Summer term 2024 Dr. Andreas Mihatsch
Elliptic curves and their moduli spaces

Homework problems (due April 24)

Problem 1

Let k be a field and A a (not necessarily commutative) finite-dimensional k-algebra.

(a) Prove that the functor $\underline{A}$ on k-schemes given by

$$\underline{A} = \mathcal{O}_T(T) \otimes_k A$$

is representable by $\mathbb{A}_k^n$, where $n = \dim_k(A)$. Show that there exists an open subscheme $\underline{A}^\times \subset \underline{A}$ that represents the functor

$$\underline{A}^\times(T) = (\mathcal{O}_T(T) \otimes_k A)^\times . \tag{1}$$

(b) Define a group scheme structure on $\underline{A}^\times$ such that (1) becomes an isomorphism of groups for every k-scheme T. (If you do this via Yoneda, then give a short argument for how it applies.)

(c) Consider the case $k = \mathbb{R}$ and $A = \mathbb{C}$; set $G = \underline{A}^\times$. Define a group scheme morphism $N : G \to \mathbb{G}_{m,\mathbb{R}}$ such that

$$N(\mathbb{R}) : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$$

is the norm map $z \mapsto z\bar{z}$. Describe the affine scheme $\ker(N)$ by equations.

Problem 2

Let k be a field. Recall that $\mathbb{G}_{a,k} = \operatorname{Spec} k[t]$ with addition law $a^*(t) = t \otimes 1 + 1 \otimes t$.

(a) Assume that $\operatorname{char}(k) = 0$. Show that $k \xrightarrow{\sim} \operatorname{End}(\mathbb{G}_{a,k})$ via

$$\lambda \longmapsto \operatorname{Spec}(t \mapsto \lambda t).$$

(b) Now assume that $\operatorname{char}(k) = p$. Show that $f = \operatorname{Spec} f^*$, where $f^* : k[t] \to k[t]$ is any k-algebra morphism, lies in $\operatorname{End}(\mathbb{G}_{a,k})$ if and only if $f^*(t)$ is of the form

$$f^*(t) = a_n t^{p^n} + a_{n-1} t^{p^{n-1}} + \ldots + a_1 t^p + a_0 t$$

for some $n \geq 0$ and coefficients $a_0, \ldots, a_n \in k$.