LECTURE 14

1. Suggested additional reading:

We suggest reading on the stacks project about Cech cohomology, in particular Tag 01EM (Lemma 10.4) was used without proof on lecture. We gather this in pdf version below.

Proof. The sheaf condition says that the kernel of $(1, -1) : \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is equal to the image of $\mathcal{F}(X)$ by the first map for any abelian sheaf $\mathcal{F}$. Lemma 8.1 above implies that the map $(1, -1) : \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is surjective whenever $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. Hence if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, then we get a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

Taking cohomology gives the result (use Homology, Lemma 13.12). We omit the proof of the functoriality of the sequence. □

01EC **Lemma 8.3** (Relative Mayer-Vietoris). *Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

Namely, for any open $W \subset Y$, and for any $n \geq 0$ the corresponding sequence of groups of sections over W

$$0 \to \mathcal{I}^n(f^{-1}(W)) \to \mathcal{I}^n(U \cap f^{-1}(W)) \oplus \mathcal{I}^n(V \cap f^{-1}(W)) \to \mathcal{I}^n(U \cap V \cap f^{-1}(W)) \to 0$$

was shown to be short exact in the proof of Lemma 8.2. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U \cap V}$ see Lemma 7.1. □

9. The Čech complex and Čech cohomology

01ED Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering, see Topology, Basic notion (13). As is customary we denote $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ for the $(p+1)$-fold intersection of members of $\mathcal{U}$. Let $\mathcal{F}$ be an abelian presheaf on X. Set

$$\check{C}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0 \ldots i_p}).$$

This is an abelian group. For $s \in \check{C}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in $\mathcal{F}(U_{i_0 \ldots i_p})$. Note that if $s \in \check{C}^1(\mathcal{U}, \mathcal{F})$ and $i, j \in I$ then s_{ij} and s_{ji} are both elements of $\mathcal{F}(U_i \cap U_j)$ but there is no imposed relation between s_{ij} and s_{ji}. In other words, we are *not* working with alternating cochains (these will be defined in Section 23). We define

$$d : \check{C}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

01EE (9.0.1)
$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}|_{U_{i_0 \ldots i_{p+1}}}$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

01EF **Definition 9.1.** Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Čech cohomology groups* associated to $\mathcal{F}$ and the covering $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

01EG **Lemma 9.2.** *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. The following are equivalent*

(1) *$\mathcal{F}$ is an abelian sheaf and*
(2) *for every open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ the natural map*

$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$

is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every open covering. $\square$

0G6S **Lemma 9.3.** *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. If $U_i = U$ for some $i \in I$, then the extended Čech complex*

$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

obtained by putting $\mathcal{F}(U)$ in degree -1 with differential given by the canonical map of $\mathcal{F}(U)$ into $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F})$ is homotopy equivalent to 0.

Proof. Fix an element $i \in I$ with $U = U_i$. Observe that $U_{i_0 \ldots i_p} = U_{i_0 \ldots \hat{i}_j \ldots i_p}$ if $i_j = i$. Let us define a homotopy

$$h : \prod\nolimits_{i_0 \ldots i_{p+1}} \mathcal{F}(U_{i_0 \ldots i_{p+1}}) \longrightarrow \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}(U_{i_0 \ldots i_p})$$

by the rule

$$h(s)_{i_0 \ldots i_p} = s_{i i_0 \ldots i_p}$$

In other words, $h : \prod_{i_0} \mathcal{F}(U_{i_0}) \to \mathcal{F}(U)$ is projection onto the factor $\mathcal{F}(U_i) = \mathcal{F}(U)$ and in general the map h equals the projection onto the factors $\mathcal{F}(U_{i i_1 \ldots i_{p+1}}) = \mathcal{F}(U_{i_1 \ldots i_{p+1}})$. We compute

$$\begin{aligned}
(dh + hd)(s)_{i_0 \ldots i_p} &= \sum\nolimits_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i i_0 \ldots i_p} \\
&= \sum\nolimits_{j=0}^{p} (-1)^j s_{i i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum\nolimits_{j=0}^{p} (-1)^{j+1} s_{i i_0 \ldots \hat{i}_j \ldots i_p} \\
&= s_{i_0 \ldots i_p}
\end{aligned}$$

This proves the identity map is homotopic to zero as desired. $\square$

10. Čech cohomology as a functor on presheaves

01EH Warning: In this section we work almost exclusively with presheaves and categories of presheaves and the results are completely wrong in the setting of sheaves and categories of sheaves!

Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. We have the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ just by thinking of $\mathcal{F}$ as a presheaf of abelian groups. However, each term $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ has a natural structure of a $\mathcal{O}_X(U)$-module and the differential is given by $\mathcal{O}_X(U)$-module maps. Moreover, it is clear that the construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

01EI (10.0.1) $$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow \text{Comp}^+(\text{Mod}_{\mathcal{O}_X(U)})$$

see Derived Categories, Definition 8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 13.9.

01EJ **Lemma 10.1.** *The functor given by Equation (10.0.1) is an exact functor (see Homology, Lemma 7.2).*

Proof. For any open $W \subset U$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PMod(\mathcal{O}_X)$ to $\text{Mod}_{\mathcal{O}_X(U)}$. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. ☐

01EK **Lemma 10.2.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules (see Homology, Definition 12.1).*

Proof. By Lemma 10.1 a short exact sequence of presheaves of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathcal{O}_X(U)$-modules. Hence we can use Homology, Lemma 13.12 to get the boundary maps $\delta_{\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. ☐

In the formulation of the following lemma we use the functor $j_{p!}$ of extension by 0 for presheaves of modules relative to an open immersion $j : U \to X$. See Sheaves, Section 31. For any open $W \subset X$ and any presheaf $\mathcal{G}$ of $\mathcal{O}_X|_U$-modules we have

$$(j_{p!}\mathcal{G})(W) = \begin{cases} \mathcal{G}(W) & \text{if } W \subset U \\ 0 & \text{else.} \end{cases}$$

Moreover, the functor $j_{p!}$ is a left adjoint to the restriction functor see Sheaves, Lemma 31.8. In particular we have the following formula

$$\text{Hom}_{\mathcal{O}_X}(j_{p!}\mathcal{O}_U, \mathcal{F}) = \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U).$$

Since the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor on the category of presheaves we conclude that the presheaf $j_{p!}\mathcal{O}_U$ is a projective object in the category $PMod(\mathcal{O}_X)$, see Homology, Lemma 28.2.

Note that if we are given open subsets $U \subset V \subset X$ with associated open immersions j_U, j_V, then we have a canonical map $(j_U)_{p!}\mathcal{O}_U \to (j_V)_{p!}\mathcal{O}_V$. It is the identity on sections over any open $W \subset U$ and 0 else. In terms of the identification $\text{Hom}_{\mathcal{O}_X}((j_U)_{p!}\mathcal{O}_U, (j_V)_{p!}\mathcal{O}_V) = (j_V)_{p!}\mathcal{O}_V(U) = \mathcal{O}_V(U)$ it corresponds to the element $1 \in \mathcal{O}_V(U)$.

01EL **Lemma 10.3.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Denote $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ the open immersion. Consider the chain complex $K(\mathcal{U})_\bullet$ of presheaves of $\mathcal{O}_X$-modules*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} (j_{i_0 i_1 i_2})_{p!} \mathcal{O}_{U_{i_0 i_1 i_2}} \to \bigoplus_{i_0 i_1} (j_{i_0 i_1})_{p!} \mathcal{O}_{U_{i_0 i_1}} \to \bigoplus_{i_0} (j_{i_0})_{p!} \mathcal{O}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$(j_{i_0 \ldots i_{p+1}})_{p!} \mathcal{O}_{U_{i_0 \ldots i_{p+1}}} \longrightarrow (j_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})_{p!} \mathcal{O}_{U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \mathrm{Ob}(PMod(\mathcal{O}_X))$.

Proof. We saw in the discussion just above the lemma that

$$\mathrm{Hom}_{\mathcal{O}_X}((j_{i_0 \ldots i_p})_{p!} \mathcal{O}_{U_{i_0 \ldots i_p}}, \mathcal{F}) = \mathcal{F}(U_{i_0 \ldots i_p}).$$

Hence we see that it is indeed the case that the direct sum

$$\bigoplus\nolimits_{i_0 \ldots i_p} (j_{i_0 \ldots i_p})_{p!} \mathcal{O}_{U_{i_0 \ldots i_p}}$$

represents the functor

$$\mathcal{F} \longmapsto \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}(U_{i_0 \ldots i_p}).$$

Hence by Categories, Yoneda Lemma 3.5 we see that there is a complex $K(\mathcal{U})_\bullet$ with terms as given. It is a simple matter to see that the maps are as given in the lemma. ☐

