Organizational:

- Sign to exercise session ecampus.
- 50 % for exercises.
- Website for the course.
- Register to basis for exam.
- start studying homological algebra.

Motivation (Informal)

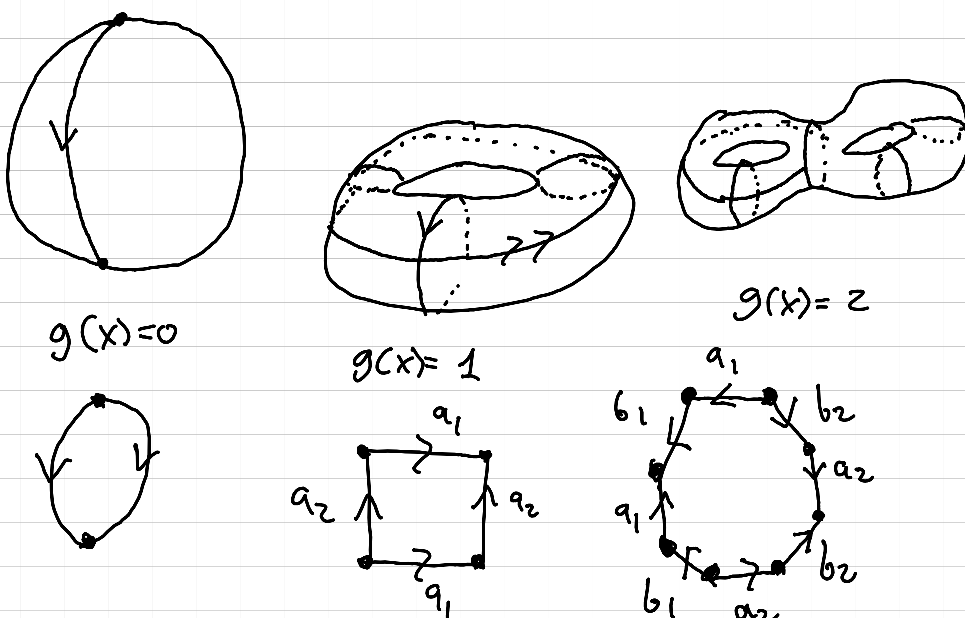
Let X be a compact connected Riemann surface. We can think of it as a locally ringed space $X := (|X|, \mathcal{O}_X)$ such that $|X|$ is compact Hausdorff connected and for all $x \in X$ there is an open $U \subseteq |X|$ such that

$$(U, \mathcal{O}_{X|U}) \simeq \mathbb{D} \quad \text{(open unit ball)}$$

$$|\mathbb{D}| = \{ x \in \mathbb{C} \mid |x| < 1 \}$$

$$\mathcal{O}_{\mathbb{D}}(U) = \{ f : U \to \mathbb{C} \mid f \text{ is holomorphic} \}$$

In particular, $|X|$ is a compact, orientable, surface.
These are classified by their genus $g(X)$.



Equivalently, by their Euler characteristic

$$\chi(X) = \#V - \#E + \#F = 2 - 2g(X).$$

Moreover, we can consider holomorphic maps $\pi : X \to Y$ of Riemann surfaces.
(These are just maps of locally ringed spaces).

They admit an easy "local description".

<u>Proposition</u> (See HWK 1)

Let $\pi : X \to Y$ be a non-constant holomorphic map of compact, connected Riemann surfaces. Let $x \in X$ and $y = \pi(x)$, then there exists a commutative diagram

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "z \mapsto z^{n_x}"] \arrow[d, "f"'] & \mathbb{D} \arrow[d, "g"] \\
X \arrow[r, "\pi"'] & Y
\end{tikzcd}$$

where f and g are open immersions of locally ringed spaces.

The number n_x is the valency of π at x. We let $N_\pi(x) = n_x$ as above.

Proposition / Definition: Given a non-constant map of compact, connected Riemann surfaces $\pi : X \to Y$ we define the degree $\deg(\pi) = \sum_{x \in \pi^{-1}(y)} N_\pi(x)$. This quantity is independent of $y \in Y$. Moreover, for all but finitely $y \in Y$ $N_\pi(x) = 1$ for all $x \in f^{-1}(y)$ so that $\deg(\pi) = |f^{-1}(y)|$.

<u>Example</u>: Holomorphic maps

$\pi : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ are given by meromorphic functions $f_\pi(z) = \dfrac{p(z)}{q(z)}$.

$\deg(\pi) = \max(\deg(p), \deg(q))$.

