Algebraic geometry 1
Exercise sheet 9

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Exercise 2. source

1. (version 1) On the right side, we are given transition maps.

   We have that

$$\alpha^{-1}_{|U_0\cap U_1} \circ \beta_{|U_0\cap U_1}$$

   is invertible, because by assumption α and β are isomorphisms. To see injectivity, remember that given sheaves on a cover and transition maps, we can uniquely (up to isomorphism) glue them to get a sheaf on the whole space.

   Well definedness of this map comes from the fact that if two vector bundles $V_1 \cong V_2$ are ismorphic, then the transition map is the same.

   It remains to show surjectivity.

2. (version 2) Suppose we have a vector bundle of rank n on $\mathbb{P}^1_k$. How do we construct a matrix $\in \mathrm{GL}_n(k[T^{\pm 1}])$?

   Take a rank n vector bundle $\mathcal{E}$. Since Picard group of U_0 and U_1 are trivial, we have isomorphisms α, β. So on $\mathrm{Spec}(k[T^{\pm 1}]) \subseteq U_0$ we have an isomorphism $\Gamma(U_0 \cap U_1, \mathcal{O}^n_{U_0}) = (k[T^{\pm 1}])^n \cong \mathcal{E}(U_0 \cap U_1)$.

   Combining this with an isomorphism $\mathcal{E}(U_0 \cup U_1) \cong (k[T^{\pm 1}])^n = \Gamma(U_0 \cap U_1, \mathcal{O}^n_{U_1})$, we get an isomorpism $(k[T^{\pm 1}])^n \cong (k[T^{\pm 1}])^n$.

   Let $\mathcal{D}$ be another rank n vector bundle on $\mathbb{P}^1_k$, and let $\varphi\colon \mathcal{E} \to \mathcal{D}$ be an isomorphism between them. On U_0 and U_1 we get induced isomorphisms

$$(k[T])^n = \mathcal{E}(U_0) \to \mathcal{D}(U_0) = (k[T])^n$$

   and

$$(k[T^{-1}])^n = \mathcal{E}(U_1) \to \mathcal{D}(U_1) = (k[T^{-1}])^n$$

2. Take $G = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \in \mathrm{GL}_n(k[T^{\pm 1}])$. We can writte each $p_i = \frac{g_i}{T^{k_i}}$ for some $g_i \in k[T]$. We can take $k_i = k$ to be all the same. Then

$$\begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} = \begin{pmatrix} T^{-k} & 0 \\ 0 & T^{-k} \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \tag{1}$$

2. second version By left multiplication we get a diagonal matrix we get that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathrm{GL}_2(k[x]),$$

so assume this without loss of generality.

Next, observe that the determinant of A in this case is in $k[x, x^{-1}] \cap k[x]$, so we can write

$$det(A) = cx^n.$$

Since $k[x]$ is an euclidean domain, we can now assume that $a_{12} = 0$. Using our observation regarding the derterminant, we conclude that a_{11} and a_{22} are monomials (with non-negative degree).

We think that we can also assume w.l.o.g that $\deg(a_{11}) \geq \deg(a_{22})$.

Now we can eliminate a_{21} by adding $k[x]$ mulitples of the second column to the first column, to eliminate all the terms in a_{21} with degree greater $\deg(a_{22})$ and all terms in a_{12} with degree smaller $\deg(a_{21})$ by adding $k[x^{-1}]$ multiples of the first row to the second row (to eliminate all terms in a_{12} with degree smaller $\deg(a_{11})$).

3. Claim: Every line bundle on $\mathbb{P}^1$ can be written as

$$\mathcal{O}_{\mathbb{P}^1}(d_n) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n).$$

Proof of claim:

By part 1 of this exercise, we can characterize the isomorphism classes of rank n vector bundles by looking at the transition functions.

In the second part of this exercise, we showed that (for $n = 2$, but actually inductively for all n) these transition functions can be writte as T^d. The claim now follows from observing that the transition matrix of

$$\mathcal{O}_{\mathbb{P}^1}(d_n) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n)$$

is given by

$$T^{(d_1, \ldots, d_n)}.$$