Grothendieck vanishing:

Theorem

Let X be a Noetherian topological space. If $\dim(X) = d$, then $H^p(X, \mathcal{F}) = 0$ for any sheaf of abelian groups and $p > d$.

Rmk Any ringed space $(X, \mathcal{O}_X)$ there is a forgetful functor

$$f : (X, \mathcal{O}_X) \longrightarrow (X, \underline{\mathbb{Z}})$$

$$R^i f_* = 0 \quad \text{for} \quad i \geq 1$$

$$H^i(X, \mathcal{F}) \cong H^i(X, f_* \mathcal{F})$$

(i.e. cohomology is the same if treated as $\mathcal{O}_X$-module or sheet of abelian groups.)

Lemma: Let $i: Z \to X$ be a closed immersion of topological spaces. Given $\mathcal{F}$ a sheaf of abelian groups on Z we have
$$H^p(Z, \mathcal{F}) \simeq H^p(X, i_* \mathcal{F})$$

proof $i_*: \mathrm{Shv}(Z, \underline{\mathbb{Z}}) \to \mathrm{Shv}(X, \underline{\mathbb{Z}})$ is an exact functor (check stalks). and sends injectives to Γ-acyclic.

Lemma If X is an irreducible topological space and A is an abelian group then $H^p(X, \underline{A}) = 0$ for $p \geq 1$.

Proof Every open $U \subseteq X$ is connected so $\underline{A}(U) = A$ and $\underline{A}$ is flasque hence Γ-acyclic.

Rmk: This shows $H^1(X, \underline{\mathbb{Z}}) = 0$ for all smooth proper curves ∇. Motivated Grothendieck to introduce étale topology.

Sketch of proof of Thm:

Step 1 WLOG X is irreducible:

If $i: Z \subseteq X$ closed immersion and

$U := X \setminus Z$ $\quad j: U \subseteq X$ open

then we have SES

$$\boxed{0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0}$$

the support of $j_! j^* \mathcal{F}$ is

$\overline{U} \subseteq X$ i.e. $j_! j^* \mathcal{F} = i_{\overline{U}*} \mathcal{G}$

for some $\mathcal{G}$.

Apply this to Z an irreducible component, then $\dim(Z), \dim(\overline{U}) \leq \dim(X)$

and # components $Z, \overline{U}$ < # component X

$$\begin{array}{ccccc} H^p(X, j_! j^* \mathcal{F}) & \to & H^p(X, \mathcal{F}) & \to & H^p(X, i_* i^* \mathcal{F}) \\ \| & & & & \| \\ H^p(\overline{U}, \mathcal{G}) & & & & H^p(Z, i^* \mathcal{F}) \\ \| & & & & \| \\ 0 & \sim\sim & \text{by induction} & \sim & 0 \end{array}$$

Step 2 Dimensional induction:

If $\dim(X)=0$ and X is irreducible then $X=\{*\}$, $Shv_{(X,\mathbb{Z})}=Ab$ and $\Gamma(X,-)$ is an equivalence.

Let $j: U \subseteq X$ any open with complement $Z = X \setminus U$ $i: Z \subseteq X$.

We have SES:

$$0 \to j_!^U \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to i_* \underline{\mathbb{Z}} \to 0$$

and $\dim(Z) \leq d-1$.

$$\begin{array}{ccccc} H^{p-1}(Z, i_*\mathbb{Z}) & \longrightarrow & H^p(X, j_!^U \underline{\mathbb{Z}}) & \longrightarrow & H^p(X, \underline{\mathbb{Z}}) \\ \| & & & & \| \\ 0 & & & & 0 \\ \text{induction.} & & & & (\text{flasque}) \end{array}$$

Intuition: The $\{j_!^U \underline{\mathbb{Z}}\}_{U \subseteq X}$ as U ranges over open subsets "generate" $Shv_{(X,\mathbb{Z})}$

Step 3: From general to locally generated by one section.

Let B denote the set of local sections i.e.

$$B = \{(U,s) \mid U \subseteq X \text{ open and } s \in \Gamma(U, \mathcal{F}|_U)\}.$$

We have

$$\bigoplus_{(U,s)\in B} j_!^U \mathbb{Z} \longrightarrow \mathcal{F}$$

is surjective. Given $S \subseteq B$ let $\mathcal{F}_S \subseteq \mathcal{F}$ the image of

$$\bigoplus_{(U,s)\in S} j_!^U \mathbb{Z} \longrightarrow \mathcal{F}$$

then

$$\mathcal{F} = \operatorname*{colim}_{\substack{S \subseteq B \\ S \text{ finite}}} \mathcal{F}_S$$

and

$$\underbrace{H^p(X, \mathcal{F}) = \operatorname*{colim}_{\substack{S \subseteq B \\ S \text{ finite}}} H^p(X, \mathcal{F}_S)}_{\text{Homework.}}$$

If S is finite $S = \{(u_1, s_1), \dots (u_n, s_n)\}$

$T = \{(u_1, s_1), \dots, (u_{n-1}, s_{n-1})\}$.

