Proposition

If $H^1(\mathrm{Spec}\, A, \mathcal{I}) = 0$ for any ring A and any quasicoherent sheaf $\mathcal{I}$ $\Rightarrow$ formal smoothness is local on source.

H^1 vs. Ext^1

Let X be a scheme and $\mathcal{M} \in \mathrm{QCoh}$.

Definition - We let $\mathrm{Exten}_{\mathcal{M}}$ be the category of extensions where objects are exact sequences:

$$0 \to \mathcal{M} \to \tilde{\mathcal{M}} \to \mathcal{O}_X \to 0$$

and maps are

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{M}}"'] & \tilde{\mathcal{M}}_1 \arrow[r] \arrow[d, "f"] & \mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{O}_X}"] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r] & \tilde{\mathcal{M}}_2 \arrow[r] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

- We let $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{M})$ denote the set of isomorphism classes of extensions.

Rmk $\mathrm{Ext}^1(\mathcal{O}_X, \mu)$ is a pointed set with the trivial extension

$$0 \to \mu \to \mu \oplus \mathcal{O}_X \to \mathcal{O}_X \to 0$$

being the distinguished element.
(Later $\mathrm{Ext}^1(\mathcal{O}_X, \mu)$ will be a group).

Proposition: $H^1(X, \mu) \simeq \mathrm{Ext}^1(\mathcal{O}_X, \mu)$.

Proof We construct an equivalence of categories

$$\Phi : \mathrm{Exten}_\mu \longrightarrow \mu\text{-torsors}.$$

If $\mathcal{E} = \left[0 \to \mu \to \tilde{\mu} \xrightarrow{p_\mathcal{E}} \mathcal{O}_X \to 0\right]$

$$\Phi(\mathcal{E})[u] = \{ s \in \tilde{\mu}(u) \mid p_\mathcal{E}(s) = 1 \}$$

μ acts on $\Phi(\mathcal{E})$ by

$$(m, s) \longmapsto m + s$$

This well defined since $p_\mathcal{E}(m) = 0$.

Since taking global sections is left exact

$$0 \to \mathcal{M}(U) \to \tilde{\mathcal{M}}(U) \xrightarrow{p} \mathcal{O}_X(U)$$

if there is $t, s \in \tilde{\mathcal{M}}(U)$ with $p(s) = 1 = p(t)$ then $p(t-s) = 0$ so $t = s + m$.

Moreover, if $U = \operatorname{Spec} A$ is affine then $0 \to M \to \tilde{M} \to A \to 0$ is exact so $\Phi(\mathcal{M})[U] \neq \emptyset$.

<u>Claim:</u> Φ is fully-faithful:

We have

$$\operatorname{Hom}_{\text{Exten}}(\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2) \xrightarrow[\Phi]{} \operatorname{Hom}_{\text{Tors}}(\Phi(\tilde{\mathcal{M}}_1), \Phi(\tilde{\mathcal{M}}_2))$$

and both promote to sheaves of sets.

To show Φ is an isomorphism we can do this locally, Restricting to $U \in X$ affine

we see
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d, equal] & \tilde{\mathcal{M}}_1 \arrow[r] \arrow[d, "\alpha", "\simeq"'] & A \arrow[r] \arrow[d, equal] & 0 & \mathcal{E}_1 \\ 0 \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{M} \oplus A \arrow[r] & A \arrow[r] & 0 & \mathcal{E}_{Triv} \\ 0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[u, equal] & \tilde{\mathcal{M}}_2 \arrow[r] \arrow[u, "\beta", "\simeq"'] & A \arrow[r] \arrow[u, equal] & 0 & \mathcal{E}_2 \end{tikzcd}$$

since A is a projective A-module.

