Algebraic geometry 1
Exercise sheet 4

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Exercise 1.

1. Let X be a finite set that is irreducible with respect to some topology $\mathcal{F}$ on X. Then we get $\mid \mathcal{F} \mid < \infty$ and since finite unions of closed sets are closed again, we get that

$$X' := \bigcup_{U \subsetneq X \text{closed}} U$$

is closed in X. Since X is by assumption irreducible, $X \neq X'$, so we can pick $x_0 \in X \backslash X'$, which is by construction generic. For the second part of the exercise we use part 2 of Hochster's Theorem. As a finite set, X is quasicompact and as a basis $\mathcal{B}$ consisting of quasicompact open sets stable under finite intersections take all of the open sets.
It remains to show that X is sober. We need to check that every irreducible subset of X has a unique generic point. The existence of a generic point comes from part of of this exercise.
Uniqueness of this point is due to the fact that generic points in T_0 spaces are unique if they exist, which follows directly from the definition of T_0.

1. Let X be a finite irreducible topological space. Since

$$X = \bigcup_{x \in X} \overline{\{x\}}$$

is a finite decomposition in closed sets, we must have $\overline{\{x\}} = X$ for some $x \in X$. This x is a generic point of X.

If we additionally assumed X is T_0, then this point x would be unique, since in a T_0 space we have $\overline{\{x\}} \neq \overline{\{y\}}$ for $x \neq y$. Also in a finite space the conditions of quasicompactness and the basis being stable under finite intersections are clearly fulfilled. So finite T_0 spaces are spectral.

2. Let us first describe what $\text{Spec}(\mathbb{Z})$ looks like. It is a PID with prime ideals being those (a) for which $a \in \mathbb{Z}$ is a prime number or $a = 0$. So

$$\text{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime}\} \cup \{(0)\}.$$

In this space (0) is a unique generic point.

Closed sets in $\text{Spec}(\mathbb{Z})$ are by definiton

$$V((a)) = \{(p) \in \text{Spec}(\mathbb{Z}) \mid (a) \subseteq (p)\} = \{(p) \in \text{Spec}(\mathbb{Z}) \mid p \text{ divides } a\}.$$

So if $a = \Pi_i p_i^{k_i}$, then $V((a)) = \{(p_i)\}_i \subseteq \text{Spec}(\mathbb{Z})$. Since any $a \in \mathbb{Z}$ is only divisible by finitely many prime numbers, we get the finite completement topology on $\text{Spec}(\mathbb{Z}) \setminus \{(0)\}$.

Adding generic point (0) to $\text{Spec}(\mathbb{Z}) \setminus \{(0)\}$ is actually the construction $X \to X^{\text{sob}}$ which we did last week.

Exercise 2. Denote $A = \lim A_i$, $B = \lim B_i$ and $C = \lim C_i$. Also denote maps $A_i \to A$ with f_i, $B_i \to B$ with g_i and $C_i \to C$ with h_i.

By composing α_i and g_i we get $A_i \to B$ defined as $g_i \circ \alpha_i$. Then by the definition of a colimit we have a unique map $\alpha : A \to B$, such that $g_i \circ \alpha_i = \alpha \circ f_i$. In the same way we obtain $\beta \colon B \to C$. With these definitions the whole diagram commutes.

Exercise 3.

1. Let $\mathcal{F}, \mathcal{G}$ be presheaves on X and $(\phi_U)_{U \in Ouv(X)}, (\psi_U)_{U \in Ouv(X)}$ morphism of presheaves from $\mathcal{F}$ to $\mathcal{G}$.
   We define for $U \in Ouv(X)$

   $$(\phi + \psi)(U) := \phi(U) + \psi(U).$$

   This is indeed a map of presheaves again, because we can restrict ϕ and ψ independently.
   The zero object in this category is the presheaf that sends every open set $U \in Ouv(X)$ to the trivial group $(0, +)$. It is inital and terminal, because group maps send 0 to 0. For this reason the category of presheaves is an additive category and since morphism of sheaves are defined using their underlying presheaves, also the category of sheaves form an additive category.
   We still need to check that kernels and cokernels exist in the category of Sheaves, so from now on $\mathcal{F}, \mathcal{G}$ and maps between them we be of sheaves.

   Define

   $$(ker\phi)(U) := ker\phi(U).$$

We check that this is indeed a presheaf. To do that one has to check that restrictions behave well.

Next, one checks that this is indeed a sheaf.

