Flatness

Commutative algebra:

Definition:

An A-module M is flat if $(-\otimes_A M): \mathrm{Mod}_A \to \mathrm{Mod}_A$ is an exact functor.

Remark Since $(-\otimes_A M, \mathrm{Hom}_A(M,-))$ is an adjoint pair, $-\otimes_A M$ is left exact.

Example a) $M = A^{\oplus n}$ and

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

then

$$\begin{array}{c} 0 \to N_1 \otimes_A M \to N_2 \otimes_A M \to N_3 \otimes_A M \to 0 \\ \| \\ 0 \to N_1^{\oplus n} \to N_2^{\oplus n} \to N_3^{\oplus n} \to 0 \end{array}$$

also exact.

b) Direct sums of flat modules are also flat.

c) Projective modules are flat

d) Filtered colimits of flat modules are flat.

e) If $A = k$ a field then all modules are flat.

f) If A is a valuation ring then M is flat <u>iff</u> it is torsion free.

<u>Theorem</u> (Lazard)

An A-module is flat <u>iff</u> it is a filtered colimit of finite free A-modules.

An A-algebra B is flat if it is flat as a A-module.

Example

1) $A[t_1, t_2, \ldots, t_n]$ is A-flat.

2) $S^{-1}A$ is A-flat.

since $S^{-1}A = \operatorname{colim}_{s \in A} A[s^{-1}]$

3) If A is Noetherian $I \subseteq A$ then the completion

$$\hat{A}_I := \varprojlim A/I^n$$

is A-flat.

Proposition Let $A \in \mathrm{Rings}$, $B \in \mathrm{Alg}_A$, $M \in \mathrm{Mod}_A$. and $N \in \mathrm{Mod}_B$.

The following hold:

-1) M is flat iff for all finitely generated ideals $I \subseteq A$

$I \otimes_A M \to I \cdot M$ is an isomorphism.

0) M is flat iff $\forall \mathfrak{p} \in \mathrm{Spec}\, A$ $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$.

1) If M is flat, then $M \otimes_A B$ is flat over B.

2) If B is flat over A and N is flat over B, then N is flat over A.

3) If M is flat and finitely presented then M is projective.

Let $f: X \to Y$ be a map of locally rinsed spaces and $\mathcal{M}$ a $\mathcal{O}_X$-module.

<u>Definition</u> a) We say that $\mathcal{M}$ is flat over Y at $x \in X$ if $\mathcal{M}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module.

b) We say that $\mathcal{M}$ is f-flat if it is flat over Y at all points of $x \in X$.

c) We say that f is flat if $\mathcal{O}_X$ is f-flat.

d) When $X = Y$ $f = id_Y$ we say $\mathcal{M}$ is a flat $\mathcal{O}_Y$-module if it is id_Y-flat.

Proposition Let $A \in \mathrm{Ring}$, $M \in \mathrm{Mod}\,A$, $Y = \mathrm{Spec}\,A$. Then $\tilde{M} \in \mathrm{QCoh}_X$ is flat **iff** M is flat over A.

proof

$$\tilde{M}\ \mathcal{O}_Y\text{-flat} \iff (\tilde{M})_y\ \mathcal{O}_{Y,y}\text{-flat}\ \forall x \in X$$
$$\iff M_{\mathfrak{p}}\ A_{\mathfrak{p}}\text{-flat}\ \forall \mathfrak{p} \in \mathrm{Spec}\,A.$$

Proposition Let $f: X \to Y$ be a map of schemes, let $M \in \mathrm{QCoh}(Y)$ and $N \in \mathrm{QCoh}(X)$.

a) Flatness is local on the source and target.

1) If M is flat over X, then f^*M is flat over Y.

1') Flatness is stable under base change

2) If f is flat and N is flat over Y then N is f-flat.

2') Flatness is stable under composition.

3) If M is finitely presented flat $\mathcal{O}_X$-module, then M is a vector bundle over X.

Proof 0) Given $x \in X$ with $y = f(x) \in Y$ we can find

$$\begin{array}{ccc} x \in \operatorname{Spec} B & \xrightarrow{f} & \operatorname{Spec} A \ni y \\ \text{open} \cap & & \cap \text{ open} \\ X & \xrightarrow{f} & Y \end{array}$$

if B is a flat A-algebra then $B_{\mathfrak{q}_x} = \mathcal{O}_{X,x}$ is a flat $A_{\mathfrak{p}_y} = \mathcal{O}_{Y,y}$-algebra.

By 0) all claims 1), 1'), 2), 2') can be reduced to the affine case.

3) Follows from the fact that finite projective modules are locally free of finite constant rank.

<u>Proposition</u> If Y is a connected locally Noetherian scheme and $f: X \to Y$ is finite flat then there is $n \geq 0$ such that for every $y \in Y$ the fiber

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} k(y) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is of the form $\operatorname{Spec} B(y)$ for a n-dimensional $k(y)$-algebra $B(y)$.

<u>Definition</u> With the setup as above we let $\deg(f) := n$.

