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1)

a) By the Baire category theorem $(X, \mathcal{T})$ is a Baire topological space. Assume that there exists a countable Hamel basis $(a_n)_{n \in \mathbb{N}}$ and define

$$X_n = \operatorname{Span}(a_1, \dots, a_n).$$

One can check that X_n is closed for all $n \in \mathbb{N}$, because it is a proper subspace of X.

We also claim that proper subspaces of normed spaces have empty interior.
Proof of claim:
Suppose X_n has nonempty. Then there exist $x \in X_n$ open, i.e. some $r > 0$ with $B_r(x) \subseteq X_n$.
Now take any $z \in X_n$ and define
$y = x + \frac{r}{2\|z\|} \cdot z \in B_r(x) \subseteq X_n$.

This implies that

$$z = \frac{2\|z\|}{r}(y - x) \in X_n,$$

as X_n is a subspace. This is a contradiction to the properness of X_n.

$\square$ claim

Hence $X \neq \bigcup_{n \in \mathbb{N}} X_n$ as X is a Baire topolgical space. $\lightning$

$\square$

b) "$\Rightarrow$": (Here we don't need T to be surjective)

Suppose that $T(A)$ is closed. By continuity of T, we get that

$$T^{-1}(T(A)) = A + \ker T$$

is closed.

"$\Leftarrow$":

Assume that $A + \ker T \subseteq X$ is closed.
Then $(A + \ker T)^c = \{x \in X : T(x) \notin T(A)\}$
is open.
By the Open mapping theorem T is open, so

$$T((A + \ker T)^c) = T(A)^c \subseteq Y$$

is open, showing that $T(A) \subseteq Y$ is closed.

4)

a) We want to show that A is closed, which will then imply by the closed graph theorem that A is continuous, or equivalently, bounded.

Suppose $(x_n, A x_n) \to (x, y)$.
We want to show that $(x, y) \in \mathcal{G}(A)$.

$$(y - Ax \quad , \quad y - Ax)$$

$$\lim_{n\to\infty} (A x_n - Ax, A x_n - Ax)$$

$$= \lim_{n\to\infty} (x_n - x \quad , \quad A^2 (x_n - x))$$

$$= \Big(\underbrace{\lim_{n\to\infty} x_n - x}_{=0} \quad , \quad \lim_{n\to\infty} A^2 (x_n - x) \Big) = 0$$

$$\Rightarrow (x, y) = (\lim_{n\to\infty} x_n, A \lim_{n\to\infty} x_n) \in \mathcal{G}(A).$$

$\square$

b)

JT is continuous, because it is bounded.
Hence, by the closed graph theorem

$$\mathcal{G}(JT) \text{ is closed.}$$

Also J is continuous and by injectivity there exists a left inverse J^{-1} of J.
Hence

$$\{(x, J^{-1}JTx) : x \in X\} = \{(x, Tx) : x \in X\}$$

is closed showing again by the closed graph theorem that T is continuous, or equivalently, bounded.

$\square$

3)

Since B is linear in each component it is bilinear.

Since for all $x \in X$ $B(x, \cdot) \in L(Y, Z)$, there exists $c_x \in \mathbb{R}$ s.t. for all $y \in Y$ with $\|y\| = 1$

$$B(x, y) \leq c_x$$

Consider the set $\mathcal{A} := \{ B(\cdot, y) : \|y\| = 1 \}$.
By Banach-Steinhaus, there exists a global constant $M \in \mathbb{R}$, s.t. for all

$$A \in \mathcal{A} \quad \text{we have} \quad \|A\| \leq M.$$

By linearity in the second component we get that

$$\|B(x, y)\| \leq M \|y\| \|x\|$$

for all $(x, y) \in (X \times Y)$.

2)

i) Take a converging sequence

$$f_k \xrightarrow{k\to\infty} f$$

with $f_k \in M_n \quad \forall k \in \mathbb{N}$.

Then for f_k there exist $t_k \in [0, 1-\frac{1}{n}]$

$$\frac{|f_k(t_k+h) - f(t)|}{h} \leq n$$

for all $h \in (0, \frac{1}{n})$.

By Bolzano-Weierstraß there exists a subsequence

$$(t_{k_j})_{j\in\mathbb{N}} \text{ of } (t_k)_{k\in\mathbb{N}},$$

s.t. $(t_{k_j}) \xrightarrow{j\to\infty} t$.

$$\left| \frac{f(t+h) - f(t)}{h} \right| = \lim_{k\to\infty} \left| \frac{f_k(t_k+h) - f_k(t_k)}{h} \right| \leq n$$

$\Rightarrow f \in M_n$.

ii.)

It is enough to show that for all $\varepsilon > 0$ and $f \in C[0,1]$

$$M_n^c \cap B_\varepsilon(f) \neq \emptyset .$$

Define $PL([0,1])$ as the piecewise linear functions on $[0,1]$. It can be shown using the fact that continuous function on the compact set $[0,1]$ are uniformly continuous that $PL([0,1])$ is dense in $C[0,1]$.
Pick a piecewise linear function p that is in $B_\varepsilon(f)$ for some $\varepsilon > 0$.
Define M as the maximal slope of p (where this is defined).
Define $q : [0,1] \to \mathbb{R}$ by setting

$$q\left(\frac{j}{M'}\right) = \begin{cases} 1 & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases}$$

and linear in between.
Then

$$h := f + \frac{1}{2}\varepsilon q$$

satisfies $d(f,h) < \varepsilon$

and for some M' large enough

we have slope of h around every point bigger than n.

$$\left(\text{for } M' = \left\lceil 2(M+n)/\varepsilon \right\rceil\right)$$

$\square$

iii) By the Baire category theorem $C[0,1]$ is a Baire topological space.

Hence,

$$\left(\bigcup_{n\in\mathbb{N}} M_n\right)^{\circ} = \emptyset .$$

Notice that a function $f \in \left(\bigcup_{n\in\mathbb{N}} M_n\right)^{c} := M^{c}$ is nowhere differentiable.

This shows that M^{c} is dense in $C[0,1]$.

In particular, M^{c} is not empty.

$\square$