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Solutions, Algebraic Geometry I Exercise Sheets

Note for usage of these solutions: Use care while using the following solutions. Some of them may contain mathematical and/or linguistic errors. Please do not distribute these solutions to anybody who is not enrolled in this class.

Exercise 1. *Direct sum of sheaves* (3 points)
Let $\mathcal{F}$, $\mathcal{G}$ be two sheaves of abelian groups on a topological space X. Show that $\mathcal{F} \oplus \mathcal{G} \colon U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ defines a sheaf.

Solution. $\mathcal{F} \oplus \mathcal{G}$ is a presheaf:

1. For all $U \subseteq X$ open, $\mathcal{F}(U) \oplus \mathcal{G}(U)$ is again an abelian group under componentwise addition.
2. For all $V \subseteq U \subseteq X$ open, let a morphism $\rho_{UV} : \mathcal{F}(U) \oplus \mathcal{G}(U) \to \mathcal{F}(V) \oplus \mathcal{G}(V)$ be a morphism of abelian groups given by $\rho^{\mathcal{F}}_{UV} \oplus \rho^{\mathcal{G}}_{UV}$.

Then we check

(i) For all $U \subseteq X$ open, $\rho_{UU} : \mathcal{F}(U) \oplus \mathcal{G}(U) \to \mathcal{F}(U) \oplus \mathcal{G}(U)$ is the identity, simply because it is so on each component.

(ii) For all $W \subseteq V \subseteq U \subseteq X$ open, we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$, that is, the following diagram formed by componentwise direct sum commutes

$$\begin{tikzcd} \mathcal{F}(U) \oplus \mathcal{G}(U) \arrow[r, "\rho_{UV}"] \arrow[rr, bend left, "\rho_{UW}"] & \mathcal{F}(V) \oplus \mathcal{G}(V) \arrow[r, "\rho_{VW}"] & \mathcal{F}(W) \oplus \mathcal{G}(W) \end{tikzcd}$$

Additionally for every $U \subseteq X$ open and every open covering $U = \cup_{i \in I} U_i$ of U the following conditions are satisfied

(i) For all $s, t \in \mathcal{F}(U) \oplus \mathcal{G}(U)$: If $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$. Indeed, we can write $s = s^{\mathcal{F}} \oplus s^{\mathcal{G}}$ for $s^{\mathcal{F}} \in \mathcal{F}(U)$ and $s^{\mathcal{G}} \in \mathcal{G}(U)$, and similarly for t. The equality of restrictions induces the equalities $s^{\mathcal{G}} = t^{\mathcal{G}}$ and $s^{\mathcal{F}} = t^{\mathcal{F}}$.

(ii) For all $\{s_i\}_{i \in I}$ with $s_i \in \mathcal{F}(U_i) \oplus \mathcal{G}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists an $s \in \mathcal{F}(U) \oplus \mathcal{G}(U)$ such that $s|_{U_i} = s_i$. Indeed, this can be argues similarly.

Exercise 2. *Sheaf $\mathcal{H}om$* (4 points)
For a (pre-)sheaf $\mathcal{F}$ on a topological space X and an open subset $U \subset X$ one defines the restriction $\mathcal{F}|_U$ to be the (pre-)sheaf on the topological space U given by $\mathcal{F}|_U(V) := \mathcal{F}(V)$ for any open subset $V \subset U$. Then for two sheaves $\mathcal{F}, \mathcal{G}$ of abelian groups, $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ denotes the abelian group of all sheaf homomorphisms $\mathcal{F}|_U \to \mathcal{G}|_U$. Show that this naturally defines a pre-sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \colon U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ which is in fact a sheaf.

Solution. Recall that any $\varphi_U \in \mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$ satisfies

$$\begin{tikzcd}
\mathcal{F}|_U(W) \arrow[r, "\varphi_U(W)"] \arrow[d, "\rho_{\mathcal{F},WV}"] & \mathcal{G}|_U(W) \arrow[d, "\rho_{\mathcal{G},WV}"] \\
\mathcal{F}|_U(V) \arrow[r, "\varphi_U(V)"] & \mathcal{G}|_U(V)
\end{tikzcd}$$

for all opens $V \subset W \subset U$. Now let $V \subset U$, then $\varphi_V := \varphi_U|_V : \mathcal{F}|_V \to \mathcal{G}|_V$ is defined simply by sending $\varphi_V(W) = \varphi_U(W)$ for all open subset $W \subset V \subset U$. The commutativity of the similar diagram for φ_V as above then follows from the commutativity for φ_U.

To check the sheaf property let $U = \cup_{i\in I} U_i$ be an open covering of U. Consider $\varphi, \psi \in \mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$: If $\varphi_{U_i} = \psi_{U_i}$ for all $i \in I$, then to see $\varphi = \psi$, let $\tau = \varphi - \psi$. Then for all $s \in \mathcal{F}(U)$ we have $\tau_{U_i}(s) = 0 \in \mathcal{G}(U_i)$ for all $i \in I$. Since $\mathcal{G}$ is a sheaf, this implies that $\tau(s) = 0$.

For the second condition let $\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i \cap U_j}$ for $\varphi_i \in \mathcal{H}om(\mathcal{F}_{U_i}, \mathcal{G}_{U_i})$. Then define $\varphi \in \mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$ as follows: let $s \in \mathcal{F}_U(V)$. Let $t_i = \varphi_i(V \cap U_i)(s)$. By hypothesis $t_i|_{U_i\cap U_j} = t_j|_{U_i\cap U_j}$ and hence the sheaf property of $\mathcal{G}|_U$ ensures the unique existence of $t \in \mathcal{G}|_U(V)$. Define $\varphi(V)(s) = t$ and note that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$.

Note that the local sheaf condition ensures uniqueness of the section that the gluing condition gives.

Exercise 3. *Gluing of sheaves* (5 points)
Let X be a topological space and let $X = \bigcup U_i$ be an open covering. We use the shorthand $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$.

Consider sheaves $\mathcal{F}_i$ on U_i and isomorphisms (*gluings*) $\varphi_{ij} : \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$. Show that if the *cocycle condition* $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on U_{ijk} is satisified, then there exists a sheaf $\mathcal{F}$ on X together with isomorphisms $\varphi_i : \mathcal{F}|_{U_i} \cong \mathcal{F}_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$ on U_{ij}. The sheaf $(\mathcal{F}, \varphi_i)$ is unique up to unique isomorphism.

Solution. not provided.

Exercise 4. *Exponential map* (4 points)
Consider $X = \mathbb{C} \setminus \{0\}$ with its usual topology and let $\mathcal{O}_X$ be the sheaf of holomorphic functions, i.e. $\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}$. Similarly, let $\mathcal{O}_X^*$ be the sheaf of holomorphic functions without zeroes. (Throughout, you may work with differentiable functions instead of holomorphic ones if you prefer.)

Show that the exponential map defines a morphism of sheaves (of abelian groups)

$$\exp : \mathcal{O}_X \to \mathcal{O}_X^*, \mathcal{O}_X(U) \ni f \mapsto \exp(f) \in \mathcal{O}_X^*(U).$$

Find a basis of the topology on X such that $\exp(U) : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$ is surjective for all U in this basis. Note that $\mathcal{O}_X(X) \to \mathcal{O}_X^*(X)$ is not surjective. Describe the kernel of $\exp_U$.

Solution. First we show that $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ is a morphism of sheaves of abelian groups. Over any open subset $U \subset X$, it is given by $\exp(U) : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U) : f \mapsto \exp(f)$ which is a morphism of abelian groups since

$$\exp(U)(f + g) = \exp(f + g) = \exp(f) \cdot \exp(g) = \exp(U)(f) \cdot \exp(U)(g)$$

for all $f, g \in \mathcal{O}_X(U)$.

Given any two open subsets $V \subset U \subset X$, commutativity of the usual square simply follows from the equality $\exp(f) \mid_V = \exp(f \mid_V)$ for all $f \in \mathcal{O}_X(U)$. This concludes the proof that exp is a morphism of sheaves.

Note that X is locally simply-connected. Thus the collection $\mathcal{B}$ of all simply-connected open subsets of $X = \mathbb{C} - \{0\}$ forms a basis for its topology (alternatively, the collection of all balls in $\mathbb{C}$ not containing 0 works as well). Given $U \in \mathcal{B}$, we claim that $\exp_U$ is surjective. A section $f \in \mathcal{O}_X^*(U)$ is a holomorphic function without zeroes whose domain U is simply-connected, so it admits a holomorphic logarithm, i.e. a holomorphic function $F\colon U \to \mathbb{C}$ such that $\exp \circ F = f$ (explicitly, fix $a \in U$ and $b \in \mathbb{C}$ such that $\exp(b) = f(a)$, then set $F(z) = b + \int_a^z f'(w)/f(w)dw$ for all $z \in U$), i.e. a section $F \in \mathcal{O}_X(U)$ such that $\exp_U(F) = f$. This shows that $\exp_U$ is surjective for all basis element U in $\mathcal{B}$.

Note that $\exp_X\colon \mathcal{O}_X(X) \to \mathcal{O}_X^*(X)$ is not surjective. Indeed, the section $f \in \mathcal{O}_X^*(X)$ defined by $f(z) = z$ for all $z \in X$ (it is holomorphic and without zeroes in X) has no preimage along $\exp_X$, for a preimage $F \in \mathcal{O}_X(X)$ would be a holomorphic branch of the logarithm on the non-simply-connected subset $\mathbb{C} - \{0\}$, which does not exist by a result from Complex Analysis. More precisely, restricting F to the unit circle $S^1 \subset X$ would give a continuous function $\frac{1}{i}F|_{S^1}\colon S^1 \to \mathbb{R}$ (the *argument* function), which must be strictly increasing as we turn around the circle in a clockwise manner. This cannot exist.

Given $U \subset X$ open, the kernel of $\exp_U$ consists of those holomorphic functions $f\colon U \to \mathbb{C}$ such that $\exp \circ f = 1$, i.e. the continuous functions with values in $2\pi i\mathbb{Z}$, i.e. the locally constant functions with values in $2\pi i\mathbb{Z}$. Equivalently, $\ker(\exp(U))$ consists of all the functions $f\colon U \to 2\pi i\mathbb{Z}$ which are constant on each connected component of U. Thus $\ker(\exp) = 2\pi i\underline{\mathbb{Z}}$ is a locally constant sheaf isomorphic to $\underline{\mathbb{Z}}$.

This exercise means that there is a short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

called the *exponential sequence*.

The following exercise uses stalks, which will be introduced in the second lecture.

Exercise 5. *'Espace étalé' of a presheaf* (4 points)
Let $\mathcal{F}$ be a presheaf on a topological space X. Define the set

$$|\mathcal{F}| := \bigsqcup_{x \in X} \mathcal{F}_x,$$

which comes with a natural projection $\pi : |\mathcal{F}| \to X$, $(s \in \mathcal{F}_x) \mapsto x$. Then any $s \in \mathcal{F}(U)$ defines a section of π over U by $x \mapsto s_x$. One endows $|\mathcal{F}|$ with the strongest topology such that all $s \in \mathcal{F}(U)$ define continuous sections $x \mapsto s_x$. Show that the sheafification $\mathcal{F}^+$ can be described as the sheaf of continuous sections of $|\mathcal{F}| \to X$.

Solution. Not provided.

Exercise 6. (Extra Point)

Exercise 7. *Sheafification* (3 points)
Describe examples of presheaves (of abelian groups) $\mathcal{F}$ for which the sheafification $\mathcal{F} \to \mathcal{F}^+$ is not injective resp. not surjective on some open set. Find an example with $\mathcal{F} \neq 0$ but $\mathcal{F}^+ = 0$.

As in Exercise 4 in the previous sheet, let $X = \mathbb{C} \setminus \{0\}$ and $\mathcal{O}_X$ the sheaf of holomorphic functions. Consider the exponential map $\mathcal{O}_X \to \mathcal{O}_X^*$ given by sending a holomorphic function $f \in \mathcal{O}_X(U)$ to $\exp(f)$. The image presheaf gives an example where the sheafification is not surjective and the cokernel presheaf gives an example where the sheafification is not injective.

To elaborate, call the cokernel $\mathcal{Q}$. It is a presheaf such that $\mathcal{Q}(X) \neq 0$ since $t \in \mathcal{O}_X^*(X)$ given by the coordinate of $\mathbb{C} \setminus \{0\}$ has no preimage under the exponential map. But $t = 0$ in $\mathcal{Q}^+(X)$ since for every $x \in X$, there is an open subset $U \ni x$ on which the branch of logarithm $\log_U$ is a well-defined holomorphic function and $t_x \in \mathcal{O}_X^*(U)$ admits a preimage given by $\log_U(t)$. Hence $t = 0 \in \mathcal{Q}^+$. Thus, the map $\mathcal{Q}(X) \to \mathcal{Q}^+(X)$ sends $0 \neq t \in \mathcal{Q}(X)$ to zero in $\mathcal{Q}^+(X)$, hence it is not injective.

In fact, $\mathcal{Q}^+ = 0$. To show this, it suffices to show that every stalk of $\mathcal{Q}$ is zero. This directly follows from exercise 4 in the previous sheet: any section of $\mathcal{O}_X^*$ has a preimage under exp if we restrict it to a simply-connected open subset. Thus, in this case, we have $0 \neq \mathcal{Q} \to \mathcal{Q}^+ = 0$.

On the other hand, let us denote the image presheaf by $\mathcal{I} \subset \mathcal{O}_X^*$. Since $\mathcal{O}_X \to \mathcal{O}_X^*$ is surjective, $\mathcal{I}^+ \simeq \mathcal{O}_X^*$. However on X, $\mathcal{O}_X^*(X)/\mathcal{I}(X) \simeq \mathcal{Q}(X) \neq 0$, which means that $\mathcal{I}(X) \to \mathcal{I}^+(X) = \mathcal{O}_X^*(X)$ is not surjective.

Solution. (Alternative Solution) Let $X = x_1 \sqcup x_2$, i.e. disjoint union of two distinct points endowed with the discrete topology. Consider the constant presheaf with value in an abelian group A (or any object in some category $\mathcal{C}$) denoted by A' with $A'(U) = A$ for all open set $U \subset X$. In particular $A'(X) = A$. Let $\underline{A} := A'^+$ be the constant sheaf. Then by definition $\underline{A}(X) = \{s \colon X \to A \times A \mid s \text{ satisfies conditions (i) and (ii)}\}$ where

(i) For all $x \in X$, $s(x) \in A'_x \simeq A$.

(ii) For all $x \in X$, there exists $V \subseteq X$ open with $x \in V$ and $t \in A'(V) = A$ such that $t_y = s(y)$ for all $y \in V$.

Let $a_1, a_2 \in A$ be elements, such that $a_1 \neq a_2$, define $s(x_i) = a_i$ for $i = 1, 2$. Then s satisfies both (i) and (ii). Indeed, (i) is by definition and for (ii), let $V_i = \{x_i\}$ for $i = 1, 2$ and let $t_i \in A'(V_i)$. Then $t_{i_{x_i}} = s(x_i)$ for both $i = 1, 2$. Hence $\underline{A}(X) = A \times A$.

Hence $A'(X) \to \underline{A}(X)$ is not surjective. Similarly when $A'(\emptyset) = A$ but $\underline{A}(\emptyset) = 0$. Hence the sheafification map is not injective.

On the same topological space if we let $\mathcal{F}(U) = 0$ for all open $U \subsetneq X$ but $\mathcal{F}(X) = A$. Then $\mathcal{F}^+ = 0$.

Exercise 8. *Subsheaf with support and left-exactness* (6 points)
Let $\mathcal{F}$ be a sheaf on a topological space X and let $Z \subseteq X$ be a closed subset.

(i) For $U \subseteq X$ open and $s, t \in \mathcal{F}(U)$, show that the set of points $x \in U$ with $s_x = t_x$ in $\mathcal{F}_x$ is open in U. In particular, if $\mathcal{F}$ is a sheaf of abelian groups, then the *support of* s defined as
$$\operatorname{supp}(s) = \{x \in U \mid s_x \neq 0\}$$
is closed in U.

(*Warning:* Observe that, in the situation of exercise 9 for $f \in \mathcal{C}^0(U)$, the zero locus does not necessarily coincide with $U - \operatorname{supp}(f)$. This is because $f_x = 0$ implies that f is zero in a neighbourhood of x, which is much stronger than just having $f(x) = 0$.)

(ii) Show that the association
$$\mathcal{H}_Z^0(\mathcal{F}) : U \mapsto \Gamma_{Z \cap U}(U, \mathcal{F}) := \{s \in \mathcal{F}(U) \mid \operatorname{supp}(s) \subseteq Z \cap U\}$$
defines a subsheaf of $\mathcal{F}$.

(A sheaf $\mathcal{F}$ of abelian groups is said to be *supported on* Z if $\mathcal{H}_Z^0(\mathcal{F}) = \mathcal{F}$.)

