Algebraic geometry 1
Exercise sheet 10

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21. Dezember 2023

Exercise 1.

1. Since X is closed and irreducible, it is of the form $X = \overline{\{p_0\}}$ for some (Eric thinks unique) $p_0 \in \mathbb{A}_k^n$. That means $X \cong \mathrm{Spec}(k[x_1, \dots x_n]/p_0)$. Denote $A = k[x_1, \dots x_n]/p_0$.

   By assumption there is a chain of specializations $p_0 \subset \cdots \subset p_d$ inside X.

   Let $Z \subseteq X \cap V(f_1, \dots, f_r)$ be a irreducible component. Thus it is the closure of a minimal prime ideal in $A/(f_1, \dots, f_r)$.

   By Krull's principal ideal theorem we have $\dim(A/(f_1, \dots, f_r)) \geq d - r$.

   Denote minimal prime ideals in $A/(f_1, \dots, f_r)$ with $q_1, \dots q_l$.

   (Eric thinks that there is a unique generic point here again, since X is sober, so there should only be one of these prime ideals, right?)

   We argue that

   $$\dim(A/(f_1, \dots, f_r)/q_j) = \dim(A/(f_1, \dots, f_r)).$$

   for any j.

   That follows from A being catenary. If there existed a maximal chain in $A/(f_1, \dots, f_r)$ that starts at q_j we could simply extend it below to get a maximal chain in A. Since all maximal chains in A are of the same length, we get that all maximal chains in $A/(f_1, \dots, f_r)$ are also of the same length.

   Since Z is an irreducible component, we have $Z = \overline{\{q_i\}} \subseteq \mathrm{Spec}(A/(f_1, \dots, f_r))$.

   Therefore any maximal chain in Z is exactly as long as the longest chain in $A/(f_1, \dots, f_r)$. And the longest chain in $A/(f_1, \dots, f_r)$ is at least of length $d - r$.

2. The diagonal $\triangle \subseteq \mathbb{A}_k^n \times \mathbb{A}_k^n$ can be defined as $V(x_i \otimes x_i \mid i = 1, \dots n) \subseteq \mathrm{Spec}(k[x_1, \dots, x_n] \otimes_{\mathbb{Z}} k[x_1, \dots, x_n])$.

   (Should there not be a minus instead of $\otimes$ in the above expression?)

Using exercise above we get that any irreducible component of $X \cap Y \cong (X \times Y) \cap V(x_i \otimes x_i \mid i = 1, \dots n)$ has dimension at least $d + e - n$.

3. Let $\tilde{X} = \overline{f^{-1}(X)}$ and $\tilde{Y} = \overline{f^{-1}(Y)}$ as in the hint.

   We have $\dim(\tilde{X}) = d + 1$ and $\dim(\tilde{Y}) = e + 1$. By the previous exercise we have $\dim(\tilde{X} \cap \tilde{Y}) \geq d + 1 + e + 1 - (n + 1) = (d + e - n) + 1 \geq 1$.

   Therefore there there exists $0 \neq x \in \tilde{X} \cap \tilde{Y}$.

   Questions from Eric:

   Why is $\tilde{X}$ irreducible (to be able to use part 2) and why does the dimension increase by 1 when we go to affine space?

Exercise 2.

1. It is enough to show that there exists a cover $X = \cup_i \operatorname{Spec}(A_i)$ of X by open affines such that $f^{-1}(\operatorname{Spec}(A_i))$ is affine for all i. Therefore it is enough to show that the hint holds, since $X = \cup_{x \in X} U_x$, where each U_x is an open affine with $x \in U_x$.

   Take some $x \in X$. If $x \notin f(Y)$ then by continuity and since $f(X)$ is closed, there exists an affine U_x that is disjoint from $f(Y)$. In this case $f^{-1}(U_x) = \emptyset = D(1)$ the preimage under f is of affine.

   If $x \in f(Y)$, then $x \in \operatorname{Spec}(A_k)$ for some k. Now choose some principal open $D(g) \subset Z$ with $f^{-1}(x) \in D(g)$.

   We can now take a principal open $D(g') \subset f(D(g))$ such that $D(g') \subset U_k$ and $x \in D(g')$. Then we can show similarly to exercise 4.1 on sheet 8 that

   $$f^{-1}(D(g'))$$

   is a principal open again, so in particular affine.

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   Take $x \in |X|$. If $x \notin f(|Y|)$, we can find an open U_x such that $f^{-1}(U_x) = \emptyset$. So assume $x \in f(|Y|)$. Then look at $f^{-1}(x)$. Take an open affine $V_x \subseteq |Y|$ with $f^{-1}(x) \in V_x$. Since f is homeomorphism on its image, we have can take $U_x = f(V_x)$ an affine neighborhood of x such that $f^{-1}(U_x) = V_x$ is affine.

2. Assume that f is universally closed. We want to show that f is integral, surjective and universally injective. By the first part of this exercise we get that f is affine. It says on The Stacks project that maps that are affine and unversally closed are also integral. The other two properties follow immediately from the assumptions.

