Algebraic geometry 1
Exercise sheet 2

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Exercise 1. Let $I = (f_1, \ldots, f_r) \subseteq k[x_1, \ldots, x_n]$ be an ideal and $X = V(I) \subseteq \mathbb{A}^n(k)$ be its vanishing locus.

1. Closure of a subset X in a topological space is an intersection of all closed subsets that contain the set X. In this case

$$\overline{X} = V(\{h \mid h \text{ homogenous}, \forall x \in X\colon h(x) = 0\}).$$

Since $V(\{\widetilde{g} \mid g \in I\})$ is closed by definition and contains X, we only have to prove the inclusion

$$\overline{X} \supseteq V(\{\tilde{g} \mid g \in I\}). \tag{1}$$

Pick any homogenous polynomial h that vanishes on $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$. Write

$$h = x_{n+1}^m h'$$

where x_{n+1} does not divide h' and write

$$f = h(x_1, \ldots, x_n, 1).$$

Observe that homogenizing f yields h'.

By Hilberts Nullstellensatz we have $f \in \sqrt{I}$, i.e. $f^l \in I$.

Claim. Homogenizing commutes with product.

Proof of claim. Take $f, g \in k[x_1, \ldots, x_n]$ of degrees k, l respectively. We have $\widetilde{f} = x_{n+1}^k f(x_1/x_{n+1}, \ldots, x_n/x_{n+1})$ and $\widetilde{g} = x_{n+1}^l g(x_1/x_{n+1}, \ldots, x_n/x_{n+1})$. Since fg is of degree $k + l$, we have

$$\widetilde{fg} = x_{n+1}^{k+l} f(x_1/x_{n+1}, \ldots, x_n/x_{n+1}) g(x_1/x_{n+1}, \ldots, x_n/x_{n+1}).$$

So $\widetilde{fg} = \widetilde{f}\widetilde{g}$. □(of claim)

Homogenize f^l and get

$$h^l = x_{n+1}^{ml}(h')^l = x_{n+1}^m(\widetilde{f})^l = x_{n+1}^{ml}\widetilde{(f^l)}$$

which means $V(h) \supseteq V\left(\widetilde{(f^l)}\right)$ and thus inclusion 1 follows.

2. We calculate $\overline{X}$ using the first part of the exercise and take the numbering of the varieties from the first exercise sheet:

$$\begin{aligned}
\overline{X_1} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(x^2 - xz) \cap V(z) = \{(0:1:0)\}, \\
\overline{X_2} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(z^2 - xy) \cap V(z) = \{(1:0:0) \cup (0:1:0)\}, \\
\overline{X_3} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(xy) \cap V(z) = \{(1:0:0) \cup (0:1:0)\}, \\
\overline{X_4} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(x^2 - xz) \cap V(z) = \{(0:1:0)\}, \\
\overline{X_5} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(x) \cap V(z) = \{(0:1:0)\}.
\end{aligned}$$

3. Define $f(x,y) := xy - 1$ and $g(x,y) := xy$. Then we have

$$X = V(f,g) = \emptyset,$$

and since $\emptyset$ is closed in projective space as the vanishing set of a non-zero constant, we have $\overline{X} = \emptyset$. However,

$$(0:1:0) \in V^+(xy - z, xy) = V^+(\tilde{f}, \tilde{g}),$$

proving that in this case $\overline{X} \neq V(\tilde{f}, \tilde{g})$.

Exercise 2. Let $F \in k[x,y,z]$ be a non-zero homogenous polynomial of degree 2.

1. Let us first prove the statement in the hint. So let

$$T(x,y) = (ax + by + u, cx + dy + v)$$

with $ad - cb \neq 0$, be an affine linear transformation of $\mathbb{A}^2(k)$. We can extend it to an automorphism of $\mathbb{P}^2(k)$ by defining

$$\widetilde{T}(x:y:z) = (ax + by + uz : cx + dy + vz : z)$$

It is an extension of T, because we have $\widetilde{T} \circ i = i \circ T$ where $i\colon \mathbb{A}^2(k) \to \mathbb{P}^2(k)$ is given by $(x,y) \mapsto (x:y:1)$. Mapping $\widetilde{T}$ is bijective with inverse

$$\widetilde{T}^{-1}(x:y:z) = (dx - by - (du - bv)z : -cx + ay + (cu - av)z : (ad - bc)z).$$

This proves that $\widetilde{T}$ is an automorphism of $\mathbb{P}^2(k)$.