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=1}^{n-1} j_!^{u_i} \mathbb{Z} \arrow[r] \arrow[d] & \bigoplus_{i=1}^{n} j_!^{u_i} \mathbb{Z} \arrow[r] \arrow[d] & j_!^{u_n} \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_T \arrow[r] & \mathcal{F}_S \arrow[r] & \mathcal{F}_S / \mathcal{F}_T \arrow[r] & 0
\end{tikzcd}$$

$$H^p(X, \mathcal{F}_T) \to H^p(X, \mathcal{F}_S) \to H^p(X, \mathcal{F}_S/\mathcal{F}_T)$$

$$\underset{\text{induction}}{\overset{\|}{0}} \qquad\qquad \underset{\text{assumption of step.}}{\overset{\|}{0}}$$

Step 4: subsheaves of $j_!^u \mathbb{Z}$

we have

$$0 \to \mathcal{G} \to j_!^u \mathbb{Z} \to \mathcal{F} \to 0$$

let $\eta \in X$ be the generic point.

for all $x \in X$ we have

$$\begin{tikzcd}
\mathcal{G}_x \arrow[r, hook] \arrow[d] & (j_!^u \mathbb{Z})_x \arrow[d, hook] \\
\mathcal{G}_\eta \arrow[r, hook] & (j_!^u \mathbb{Z})_\eta = \mathbb{Z}
\end{tikzcd}$$

then $\mathcal{G}_{|_U} = \partial\mathbb{Z} \subseteq \mathbb{Z}$ and

$\mathcal{G} \to j_!^U \underline{\mathbb{Z}}$ factors through

$\mathcal{G} \to j_!^U \partial\underline{\mathbb{Z}} \subseteq j_!^U \underline{\mathbb{Z}}$

Consider $0 \to \mathcal{G} \to j_!^U \partial\mathbb{Z} \to \mathcal{F}' \to 0$

Suppose $\mathcal{G}(V) \neq 0$ for some

$V \subseteq U$ then

$$j_!^V \partial\underline{\mathbb{Z}} \subseteq \mathcal{G} \subseteq j_!^U \partial\mathbb{Z}.$$

this shows $\mathcal{F}'_{|V} = 0$ Let $Z = X \setminus V$

$i : Z \underset{\text{closed}}{\subseteq} X$ then $\mathcal{F}' = i_* i^* \mathcal{F}'$

by dimension induction

$$\underset{0}{\underset{\parallel}{H^{p-1}(X, \mathcal{F}')}} \to H^p(X, \mathcal{G}) \to \underset{0}{\underset{\parallel}{H^p(X, j_!^U \mathbb{Z})}}$$

$$\therefore \quad \underset{0}{\underset{\parallel}{H^p(X, j_!^U \mathbb{Z})}} \to H^p(X, \mathcal{F}) \to \underset{0}{\underset{\vee}{H^{p+1}(X, \mathcal{G})}}$$

Rf_*^i preserve Qcoh:

Proposition

Let $f : X \to Y$ be a qcqs map of schemes and let $\mathcal{F} \in \mathrm{Qcoh}(X)$. Then $R^i f_* \mathcal{F} \in \mathrm{Qcoh}(Y)$ and for all open affine $V \subseteq Y$ with $U = f^{-1}(V)$ we have

$$R^i f_* \mathcal{F}(V) = \widetilde{H^i(U, \mathcal{F})}.$$

Proof $R^i f_* \mathcal{F}$ is the sheafification of $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_V)$.

So the formula holds

$$(R^i f_* \mathcal{F})|_V = R^i f_*^V \left(\mathcal{F}|_{f^{-1}(V)}\right).$$

WLOG $V = \operatorname{Spec} B$ and we want to show

$$R^i f_* \mathcal{F} = \widetilde{H^i(X, \mathcal{F})}.$$

Let $g \in B$ it suffices to show

$$H^i(X, \mathcal{F})[g^{-1}] \simeq H^i(f^{-1}(D(g)), \mathcal{F}).$$

<u>Case 1 X is separated:</u>

write $X = \bigcup_{i=1}^n U_i$ affine cover.

$H^i(X, \mathcal{F})$ is computed by Čech complex $C^\bullet(\{U_i\}_{i=1}^n, \mathcal{F})$. with

$$C^k(\{U_i\}_{i=1}^n, \mathcal{F}) = \bigoplus_{\substack{J \subseteq \{1, \dots, n\} \\ |J| = k}} \mathcal{F}(U_J)$$

Moreover, each U_J is affine,

$f^{-1}(D(g)) = \bigcup_{i=1}^n D_{U_i}(g)$ and

$$\bigcap_{i \in J} D_{U_i}(g) = D_{U_J}(g)$$

is again affine.

