Elliptic curves and their moduli spaces
Exercise sheet 3

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Problem 1.

a) Let

$$F\colon \operatorname{Hom}_{Y-group}(Y \times_{\operatorname{Spec} k} A_1, Y \times_{\operatorname{Spec} k} A_2) \to \operatorname{Hom}_{k-group}(A_1, A_2)$$
$$g \mapsto g \mid_{\{x\} \times A_1}$$

be the map given in the exercise.

We can check injectivity and surjectivity of F by hand. Since both have hom sets have group structure, we can take f that gets mapped to 0 (i.e. $f \mid_{\{x\}\times A_1}$ is the unique map $A_1 \to A_2$ that factors through $e\colon \operatorname{Spec} k \to A_2$). In particular that means that the composition $Y \times A_1 \to Y \times A_2$ is constant when restricted to $\{x\} \times A_1$. Since all our assumptions conveniently fit Rigity theorem, we can use that to get that f factors through $Y \to Y \times A_2$, which shows that $Y \times A_1 \to Y \times A_2$ is the identity element in $\operatorname{Hom}_{Y-\text{group}}(Y \times A_1, Y \times A_2)$ (i.e. the "zero map"). Therefore F is injective. It is clearly surjective; given a map $g\colon A_1 \to A_2$ we can do base change to a map $(\mathrm{id}, g)\colon Y \times A_1 \to Y \times A_2$, which restricts to g.

b)

Problem 2.

a) Lets use primitive element theorem and write $K = k(a)$ for some $a \in K$. Written differently we have $K = \operatorname{Quot}(k[x]/f(x))$ where f is the minimal polynomial of $a \in K$. Denote $A = k[x]/f(x)$. Note that minimal polynomial is irreducible and thus A a domain.

From Kähler arithmetic we have that

$$\Omega^1_{K/k} = K \otimes_A \Omega^1_{A/k}$$

So it is enough to calculate $\Omega^1_{A/k}$.

Suppose K/k is separable. That implies x is separable and $f(x)$ has no multiple roots. Therefore $f'(x)$ and $f(x)$ are coprime and thus generate whole $k[x]$. That means $f'(x)$ is invertible as element in A.

Let M be any A-module and $\delta \in \mathrm{Der}_k(A, M)$. Derivation δ has to be k-linear, so it is uniquely defined by its value in x. Since $f(x)$ is 0 in A, we must have

$$\delta(f(x)) = f'(x)\delta(x) = 0$$

But $f'(x)$ is invertible, so we can simply multiple by its inverse and obtain $\delta(x) = 0$. We've shown that for every A-module M, there exist only derivation constantly 0. Therefore $\Omega^1_{A/k} = 0$ and thus also $\Omega^1_{K/k} = 0$.

b) So $K = \mathrm{Quot}(k[x_1, \ldots, x_n]/I)$. Denote $B = k[x_1, \ldots, x_n]/I$ and let $A = k[y_1, \ldots, y_d]$ be Noether normalization of B (so $A \to B$ is finite). By definition $d = \mathrm{trdeg}(K/k)$.

So we have maps $k \to A \to B$. Using Kähler arithmetic we get exact sequence

$$B \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k} \to \Omega^1_{B/A} \to 0.$$

Because $\mathrm{char}(k) = 0$ and $A \to B$ finite, we have by previous part $\Omega^1_{B/A} = 0$. So we get a surjection $B \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k}$. Since $A \to B$ is injective, the map $B \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k}$ is also injective. So we have

$$B \otimes_A \Omega^1_{A/k} \xrightarrow{\sim} \Omega^1_{B/k}$$

We've shown during lectures that $\Omega^1_{A/k} = A^d$. So $B \otimes_A \Omega^1_{A/k} = B^d$.

Again using Kähler arithmetic for localization we have

$$K \otimes_B \Omega^1_{B/k} \cong \Omega^1_{K/k}$$

So $\Omega^1_{K/k} \cong K^d$.

c) It suffices to find any non-empty open subscheme, as X is integral.

Suppose $\Omega^1_{X/k}$ has rank n at the generic point. Since $\mathrm{char}(k) = 0$, that is equal to the local dimension.

It has rank at least n everywhere else. And using upper semicontinuity we get that it has rank exactly n on an open neighbourhood of the generic point. So $\Omega^1_{X/k}$ is therefore locally free of rank n on some non-empty neighbourhood of the generic point, which is where X is then smooth.