<u>Definition</u> Let $\pi : X \longrightarrow Y$ be a non-constant holomorphic map between compact connected Riemann surfaces.

The <u>total branching index</u> (or branching number) b_π is defined as:

$$b_\pi = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} (\nu_\pi(x) - 1) = \underbrace{\sum_{y \in Y} \left(\deg(\pi) - |\pi^{-1}(y)| \right)}_{\text{this is a finite sum !!!}}$$

Example: $\pi: \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$

$$z \longmapsto z^2$$

then all points in $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, \infty\}$ have 2 square roots, but $0, \infty$ only have 1 so

$b_\pi = 2$. since $N_\pi(0) = 2$
$N_\pi(\infty) = 2$.

Theorem (Riemann – Hurwitz formula)

Let $\pi: X \to Y$ be a holomorphic map of compact connected Riemann surfaces. If π is not constant then

$$\chi(X) = \deg(\pi)\chi(Y) - b_\pi.$$

Equivalently: $g(X) - 1 = \deg(\pi)(g(Y) - 1) + \frac{b_\pi}{2}$.

<u>Example</u>

There are no non-constant holomorphic maps of compact connected Riemann surfaces $\pi : X \to Y$ with $g(X) = 2$ and $g(Y) = 3$

$$\boxed{1 = 2d + b_\pi/2 \quad \text{doesn't have a solution !!!}}$$

Let $\left\{ \begin{matrix} \text{Riem.} \\ \text{Surf.} \end{matrix} \right\}$ denote the category of proper connected Riemann surfaces and let $\{ \mathrm{Curv}/\mathbb{C} \}$ denote the category of smooth proper curves over Spec $\mathbb{C}$. [curve = 1-dim variety].

<u>Theorem</u> The categories

$$\left\{ \begin{matrix} \text{Riem.} \\ \text{Surf.} \end{matrix} \right\} \quad \text{and} \quad \{ \mathrm{Curv}/\mathbb{C} \}$$

are equivalent.

<u>Question</u> Can we make the above considerations algebraic? Does it work for other basefields $k \neq \mathbb{C}$?

- <u>Flatness</u>

If $\pi: X \to Y$ is a map of schemes, then for all $y \in Y$ with residue field $k(y)$ we can consider the fiber:

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} k(y) \arrow[d, hook] \\ X \arrow[r] & Y \end{tikzcd}$$

and we think of $\{X_y\}_{y \in Y}$ as a family of schemes parametrized by Y.

<u>Definition</u> $\pi : X \to Y$ map of schemes is flat if for all affines

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ \cap\!| & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

B is a flat A-module.
(i.e. $- \otimes_A B$ preserves exact sequences).

"Flatness" of π is the algebraic geometry way of saying the family $\{X_y\}_{y \in Y}$ varies ("continuously"?, "nicely"?).

<u>Non-example</u> : $Y = \mathbb{A}^2_k$

$$\coprod_{y \in |Y|} \operatorname{Spec} k(y) \longrightarrow Y.$$

is a non-flat family.

<u>Example</u>: Non-constant maps of smooth connected curves

$$\pi : X \longrightarrow Y$$

are automatically finite flat.
(we will prove this later).

This implies $X_y = \operatorname{Spec} A(y)$ where $A(y)$ is a finite $k(y)$-algebra.

Flatness gives:

$$\dim_{k(y)} A(y) \text{ is constant}$$

we can let $\deg(\pi) = \dim_{k(y)} A(y)$.

Sheaf of

- Differentials $\Omega^1_{X/k}$:

  - Allows us to talk about smooth maps, ramification and branching.

If

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr] & Y \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd}$$

is a map of smooth proper curves we get an exact sequence:

$$0 \to \pi^* \Omega^1_{Y/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/Y} \to 0$$

of coherent sheaves over X.

Moreover, we can define the "geometric genus"

$$g_{geo} := \dim_k \Gamma(X, \Omega^1_{X/k}).$$

• Cohomology:

The functor

$$\mathcal{F} \longmapsto \Gamma(X, \mathcal{F})$$
$$\left\{ \begin{array}{c} \text{Quasicoherent} \\ \text{Sheaves}/X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} k\text{-vector} \\ \text{spaces} \end{array} \right\}$$

is not exact.

