Theorem (Grothendieck spectral sequence)

Let $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that F sends injective objects to G-acyclic objects. Then for each $A \in \mathcal{A}$ there is a convergent spectral sequence

$$E_2^{p,q} = R^pG \circ R^qF \Rightarrow R^{p+q}(G \circ F)(A).$$

Definition A Cartan-Eilenberg resolution of a chain complex $C_\bullet$ is a bicomplex $C^\bullet \to I^{\bullet,\bullet}$

$$\begin{tikzcd}
\cdots \arrow[r] & I^{0,1} \arrow[r] & I^{1,1} \arrow[r] & \cdots \\
I^{-1,0} \arrow[r] \arrow[u] & I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[u] \\
C^{-1} \arrow[r] \arrow[u] & C^{0} \arrow[r] \arrow[u] & C^{1} \arrow[r] \arrow[u] & C^{2} \arrow[u]
\end{tikzcd}$$

s.t. for each p

$C^p \to I^{p,\bullet}$ is an injective resolution and taking kernels, images and cohomology horizontally in $I^{\bullet,\bullet}$ give injective resolutions of the kernels, images and cohomology of $C^\bullet$.

<u>Sketch of proof of Thm</u>:

Let $A \in \mathcal{A}$ with injective resolution $A \to I^\bullet$.

Consider $\mathcal{F}(I^\bullet)$, this is a complex of $\mathcal{G}$-acyclic objects.

We write a double complex

$$\mathcal{F}(I^\bullet) \to D^{\bullet,\bullet} \text{ a}$$

Cartan-Eilenberg resolution of

$$\mathcal{F}(I^\bullet).$$

and we let $C^{\bullet,\bullet} = \mathcal{G}(D^{\bullet,\bullet})$

we have two spectral sequences

converging to $\mathrm{Tot}(C^{\bullet,\bullet})$.

vertical cohomology gives a

E_1 - page $\quad E_1^{p,q} = RG^q(F(I^p))$

Since $F(I^q)$ is G - acyclic

this vanishes unless $q=0$

E_2 - page degenerates and

$$E_\infty^{p,q} = \begin{cases} R^p(G \circ F)(A) & q=0 \\ 0 & \text{otherwise.} \end{cases}$$

Second spectral sequence:

<u>Proposition</u> If $I^\bullet = \{ I_0 \xrightarrow{1} I_1 \xrightarrow{d_2} I_1 \ldots \to I_n \ldots \}$

1) I_j is injective
2) $\ker d_j$ is injective
3) $\mathrm{Im}\, d_j$ is injective
4) H^j is injective

then

Then $H^j(G(I^\cdot)) = G(H^j(I^\cdot))$

<u>sketch</u>: Break $I^\cdot$ into SES.

$$0 \to \ker \partial_j \to I^j \to \operatorname{im}(\partial_j) \to 0$$

$$0 \to \operatorname{im}(\partial_{j-1}) \to \ker(\partial_j) \to H^j(I^\cdot) \to 0$$

Applying G and that $R^1 G$ vanishes for injectives.

$$0 \to \ker G(\partial_j) \to G(I^j) \to G(\operatorname{im}(\partial_j)) \to 0$$

So $G(\operatorname{im}(\partial_j)) = \operatorname{im}(G(\partial_j))$

and

$$\begin{array}{ccccccccc} 0 \to & G(\operatorname{im}(\partial_{j-1})) & \to & G(\ker(\partial_j)) & \to & G(H^j(I^\cdot)) & \to 0 \\ & \| & & \| & & & \\ 0 \to & \operatorname{im}(G(\partial_{j-1})) & \to & \ker(G(\partial_j)) & \to & H^j(G(I^\cdot)) & \end{array}$$

So $G(H^j(I^\cdot)) \simeq H^j(G(I^\cdot))$

For the other spectral sequence. horizontal cohomology, by definition of Cartan-Eilenberg resolution. Gives a E_1-page with G applied to an injective resolution of $R^p\mathcal{F}(A)$ passing to the E_2-page gives $R^qG \circ R^p\mathcal{F}(A)$.

Čech cohomology:

<u>Rmk</u>: Main computational tool.

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X.

<u>Definition</u> For any presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we let $C^\bullet(\mathcal{U}, \mathcal{F})$ be the Čech complex constructed as follows:

Suppose I is well ordered.

