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Algebraic Geometry I

5. Exercise sheet

Exercise 1 (4 points):

Let $U_i, i = 1, 2$, be two schemes. Let $V_i \subseteq U_i$ be open subschemes and let $\varphi\colon V_1 \xrightarrow{\sim} V_2$ be an isomorphism.
1) Show that there exists a scheme X with open subschemes $W_i \subseteq X$ and isomorphisms $\alpha_i\colon U_i \xrightarrow{\sim} W_i$, $i = 1, 2$, such that $\alpha_i^{-1}(W_1 \cap W_2) = V_i$ and $\varphi = \alpha_2^{-1} \circ \alpha_{1|V_1}$.
2) Let A be a ring and let $X_{\pm 1}$ be the scheme obtained by glueing $U_1 = U_2 = \mathrm{Spec}(A[T])$ along the isomorphism $\varphi_{\pm 1}\colon \mathrm{Spec}(A[T, T^{-1}]) \to \mathrm{Spec}(A[T, T^{-1}])$, $T \mapsto T^{\pm 1}$.
3) Show that $X_{\pm 1}$ are not affine schemes and that X_+ is not isomorphic to X_-.
Remark: X_1 is called the affine line over A with doubled origin, X_{-1} is called the projective line over A.

Exercise 2 (4 points):

1) Let $(Y, \mathcal{O}_Y)$ be a locally ringed space. Show that $(Z, \mathcal{O}_Z) \mapsto Z$ induces a bijection between open subsets of Y and equivalence classes of open immersions $(Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces. Here, open immersions are equivalent if they are isomorphic as locally ringed spaces over $(Y, \mathcal{O}_Y)$.
2) Let k be an algebraically closed field. For a (classical) quasi-projective variety $X \subseteq \mathbb{P}^n_k(k)$ let $\mathcal{O}$ be its sheaf of regular functions $U \mapsto \mathcal{O}(U)$. Let $\pi\colon X \to X^{\mathrm{sob}}$ be the soberification (Sheet 3, Exercise 4). Show that $X^{\mathrm{sch}} := (X^{\mathrm{sob}}, \pi_*\mathcal{O})$ is a scheme over $\mathrm{Spec}(k)$ and that $X \mapsto X^{\mathrm{sch}}$ is a fully faithful functor from the category of quasi-projective varieties to the category of schemes over $\mathrm{Spec}(k)$.
Hint: Reduce to the case of affine algebraic sets by glueing morphisms of locally ringed spaces.

Exercise 3 (4 points):

1) Show that the functor

$$\Phi\colon \{\text{schemes}\} \to \mathrm{Fun}(\mathrm{Rings}, \mathrm{Sets}),\ X \mapsto (R \mapsto \mathrm{Hom}_{lrs}(\mathrm{Spec}(R), X))$$

is fully faithful by reducing to the Yoneda lemma.
2) Show that the functors $R \mapsto F_n(R) := \{x \in R \mid x^n = 1\}$ for $n \geq 1$, and $R \mapsto G(R) := \{(x, y) \in R^2 \mid R^2 \xrightarrow{(x,y)} R \text{ surjective}\}$ lie in the essential image of Φ.

Exercise 4 (4 points):

1) Show that if a functor $F\colon \mathcal{C} \to \mathcal{D}$ admits a left/right adjoint, then this adjoint is unique up to canonical isomorphism.
2) Assume that $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{\tilde{F}} \mathcal{E}$ are left adjoint functors with right adjoints G and $\tilde{G}$. Show that $\tilde{F} \circ F$ is right adjoint to $G \circ \tilde{G}$.
3) Let $f\colon Y \to X$ be a continuous map of topological spaces. Let $f^{p,*}(-)$ resp. $f^*(-)$ be the presheaf resp. sheaf pullback, and let $(-)^\sharp$ denote sheafification. Show that there exists a natural isomorphism $f^{p,*}(\mathcal{F})^\sharp \cong f^*(\mathcal{F}^\sharp)$ for any presheaf $\mathcal{F}$ (of sets/abelian groups/...) on X.

To be handed in on: Thursday, 16.11.2023 (during the lecture, or via eCampus).