01EM **Lemma 10.4.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{O}_\mathcal{U} \subset \mathcal{O}_X$ be the image presheaf of the map $\bigoplus j_{p!} \mathcal{O}_{U_i} \to \mathcal{O}_X$. The chain complex $K(\mathcal{U})_\bullet$ of presheaves of Lemma 10.3 above has homology presheaves*

$$H_i(K(\mathcal{U})_\bullet) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ \mathcal{O}_\mathcal{U} & \text{if} \quad i = 0 \end{cases}$$

Proof. Consider the extended complex $K^{ext}_\bullet$ one gets by putting $\mathcal{O}_\mathcal{U}$ in degree -1 with the obvious map $K(\mathcal{U})_0 = \bigoplus_{i_0} (j_{i_0})_{p!} \mathcal{O}_{U_{i_0}} \to \mathcal{O}_\mathcal{U}$. It suffices to show that taking sections of this extended complex over any open $W \subset X$ leads to an acyclic complex. In fact, we claim that for every $W \subset X$ the complex $K^{ext}_\bullet(W)$ is homotopy equivalent to the zero complex. Write $I = I_1 \amalg I_2$ where $W \subset U_i$ if and only if $i \in I_1$.

If $I_1 = \emptyset$, then the complex $K^{ext}_\bullet(W) = 0$ so there is nothing to prove.

If $I_1 \neq \emptyset$, then $\mathcal{O}_\mathcal{U}(W) = \mathcal{O}_X(W)$ and

$$K^{ext}_p(W) = \bigoplus\nolimits_{i_0 \ldots i_p \in I_1} \mathcal{O}_X(W).$$

This is true because of the simple description of the presheaves $(j_{i_0 \ldots i_p})_{p!} \mathcal{O}_{U_{i_0 \ldots i_p}}$. Moreover, the differential of the complex $K^{ext}_\bullet(W)$ is given by

$$d(s)_{i_0 \ldots i_p} = \sum\nolimits_{j = 0, \ldots, p+1} \sum\nolimits_{i \in I_1} (-1)^j s_{i_0 \ldots i_{j-1} i i_j \ldots i_p}.$$

The sum is finite as the element s has finite support. Fix an element $i_{\text{fix}} \in I_1$. Define a map

$$h : K^{ext}_p(W) \longrightarrow K^{ext}_{p+1}(W)$$

by the rule

$$h(s)_{i_0\ldots i_{p+1}} = \begin{cases} 0 & \text{if} \quad i_0 \neq i_{\text{fix}} \\ s_{i_1\ldots i_{p+1}} & \text{if} \quad i_0 = i_{\text{fix}} \end{cases}$$

We will use the shorthand $h(s)_{i_0\ldots i_{p+1}} = (i_0 = i_{\text{fix}})s_{i_1\ldots i_p}$ for this. Then we compute

$$\begin{aligned} & (dh + hd)(s)_{i_0\ldots i_p} \\ = \; & \sum_j \sum_{i\in I_1} (-1)^j h(s)_{i_0\ldots i_{j-1} i i_j \ldots i_p} + (i = i_0)d(s)_{i_1\ldots i_p} \\ = \; & s_{i_0\ldots i_p} + \sum_{j\geq 1}\sum_{i\in I_1} (-1)^j (i_0 = i_{\text{fix}}) s_{i_1\ldots i_{j-1} i i_j\ldots i_p} + (i_0 = i_{\text{fix}})d(s)_{i_1\ldots i_p} \end{aligned}$$

which is equal to $s_{i_0\ldots i_p}$ as desired. □

01EN **Lemma 10.5.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering of $U \subset X$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : \textit{PMod}(\mathcal{O}_X) \longrightarrow \textit{Mod}_{\mathcal{O}_X(U)}.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the right derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(\textit{PMod}(\mathcal{O}_X)) \longrightarrow D^+(\mathcal{O}_X(U))$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of presheaves of $\mathcal{O}_X$-modules has enough injectives, see Injectives, Proposition 8.5. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 20.