For finite $J \subseteq I$ let $U_J = \bigcap_{j \in J} U_j$.

$$C^i(\mathcal{U}, \mathcal{F}) = \prod_{\substack{J \subseteq I \\ |J| = i+1}} \mathcal{F}(U_J)$$

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^2(\mathcal{U}, \mathcal{F}) \cdots$$

with morphism

$$d(s)_{J = \{j_0, \dots j_{n+1}\}} := \sum_{k=0}^{n+1} (-1)^k \operatorname{res}^{U_{J \setminus \{j_k\}}}_{U_J} \left(s_{J \setminus \{j_k\}} \right)$$

Definition: For any presheaf we define the ith-cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$

$$\check{H}^i_{\mathcal{U}}(X, \mathcal{F}) := H^i(C^\bullet(\mathcal{U}, \mathcal{F})).$$

Example $X = \mathbb{P}^2_k = \qquad \mathcal{F} = \mathcal{O}_X$.

$U_1 = X \setminus V^+(x)$, $U_2 = X \setminus V^+(y)$, $U_3 = X \setminus V^+(z)$.

$C^0(\mathcal{U}, \mathcal{F})$	$C^1(\mathcal{U}, \mathcal{F})$	$C^2(\mathcal{U}, \mathcal{F})$
$k[\frac{x}{x}, \frac{z}{x}]$	$k[\frac{y}{x}, \frac{x}{y}, \frac{z}{x}, \frac{z}{y}]$	
$\times$	$\times$	
$k[\frac{x}{y}, \frac{z}{y}]$	$k[\frac{y}{x}, \frac{z}{x}, \frac{x}{z}, \frac{y}{z}]$	$k[\frac{x}{y}, \frac{y}{x}, \frac{x}{z}, \frac{z}{x}, \frac{y}{z}, \frac{z}{y}]$
$\times$	$\times$	
$k[\frac{x}{z}, \frac{y}{z}]$	$k[\frac{x}{y}, \frac{z}{y}, \frac{y}{z}, \frac{x}{z}]$	

$$\begin{tikzcd}
\mathcal{F}(U_1) \arrow[r, "-"] \arrow[dr, "-"] & \mathcal{F}(\mathcal{U}_{\{1,2\}}) \arrow[dr, "+"] & \\
\mathcal{F}(U_2) \arrow[ur, "+"] \arrow[dr, "-"] & \mathcal{F}(\mathcal{U}_{\{1,3\}}) \arrow[r, "-"] & \mathcal{F}(\mathcal{U}_{\{1,2,3\}}) \\
\mathcal{F}(U_3) \arrow[ur, "+"] \arrow[r, "+"] & \mathcal{F}(\mathcal{U}_{\{2,3\}}) \arrow[ur, "+"] &
\end{tikzcd}$$

$\check{H}^2(\mathbb{P}^2_k, \mathcal{O}_X) = 0 = \check{H}^1(\mathbb{P}^2_k, \mathcal{O}_X)$.

<u>Proposition</u> There are boundary maps δ^i making $(\check{H}^i_{\mathcal{U}}, \delta^i)$ a universal cohomological δ-functor extending the left-exact functor $\check{H}^0_{\mathcal{U}}$.

<u>Sketch</u>: Given a short exact sequence of presheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

we get a short exact sequence of Čech complexes.

$$0 \to C(\mathcal{U}^\bullet, \mathcal{F}_1) \to C(\mathcal{U}^\bullet, \mathcal{F}_2) \to C(\mathcal{U}^\bullet, \mathcal{F}_3) \to 0$$

which gives LES

$$0 \to \check{H}^0_{\mathcal{U}}(X, \mathcal{F}_1) \to \check{H}^0_{\mathcal{U}}(X, \mathcal{F}_2) \to \check{H}^0_{\mathcal{U}}(X, \mathcal{F}_3) \xrightarrow{\delta} \check{H}^1_{\mathcal{U}}(X, \mathcal{F}_1) \to \cdots$$

we show $\check{H}^i_{\mathcal{U}}(X, I) = 0$ for injective I (making it erasable).

Recall the extension by 0 presheaf

$$j_{U_J,!}^{pre} \mathcal{O}_{U_J}[V] = \begin{cases} \mathcal{O}_{U_J}(V) & \text{if } V \subseteq U_J \\ 0 & \text{if not.} \end{cases}$$

We define a complex $K^\bullet$

$$\underset{\substack{J \subseteq I \\ |J|=3}}{////} \longrightarrow \bigoplus_{\substack{J \subseteq I, \\ |J|=2}} (j_{U_J})_!^{pre} \mathcal{O}_{U_J} \to \bigoplus_{\substack{J \subseteq I \\ |J|=1}} (j_{U_J})_!^{pre} \mathcal{O}_{U_J} \to 0$$

with natural map coming from $(j_{V_1})_!^{pre} \mathcal{O}_{V_1} \to (j_{V_2})_!^{pre} \mathcal{O}_{V_2}$ for $V_1 \subseteq V_2$.

