Summer term 2024 Dr. Andreas Mihatsch
Elliptic curves and their moduli spaces

In-class problems (April 15/16)

Problem 1

Let k be a field of characteristic 0 and let $h \in k[x]$ be a separable polynomial of degree $m \geq 4$. Let $F \in k[X, Y, Z]$ be the homogenization of

$$f(x, y) = y^2 - h(x).$$

Show that $V_+(F) = V(f) \cup \{[0 : 1 : 0]\}$ and that $V(f)$ is smooth.[1]

Show further that $V_+(F)$ is not smooth in $[0 : 1 : 0]$.

Problem 2

Let k be a field with $\mathrm{char}(k) \neq 2$. Consider a smooth simplified Weierstrass equation

$$f(x, y) = y^2 - x^3 - ax - b.$$

Find an equation for the line T_P tangent to $V(f)$ in a point $P = (x_0, y_0) \in V(f)(k)$.

Bonus: Determine the intersection $T_P \cap V(f)$ and obtain a formula for $P + P$.

Homework problem (Hand in April 17)

Problem 3

Let $f \in k[x, y]$ be a non-zero polynomial and set $X = V(f) \subset \mathbb{A}^2_k$. Let $(x_0, y_0) \in X(k)$ be a rational point and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$. Prove that

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1 \quad \Leftrightarrow \quad (\partial f/\partial x,\ \partial f/\partial y)(x_0, y_0) \ \neq\ (0, 0).$$

Deduce that smoothness and normality[2] are equivalent for plane curves when k is algebraically closed.

Hint: First show that $\mathfrak{m}/\mathfrak{m}^2$ is the image of $(x - x_0, y - y_0)/(x - x_0, y - y_0)^2$ in $k[x, y]/(f)$. Then determine the image of f in $(x - x_0, y - y_0)/(x - x_0, y - y_0)^2$.

[1] Here, we have identified $V_+(F) \cap D_+(Z) = V(f)$ via $x = X/Z$ and $y = Y/Z$.
[2] See §20 and §21 of last semester's algebraic geometry for a recap on normality.