(iii) Prove that $\Gamma_{Z\cap U}(U, \ ) \colon \mathrm{Sh}(X) \to (Ab)$ is a left exact functor, i.e. for any short exact sequence of sheaves
$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
the sequence
$$0 \to \Gamma_{Z\cap U}(U, \mathcal{F}_1) \to \Gamma_{Z\cap U}(U, \mathcal{F}_2) \to \Gamma_{Z\cap U}(U, \mathcal{F}_3)$$
is exact. Note the important special case $Z = X$, where $\Gamma_{Z\cap U}(U, -) = \Gamma(U, -)$.

(*Warning:* But usually $\Gamma_{Z\cap U}(\mathcal{F}_2) \to \Gamma_{Z\cap U}(\mathcal{F}_3)$ is not surjective, i.e. $\Gamma_{Z\cap U}$ is not exact.)

Solution. Not provided.

Exercise 9. *Local rings of continuous functions* (3 points)
Let X be a topological space and let $\mathcal{C}^0$ be the sheaf of continuous functions on X. Consider for a point $x \in X$ the stalk $\mathcal{C}^0_x$. Show that the map $\mathcal{C}^0_x \to \mathbb{R} \colon f \mapsto f(x)$ is well-defined and that $\mathcal{C}_x$ is a local ring with maximal ideal $\mathfrak{m}_x := \{f \in \mathcal{C}^0_x \mid f(x) = 0\}$. Describe similar situations involving differentiable or holomorphic functions.

Solution. By definition of the stalk, $\mathcal{C}^0_x$ is the direct limit of all the $\mathcal{C}^0(U)$ where U runs over all neighborhoods of x. Concretely, it can be written as

$$\begin{aligned} \mathcal{C}^0_x &\cong \{(U, f) \mid x \in U \subset X \text{ open and } f \in \mathcal{C}^0(U)\}/\sim \\ &= \{[U, f] \mid x \in U \subset X \text{ open and } f \in \mathcal{C}^0(U)\} \end{aligned}$$

where $\sim$ is the following equivalence relation: $(U, f) \sim (V, g)$ if there exists a smaller neighborhood $W \subset U \cap V$ of x such that $f_{|W} = f_{|W}$. The equivalence class of (U, f) is denoted by $[U, f]$. The addition in the ring $\mathcal{C}^0_x$ is then described as $[U, f] + [V, g] = [U \cap V, f_{|U\cap V} + g_{|U\cap V}]$, and similarly for the multiplication.

For $[U, f] \in \mathcal{C}^0_x$, the value $f(x)$ does not depend on the choice of representative of the class: if $[U, f] = [V, g]$, then f and g coincide around x, in particular $f(x) = g(x)$. Thus the map *evaluation at* x $\mathrm{ev}_x \colon \mathcal{C}^0_x \to \mathbb{R} \colon f \mapsto f(x)$ is well-defined. Moreover, it is clear from the above description that ev_x is a ring homomorphism. It is surjective: any real $c \in \mathbb{R}$ is the image of $[U, y \mapsto c]$, where $y \mapsto c$ is the constant function with value c on U. Note that $\mathfrak{m}_x = \ker(\mathrm{ev}_x)$, which is then a maximal ideal since $\mathbb{R}$ is a field.

It remains to show that $\mathcal{C}^0_x$ is local, i.e. $\mathfrak{m}_x$ is the unique maximal ideal in the stalk $\mathcal{C}^0_x$. This is equivalent to showing that any element in $\mathcal{C}^0_x - \mathfrak{m}_x$ is invertible. Let $[U, f] \in \mathcal{C}^0_x - \mathfrak{m}_x$, i.e. $f(x) = \mathrm{ev}_x(f) \neq 0$. Then, by continuity of f, there exists a neighborhood $V \subset U$ of x on which f does not vanish. Therefore, the function $\frac{1}{f} \colon V \to \mathbb{R} \colon y \mapsto \frac{1}{f(x)}$ is a well-defined continuous function around x and we have

$$[U, f] \cdot [V, \tfrac{1}{f}] = [V, f] \cdot [V, \tfrac{1}{f}] = [V, f\tfrac{1}{f}] = [V, 1] = 1_{\mathcal{C}^0_x},$$

in the stalk $\mathcal{C}^0_x$, i.e. $[U, f]$ is invertible. This concludes the proof that $\mathfrak{m}_x$ is the unique maximal ideal of $\mathcal{C}^0_x$.

In the case of differentiable and holomorphic functions, the argument is exactly the same. Indeed, if f is differentiable (resp. holomorphic) and does not vanish on an open subset V, then its inverse $\frac{1}{f}$ is also differentiable (resp. holomorphic).

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 10. *Functor of points and the Yoneda lemma* (4 extra points)
Let $\mathcal{C}$ be a category with sets of morphisms between two objects X, Y denoted $\mathrm{Mor}(X,Y)$. Then every object X in $\mathcal{C}$ induces a functor

$$h_X \colon \mathcal{C}^{\mathrm{op}} \to (Sets),\ Y \mapsto h_X(Y) \coloneqq \mathrm{Mor}(Y,X).$$

Observe that $h_X(X)$ contains a distinguished element.

(i) Consider the three categories $\mathcal{C} \coloneqq (Top)$ (of topological spaces); $\mathcal{C} \coloneqq (Ab)$ (of abelian groups); $\mathcal{C} \coloneqq (Rings)$ (of rings) and denote for each object X in $\mathcal{C}$ by $|X|$ the underlying set (the set of points). Show that in all three cases there exists an object Z in $\mathcal{C}$ such that for all X the set of points $|X|$ can be recovered as $|X| = h_X(Z)$.

(ii) Consider the category of affine schemes $\mathcal{C} \coloneqq (AffSch)$. Does there exist an object as in (i) in this case?

(iii) For an arbitrary category $\mathcal{C}$, denote by $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (Sets))$ the category of functors $\mathcal{C}^{\mathrm{op}} \to (Sets)$ and consider the functor

$$\begin{aligned} h\colon \quad \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (Sets)) \\ X &\mapsto h_X. \end{aligned}$$

The Yoneda lemma then asserts that h is a fully faithful embedding, in other words h defines an equivalence of categories between $\mathcal{C}$ and a full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (Sets))$. Spell out what this means and try to prove it. Check Vakil's notes on the subject (or any other source). Objects in the image of h (or, more precisely, objects isomorphic to objects in the image) are called *representable functors*.

Exercise 11. *Direct and inverse image are adjoint* (6 points)
Let $f\colon X \to Y$ be a continuous map. Show that $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is right adjoint to $f^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ (one writes $f^{-1} \dashv f_*$), i.e. for all $\mathcal{F} \in \mathrm{Sh}(X)$ and $\mathcal{G} \in \mathrm{Sh}(Y)$, there exists an isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Show that, in particular, there exist natural homomorphisms

$$\mathcal{G} \to f_*f^{-1}\mathcal{G} \text{ and } f^{-1}f_*\mathcal{F} \to \mathcal{F}.$$

Verify also that for the composition of two continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ one has $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Solution.
Not provided.

Exercise 12. *Stalks of direct image sheaf* (3 points)
Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of abelian groups on X.

(i) Show that if $y \in Y$ is not in the closure of $f(X)$ in Y, then $(f_*\mathcal{F})_y = 0$.

(ii) Show that if f is the inclusion of a subspace (i.e. f is injective and $U \subseteq X$ is open if and only if there exists $V \subseteq Y$ open with $f^{-1}(V) = U$) and $y = f(x)$ for some $x \in X$, then $(f_*\mathcal{F})_y \cong \mathcal{F}_x$.

(iii) Give an example where f is as in (ii), but y is not in $f(X)$, and $(f_*\mathcal{F})_y \neq 0$.

Solution.

(i) Let $U \subset Y$ be an open neighborhood of y such that $f(X) \cap Y = \emptyset$. Then $f_*\mathcal{F}(U) = \mathcal{F}(\emptyset) = 0$ and hence $(f_*\mathcal{F})_y = 0$.

(ii) There is an obvious homomorphism of groups

$$(f_*\mathcal{F})_y \to \mathcal{F}_x, \quad [s, U] \mapsto [s, f^{-1}(U)],$$

where $\tilde{s} \in \mathcal{F}(f^{-1}(U))$ is the secion corresponding to s under the identification $\mathcal{F}(f^{-1}(U)) = f_*\mathcal{F}(U)$. We show that this homomorphism is actually an isomorphism.

(a) Injectivity: suppose $[s, U] \in (f_*\mathcal{F})_y$ with $[\tilde{s}, f^{-1}(U)] = 0$. This means that there exists $V \subset f^{-1}(U)$ open such that $\tilde{s}_{|V} = 0$. Let $W \subset U$ open such that $f^{-1}(W) = V$. Then the above implies $s_{|W} = 0$.

(b) Surjectivity: given $[\tilde{s}, V] \in \mathcal{F}_x$. Let $U \subset Y$ such that $V = f^{-1}(U)$ and let $s \in f_*\mathcal{F}(U)$ the section corresponding to $\tilde{s}$. Then by definition $\varphi([s, U]) = [\tilde{s}, V]$.

(iii) Let $X = \{p\}$ be a single point and $Y = \{x, y\}$ be the topological space with open sets $\emptyset, \{x\}, Y$. Let $f : X \to Y, f(p) = x$, then f is an embedding. Given a non-zero sheaf $\mathcal{F}$ on X, then $(f_*\mathcal{F})_y = \mathcal{F}(p) \neq 0$.

Exercise 13. *Direct image under point inclusion* (3 points)
Let $x \in X$ be an arbitrary point (not necessarily closed) of a topological space X. Is the direct image $(i_x)_*\colon \mathrm{Sh}(\{x\}) \to \mathrm{Sh}(X)$ associated with the inclusion $i_x\colon \{x\} \hookrightarrow X$ exact? Give a proof or a counterexample.

Solution.

We are going to prove that the direct image functor $(i_x)_*$ is exact.

Consider a short exact sequence of sheaves over $\{x\}$

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0.$$

Since it consists of sheaves over a 1-point space, its exactness is equivalent to the exactness of the corresponding sequence of stalks at x, which is just

$$0 \to \mathcal{F}_1(\{x\}) \to \mathcal{F}_2(\{x\}) \to \mathcal{F}_3(\{x\}) \to 0. \tag{1}$$

We want to prove that the sequence obtained by applying $(i_x)_*$ remains exact. To do that, we show that it is exact at each stalk. Let $y \in X$ be any point and consider the sequence

$$0 \to ((i_x)_*\mathcal{F}_1)_y \to ((i_x)_*\mathcal{F}_2)_y \to ((i_x)_*\mathcal{F}_3)_y \to 0. \tag{2}$$

We have:

- If y is not in the closure of $\{x\}$ in X, then each term of that sequence is zero (exercise 12(i)), hence it is exact.
- Else, then any open subset containing y also contains x, thus for any sheaf $\mathcal{F}$ on $\{x\}$ we have $((i_x)_*\mathcal{F})_y = \mathcal{F}(\{x\})$. Thus the sequence (2) equals the sequence (1), hence it is exact.

This concludes the proof that $(i_x)_*$ is an exact functor.

Remark. In fact, for any continuous map $f\colon Y \to X$, the *direct image* functor f_* is left-exact. A short "abstract" proof immediatly follows from exercise 11 and the fact: right adjoint functors are left exact (it is a good exercise to prove that statement). Here is an alternative proof. Consider a short exact sequence of sheaves over Y

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0.$$

For a subset $U \subset X$, we know that the functor "global sections over $f^{-1}(U)$" is left-exact (exercise 8). But for any sheaf $\mathcal{F}$ on Y we have $f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$, thus the following two rows are exact:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_1(f^{-1}(U)) \arrow[r] \arrow[d, equal] & \mathcal{F}_2(f^{-1}(U)) \arrow[r] \arrow[d, equal] & \mathcal{F}_3(f^{-1}(U)) \arrow[d, equal] \\ 0 \arrow[r] & f_*\mathcal{F}_1(U) \arrow[r] & f_*\mathcal{F}_2(U) \arrow[r] & f_*\mathcal{F}_3(U) \end{tikzcd}$$

Since this holds for any open subset $U \subset X$, this shows that f_* is left-exact.

In general f_* is not exact. However, if f is the inclusion of a closed subspace Y into X, then f_* is exact and exhibits an equivalence of categories between sheaves on Y and sheaves on X that are supported on Y (cfr exercise 8). This is a good exercise.

Exercise 14. *Glueing morphisms of locally ringed spaces* (4 points)
Let X and Y be locally ringed spaces. For each $U \subseteq X$ open, let $Hom(U, Y)$ be the set of morphisms $(U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces. Show that $U \mapsto Hom(U, Y)$ defines a sheaf of sets on X.

(In particular: For every open cover $X = \cup_{i \in I} U_i$, giving a morphism of locally ringed spaces $\varphi : X \to Y$ is the same as giving morphisms $\varphi_i : U_i \to Y$ which agree on intersections.)

Solution. The set $\mathrm{Hom}(U, Y)$ consists of pairs $(f, f_\flat)$ where $f : U \to Y$ is a continuous map and $f_\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_U$ is a morphism of sheaves (it is equivalent to the datum of $f^\sharp$ by the adjunction, see exercise 11). The restriction map is given by $\rho^U_V(f, f^\sharp) = (f|_V, f^\sharp|_V)$. The map $f^\sharp|_V$ is explicitly given by the composition $\iota^{-1}f^{-1}\mathcal{O}_Y \to \iota^{-1}\mathcal{O}_U = \mathcal{O}_V$, where ι is the inclusion map $V \hookrightarrow U$. The presheaf conditions are easy to verify.
For locality say given $U = \bigcup_i U_i$ an open cover of U and two pairs $(f, f^\sharp), (g, g^\sharp)$ such that $(f|_{U_i}, f^\sharp|_{U_i}) = (g|_{U_i}, g^\sharp|_{U_i})$ for all i. Then clearly $f = g$, and note $f^{-1}|_{U_i}\mathcal{O}_Y = g^{-1}|_{U_i}\mathcal{O}_Y$, so one can check $f^\sharp = g^\sharp$ on stalks.
For gluability we need to spend more effort. Now we have a bunch of pairs $(f_i, f_i^\sharp)$ defined on each U_i agreeing on $U_i \cap U_j$. It is clear that all f_i glue together to a continuous map $f : U \to Y$. We then have indeed $f_i^{-1}\mathcal{O}_Y = f^{-1}\mathcal{O}_Y|_{U_i}$ and $\mathcal{O}_{U_i} = \mathcal{O}_U|_{U_i}$. Apply the sheaf property of internal hom and we win.

Exercise 15. *Rational points* (4 points)
Let $(X, \mathcal{O}_X)$ be a scheme and let $x \in X$ with residue field $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$.

(i) Show that, for a field K, to give a morphism of schemes $(\mathrm{Spec}(K), \mathcal{O}_{\mathrm{Spec}(K)}) \to (X, \mathcal{O}_X)$ with image x is equivalent to giving a field inclusion $k(x) \hookrightarrow K$.

(ii) If $(X, \mathcal{O}_X)$ is a k-scheme for some field k, i.e. a morphism of schemes

$$(X, \mathcal{O}_X) \to (\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$$

is fixed, show that every residue field $k(x)$ is naturally a field extension $k \subset k(x)$. A point $x \in X$ is *rational* if this extension is bijective, i.e. $k = k(x)$. The *set of rational points* is denoted by $X(k)$. Show that $X(k)$ can be described as the set of k-morphisms $\mathrm{Spec}(k) \to (X, \mathcal{O}_X)$, i.e. morphisms such that the composition

$$(\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)}) \to (X, \mathcal{O}_X) \to (\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$$

is the identity of schemes.

Solution.

(i) To give a morphism of schemes is to give a continuous map of topological spaces $f : \mathrm{Spec}(K) \to X$ together with a morphism of sheaves $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{\mathrm{Spec}(K)}$. Since the underlying topological space of $\mathrm{Spec}(K)$ is just one point, the data of a continuous map f is the same as a choice of a point $x \in X$. Now, having fixed this $x \in X$, if we are given a map $f^\sharp$ as above, we get the induced map on stalks $f_x^\sharp : \mathcal{O}_{X,x} \to (f_*\mathcal{O}_{\mathrm{Spec}(K)})_x = \mathcal{O}_{\mathrm{Spec}(K)}(\mathrm{Spec}(K)) \cong K$ *or use Ex 12. (ii)*. Since we started with a map of locally ringed spaces, this map $f_x^\sharp$ is local, which means in this case that $f_x^\sharp(\mathfrak{m}_x) = 0$. Hence $f_x^\sharp$ descends to a map $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \to K$, which is the desired field extension. Conversely, given such a field extension $k(x) \hookrightarrow K$, we define a morphism of sheaves $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{\mathrm{Spec}(K)}$ by the following recipe: for any open subset $U \subset X$ containing x, we take the composite $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \twoheadrightarrow k(x) \to K \cong \mathcal{O}_{\mathrm{Spec}(K)}(\mathrm{Spec}(K)) = (f_*\mathcal{O}_{\mathrm{Spec}(K)})(U)$; if U doesn't contain x, we take the zero map (we don't really have much choice as $(f_*\mathcal{O}_{\mathrm{Spec}(K)})(U) = 0$). Now, one easily sees that these constructions are inverse to each other.

