Algebraic geometry 1
Exercise sheet 3

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Exercise 1.

1. Let X be a finite set that is irreducible with respect to some topology $\mathcal{F}$ on X. Then we get $\mid \mathcal{F} \mid < \infty$ and since finite unions of closed sets are closed again, we get that

$$X' := \bigcup_{U \subsetneq X \text{closed}} U$$

is closed in X. Since X is by assumption irreducible, $X \neq X'$, so we can pick $x_0 \in X \backslash X'$, which is by construction generic. For the second part of the exercise we use part 2 of Hochster's Theorem. As a finite set, X is quasicompact and as a basis $\mathcal{B}$ consisting of quasicompact open sets stable under finite intersections take all of the open sets.
It remains to show that X is sober. We need to check that every irreducible subset of X has a unique generic point. The existence of a generic point comes from part of of this exercise.
Uniqueness of this point is due to the fact that generic points in T_0 spaces are unique if they exist, which follows directly from the definition of T_0.

1. Let X be a finite irreducible topological space. Since

$$X = \bigcup_{x \in X} \overline{\{x\}}$$

is a finite decomposition into closed sets, we must have $\overline{\{x\}} = X$ for some $x \in X$. This x is a generic point of X.

If we additionally assumed X is T_0, then this point x would be unique, since in a T_0 space we have $\overline{\{x\}} \neq \overline{\{y\}}$ for $x \neq y$. Also in a finite space the conditions of quasicompactness and the basis being stable under finite intersections are clearly fulfilled. So finite T_0 spaces are spectral.

2. Let us first describe what $\mathrm{Spec}(\mathbb{Z})$ looks like. The ring $\mathbb{Z}$ a PID with prime ideals being those (a), for which $a \in \mathbb{Z}$ is either a prime number or $a = 0$. So

$$\mathrm{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime}\} \cup \{(0)\}.$$

In this space (0) is a unique generic point.

Closed sets in $\mathrm{Spec}(\mathbb{Z})$ are by definiton

$$V((a)) = \{(p) \in \mathrm{Spec}(\mathbb{Z}) \mid (a) \subseteq (p)\} = \{(p) \in \mathrm{Spec}(\mathbb{Z}) \mid p \text{ divides } a\}.$$

So if $a = \Pi_i p_i^{k_i}$, then $V((a)) = \{(p_i)\}_i \subseteq \mathrm{Spec}(\mathbb{Z})$. Since any $a \in \mathbb{Z}$ is only divisible by finitely many prime numbers, we get the cofinite topology on $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$.

Adding generic point (0) to $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$ is the construction $X \to X^{\mathrm{sob}}$ which we did last week.

So closed sets in $\mathrm{Spec}(\mathbb{Z})$ are exactly all finite subsets of $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$, the whole space and the empty set.

Define the diagram with objects

$$F_i := \{(p_1), \ldots, (p_i)\} \cup \{(0)\}$$

with subspace topology, as $F_i \subseteq \mathrm{Spec}(\mathbb{Z})$, and morphisms

$$f_i \colon F_{i+1} \to F_i$$

mapping $f_i((p_j)) = (p_j)$ for $j \leq i$ and $f_i((p_{i+1})) = f_i((0)) = (0)$.

We view F_i as a subspace of F_j for $i < j$.

First we have to check that f_i are continuous. Take a closed subset $U \subseteq F_i$. If $(0) \in U$, then $U = F_i$ and $f_i^{-1}(U) = F_{i+1}$. And if $U \subseteq \{(p_1), \ldots, (p_i)\}$, then $f_i^{-1}(U) = \{(p_1), \ldots, (p_i)\} \subseteq F_{i+1}$.

We claim $\mathrm{Spec}(\mathbb{Z})$ is the limit of this diagram. For that we define $\alpha_i \colon \mathrm{Spec}(\mathbb{Z}) \to F_i$ by $\alpha_i((p_j)) = (p_j)$ for $j \leq i$ and $\alpha_i((p_j)) = \alpha_i((0)) = (0)$ for $j > i$. Clearly we have $\alpha_i = f_i \circ \alpha_{i+1}$.

Suppose now there exists an object B with $\beta_i \colon B \to F_i$. Define $\alpha \colon B \to \mathrm{Spec}(\mathbb{Z})$ in the following way. Take $b \in B$ and look at $\beta_i(b)$ for different $i = 1, 2, \ldots$. If $\beta_i(b) = (0)$ for all i, then define $\alpha(b) = (0) \in \mathrm{Spec}(\mathbb{Z})$. If $\beta_{i_0}(b) \neq (0)$ for some i_0, then $\beta_j(b) = \beta_{i_0}(b)$ for all $j > i_0$ using commutativity of the diagram.

Lets show that α is continuous. Take a closed subset $A \subseteq \mathrm{Spec}(\mathbb{Z})$. If $(0) \in A$, then $\alpha^{-1}(A) = B$. Else A is finite subset of $\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$. Then A is contained in some F_i (pick i to be largest such that $(p_i) \in A$) and $(\alpha_i^{-1} \circ \alpha_i)(A) = A$ holds. Therefore

$$\alpha^{-1}(A) = (\alpha^{-1} \circ \alpha_i^{-1} \circ \alpha_i)(A) = (\beta_i^{-1} \circ \alpha_i)(A) = \beta_i^{-1}(\alpha_i(A)).$$

The set $\alpha_i(A)$ is a finite subset of F_i that does not contain (0), so it is closed. Since β_i is continuous, the set $\alpha^{-1}(A)$ is then closed, which proves continuity of α.

The map α was defined using condition that the diagram must commute. It is clear that any other definition of α would not satisfy commutativity conditions $\beta_i = \alpha_i \circ \alpha$ for all i. So α was unique and thus $\mathrm{Spec}(\mathbb{Z})$ is the limit of the constructed diagram.

Exercise 2. Denote $A = \lim A_i$, $B = \lim B_i$ and $C = \lim C_i$. Also denote maps $A_i \to A$ with f_i, $B_i \to B$ with g_i and $C_i \to C$ with h_i.

By composing α_i and g_i we get $A_i \to B$ defined as $g_i \circ \alpha_i$. Then by the definition of a colimit we have a unique map $\alpha : A \to B$, such that $g_i \circ \alpha_i = \alpha \circ f_i$. In the same way we obtain $\beta \colon B \to C$. With these definitions the whole diagram commutes.

Exercise 3.

1. Let $\mathcal{F}, \mathcal{G}$ be presheaves on X and $(\phi_U)_{U \in Ouv(X)}, (\psi_U)_{U \in Ouv(X)}$ morphism of presheaves from $\mathcal{F}$ to $\mathcal{G}$.
   We define for $U \in Ouv(X)$

   $$(\phi + \psi)(U) := \phi(U) + \psi(U).$$

   This is indeed a map of presheaves again, because we can restrict ϕ and ψ independently.
   The zero object in this category is the presheaf that sends every open set $U \in Ouv(X)$ to the trivial group $(0, +)$. It is inital and terminal, because group maps send 0 to 0. For this reason the category of presheaves is an additive category and since morphism of sheaves are defined using their underlying presheaves, also the category of sheaves form an additive category.
   We still need to check that kernels and cokernels exist in the category of Sheaves, so from now on $\mathcal{F}, \mathcal{G}$ and maps between them we be of sheaves.
   Define

   $$(ker\phi)(U) := ker\phi(U).$$

   We check that this is indeed a presheaf. To do that one has to check that restrictions behave well.
   Next, one checks that this is indeed a sheaf.