Algebraic geometry 1
Exercise sheet 8

Solutions by: Eric Rudolph and David Čadež

7. Dezember 2023

Exercise 1.

1. Let $I \subseteq A$ be an ideal, which is finite locally free. Take some $f = up_1^{a_1} \dots p_r^{a_r} \in I$.

   Pick some $D(g_i)$ such that $(p_i) \in D(g_i)$ and $(p_j) \notin D(g_i)$ (for all $j \neq i$) and I_{g_i} finitely generated A_{g_i}-module (not sure if free-ness is preserved by making neighborhoods smaller). We can do that by taking g_i' to be such that $I_{g_i'}$ is a finite free module and then letting $g_i = g_i' \prod_{j \neq i} p_j$.

   That means I_{g_i} is finitely generated. Note that we can also assume, that $D(g_i)$ is small enough that I_{g_i} is generated by a single element, while still $(p_i) \in D(g_i)$. We could do this for example by taking generators of I_{g_i}, writing prime decomposition of each one and then $g_i := pg_i$ for each prime $p \neq p_i$ that appears in some decomposition. So $I_{g_i} = (h_i)$ for some $h_i \in A_{g_i}$. Since $f = u'p_i^{a_i} \in I_{g_i}$, we must have $h_i = p_i^{c_i}$ (up to multiplication with a unit) for some c_i.

   This way we obtain open subsets $D(g_i)$ each of which contains only their respective $(p_i) \in \operatorname{Spec}(A)$. To get an open cover, we need to add principal opens $D(g')$, which can be chosen to not contain any (p_i). Therefore localizations $I_{g'}$ will be equal to $A_{g'}$, because they invert $f \in I$. So on $D(g')$ any element $\in A$ will satisfy the condition.

   Now we show that it is enough to check whether $a \in I$ (for element $a \in A$) on a cover with principal opens.

   We use the following result: Let $(g_1, \dots, g_n) = A$ be an open cover. If for an A-module M, the localizations $M_{g_i} = 0$ for every i, then $M = 0$.

   We apply it to this case: If we have an element $a \in A$ and we want to check if $a \in I$, we can set $M = A_a/I_a$ an A_a-module and $(g_1, \dots, g_n) = A$ an open cover. If we know $a \in I_{g_i}$ for every i, then $A_{ag_i}/I_{ag_i} = 0$ for every i, and thus $A_a/I_a = 0$, so $a \in I$.

   This demonstrates that $I \supseteq (p_1^{c_1} \dots p_r^{c_r})$, and the other inclusion is obvious.

Now it is clear that $a \in I$ if and only if $p_i^{c_i} \mid a$ for every $i = 1, \ldots, r$. This proves that $I = (p_1^{c_1} \ldots p_r^{c_r})$.

2. Pick any invertible A-module M.

   Then M is finite locally free.

   Pick any non-zero homomorphism $\varphi \in \mathrm{Hom}_A(M, A)$. We first show that it is injective (using stackexchange): Let $K = \mathrm{Quot}(A)$. Then $K \otimes M \cong K$, since M is locally free of rank 1 by assumption. Then $\varphi\colon M \to A$ induces $\varphi \otimes \mathrm{id}\colon M \otimes K \to A \otimes K$. Since M is torsion free, we have an embedding $M \to M \otimes K$. Since φ is non-zero, so is $\varphi \otimes \mathrm{id}$. Since $M \otimes K$ and $A \otimes K$ are 1-dimensional vector spaces and $\varphi \otimes \mathrm{id}$ a bijection, φ is injective.

   The image $\varphi(M)$ is an ideal in A. Since M is finite locally free and φ injective, the image $\varphi(M)$ is also finite locally free. By the previous part, we get that $\varphi(M)$ is principal and thus isomorphic to A (since A is a domain). So $M \cong \varphi(M) \cong A$. Since every invertible ideal is isomorphic to A, we have that $\mathrm{Pic}(A) = 0$.

Exercise 2. Note that for a unique factorization domain A we get by Gauss that also $A[x_1, \ldots, x_n]$ is a unique factorization domain. This means that by construction of $\mathbb{P}_A^n$ its local rings are UFD's. Using stacks project, we infer that $\mathrm{Pic}(\mathbb{P}_A^n) \cong \mathrm{CL}(\mathbb{P}_A^n) = \mathbb{Z}$.

We now want to give a concrete argument using the given map.

Note that by definion $\mathcal{O}_A^n(0)$ is just the structure sheaf and since maps of groups send 1 to 1, we found the neutral element of this group. One can also check locally that

$$O_{\mathbb{P}_A^n}(m) \otimes_{O_{\mathbb{P}_A^n}} O_{\mathbb{P}_A^n}(n) = O_{\mathbb{P}_A^n}(m+n).$$

This also proves that the given map maps to $\mathrm{Pic}(\mathbb{P}_A^n)$.

