

# Algebraic geometry 1

## Exercise sheet 6

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### Exercise 1.

1. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram

$$\begin{array}{ccccc}
 U_i \times_{S_{i,j}} V_j & & & & \\
 \searrow \scriptstyle p & \dashrightarrow & X \times_S Y & \xrightarrow{\pi_2} & V_j \\
 & & \downarrow \scriptstyle \pi_1 & & \downarrow \scriptstyle \psi \\
 & & U_i & \xrightarrow{\phi} & S_{i,j}
 \end{array}$$

(Note: A curved arrow labeled  $q$  also points from  $U_i \times_{S_{i,j}} V_j$  to  $V_j$ )

Therefore, on the level of sets,

$$U_i \times_{S_{i,j}} V_j \subset X \times_S Y,$$

but in exercise 5.2.1, we showed that this induces an open immersion as locally ringed spaces.

Now observe that

$$\begin{array}{ccc}
 \bigcup_{i,j} (U_i \times_{S_{i,j}} V_j) & \longrightarrow & Y \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & S
 \end{array}$$

commutes, because  $S = \bigcup_{i,j} S_{i,j}$ . Now by uniqueness of the pullback,

$$\bigcup_{i,j} (U_i \times_{S_{i,j}} V_j) \cong U_i \times_{S_{i,j}} V_j.$$

I guess this is a good step in the direction of understanding why the pullback in the category of sheaves exists, right? If we assume  $X, Y, S$  to be

sheaves and  $U_i, V_j, S_{i,j}$  to be affine schemes, then by the above argument we found a cover of  $X \times_S Y$  by affine schemes.

- 1 (alternative) Let  $U \subseteq X$ ,  $V \subseteq Y$  and  $W \subseteq S$  open subschemes. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram

$$\begin{array}{ccccc}
 U \times_W V & \xrightarrow{q} & V & & \\
 \downarrow p & \searrow & \swarrow & & \\
 U & & X \times_S Y & \xrightarrow{\pi_2} & Y \\
 & \searrow & \downarrow \pi_1 & & \downarrow \psi \\
 & & X & \xrightarrow{\phi} & S
 \end{array}$$

so we get a unique map  $U \times_W V \rightarrow X \times_S Y$ .

Observe the concrete space  $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$  with inclusion

$$\pi_1^{-1}(U) \cap \pi_2^{-1}(V) \hookrightarrow X \times_S Y$$

also satisfies the universal property of being a fibred product  $U \times_S V$ . If  $T \rightarrow U$  and  $T \rightarrow V$  such that  $T \rightarrow U \rightarrow W = T \rightarrow V \rightarrow W$ , then we can create the following diagram

$$\begin{array}{ccccc}
 T & \xrightarrow{q} & V & & \\
 \downarrow p & \searrow & \swarrow & & \\
 U & & X \times_S Y & \xrightarrow{\pi_2} & Y \\
 & \searrow & \downarrow \pi_1 & & \downarrow \psi \\
 & & X & \xrightarrow{\phi} & S
 \end{array}$$

from which we get a unique map  $T \rightarrow X \times_S Y$ . Since its image is contained in  $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ , it factors uniquely through

$$T \rightarrow \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \hookrightarrow X \times_S Y$$

Therefore, the fibre product  $U \times_W V$  can be identified as an open subspace  $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) \subseteq X \times_S Y$ .

Then clearly for coverings  $X = \cup_i U_i$ ,  $Y = \cup_i V_i$  and  $S = \cup_{i,j} S_{i,j}$ , we have a covering

$$\bigcup_{i,j} U_i \times_{S_{i,j}} V_j = \bigcup_{i,j} (\pi_1^{-1}(U_i) \cap \pi_2^{-1}(V_j)) = X \times_S Y.$$

I guess this is a good step in the direction of understanding why the pullback in the category of schemes exists, right? If we assume  $X, Y, S$  to be sheaves and  $U_i, V_j, S_{i,j}$  to be affine schemes, then by the above argument we found a cover of  $X \times_S Y$  by affine schemes.

2. If  $X = \text{Spec}(A), Y = \text{Spec}(B), Z = \text{Spec}(R)$  were affine schemes, we would have homeomorphism, since  $\text{Spec}(A \otimes_R B) = \text{Spec}(A) \times_{\text{Spec}(R)} \text{Spec}(B)$ . But in schemes (using that the fiber product is also a scheme) we locally have an isomorphism, which means the map must be surjective.

