Dr. I. Gleason **SS 2024**
Dr. J. Anschütz

Algebraic Geometry II

9. Exercise sheet

Exercise 1 (4 points):

Let $f\colon Y \to X$ be a morphism of ringed spaces. Let $\mathcal{M}$ be an $\mathcal{O}_Y$-module.
i) Show that $R^i f_*(\mathcal{M})$ is the sheafification of the presheaf $U \mapsto H^i(f^{-1}(U), \mathcal{M}_{|f^{-1}(U)})$.
ii) Show that flasque $\mathcal{O}_Y$-modules are acyclic for f_*. Deduce that if $g\colon Z \to Y$ is a morphism of ringed spaces, then g_* sends injective $\mathcal{O}_Z$-modules to acyclic objects for f_*.
Hint/Remark: In part i) use that restricting to open subsets preserves injective $\mathcal{O}$-modules. Part ii) implies the existence of the Leray spectral sequence $E_2^{ij} = R^i f_*(R^i g_*(-)) \Rightarrow R^{i+j}(f \circ g)_*(-)$.

Exercise 2 (4 points):

Use the horseshoe lemma to prove the existence of Cartan–Eilenberg resolutions as defined in class.

Exercise 3 (4 points):

Let k be a field. Using a suitable open affine cover of X, show that $H^1(X, \mathcal{O}_X)$ is an infinite-dimensional k-vector space in the following cases:
1) X is the affine line over k with doubled origin.
2) $X = \mathbb{A}_k^2 \setminus \{(0,0)\}$.

Exercise 4 (4 points):

Let A be a ring and M, N two A-modules with flat resolutions $P^\bullet$ resp. $Q^\bullet$ (thus $Q^i = P^i = 0$ for $i > 0$). Use the spectral sequence associated to the double complex $C^{\bullet,\bullet}$ with $C^{i,j} = P^i \otimes Q^j$ to show that for $i \in \mathbb{Z}$ there are natural isomorphism

$$\mathrm{Tor}_{-i}^A(M, N) := H^i(P^\bullet \otimes N) \cong H^i(\mathrm{Tot}(C^{\bullet,\bullet})) \cong H^i(M \otimes Q^\bullet).$$

In particular, Tor-functors can be computed via flat resolutions in either variable.

To be handed in on: Thursday, 20.06.2024 (during the lecture or via eCampus).