<u>Fact:</u> $K^\bullet$ is a complex of presheaves with $\underbrace{H_i(K^\bullet)}_{\text{homology.}} = 0$ for $i \geq 1$

Fact $C^\bullet(\mathcal{U}, I) = \mathrm{Hom}_{psh}(K^\bullet, \tilde{I})$.

Then $\check{H}^i_{\mathcal{U}}(X, I) = H^i(\mathrm{Hom}_{psh}(K^\bullet, \tilde{I}))$

since $\tilde{I}$ is injective it preserves

$\mathrm{Hom}(-, I)$ preserves exactness

so $H^i(\mathrm{Hom}_{psh}(K^\bullet, I)) = 0$ for $i \geq 1$.

This shows each $\check{H}^i_{\mathcal{U}}$ is erasable

so $(\check{H}^i_{\mathcal{U}}, \delta^i)$ is universal.

Lemma Let $R : \mathcal{C} \to \mathcal{D}$ additive functor of abelian categories. Suppose it has an exact left adjoint $L : \mathcal{D} \to \mathcal{C}$

If $I \in \mathcal{C}$ is injective, then $R(I)$ is also injective.

Proof

$\mathrm{Hom}(-, RI) \simeq \mathrm{Hom}(L(-), I)$

which by hypothesis is adjoint.

Corollary: $\mathrm{Forget}: \mathrm{Shv}_{(X,\mathcal{O}_X)} \to \mathrm{PSh}_{(X,\mathcal{O}_X)}$ preserves injectives.
(sheafification is exact).

Observation: For all covers $\mathcal{U}$ of X.

$$\Gamma(X,-) \simeq \check{H}^0_{\mathcal{U}}(X, \mathrm{Forget}(-))$$

as functors $\mathrm{Shv}_{(X,\mathcal{O}_X)} \longrightarrow \mathrm{Ab}$.

$\therefore$ we get spectral sequence

$$\check{H}^p_{\mathcal{U}}(X, \underline{H^q(X,\mathcal{F})}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Čech-to-cohomology spectral sequence.

Corollary: Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of X. Suppose that $\mathcal{F}$ is $\Gamma(U_J, -)$-acyclic for all $J \subseteq I$ finite. Then

$$H^i(X, \mathcal{F}) \simeq \check{H}^i_{\mathcal{U}}(X, \mathcal{F}).$$

proof E_2-page:

$$\operatorname{Ker}\Big(\underbrace{\prod_{\substack{J \subseteq I \\ |J|=1}} H^1(U_J, \mathcal{F})}_{=0} \to \underbrace{\prod_{\substack{J \subseteq I \\ |J|=2}} H^1(U_J, \mathcal{F})}_{=0} \Big)$$

$$\| $$

$$\check{H}^0_{\mathcal{U}}(X, \underline{H^1(X, \mathcal{F})}) \qquad \check{H}^1_{\mathcal{U}}(X, H^1(X, \mathcal{F}))$$

$$\check{H}^0_{\mathcal{U}}(X, \underline{H^0(X, \mathcal{F})}) \qquad \check{H}^1_{\mathcal{U}}(X, \underline{H^0(X, \mathcal{F})})$$

$$E_2^{p,q} = \begin{cases} \check{H}^p_{\mathcal{U}}(X, \mathcal{F}) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

then $\check{H}^p_{\mathcal{U}}(X, \mathcal{F})$ is the only non-zero term with $p+q=p$

so $\check{H}^p_{\mathcal{U}}(X, \mathcal{F}) = H^p(X, \mathcal{F})$.

<u>Theorem:</u> If X is a separated scheme, $\mathcal{F}$ is a quasicoherent sheaf and $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover with each U_i affine, then

$$H^i(X, \mathcal{F}) \simeq \check{H}^i_{\mathcal{U}}(X, \mathcal{F})$$

<u>proof</u>

<u>sketch</u>
- Separated $\Rightarrow$ U_J affine.
- We know $\underbrace{H^1(U_J, \mathcal{F}) = 0}_{\text{Ext}^1\text{-interpretation.}}$

- The point is showing
  $H^n(\mathrm{Spec}\,A, \mathcal{F}) = 0$ $\forall$ rings A
  and quasicoherent sheaves $\mathcal{F}$.

But on the category of quasicoherent sheaves $\Gamma(\mathrm{Spec}\,A, -)$ is exact.

Subtlety:

$\mathrm{QCoh}_{\mathrm{Spec}\,A} \hookrightarrow \mathcal{O}_X\text{-modules}$ ∇

doesn't preserve injectives. $\mathcal{O}$.