Algebraic geometry 1
Exercise sheet 7

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Exercise 1.

1. We have the following bijection
$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_x}(\widetilde{\mathcal{N}_{|A}}, f_*\widetilde{\mathcal{N}}) &\cong \mathrm{Hom}_A(\widetilde{\mathcal{N}_{|A}}(B), f_*\widetilde{\mathcal{N}}(B))\\ &= \mathrm{Hom}_A(N_{|A}, \widetilde{\mathcal{N}}(A)) \cong \mathrm{Hom}_{\mathcal{O}_x}(\widetilde{\mathcal{N}_{|A}}, \widetilde{\mathcal{N}_{|A}}).\end{aligned}$$
By the Yoneda lemma, this implies that $f_*\widetilde{N} \cong \tilde{N}_{|A}$.
2. For the second part of this exercise, we extend the first part as follows, using that f_* is left-adjoint to f^*
$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_y}(f^*\tilde{\mathcal{M}}, \tilde{N}) &\cong \mathrm{Hom}_{\mathcal{O}_x}(\widetilde{\mathcal{M}}, f_*\widetilde{N}) \cong \mathrm{Hom}_A(\widetilde{\mathcal{M}}(B), f_*\widetilde{N}(B))\\ = \mathrm{Hom}_A(M, \widetilde{\mathcal{N}}(A)) &\cong \mathrm{Hom}_B(N_{|A} \otimes_A B, \tilde{N}(A)) \cong \mathrm{Hom}_{\mathcal{O}_y}(\widetilde{\mathcal{M} \otimes_A B}, \widetilde{\mathcal{N}}).\end{aligned}$$
Now, by the Yoneda lemma we again obtain that
$$\widetilde{\mathcal{M} \otimes_A B} \cong f^*\tilde{\mathcal{M}}.$$
Next, we want to show that we can extend this exercise from affine schemes to schemes.

   Let S_i with $i \in I$ be a cover of S by open affines. Then for each $i \in I$ we get that $g^{-1}(S_i)$ is a subscheme of $Z_i \subset Z$ (unfortunately not necessarilary affine). Now, we cover each of these subschemes Z_i by open affines Z_{ij}. By construction g maps Z_{ij} into S_i. Hence,
$$(g^*\mathcal{M})_{Z_{ij}} = f^*\mathcal{M}_{Z_{ij}} \cong \widetilde{M \otimes_A B},$$
showing that g^* preserves quasi-coherence.

Exercise 2. For every homogenous polynomial $F(X_0, \ldots, X_n)$ of degree m we attach $\{f_i\}_{i=0,\ldots,n}$, where $f_i(X_{0/i}, \ldots, X_{n/i})$ is the unique polynomial such that

$\beta_i(f_i) = \frac{F(X_0,\ldots,n_n)}{X_i^m}$, where

$$\beta_i \colon \mathbb{Z}[X_{0/i},\ldots,X_{n/i}] \to \mathbb{Z}[X_0,\ldots,X_n,X_i^{-1}]$$
$$X_{j/i} \mapsto \frac{X_j}{X_i}$$

Injectivity: If $f_i = 0$

Exercise 3.

1. We have a cover $\mathbb{P}^n_{\mathbb{Z}} = \cup_i U_i$, where $U_i = \operatorname{Spec}(\mathbb{Z}[X_{j/i}, j \neq i])$. We defined $\mathbb{P}^n_k$ to be simply the fibered product $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(k)$. We can use 1st exercise from sheet 6, to get a cover

$$\begin{aligned}
\mathbb{P}^n_k &= \bigcup_i U_i \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(k) \\
&= \bigcup_i \operatorname{Spec}(\mathbb{Z}[X_{j/i}, j \neq i]) \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(k) \\
&= \bigcup_i \operatorname{Spec}(\mathbb{Z}[X_{j/i}, j \neq i] \otimes_{\mathbb{Z}} k) \\
&= \bigcup_i \operatorname{Spec}(k[X_{j/i}, j \neq i]).
\end{aligned}$$

   Define morphism $\mathbb{P}^n_k \to (\mathbb{P}^n_k(k))^{\text{sob}}$ on the cover.

   We can show that soberification of $(\mathbb{P}^n_k(k))^{\text{sob}}$ is same as soberification on each open set of the cover and then gluing.

   Lemma 06N9 We have that for a space X and a covering $X = \bigcup_i X_i$, the space X is sober if and only if X_i is sober for every i.

   We showed on sheet 3 that soberification of an $\mathbb{A}^n_k(k)$ is $\operatorname{Spec}(k[X_1,\ldots,X_n])$.

   So we have $(\mathbb{P}^n_k(k))^{\text{sob}} = \bigcup_i (\mathbb{A}^n_k(k))^{\text{sob}} = \bigcup_i \operatorname{Spec}(k[X_1,\ldots,X_n])$.

   Define morphism

$$\mathbb{P}^n_k = \bigcup_i \operatorname{Spec}(k[X_{j/i}, j \neq i]) \to (\mathbb{P}^n_k(k))^{\text{sob}} = \bigcup_i \operatorname{Spec}(k[X_1,\ldots,X_n])$$

   with the obvious isomorphism for every i.

2. We defined $V(s) \subseteq \mathbb{P}^n_k$ locally on affine subschemes. Our definition assumed we have a line bundle $\mathcal{L}$ on $(X, \mathcal{O}_X)$.

   In our case $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_k}(d)$

   Locally on $U_i = \operatorname{Spec}(k[X_{j/i}, j \neq i])$ we have isomorphism $\mathcal{O}_{\mathbb{P}^n_k}(d)|_{U_i} \cong \mathcal{O}_{U_i}$.

   So we have $V(s)|_{U_i} = \operatorname{Spec}(k[X_{j/i}, j \neq i]/t())$.