Lets consider the polynomial F now. We looks at its zero-set on $\mathbb{A}^2(k)$, which is exactly $V(F) \cap V(z-1) = V(F, z-1) = V(F(x,y,1))$.

This set could be empty, in which case $F(x,y,1)$ must be a non-zero constant (because k is algebraically closed). A general form of F would be $F(x,y,z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$. If we set $z = 1$

and F becomes a non-zero constant, we get that a_3 is a non-zero constant and other a_i are 0. So $F(x,y,z) = az^2$, which means $V(F)$ is isomorphic to the projective line $V(x) = V(x^2)$.

Suppose now $V := V(F(x,y,1)) \subseteq \mathbb{A}^2(k)$ is not empty. Note that $F(x,y,1)$ is of the form we treated in exercise 3 of sheet 1. Then we can use an affine linear transformation T on $\mathbb{A}^2(k)$ so that $V \subseteq \mathbb{A}^2(k)$ assumes the forms of one of the 5 curves we treated in exercise 3 sheet 1, namely $V(x), V(y-x^2), V(xy), V(xy-1), V(x(x-1))$. We showed earlier that his transformation can be extended to an isomorphism $\widetilde{T}$ of $\mathbb{P}^2(k)$.

Loosely speaking, we could have "lost some information" when we took the intersection $\mathbb{A}^2(k) \cap V(F)$, but it turns out that we didnt, if we take into account that F is a non-zero homogenous polynomial of degree 2. We are able to "retrieve all of it back" by taking the closure and remembering that original curve was of degree 2.

Since the projective algebraic variety $V(F)$ is by definition closed, the projective algebraic variety $\widetilde{T}(V(F)) \subseteq \mathbb{P}^2(k)$ will also be closed. By construction we have one of the following cases

(a) $V(x) \cap \mathbb{A}^2(k) = V(x^2) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$
(b) $V(xy) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$
(c) $V(xy-1) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$
(d) $V(y-x^2) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$
(e) $V(x(x-1)) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$

In each case we can take the closure of the left side and, since right side is closed, the inclusion will still hold. We use 1st exercise of this sheet to calculate the closure, so by homogenizing the polynomial. We get one of the following cases

(a) $V(x) = V(x^2) \subseteq \widetilde{T}(V(F))$
(b) $V(xy) \subseteq \widetilde{T}(V(F))$
(c) $V(xy-z^2) \subseteq \widetilde{T}(V(F))$
(d) $V(yz-x^2) \subseteq \widetilde{T}(V(F))$
(e) $V(x^2-xz) \subseteq \widetilde{T}(V(F))$

All that is left is to show these inclusions are in fact equalities.

Polynomial F does not have a well defined value on projective space $\mathbb{P}^2(k)$, but its zeroes are. So let us denote $\widetilde{F} := F \circ \widetilde{T}^{-1}$ a polynomial for which the following holds $V(\widetilde{F}) = \widetilde{T}(V(F))$. We can do this, for example, by changing variables in F by

$$\begin{aligned}
x &\mapsto dx - by - (du - bv)z \\
y &\mapsto -cx + ay + (cu - av)z \\
z &\mapsto (ad - bc)z
\end{aligned}$$

This is just one example of changing variables that would work. In this case we used that $\widetilde{T}$ is an extension of affine linear transformation in $\mathbb{A}^n(k)$.

In all cases above the polynomial $\widetilde{F}$ must lie in the radical of the ideal generated by the polynomial on the left.

(a) $\widetilde{F} \in \sqrt{(x^2)} = (x)$, so $\widetilde{F}$ is in this case divisible by x. In general $\widetilde{F}$ is then of the form $F = x(\alpha x + \beta y + \gamma z)$. But since $V(\widetilde{F}(x, y, 1)) = V(x(\alpha x + \beta y + \gamma)) = V(x)$, we have $V(\alpha x + \beta y + \gamma) \subseteq V(x)$.
That gives two options, either
- $\alpha = \beta = 0$ and $\gamma \neq 0$; in this case $\widetilde{F} = \gamma xz$ which gives $\widetilde{T}(V(F)) = V(xz)$ (which is isomorphic to $V(xy)$).
- $\alpha \neq 0$ and $\beta = \gamma = 0$; in this case $\widetilde{F} = \alpha x^2$ which gives $\widetilde{T}(V(F)) = V(x)$.

