Summer term 2024 Dr. Andreas Mihatsch
Elliptic curves and their moduli spaces

Homework problems (due May 8)

Problem 1 (Differential forms on elliptic curves)

Let k be a field of characteristic $\neq 2$. Consider the plane smooth cubic $E = V_+(F) \subset \mathbb{P}^2_k$ defined by the simplified Weierstrass equation

$$F(X, Y, Z) = Y^2Z - X^3 - aXZ^2 - bZ^3.$$

On the open chart $D_+(Z)$, this curve is described by

$$E \cap D_+(Z) = V(y^2 - x^3 - ax - b) \subset \mathbb{A}^2_k.$$

Prove that the differential forms dx/y and $dy/(3x^2 - a)$ glue to a differential form on $E \cap D_+(Z)$. Show further that this form extends to a global section $\omega \in \Omega^1_{E/k}(E)$. Give an argument why ω is necessarily translation invariant.

Hint: The smoothness of E implies that $D(y)$ and $D(3x^2 - a)$ cover $E \cap D_+(Z)$. Moreover, you know that $d(y^2 - x^3 - ax - b)|_{E \cap D_+(Z)} = 0$.

Problem 2 (Left vs. right translation)

Let $\pi : G \to S$ be a group scheme and let $\omega \in \Omega^1_{G/S}(G)$ be left translation invariant.

(a) Let $g \in G(S)$ be an S-valued point. Show that the right translate $r_g^*(\omega) \in \Omega^1_{G/S}(G)$ is again left translation invariant.

(b) Let $i : G \to G$ be the inverse morphism. Prove that $i^*(\omega)$ is right translation invariant.

Further Problems

Problem 3 (Translation invariant forms on $\mathbb{G}_m \ltimes \mathbb{G}_a$)

Let k be a field and let $G \subseteq GL_{2,k}$ be the closed subgroup k-scheme

$$G = \left\{ \begin{pmatrix} x & y \\ & x \end{pmatrix} \right\}.$$

Determine all left and all right translation invariant differential forms on G. That is, find all polynomials $f(x, y), g(x, y) \in k[x^{\pm 1}, y]$ such that the differential form

$$f(x, y)dx + g(x, y)dy \in \Omega^1_{G/k}(G)$$

is left (resp. right) translation invariant.