Algebraic geometry 2
Exercise sheet 5

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Exercise 1.

1. Flatness is local on the target, so we can check it on affine cover of $\mathbb{P}^1_A$.

   Lets first look at $D(z) \to D(z)$. We have a map of rings $A[y] \to A[x,y]/(y^2 - g(x))$ mapping $y \mapsto y$. We have to check the target is a flat $A[y]$-module. We can let $B := A[y]$, then $B \to B[x]/(b - g(x))$ (where $b = y^2 \in B$). Writing it this way makes it clear that the target is isomorphic to B^d as a B-module.

   And now observe $D(y) \to D(y)$. We similarly get a ring map $A[z] \to A[x,z]/(z^{d-2} - z^d g(x/z))$ with $z \mapsto z$. Again, setting $B := A[z]$ makes it clear that $B \to B[x]/(f)$ (f some polynomial in $B[x]$ of degree d) makes the target a finite free B-module.

   So $X \to \mathbb{P}^1_A$ is flat.

   Morphism $\mathbb{P}^1_A \to \mathrm{Spec}(A)$ is flat, because maps $A \to A[t]$ are flat.

   Composition of flat morphisms is flat, so $X \to \mathbb{P}^1_A \to S = \mathrm{Spec}(A)$ is flat.

2. On discord somebody wrote an example that they think is a counterexample to this statement. And to me it seems like its valid.

3. In the first case the discriminant is $a^2 - 4b$ and in the second it is $-4a^3 - 27b^2$.

Exercise 3. Because curves X and Y are smooth, local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are geometrically regular. In our case k is algebraically closed, so they are already regular without any base change necessary. And regular 1-dimensional local rings (and integral) are DVRs, so we can pick t_x and t_y to be their respective uniformizers.

Then, using commutative algebra facts, we get that $\widehat{\mathcal{O}_{X,x}} = k[[t_x]]$ and $\widehat{\mathcal{O}_{Y,y}} = k[[t_y]]$.

By our assumptions we have $f_y^*(t_x) = st_y^e$ for some $s \in \mathcal{O}_{Y,y}^\times$. By previous exercise we know s admits an e-th root. Then we can go back to the start and pick t_y to be $\sqrt{e}yt_y$, which is still a uniformizer as $\sqrt{e}y \in \mathcal{O}_{Y,y}^\times$. So we can assume $f_y^*(t_x) = t_y^e$. Map f_y^* induces a map between diagrams $\{\mathcal{O}_{X,x}/m_{X,x}^i\}_i \to \{\mathcal{O}_{Y,y}/m_{Y,y}^i\}_i$, which in turn gives a map between limits of these diagrams, i.e. the completions. This induces map between completions is clearly given by the same rule $t_x \mapsto t_y^e$.