Algebraic geometry 2
Exercise sheet 2

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Exercise 4. For $i \leq -1$ we define $h_i = 0$.

Since the chain is exact, the map d_1 must be surjective. Therefore we can define h_0 as the lift of $\mathrm{id}\colon C_0 \to C_0$ along surjection $d_1\colon C_1 \to C_0$. (Though there was no need to treat this case separately)

Let now $i \geq 1$ and suppose h_j for $j < i$ exist with property as in the exercise. Observe that $\mathrm{id}_{C_i} - h_{i-1} \circ d_i\colon C_i \to C_i$ factors through $\ker(d_i)$, since

$$\begin{aligned} d_i \circ (\mathrm{id}_{C_i} - h_{i-1} \circ d_i) &= d_i - d_i \circ h_{i-1} \circ d_i \\ &= d_i - (id_{C_{i-1}} - h_{i-2} \circ d_{i-1}) \circ d_i \\ &= d_i - d_i \\ &= 0. \end{aligned}$$

Using exactness ($\mathrm{im}(d_{i+1}) = \ker(d_i)$), we can lift $C_i \to \mathrm{im}(d_{i+1})$ along the surjection $C_{i+1} \to \mathrm{im}(d_{i+1})$ to obtain $h_i\colon C_i \to C_{i+1}$ for which $d_{i+1} \circ h_i = \mathrm{id}_{C_i} - h_{i-1} \circ d_i$.