$$\mathrm{Hom}_{\mathrm{Extn}}(\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2)[U] \xrightarrow[\simeq]{\beta f \alpha^{-1}} \left[\begin{pmatrix} id_{\mathcal{M}} & \mathrm{Hom}(\mathcal{O}_X, \mathcal{M}) \\ 0 & id_{\mathcal{O}_X} \end{pmatrix}\right] \simeq \underset{\mathrm{Aut}(\mathcal{E}_{Triv})}{\overset{\parallel}{\mathcal{M}(U)}}$$

the identification also shows
$$\Phi(\mathcal{M}_1) \overset{\alpha}{\simeq} \mathcal{M} \overset{\beta}{\simeq} \Phi(\mathcal{M}_2) \quad \text{as}$$
$\mathcal{M}$-torsors. and
$$\mathrm{Hom}_{\mathcal{M}\text{-tors}}(\Phi(\mathcal{M}_1), \Phi(\mathcal{M}_2))[U] \xrightarrow[\simeq]{\beta g \alpha^{-1}} \mathcal{M}(U) = \mathrm{Aut}(\Phi(\mathcal{E}_{Triv}))$$

this shows Φ is fully-faithful.

Essential Surjectivity:

Objects in the categories Exten_μ μ-Tors. glue.

That is if $\bigcup_{i\in I} U_i$ is a cover of X a G-torsor P can be specified by finding

$$\left(\{P_{U_i}\}_{i\in I},\ \alpha_{ij}: P_{U_i}|_{U_j} \to P_{U_j}|_{U_i}\right)$$

s.t. $\alpha_{jk}\circ\alpha_{ij} = \alpha_{ik}$. (similarly for Exten_μ).

· Since $\mathrm{Exten}_\mu \to \mu$-tors is fully-faithful to show it is essentially surjective can be proved locally.

But locally every μ-torsor is trivial and the trivial μ-torsor is $\Phi(0 \to \mu_s\ \mu\oplus A \to A \to 0)$.

Towards derived functors

Proposition Let T be a topological space and $0 \to F_1 \to F_2 \xrightarrow{f} F_3 \to 0$ an exact sequence of sheaves over T. Then we get an exact sequence

$$0 \to \Gamma(T, F_1) \to \Gamma(T, F_2) \xrightarrow{f} \Gamma(T, F_3) \xrightarrow[\delta]{} H^1(T, F_1)$$

proof Given $s \in \Gamma(T, F_3)$ we attach an F_1-torsor P_s defined as

$$P_s : U \longmapsto \{ t \in F_2(U) \mid f(t) = s_{|U} \}$$

with F_1-action given by

$$F_1 \times F_2 \longrightarrow F_2$$
$$(t_1, t_2) \longmapsto (t_1 + t_2).$$

We let $\delta(s) = [P_s] \in \{F_1\text{-torsors}\}/\simeq$.

$$\delta(s) = * \iff P_s \simeq F' \iff P_s(T) \neq \emptyset$$
$$\iff s \in f(\Gamma(T, F_3)).$$

Definition A functor of abelian categories $F: C \to D$ is additive if $\forall A, B \in C$ the map $F: \mathrm{Hom}_C(A,B) \longrightarrow \mathrm{Hom}_D(F(A), F(B))$ is a map of groups.

Definition: Let $F: C \to D$ be a left exact functor between abelian categories. A cohomological δ-functor extending F is a sequence of additive functors $F^i: C \to D$ together with functorial maps

$$\delta^i : F^i(C) \to F^{i+1}(A)$$

for all short exact sequences

$$0 \to A \to B \to C \to 0.$$

s.t.

$$\cdots \xrightarrow{\delta^{i-1}} F^i(A) \to F^i(B) \to F^i(C) \xrightarrow{\delta^i} F^{i+1}(A) \to \cdots$$

is exact. and $F^0 = F$

Rmk Functoriality of δ^i means that for

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

the induced diagram

$$\begin{array}{ccc} F^i(C) & \xrightarrow{\delta^i} & F^{i+1}(A) \\ \downarrow & & \downarrow \\ F^i(C') & \xrightarrow[\delta^i]{} & F^{i+1}(A') \end{array}$$

commutes.

We can form category of data (F^i, δ^i) of natural transformations compatible with δ.

Definition Let $F: \mathcal{C} \to \mathcal{D}$ be a left exact functor. A coh. δ-functor extending F is universal if it is initial in the category of coh. δ-functors

Definition:

Let $F: \mathcal{C} \to D$ be left exact.