It is also quite clear by definition that for $m \neq n$ we have

$$O_{\mathbb{P}_A^n}(m) \ncong O_{\mathbb{P}_A^n}(n). \tag{1}$$

which gives us injectivity. It remains to show surjectivity of this map.

We have to show that any invertible $\mathcal{O}_{\mathbb{P}_k^n}$-module is $\mathcal{O}_{\mathbb{P}_k^n}(d)$ for some $d \in \mathbb{Z}$. By the previous exercise, for every invertible sheaf $\mathcal{M}$ we have $\mathcal{M}(U_i) \cong \mathcal{O}_{U_i}$, where $\mathcal{O}_{U_i}$ is the structure sheaf on affine U_i. This mean that $\mathcal{M}$ will be defined by gluing rules. We need to define ring maps $\mathcal{M}(U_{i,j}) \to \alpha_{i,j}^* \mathcal{M}(U_{j,i})$. Since they are both rings and the map has to be $\mathcal{O}_{U_i}(U_i)$-linear, they are uniquely defined by where they map the unit. Since only invertible elements in $\mathcal{O}_{U_i}(U_{j/i})$ are $X_{j/i}$, element 1 has to be mapped to some power of it. Lastly we show that gluing U_i and U_j already defines all other gluing rules, because they need to satisfy the cocycle condition. So we get that $\mathcal{M}$ is isomorphic to the twisting sheaf $\mathcal{O}_{\mathbb{P}_k^n}(m)$, where m is the power of $X_{j/i}$ that we chose above.

Exercise 3.

1. In exercise 2 we showed that all invertible quasicoherent sheaves on $\mathbb{P}^n_k$ are isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$. So we have to show $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is an invertible sheaf.

   Since invertible $\mathcal{O}_{\mathbb{P}^n_k}$-modules are same as line bundles, we have to show that locally $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^m_k}$.

   By definition $f^*\mathcal{O}_{\mathbb{P}^m_k}(1) = f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^m_k}} \mathcal{O}_{\mathbb{P}^n_k}$. Pick some $x \in \mathbb{P}^n_k$. Pick small enough affine neighborhood $f(x) \in U \subseteq \mathbb{P}^m_k$ such that $\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^m_k}$ on U. Now pick neighborhood $x \in W \subseteq \mathbb{P}^m_k$ such that $f(W) \subseteq U$.

   Then

$$\begin{aligned} f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)(W) &= \operatorname{colim}_{f(W)\subseteq V} \mathcal{O}_{\mathbb{P}^m_k}(1)(V) \\ &= \operatorname{colim}_{f(W)\subseteq V\subseteq U} \mathcal{O}_{\mathbb{P}^m_k}(1)(V) \\ &\cong \operatorname{colim}_{f(W)\subseteq V\subseteq U} \mathcal{O}_{\mathbb{P}^m_k}(V) \\ &\cong f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(W). \end{aligned}$$

   So locally $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}$, so $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^m_k}} \mathcal{O}_{\mathbb{P}^n_k}$ is locally isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}$, which proves that $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is an invertible $\mathcal{O}_{\mathbb{P}^n_k}$-module and thus isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$.

2. At first it was not completely clear to us what the map $f^* \colon \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1)) \to \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ is. So we assumed it is the following:

   For a global section $s \in \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1))$ we first map it with the restriction

$$\Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1)) \to \Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)).$$

   Denote its image with s'. By definition we have

$$\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1)) = \Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)) \otimes_{\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k})} \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})$$

   So include s' into $\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1))$ as $s' \otimes 1$. By part 1 we have an isomorphism $\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1)) \cong \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$. We map $s' \otimes 1$ with this isomorphism to obtain $f^*(s)$.

   The polynomials $y_0, \ldots, y_n$ generate $\Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1))$, which is isomorphic to the module of homogenous polynomials of degree 1. So their restrictions generate $\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1))$. Their images in the tensor product

$$\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)) \otimes_{\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k})} \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})$$

   then also stay generators. And finally isomorphism $\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1)) \cong \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ also preserves generating set.

   So $g_i = f^*(y_i) \in \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ are generators.

If $d \geq 1$, then g_i always vanish at $0 \in \mathbb{A}_k^{n+1}$.

Take some $(a_0, \ldots, a_n) \in V(g_0, \ldots, g_m) \subseteq \mathbb{A}_k^{n+1}$. If $a_i \neq 0$ for some i, then the line going through $(a_0, \ldots, a_n)$ and 0 would lie in $V(g_0, \ldots, g_m)$. Then $(g_0, \ldots, g_m)$ would be contained in the set of equations parametrizing this line. Therefore it wouldn't be generating the whole module.