(ii) Given a morphism of schemes $g : (X, \mathcal{O}_X) \to (\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$, we in particular have a map $g^\sharp : \mathcal{O}_{\mathrm{Spec}(k)} \to g_*\mathcal{O}_X$. Evaluating the latter at $\mathrm{Spec}(k)$, we get a map $k \cong \mathcal{O}_{\mathrm{Spec}(k)}(\mathrm{Spec}(k)) \to (g_*\mathcal{O}_X)(\mathrm{Spec}(k)) = \mathcal{O}_X(X)$, which we can compose with the natural map $\mathcal{O}_X(X) \to \mathcal{O}_{X,x} \twoheadrightarrow k(x)$ to get a field extension $k \subset k(x)$, for any $x \in X$.

Assume now that we are given a k-morphism $f : \mathrm{Spec}(k) \to (X, \mathcal{O}_X)$, i.e. we have a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "f"] \arrow[dr, "\mathrm{id}"'] & X \arrow[d, "g"] \\ & \mathrm{Spec}(k) \end{tikzcd}$$

This in particular yields a diagram of sheaves on $\mathrm{Spec}(k)$

$$\begin{tikzcd} \mathcal{O}_{\mathrm{Spec}(k)} & g_*\mathcal{O}_X \arrow[l, "g_*(f^\sharp)"'] \\ & \mathcal{O}_{\mathrm{Spec}(k)} \arrow[ul, "\mathrm{id}"] \arrow[u, "g^\sharp"'] \end{tikzcd}$$

Taking global sections, this last diagram gives

$$\begin{tikzcd} k & \mathcal{O}_X(X) \arrow[l] \\ & k \arrow[ul, "\mathrm{id}"] \arrow[u] \end{tikzcd}$$

Now, by the construction in part (i), we know that the upper horizontal map in this diagram is in fact the composite $\mathcal{O}_X(X) \to \mathcal{O}_{X,x} \twoheadrightarrow k(x) \to k$ (where $k(x) \to k$ is the field extension induced by the map of schemes $\operatorname{Spec}(k) \to X$), and the composition of the right vertical map $k \to \mathcal{O}_X(X)$ with the first part of this composite $\mathcal{O}_X(X) \to \mathcal{O}_{X,x} \twoheadrightarrow k(x)$ is exactly the field extension $k \subset k(x)$, induced by the structure map $X \to \operatorname{Spec}(k)$. Therefore, we finally arrive at the diagram

$$\begin{tikzcd} k & k(x) \arrow[l] \\ & k \arrow[ul, "\mathrm{id}"] \arrow[u] \end{tikzcd}$$

which shows that the field extension $k \subset k(x)$ is trivial, i.e. x is a rational point.

Conversely, if we are given a rational point x, i.e. $k = k(x)$, we just define the map of schemes $\operatorname{Spec}(k) \to X$ with image x that corresponds (by part (i)) to the trivial field extension $k(x) \overset{\mathrm{id}}{\to} k$. This defines a k-morphism $\operatorname{Spec}(k) \to X$.

Finally, you can easily convince yourself that these two constructions are inverse to each other.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 16. *Small non-affine schemes* (3 extra points)
Construct an example of a scheme $(X, \mathcal{O}_X)$ which is not affine and for which X is finite and as small as possible.

Solution. The smallest example has three points. Any scheme with two points is affine: Let X = {p,q}. If X is not discrete, then every open cover of X must contain X itself, hence X is affine. So assume X is discrete. Then $R_p = \mathcal{O}_X(\{p\})$ and $R_q = \mathcal{O}_X(\{q\})$ are local rings (since they are equal to the stalks of the structure sheaf). The product of the restriction maps $\mathcal{O}_X(X) \to R_p \times R_q$ is an isomorphism. Then the map $\operatorname{Spec} R_p \times R_q \to X$ is an isomorphism (not immediately obvious, you can check it).

Smallest example has three points. Idea: glue two copies of $\operatorname{Spec}(k[x]_{(x)})$ along the complement of the closed point. This is the same idea as the affine line with two origins, but restricted to the neighbourhood of the origin(s).

Exercise 17. *Zariski tangent space* (4 points)
Let $(X, \mathcal{O}_X)$ be an scheme. For a point $x \in X$ the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$ is considered as a vector space over the residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. The *Zariski tangent space* T_x of X at $x \in X$ is defined as the dual of this vector space, i.e.

$$T_x \coloneqq (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = \operatorname{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

Assume $(X, \mathcal{O}_X)$ is a k-scheme, where k is a field, and denote the *ring of dual numbers* $k[t]/(t^2)$ by $k[\varepsilon]$.

(i) Show that giving a morphism $(\mathrm{Spec}(k[\varepsilon]), \mathcal{O}_{\mathrm{Spec}(k[\varepsilon])}) \to (X, \mathcal{O}_X)$ that is compatible with the morphisms to $(\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$ is equivalent to giving a rational point $x \in X$ and an element $v \in T_x$.

(ii) Calculate T_0 for $\mathbb{A}^n_k$ and for $\mathrm{Spec}(k[x,y]/(y^2 + x^3)) \subseteq \mathbb{A}^2_k$.

Solution.

(i) Let M be the set of morphisms $(\mathrm{Spec}(k[\varepsilon]), \mathcal{O}_{\mathrm{Spec}(k[\varepsilon])}) \to (X, \mathcal{O}_X)$ that are compatible with the induced morphisms to $(\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$ and

$$N = \{(x, v) \mid x \in X \text{ rational point, } v \in T_x\}.$$

We now construct inverse maps $\phi : M \to N$ and $\psi : N \to M$.

Given $(f, f^\sharp) \in M$. Let $\eta \in \mathrm{Spec}(k[\varepsilon])$ be the point and let $x = f(\eta)$. By assumption, $f^\sharp$ induces a homomorphism of local k-algebras

$$f^\sharp_x : \mathcal{O}_{X,x} \to k[\varepsilon].$$

In particular, $f^\sharp_x(\mathfrak{m}_x) \subset k\varepsilon$ and hence $f^\sharp_x(\mathfrak{m}_x^2) = 0$. Hence $f^\sharp_x$ induces a homomorphism of k-vector spaces $\tilde{f}^\sharp_x \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, where $\tilde{f}$ is defined via

$$f^\sharp_x(m) = \tilde{f}^\sharp_x([m])\varepsilon, \quad \text{for all } m \in \mathfrak{m}_x.$$

We set $\phi(f, f^\sharp) := (x, \tilde{f}^\sharp_x)$.

Conversely, given $(x, v) \in N$. As x is rational, we have a splitting of k-vector spaces $\mathcal{O}_{X,x} = k1 \oplus \mathfrak{m}_x$. There exists a unique homomorphism of local k-algebras

$$u^\sharp : \mathcal{O}_{X,x} \to k[\varepsilon]$$

such that $u(m) = v([m])\varepsilon$ for all $m \in \mathfrak{m}_x$. Indeed, this gives a morphism of vector spaces respecting the unit and $u^\sharp(\mathfrak{m}_x) \subset k\varepsilon$. To see that it is multiplicative let $m_1, m_2 \in \mathfrak{m}_x, \lambda_1, \lambda_2 \in k$. Then, by definition we have

$$\begin{aligned} u^\sharp((\lambda_1 \cdot 1 + m_1)(\lambda_2 \cdot 1 + m_2)) &= \lambda_1\lambda_2 \cdot 1 + v([\lambda_1 m_2 + \lambda_2 m_1 + m_1 m_2])\varepsilon \\ &= \lambda_1\lambda_2 \cdot 1 + v([\lambda_1 m_2 + \lambda_2 m_1])\varepsilon \\ &= u^\sharp(\lambda_1 \cdot 1 + m_1) u^\sharp(\lambda_2 \cdot 1 + m_2). \end{aligned}$$

ALTERNATIVELY: One can define $u^\sharp \colon \mathcal{O}_{X,x} \to k[\epsilon]$ by $f \mapsto \overline{f} + v(f - \overline{f})\epsilon$ where $\overline{f} \in \mathcal{O}_{X,x}/\mathfrak{m} \simeq k$ is the image of f. Note that $f - \overline{f} \in \mathfrak{m}$. To see why this is a map of k-algebras, Note that

$$\begin{aligned} u^\sharp(f)u^\sharp(g) &= (\overline{f} + v(f - \overline{f})\epsilon)(\overline{g} + v(g - \overline{g})\epsilon) \\ &= \overline{f}\overline{g} + (\overline{g}v(f - \overline{f}) + \overline{f}v(g - \overline{g}))\epsilon \\ &= \overline{f}\overline{g} + v(\overline{g}f - \overline{g}\overline{f} + \overline{f}g - \overline{f}\overline{g})\epsilon \\ &= \overline{f}\overline{g} + v(fg - \overline{f}\overline{g} - (g - \overline{g})(f - \overline{f}))\epsilon \\ &= \overline{f}\overline{g} + v(fg - \overline{f}\overline{g})\epsilon \\ &= u^\sharp(fg) \end{aligned}$$

The penultimate equality follows from the fact that $(g - \overline{g})(f - \overline{f}) \in \mathfrak{m}^2$.

Thus, we get a morphism of affine schemes

$$(u, u^\sharp) : (\mathrm{Spec}(k[\varepsilon]), \mathcal{O}_{\mathrm{Spec}(k[\varepsilon])}) \to (\mathrm{Spec}(\mathcal{O}_{X,x}), \mathcal{O}_{\mathrm{Spec}(\mathcal{O}_{X,x})}).$$

Define $\psi(x, v)$ to be the composition of $(u, u^\sharp)$ with the canonical morphism of schemes

$$(\mathrm{Spec}(\mathcal{O}_{X,x}), \mathcal{O}_{\mathrm{Spec}(\mathcal{O}_{X,x})}) \to (X, \mathcal{O}_X).$$

By construction, ϕ and ψ are inverse maps.

(ii) (a) The tangent space of $\mathbb{A}_k^n$ at the origin is given by

$$T_0 = \mathrm{Hom}_k(\mathfrak{m}_0/\mathfrak{m}_0^2, k) = \mathrm{Hom}_k(\mathrm{span}_k(x_1, \dots, x_n), k).$$

In particular, T_0 is of dimension n over k.

(b) We use the isomorphism of k-algebras

$$k[x, y]/(y^2 + x^3) \cong k[t^2, t^3], \quad x \mapsto -t^2, y \mapsto t^3.$$

Then, $\mathfrak{m}_0 = (t^2, t^3)$. In particular, $\mathfrak{m}_0$ has basis $(t^i | i \geq 2)$ and $\mathfrak{m}_0^2$ has basis $(t^i | i \geq 4)$. Hence, $\mathfrak{m}_0/\mathfrak{m}_0^2$ has a k-basis given by (t^2, t^3). So we see that T_0 is of dimension 2.

Exercise 18. *Noetherian topological spaces* (4 points)
A topological space X is called *Noetherian* if it satisfies the descending chain condition for closed subsets, i.e., if for every chain of closed subsets $V_1 \supseteq \dots \supseteq V_i \supseteq \dots$, there exists an $r \geq 1$ such that $V_i = V_r$ for all $i \geq r$.

(i) Show that X is Noetherian if and only if every open subset $U \subseteq X$ is quasi-compact. In particular, Noetherian spaces are quasi-compact and quasi-separated.

(ii) Show that any subset (with the subspace topology) of a Noetherian space X is Noetherian.

(iii) An *irreducible component* of a topological space X is an irreducible subset which is maximal with respect to inclusion of irreducible subsets. Show that, in general, irreducible components are closed. Show that a Noetherian space has finitely many irreducible components.

(iv) Show that the underlying topological space of a Noetherian scheme $(X, \mathcal{O}_X)$ is Noetherian. Give an example that shows that the converse is not true in general.

Solution. First, we remark that a topological space X is Noetherian if and only if it satisfies the ascending chain condition for open subsets, i.e. every increasing chain of open subsets $U_1 \subset U_2 \subset \dots$ eventually stabilizes. This directly follows from the descending chain for closed subsets by taking complements.

(i) Let $U \subset X$ be an open subset and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of U. Assume that it has no finite subcover. Pick any $i_1 \in I$ and let $\Omega_1 := U_{i_1}$. Since $\mathcal{U}$ has no finite subcover, Ω_1 cannot cover U. Thus there exists a point $x_2 \in U - \Omega_1$ and $i_2 \in I$ with $x_2 \in U_{i_2}$. Let $\Omega_2 := U_{i_1} \cup U_{i_2}$. Again, Ω_2 cannot cover U, and a point $x_3 \in U - \Omega_2$ is covered by some U_{i_3} with $i_3 \in I$. Inductively, we construct a strictly increasing chain of open subsets $\Omega_1 \subset \Omega_2 \subset \dots$ where each $\Omega_n = U_{i_1} \cup \dots \cup U_{i_n}$ contains a point x_n not in Ω_{n-1}. This shows that X is not Noetherian.

Conversely, assume that X is not Noetherian. Then there exists a strictly increasing chain of open subsets

$$U_1 \subset U_2 \subset \dots$$

Then $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ is an open cover of the open subset $U := \bigcup_i U_i$, which has no finite subcover (else the chain would stabilize). Thus U is not quasi-compact.

(ii) Let $Y \subset X$ be a subspace of the Noetherian space X. We show that Y is Noetherian by showing that it satisfies the descending chain condition for closed subsets. Let

$$Y_1 \supset Y_2 \supset \dots \tag{3}$$

be a decreasing chain of closed subsets in Y. Then for each n, we have $Y_n = Y \cap Y'_n$ for a closed subset $Y'_n \subset X$. In particular, $X_n := \bigcap_{i=1}^n Y'_n$ form a decreasing chain of closed subsets of X, hence it must stabilize from rank r on. This means that for all $n \geq r$, we have $X_n = X_r$. But

$$X_n \cap Y = \bigcap_{i=1}^{n} (Y'_n \cap Y) = \bigcap_{i=1}^{n} Y_i = Y_n,$$

hence $Y_n = Y_r$ for all $n \geq r$. This shows that Y is Noetherian.

(iii) First, we show that irreducible components are closed. Let $Y \subset X$ be an irreducible component, i.e. a maximal irreducible subset. Let $x \in X - Y$, then $Y \cup \{x\}$ is not irreducible anymore. Thus there exist closed subsets $C_1, C_2 \subset X$ with $Y \cup \{x\} \subset C_1 \cup C_2$ but $Y \cup \{x\} \not\subset C_i$ for $i = 1, 2$. Since Y is irreducible, it is entirely contained in one of C_1 or C_2, assume $Y \subset C_1$. Then $x \notin C_1$, otherwise we would have $Y \cup \{x\}$ contained in C_1. Therefore, $X - C_1$ is an open neighborhood of x which is disjoint from Y. This shows that Y is closed.

Now, we show that a Noetherian space has finitely many irreducible components. We write $I(X)$ for the set of irreducible components of X.

Assume that X has infinitely many irreducible components. In particular, X is not irreducible, hence we can write $X = X_1 \cup X'_1$ for two proper closed subsets X_1, X'_1 of X. An irreducible component of X is either entirely contained in X_1 or in X'_1. Thus $I(X) = I(X_1) \cup I(X'_1)$, thus one of $I(X_1)$ or $I(X'_1)$ must be infinite, say $I(X_1)$ is infinite.

Then X_1 is not irreducible, so we can write $X_1 = X_2 \cup X'_2$ for two proper closed subsets X_2, X'_2 of X_1. Again, $I(X_1) = I(X_2) \cup I(X'_2)$ and $I(X_2)$ is infinite (up to interchanging X_2 and X'_2).

Pursuing this process inductively, we find a strictly decreasing chain of closed subsets

$$X_1 \supset X_2 \supset X_3 \supset \dots$$

in X. Therefore X is not Noetherian.

(iv) Let $(X, \mathcal{O}_X)$ be a Noetherian scheme. This means that X is quasi-compact and is locally Noetherian, i.e. for all open affine $\mathrm{Spec}(A) \subset X$, the ring A is Noetherian. We show that X is a Noetherian topological space in two steps:

1. For a Noetherian ring A, the space $\mathrm{Spec}(A)$ is Noetherian;

2. If X has a finite cover by Noetherian open subspaces, then X is a Noetherian space.

It is clear that (1) and (2) together imply the desired result. Indeed, X has a finite cover by open affine subschemes $\mathrm{Spec}(A_i)$ because it is quasi-compact.