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We define for $U \in Ouv(X)$

$$(\phi + \psi)(U) := \phi(U) \oplus \psi(U),$$

where $\oplus$ is the usual direct sum of groups. This is indeed a map of presheaves again, because we can restrict ϕ and ψ independently.
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Next, one checks that this is indeed a sheaf.
The construction of limits and colimits just works using limits and colimits of the abused groups, but these are a priori just presheaves. To see that the colimit of sheaves is indeed a sheaf, just remember that by part 4 of proposition 5.7 sheaffification commutes with all colimits, because it is left-ajoint functor.
Why are the limits already sheaves??

2. Assume $f\colon \mathcal{F} \to \mathcal{G}$ is surjective on each stalk. Then we can show that $f(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for each $U \in \mathrm{Ouv}_X^{\mathrm{op}}$. Pick $s \in \mathcal{G}(U)$. Then $s_x \in \mathcal{G}_x$ the germ of s at x. Since $f_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective we have a function $t_x \in \mathcal{F}(U)$ such that $f(U)(t_x)$ has the same germ at x as s. So $t_x \in \mathcal{F}(U)$ is mapped to a function that is locally at x same as s. Denote V_x the open subset with $x \in V_x$ for which $f(U)(t_x) \mid_{V_x} = s \mid_{V_x}$.

   We were inspired by some stack exchange site and only worked out some of the details. Let $\mathcal{F}, \mathcal{G}$ be sheaves on X and $\phi : \mathcal{F} \to \mathcal{G}$ an epimorphism on sheaves. We want to show that ϕ_x is surjective on each stalk.
   Let $\mathcal{F}$ be the skyscraper sheaf at x with value $\mathcal{G}_x / Im(\phi)$, i.e. for $U \subset X$ open, we have

$$\mathcal{H}(U) = \begin{cases} \mathcal{G}_x / Im(\phi_x) & \text{if } x \in U \\ (0, +) & \text{else.} \end{cases}$$

Also, define

$$\begin{aligned} \psi : \mathcal{G} &\to \mathcal{H} \\ f &\mapsto \bar{f}_x \end{aligned}$$

if f is a section over a set containig x.
Now assume, that ϕ_x is not surjective for all $x \in X$. Then there exists $x \in X$ such that ϕ_x is not surjective, meaning $\mathcal{G}_x / Im(\phi_x)$ is not the trivial group. In this case, the two maps

$$(\psi, 0), (0, \psi) : \mathcal{G} \to \mathcal{H} \oplus \mathcal{H}$$

are not the same since $\mathcal{H}$ is not trivial. However, by construction they are the same if precomposed with ϕ showing that ϕ is not an epimorphism of sheaves.

3. First prove that $\mathcal{F}$ and $\mathcal{G}$ are sheaves.

   Define a morphism $f \colon \mathcal{F} \to \mathcal{G}$ with $f(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$ being the map $s \oplus t \mapsto s \mid_{D_- \cap D_+}$. Note that if $U \cap D_-$ is connected, the map $s \mid_{D_- \cap D_+}$ must assume same value on whole $U \cap D_- \cap D_+$.

   Using part 2 it suffices to check that f_x is surjective at every $x \in X$. For $x \notin D_- \cap D_+$, then $\mathcal{G}_x$ is a null group in *AbGrp*, so f_x is surjective. Let now $x = (-1, 0)$.

   Define $U_\epsilon = B_\epsilon(-1, 0) \cap X$. Every neighborhood contains some U_ϵ for small enough ϵ. So every $\mathcal{F}(U) \to F_x$ factors through $\mathcal{F}(U_\epsilon)$ for small enough ϵ. We have $\mathcal{F}(U_\epsilon) = \mathbb{Z} \oplus \{0\}$ for small $\epsilon > 0$, so $\mathcal{F}_x = \mathbb{Z}$ with morphisms

   $$\mathcal{F}(U_\epsilon) \to \mathcal{F}_x$$

   being $a \oplus 0 \mapsto a$. We do exactly the same for $\mathcal{G}$ and we get $\mathcal{G}_x = \mathbb{Z}$. The identity id is the unique map that makes

   $$\begin{array}{ccc} \mathcal{F}(U_\epsilon) & \xrightarrow{f(U_\epsilon)} & \mathcal{G}(U_\epsilon) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{\text{id}} & \mathcal{G}_x \end{array}$$

   commute, so f_x is surjective for $x = (-1, 0)$. We do exactly the same for $x = (1, 0)$. So f_x is surjective for every $x \in X$. By part 2, $\mathcal{F}$ is an epimorphism.

   We have $\mathcal{F}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\mathcal{G}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the map $f(X)$ as defined above. The image $\operatorname{im} f(X)$ is the diagonal $\triangle \mathbb{Z} \oplus \mathbb{Z}$, so $f(X)$ is not surjective.