Dr. I. Gleason **SS 2024**
Dr. J. Anschütz

Algebraic Geometry II

6. Exercise sheet

Exercise 1 (4 points):
Let $f : Z \to S$ and $g : X \to S$ be two smooth morphisms of schemes, and let $i : Z \to X$ be a closed immersion over S. Prove that for all $z \in Z$ there exists an open neighborhood $U \subseteq X$ of $i(z)$ and sections $f_1, ..., f_n \in \mathcal{O}_X(U)$ such that $Z \cap U = V(f_1, ..., f_d)$ for some $d \leq n$ and the induced diagram

$$\begin{tikzcd} Z \cap U \arrow[r] \arrow[d, "(\overline{f}_{d+1},...,\overline{f}_n)"'] & U \arrow[d, "(f_1,...,f_n)"] \\ \mathbb{A}^{n-d}_S \arrow[r, "\alpha"] & \mathbb{A}^n_S \end{tikzcd}$$

with $\alpha((x_{d+1}, ..., x_n)) = (0, ..., 0, x_{d+1}, ..., x_n)$ is cartesian with both vertical arrows étale.
Hint: Let $\mathcal{I}$ be the ideal sheaf of Z in X. Prove that $0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0$ is a short exact sequence of vector bundles and choose, locally around z, sections $f_1, ..., f_n \in \mathcal{O}_X$ such that the differentials $df_1, ..., df_n$ form an adapted basis of $i^*\Omega^1_{X/S}$).

Exercise 2 (4 points):
Let $g\colon S' \to S$ be a faithfully flat morphism. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_S$-module.
i) Assume g is quasi-compact. Show that $\mathcal{M}$ is locally finitely generated (resp. locally of finite presentation resp. flat resp. finite locally free) if and only if $g^*\mathcal{M}$ is.
ii) Give an example where $g^*\mathcal{M}$ is locally finitely generated, but $\mathcal{M}$ not.

Exercise 3 (4 points):
Let $g\colon S' \to S$ be a faithfully flat and quasi-compact morphism. Let $f\colon Y \to X$ be a morphism of schemes over S with base change $f'\colon Y' \to X'$ to S'. Use exercise 2 and the Jacobian criterion to show that f is smooth if and only if f' is smooth.

Exercise 4 (4 points):
Let A be a ring and let $f_\bullet\colon C_\bullet \to D_\bullet$ be a morphism of complexes of A-modules. Let $E_\bullet$ be another complex of A-modules.
i) Show that

$$\ldots \xrightarrow{f_\bullet[1]^*} \mathrm{Hom}_{\mathcal{K}(A)}(C_\bullet[1], E_\bullet) \to \mathrm{Hom}_{\mathcal{K}(A)}(C(f_\bullet), E_\bullet) \to \mathrm{Hom}_{\mathcal{K}(A)}(D_\bullet, E_\bullet) \xrightarrow{f_\bullet^*} \mathrm{Hom}_{\mathcal{K}(A)}(C_\bullet, E_\bullet) \to \ldots$$

is exact. Here, $\mathcal{K}(A)$ denotes the *homotopy category* of the abelian category of A-modules *(see [Tag 05RN, Stacks project], up to a change of indexing of complexes)*.
ii) Assume that $f_\bullet$ is a quasi-isomorphism *([Tag 010Z, Stacks project])* of bounded to the right complexes consisting of projective A-modules. Show that $f_\bullet$ is a homotopy equivalence.
Hint: Combine i) with the material of the exercises 4 from sheets 1,2,3,4.

To be handed in on: Thursday, 30.05.2024 (during the lecture or via eCampus).