Let us show (1). Recall that there is an order-reversing[1] bijection between closed subsets of $\mathrm{Spec}(A)$ and radical ideals of A. Thus, a descending chain of closed subsets of $\mathrm{Spec}(A)$ corresponds to an ascending chain of radical ideals of A, which much stabilize if A is Noetherian. This shows that $\mathrm{Spec}(A)$ is a Noetherian topological space.

[1] The order is given by inclusion on both sides.

Let us now show (2). Assume that X can be covered by the open subspaces $X_1, \dots, X_n$, each of which is Noetherian. Consider an ascending chain of open subsets $U_1 \subset U_2 \subset \dots$. Then, for each $j = 1, \dots, n$, the intersection of this chain with X_j

$$X_j \cap U_1 \subset X_j \cap U_2 \subset \dots$$

is an ascending chain of open subsets of X_j, hence it stabilizes from some rank $r_j \in \mathbb{N}$ on (because X_j is Noetherian). Let $r := \max\{r_1, \dots, r_n\}$. For all $i \geq r$, we have

$$U_i = \left(\bigcup_{j=1}^{n} X_j\right) \cap U_i = \left(\bigcup_{j=1}^{n} X_j \cap U_i\right) = \left(\bigcup_{j=1}^{n} X_j \cap U_r\right) = U_r,$$

i.e. the chain $U_1 \subset U_2 \subset \dots$ stabilizes. This shows that X is a Noetherian topological space and concludes the proof.

Example that the converse is not true: Let $X = \operatorname{Spec} k[x_1, x_2, x_3, \cdots]/(x_1^2, x_2^2, x_3^2, \cdots)$, i.e. the origin in $\mathbb{A}^\infty$ with a non-reduced scheme structure. The topological space is a point and hence is noetherian, however the chain of ideals $(x_2) \subset (x_2, x_3) \subset \cdots$ defines an infinite chain of closed subschemes that does not stabilize.

Exercise 19. *A non-affine open subscheme of an affine scheme* (3 points)
Let k be a field and consider the affine plane $\mathbb{A}_k^2 = \operatorname{Spec}\ k[x, y]$. Let $0 \in \mathbb{A}_k^2$ be the closed point corresponding to the maximal ideal (x, y) and let $U := \mathbb{A}_k^2 \setminus 0$.

(i) Show that restriction of sections induces an isomorphism $H^0(\mathbb{A}_k^2, \mathcal{O}_{\mathbb{A}_k^2}) \xrightarrow{\sim} H^0(U, \mathcal{O}_U)$.

(ii) Deduce that U is not an affine scheme.

Solution. Not provided.

Exercise 20. *Glueing schemes and morphisms.* (5 points)
A *glueing datum* is a quadruple $(I, \{X_i\}_{i \in I}, \{U_{ij}\}_{i,j \in I}, \{\varphi_{ij}\}_{i,j \in I})$ where I is an index set, each X_i is a scheme, each $U_{ij} \subseteq X_i$ is an open subscheme, and each $\varphi_{ij} : U_{ij} \to U_{ji}$ is an isomorphism of schemes such that for all $i, j, k \in I$ the following three conditions hold:

(a) $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$.

(b) $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$.

(c) $\varphi_{ik}|_{U_{ij} \cap U_{ik}} = \varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}}$

For $x \in X_i$ and $x' \in X_j$ we write $x \sim x'$ if and only if $x \in U_{ij}$, $x' \in U_{ji}$, and $\varphi_{ij}(x) = x'$.

(i) Show that $\sim$ is an equivalence relation. Define $X := (\coprod X_i)/\sim$ and let $\varphi_i : X_i \to X$ be the natural map.

(ii) Define a subset $U \subseteq X$ to be open if and only if $\varphi_i^{-1}(U)$ is open for all i. Show that this defines a topology on X such that each φ_i is a homeomorphism onto its image. Define $U_i := \varphi_i(X_i)$.

(iii) Define $\mathcal{O}_{U_i} := \varphi_{i,*}\mathcal{O}_{X_i}$. Use the φ_{ij} to glue the $\mathcal{O}_{U_i}$ to a sheaf of rings $\mathcal{O}_X$ on X such that $(X, \mathcal{O}_X)$ is a scheme.

(Hint: Recall Exercise 3.)

(iv) Show that the scheme X satisfies the following universal property: For every scheme Y and every collection of morphisms of schemes $f_i : X_i \to Y$ such that $f_j|_{U_{ji}} \circ \varphi_{ij} = f_i|_{U_{ij}}$ for all $i, j \in I$, there exists a unique morphism of schemes $f : X \to Y$ such that $f_i = f \circ \varphi_i$ for all $i \in I$.

(Remark: As a special case, note that if X' is a scheme, $X' = \cup_{i \in I} X_i$ is an open cover with $U_{ij} := X_i \cap X_j$ and $\varphi_{ij} := \mathrm{id}_{U_{ij}}$, then the inclusions $X_i \to X'$ define an isomorphism $f : X \to X'$ by Exercise 14.)

Solution.

(i) We only need to check transitivity: Let $x \in X_i$, $x' \in X_j$ and $x'' \in X_k$ such that $x \sim x'$ and $x' \sim x''$. By definition, $x \in U_{ij}$, $x' \in U_{ji} \cap U_{jk}$, $x'' \in U_{kj}$ such that $\varphi_{ij}(x) = x'$ and $\varphi_{jk}(x') = x''$.

Using *condition (b)*, we obtain that $x = \varphi_{ij}^{-1}(x') \in U_{ij} \cap U_{ik}$. Similarly, $x'' \in U_{ki} \cap U_{kj}$.

By the cocycle condition in (c), we conclude that

$$x'' = \varphi_{jk}|_{U_j i \cap U_j k}(x') = \varphi_{jk}|_{U_j i \cap U_j k} \circ \varphi_{ij}|_{U_i j \cap U_{ik}}(x) = \varphi_{ik}|_{U_{ij} \cap U_{ik}}(x) = \varphi_{ik}(x).$$

(ii) It is immediate that both $\emptyset$ and X are open in this topology.Let U_α for $\alpha \in J$, an index set be open subsets of X. Let $U := \bigcup U_\alpha$. Since $\varphi_i^{-1}(U) = \bigcup \varphi_i^{-1}(U_\alpha)$, $\varphi_i^{-1}(U)$ is open in X_i and hence U is also open. A similar argument ensures that finite intersection of open sets is open, thereby defining a topology.

The maps $\varphi_i \colon X_i \to X$ is continuous and bijective on to its image by definition. *We use the fact that a continuous bijective map is a homeomorphism if and only if it is open.* Needless to say that $U_i := \varphi_i(X_i)$ is open, since $\varphi_j^{-1}(U_i) = U_{ji}$ which by hypothesis is open in X_j for all j. For all other open subset $U \subset X_i$, $\varphi_j^{-1}(\varphi_i(U)) = \varphi_{ij}^{-1}(U_{ij} \cap U)$ which is open in X_j since φ_{ij} is an isomorphism of schemes and hence in particular a homeomorphism.

(iii) On U_i, we define $\mathcal{O}_{U_i} := \varphi_{i*}\mathcal{O}_{X_i}$. Note that in X, $U_i \cap U_j \simeq U_{ij} \simeq U_{ji}$; more precisely $\varphi_i(U_{ij}) = \varphi_j(\varphi_{ji}(U_{ji})) = U_i \cap U_j$. According to Exercise 3, we need a *gluing data* $\psi_{ij} \colon \mathcal{O}_{U_i}|_{U_i \cap U_j} \to \mathcal{O}_{U_j}|_{U_i \cap U_j}$. To this end, notice that we have an isomorphism $\varphi_{ji}^\sharp \colon \mathcal{O}_{U_{ij}} \xrightarrow{\sim} \varphi_{ji*}\mathcal{O}_{U_{ji}}$. Now define

$$\psi_{ij} \colon \mathcal{O}_{U_i}|_{U_i \cap U_j} := \varphi_{i*}(\mathcal{O}_{U_{ij}}) \xrightarrow{\varphi_{ji}^\sharp} \varphi_{i*}(\varphi_{ji*}\mathcal{O}_{U_{ji}}) \simeq \varphi_{j*}(\mathcal{O}_{U_{ji}}) =: \mathcal{O}_{U_j}|_{U_i \cap U_j}.$$

One should now check that ψ_{ij}'s satisfy the cocycle condition and apply Exerc 3.

(iv) Since X_i is isomorphic to U_i as schemes, by abuse of notation we let $f_i \in Hom(U_i, Y)$. Since $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, we obtain a global section f of the sheaf on X given by $U \mapsto Hom(U, Y)$, i.e. $f \in Hom(X, Y)$. By definition of sheaf, such f is *unique.*

An Example(The Twisted Cubic): $\mathbb{P}^1 \to \mathbb{P}^3$ given by sending $(s : t) \to (s^3 : s^2 t : s t^2 : t^3)$. Show that $t \mapsto (1 : t : t^2 : t^3)$ and $s \mapsto (s^3 : s^2 : s : 1)$ from $\mathbb{A}^1 \to \mathbb{P}^3$ glue to give the parametrization of the twisted cubic $\mathbb{P}^1 \to \mathbb{P}^3$.

Exercise 21. *Reduced schemes and reduction of schemes* (4 points)
Let X be a scheme. The *reduction* of X is a reduced scheme X_{red} together with a morphism $\iota : X_{\mathrm{red}} \to X$ such that every morphism $Z \to X$ from a reduced scheme Z factors uniquely through ι.

(i) Show that if $X = \mathrm{Spec}(A)$ is affine, then $\mathrm{Spec}(A/\mathfrak{N}(A))$, where $\mathfrak{N}(A)$ is the nilradical of A, together with the morphism ι induced by the quotient map $A \to A/\mathfrak{N}(A)$ is the reduction of X.

(ii) Show that every scheme admits a (necessarily unique) reduction. Show that ι is a homeomorphism of topological spaces.

Solution.

(i) Let Z be a reduced scheme and $f : Z \to \mathrm{Spec}(A)$ a morphism. By adjunction, to it corresponds a morphism $A \to \Gamma(Z, \mathcal{O}_Z)$. Now, $\Gamma(Z, \mathcal{O}_Z)$ is reduced, therefore the latter morphism factors uniquely through $A/\mathfrak{N}(A)$, i.e. we have the following commutative triangle

$$\begin{tikzcd} \Gamma(Z, \mathcal{O}_Z) & A \arrow[l] \arrow[d] \\ & A/\mathfrak{N}(A) \arrow[ul, dashed] \end{tikzcd}$$

Employing the adjunction again, we get

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, dashed] & \mathrm{Spec}(A) \\ & \mathrm{Spec}(A/\mathfrak{N}(A)) \arrow[u] \end{tikzcd}$$

which is what we claimed.

Note also that the map $\mathrm{Spec}(A/\mathfrak{N}(A)) \to \mathrm{Spec}(A)$ is a homeomorphism because every prime ideal of A contains $\mathfrak{N}(A)$.

(ii) In the general case, we construct the reduction by gluing. Take an open affine cover $X = \cup_{i \in I} \tilde{X}_i$ with $\tilde{X}_i \cong \mathrm{Spec}(A_i)$. Put $X_i := \mathrm{Spec}(A_i/\mathfrak{N}(A_i))$. Then, for each $i \in I$, we have the reduction map $\varphi_i : X_i \to \tilde{X}_i \subset X$. Put $\tilde{U}_{ij} := \tilde{X}_i \cap \tilde{X}_j$ and $\tilde{\varphi}_{ij} : \tilde{U}_{ij} \xrightarrow{\mathrm{id}} \tilde{U}_{ji}$.

Next, we take $U_{ij} := \varphi_i^{-1}(\tilde{U}_{ij})$ and define (for each i and j) $\varphi_{ij} : U_{ij} \to U_{ji}$ to be the unique morphism of schemes filling the square

$$\begin{tikzcd} U_{ij} \arrow[r, "\varphi_i"] \arrow[d, dashed, "\varphi_{ij}"] & \tilde{U}_{ij} \arrow[d, "\tilde{\varphi}_{ij}=\mathrm{id}"] \\ U_{ji} \arrow[r, "\varphi_j"] & \tilde{U}_{ji} \end{tikzcd} \tag{4}$$

Note that to construct φ_j we are using Ex. 23.

Indeed, this morphism exists as U_{ij} is a reduced scheme (as an open subscheme of a reduced scheme), and the morphism $\varphi_j : U_{ji} \to \tilde{U}_{ji}$ satisfies the universal property of a reduction of a scheme, as it is just the restriction (on the target) of the reduction map $\varphi_i : X_i \to \tilde{X}_i$. Then, of course, since φ_{ij} is constructed in this way for all i, j, we must have that φ_{ij} and φ_{ji} are inverses of each other, so φ_{ij} is an isomorphism for all $i, j \in I$.

(Note also that all the morphisms $\varphi_i : U_{ij} \to \tilde{U}_{ij}$ are homeomorphisms on the underlying topological spaces.)

Therefore, we get a gluing datum $(I, \{X_i\}_{i \in I}, \{U_{ij}\}_{i,j \in I}, \{\varphi_{ij}\}_{i,j \in I})$. Let's check that it is indeed a gluing datum.

(a) $U_{ii} = \varphi_i^{-1}(\tilde{U}_{ii} = \tilde{X}_i) = X_i$.
(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ holds by definition of U_{ij} and by staring at diagram (4).
(c) We can also consider the following modification of diagram (4):

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[r, "\varphi_i"] \arrow[d, dashed, "?"] & \tilde{U}_{ij} \cap \tilde{U}_{ik} \arrow[d, "\tilde{\varphi}_{ij}=\mathrm{id}"] \\ U_{ji} \cap U_{jk} \arrow[r, "\varphi_j"] & \tilde{U}_{ji} \cap \tilde{U}_{jk} \end{tikzcd}$$

(note that here $\tilde{U}_{ij} \cap \tilde{U}_{ik} = \tilde{U}_{ji} \cap \tilde{U}_{jk} = \tilde{X}_i \cap \tilde{X}_j \cap \tilde{X}_k$) and by the same reasoning conclude that there exists a unique dashed arrow filling this square. But we notice (implicitly using (*b*) here, so that $\varphi_{jk}|_{U_{ji}\cap U_{jk}} \circ \varphi_{ij}|_{U_{ij}\cap U_{ik}}$ is well-defined on $U_{ij} \cap U_{ik}$) that both $\varphi_{ik}|_{U_{ij}\cap U_{ik}}$ and $\varphi_{jk}|_{U_{ji}\cap U_{jk}} \circ \varphi_{ij}|_{U_{ij}\cap U_{ik}}$ qualify for the dashed arrow and hence, they are equal.

Therefore, we get a glued scheme X_{red} and, by the universal property of gluing of schemes, we get a unique morphism of schemes $\varphi : X_{\mathrm{red}} \to X$ such that for all $i \in I$, the restriction $X_i \subset X_{\mathrm{red}} \xrightarrow{\varphi} X$ is equal to $\varphi_i : X_i \to \tilde{X}_i \subset X$.

Clearly, X_{red} is reduced, since reducedness can be checked on stalks.

Also, $\varphi : X_{\mathrm{red}} \to X$ is a homeomorphism, as being a homeomorphism is a property local on the target and $\varphi^{-1}(\tilde{X}_i) = X_i$, and we've already noted in part (i) that $X_i \xrightarrow{\varphi|_{X_i}=\varphi_i} \tilde{X}_i$ is a homeomorphism.

Finally, let's check that $\varphi : X_{\mathrm{red}} \to X$ satisfies the desired universal property. Let $g : Z \to X$ be any morphism, where Z is a reduced scheme. Denoting $Z_i := g^{-1}(\tilde{X}_i)$, we can then consider the restrictions $g|_{Z_i} : Z_i \to \tilde{X}_i$, for each $i \in I$. By part (i), we obtain a unique morphism $Z_i \to X_i$ making the left triangle in the following diagram commute (the right square commutes by construction)

$$\begin{tikzcd} & X_i \arrow[r, hook] \arrow[d, "\varphi_i"] & X_{\mathrm{red}} \arrow[d, "\varphi"] \\ Z_i \arrow[ur, dashed] \arrow[r, "g|_{Z_i}"'] & \tilde{X}_i \arrow[r, hook] & X \end{tikzcd}$$

Composing the dashed arrow with the open immersion $X_i \hookrightarrow X_{\mathrm{red}}$, we get morphisms $Z_i \to X_{\mathrm{red}}$, for each $i \in I$, which after postcomposition with φ are equal to $g|_{Z_i} : Z_i \to X$. For different i and j, these morphisms agree on the overlaps $Z_i \cap Z_j$ because again, we can consider the following modified diagram

$$\begin{tikzcd} & X_i \cap X_j \arrow[r, hook] \arrow[d] & X_{\mathrm{red}} \arrow[d, "\varphi"] \\ Z_i \cap Z_j \arrow[ur, dashed, "?"] \arrow[r, "g|_{Z_i \cap Z_j}"'] & \tilde{X}_i \cap \tilde{X}_j \arrow[r, hook] & X \end{tikzcd}$$

and ask for existence of the dashed arrow (which we know exists and is unique), and then notice that both our candidates qualify for the dashed arrow, hence are equal. Therefore, the morphisms $Z_i \to X_{\mathrm{red}}$ glue to a morphism $h : Z \to X_{\mathrm{red}}$ such that $\varphi \circ h = g$. And of course such a morphism if unique, because otherwise we get for some $i \in I$, two different morphisms $Z_i \to X_i$, which was excluded above. This finishes the verification that $X_{\mathrm{red}} \to X$ has the desired universal property and thus also the exercise.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 22. *Right-adjoint to global sections* (3 extra points)
In the lecture, we have seen that Spec$(-)$ is right-adjoint to the functor that maps a locally ringed space to its ring of global sections. Can you find the right-adjoint to the functor that maps a ringed space to its ring of global sections?

