Algebraic geometry 2
Exercise sheet 2

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Exercise 2. Let $f\colon \tilde{X} \to X$ be the normalization. We have to show that if $\mathcal{O}_{\tilde{X},x}$ is a flat $\mathcal{O}_{X,f(x)}$-module, then it is an isomorphism. By construction $\mathcal{O}_{\tilde{X},x}$ is the normalization of the ring $\mathcal{O}_{X,f(x)}$. Since X is integral scheme, $\mathcal{O}_{X,f(x)}$ is an integral domain.

So it all boils down to showing: If an integral closure B of an integral domain A inside fraction field $\mathrm{Quot}(A)$ is a finitely generated and flat A-module, then $A = B$.

We actually don't even need B to be finitely generated, since we know flatness can be checked on all finitely generated submodules.

Exercise 4. For $i \leq -1$ we define $h_i = 0$.

Since the chain is exact, the map d_1 must be surjective. Therefore we can define h_0 as the lift of $\mathrm{id}\colon C_0 \to C_0$ along surjection $d_1\colon C_1 \to C_0$. (Though there was no need to treat this case separately)

Let now $i \geq 1$ and suppose h_j for $j < i$ exist with property as in the exercise. Observe that $\mathrm{id}_{C_i} - h_{i-1} \circ d_i \colon C_i \to C_i$ factors through $\ker(d_i)$, since

$$\begin{aligned} d_i \circ (\mathrm{id}_{C_i} - h_{i-1} \circ d_i) &= d_i - d_i \circ h_{i-1} \circ d_i \\ &= d_i - (id_{C_{i-1}} - h_{i-2} \circ d_{i-1}) \circ d_i \\ &= d_i - d_i \\ &= 0. \end{aligned}$$

Using exactness ($\mathrm{im}(d_{i+1}) = \ker(d_i)$), we can lift $C_i \to \mathrm{im}(d_{i+1})$ along the surjection $C_{i+1} \to \mathrm{im}(d_{i+1})$ to obtain $h_i\colon C_i \to C_{i+1}$ for which $d_{i+1} \circ h_i = \mathrm{id}_{C_i} - h_{i-1} \circ d_i$.