Concretely:

**Exercise 2.**

1. First let  $f: X \rightarrow S$  be open immersion. In this case we can directly use previous exercise on the following fibred product

$$\begin{array}{ccc} S & \xrightarrow{\text{id}} & S \\ p \uparrow & & \uparrow g \\ S \times_S S' & \xrightarrow{q} & S' \end{array}$$

by taking subset of  $X \subseteq S$  and immediately getting open immersion  $X \times_S S' \rightarrow S \times_S S'$ , which we postcompose with canonical isomorphism  $S \times_S S' \rightarrow S'$  and get that  $X \times_S S' \rightarrow S'$  is open immersion.

Now suppose  $f: X \rightarrow S$  is a closed immersion. So we have the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ p \uparrow & & \uparrow g \\ X \times_S S' & \xrightarrow{q} & S' \end{array}$$

We want to show  $X \times_S S' \rightarrow S'$  is also a closed immersion. For that it satisfies to find an open covering of  $S'$  with affine subschemes such that preimages with be also affine schemes and induced maps of rings surjective.

Take  $s \in S'$  and a neighborhood  $g(s) \in \text{Spec}(R) = U \subseteq S$ . Preimage  $f^{-1}(U) = \text{Spec}(A)$  already is affine, since  $f$  is closed immersion, and for  $g^{-1}(U)$  we have to take some smaller affine neighborhood of  $s$ . So we get  $s \in \text{Spec}(B) \subseteq g^{-1}(U)$ . Then use previous exercise on these open sets and obtain open immersion

$$\text{Spec}(A) \times_{\text{Spec}(R)} \text{Spec}(B) = \text{Spec}(A \otimes_R B) \rightarrow X \times_S S'.$$

By remark at the start we have

$$\text{Spec}(A \otimes_R B) = p^{-1}(\text{Spec}(A)) \cap q^{-1}(\text{Spec}(B)) = q^{-1}(\text{Spec}(B)).$$

Only thing to argue is why the map  $B \rightarrow A \otimes_R B$  is surjective. Since  $R \twoheadrightarrow A$  surjective,  $R/I \cong A$  and thus  $A \otimes_R B = R/I \otimes_R B = B/IB$ . Clearly  $B \rightarrow B/IB$  is then surjective.

**Exercise 3.** By definition we have to compute a fibred product of  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  and  $\text{Spec}(k(p)) \rightarrow \text{Spec}(A)$  (where  $k(p)$  is the residue field of  $p \in \text{Spec}(A)$  and  $\rightarrow$  is the canonical inclusion). Since we are dealing with affine schemes, we can express it concretely as  $\text{Spec}(B \otimes_A k(p))$ . Note that  $B$  has the structure of an  $A$ -algebra, which is induced by the starting morphism of schemes  $\text{Spec}(B) \rightarrow \text{Spec}(A)$ . So this exercise reduces to computing these tensor products.

We also observe that  $k[T]$  is a PID, which means every non-zero prime ideal is a maximal ideal. This will be handy when computing residue fields, because after quotienting with a non-zero ideal we already get a field (we do not have to further take the quotient field).

1. In the first example we do now even have to calculate the tensor product, because we can rewrite  $k[T, U]/(TU - 1) = k[T, T^{-1}]$ , so this is just a localization of  $k[T]$ . Morphism of spectrums, induced by inclusion into localization, is an open immersion, so fibers will be singletons if  $x \in D(T)$  and empty sets otherwise. And the structure sheaf is also clear, it is just the restriction of structure sheaf  $\mathcal{O}_{\text{Spec}(k[T])}$ .
- 2.
- 3.
- 4.

**Exercise 4.** Take  $U = D(f)$  for some  $f \in A$  and let  $U = \cup_i D(f_i)$  be some cover. We have to check that

$$M[f^{-1}] \rightarrow \text{Eq} \left[ \prod_i M[f_i^{-1}] \rightrightarrows \prod_{i,j} M[(f_i f_j)^{-1}] \right]$$

is isomorphism.

This proof is exactly the same as when we proved that  $\mathcal{O}_{\text{Spec}(A)}$  is a sheaf, after we defined it the basis of principal opens.

Then proved that  $A = \text{Eq} \left[ \prod_i A[f_i^{-1}] \rightrightarrows \prod_{i,j} A[(f_i f_j)^{-1}] \right]$  where  $\text{Spec}(A) = \cup_i D(f_i)$  is a cover.