Algebraic geometry 1
Exercise sheet 11

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Exercise 3.

1. Since k is algebraically closed, the only irreducible polynomials $f \in k[x, y]$ are of degree 1.

   Hence, we can write
   $$f_r = l_1 \dots l_r,$$
   where $l_i \in k[x, y]$ is of degree 1. From the assumption that f_r is homogenous it follows that the l_i are homogenous.

   Therefore, we can write
   $$Z = V(f_r) = V(l_1 \dots l_r) = \cup_i V(l_i)$$
   and since $V(l_i)$ is a line through the origin, Z can be written as the finite union of lines through the origin.

2. We first want to prove that $\dim(\mathcal{O}_{X,(x,y)}) = 1$ for all r. The prime ideals p in this ring fulfil $(f) \subset p \subset (x, y)$. Remember that we can write down these prime ideals explicitely as in "What do primes of k[x,y] look like". From this the claim follows.

   We know that $\dim_k(m_{\mathcal{O}_{X,(x,y)}}/m^2_{\mathcal{O}_{X,(x,y)}})$ is the number of generators of $m_{\mathcal{O}_{X,(x,y)}}$.

   Now if $r = 1$, then we can write $f = g(x, y)x + h(x, y)y$ and w.l.o.g. we have $g(0, 0) = 1$, meaning that it is invertible (after localizing). Therefore $f = x + h(x, y)y$, so $y \mid x$ meaning $(x, y) = (y)$ On the other hand, if $r > 1$, then $x \nmid y$ and $y \nmid x$ meaning that m is no principal ideal showing that X is singular at zero in this case. (This can be seen by writing f as $f = x^2h_1(x, y) + xyh_2(x, y) + y^2h_3(x, y)$).

3. By part two of this exercise, all the schemes have a singular point at the origin. I don't know why they do not have singular points anywhere else.