Smoothness

<u>Intuition:</u> If X is a k-variety then $X(k[\varepsilon]/\varepsilon^2)$ parametrizes a k-point $x \in X(k)$ and a "tangent" direction.

$\mathbb{A}^1_k$ (affine line)

$\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ (infinitesimal affine line).

Given $x \in X(k)$ we obtain a

$$\begin{array}{ccc} T_x X & \longrightarrow & * \\ \downarrow & & \downarrow \\ X(k[\varepsilon]/\varepsilon^2) & \longrightarrow & X(k) \end{array}$$

Given a map of varieties $f : X \to Y$ we set

$$\begin{array}{ccc} df : T_x X & \longrightarrow & T_{f(x)} Y \\ \wr\| & & \wr\| \\ X(k[\varepsilon]/\varepsilon^2) & \longrightarrow & Y(k[\varepsilon]/\varepsilon^2) \end{array}$$

Definition A closed immersion $i: S_0 \to S$ of schemes is a *first order thickening* if the ideal sheaf $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_S \to i_* \mathcal{O}_{S_0})$ satisfies $\mathcal{I}^2 = 0$.

A first order thickening is *split* if there is $s: S \to S_0$ with $s \circ i = \mathrm{Id}_{S_0}$.

Example $\mathrm{spec}\, k \to \mathrm{spec}\, k[\varepsilon]/\varepsilon^2 \to \mathrm{spec}\, k$ is split.

$\mathrm{spec}\, \mathbb{F}_p \hookrightarrow \mathrm{spec}\, \mathbb{Z}/p^2\mathbb{Z}$ is not split.

Remark Given a quasicoherent $\mathcal{O}_{S_0}$-module $\mathcal{M}$ we can consider a f.o.t. by letting $S = \underline{\mathrm{Spec}}(\mathcal{O}_{S_0} \oplus \mathcal{M})$ with ring structure given by

$$(f, m) \cdot (g, m') = (fg, fm' + gm).$$

All split f.o.t. arise in this way.

Definition Let $f: X \to Y$ be a map of schemes. Let $S_0 \to S$ be a f.o.t. of affine schemes.

Suppose we are given a commutative diagram

$$C = \begin{tikzcd} S_0 \arrow[r, "u_0"] \arrow[d, hook] & X \arrow[d, "f"] \\ S \arrow[r] \arrow[ur, dotted, "u"] & Y \end{tikzcd}$$

We let $\mathrm{Lift}(C)$ denote the set of u making the diagram commutative.

1) f is formally smooth if for all such C $|\mathrm{Lift}(C)| \geq 1$ $(\exists)$

2) f is formally étale if $|\mathrm{Lift}(C)| = 1$ $(\exists !)$

3) f is unramified if $|\mathrm{Lift}(C)| \leq 1$ $(\ \ !)$

Remark $\mathrm{Lift}(C)$ can be computed on the basechange

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d] & X \times_Y S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ S \arrow[r, equal, "\mathrm{id}"'] \arrow[ur, dotted] & S \arrow[r, squiggly] & Y \end{tikzcd}$$

in particular the properties are stable under basechange.

Example: 1) Open immersions are formally étale.

2) Closed immersions are formally unramified.

3) $\mathbb{A}^n_S \to S$ is formally smooth

If $S = \mathrm{Spec}\, R$ $S_0 = \mathrm{Spec}\, R_0$ and $S_0 \to S$ is a f.o.t. the map $R[x_1 \ldots x_n] \to R_0$ can be lifted

$$\begin{tikzcd} R[x_1 \ldots x_n] \arrow[r] \arrow[dr] & R_0 \\ & R \arrow[u] \end{tikzcd}$$

since $R \to R_0$ is surjective.

Proposition Formally smooth, étale, unramified satisfy BC, COMP, LOCS, LOCT.

Remark 1) BC, COMP are easy

Example:

$$\begin{tikzcd}
S_0 \arrow[r] \arrow[dr] \arrow[dd] & X_1 \arrow[d, "\text{f. smooth}"] \\
& X_2 \arrow[d, "\text{f. smooth}"] \\
S \arrow[r] \arrow[ur, dashed, "\exists \; 1"] \arrow[uur, dashed, "\exists \; 2"] & X_3
\end{tikzcd}$$

2) LOCS, LOCT for f. étale and f. unramified are also easy.

Key point: uniqueness allows us to glue.

Example:

$$\begin{tikzcd}
T_0 \arrow[rr] \arrow[dr] \arrow[dd] & & U \arrow[d, "\text{open cover}"] \arrow[ddd, bend left, "\text{f. étale}"] \\
& S_0 \arrow[r] \arrow[dd] & X \arrow[dd] \\
T \arrow[rruu, dashed, bend left, "\exists!"] \arrow[dr, "\text{affine open cover}"'] & & \\
& S \arrow[r] & Y
\end{tikzcd}$$

the map $T \longrightarrow X$ glues by uniqueness.

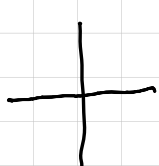
Definition $f: X \to Y$ is

1) Smooth (resp. étale) if it is f. smooth (resp. f. étale) and finitely presented.

2) Unramified if it is f. unramified and of finite type.

