Algebraic geometry 1
Exercise sheet 11

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11. Januar 2024

Exercise 3.

1. Since k is algebraically closed, the only irreducible polynomials $f \in k[x, y]$ are of degree 1.

   Hence, we can write
   $$f_r = l_1 \dots l_r,$$
   where $l_i \in k[x, y]$ is of degree 1. From the assumption that f_r is homogenous it follows that the l_i are homogenous.

   Therefore, we can write
   $$Z = V(f_r) = V(l_1 \dots l_r) = \cup_i V(l_i)$$
   and since $V(l_i)$ is a line through the origin, Z can be written as the finite union of lines through the origin.

2. Assume that $r = 1$, i.e. $X = V(f_1)$. We want to show that for $x = \{0\}$ we have that$\mathcal{O}_{X,x}$ is a field, which would imply that it is regular.

   We calculate
   $$\mathcal{O}_{X,x} = \varinjlim_{x \in D(g)} k[x, y]/f_1[g^{-1}] = \varinjlim_{x \in D(g)} k[u][g^{-1}] = K(A),$$
   using that f_1 is homogenous of degree 1 and where $K(A)$ is the field of fractions of A.