Solution. Not Provided

Exercise 23. *Factoring morphisms through subschemes* (4 points)
Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes.

(i) Let $(\iota, \iota^\sharp) : (U, \mathcal{O}_U) \to (Y, \mathcal{O}_Y)$ be the open immersion of an open subscheme. Show that $(f, f^\sharp)$ factors through $(\iota, \iota^\sharp)$ if and only if $f(X) \subseteq U$ (as sets).

(ii) Let $(\iota, \iota^\sharp) : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ be the closed immersion of a closed subscheme. Show that $(f, f^\sharp)$ factors through $(\iota, \iota^\sharp)$ if and only if $f^\sharp$ factors through $\iota^\sharp$. Show that this is automatically satisfied if X is reduced and $f(X) \subseteq Z$ (as sets).

(Remark: The smallest closed subscheme of Y through which $(f, f^\sharp)$ factors is also called *scheme-theoretic image* of $(f, f^\sharp)$)

(iii) (+1 extra point) Give an example of a morphism of schemes such that the underlying set of its scheme-theoretic image is not the closure of its set-theoretic image.

Solution.

(i) Suppose that $f(X) \subset U$ as sets. Define a morphism of schemes $(g, g^\sharp) : (X, \mathcal{O}_X) \to (U, \mathcal{O}_U)$ as follows. On the level of topological spaces take $g = f$, which is possible as $f(X) \subset U$; we clearly then have $f = \iota \circ g$ on the level of topological spaces. Define $g^\sharp : \mathcal{O}_U \to g_*\mathcal{O}_X$ by taking on each $V \subset U$ the composite

$$\mathcal{O}_U(V) \xrightarrow{(\iota^\sharp)^{-1}(V)} \mathcal{O}_Y(V) \xrightarrow{f^\sharp(V)} \mathcal{O}_X(g^{-1}(V))$$

Then we get $(\iota, \iota^\sharp) \circ (g, g^\sharp) = (f, \iota_*(g^\sharp) \circ \iota^\sharp) = (f, f^\sharp)$, where the last equality is literally by definition of $g^\sharp$.

One could also write the same without using V: The adjoint of $\iota^\sharp$ induces an isomorphism by definition $\iota^{-1}(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_U$. Then define $g^\sharp$ to be the adjoint of $g^{-1}(\mathcal{O}_U) \to g^{-1}\iota^{-1}(\mathcal{O}_Y) \simeq f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ where the last map is given by the image of $f^\sharp$ under the adjunction.

(ii) "Factors through $\iota^\sharp$" here means that there exists a morphism of sheaves of rings $\phi : \iota_*\mathcal{O}_Z \to f_*\mathcal{O}_X$ such that $f^\sharp = \phi \circ \iota^\sharp$. First, existence of such a morphism implies that $f_*\mathcal{O}_X$ is supported on Z, since $\iota_*\mathcal{O}_Z$ is and since for each $y \in Y$, the induced morphism on stalks $\phi_y : (\iota_*\mathcal{O}_Z)_y \to (f_*\mathcal{O}_X)_y$ is a ring map, i.e. sends 1 to 1. This means that $f(X) \subset Z$, therefore, on the level of topological spaces, $f = \iota \circ g$ with $g : X \to Z$. Because of this and because Z has the subspace topology of X, we can define a map of sheaves $g^\sharp : \mathcal{O}_Z \to g_*\mathcal{O}_X$ by taking for $V \subset Y$ the map:

$$\mathcal{O}_Z(V \cap Z) = (i_*\mathcal{O}_Z)(V) \xrightarrow{\phi(V)} (f_*\mathcal{O}_X)(V) = (g_*\mathcal{O}_X)(V \cap Z)$$

which is well-defined as ϕ is a morphism of sheaves, i.e. is compatible with the restrictions. Finally, we note that by definition of $g^\sharp$, we have $\phi = \iota_*(g^\sharp)$ and hence $f^\sharp = \iota_*(g^\sharp) \circ \iota^\sharp$, proving that $(f, f^\sharp) = (\iota, \iota^\sharp) \circ (g, g^\sharp)$.

Finally finally, the locality of g comes from the fact that both ι and f are local, and because $\iota^\sharp : \mathcal{O}_Y \to \iota_*\mathcal{O}_Z$ is surjective (hence also surjective on stalks), and of course using that $(\iota_*\mathcal{O}_Z)_y = \mathcal{O}_{Z,y}$ for any $y \in Z$.

Let now X be a reduced scheme and $f(X) \subset Z$ as sets. Again, we have $f = \iota \circ g$ set-theoretically. Denote $\mathcal{I} := \ker(\iota^\sharp : \mathcal{O}_Y \to \iota_*\mathcal{O}_Z)$. To see the claim, we have to show that for any open $V \subset Y$, $\mathcal{I}(V) = \ker(\iota^\sharp(V)) : \mathcal{O}_Y(V) \to (\iota_*\mathcal{O}_Z)(V) = \mathcal{O}_Z(V \cap Z)$ is contained in $\ker(f^\sharp(V)) : \mathcal{O}_Y(V) \to (f_*\mathcal{O}_X)(V) = (g_*\mathcal{O}_X)(V \cap Z)$, using that the map $\iota^\sharp : \mathcal{O}_Y \to \iota_*\mathcal{O}_Z$ is surjective. But we see that for every $s \in \mathcal{I}(V)$ and affine open $U = \operatorname{Spec}(B) \subset X$ with $f(U) \subset V$, the section $t := f^\sharp(V)(s)|_U$ maps to zero in the residue field $k(x)$ for every $x \in U$ (since $s_x \in \mathcal{I}_{f(x)} \subset \mathfrak{m}_{f(x)}$ and hence by locality $t_x \in \mathfrak{m}_x$), or equivalently $t \in \bigcap_{\mathfrak{p} \subset B} \mathfrak{p} = \mathfrak{N}(B) = 0$, where the last equality holds as B is reduced. We win.

(iii) For every $n \in \mathbb{N}$, we have the closed immersion $\operatorname{Spec}(k[x]/(x^n)) \hookrightarrow \mathbb{A}^1$. Therefore, we get a map $\coprod_{n\in\mathbb{N}} \operatorname{Spec}(k[x]/(x^n)) \to \mathbb{A}^1$. It is now clear that whatever closed subscheme structure we give to the point $0 \in \mathbb{A}^1$, this map will not factor through that subscheme structure (since all subscheme structures on $0 \in \mathbb{A}^1$ are precisely given by $\operatorname{Spec}(k[x]/(x^n))$ for some $n \in \mathbb{N}$, and for $m > n$ the map $k[x] \twoheadrightarrow k[x]/(x^m)$ doesn't factor through $k[x]/(x^n)$).

Exercise 24. *Integral and irreducible fibres* (4 points)
Find examples for the following phenomena:

(i) Show that there exist surjective morphisms $X \to Y$ with Y integral and such that all fibres X_y are irreducible without X being irreducible.

(ii) Show that for every algebraically closed field k, there exist morphisms $X \to \operatorname{Spec}(k[x])$ with X integral, the generic fibre X_η non-empty and integral, but no closed fibre integral.

(iii) Show that there exist morphisms $X \to \operatorname{Spec}(\mathbb{Q}[x])$ with X integral and infinitely many irreducible and infinitely many reducible closed fibres. What happens for the geometric closed fibres in your example?

(iv) Show that there exist morphisms $f : X \to Y$ with X and Y integral whose geometric generic fiber is not reduced.

Solution.

(i) Take X to be the union of the axes in $\mathbb{A}^2$ and project onto the x-axis. More precisely, $f\colon X \to Y = \operatorname{Spec} k[x]$, defined by $k[x] \hookrightarrow k[x,y] \to k[x,y]/(xy)$. The fibre over $0 = \operatorname{Spec} \frac{k[x]}{(x)}$ is $\mathbb{A}^1$: since

$$\frac{k[x,y]}{(xy)} \otimes_{k[x]} \frac{k[x]}{(x)} \simeq k[y].$$

But $k[x,y]/(xy) \otimes_{k[x]} k[x]/(x-\lambda) \simeq k[y]/(\lambda y) \simeq k$. Thus, the other fibres are points.

(ii) Take $X = \operatorname{Spec} k[x,y]/(y^2 - x)$. Define the morphism via $f\colon k[x] \to k[x,y]/(y^2 - x)$. The generic fibre, $\operatorname{Spec} k[x,y]/(y^2-x) \otimes_{k[x]} k(x) \simeq \operatorname{Spec} k(x)[y]/(y^2-x)$ is integral since $\sqrt{x} \notin k(x)$ and hence $y^2 - x$ is irreducible over $k(x)$. But the closed fibres are given by

$$\operatorname{Spec} \frac{k[x,y]}{(y^2-x)} \otimes_{k[x]} \frac{k[x]}{(x-\lambda)} \simeq \frac{k[y]}{(y^2-\lambda)}.$$

Since k is algebraically closed, $\sqrt{\lambda} \in k$ and hence the fibre is either two points if $\lambda \neq 0$ or one point with non-reduced scheme structure $k[y]/(y^2)$.

(iii) Take $X = Spec\mathbb{Q}[x,y]/(y^2 - x)$. Using similar tensor product as before one observes that the fibre over $\lambda \in \mathbb{Q}$ is integral whenever $\sqrt{\lambda} \notin \mathbb{Q}$ and is not integral otherwise. The geometric fibres, i.e. fibres over $\overline{\mathbb{Q}}[x]/(x - \lambda)$ are either not irreducible or not reduced.

(iv) Let $\operatorname{Char} k = 2$ and let $Y = \operatorname{Spec} k[x]$ and $Y = \operatorname{Spec} k[x,y]/(y^2 - x)$. The geometric generic fibre is given by

$$\frac{k[x,y]}{(y^2+x)} \otimes_{k[x]} \overline{k(x)} \simeq \frac{\overline{k(x)}[y]}{(y^2+x)}.$$

Spectrum of this consists of a point with non-reduced scheme structure since $\operatorname{Char} k = 2$ and $(y + \sqrt{x})^2 = y^2 + x$.

Exercise 25. *Immersions and base change* (4 points)
Let $f : X \to Y$ be a morphism of schemes.

(i) Show that f is an open (resp. closed) immersion if and only if there exists an open affine cover $Y = \cup_{i \in I} U_i$ such that each $f^{-1}(U_i) \to U_i$ is an open (resp. closed) immersion.

(ii) Show that f is an open (resp. closed) immersion if and only if for all morphisms of schemes $g : Z \to Y$, the induced morphism $Z \times_Y X \to Z$ is an open (resp. closed) immersion.

Solution. Not provided.

Exercise 26. *Generic points and dominant rational maps* (3 points)
A point $\eta \in X$ of a topological space X is called *generic* if $\overline{\eta} = X$ and a continuous map $f : X \to Y$ is called *dominant* if $f(X)$ is dense in Y.

(i) Show that every integral scheme X admits a unique generic point η_X. Show that the local ring $\mathcal{O}_{X,\eta_X}$ is a field. This field is called *function field* of X and is denoted by $k(X)$.

(ii) Show that a morphism $f : X \to Y$ between integral schemes is dominant if and only if $f(\eta_X) = \eta_Y$. Deduce that a dominant morphism of integral schemes induces a field extension $k(Y) \hookrightarrow k(X)$.

Solution.

(i) First, we assume that $X = \operatorname{Spec}(A)$ is an integral affine scheme, i.e. A is an integral domain. Let $\eta = (0) \in \operatorname{Spec}(A)$ be the zero ideal (which is prime because A is integral). We claim that η is the unique generic point of X. To that end, we show the following result:

- **For $\mathfrak{p} \in \operatorname{Spec}(A)$, we have $\overline{\mathfrak{p}} = V(\mathfrak{p})$.** First, $V(\mathfrak{p}) = \{\mathfrak{q} \mid \mathfrak{p} \subset \mathfrak{q}\}$ is a closed subset that contains $\mathfrak{p}$, hence it contains $\overline{\mathfrak{p}}$. Conversely, let $\mathfrak{q} \in V(\mathfrak{p})$, i.e. $\mathfrak{p} \subset \mathfrak{q}$. Any nonempty open subset U containing $\mathfrak{q}$ contains a principal open subset $D(f)$ with $0 \neq f \in A$. Thus $f \in A - \mathfrak{q} \subset A - \mathfrak{p}$, hence $\mathfrak{p}$ is also in $D(f)$. This shows that $\mathfrak{q} \in \overline{\mathfrak{p}}$ and concludes the proof.

From the above, we directly get $\overline{\eta} = V((0)) = X$. For any other point $(0) \neq \mathfrak{p} \in \operatorname{Spec}(A)$, the closure $\overline{\mathfrak{p}} = V(\mathfrak{p})$ does not contains (0) since $\mathfrak{p} \not\subset (0)$. Thus $\eta = (0)$ is the unique generic point in $X = \operatorname{Spec}(A)$.

Now let X be any integral scheme. Let $U = \operatorname{Spec}(A) \subset X$ be an open affine, then $A = \mathcal{O}_X(U)$ is an integral domain. Let $\eta = (0) \in \operatorname{Spec}(A) \subset X$. Since η is dense in U (by the affine case above) and U is dense in X (because X is irreducible), we get that η is dense in X. It remains to show that it is the unique dense point of X. If $U = X$, we are done by the affine case. Else, let x be a point in X. If $x \notin U$, then $\overline{x} \subset X - U \neq X$ and x is not a dense point. If $x \in U = \operatorname{Spec}(A)$, then it is not dense unless $x = \eta$. This concludes the proof that X has a unique generic point.

Now we show that the local ring $\mathcal{O}_{X,\eta}$ is a field. We have $\mathcal{O}_{X,\eta} = \mathcal{O}_{U,\eta} \cong A_{(0)} = \operatorname{Frac}(A)$ is the fraction field of A.

(ii) Let $f\colon X \to Y$ be a morphism between integral schemes, and denote by η_X and η_Y their respective generic point. We show that f is dominant if and only if $f(\eta_X) = \eta_Y$.

- Suppose that $f(\eta_X) = \eta_Y$. Then

$$f(X) = f(\overline{\eta_X}) \subset \overline{f(\eta_X)} = \overline{\eta_Y} = Y$$

  where we used continuity of f for the middle inclusion.

- Now suppose that f is dominant, then

$$Y = \overline{f(X)} = \overline{f(\overline{\eta_X})} \subset \overline{\overline{f(\eta_X)}} = \overline{f(\eta_X)},$$

  hence $f(\eta_X)$ is a dense point in Y, thus it must be the generic point η_Y by uniqueness.

Now assume that f is dominant, so $f(\eta_X) = \eta_Y$. Then it induces a morphism between stalks $f^\#\colon \mathcal{O}_{Y,\eta_Y} \to \mathcal{O}_{X,\eta_X}$ which is a local homomorphim of rings between fields (by (i)), hence it is injective. Therefore, f induces a field extension $k(Y) \xrightarrow{\subset} k(X)$

Exercise 27. *Normalization* (5 points)
A scheme X is *normal* if all its local rings $\mathcal{O}_{X,x}$ are integrally closed domains. The *normalization* of an integral scheme X is an irreducible normal scheme $\tilde{X}$ together with a dominant morphism $\nu : \tilde{X} \to X$ such that every dominant morphism $Z \to X$ from an irreducible normal scheme Z factors uniquely through ν.

(i) Let $X = \operatorname{Spec}(A)$ for an integral domain A, let $\tilde{X} = \operatorname{Spec}(\tilde{A})$, where $\tilde{A}$ is the integral closure of A in its field of fractions $K(X)$, and let $\nu : \tilde{X} \to X$ be the morphism induced by the inclusion $A \subset \tilde{A}$. Show that $\tilde{X}$ together with ν is the normalization of X.

(ii) Show that every integral scheme admits a (necessarily unique) normalization.

