Algebraic geometry 1
Exercise sheet 4

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Exercise 1.

1. Let X be a finite irreducible topological space. Since
$$X = \bigcup_{x \in X} \overline{\{x\}}$$
is a finite decomposition in closed sets, we must have $\overline{\{x\}} = X$ for some $x \in X$. This x is a generic point of X.

   If we additionally assumed X is T_0, then this point x would be unique, since in a T_0 space we have $\overline{\{x\}} \neq \overline{\{y\}}$ for $x \neq y$. Other conditions are also fulfilled, namely a finite space is always quasicompact. For the basis we can take B to contain all open sets, it is clearly closed under finite intersections and every element is quasicompact. So finite T_0 spaces are spectral.

2.

Exercise 2. Denote $A = \lim A_i$, $B = \lim B_i$ and $C = \lim C_i$. Also denote maps $A_i \to A$ with f_i, $B_i \to B$ with g_i and $C_i \to C$ with h_i.

By composing α_i and g_i we get $A_i \to B$ defined as $g_i \circ \alpha_i$. Then by the definition of a colimit we have a unique map $\alpha : A \to B$, such that $g_i \circ \alpha_i = \alpha \circ f_i$. In the same way we obtain $\beta \colon B \to C$. With these definitions the whole diagram commutes.

First look at maps $A_i \to B_i \to C_i \to C$. Since bottom rows are all exact, they are 0 for all i. Since $\beta \circ \alpha$ makes this commute and 0 does well, by uniqueness of the map $A \to C$, we get $\beta \circ \alpha = 0$. In this argument we did not use that the colimit is filtered in our case, because it is true in general that colimits are right-exact.

For left exactness we need to use that we have a filtered colimit. For this part we used Proposition 2 from page 212 of Mac Lane's "Categories For the

Working Mathematician," stating "The forgetful functor $Grp \to Set$ creates filtered colimits." Now take $b \in B$ with $\beta(b) = 0$. Using the proposition we know there exists an i and $\overline{b} \in B_i$ for which $g_i(\overline{b}) = b$. Since diagram commutes we have $(h_i \circ \beta_i)(b) = 0$. But since h_i maps an element to 0 if and only if there exists a C_j and $C_i \to C_j$ (in a diagram) that maps it to 0, we must have C_j and $\gamma\colon C_i \to C_j$ (in a diagram) with $\gamma(\beta_i(b)) = 0$. Take B_j and the map $\delta\colon B_i \to B_j$ with $(\beta_j \circ \delta)(\overline{b}) = 0$. Since bottom rows are exact, we have $\overline{a} \in A_j$ with $\alpha_j(\overline{a}) = \delta(\overline{b})$. Then we have $(g_j \circ \alpha_j)(\overline{a}) = b$ and using commutativity of the diagram we get $\alpha(f_j(\overline{a})) = b$, so $b \in \mathrm{im}(\alpha)$.

This completes the proof of exactness. In it we a little implicitly used the facts that follow from the construction of a colimit in abelian category by first computing the colimit in sets and then defining a structure on that set. But we could not find a way to do this more elegantly without chasing an explicit element around the diagram.

Exercise 3.

1. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$ and $\phi, \psi\colon \mathcal{F} \to \mathcal{G}$ morphisms of sheaves.

   We define $\phi + \psi$ to be a morphism of sheaves defined by

   $$(\phi + \psi)(U) := \phi(U) + \psi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U).$$

   for every $U \in \mathrm{Ouv}_X^{\mathrm{op}}$. So its just a sum of homomorphisms of abelian groups. It is clear that this still defines a natural transformation (i.e. a morphism) in $\mathrm{Sh}_{\mathrm{Ab}}(X)$.

   The zero object in this category is the sheaf that sends every open set $U \in \mathrm{Ouv}_X^{\mathrm{op}}$ to the trivial group $(0, +)$. It is inital and terminal, because the trivial group is a zero object in the category of abelian groups. For this reason the category of sheaves is an additive category.

   Instead of checking for kernels and cokernels, we simply check for all limits and colimits, of which they are a special case.

   Let us check that it has limits and colimits. Let F_i be some diagram in $\mathrm{Sh}_{\mathrm{Ab}}(X)$. Define a limit of this diagram to be a sheaf F which maps $U \in \mathrm{Ouv}_X^{\mathrm{op}}$ to $\lim_i F_i(U)$. So for each U we let $F(U)$ be the limit of a diagram $F_i(U)$ (which we know exist because its in the category of abelian groups).

   Check that F this is actually a limit in $\mathrm{Sh}_{\mathrm{Ab}}(X)$. Let G be some other presheaf with maps $G \to F_i$. Then for every $U \in \mathrm{Ouv}_X^{\mathrm{op}}$ we have maps $G(U) \to F_i(U)$, so this cone factors uniquely through $F(U)$. This gives rise to a morphism $G \to F$. So F is indeed a limit.

   Now we check that F is a presheaf. Let $U \to V$. Then we have maps $F_i(U) \to F_i(V)$. Composing them with $F(U) \to F_i(U)$ gives maps $F(U) \to F_i(V)$. By the definition of a limit there exists a unique map $F(U) \to F(V)$. This proves that F is indeed a presheaf.

Now sheafify F to get a sheaf $\overline{F} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$. Since F_i are sheaves, the maps $F \to F_i$ uniquely factors through $\overline{F}$. For any sheaf G and a cone $G \to F_i$, we can look at this inside the category of presheaves and find a unique map $F \to G$. But since G is a sheaf, it factors through $\overline{F}$ and we get a unique map $\overline{F} \to G$. This proves that $\overline{F}$ is a limit of $F_i \in \mathrm{Sh}_{\mathrm{Ab}}(X)$.