One defines
$$H^0(X, \mathcal{F}) := \Gamma(X, \mathcal{F})$$
and
$$H^i(X, \mathcal{F}) \qquad i > 0$$
capturing the failure of exactness:

If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$
is SES (short exact sequence),
then we get a LES (long exact sequence)

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \xrightarrow{\delta} H^1(X, \mathcal{F}_1) \to H^1(X, \mathcal{F}_2)$$
$$\to H^1(X, \mathcal{F}_3) \xrightarrow{\delta} H^2(X, \mathcal{F}_1) \to \cdots$$

Vanishing: If $\mathcal{F}$ is a coherent sheaf and $\dim(\operatorname{supp}(\mathcal{F})) < n$ then $H^k(X, \mathcal{F}) = 0$ for $n \leq k$.

Finiteness: When X is proper over $\operatorname{spec} k$ and $\mathcal{F}$ is coherent, then $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space.

We let $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$.

We have an Euler characteristic for coherent sheaves

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i h^i(X, \mathcal{F})$$

Fact If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is SES of coherent sheaves then $\chi(\mathcal{F}_2) = \chi(\mathcal{F}_3) + \chi(\mathcal{F}_1)$.

Going back to non-constant maps of smooth proper curves

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr] & Y \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd}$$

from

$$0 \to \pi^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$$

we deduce

$$\chi(X, \Omega^1_{X/k}) = \chi(X, \pi^*\Omega^1_{Y/k}) + \chi(\Omega^1_{X/Y})$$

$$\underbrace{\chi(X, \Omega^1_{X/k}) - \chi(X, \mathcal{O}_X)}_{\deg(\Omega^1_{X/k})} = \underbrace{\chi(X, \pi^*\Omega^1_{Y/k}) - \chi(X, \mathcal{O}_X)}_{\deg(\pi^*\Omega^1_{Y/k})} + \chi(\Omega^1_{X/Y})$$

$\deg(\Omega^1_{X/k})$ (as line bundles)

$\deg(\pi^*\Omega^1_{Y/k}) = \deg(\pi) \cdot \deg(\Omega^1_{Y/k})$

<u>Claim</u> $\deg(\Omega^1_{X/k}) = 2g_{geo}(X) - 2$

We compute

$$h^0(X,\Omega^1_{X/k}) - h^1(X,\Omega^1_{X/k}) + h^0(X,\mathcal{O}_X)$$
$$- h^1(X,\mathcal{O}_X).$$

Serre duality:

For all smooth proper schemes X over $\operatorname{Spec} k$ of dimension n and vector bundles $\mathcal{E}$ over X we have an isomorphism

$$H^i(X,\mathcal{E}) = \operatorname{Hom}_k\left(H^{n-i}(X,\Omega^n_{X/k}\otimes\mathcal{E}^\vee),k\right)$$

In our case $n=1$:

$$h^1(X,\Omega^1_{X/k}) = h^0\left(X,\Omega^1_{X/k}\otimes(\Omega^1_{X/k})^\vee\right) = 1$$

$$h^1(X,\Omega^1_{X/k}) = h^0(X,\Omega^1_{X/k}\otimes\mathcal{O}_X^\vee) = g_{geo}(X)$$

$$\deg(\Omega^1_{X/k}) = 2g_{geo}(X) - 2$$

So far

$$g_{geo}(X) - 1 = (\deg \pi)(g_{geo}(Y) - 1) + \frac{\chi(X, \Omega^1_{X/Y})}{2}$$

we only need to show that the "total branching number"

$$b_\pi = \chi(X, \Omega^1_{X/Y}) = h^0(X, \Omega^1_{X/Y})$$

when $k = \mathbb{C}$.

Given $x \in X(\mathbb{C})$ with $y = \pi(x) \in Y(\mathbb{C})$

we get a map of DVR

$$\mathcal{O}_{X,x} \xleftarrow{\pi^*} \mathcal{O}_{Y,y}$$

let $t_y \in \mathcal{O}_{Y,y}$ and $t_x \in \mathcal{O}_{X,x}$ be uniformize

then $\pi^*(t_y) = u\, t_x^{n_x}$

$n_x \sim N_\pi(x)$

Roughly: $\chi(X, \Omega^1_{X/Y}) = h^0(X, \Omega^1_{X/Y})$

since $\mathrm{supp}(\Omega^1_{X/Y})$ is finite.

and if $x \in X$ and $y = \pi(x) \in Y$

then

$$dt_y \mapsto n_x t^{n_x - 1} dt_x$$

$$0 \to \mathcal{O}_{X,x} \cdot dt_y \to \mathcal{O}_{X,x} \cdot dt_x \to (\Omega^1_{X/Y})_{\mathcal{O}_{X,x}} \to 0$$