Solution. We will imply the following characterization of dominant morphisms:

Lemma 1 (Liu, Exo.2.4.11)**.** Let $f\colon Y \to X$ be a morphism of irreducible schemes, and denote the generic points of X and Y by ξ_X respectively ξ_Y. Then the following are equivalent:

1. f is dominant;
2. $f^\#\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective;

3. for every open subset V of X and every open subset $U \subseteq f^{-1}(V)$, the map $\mathcal{O}_X(V) \to \mathcal{O}_Y(U)$ is injective;
4. $f(\xi_Y) = \xi_X$;
5. $f_X \in f(Y)$.

1. Recall that being normal can be checked on an affine covering Also, *note that an affine scheme $X = \mathrm{Spec}(A)$ is integral if and only if A is an integral domain.* In particular, an affine integral scheme $X = \mathrm{Spec}(A)$ is normal if and only if A is a normal ring, i.e. $A \subseteq K(A)$ is integrally closed, here $K(A)$ denotes the field of fractions of A. This motivates the definition of the normalization in the affine case: Let $\tilde{A} \subseteq K(A)$ be the integral closure of A, which is defined as

$$\tilde{A} \coloneqq \{x \in K(A) \mid \exists \text{ monic } \mu \in A[X] \text{ with } \mu(x) = 0\} \subseteq K(A)$$

It is a classical fact from commutative algebra that $\tilde{A}$ is an integrally closed subring of $K(A)$. We claim that $\tilde{X} \coloneqq \mathrm{Spec}(\tilde{A})$, together with the morphism $\nu \colon \tilde{X} \to X$ induced by the inclusion $A \hookrightarrow \tilde{A}$, is a normalization of $\mathrm{Spec}(A)$.

- *ν is dominant*: We have that since $\tilde{A} \subseteq K(A)$ is a subring of a field, it is in particular an integral domain, and so $\tilde{X} = \mathrm{Spec}(\tilde{A})$ is an integral scheme. So $\tilde{X}$ has a unique generic point, corresponding to (0). Then the claim follow from the topological fact that any morphism $f \colon X \to S$ with S irreducible with $\{\nu \in S \mid \nu \text{ generic}\} \subseteq \mathrm{Im}(f)$ is automatically dominant. (`01RK`)
- *Every dominant morphism $Z \to X$ from an irreducible normal scheme factors over ν:* Let $g \colon Z \to X$ be a dominant morphism from a normal scheme. We which to produce a factorization of the form

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, dashed, "?"'] & & X \\ & \tilde{X} \arrow[ur] & \end{tikzcd}$$

Using the characterization of a dominant morphism from the lemma above, we have that the map $A \to \mathcal{O}_Z(Z)$ is injective. To produce a factorization of the form

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, hook] & & \mathcal{O}_Z(Z) \\ & \tilde{A} \arrow[ur, hook, dashed] & \end{tikzcd}$$

we note that we can use the polynomial relations to get the extensions to $\tilde{A}$. Moreover, the induced map $Z \to \mathrm{Spec}(\tilde{A})$ is automatically dominant, again by the above lemma.

2. This can be done by glueing. The idea is to cover the integral scheme X as $X = \bigcup U_i$ with U_i integral affine schemes. Then each of the U_i admits a (unique) normalization $\nu_i \colon \tilde{U}_i \to U_i$ by what we did above. By the uniqueness of the normalization, we can glue these to a morphism $\tilde{X} \to X$.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 28. *Monomorphisms of schemes* (+ 3 extra points)
Recall that a *monomorphism* in a category $\mathcal{C}$ is a morphism $f : X \to Y$ such that for any two morphisms $g_1, g_2 : Y \to Z$, the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

(i) Show that $f : X \to Y$ is a monomorphism in $\mathcal{C}$, if and only if the fiber product $X \times_Y X$ exists and the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.

(ii) Let $(f, f^\sharp) : X \to Y$ be a morphism of schemes. Assume that f is injective and for all points $x \in X$, the morphism $f_x^\sharp$ is surjective. Show that $(f, f^\sharp)$ is a monomorphism in (Sch). Deduce that open and closed immersions of schemes are monomorphisms.

(iii) Find a monomorphism of schemes which is not a composition of open and closed immersions.

Solution. Not Provided.

Exercise 29. *Properties of diagonals* (4 points)
Let $\mathcal{P}$ be a property of morphisms of schemes that is satisfied by isomorphisms. We say that a morphism $f : X \to Y$ satisfies $\Delta_{\mathcal{P}}$ if its diagonal satisfies $\mathcal{P}$. Show the following:

(i) If $\mathcal{P}$ satisfies (BC), then $\Delta_{\mathcal{P}}$ satisfies (BC).

(ii) If $\mathcal{P}$ satisfies (BC) and (COMP), then $\Delta_{\mathcal{P}}$ satisfies (COMP).

(iii) If $\mathcal{P}$ satisfies (LOCT), then $\Delta_{\mathcal{P}}$ satisfies (LOCT).

(iv) If $\mathcal{P}$ satisfies (BC) and (COMP), $f : X \to Y, g : X \to Z$ are morphisms of schemes over S that satisfy $\mathcal{P}$ and $X \to S$ satisfies $\Delta_{\mathcal{P}}$, then $(f, g) : X \to Y \times_S Z$ satisfies $\mathcal{P}$.

Solution.

i)

Let X, Y, X', Y' be objects in the category such that the following diagram is cartesian.

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

Then we claim the following diagram is also cartesian.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ X' \times_{Y'} X' \arrow[r] & X \times_Y X \end{tikzcd}$$

Let W be an object with morphisms $g : W \to X$ and $h_1, h_2 : W \to X'$ such that $f'h_1 = f'h_2$. We get a diagram.

$$\begin{tikzcd} W \arrow[drr, "g"] \arrow[ddr, "(h_1{,}h_2)"'] & & \\ & X' \arrow[r] \arrow[d] & X \arrow[d] \\ & X' \times_{Y'} X' \arrow[r] & X \times_Y X \end{tikzcd}$$

It suffices then to show $h_1 = h_2$, but we see both h_1, h_2 fit into the diagram below.

$$\begin{tikzcd}
W \arrow[dr, dashed] \arrow[drr, "g"] \arrow[ddr, "f'h_1=f'h_2"'] & & \\
& X' \arrow[r] \arrow[d] & X \arrow[d] \\
& Y' \arrow[r] & Y
\end{tikzcd}$$

Hence $h_1 = h_2$ and we are done.

(ii)

The result follows from the magic diagram:

$$\begin{tikzcd}
X \arrow[r, "\Delta"] & X \times_y X \arrow[r] \arrow[d] & X \times_Z X \arrow[d] \\
& Y \arrow[r] & Y \times_Z Y
\end{tikzcd}$$

(iii)

Let $f : X \to Y$ be a morphism and $Y = \bigcup_i U_i$ an open covering. We see that $f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$ form an open cover of $X \times_Y X$. We win.

(iv)

Not provided.

Exercise 30. *Topological properties of morphisms* (4 points)
Show the following:

(i) "surjective" satisfies (BC).

(ii) "injective","bijective" do not satisfy (BC).

(iii) (+ 1 extra point) "finite fibers" does not satisfy (BC).

(iv) "closed" does not satisfy (BC).

(v) "quasi-compact" and "quasi-separated" do not satisfy (LOCS).

Solution.

(i) Let $f\colon X \to Y$ be a surjective morphism. For any field k, and a map $\operatorname{Spec} k \to Y$, the fibre product map $f_k\colon X_k \to \operatorname{Spec} k$ is surjective. Indeed, let the image of $\operatorname{Spec} k$ be $\operatorname{Spec} k(y)$ for some $y \in Y$ and let B be a $k(y)$-algebra such that $\operatorname{Spec} B$ is an affine open in $X_{k(y)}$. The map $\operatorname{Spec} B \otimes_{k(y)} k$ is a non-trivial *because f is surjective* k-algebra and hence $X_k \to \operatorname{Spec} k$ is surjective.

Now let $Z \to Y$ be any non-trivial map of schemes. Then for every point $z \in Z$, the fibre $f^{-1}(z) \simeq X_{k(z)}$ is non-trivial and hence $X_Z \to Z$ is surjective.

(ii) "injective","bijective" do not satisfy (BC). For instance, consider the bijection $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$. The base change $\operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \to \operatorname{Spec}(\mathbb{C})$ is not injective, since there is an isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \otimes_{\mathbb{R}} \frac{\mathbb{R}[x]}{(x^2+1)} \simeq \frac{\mathbb{C}[x]}{(x^2+1)} \simeq \frac{\mathbb{C}[x]}{(x-i)} \times \frac{\mathbb{C}[x]}{(x+i)} \simeq \mathbb{C} \times \mathbb{C}$$

and hence $\operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ has two points.

(iii) (+ 1 extra point) "finite fibers" does not satisfy (BC). For instance, consider the bijection $\operatorname{Spec}\mathbb{Q}' \to \operatorname{Spec}\mathbb{Q}$ where $\mathbb{Q}'$ is the extension of $\mathbb{Q}$ after adjoining all roots of unity. For any prime p, let ξ_p be the p-th root of unity, then $\mathbb{Q}[\xi] \otimes_{\mathbb{Q}} \mathbb{Q}[\xi] \simeq \prod_{i=1}^{p} \mathbb{Q}'$. This isomorphism can be seen exactly as in (ii). Thus, $\mathbb{Q}' \otimes_{\mathbb{Q}} \mathbb{Q}' \simeq \prod \mathbb{Q}'$ where the product is an infinite product. Thus $\operatorname{Spec}(\mathbb{Q}' \otimes_{\mathbb{Q}} \mathbb{Q}') \to \operatorname{Spec}\mathbb{Q}'$ does not have finite fibre.

(iv) "closed" does not satisfy (BC) Recall that the property "closed" by definition can be checked at the level of topological spaces. Thus the morphism $\mathbb{A}^1 \to \operatorname{Spec} k$ is closed. But the fibre product map $\mathbb{A}^2 \simeq \mathbb{A}^1 \times_k \mathbb{A}^1 \to \mathbb{A}^1$ is not closed, since the image of the closed subset $(xy = 1)$ maps surjectively to $\mathbb{A}^1 \setminus \{0\}$.

(v) "quasi-compact" and "quasi-separated" do not satisfy (LOCS). For instance, consider $\mathbb{A}^\infty \setminus \{0\} \to \operatorname{Spec} k$ is not quasi-compact because the open affine cover given by $\operatorname{Spec} k[x_1, x_2, \cdots][x_i^{-1}]$ does not have a finite subcover, where $\mathbb{A}^\infty \simeq \operatorname{Spec} k[x_1, x_2, \cdots]$. Nonetheless, $\operatorname{Spec} k[x_1, x_2, \cdots][x_i^{-1}]$ is affine and hence is quasi-compact.

For quasi-separatedness, consider X to be the k-scheme given by $\mathbb{A}^\infty_k$ with double origin. This is not quasi-separated. To see this, first recall that if the diagonal $\Delta_{X/k}$ is quasi-compact then any affine cover $\{U_i\}_{i \in I}$ of X should have quasi-compact intersection[2]. *This is Remark 9.12 in the lecture notes.* Let the affine cover be given by two copies U_1 and U_2 of $\mathbb{A}^\infty$ containing one or the other origin. The intersection $U_1 \cap U_2 \simeq \mathbb{A}^\infty \setminus \{0\}$ which as we saw above is not quasi-compact. Nonetheless, each $U_1 \to \operatorname{Spec} k$ and $U_2 \to \operatorname{Spec} k$ is an affine scheme and hence quasi-separated.

Exercise 31. *Morphisms locally of finite type* (5 points)
Consider the property $\mathcal{P}$ = "locally of finite type":

(i) Show that "f_V is locally of finite type" is an affine-local property of open affine subschemes $V \subseteq Y$. Observe that this implies that $\mathcal{P}$ satisfies (LOCT).

(ii) Show that $\mathcal{P}$ satisfies (COMP), (BC), and (CANC).

(iii) Show that $\mathcal{P}$ satisfies (LOCS).

Solution.

1. The statement of this part will obviously follow if we show that, for a map $f : X \to Y$, the following definitions of "locally of finite type" are equivalent:

   (a) There exists an open affine cover $Y = \cup_{i \in I} \operatorname{Spec} A_i$ and, for every $i \in I$, an open affine cover $f^{-1}(\operatorname{Spec} A_i) = \cup_{j \in J_i} \operatorname{Spec} B_{ij}$ such that B_{ij} is a finite type A_i-algebra.

   (b) For every open affine $\operatorname{Spec} A \subset Y$, there exists an open affine cover $f^{-1}(\operatorname{Spec} A) = \cup_{j \in J} \operatorname{Spec} B_j$ such that B_j is a finite type A-algebra.

[2] This condition is also necessary.

(c) For every open affine $\operatorname{Spec} A \subset Y$ and $\operatorname{Spec} B \subset f^{-1}(\operatorname{Spec} A)$, we have that B is a finite type A-algebra.

Let's show that (a) implies (b). For this, by Lemma 6.15, we have to show that the property "there exists an open affine cover $f^{-1}(\operatorname{Spec} A) = \cup_{j\in J} \operatorname{Spec} B_j$ such that B_j is a finite type A-algebra" is an affine-local property of affine open subschemes $\operatorname{Spec} A \subset X$. To this end, let $V = \operatorname{Spec} A \subset Y$ be an open affine subscheme and let $a_1, \ldots, a_n \in A$ be such that $(a_1, \ldots, a_n) = A$. Assume we have an open affine covering $f^{-1}(V) = \cup_{j\in J} \operatorname{Spec} B_j$ with B_j finite type A-algebras; denote the corresponding ring maps $\varphi_j : A \to B_j$. Then we have an affine open covering $f^{-1}(\operatorname{Spec} A_{a_j}) = \cup_{j\in J} \operatorname{Spec}(B_j)_{\varphi_j(a)}$ and $(B_j)_{\varphi_j(a)}$ is a finite type A_a-algebra for any $j \in J$.

Conversely, assume that for all $i \in \{1, \ldots, n\}$ we have open affine covers $f^{-1}(\operatorname{Spec} A_{a_i}) = \cup_{j\in J_i} \operatorname{Spec} B_{ij}$ with B_{ij} a finite type A_{a_i}-algebra. But then we get an open affine cover $f^{-1}(\operatorname{Spec} A) = f^{-1}(\cup_i \operatorname{Spec} A_{a_i}) = \cup_{i,j} \operatorname{Spec} B_{ij}$ (where the first equality holds because $(a_1, \ldots, a_n) = A$) and B_{ij} is a finite type A-algebra for every i, j.

In a similar fashion (by fixing $\operatorname{Spec} A \subset Y$ and considering the property "B is a finite type A-algebra" for affine open subschemes $\operatorname{Spec} B \subset f^{-1}(\operatorname{Spec} A)$), the implication $(b) \implies (c)$ follows by Lemma 6.15 and the following commutative algebra lemma

Lemma 2 (Vakil 5.3.3.)**.** Let B be an A-algebra and $(b_1, \ldots, b_n) = B$. Then B is a finite type A-algebra if and only if all B_{b_i} are finite type A-algebras.

2. Let's show that $\mathcal{P}$ satisfies (COMP). Let $f : X \to Y$ and $g : Y \to Z$ be locally of finite type. Take any open affine $\operatorname{Spec} A \subset Z$. As g has $\mathcal{P}$, there exists an open affine cover $g^{-1}(\operatorname{Spec} A) = \cup_{i\in I} \operatorname{Spec} B_i$ with B_i a finite type A-algebra for any $i \in I$. As f has $\mathcal{P}$, for any $i \in I$ we have a cover $f^{-1}(\operatorname{Spec} B_i) = \cup_{j\in J_i} \operatorname{Spec} C_j$ with C_j a finite type B_i-algebra for any $i \in I$, $j \in J_i$. Therefore we get a cover $(g \circ f)^{-1}(\operatorname{Spec} A) = \cup_{i\in I, j\in J_i} \operatorname{Spec} C_j$ and C_j is a finite A-algebra for any $i \in I$, $j \in J$, i.e. $g \circ f$ has $\mathcal{P}$.

   Let's show that $\mathcal{P}$ satisfies (BC). Let $f : X \to S$ have $\mathcal{P}$ and $g : S' \to S$ be any morphism. Consider a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

   Pick any open affine cover $S = \cup_{i\in I} \operatorname{Spec} A_i$ and for each $i \in I$, open affine covers $f^{-1}(\operatorname{Spec} A_i) = \cup_{j\in J_i} \operatorname{Spec} B_j$ and $g^{-1}(\operatorname{Spec} A_i) = \cup_{k\in K_i} \operatorname{Spec} C_k$, such that B_j is a finite type A_i-algebra for any $i \in I$, $j \in J$. Then, by construction of fiber products, $X' = \cup_{i\in I, j\in J_i, k\in K_i} \operatorname{Spec}(C_k \otimes_{A_i} B_j)$ is an affine cover of X', and moreover, $C_k \otimes_{A_i} B_j$ is a finite type C_k-algebra, i.e. f' has $\mathcal{P}$.