(b) $\widetilde{F} \in \sqrt{(xy)} = (xy)$, which already proves that $\widetilde{T}(V(F)) = V(xy)$.

(c) $\widetilde{F} \in \sqrt{(xy - z^2)} = (xy - z^2)$, which already proves that $\widetilde{T}(V(F)) = V(xy - z^2)$.

(d) $\widetilde{F} \in \sqrt{(yz - x^2)} = (yz - x^2)$, which already proves that $\widetilde{T}(V(F)) = V(yz - x^2)$ (of course $V(x^2 - yz) \cong V(xy - z^2)$).

(e) $\widetilde{F} \in \sqrt{(x^2 - xz)} = (x^2 - xz)$, which already proves that $\widetilde{T}(V(F)) = V(x^2 - xz)$, where $V(x^2 - xz)$ is isomorphic to $V(xy)$ by the isomorphism $(x : y : z) \mapsto (x : z : x - y)$.

Exercise 3.

1. We define a closed subset $V(f - g) \subseteq X$, which contains the open set $\mathcal{U} \subseteq V(f-g)$. By definition the completent $\mathcal{U}^C$ is closed and $V(f-g) \cup \mathcal{U}^C = X$. Since X is irreducible and $\mathcal{U}$ is non-empty, we have $f = g$.

2. Lets show first that $\chi \colon A \mapsto \chi_A(A)$ vanishes on diagonalizable matrices with pairwise different eigenvalues: For $A = TDT^{-1}$ we have

$$\chi_A(A) = \sum_{i=0}^{n} \alpha_i A^i = \sum_{i=0}^{n} \alpha_i (TDT^{-1})^i = T \left(\sum_{i=0}^{n} \alpha_i D^i \right) T^{-1}.$$

Denote $D = \operatorname{diag}(d_1, \ldots, d_n)$ and notice that

$$\sum_{i=0}^{n} \alpha_i D^i = \sum_{i=0}^{n} \operatorname{diag}(\alpha_i d_1^i, \ldots, \alpha_i d_n^i) = \operatorname{diag}(\chi_A(d_1), \ldots, \chi_A(d_n)) = 0$$

because characteristic polynomial vanishes on eigenvalues d_i.

Since $\mathbb{A}^{n \times n}(L)$ is irreducible and χ vanishes on open subset of it, namely on diagonalizable matrices with pairwise different eigenvalues, χ must be 0 on whole $\mathbb{A}^{n \times n}(L)$.

Exercise 4.

1. First, we note that points in projective space are closed. To see this, take some point $a = (a_0 : \cdots : a_n) \in \mathbb{P}_k^n$. Then $a = V(\{f \in k[x_0, \ldots, x_n] : f = x_i a_j - x_j a_i\})$. Now, we let $X \subseteq \mathbb{P}_k^n$ be some quasi-projective variety and $x, y \in X$ with $x \neq y$. Define $U := \{x\}^c$, which is open as the complement of a closed set and by definition fulfils the properties $x \notin U$ and $y \in U$.

2. Finite T_1 spaces are discrete, in particular they are Hausdorff. And by first part of the exercise, X is T_1.

   A quasi-projective variety is an open subset of a projective variety with subspace topology. Let Y be a projective variety, with $X \subseteq Y$ an open subset. Take now any two points $x, y \in X$ with $x \neq y$. Suppose there exist open subsets $U, V \subseteq X$ with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. By definition of subspace topology we have $\tilde{U}, \tilde{V} \subseteq Y$ open such that $\tilde{U} \cap X = U$ and $\tilde{V} \cap X = V$. We can then cover Y with closed sets $\tilde{U}^c$, $\tilde{V}^c$ and X^c. However, none of them is equal to Y. Concretely, $x \notin \tilde{U}^c$, $y \notin \tilde{V}^c$ and $x, y \notin X^c$. But because Y is by definition irreducible, we arrive at a contradiction with the existance of such U and V. So x, y cannot be separated with disjoint open sets and X is not Hausdorff.

   Comment: In this solution we did not use that there are infinitely many points in X, only that there are at least two and that we can separate them with open subsets. I think that is because a quasi-projective variety cannot be a finite set unless if it is a single point. Otherwise it would not be irreducible. I hope this thinking is correct. This exercise was a little bit confusing.