Algebraic geometry 1
Exercise sheet 8

Solutions by: Eric Rudolph and David Čadež

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Exercise 1.

Exercise 2.

Exercise 3.

1. In exercise 2 we showed that all invertible quasicoherent sheaves on $\mathbb{P}^n_k$ are isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$. So we have to show $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is an invertible sheaf.

   Since invertible $\mathcal{O}_{\mathbb{P}^n_k}$-modules are same as line bundles, we have to show that locally $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^m_k}$.

   By definition $f^*\mathcal{O}_{\mathbb{P}^m_k}(1) = f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^m_k}} \mathcal{O}_{\mathbb{P}^n_k}$. Pick some $x \in \mathbb{P}^n_k$. Pick small enough affine neighborhood $f(x) \in U \subseteq \mathbb{P}^m_k$ such that $\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^m_k}$ on U. Now pick neighborhood $x \in W \subseteq \mathbb{P}^m_k$ such that $f(W) \subseteq U$.

   Then

$$\begin{aligned}
f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)(W) &= \operatorname{colim}_{f(W)\subseteq V} \mathcal{O}_{\mathbb{P}^m_k}(1)(V) \\
&= \operatorname{colim}_{f(W)\subseteq V\subseteq U} \mathcal{O}_{\mathbb{P}^m_k}(1)(V) \\
&\cong \operatorname{colim}_{f(W)\subseteq V\subseteq U} \mathcal{O}_{\mathbb{P}^m_k}(V) \\
&\cong f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(W).
\end{aligned}$$

   So locally $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}$, so $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^m_k}} \mathcal{O}_{\mathbb{P}^n_k}$ is locally isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}$, which proves that $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is an invertible $\mathcal{O}_{\mathbb{P}^n_k}$-module and thus isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$.