structure sheaf  $\mathcal{O}_{X|U_1} = \mathcal{O}_{x_i}$ . Here we finally used, as promised, that  $\pi_i$  is a homeomorphism onto open subsets of  $X$ .