

Algebraic geometry 1

Exercise sheet 6

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Exercise 1.

1. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram

$$\begin{array}{ccccc}
 U_i \times_{S_{i,j}} V_j & & & & \\
 & \searrow \text{dotted} & & \nearrow q & \\
 & X \times_S Y & \xrightarrow{\pi_2} & V_j & \\
 & \downarrow \pi_1 & & \downarrow \psi & \\
 & U_i & \xrightarrow{\phi} & S_{i,j} &
 \end{array}$$

(Note: A curved arrow labeled p also goes from $U_i \times_{S_{i,j}} V_j$ to U_i)

Therefore, on the level of sets,

$$U_i \times_{S_{i,j}} V_j \subset X \times_S Y,$$

but in exercise 5.2.1, we showed that this induces an open immersion as locally ringed spaces.

Now observe that

$$\begin{array}{ccc}
 \bigcup_{i,j} (U_i \times_{S_{i,j}} V_j) & \longrightarrow & Y \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & S
 \end{array}$$

commutes, because $S = \bigcup_{i,j} S_{i,j}$. Now by uniqueness of the pullback,

$$\bigcup_{i,j} (U_i \times_{S_{i,j}} V_j) \cong U_i \times_{S_{i,j}} V_j.$$

Exercise 3. By definition we have to compute a fibred product of $\text{Spec}(B) \rightarrow \text{Spec}(A)$ and $\text{Spec}(k(p)) \rightarrow \text{Spec}(A)$ (where $k(p)$ is the residue field of $p \in \text{Spec}(A)$ and \rightarrow is the canonical inclusion). Since we are dealing with affine schemes, we can express it concretely as $\text{Spec}(B \otimes_A k(p))$. Note that B has the structure of an A -algebra, which is induced by the starting morphism of schemes $\text{Spec}(B) \rightarrow \text{Spec}(A)$. So this exercise reduces to computing these tensor products.

We also observe that $k[T]$ is a PID, which means every non-zero prime ideal is a maximal ideal. This will be handy when computing residue fields, because after quotienting with a non-zero ideal we already get a field (we do not have to further take the quotient field).

1. In the first example we do now even have to calculate the tensor product, because we can rewrite $k[T, U]/(TU - 1) = k[T, T^{-1}]$, so this is just a localization of $k[T]$. Morphism of spectrums, induced by inclusion into localization, is an open immersion, so fibers will be singletons if $x \in D(T)$ and empty sets otherwise. And the structure sheaf is also clear, it is just the restriction of structure sheaf $\mathcal{O}_{\text{Spec}(k[T])}$.
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Exercise 4. Take $U = D(f)$ for some $f \in A$ and let $U = \cup_i D(f_i)$ be some cover. We have to check that

$$M[f^{-1}] \rightarrow \text{Eq} \left[\prod_i M[f_i^{-1}] \rightrightarrows \prod_{i,j} M[(f_i f_j)^{-1}] \right]$$

is isomorphism.

This proof is exactly the same as when we proved that $\mathcal{O}_{\text{Spec}(A)}$ is a sheaf, after we defined it the basis of principal opens.

Then proved that $A = \text{Eq} \left[\prod_i A[f_i^{-1}] \rightrightarrows \prod_{i,j} A[(f_i f_j)^{-1}] \right]$ where $\text{Spec}(A) = \cup_i D(f_i)$ is a cover.

We can simply tensor the whole diagram and, since tensor product commute with direct limits, we have that

$$M = \text{Eq} \left[\prod_i M \otimes_A A[f_i^{-1}] \rightrightarrows \prod_{i,j} M \otimes_A A[(f_i f_j)^{-1}] \right].$$