   Let's show that $\mathcal{P}$ satisfies (CANC). This is in the same spirit and follows from the fact that if a composition of ring maps $A \to B \to C$ is of finite type, then the second map $B \to C$ is of finite type.

3. One direction follows from an easy observation that open immersions have $\mathcal{P}$. Let's show the more difficult direction. To this end, take a morphism $f : X \to Y$ and let $X = \cup_{i\in I} U_i$ be an open cover such that $f|_{U_i} : U_i \to Y$ has $\mathcal{P}$ for all $i \in I$. As $\mathcal{P}$ satisfies (LOCT), WLOG can assume that $Y = \operatorname{Spec} A$ (also uses the forward direction of (iii)). For each $i \in I$, we can now find an cover $U_i = \cup_{j\in J_i} \operatorname{Spec} B_j$ with B_j being a finite type A-algebra. Then $X = \cup_{i\in I, j\in J_i} \operatorname{Spec} B_j$ is a cover of X and all of the B_j are finite type A-algebras, i.e. f has $\mathcal{P}$.

Exercise 32. *Diagonals are immersions* (4 points)
Let $f : X \to Y$ be a morphism of schemes.

1. Show that if f is affine, then $\Delta_{X/Y}$ is a closed immersion.
2. Show that $\Delta_{X/Y}$ is an immersion. Deduce that $\Delta_{X/Y}$ is a closed immersion if and only if its set-theoretic image is closed.
3. Show that if f is an immersion, then $\Delta_{X/Y}$ is an isomorphism. [3]
4. For each of the following properties $\mathcal{P}$, show that if $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ satisfies $\mathcal{P}$ and g is separated, then f satisfies $\mathcal{P}$: "quasi-compact", "quasi-separated", "closed immersion", "of finite type", "quasi-finite", "affine", "finite", "separated", "proper"

Solution.

1. Note that if $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a morphism of affine schemes, then the diagonal on global sections corresponds to the map
$$B \otimes_A B \to B,\ b_1 \otimes b_2 \mapsto b_1 b_2.$$
This map is surjective. In particular, we have $B \cong (B \otimes_A B)/J$ for some ideal $J \subseteq B \otimes_A B$, and thus $\Delta_{\operatorname{Spec}(B)/\operatorname{Spec}(A)}$ is a closed immersion. The general statement for affine maps now follows from closed immersion being a local property on the target.

2. Let
$$W := \bigcup_{\substack{U \subset Y \text{ open} \\ V \subset X \text{ open} \\ f(V) \subseteq U}} V \times_U V \subseteq X \times_Y X.$$
Recall from the construction of the fiber product in the lecture that each of the $V \times_U V \subseteq X \times_Y X$ is an affine open; in particular, W is an open subset of $X \times_Y X$. Moreover, $\Delta_{X/Y}$ clearly (?) factors over W. Now being a closed immersion is local on the target, so it suffices to see that the restriction
$$V \to V \times_U V$$
is a closed immersion, which is part (i) of the exercise.

3. Not provided

4. All of the given properties are closed under composition and base change, and a closed immersion has all of these properties. Consider now the diagram
$$\begin{tikzcd}
X \arrow[r, equal] & X \times_Y Y \arrow[r, "\mathrm{id}\times\mathrm{id}"] \arrow[dl, "\mathrm{pr}"'] & X \times_Z Y \arrow[r, "(g\circ f)\times\mathrm{id}"] \arrow[dl, "f\times\mathrm{id}"] \arrow[dr, "\mathrm{pr}_1"'] & Z \times_Z Y \arrow[r, equal, "\mathrm{pr}_2"] \arrow[dr, "\mathrm{pr}_1"] & Y \\
Y \arrow[r, "\Delta_{Y/Z}"'] & Y \times_Z Y & & X \arrow[r, "g\circ f"'] & Z
\end{tikzcd},$$
where both squares are cartesian. By assumption, the bottom two arrows have $\mathcal{P}$, and since $\mathcal{P}$ is stable under base change, so do the induced maps in the cartesian squares. As $\mathcal{P}$ is stable under the composition, we can conclude that the morphism from the upper row satisfies $\mathcal{P}$ altogether. It now remains that it is given by f, which can be done by hand.

[3] This occurred as a special case of the bonus exercise on the previous sheet, but you can prove it directly.

Exercise 33. *Morphisms into separated schemes* (4 points)
Let $f_1, f_2 : X \to Y$ be morphisms of schemes over S. In this exercise, we want to study the locus of points in X where f_1 and f_2 coincide and endow it with a scheme structure.

(i) Let $\iota : \mathrm{eq}(f_1, f_2) \to X$ be the equalizer of f_1 and f_2, i.e., $\mathrm{eq}(f_1, f_2)$ is a scheme and ι is a morphism with $f_1 \circ \iota = f_2 \circ \iota$ satisfying the following universal property:

For all $g : Z \to X$ such that $f_1 \circ g = f_2 \circ g$, the morphism g factors uniquely through ι.

Show that ι exists and that it is an immersion.

(Hint: Rephrase the universal property of ι as a universal property of a fiber product)

(ii) Assume that Y is separated over S. Show that ι is a closed immersion.

(iii) Conclude that if X is reduced, Y is separated, and f_1 and f_2 coincide on an open dense subset of X, then $f_1 = f_2$.

(iv) Give a counterexample to (iii) if Y is not separated.

Solution.

1. We define the pair $(\mathrm{eq}(f_1, f_2), \iota)$ as the fiber product:

$$\begin{tikzcd} \mathrm{eq}(f_1,f_2) \arrow[r, "\phi"] \arrow[d, "\iota"] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "f_1 \times_S f_2"] & Y \times_S Y \end{tikzcd}$$

Let $g : Z \to X$ s.t. $f_1 \circ g = f_2 \circ g =: p$. It follows

$$\Delta_Y \circ p = (f_1 \times_S f_2) \circ g.$$

Hence there exists a unique $q : Z \to \mathrm{eq}(f_1, f_2)$ such that $g = \iota \circ q$ and $p = \phi \circ q$. Next, we show that q is already unique *when we just assume $g = \iota \circ q$*. So suppose $q' : Z \to \mathrm{eq}(f_1, f_2)$ also satisfies $g = \iota \circ q'$. Recall that Δ_Y is an immersion and hence separated. It follows that Δ_Y is a monomorphism. Thus, using

$$\Delta_Y \circ \phi \circ q' = (f_1 \times_S f_2) \circ \iota \circ q' = (f_1 \times_S f_2) \circ \iota \circ q = \Delta_Y \circ \phi \circ q,$$

we deduce $p = \phi \circ q = \phi \circ q'$. This implies $q = q'$ by the universal property of fiber products.

Since Δ_Y is an immersion *by Ex. 32 (ii) above* and immersions are stable under base change, ι is also an immersion.

2. By definition of separatedness, Δ_Y is a closed immersion. As closed immersions are stable under base change (BC) we deduce that also ι is a closed immersion.

3. Suppose $U \subset X$ open dense with $f_1 = f_2$ on U. Then, (ii) gives that $\mathrm{eq}(f_1, f_2)$ is a closed subscheme of X containing U. Thus, the underlying topological space of $\mathrm{eq}(f_1, f_2)$ is X. Since X is reduced we conclude that $\mathrm{eq}(f_1, f_2) = X$ as schemes.

4. Let L be the affine line with two origins and $f_1, f_2 : \mathbb{A}^1 \to L$ be the two standard inclusions. Then, f_1 and f_2 coincide on $\mathbb{A}^1 \setminus 0$, but $f_1 \neq f_2$.

5. In our discussions, the question came up if the assumption X is reduced is necessary for this to hold. It should be necessary. First, recall that we say that an open subscheme $U \subseteq X$ is called *schemetically dense* if for every open subscheme $V \subseteq X$, the scheme-theoretic closure of $U \cap V$ in V is given by all of V. In particular, we can extend Tils solution to see that if f and g agree on a *schematically* dense open subscheme, then they are already equal. Moreover, if the scheme is reduced and locally noetherian, then for a subscheme to be schematically dense it suffices that it is (topologically) dense as a subset.

   To find a counterexample in the case for of a non-reduced scheme, we thus need to find a scheme that is dense but not schematically dense: Let $X = \mathrm{Spec}(k[x,y]/(x^2, xy))$, and consider the embedding of $U = D(y)$. Then we have the factorization

$$U \hookrightarrow \mathrm{Spec}(k[y]) \hookrightarrow X.$$

   Now the maps

$$f_1, f_2 \colon X \rightrightarrows \mathrm{Spec}(k[t])$$

   induced by $t \mapsto y$ and $t \mapsto x + y$ agree on $D(y)$ but not on all of X.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 34. *Base change as a functor* (+ 3 extra points)
Consider a morphism of schemes $S \to T$. Every S-scheme X, i.e. every morphism $X \to S$, yields via composition with $S \to T$ a T-scheme $X \to S \to T$ which we shall denote ${}_TX$. Conversely, to every T-scheme $Y \to T$ base change defines an S-scheme $Y_S \coloneqq S \times_T Y \to S$.

(i) Show that this defines two functors

$${}_T(\ )\colon (Sch/S) \to (Sch/T),\ X \mapsto {}_TX \text{ and } (\ )_S \colon (Sch/T) \to (Sch/S),\ Y \mapsto Y_S$$

which are adjoint to each other. More precisely, ${}_T(\ )$ is left adjoint to $(\ )_S$, i.e. ${}_T(\ ) \dashv (\ )_S$.

For any S-scheme X consider the functor

$$\begin{aligned} \mathbf{Res}_{S/T}(X)\colon \quad (Sch/T)^{\mathrm{op}} &\to (Sets) \\ Y &\mapsto \mathrm{Mor}_S(Y_S, X). \end{aligned}$$

If this functor is representable by a T-scheme, which will be denoted by $\mathrm{Res}_{S/T}(X)$, it is called the *Weil restriction* and satisfies $h_{\mathrm{Res}_{S/T}(X)} \cong \mathbf{Res}_{S/T}(X)$.

One can show that the Weil restriction exists for finite field extensions $S = \mathrm{Spec}(K) \to T = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(A)$, where A is a finite type K-algebra. In this case, one writes $\mathrm{Res}_{K/k}(X)$ for the Weil restriction.

(ii) Set $\mathbb{S} \coloneqq \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_{\mathbb{C}} \setminus 0)$. Show that the rational points of $\mathbb{S}$ can be described as

$$\mathbb{S}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R},\ a^2 + b^2 \neq 0 \right\}.$$

(Remark: Usually, i.e. for general $S \to T$ and X, the Weil restriction does not exist.)

Solution. Not provided.

Recall:

Let $\mathcal{P}$ be a property of morphisms of schemes. Then, we say that $\mathcal{P}$ satisfies

(COMP) if it is *stable under composition*, that is, if for all morphisms $f : X \to Y$ and $g : Y \to Z$ that satisfy $\mathcal{P}$, also $g \circ f$ satisfies $\mathcal{P}$.

(CANC) if it is *cancellable*, that is, if for all morphisms $f : X \to Y$ and $g : Y \to Z$ such that $g \circ f$ satisfies $\mathcal{P}$, also f satisfies $\mathcal{P}$.

(BC) if it is *stable under base change*, that is, if for all morphisms $f : X \to Y$ and $g : Y' \to Y$ such that f satisfies $\mathcal{P}$, also $f_{Y'}$ satisfies $\mathcal{P}$.

(LOCT) if it is (Zariski-)*local on the target*, that is, if for all morphisms $f : X \to Y$ and all open covers $Y = \cup_{i \in I} V_i$, the morphism f satisfies $\mathcal{P}$ if and only if f_{V_i} satisfies $\mathcal{P}$ for all $i \in I$.

(LOCS) if it is (Zariski-)*local on the source*, that is, if for all morphisms $f : X \to Y$ and all open covers $X = \cup_{i \in I} U_i$, the morphism f satisfies $\mathcal{P}$ if and only if $f|_{U_i}$ satisfies $\mathcal{P}$ for all $i \in I$.

Note that $\Delta_{\mathcal{P}}$ is also a property of a morphism $f\colon X \to Y$. For example, $\Delta_{\mathcal{P}}$ satisfies (BC) means, given any base change $f_{Y'}\colon X_{Y'} \to Y'$ of f, $\Delta_{X_{Y'}/Y'}$ satisfies $\mathcal{P}$. Similarly $\Delta_{\mathcal{P}}$ satisfies (LOCT) means that for every open covering $Y = \cup_{i \in I} V_i$, the map $\Delta_{X/Y}$ satisfies $\mathcal{P}$ if and only if the map $\Delta_{X_{V_i}/V_i}$ satisfies $\mathcal{P}$ for all $i \in I$ and so on.

Exercise 35. *Valuation rings* (5 points)
In this exercise, we want to understand why valuation rings are called "valuation" rings and why discrete valuation rings (DVRs) are indeed "discrete".

1. Let A be a subring of a field K. Then A is a valuation ring in K if and only if for every $x \in K^\times$, either $x \in A$ or $x^{-1} \in A$. Show that the condition is sufficient to ensure that A is a valuation ring.

   [Bonus Exercise (+2 extra points): Show that the condition is also necessary].

2. Let A be a valuation ring in a field K. Let $\Gamma = K^\times/A^\times$ be the group (called the *value group)* formed by quotienting the multiplicative group $K^\times$ by the group $A^\times$ of invertible elements of A. Let $v : K^\times \to \Gamma$ be the quotient map.

   We will see that Γ is a *totally ordered* abelian group (i.e. an abelian group with a total order $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ for any $a, b, c \in \Gamma$) with the order $(\leq)$ defined as follows: for $\gamma, \gamma' \in \Gamma$, represented by $x, x' \in K^\times$, define $\gamma \leq \gamma'$ if $\frac{x'}{x} \in A$.

   Show that $\leq$ is a well-defined total order on Γ. Show that v is a *valuation* (i.e. a group homomorphism satisfying $v(a + b) \geq \min\{v(a), v(b)\}$).

   [Hint: Use the new definition of valuation ring as in (i).]

3. Conversely, let Γ be a totally ordered abelian group and let $v : K^\times \to \Gamma$ be a valuation. Show that $A = \{a \in K^\times \mid v(a) \geq 0\} \cup \{0\}$ is a valuation ring in K with maximal ideal $\mathfrak{m}_A := \{a \in K^\times \mid v(a) > 0\} \cup \{0\}$.

4. Show that in a valuation ring every finitely generated ideal is principal. Thus a Noetherian valuation ring is a local principal ideal domain.

5. Show that a valuation ring A is Noetherian if and only if the associated valuation factors through a cyclic subgroup.

   [Hint: Krull's Intersection Theorem]

   (Note: In particular, either $v(a) = 0$ for all $a \in A$ and then $A = K$, or $v(a) > 0$ for some $a \in A$ and then v factors through a group isomorphic to $\mathbb{Z}$, hence the name "discrete" valuation ring.)

Solution.

1. Recall that the lecture defined a “valuation ring” to be a local integral domain that is maximal in the domination order of its residue field. We have the condition

$$\textit{for } x \in K^\times, \textit{ either } x \in A \textit{ or } x^{-1} \in A \tag{5}$$

from the task.

   - Assume A is a valuation ring in the sense of the lecture. Let $x \in K$ and assume that $x \notin A$. Write A' for the subring generated by A and x, then $A \subsetneq A'$. Because A is a valuation ring, we have that A' has no proper prime ideal that pulls back to $\mathfrak{m}_A$, and thus there is no prime ideal lying over $\mathfrak{m}_A A'$, i.e. $\mathfrak{m}_A A' = A'$. So $x^{-1} \in K$ is integral over A, and the subring A'' generated by A and x^{-1} is an integral extension of A. By “lying over for integral extensions”, there is a prime ideal $\mathfrak{p} \subset A''$ such that $\mathfrak{p} \cap A = \mathfrak{m}_A$, and thus $A = A''$, i.e. $x^{-1} \in A$.
   - Conversely, assume that $A \subseteq K$ is a subring that satisfies (5). Note that the ideals of A are lineraly ordered: if for $a, b \in R$, we have $ab^{-1} \in R$ then $(a) \subseteq (b)$ and vice versa. So A has at most one maximal ideal. Since every non-zero ring has at least one maximal ideal, it follows that A is local. It remains to see that A is maximal for the domination order: Let $A \subseteq B$ be a ring extension, and assume there is a $x \in B \setminus A$. Then $x \notin A$, so by assumption, $x^{-1} \in A$, and since x^{-1} is not a unit in A (precisely because $x \notin A$), we have in fact $x^{-1} \in \mathfrak{m}_A$. So $1 \in \mathfrak{m}_A B$, and so the maximal ideal of B cannot lie over $\mathfrak{m}_A$.

