Dr. I. Gleason
Dr. J. Anschütz **SS 2024**

Algebraic Geometry II

8. Exercise sheet

Exercise 1 (4 points):

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{B} := \mathrm{Fun}(\bullet \to \bullet, \mathcal{A})$ be the category of arrows in $\mathcal{A}$. Set

$$F\colon \mathcal{B} \to \mathcal{A},\ (f\colon X \to Y) \mapsto \ker(f).$$

Use the snake lemma to find a universal δ-functor for F, and show that $R^1F(f\colon X \to Y) \cong \mathrm{coker}(f)$ while $R^iF = 0$ for $i \geq 2$.

Exercise 2 (4 points):

Prove the horseshoe lemma as presented in class.

Exercise 3 (4 points):

Let $(X, \mathcal{O}_X)$ be a ringed space, let $V \subseteq X$ be an open subset and set $\mathcal{O}_V := \mathcal{O}_{X|V}$. Let $j\colon (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ be the canonical morphism of ringed spaces.
i) Let $\mathcal{F}$ be an $\mathcal{O}_V$-module and set $j_!$ as the sheafification of the presheaf

$$U \subseteq X \mapsto \begin{cases} \mathcal{F}(U) & \text{if } U \subseteq V \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the functor $j_!$ from $\mathcal{O}_V$-modules to $\mathcal{O}_X$-modules is left adjoint to the restriction functor j^*.
ii) Let $\mathcal{F}$ be an injective $\mathcal{O}_X$-module. Prove that the restriction $\mathcal{F}_{|V}$ is an injective $\mathcal{O}_V$-module.
iii) If $(X, \mathcal{O}_X)$ is a scheme, show that in general $j_!\mathcal{O}_V$ is not quasi-coherent.

Exercise 4 (4 points):

1) Let $\iota\colon \mathrm{Mod}_{\mathbb{Z}/2} \to \mathrm{Mod}_{\mathbb{Z}}$ be the natural forgetful functor and set $F\colon \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}},\ A \mapsto \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, A)$. Show that in general $R^i(F \circ \iota) \ncong R^iF \circ \iota$.
2) Let $(X, \mathcal{O}_X)$ be a ringed space. Use exercise 3 to show that each injective $\mathcal{O}_X$-module is a flasque abelian sheaf. Let $|-|\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Sh}_{\mathrm{Ab}}(X)$ be the forgetful functor to abelian sheaves. Conclude that for $\mathcal{M} \in \mathrm{Mod}_{\mathcal{O}_X}$ we have $H^i(X, \mathcal{M}) \cong H^i(X, |\mathcal{M}|)$ for $i \geq 0$.

To be handed in on: Thursday, 13.06.2024 (during the lecture or via eCampus).