Let $\mathcal{I}$ be a injective presheaf of $\mathcal{O}_X$-modules. In this case the functor $\text{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact on $\textit{PMod}(\mathcal{O}_X)$. By Lemma 10.3 we have

$$\text{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 10.4 we have that $K(\mathcal{U})_\bullet$ is quasi-isomorphic to $\mathcal{O}_\mathcal{U}[0]$. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 10.2) satisfies the assumptions of Homology, Lemma 12.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 20.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 12.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any presheaf of $\mathcal{O}_X$-modules. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $\textit{PMod}(\mathcal{O}_X)$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Consider the associated total complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$, see Homology, Definition 18.3. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 25.4. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 10.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathcal{O}_X(U))$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. □

11. Čech cohomology and cohomology

01EO

01EP **Lemma 11.1.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. An injective $\mathcal{O}_X$-module is also injective as an object in the category *PMod*($\mathcal{O}_X$) (for example since sheafification is an exact left adjoint to the inclusion functor, using Homology, Lemma 29.1). Hence we can apply Lemma 10.5 (or its proof) to see the result. □

01EQ **Lemma 11.2.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors Mod($\mathcal{O}_X$) $\to D^+(\mathcal{O}_X(U))$. In particular this provides canonical maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over Mod($\mathcal{O}_X$).

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 25.4 to see that α is a quasi-isomorphism. Namely, Lemma 11.1 implies that the qth row of the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathcal{O}_X(U))$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

0B8R **Lemma 11.3.** *Let X be a topological space. Let $\mathcal{H}$ be an abelian sheaf on X. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. The map*

$$\check{H}^1(\mathcal{U}, \mathcal{H}) \longrightarrow H^1(X, \mathcal{H})$$

is injective and identifies $\check{H}^1(\mathcal{U}, \mathcal{H})$ via the bijection of Lemma 4.3 with the set of isomorphism classes of $\mathcal{H}$-torsors which restrict to trivial torsors over each U_i.

Proof. To see this we construct an inverse map. Namely, let $\mathcal{F}$ be a $\mathcal{H}$-torsor whose restriction to U_i is trivial. By Lemma 4.2 this means there exists a section $s_i \in \mathcal{F}(U_i)$. On $U_{i_0} \cap U_{i_1}$ there is a unique section $s_{i_0i_1}$ of $\mathcal{H}$ such that $s_{i_0i_1} \cdot s_{i_0}|_{U_{i_0}\cap U_{i_1}} = s_{i_1}|_{U_{i_0}\cap U_{i_1}}$. A computation shows that $s_{i_0i_1}$ is a Čech cocycle and that its class is well defined (i.e., does not depend on the choice of the sections s_i). The inverse maps the isomorphism class of $\mathcal{F}$ to the cohomology class of the cocycle $(s_{i_0i_1})$. We omit the verification that this map is indeed an inverse. □

01ER **Lemma 11.4.** *Let X be a ringed space. Consider the functor $i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X)$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^p(U) \to \mathcal{I}^{p+1}(U))}{\mathrm{Im}(\mathcal{I}^{p-1}(U) \to \mathcal{I}^p(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

01ES **Lemma 11.5.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 22.2) for the functors

$$i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(U)}.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 9.2. We have that $i(\mathcal{I})$ is Čech acyclic by Lemma 11.1. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $PMod(\mathcal{O}_X)$ by Lemma 10.5. Putting everything together gives the lemma. □

01ET **Lemma 11.6.** *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that $H^i(U_{i_0\ldots i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$ as $\mathcal{O}_X(U)$-modules.*

Proof. We will use the spectral sequence of Lemma 11.5. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

01EU **Lemma 11.7.** *Let X be a ringed space. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subset. If there exists a cofinal system of open coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose an open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find $\mathcal{U}$ such that (b) holds it follows from the assumptions of the lemma that we can find $\mathcal{U}$ such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} - s_{i_0}|_{U_{i_0 i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0 i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0 i_1}} - t_{i_0}|_{U_{i_0 i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. $\square$

01EV **Lemma 11.8.** *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$$

for all $p > 0$ and any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open of X. Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any open $U \subset X$.

Proof. Let $\mathcal{F}$ be a sheaf satisfying the assumption of the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any open covering". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 11.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any open covering. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 11.7 and our assumptions this sequence is actually exact as a sequence of presheaves! In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U}$, see Lemma 10.2 for example. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all open coverings.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\ & H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\ & \ldots & \ldots & \ldots \end{tikzcd}$$

for any open $U \subset X$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. $\square$

01EW **Lemma 11.9.** *(Variant of Lemma 11.8.) Let X be a ringed space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there exists a set of open coverings Cov with the following properties:*

(1) *For every* $\mathcal{U} \in$ *Cov with* $\mathcal{U} : U = \bigcup_{i \in I} U_i$ *we have* $U, U_i \in \mathcal{B}$ *and every* $U_{i_0 \ldots i_p} \in \mathcal{B}$.
(2) *For every* $U \in \mathcal{B}$ *the open coverings of* U *occurring in Cov is a cofinal system of open coverings of* U.
(3) *For every* $\mathcal{U} \in$ *Cov we have* $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ *for all* $p > 0$.