3. If $m < n$, then $\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$, which can be identified as a k-module of homogenous polynomials of degree d, cannot be generated by m elements. It is a vector space of dimension $\binom{n+d}{n}$ and $\binom{n+d}{n} > m$ for $d > 0$. Therefore $d = 0$.

   Now we show that f must be constant. Suppose $f(\mathbb{P}_k^n)$ has two points. Then we can separate these two point with two independent polynomials $s, t \in k[y_0, \ldots, y_m]_1$. Then $s \otimes 1$ and $t \otimes 1$ are independent elements of $\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n})$-module

$$\Gamma(\mathbb{P}_k^n, f^{-1}\mathcal{O}_{\mathbb{P}_k^m}(1)) \otimes_{\Gamma(\mathbb{P}_k^n, f^{-1}\mathcal{O}_{\mathbb{P}_k^m})} \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}).$$

   But $\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(0))$ is the trivial line bundle, so it cannot contain two elements which are independent over global sections of line bundle itself.

Exercise 4.

1. Let $U_i = \operatorname{Spec}(A_i)$.

   Take a point $x \in U_1 \cap U_2$.

   Take a principal open $x \in D(f) \subseteq U_1$ ($f \in U_1$). Then find a smaller principal open $x \in D(g) \subseteq D(f) \subseteq U_2$ ($g \in U_2$).

   Now we show that $D(g)$ is also a principal open in U_1.

   Since $D(f) \subseteq U_2$ open, we have a map $\mathcal{O}(U_2) \to \mathcal{O}(D(f))$, which induces $A_2 \to (A_1)_f$. Denote by $g' = g|_{\operatorname{Spec}((A_1)_f)}$ the image of g under this map. Since $g' \in (A_1)_f$, we can write it as $g' = \frac{h}{f^n}$. Then $D(g) = D(g) \cap D(f) = D(g') \cap D(f) = D(h) \cap D(f) = D(hf)$, where $h, f \in A_1$. This shows that $D(g)$ is also principal open in U_1.

2. We have to show that the property of being of finite presentation is a local property and that f as defined above is locally of finite presentation.

   Let $\operatorname{Spec}(B) \subseteq X$ and $\operatorname{Spec}(A) \subseteq S$ open affines. Pick a point $x \in \operatorname{Spec}(B)$. Then $x \in \operatorname{Spec}(B) \cap \operatorname{Spec}(B_i)$ for some i. Pick some neighborhood $x \in U \subseteq \operatorname{Spec}(B) \cap \operatorname{Spec}(B_i)$ such that U is principal open in $\operatorname{Spec}(B)$ and in $\operatorname{Spec}(B_i)$.

   Now take a neighborhood $f(x) \in V \subseteq f(U)$ so that V is principal open in $\operatorname{Spec}(A)$ and in $\operatorname{Spec}(A_i)$. Now take another smaller neighborhood $x \in U' \subseteq f^{-1}(V)$ such that U' is principal open in $\operatorname{Spec}(B)$ and in $\operatorname{Spec}(B_i)$.

So we have $U' \to V$, where both U' and V are principal opens of $\text{Spec}(B_i)$ and $\text{Spec}(A_i)$ respectively. Since $A_i \to B_i$ is of finite presentation, then localizations $(A_i)_f \to (B_i)_g$ (for some $f \in A_i$ and $g \in B_i$) are as well.

So for every point $x \in \text{Spec}(B)$ we can find a principal open neighborhood in $x \in D(f_x)$ and a principal open neighborhood $f(x) \in D(g_x)$ such that $A_{g_x} \to B_{f_x}$.

Since $\text{Spec}(B)$ is quasi-compact, we have $\text{Spec}(B) = D(f_1) \cup \cdots \cup D(f_n)$. Denote $g_1, \ldots, g_n \in A$ be the respective elements in A.

We have composition $\text{Spec}(B_{f_i}) \to \text{Spec}(A_{g_i}) \hookrightarrow \text{Spec}(A)$, which induces a map of rings $A \to A_{g_i} \to B_{f_i}$. Since $A_{g_i} \cong A[X]/(Xg_i - 1)$ and $A_{g_i} \to B_{f_i}$ are of finite presentation by assumption, and being of finite presentation is stable under compositions, we have that $A \to B_{f_i}$ are of finite presentation for every i.

Now its just commutative algebra to show that $A \to B$ is of finite presentation as well, so I hope its okay to assume this part. Otherwise we could just rewrite something like Lemma 00EP.