__Proof__ Since f is finite then $X = \underline{\mathrm{Spec}}_{\mathcal{O}_Y} \mathcal{B}$ for a finite type quasicoherent $\mathcal{O}_Y$-algebra. Then $X_y = \mathrm{Spec}(\mathcal{B}_y \otimes_{\mathcal{O}_{Y,y}} k(y))$, we let $n_y = \dim_{k(y)} \mathcal{B}(y)$, we show that the function $n_y : |Y| \longrightarrow \mathbb{Z}$ is locally constant.

Since Y is locally Noetherian $\mathcal{B}$ is finitely presented as $\mathcal{O}_Y$-module

Since f is flat $\mathcal{B}$ is a flat $\mathcal{O}_Y$-module.

Consequently, it is locally free of finite rank. If $y \in Y$ and $y \in U \subseteq Y$ such that $\mathcal{B}|_U$ is a free $\mathcal{O}_U$-module then $n_y = \mathrm{rank}_{\mathcal{O}_U} \mathcal{B}|_U$.

Definition An A-module M is faithfully flat if for all complexes of A-modules

$C^\bullet = [N_1 \to N_2 \to N_3]$

$C^\bullet$ is a SES $\iff$ $M \otimes_A C^\bullet$ is a SES

Proposition Let M be a flat A-module. TFAE:

(1) M is faithfully flat

(2) For all $N \in \mathrm{Mod}\, A$, $N = 0 \iff N \otimes_A M = 0$

(3) For all $\mathfrak{p} \in \mathrm{Spec}\, A$ $M \otimes_A k(\mathfrak{p}) \neq 0$

(4) For all maximal ideal $\mathfrak{m} \subseteq A$

$M \otimes_A k(\mathfrak{m}) \neq 0$.

<u>Proof</u> $\underline{(1) \Rightarrow (2)}$

Let $C^\bullet = [0 \to N \to 0]$.

If $N \otimes_A M = 0 \Rightarrow C^\bullet \otimes_A M$ is exact

$\Rightarrow C^\bullet$ is exact $\Rightarrow N = 0$.

$\underline{(2) \Rightarrow (3) \Rightarrow (4)}$ Obvious.

$\underline{(4) \Rightarrow (1)}$

Let $C^\bullet = [N_{-1} \to N_0 \to N_1]$ a complex of A-modules.

suppose $C^\bullet \otimes_A M$ is exact.

Let $H = H^0(C^\bullet) = \dfrac{\ker(N_0 \to N_1)}{\operatorname{im}(N_{-1} \to N_0)}$.

We want to show $H = 0$.

since M is flat

$$H \otimes_A M = H^0(C^\bullet \otimes_A M) = 0$$

Let $x \in H$ let

$I = \mathrm{Ann}(x) = \{ a \in A \mid a \cdot x = 0 \}$

$$0 \to A/I \xrightarrow{i} H \to \mathrm{coker}(i) \to 0$$

then

$$0 \to A/I \otimes_A M \to H \otimes_A M$$

this gives $M/IM = 0$

if $I \neq A$ there is $I \subseteq m$

contradicting $M \otimes_A A/m = M/m \cdot M \neq 0$.

Corollary If B is a flat A-algebra then it is faithfully flat if and only if $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is surjective.

Corollary A flat morphism of local rings is faithfully flat iff it is flat and a local map.

Definition Let $f: X \to Y$ be a map of schemes.

We say f is faithfully flat if f is flat and $|f|: |X| \to |Y|$ is surjective.

Topological properties of flatness:

Proposition: If $f: X \to Y$ is a flat map of schemes, then $f(|X|) \subseteq Y$ is generalizing.

proof Let $y = f(x)$ an let z be a generization of y.

We get a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O}_{Y,y} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

The map $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{Y,y}$ is flat, here faithfully flat and surjective. Since $z \in \operatorname{Spec} \mathcal{O}_{Y,y}$ it is in the image of f.

<u>Definition</u> Let X be a spectral space $S \subseteq X$ is constructible if it belongs to the boolean algebra generated by qc-open subsets of X.

<u>Theorem</u> (Chevalley) If $f: \operatorname{Spec} B \to \operatorname{Spec} A$

is finitely presented, then f sends constructible sets to constructible sets.

<u>Non-example</u>: $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ is not of finite presentation

the set only containing the generic point is not constructible.

<u>sketch</u> • Reduce to affine case.

• Present $B = A[x_1, x_2, \dots x_n]/(f_1, f_2, \dots f_r)$

• Finitely presented closed immersions are by definition constructible

• Key case $B = A[x]$, this is lengthy complicated commutative algebra.

<u>Proposition</u> If $f: X \to Y$ is finitely presented and flat then it is universally open.

<u>proof</u> Since finite presentation and flatness are preserved under basechange it suffices to show $|f|: |X| \to |Y|$ is open.

WLOG $Y = \operatorname{Spec} A$. Let $U \subseteq X$ we want to show $f(U) \subseteq |Y|$ is open.

WLOG $U = \operatorname{Spec} B$. By Chevalley $f(U) \subseteq |Y|$ is constructible, by flatness it is generalizing.

But a set $T \subseteq \operatorname{Spec} A$ is open <u>iff</u> it is generalizing and open for constructible topology.