(F^i, δ^i) extending F is **erasable** if $\forall i \geq 1$ $\forall A \in \mathcal{C}$ there exists a monomorphism

$$A \longrightarrow B \quad \text{s.t.} \quad F^i(B) = 0.$$

Theorem Every erasable δ-functor extending F is universal.

Proof Suppose (F^i, δ_F^i) is erasable and (G^i, δ_G^i) also extends F.

We want to construct unique $\phi^i: F^i \longrightarrow G^i$. We induct on i

$$\begin{array}{ccc} \phi^0: F^0 & \longrightarrow & G^0 \\ \| & & \| \\ F & \xrightarrow{id} & F \end{array}$$

Given $A \in \mathcal{C}$ there is $B \in \mathcal{C}$ with $0 \to A \to B \xrightarrow[\varphi]{} C \to 0$ and $F^i(B) = 0$

We have by induction

$$\begin{tikzcd}
\longrightarrow F^{i-1}(B) \arrow[r, "F^{i-1}(\varphi)"] \arrow[d, "\phi^{i-1}"] & F^{i-1}(C) \arrow[r, "\delta_F^i"] \arrow[d, "\phi^{i-1}"] & F^i(A) \arrow[r] & F^i(B)=0 \\
\longrightarrow G^{i-1}(B) \arrow[r, "G^{i-1}(\varphi)"] & G^{i-1}(C) \arrow[r, "\delta_G^i"'] & G^i(A) \arrow[r] & G^i(B)
\end{tikzcd}$$

Since $\delta_G^i \circ \phi^{i-1} \circ F^{i-1}(\varphi) = 0$ this produces a unique map

$$\phi^i : F^i(A) \longrightarrow G^i(A).$$

Book keepings:

a) The construction of ϕ^i doesn't depend on B.

b) $\phi^i : F^i(A) \longrightarrow G^i(A)$ is functorial

c) we have commutative diagram

$$\begin{tikzcd}
F^{i-1}(L) \arrow[r, "\delta_F^i"] \arrow[d, "\phi_K^{i-1}"'] & F^i(K) \arrow[d, "\phi_L^i"] \\
G^{i-1}(L) \arrow[r, "\delta_G^i"'] & G^i(K)
\end{tikzcd}$$

for exact sequence

$$0 \to K \to J \to L \to 0$$

Trick for a):

$$\underbrace{\begin{matrix} A \hookrightarrow B \\ A \hookrightarrow B' \end{matrix}}_{\text{Two different injections}} \rightsquigarrow A \hookrightarrow B \oplus B'$$

$$\begin{aligned} &F^i(B \oplus B') \\ &= F^i(B) \oplus F^i(B') \\ &= 0 \end{aligned}$$

$$\begin{tikzcd}
F^{i-1}(B/A) \arrow[dr] & & \\
& F^{i-1}(B \oplus B'/A) \arrow[r, two heads] \arrow[dr] & F^i(A) \\
F^i(B'/A) \arrow[ur] & & G^i(A)
\end{tikzcd}$$

Trick for b): $f: A \to A'$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d, hook] & C \arrow[r] \arrow[d, equal] & {} \\
0 \arrow[r] & A' \arrow[r] \arrow[d, equal] & B \oplus A'/A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
F^{i-1}(C) \arrow[rr, two heads] \arrow[dr] \arrow[dd] & & F^i(A) \arrow[dr, dashed] \arrow[dd] & \\
& G^{i-1}(C) \arrow[rr] \arrow[dd] & & G^i(A) \arrow[dd] \\
F^{i-1}(C') \arrow[rr, two heads] \arrow[dr] & & F^i(A') \arrow[dr, dashed] & \\
& G^{i-1}(C') \arrow[rr] & & G^i(A')
\end{tikzcd}$$

Trick for c) :

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & J \arrow[r] \arrow[d, hook] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & B \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
F^{i-1}(L) \arrow[rr] \arrow[dr] \arrow[dd] & & F^{i}(K) \arrow[dd] \\
& F^{i-1}(S) \arrow[ur] \arrow[dd] & \\
G^{i-1}(L) \arrow[rr] \arrow[dr] & & G^{i}(K) \\
& G^{i-1}(S) \arrow[ur] &
\end{tikzcd}$$