Algebraic geometry 1
Exercise sheet 3

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31. Oktober 2023

Exercise 1.

1. Define

$$\pi^{-1} : U \longrightarrow \pi^{-1}(U)$$
$$(x_1, ..., x_n) \mapsto (x_1, ..., x_n)[x_1 : ... : x_n].$$

This is well-defined, because by definition of U, not all x_i can be zero at the same time, so $[x_1 : ... : x_n]$ is actually a point in projective space. We also have $(x_1, ..., x_n)[x_1 : ... : x_n] \in Z$ for $(x_1, ..., x_n) \in U$, because $x_i x_j = x_j x_i$ for all $1 \leq i, j \leq n$. To see injectivity of π^{-1}, let $(x_1, ..., x_n) \in U$ with $x_j \neq 0$. Then we have $y_j \neq 0$, because if we assume $x_j \neq 0$ and $y_j = 0$, then for some $y_i \neq 0$ (which exists since $[y_1 : ... : y_n]$ is a point in projective space) we have $0 \neq x_j y_i = x_i y_j = 0$. Therefore, we can just set $y_j = 1$. Then

$$x_i y_j = x_j y_i \implies y_1 = \frac{x_1 y_j}{x_j} = \frac{x_1}{x_j},$$

showing that all the y_i are fixed up to a scalar after fixing all the x_i.

2. Define

$$\phi : V_i \to \mathbb{A}^k_n$$
$$(x, y) \mapsto (\frac{x_1}{y_i}, \ldots, x_i, \ldots, \frac{x_n}{y_i}),$$

where the inverse map is given by

$$\phi^{-1} : \mathbb{A}^k_n \to V_i$$
$$(x_1, \ldots, x_n) \mapsto (x_1 x_i, \ldots, x_i, \ldots, x_n x_i)[x_1 : \cdots : x_{i-1} : 1 : \cdots : x_n].$$

Exercise 4.

1. Lets first prove that V_U are stable under intersections:

   Claim. Take $U, W \subseteq X$ open subsets. Then $V_{U \cap W} = V_U \cap V_W$.

   Proof of claim. Inclusion $V_{U \cap W} \subseteq V_U \cap V_W$ is clear.

   For the other inclusion take $Z \in V_U \cap V_W$. By definition $Z \cap U \neq \emptyset$ and $Z \cap V \neq \emptyset$. Suppose $Z \cap (U \cap V) = \emptyset$. Then $(Z \cap U)^c \cup (Z \cap V)^c = X$. But since Z is irreducible, and is covered by $U^c \cup V^c$, we must have (WLOG) $Z \subseteq U^c$. That is in contradiction with $Z \cap U \neq \emptyset$. $\square$(of claim)

   It also behaves well under unions:

$$\begin{aligned} V_{U \cup W} &= \{Z \text{ cl. irred.} \mid Z \cap (U \cup W) \neq \emptyset\} \\ &= \{Z \text{ cl. irred.} \mid (Z \cap U) \neq \emptyset \text{ or } (Z \cap W) \neq \emptyset\} \\ &= \{Z \text{ cl. irred.} \mid (Z \cap U) \neq \emptyset\} \cup \{Z \text{ cl. irred.} \mid (Z \cap W) \neq \emptyset\} \\ &= V_U \cup V_W \end{aligned}$$

   and practically same argument applies to infinite unions.

   Therefore every open subset of X^{sob} can be written as V_U for some open $U \subseteq X$.

   Claim. Closed irreducible subsets of X^{sob} are exactly V_U^c for open $U \subseteq X$ such that (closed) subset $U^c \subseteq X$ is irreducible.

   Proof of claim. Take V_U^c such that U^c is not irreducible. Then there exist closed subsets $U_1^c, U_2^c \subseteq X$ with $U^c = U_1^c \cup U_2^c$ meanwhile $U^c \neq U_1^c$ and $U^c \neq U_2^c$. Then $V_U = V_{U_1 \cup U_2} = V_{U_1} \cup V_{U_2}$ and we can thus cover V_U with V_{U_1} and V_{U_2} but $V_U \neq V_{U_1}$ and $V_U \neq V_{U_2}$. $\square$(of claim)

   Let us show X^{sob} is sober. Let V_U^c be closed irreducible. Then by last claim U^c is closed and irreducible. This U^c will be the generic point with $\overline{\{U^c\}} = V_U^c$.