Algebraic geometry 1
Exercise sheet 7

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Exercise 1.

1. We have the following bijection

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_x}(\widetilde{\mathcal{N}_{|A}}, f_*\widetilde{\mathcal{N}}) &\cong \mathrm{Hom}_A(\widetilde{\mathcal{N}_{|A}}(B), f_*\widetilde{\mathcal{N}}(B)) \\ &= \mathrm{Hom}_A(N_{|A}, \widetilde{\mathcal{N}}(A)) \cong \mathrm{Hom}_{\mathcal{O}_x}(\widetilde{\mathcal{N}_{|A}}, \widetilde{\mathcal{N}_{|A}}).\end{aligned}$$

   By the Yoneda lemma, this implies that $f_*\widetilde{N} \cong \tilde{N}_{|A}$.

2. For the second part of this exercise, we extend the first part as follows, using that f_* is left-adjoint to f^*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_y}(f^*\tilde{\mathcal{M}}, \tilde{N}) &\cong \mathrm{Hom}_{\mathcal{O}_x}(\widetilde{\mathcal{M}}, f_*\tilde{\mathcal{N}}) \cong \mathrm{Hom}_A(\widetilde{\mathcal{M}}(B), f_*\widetilde{\mathcal{N}}(B)) \\ = \mathrm{Hom}_A(M, \widetilde{\mathcal{N}}(A)) &\cong \mathrm{Hom}_B(N_{|A} \otimes_A B, \tilde{N}(A)) \cong \mathrm{Hom}_{\mathcal{O}_y}(\widetilde{\mathcal{M} \otimes_A B}, \widetilde{\mathcal{N}}).\end{aligned}$$

   Now, by the Yoneda lemma we get again that

$$\widetilde{\mathcal{M} \otimes_A B} \cong f^*\tilde{\mathcal{M}}.$$

   Next, we want to show that we can extend this exercise from affine schemes to schemes.

   Let S_i with $i \in I$ be a cover of S by open affines. Then for each $i \in I$ we get that $g^{-1}(S_i)$ is a subscheme of $Z_i \subset Z$ (unfortunately not necessarilary affine). Now, we cover each of these subschemes Z_i by open affines Z_{ij}. By construction Z_{ij} maps into S_i.