   On the other hand, assume that f is integral, surjective and universally injective. We now from algebra 1 that integral maps are closed and we learned in this course that the property of a morphism being integral is stable under base change. If you put these two facts together you get that integral maps are universally closed.

We also know from The Stacks project that the property of a map of schemes being surjective is stable under base change, so f surjective already implies f unviversally surjective. All in all, we get that f is unviversally bijective.

Exercise 3.

1. We have a map $k \to \Gamma(X, \mathcal{O}_X)$. For any $f \in \Gamma(X, \mathcal{O}_X)$ we can define $k[x] \to \Gamma(X, \mathcal{O}_X)$ by $x \mapsto f$.

   Pick some $f \in \Gamma(X, \mathcal{O}_X)$ and define the induced $g \colon X \to \mathbb{A}^1_k$.

   First observe that: $g(X)$ does not contain the generic point of $\mathbb{A}^1_k$ if and only if $k[x] \to \Gamma(X, \mathcal{O}_X)$ with $x \mapsto f$ is not injective.

   We have a composition $k \to k[x] \to \Gamma(X, \mathcal{O}_X)$. So also $X \to \mathbb{A}^1_k \to \mathrm{Spec}(k)$.

   Map $X \to \mathrm{Spec}(k)$ is proper.

   Map $\mathbb{A}^1_k \to \mathrm{Spec}(k)$ is separated, since it is a map of affine schemes. (Follows from the fact that $k[x] \otimes_k k[x] \to k[x]$ is surjective, and thus $\mathbb{A}^1_k \to \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ a closed immersion.)

   Thus by the proposition from the lectures, the map $g \colon X \to \mathbb{A}^1_k$ is proper. In particular it is closed. Since X is connected, the image $g(X)$ is connected as well.

   Using the hint, we can postcompose with open inclusion $\mathbb{A}^1_k \subseteq \mathbb{P}^1_k$ to obtain $X \to \mathbb{A}^1_k \to \mathbb{P}^1_k$. Now the conclusion should be that the image of X in $\mathbb{P}^1_k$ is also closed. Since $\mathbb{A}^1_k \subseteq \mathbb{P}^1_k$ is not closed, the image of X cannot be whole $\mathbb{A}^1_k$. Therefore it must be a single point.

   Since we did not exactly understand why should composition $X \to \mathbb{P}^1_k$ be closed, we decided to rather show that $X \to \mathbb{A}^1_k$ cannot be surjective, as that would imply $\mathbb{A}^1_k$ being universally closed over $\mathrm{Spec}(k)$ (which we've shown during the lectures to be false).

   Instead of doing it abstractly, we can show that $X \to \mathbb{A}^1_k$ being surjective would imply $\mathbb{A}^2_k \to \mathbb{A}^1_k$ being closed.

   By the universal property of $\mathbb{A}^2_k$ we get a map $X \to \mathbb{A}^2_k$, induced by $X \to \mathbb{A}^1_k$. So we have a map $X \to \mathbb{A}^2_k \to \mathbb{A}^1_k$. Denote $\alpha \colon X \to \mathbb{A}^2_k$ and $\beta \colon \mathbb{A}^2_k \to \mathbb{A}^1_k$. If α would be surjective, then for any $U \subseteq \mathbb{A}^2_k$ we would have $(\beta \circ \alpha)(\alpha^{-1}(U)) = \beta(U)$. Since $\beta \circ \alpha$ is closed by assumption, this would prove that β is closed. That is not true, so $g = \beta \circ \alpha$ is not surjective.

We've shown that the image of $X \to \mathbb{A}^1_k$ is a single point. Since this point is closed, it is not the generic point. This shows that $k[x] \to \Gamma(X, \mathcal{O}_X)$ induced by $f \in \Gamma(X, \mathcal{O}_X)$ is not injective.

2. We have a map $k \to \Gamma(X, \mathcal{O}_X)$. It cannot be 0, since X is locally finite type over $\mathrm{Spec}(k)$. So it is injective.

   It is also surjective, since for any $f \in \Gamma(X, \mathcal{O}_X)$ the map $k[x] \to \Gamma(X, \mathcal{O}_X)$ defined by $x \mapsto f$ is not injective. Therefore $k \cong \Gamma(X, \mathcal{O}_X)$.

Exercise 4.

1. Let $\{Z_i\}_{i \in I}$ be all closed subschemes Z_i such that $f\colon X \to S$ factors through Z_i. Since equalizers exist in category of schemes, we take $\mathrm{im}(f)$ to be the equalizer $\mathrm{eq}(Z_i \rightrightarrows S)$.

   Let $\mathrm{im}(f)$ be the schematic image of $f\colon X \to S$. We have a factorization $f = i \circ f'$, where $f'\colon X \to \mathrm{im}(f)$ and $i\colon \mathrm{im}(f) \to S$ closed immersion.

