Algebraic geometry 1
Exercise sheet 6

Solutions by: Eric Rudolph and David Čadež

23. November 2023

Exercise 1.

1. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram

$$\begin{tikzcd}
U_i \times_{S_{i,j}} V_j \arrow[dr, dotted] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & V_j \arrow[d, "\psi"] \\
& U_i \arrow[r, "\phi"] & S_{i,j}
\end{tikzcd}$$

   Therefore, on the level of sets,

$$U_i \times_{S_{i,j}} V_j \subset X \times_S Y,$$

   but in exercise 5.2.1, we showed that this induces an open immersion as locally ringed spaces.

   Now observe that

$$\begin{tikzcd}
\bigcup_{i,j} \left(U_i \times_{S_{i,j}} V_j\right) \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

   commutes, because $S = \bigcup_{i,j} S_{i,j}$. Now by uniqueness of the pullback,

$$\bigcup_{i,j} \left(U_i \times_{S_{i,j}} V_j\right) \cong U_i \times_{S_{i,j}} V_j.$$

   I guess this is a good step in the direction of understanding why the pullback in the category of sheaves exists, right? If we assume X, Y, S to be sheaves and $U_i, V_j, S_{i,j}$ to be affine schemes, then by the above argument we found a cover of $X \times_S Y$ by affine schemes.

1. (alternative) Let $U \subseteq X$, $V \subseteq Y$ and $W \subseteq S$ open subschemes. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram

$$\begin{tikzcd}
U \times_W V \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, dotted] & V \arrow[dr, hook] & \\
U \arrow[dr, hook] & X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "\psi"] \\
 & X \arrow[r, "\phi"] & S
\end{tikzcd}$$

so we get a unique map $U \times_W V \to X \times_S Y$.

Observe the concrete space $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ with inclusion

$$\pi_1^{-1}(U) \cap \pi_2^{-1}(V) \hookrightarrow X \times_S Y$$

also satisfies the universal property of being a fibred product $U \times_S V$. If $T \to U$ and $T \to V$ such that $T \to U \to W = T \to V \to W$, then we can create the following diagram

$$\begin{tikzcd}
T \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, dotted] & V \arrow[dr, hook] & \\
U \arrow[dr, hook] & X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "\psi"] \\
 & X \arrow[r, "\phi"] & S
\end{tikzcd}$$

from which we get a unique map $T \to X \times_S Y$. Since its image is contained in $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$, it factors uniquely through

$$T \to \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \hookrightarrow X \times_S Y$$

Therefore, the fibre product $U \times_W V$ can be identified as an open subspace $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) \subseteq X \times_S Y$.

Then clearly for coverings $X = \cup_i U_i$, $Y = \cup_i V_i$ and $S = \cup_{i,j} S_{i,j}$, we have a covering

$$\bigcup_{i,j} U_i \times_{S_{i,j}} V_j = \bigcup_{i,j} (\pi_1^{-1}(U_i) \cap \pi_2^{-1}(V_j)) = X \times_S Y.$$

I guess this is a good step in the direction of understanding why the pullback in the category of schemes exists, right? If we assume X, Y, S to be sheaves and $U_i, V_j, S_{i,j}$ to be affine schemes, then by the above argument we found a cover of $X \times_S Y$ by affine schemes.

2. Surjectivity follows, because a pullback of schemes in partiular makes

$$\begin{tikzcd} \mid X \times_S Y \mid \arrow[r] \arrow[d] & \mid X \mid \arrow[d, "\psi"] \\ \mid Y \mid \arrow[r, "\phi"] & \mid S \mid \end{tikzcd}$$

commute for all ψ, ϕ from maps of schemes.

2. (alternative) If $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B), Z = \operatorname{Spec}(R)$ were affine schemes, we would have homeomorphism, since $\operatorname{Spec}(A \otimes_R B) = \operatorname{Spec}(A) \times_{\operatorname{Spec}(R)} \operatorname{Spec}(B)$. But in schemes (using that the fiber product is also a scheme) we only locally have isomorphisms, which is means the map must be surjective.

Concretely: For every $(x, y) \in |X| \times_{|S|} |Y|$ we have $x \in \operatorname{Spec}(A) \subseteq |X|$ and $y \in \operatorname{Spec}(B) \subseteq |Y|$ such that $|X \times_S Y| \to |X| \times_{|S|} |Y|$ restricted to $\operatorname{Spec}(A \otimes_R B)$ will be isomorphism (with (x, y) in its image).

We now want to show that the given map is not injective in general, not even for affine schemes. To see this, define the following affine schemes

$$X = Y = \operatorname{Spec}(\mathbb{C}), S = \operatorname{Spec}(\mathbb{R}).$$

Now $|\operatorname{Spec}(X \times_S Y)| \cong |\operatorname{Spec}(X \otimes_S Y)| \cong |\operatorname{Spec}(\mathbb{C} \times \mathbb{C})| = \{(0, 1), (1, 0)\}$.

On the other hand, $\operatorname{Spec}(|X| \times_{|S|} |Y|) = \{*\}$ and clearly there is no injection from a set consisting of two points, to a set consisting of one point.

Exercise 2.