Then $H^p(U, \mathcal{F}) = 0$ *for all* $p > 0$ *and any* $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in \text{Cov}$". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 11.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in \text{Cov}$. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 11.7 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in \text{Cov}$ we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U} \in \text{Cov}$. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\
& H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\
& \ldots & \ldots & \ldots
\end{tikzcd}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

01EX **Lemma 11.10.** *Let* $f : X \to Y$ *be a morphism of ringed spaces. Let* $\mathcal{I}$ *be an injective* $\mathcal{O}_X$*-module. Then*

(1) $\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ *for all* $p > 0$ *and any open covering* $\mathcal{V} : V = \bigcup_{j \in J} V_j$ *of* Y.
(2) $H^p(V, f_*\mathcal{I}) = 0$ *for all* $p > 0$ *and every open* $V \subset Y$.

In other words, $f_*\mathcal{I}$ *is right acyclic for* $\Gamma(V, -)$ *(see Derived Categories, Definition 15.3) for any* $V \subset Y$ *open.*

Proof. Set $\mathcal{U} : f^{-1}(V) = \bigcup_{j \in J} f^{-1}(V_j)$. It is an open covering of X and

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This is true because

$$f_*\mathcal{I}(V_{j_0\ldots j_p}) = \mathcal{I}(f^{-1}(V_{j_0\ldots j_p})) = \mathcal{I}(f^{-1}(V_{j_0}) \cap \ldots \cap f^{-1}(V_{j_p})) = \mathcal{I}(U_{j_0\ldots j_p}).$$

Thus the first statement of the lemma follows from Lemma 11.1. The second statement follows from the first and Lemma 11.8. □

The following lemma implies in particular that $f_* : \textit{Ab}(X) \to \textit{Ab}(Y)$ transforms injective abelian sheaves into injective abelian sheaves.

02N5 **Lemma 11.11.** *Let $f : X \to Y$ be a morphism of ringed spaces. Assume f is flat. Then $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module for any injective $\mathcal{O}_X$-module $\mathcal{I}$.*

Proof. In this case the functor f^* transforms injections into injections (Modules, Lemma 20.2). Hence the result follows from Homology, Lemma 29.1. □

0D0A **Lemma 11.12.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$ let $\mathcal{F}_i$ be an $\mathcal{O}_X$-module. Let $U \subset X$ be open. The canonical map*

$$H^p(U, \prod\nolimits_{i \in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i \in I} H^p(U, \mathcal{F}_i)$$

is an isomorphism for $p = 0$ and injective for $p = 1$.

Proof. The statement for $p = 0$ is true because the product of sheaves is equal to the product of the underlying presheaves, see Sheaves, Section 29. Proof for $p = 1$. Set $\mathcal{F} = \prod \mathcal{F}_i$. Let $\xi \in H^1(U, \mathcal{F})$ map to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 7.2, there exists an open covering $\mathcal{U} : U = \bigcup U_j$ such that $\xi|_{U_j} = 0$ for all j. By Lemma 11.3 this means ξ comes from an element $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$. Since the maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all i (by Lemma 11.3), and since the image of ξ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 32.1 we conclude that $\check{\xi} = 0$ as desired. □

12. Flasque sheaves

09SV Here is the definition.

09SW **Definition 12.1.** Let X be a topological space. We say a presheaf of sets $\mathcal{F}$ is *flasque* or *flabby* if for every $U \subset V$ open in X the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

We will use this terminology also for abelian sheaves and sheaves of modules if X is a ringed space. Clearly it suffices to assume the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for every open $U \subset X$.

09SX **Lemma 12.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Then any injective $\mathcal{O}_X$-module is flasque.*

Proof. This is a reformulation of Lemma 8.1. □

09SY **Lemma 12.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Any flasque $\mathcal{O}_X$-module is acyclic for $R\Gamma(X, -)$ as well as $R\Gamma(U, -)$ for any open U of X.*