For colimits everything works exactly the same, except all arrows are reversed.

Maybe some parts of this proof follow from some more elegant argument using adjoints and forgetful functors.

2. Assume $f\colon \mathcal{F} \to \mathcal{G}$ is surjective on each stalk. Then we can show that $f(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for each $U \in \mathrm{Ouv}_X^{\mathrm{op}}$. Pick $s \in \mathcal{G}(U)$. Then $s_x \in \mathcal{G}_x$ the germ of s at x. Since $f_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective we have a function $t_x \in \mathcal{F}(U)$ such that $f(U)(t_x)$ has the same germ at x as s. So $t_x \in \mathcal{F}(U)$ is mapped to a function that is locally at x same as s. Denote V_x the open subset with $x \in V_x$ for which $f(U)(t_x) \mid_{V_x} = s \mid_{V_x}$. So we have a cover $X = \cup_x V_x$. Using that $\mathcal{F}$ is a sheaf we can find an element t such that $f(U)(t) = s$. That proves one implication.

   We were inspired by some stack exchange site and only worked out some of the details. Let $\mathcal{F}, \mathcal{G}$ be sheaves on X and $\phi : \mathcal{F} \to \mathcal{G}$ an epimorphism on sheaves. We want to show that ϕ is surjective on each stalk.
   Let $\mathcal{F}$ be the skyscraper sheaf at x with value $\mathcal{G}_x / \operatorname{im}(\phi_x)$, i.e. for $U \subset X$ open, we have

$$\mathcal{F}(U) = \begin{cases} \mathcal{G}_x / \operatorname{im}(\phi_x) & \text{if } x \in U \\ (0,+) & \text{else.} \end{cases}$$

   Also, define

$$\begin{aligned} \psi : \mathcal{G} &\to \mathcal{H} \\ f &\mapsto \bar{f}_x \end{aligned}$$

   if f is a section over a set containig x.
   Now assume, that ϕ_x is not surjective for all $x \in X$. Then there exists $x \in X$ such that ϕ_x is not surjective, meaning $\mathcal{G}_x / \operatorname{im}(\phi_x)$ is not the trivial group. In this case, the two maps

$$(\psi, 0), (0, \psi) : \mathcal{G} \to \mathcal{H} \oplus \mathcal{H}$$

   are not the same. However, by construction they are the same if precomposed with ϕ showing that ϕ is not an epimorphism of sheaves.

3. First prove that $\mathcal{F}$ and $\mathcal{G}$ are sheaves.

   Define a morphism $f\colon \mathcal{F} \to \mathcal{G}$ with $f(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ being the map $s \oplus t \mapsto s \mid_{D_- \cap D_+}$. Note that if $U \cap D_-$ is connected, the map $s \mid_{D_- \cap D_+}$ must assume same value on whole $U \cap D_- \cap D_+$.

Using part 2 it suffices to check that f_x is surjective at every $x \in X$. For $x \notin D_- \cap D_+$, then $\mathcal{G}_x$ is a null group in *AbGrp*, so f_x is surjective. Let now $x = (-1, 0)$.

Define $U_\epsilon = B_\epsilon(-1,0) \cap X$. Every neighborhood contains some U_ϵ for small enough ϵ. So every $\mathcal{F}(U) \to F_x$ factors through $\mathcal{F}(U_\epsilon)$ for small enough ϵ. We have $\mathcal{F}(U_\epsilon) = \mathbb{Z} \oplus \{0\}$ for small $\epsilon > 0$, so $\mathcal{F}_x = \mathbb{Z}$ with morphisms

$$\mathcal{F}(U_\epsilon) \to \mathcal{F}_x$$

being $a \oplus 0 \mapsto a$. We do exactly the same for $\mathcal{G}$ and we get $\mathcal{G}_x = \mathbb{Z}$. The identity id is the unique map that makes

$$\begin{tikzcd}
\mathcal{F}(U_\epsilon) \arrow[r, "f(U_\epsilon)"] \arrow[d] & \mathcal{G}(U_\epsilon) \arrow[d] \\
\mathcal{F}_x \arrow[r, "\mathrm{id}"] & \mathcal{G}_x
\end{tikzcd}$$

commute, so f_x is surjective for $x = (-1, 0)$. We do exactly the same for $x = (1, 0)$. So f_x is surjective for every $x \in X$. By part 2, $\mathcal{F}$ is an epimorphism.

We have $\mathcal{F}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\mathcal{G}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the map $f(X)$ as defined above. The image $\operatorname{im} f(X)$ is the diagonal $\triangle(\mathbb{Z} \oplus \mathbb{Z})$, so $f(X)$ is not surjective.

Exercise 4. Denote the maps

$$\varphi\colon \operatorname{Sh}(X) \to \operatorname{Sh}_B(X)$$

and

$$\mu\colon \operatorname{Sh}_B(X) \to \operatorname{Sh}(X).$$

The composite $\varphi \circ \mu$ is the identity on $\operatorname{Sh}_B(X)$.

We have to prove that $\mu \circ \varphi\colon \operatorname{Sh}(X) \to \operatorname{Sh}(X)$ mapping a functor F to a functor $(U \mapsto \lim_{B \subseteq U} F(B))$ is equivalent to the identity.

Define a natural transformation

$$\zeta\colon \mathrm{id} \to \mu \circ \varphi$$

which for every $F \in \operatorname{Sh}(X)$ defines a map

$$\mathrm{id}(F) = F \longrightarrow (U \mapsto \lim_{B \subseteq U} F(B))$$

being the unique natural transformation $F(U) \to \lim_{B \subseteq U} F(B)$ (which exists $F(U) \to F(B)$ exist for every $B \subseteq U$).