1. First let $f \colon X \to S$ be open immersion. In this case we can directly use previous exercise on the following fibred product

$$\begin{tikzcd} S \arrow[r, "\mathrm{id}"] & S \\ S \times_S S' \arrow[u, "p"] \arrow[r, "q"] & S' \arrow[u, "g"] \end{tikzcd}$$

by taking subset of $X \subseteq S$ and immediately getting open immersion $X \times_S S' \to S \times_S S'$, which we postcompose with canonical isomorphism $S \times_S S' \to S'$ and get that $X \times_S S' \to S'$ is open immersion.

Now suppose $f \colon X \to S$ is a closed immersion. So we have the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] & S \\ X \times_S S' \arrow[u, "p"] \arrow[r, "q"] & S' \arrow[u, "g"] \end{tikzcd}$$

We want to show $X \times_S S' \to S'$ is also a closed immersion. For that it satisfies to find an open covering of S' with affine subschemes such that preimages with be also affine schemes and induced maps of rings surjective.

Take $s \in S'$ and a neighborhood $g(s) \in \mathrm{Spec}(R) = U \subseteq S$. Preimage $f^{-1}(U) = \mathrm{Spec}(A)$ already is affine, since f is closed immersion, and for $g^{-1}(U)$ we have to take some smaller affine neighborhood of s. So we get $s \in \mathrm{Spec}(B) \subseteq g^{-1}(U)$. Then use previous exercise on these open sets and obtain open immersion

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_R B) \to X \times_S S'.$$

By remark at the start we have

$$\mathrm{Spec}(A \otimes_R B) = p^{-1}(\mathrm{Spec}(A)) \cap q^{-1}(\mathrm{Spec}(B)) = q^{-1}(\mathrm{Spec}(B)).$$

Only thing to argue is why the map $B \to A \otimes_R B$ is surjective. Since $R \twoheadrightarrow A$ surjective, $R/I \cong A$ and thus $A \otimes_R B = R/I \otimes_R B = B/IB$. Clearly $B \to B/IB$ is then surjective.

2.

Exercise 3. By definition we have to compute a fibred product of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and $\mathrm{Spec}(k(p)) \to \mathrm{Spec}(A)$ (where $k(p)$ is the residue field of $p \in \mathrm{Spec}(A)$ and $\to$ is the canonical inclusion). Since we are dealing with affine schemes, we can express it concretely as $\mathrm{Spec}(B \otimes_A k(p))$. Note that B has the structure of an A-algebra, which is induced by the starting morphism of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. So this exercise reduces to computing these tensor products.

We also observe that $k[T]$ is a PID, which means every non-zero prime ideal is a maximal ideal. This will be handy when computing residue fields, because after quotienting with a non-zero ideal we already get a field (we do not have to further take the quotient field).

1. In the first example we do now even have to calculate the tensor product, because we can rewrite $k[T,U]/(TU-1) = k[T,T^{-1}]$, so this is just a localization of $k[T]$. Morphism of spectrums, induced by inclusion into localization, is an open immersion, so fibers will be singletons if $x \in D(T)$ and empty sets otherwise. And the structure sheaf is also clear, it is just the restriction of structure sheaf $\mathcal{O}_{\mathrm{Spec}(k[T])}$.

2.

3.

4.

Exercise 4. This proof is the same as when we proved that $\mathcal{O}_{\mathrm{Spec}(A)}$ is a sheaf, after we defined it on the basis of principal opens.

Take $U = D(f)$ for some $f \in A$ and let $U = \cup_i D(f_i)$ be some cover. We have to check that

$$M[f^{-1}] \to \mathrm{Eq}\left[\prod_i M[f_i^{-1}] \rightrightarrows \prod_{i,j} M[(f_i f_j)^{-1}]\right]$$

is isomorphism.

We make following simplifications, namely we can set $M := M[f^{-1}]$ and assume I is finite, say $I = \{1, \ldots, n\}$ (we can do that since $\mathrm{Spec}(A)$ is quasi-compact).

So we are trying to show M is isomorphic to a submodule of $\prod_i M[f_i^{-1}]$ defined as $\{(m_1, \ldots, m_n) \in \prod_i M[f_i^{-1}] \mid m_i = m_j \in M[(f_i f_j)^{-1}]\}$.

Injectivity: Take $m \in M$. Since $m = 0 \in M[f_i^{-1}]$, we have $f_i^{k_i} m = 0$ for every i for some k_i. Since I is finite, take $k = \max_i k_i$. Since $D(f_i^k)$ is still a cover, we have $1 = \sum_i a_i f_i^k$ for some $a_i \in A$. Then $1m = \sum_i a_i f_i^k m = 0$, so $m = 0$.

Surjectivity: Let $(m_1, \ldots, m_n) \in \prod_i M[f_i^{-1}]$ with $m_i = m_j \in M[(f_i f_j)^{-1}]$ for all i, j. Write $m_i = \frac{a_i}{f_i^{k_i}}$. WLOG $k = \max_i k_i$. For every pair there exists $l \in \mathbb{N}$ such that $(f_i f_j)^l (m_i - m_j) = 0$. Take l again to be maximum over all pairs. Because $D(f_i^l)$ is still a cover, we have $1 = \sum_i b_i f_i^l$. Then define $s = \sum_i b_i a_i$. Clearly $f_j^l s = \sum_i b_i f_j^l a_i = \sum_i b_i f_i^l a_j = a_j$. So $(m_1, \ldots, m_n)$ is the image of s.