$H^i(f^{-1}(D(g)), \mathcal{F})$ is computed by Čech complex $C^\bullet(\{D_{U_i}(g)\}_{i=1}^n, \mathcal{F})$

Note that

$$C^{\bullet}\left(\{D_{u_i}(s)\}_{i=1}^{n}, \mathcal{F}\right) = C^{\bullet}\left(\{u_i\}_{i=1}^{n}, \mathcal{F}\right)\left[\tfrac{1}{s}\right]$$

Since localization is exact, it commutes with cohomology

$$\text{so } \check{H}^{i}\left(f^{-1}(D(s)), \mathcal{F}\right) = \check{H}^{i}\left(X, \mathcal{F}\right)\left[\tfrac{1}{s}\right]$$

$$\begin{array}{ccc} \| & & \| \\ H^{i}(f^{-1}(D(s)), \mathcal{F}) & & H^{i}(X, \mathcal{F})\left[\tfrac{1}{s}\right]. \end{array}$$

<u>Case 2, several X:</u>

We only assume X is qcqs over $Y = \operatorname{Spec} B$.
We have finite covers by affines
$X = \bigcup_{i=1}^{n} U_i$ and the U_J are
quasicompact separated schemes
but not necessarilly affine,
in particular $H^p(U_J, \mathcal{F})$ might not
vanish for $p \geq 1$.

We still have Čech-to-cohomology spectral sequence.

we have a map of Čech complexes

$$C^\bullet(\{U_i\}_{i=1}^n, \underline{H^i(-, \mathcal{F})})$$
$$\downarrow$$
$$C^\bullet(\{D_{U_i}(g)\}_{i=1}^n, \underline{H^i(-, \mathcal{F})})$$ and

by the separated case this map induces an isomorphism.

$$C^\bullet(\{D_{U_i}(g)\}_{i=1}^n, \underline{H^i(-, \mathcal{F})})$$
$$= C^\bullet(\{U_i\}_{i=1}^n, \underline{H^i(-, \mathcal{F})})\left[\frac{1}{g}\right]$$

Let $\mathcal{U} = \{U_i\}_{i=1}^n$ and

$\mathcal{V} = \{D_{U_i}(g)\}$

we have functors

$$F_X : Shv_{(X,\mathcal{O}_X)\text{-mod}} \longrightarrow PSh_X$$

$$\mathcal{F} \longmapsto [U \mapsto H^i(U, \mathcal{F})]$$

$$F_{D(g)} : Shv_{(X,\mathcal{O}_X)} \longrightarrow PSh_X$$

$$\mathcal{F} \longmapsto [U \mapsto H^i(U \cap f^{-1}(D(g)), \mathcal{F})]$$

and natural transformation

$$F_X \Longrightarrow F_{D(g)}$$

Moreover, $\check{H}^0_{\mathcal{U}} \circ F_X \Longrightarrow \check{H}^0 \circ F_{D(g)}$

$$\check{H}^0_{\mathcal{U}} \circ F_X = H^0(X, -) \qquad \check{H}^0 \circ F_{D(g)} = H^0(D(g), -)$$

We get morphism of convergent Grothendieck spectral sequences

$${}^{\mathcal{U}}E_2^{p,q} = \check{H}^p_{\mathcal{U}}(X, H^q(\mathcal{U}_J, \mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

$${}^{\mathcal{V}}E_2^{p,q} = \check{H}^p_{\mathcal{V}}(f^{-1}D(g), H^q(f^{-1}D(g) \cap \mathcal{U}_J, \mathcal{F}) \Rightarrow H^{p+q}(f^{-1}D(g), \mathcal{F})$$

By the separated case

$${}^{\mathcal{U}}E_2^{p,q}\left[\tfrac{1}{g}\right] \xrightarrow{\sim} {}^{\mathcal{V}}E_2^{p,q}$$

since each $\mathcal{U}_J$ is separated. Since localization is exact this shows ${}^{\mathcal{U}}E_r^{p,q}\left[\tfrac{1}{g}\right] \xrightarrow{\sim} {}^{\mathcal{V}}E_2^{p,q}$ $\forall\, 2 \le r \le \infty$.

By construction

$$H^{p+q}(X, \mathcal{F})\left[\tfrac{1}{g}\right] \longrightarrow H^{p+q}(f^{-1}(D(g)), \mathcal{F})$$

is compatible with filtration and isomorphism on graded pieces since it is at $E_\infty^{p,q}$.