Ex. 4) (i) $\dots \to C(f)_{n+1} \longrightarrow C(f)_n \longrightarrow C(f)_{n-1} \to \dots$

$$D_{n+1} \oplus C_n \longrightarrow D_n \oplus C_{n-1} \longrightarrow D_{n-1} \oplus C_{n-2}$$

$$\begin{pmatrix} d_{n+1} \\ c_n \end{pmatrix} \longmapsto \begin{pmatrix} \partial^D_{n+1}(d_{n+1}) + f_n(c_n) \\ -\partial^C_n(c_n) \end{pmatrix} \longmapsto \begin{pmatrix} T \\ -\partial^C_{n-1}(-\partial_n(c_n)) \end{pmatrix} = \begin{pmatrix} T \\ 0 \end{pmatrix}$$

where $T = \partial^D_n(\partial^D_{n+1}(d_{n+1}) + f_n(c_n)) + f_{n-1}(-\partial^C_n(c_n))$

$$= \cancel{\partial^D_n \circ \partial^D_{n+1}(d_{n+1})} + \cancel{\partial^D_n \circ f_n(c_n) - f_{n-1} \circ \partial^C_n(c_n)}$$

$= 0$ because $D_\bullet$ is a complex and f a complex morphism.

(ii) Let us state snake lemma applied to the short exact seq. of the hint:

$$
\begin{tikzcd}
 & \vdots & \vdots & C_{n-1} \arrow[d] & \\
0 \arrow[r] & D_{n+1} \arrow[r, "\iota_{n+1}"] \arrow[d, "\partial^D_{n+1}"] & D_{n+1} \oplus C_n \arrow[r, "\pi_n"] \arrow[d, "\partial^D_{n+1} \oplus \partial^C_n"] & C_n \arrow[r] \arrow[d, "\partial^C_n"] & 0 \\
0 \arrow[r] & D_n \arrow[r] \arrow[d, "\partial^D_n"] & D_n \oplus C_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \\
 & D_{n-1} & \vdots & \vdots &
\end{tikzcd}
$$

which gives :

$$
\begin{tikzcd}
0 \arrow[r] & \ker(\partial^D_{n+1}) \arrow[r, "\iota^*_{n+1}"] & \ker(\partial^D_{n+1} \oplus \partial^C_n) \arrow[r, "\pi_n^*"] & \ker(\partial^C_n) \arrow[dll, "\mu"] \\
 & \operatorname{coker}(\partial^D_{n}) \arrow[r, "\iota^*_n"'] & \operatorname{coker}(-) \arrow[r] & \operatorname{coker}(\partial^C_n) \arrow[r] & 0
\end{tikzcd}
$$

this gives on homology the following :

$$H_{n+1}(D_\bullet) \longrightarrow H_{n+1}(C(g)_\bullet) \longrightarrow H_n(C_\bullet) \longrightarrow$$

$$\hookrightarrow H_n(D_\bullet) := \frac{\ker(\partial^D_n)}{\mathrm{im}(\partial^D_{n+1})} \cong \mathrm{im}(\mu) \longrightarrow H_n(C(f)_\bullet) \longrightarrow \cdots$$

Exactness at every point of the seq. is clear but at $H_n(D_\bullet)$.

Since $\mu(m) \xmapsto[t^*_N]{} \in \mathrm{im}(\partial^D_{n+1} \oplus \partial^C_n) \equiv 0$ in $H_n(C(f)_\bullet)$

we have exactness there too. Repeating this process creates

the desired long exact seq. from the short one.