Example $\mathrm{Spec}\, k[[t]] \to \mathrm{Spec}\, k$ is formally smooth, but it is not smooth (not finite type).

Example: Let $X = \mathrm{Spec}\, k[t_1, t_2]/t_1 t_2$



is it smooth over $\mathrm{Spec}\, k$?

$$
C = \begin{tikzcd}
\operatorname{Spec} k \arrow[r, "{x=0,\ y=0}"] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[r] & \operatorname{Spec} k
\end{tikzcd}
$$

$\mathrm{Lift}(C)$ $\qquad t_1 \longmapsto a_1 \varepsilon$

$\qquad\qquad t_2 \longmapsto a_2 \varepsilon$

This suggest it might be smooth

We need all f.o.t !!!

$$
\text{f.o.t.} \quad \begin{tikzcd}
\operatorname{Spec} k[x,y]/(x^2,xy,y^2) \arrow[r, "u_0"] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} k[x,y]/(x^3,x^2y,x^2y,y^3) \arrow[r] \arrow[ur, dashed, "u"] & \operatorname{Spec} k
\end{tikzcd}
$$

$u_0 : \ t_1 \longmapsto x$

$\qquad t_2 \longmapsto y$

$u : \ t_1 \longmapsto x + p(x,y)$

$\qquad t_2 \longmapsto y + q(x,y)$

p, q homogeneous of degree 2.

but $u(t_1) \cdot u(t_2) = xy \neq 0.$

Differentials:

Intuition:

Given $x \in U \underset{open}{\subseteq} X$ and $f \in \Gamma(U, \mathcal{O}_U)$ with $f(x) = 0$ we get a derivative

$$df_x : T_x \longrightarrow T_0 \mathbb{A}^1_k = k$$

A differential is an element $d \in \mathrm{Hom}_k(T_x, k)$. and we have a pairing $\langle \, , \, \rangle \; T_x^\vee \times T_x \longrightarrow k$ and a tangent vector $t \in T_x$ is <u>determined</u> by the transformation $\langle df, t \rangle : T_x^\vee \longrightarrow k$. as f varies over functions $f \in \Gamma(U, \mathcal{O}_U)$.

<u>Example</u> If A is a k-algebra $x: \mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\, A$ a point.

then $t \in T_x \mathrm{Spec}\, A$ corresponds to a map

$$t^*: A \longrightarrow k[\varepsilon]/\varepsilon^2$$

$$t^*(f) = f(x) + df_x(t)\,\varepsilon$$

For this rule to be an algebra map we need:

1) $t^*(fg) = (f(x) + df_x(t)\,\varepsilon)(g(x) + dg_x(t)\,\varepsilon)$

$= f(x)g(x) + [g(x)\,df_x(t) + f(x)\,dg_x(t)]\,\varepsilon$

2) $t^*(f+g) = f(x) + g(x) + [df_x(t) + dg_x(t)]\,\varepsilon$

3) $t^*(\lambda) = \lambda + 0 \cdot \varepsilon$

In other words, given fixed $x \in \operatorname{Spec} A$ we can treat k as an A-module through the surjection $A \twoheadrightarrow k$ and finding $t \in T_x \operatorname{Spec} A$ is equivalent to finding $\partial := d(\ )_x[t] : A \longrightarrow k$

1) Leibniz rule: 1) $\partial(fg) = g \cdot \partial f + f \cdot \partial g$

k-linearity $\left\{ \begin{array}{l} 2)\ \partial(f+g) = \partial f + \partial g \\ 3)\ \partial(\lambda) = 0 \quad \forall \lambda \in k. \end{array} \right.$

Definition Let $R \to A$ be map of rings, M an A-module.
A derivation of A over R with values in M is an R-linear map $\partial : A \longrightarrow M$ s.t. $\forall a, b \in A$

$$\partial(ab) = a\,\partial(b) + b\,\partial(a)$$

We let $\mathrm{Der}_R(A, M)$ denote the set of derivations.
Note that $\mathrm{Der}_R(A, M)$ is an A-module.

Proposition There exists a universal derivation $d : A \longrightarrow \Omega^1_{A/R}$.
In other words,

$$\mathrm{Hom}_A(\Omega^1_{A/R}, M) \simeq \mathrm{Der}_R(A, M).$$

Proof

We let $\Omega^1_{A/k} = \mathrm{Coker}\left(G_2 \xrightarrow{\Phi} G_1\right)$

where $G_1 = \bigoplus_{f \in A} A \cdot df$ and G_2

so that 1) $\partial(fg) - f\partial g - g\partial f$

2) $\partial(f+g) - \partial f - \partial g$

3) $\partial(\lambda f) - \lambda \partial f$

generate the image $\Phi(G_2)$.

Definition The representing object $\Omega^1_{A/R}$ is the A-module of Kähler differentials.

Back to intuition: $x \in \mathrm{Spec}\, A$

then

$$T_x \mathrm{Spec}\, A = \mathrm{Der}_k(A, k) = \mathrm{Hom}_A(\Omega^1_{A/k}, k)$$
$$= \mathrm{Hom}_k(\Omega^1_{A/k} \otimes_A k, k) = (\Omega^1_{A/k} \otimes_A k)^\vee.$$