LECTURE 4 (SUMMARY)

(1) We use generic freeness to prove Chevalley's theorem. To do this we introduce two important techniques, reducing finite presentation hypothesis to the Noetherian case and Noetherian induction on the dimension.
(2) We recall the notion of codimension and recall the inequality

$$\dim(\overline{\{x\}}) + \operatorname{codim}_X(x) \leq \dim(X)$$

(3) We explain that equality fails for the following reasons:
  (a) The scheme is not irreducible.
  (b) The scheme is not catenary.
(4) We define the local dimension of X at a point $x \in X$ denoted $\dim_x(X)$.
(5) We have that for finite type k-schemes

$$\dim(\overline{x}) + \operatorname{codim}_X(x) = \dim_x(X)$$

although $\dim_x(X) < \dim(X)$ can happen if X is not equidimensional.
(6) We define upper semi-continuous functions.
(7) We state the upper semi-continuity of fiber dimensions for maps of finite type k-schemes, and sketch a proof. More precisely, that if $\pi : X \to Y$ is a map of finite type k-schemes then the function $f : |X| \to \mathbb{R}$ given by $f(x) = \dim_x(X_{\pi(x)})$ is upper semi-continuous and that if $\pi : X \to Y$ is proper then the function $g : |Y| \to \mathbb{R}$ given by $g(x) = \dim(X_y)$ is upper semi-continuous.
(8) **Warning:** With the setup as above there are examples of maps of finite type k-schemes $\pi : X \to Y$ such that the function $f(x) = \dim(X_{\pi(x)})$ so the use of $\dim_x$ is strictly necessary.
(9) We give a brief discussion of dimension theory for Noetherian local rings. The main point was to clarify that even if the Noetherian local ring is not catenary Krull's principal ideal theorem and the stronger version Krull's height theorem still hold and still give a well behaved dimension theory.

1. Suggested additional reading:

- Section 11.5 in Vakil's Foundations of algebraic geometry.
- Chapter 11 in Atiyah–Macdonald's Introduction to Commutative Algebra.