   Then $\mathcal{O}_S \to f_*(\mathcal{O}_X)$ factors as $\mathcal{O}_S \to i_*\mathcal{O}_{\mathrm{im}(f)} \to f_*(\mathcal{O}_X)$. This shows that ideal sheaf of the image is indeed contained in the kernel of $f^{\#}$.

   Show that ideal sheaf of a closed immersion is indeed a quasi-coherent sheaf.

   Let $i\colon \mathrm{im}(f) \to S$ be a closed immersion. We have a surjection $\mathcal{O}_S \to i_*\mathcal{O}_Z$.

   Pick a point $x \in S$ and an affine open neighborhood $x \in U = \mathrm{Spec}(A) \subseteq S$. We obtain a map $A \to i_*\mathcal{O}_Z(U)$.

   Denote the kernel $I = \ker(A \to i_*\mathcal{O}_Z(U))$.

   Since $\mathcal{O}_S \to i_*\mathcal{O}_Z$ surjective implies $A \to i_*\mathcal{O}_Z(U)$ surjective, we have $i_*\mathcal{O}_Z(U) \cong A/I$.

   We want to show the ideal sheaf is equal to $\tilde{I}$. Pick any $f \in A$.

   Since $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_S$-module, we have $i_*\mathcal{O}_Z(D(f)) = i_*\mathcal{O}_Z(U)[f^{-1}] = (A/I)[f^{-1}]$.

   Kernel of $A[f^{-1}] \to (A/I)[f^{-1}]$ is then $I[f^{-1}]$. This shows that ideal sheaf isomorphic to $\tilde{I}$ on U and thus a quasi-coherent sheaf.

   Now we have to show it is in fact maximal such.

   Take any quasi-coherent ideal M that factors through $\ker(f^{\#})$. Sheaf M induces a closed subscheme, locally on $\mathrm{Spec}(A) \subseteq S$ defined as a closed subscheme $V(M(U)) \subseteq \mathrm{Spec}(A)$.

   For any affine open $U = \mathrm{Spec}(A) \subseteq S$, the ideal $M(U)$ factors through kernel $I = \ker(f^{\#})(U)$: $M(U) \to I \to A \to A/I$. This implies $V(M(U)) \subseteq \mathrm{im}(f)$ locally on affine opens, so M defines a closed subscheme of $\mathrm{im}(f)$.

2. If $f_*\mathcal{O}_X$ would be quasi-coherent, then the statement would hold, since kernels of quasi-coherent sheaves are quasi-coherent.

   However, let us assume now only that f is quasi-compact. Since quasi-coherentness is a local property, we can assume that S is affine. Using that f is quasi-compact we have that

   $$X = f^{-1}(Y)$$

   is compact, so we can write

   $$X = \bigcup_{i=1}^{n} U_i$$

   as a finite union of open affines.

   This gives a map

   $$f' : \bigsqcup U_i \to X \to S.$$

   Now $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_{X'}$, so

   $$\mathcal{I} = \ker(\mathcal{O}_S \to \mathcal{O}_{X'}).$$

   Therefore, by stacks project the sheaf of ideals is quasi-coherent in this case.

   Now the scheme-theoretic image is just the closed subscheme determined by $\mathcal{I}$.

3. Denote $X = \bigsqcup_{n\geq 0} \mathrm{Spec}(\mathbb{Z}/p^n)$ and $f\colon X \to \mathrm{Spec}(Z)$. For every $n \geq 0$ we have $\mathbb{Z}/p^n$ which has a unique prime ideal, namely (0) if $0 \leq n \leq 1$ and (p) if $n \geq 2$.

   Every $\mathrm{Spec}(\mathbb{Z}/p^n)$ thus has one point. By looking at preimages of $\mathbb{Z} \to \mathbb{Z}/p^n$ we see that all of them are mapped to $(p) \in \mathrm{Spec}(\mathbb{Z})$. So topologically the image should be $\{(p)\} \subseteq \mathrm{Spec}(\mathbb{Z})$ (we thought naively at the start).

   We use previous two parts to compute ideal sheaf, from which we can infer closed subscheme $\mathrm{im}(f)$. Ideal sheaf is quasi-coherent, so corresponds to some ideal $I \subseteq Z$. We also know ideal sheaf $\tilde{I}$ is contained in the kernel $\ker(\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})} \to f_*\mathcal{O}_X)$ Applying this to global sections we get that I must be contained in the kernel of $\ker(\mathbb{Z} \to \prod_{n\geq 0} \mathbb{Z}/p^n)$. But this map is injective, since every non-zero $a \in \mathbb{Z}$ will be non-zero in some $\mathbb{Z}/p^n$ for big enough n. Therefore I is zero.

   Ideal I vanishes everywhere, so $\mathrm{im}(f)$ is topologically homeomorphism on $\mathrm{Spec}(\mathbb{Z})$ and $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})} \to \mathcal{O}_{\mathrm{im}(f)}$ is surjective map with trivial kernel, so an isomorphism. Therefore $\mathrm{im}(f) = \mathrm{Spec}(\mathbb{Z})$ (which is very surprising).