2. To see well-definedness, note that for units $u_1, u_2 \in A^\times$, we have $a/b \in A$ if and only if $u_1/u_2 \cdot a/b \in A$. The relation is antisymmetric, since $a/b \in A$ in conjuction with $b/a \in R$ implies that there is a unit $u \in A^\times$ such that $a/b = c$. Transitivity can also be checked, and totallity follows directly from the assumption (5). The relation is also compatible with the group structure, since $b/a \in A$ implies $bu/au \in A$ for all $u \in K^\times$.

   Finally, let $v\colon K^\times \to \Gamma$ be the quotient map. Then, by construction, $v(a) \leq v(b)$ if and only if $b/a \in A$. Moroever, v is indeed a valuation, since for $a/b \in A$, we have $(a+b)/b = a/b + 1 \in A$, and this $v(a+b) \geq v(b)$.

   Remark: It can be useful to extend the valuation to all of K, by using the augmented valuation $K \to \Gamma \cup \{\infty\}$.

3. Let $v\colon K^\times \to \Gamma$ be the valuation. To see that

$$\mathcal{O} = \left\{a \in K^\times \mid v(a) \geq 0\right\} \cap \{0\}$$

   is a valuation ring, we make again use of (5).

4. We note:

$$\textit{If } A \textit{ is a valuation ring with valuation } v, \textit{ then } v(a) \leq v(b) \textit{ if and only if } b \in (a). \tag{6}$$

   Indeed, for $a \neq 0$, we have $v(b/a) \geq 0$ if and only if $b/a \in A$ if and only if $b \in (a)$; if $a = 0$, then $v(b) \geq \infty$ if and only if $v(b) = \infty$ if and only $b = 0$ if and only if $b \in (0)$, as A is a domain. So for any finitely generated ideal $\mathfrak{a} = (a_1, \dots, a_n)$, there is a minimal element a_i and thus $a_j \in (a_i)$ for all $1 \leq j \leq n$, i.e. $(a_i) = \mathfrak{a}$.

5. Recall Krull's Intersection Theorem for a noetherian local ring A:

$$\textit{Let } \mathfrak{a} \subsetneq A \textit{ be a proper ideal and } M \textit{ a finite } A\textit{-module. Then } \bigcap_{n \geq 0} \mathfrak{a}^n M = 0. \tag{7}$$

- By (7), we have $\bigcap_{n\geq 0}^{\infty} \mathfrak{m}^n = \{0\}$. So in the decreasing (!) chain

$$\ldots \subsetneq \mathfrak{m}_A^3 \subsetneq \mathfrak{m}_A^2 \subsetneq \mathfrak{m}_A \subsetneq A\,,$$

each inclusion is strict. So for every $a \in A$, there is a unique minimal integer $v(a)$ such that $a \in \mathfrak{m}_A^{v(a)}$ but $a \notin \mathfrak{m}_A^{v(a)+1}$ and such that $v(a) = v(au)$ for every unit $u \in A^\times$. So we can extended the assignement $a \mapsto v(a)$ to a unique surjective group homomorphism $K^\times \to \mathbf{Z}$, which is the desired valuation on A.
- Conversely, assume that v factors over $\mathbf{Z}$. Choose a $t \in A$ such that $v(t) = 1$. Then for every $x \in \mathfrak{m}_A$, there is a $n > 0$ such that $v(x/t^n) = 0$, and hence $x = t^n u$ for a unit $u \in A^\times = \ker v$. So in particular, $\mathfrak{m}_A = (t)$. Now, let $\mathfrak{a} \subseteq A$ be any ideal, and consider the set $v(\mathfrak{a} \setminus \{0\})$. Because $\mathfrak{a} \subseteq A$, this is a bounded-below subset of $\mathbf{Z}$, so in particular, it contains a minimal element $k = v(b)$. If $k = 0$, then $b \in \ker(v) = A^\times$, and thus $\mathfrak{a} = (1) = A$. If on the other hand $k > 0$, then again by (6), we have $\mathfrak{a} = (b)$.

Exercise 36. *Morphisms from regular curves to proper schemes* (4 points)
The goal of this exercise is to get a taste of how the valuative criterion for properness is used in practice.

(i) Let S be a locally Noetherian scheme, let X and Y be schemes over S, assume Y is locally of finite type over S, and let $x \in X$ be a point. Show that every morphism of schemes Spec $\mathcal{O}_{X,x} \to Y$ over S is induced by a morphism of schemes $U \to Y$ over S, where $U \subseteq X$ is an open affine neighbourhood of x in X.

(ii) In addition to the assumptions in (i), assume that $\mathcal{O}_{X,x}$ is a valuation ring, X is integral and Y is proper over S. Show that for every open subscheme $U \subseteq X$ such that x is contained in the closure of U, any morphism of S-schemes $U \to Y$ extends uniquely to a morphism $f' \colon U' \to Y$ over S where $U \subset U' \subseteq X$ is a bigger open set containing x.

Solution.

(i) The assertion is equivalent to the following commutative algebra problem: let R be a Noetherian ring and A, B finitely generated R-algerbas. Let $\mathfrak{p} \subset B$ be a prime ideal and $\varphi : A \to B_\mathfrak{p}$ be an R-algebra homomorphism. Then there exists $b \in B \setminus \mathfrak{p}$ such that we have a factorization:

$$\begin{tikzcd} A \arrow[r, "\bar{\varphi}"] \arrow[dr, "\varphi"'] & B_b \arrow[d] \\ & B_\mathfrak{p} \end{tikzcd} \tag{8}$$

Fix a presentation $R[X_1, \ldots, X_n]/(f_1, \ldots, f_m) = A$ and let $\pi : R[X_1, \ldots, X_n] \to A$ the corresponding surjection. Write $\varphi(\pi(X_i)) = \frac{a_i}{b_i}$ for $a_i \in B, b_i \in B \setminus \mathfrak{p}$. In addition, write $f_j = \sum_\nu \lambda_\nu X^\nu$ and let $M > 0$ be a natural number satisfying $\lambda_\nu = 0$ if $|\nu| \geq M$. We set

$$c_j = \sum_\nu \lambda_\nu \prod_{i=1}^n a_i^{\nu_i} \prod_{l=1}^n b_l^{M-\nu_l}.$$

Then there exists $b'_j \in B \setminus \mathfrak{p}$ such that $b'_j c_j = 0$ as $f_j \in \ker(\pi)$. Set $b := b_1 \cdots b_n b'_1 \cdots b'_m$ and define an R-algebra homomorphism

$$\psi : R[X_1, \ldots, X_n] \to B_b, \quad X_i \mapsto \frac{a_i \prod_{j\neq i} b_j \prod_{l=1}^m b'_l}{b}.$$

By construction $\psi(f_j) = 0$ for all j and hence we get an induced homomorphism $\bar{\varphi} : A \to B_b$ making the diagram (8) commute.

(ii) Denote the morphism U by g. Let K be the field of fractions of $\mathcal{O}_{X,x}$ and apply the valuative criterion of properness to the diagram:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{X,x}) \arrow[r] & S \end{tikzcd}$$

This gives us an S-morphism $f : \mathrm{Spec}(\mathcal{O}_{X,x}) \to Y$. According to (i), we can extend f to an open neighborhood U_1 of x. Let $U' := U \cup U_1$. By construction, $f_{|\mathrm{Spec(K)}} = g_{|\mathrm{Spec(K)}}$ which implies $f = g$ on $U' \setminus \{x\}$ as K is also the field of fractions of X.

Exercise 37. *Adjoint functors $f^* \dashv f_*$* (4 points)
Consider a morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. Show that f^* is left adjoint to f_*, i.e. for all $\mathcal{F} \in \mathrm{Mod}(X, \mathcal{O}_X)$ and $\mathcal{G} \in \mathrm{Mod}(Y, \mathcal{O}_Y)$ there exists an isomorphism (functorial in $\mathcal{F}$ and $\mathcal{G}$):

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Solution. Not provided.

Exercise 38. *$M \mapsto \tilde{M}$ and adjunction* (4 points)
Let X be the affine scheme $\mathrm{Spec}(A)$ and consider an A-module M and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. Show that $(A\text{-mod}) \to \mathrm{Mod}(X, \mathcal{O}_X)$, $M \mapsto \tilde{M}$ is left adjoint to $\mathrm{Mod}(X, \mathcal{O}_X) \to (A\text{-mod})$, $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$, i.e. that there exists a functorial (in M and $\mathcal{F}$) isomorphism

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}).$$

Solution. Not Provided.

Exercise 39. *Projection formula* (3 points)
Consider a morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $\mathcal{F} \in \mathrm{Mod}(X, \mathcal{O}_X)$ and $\mathcal{G} \in \mathrm{Mod}(Y, \mathcal{O}_Y)$. Suppose $\mathcal{G}$ is locally free of finite rank. Show that there exists a natural isomorphism

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}.$$

Solution. Not provided.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 40. *Valuative criteria with DVRs* (+ 6 extra points)
The goal of this exercise is to prove a valuative criterion for morphisms of finite type between locally Noetherian schemes.

(i) Let A be a Noetherian local integral domain with field of fractions K. Let L be a finitely generated field extension of K. Show that there exists a discrete valuation ring B with field of fractions L that dominates A. You can follow the following steps:

  (a) If L is not finite over K, let $x_1, \dots, x_n$ be a transcendence basis of L over K and replace A by a suitable localization of $A[x_1, \dots, x_n]$ to reduce to the case where L is finite over K.

(b) Take any valuation ring dominating A in K and let v be the corresponding valuation. Let $x_1, \dots, x_n$ be a minimal set of generators of the maximal ideal $\mathfrak{m}$ of A and order the x_i such that $v(x_n) = \min\{v(x_1), \dots, v(x_n)\}$. Set $A' = A[\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}] \subseteq K$. Show that $\mathfrak{m}A' \neq A'$. Show that the localization of A' at a minimal prime over $\mathfrak{m}A'$ has dimension 1 and dominates A.

(c) Finish the proof by applying the Krull–Akizuki theorem. (see for example TAG 00PG on the Stacks project)

(ii) Let $f : X \to Y$ be a morphism of finite type and assume that Y is locally Noetherian.

(a) Show that f is universally closed if and only if f satisfies the existence part of the valuative criterion for DVRs.

(b) Show that f is separated if and only if f satisfies the uniqueness part of the valuative criterion for DVRs.

(c) Show that f is proper if and only if f satisfies existence and uniqueness of the valuative criterion for DVRs.

Solution. Not provided.

Exercise 41. *Support* (4 points)
Recall the notions of support of a section of a sheaf and subsheaf with supports from Exercise 8. If $\mathcal{F}$ is a sheaf of Abelian groups on a topological space X, then

$$\operatorname{supp}(\mathcal{F}) := \{x \in U \mid \mathcal{F}_x \neq 0\}.$$

(i) Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathcal{F} = \widetilde{M}$. For any $m \in M$, show that $\operatorname{supp}(m) = V(\operatorname{Ann} m)$, where $\operatorname{Ann} m = \{a \in A \mid am = 0\}$ is the annihilator of m.

(ii) Assume that A is Noetherian and M is finitely generated. Show that $\operatorname{supp}(\mathcal{F}) = V(\operatorname{Ann} M)$, where $\operatorname{Ann} M = \{a \in A \mid aM = 0\}$.

(iii) Deduce that the support of a coherent sheaf on a Noetherian scheme is closed.

(iv) For any ideal $I \subseteq A$, set $\Gamma_I(M) \subseteq M$ via $\Gamma_I(M) := \{m \in M \mid I^n m = 0 \text{ for some } n > 0\}$. Assume that A is Noetherian and M is an arbitrary A-module. Show that $\widetilde{\Gamma_I(M)} \cong \mathcal{H}^0_Z(\mathcal{F})$, where $Z = V(I)$ and $\mathcal{F} = \widetilde{M}$.

(v) Deduce that if X is a Noetherian scheme, $Z \subseteq X$ is a closed subset and $\mathcal{F}$ is a (quasi-)coherent sheaf on X, then $\mathcal{H}^0_Z(\mathcal{F})$ is also (quasi-)coherent.

Solution.

(i) $\mathfrak{p} \in \operatorname{Supp}(m) \iff \frac{m}{1} \neq 0$ in $M_\mathfrak{p} \iff \forall a \in A \setminus \mathfrak{p} : am \neq 0$ in $M \iff \operatorname{Ann}(m) = \{a \in A \mid am = 0\} \subset \mathfrak{p} \iff \mathfrak{p} \in V(\operatorname{Ann}(m))$.

(ii) *Don't need any Noetherian assumption here*. First, note that $\operatorname{Supp}(\mathcal{F}) = \operatorname{Supp}(M) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M_\mathfrak{p} \neq 0\}$. Let's say M is generated by some elements $m_1, \dots, m_n$. Then we have

$$\operatorname{Supp}(M) = \cup_{i=1}^n \operatorname{Supp}(m_i) \overset{(i)}{=} \cup_{i=1}^n V(\operatorname{Ann}(m_i)) = V(\cap_{i=1}^n \operatorname{Ann}(m_i)) = V(\operatorname{Ann}(M)).$$

(iii) Let X be a scheme and $\mathcal{F} \in \mathrm{Coh}(X)$. Take any affine open cover $X = \cup_{i=1}^n \operatorname{Spec} A_i$. Then $\mathcal{F}|_{\operatorname{Spec} A_i} \cong \tilde{M}_i$ for some finitely generated A_i-module M_i. Then

$$\operatorname{Supp}(\mathcal{F}) \cap \operatorname{Spec} A_i = \operatorname{Supp}(\mathcal{F}|_{\operatorname{Spec} A_i}) = \operatorname{Supp}(\tilde{M}_i) = V(\operatorname{Ann}_{A_i}(M_i))$$

which is closed in $\operatorname{Spec} A_i$ for every i. Since the $\operatorname{Spec} A_i$ cover X, we conclude that $\operatorname{Supp}(\mathcal{F})$ is closed in X.

(iv) By definition we have that $\mathcal{H}^0_Z(\mathcal{F}) = \ker(\mathcal{F} \to j_*(\mathcal{F}|_U))$, where $U = X \setminus Z$ and the unit morphism $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is given by $\mathcal{F}(V) \ni s \mapsto s|_{V\cap U} \in \mathcal{F}(V \cap U)$. Now, since $\operatorname{Supp}(\widetilde{\Gamma_I(M)}) = \operatorname{Supp}(\Gamma_I(M)) \subset Z$, we get that the composite $\widetilde{\Gamma_I(M)} \to \tilde{M} = \mathcal{F} \to j_*(\mathcal{F}|_U)$ is zero; therefore we get a map $\widetilde{\Gamma_I(M)} \to \mathcal{H}^0_Z(\mathcal{F})$, which we also immediately see to be injective. Since both $\widetilde{\Gamma_I(M)}$ and $\mathcal{H}^0_Z(\mathcal{F})$ are sheaves, in order to show that this map is surjective, it suffices to check that over any distinguished open $D(f)$, the corresponding map on sections is surjective.

We have $\widetilde{\Gamma_I(M)}(D(f)) \cong \Gamma_I(M)_f$ and $\mathcal{H}^0_Z(\mathcal{F})(D(f)) \cong \{x \in M_f \mid \operatorname{Supp}(x) \subset V(I)\}$, and under these identifications, surjectivity boils down to showing that $\{x \in M_f \mid \operatorname{Supp}(x) \subset V(I)\}$ is contained in $\Gamma_I(M)_f \subset M_f$.

Take any $x \in M_f$ with $\operatorname{Supp}(x) \subset V(I)$ (when writing Supp and Ann here we consider M_f as an A-module). By (i), $V(\operatorname{Ann}(x)) = \operatorname{Supp}(x) \subset V(I)$, hence $I \subset \sqrt{I} \subset \sqrt{\operatorname{Ann}(x)}$. Now, because A is Noetherian, I is finitely generated, therefore we already have $I^n \subset \operatorname{Ann}(x)$ for some $n \geq 1$, which means that $x \in \Gamma_I(M_f)$. We now conclude by noting that $\Gamma_I(M_f) = \Gamma_I(M)_f$ as subsets of M_f. Indeed, the inclusion "$\supset$" is clear. For "$\subset$", take $\frac{m}{f^k} \in M_f$ such that $I^n \frac{m}{f^k} = 0$ in M_f for some $n \geq 1$, i.e. there exists $r \geq 1$ such that $f^r I^n m = 0$ in M. But the last equality implies that $f^r m \in \Gamma_I(M)$ and hence, $\frac{m}{f^k} = \frac{f^r m}{f^{r+k}} \in \Gamma_I(M)_f$, showing the claimed inclusion.

(v) Clear from (v) and from the observation that for every open affine $U = \operatorname{Spec} A \subset X$ (also true for any open subset $U \subset X$), we have $\mathcal{H}^0_Z(\mathcal{F})|_U = \mathcal{H}^0_{Z\cap U}(\mathcal{F}|_U)$.

Exercise 42. *Fiber dimension* (4 points)
Let X be a Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. We will consider the function

$$\varphi(x) := \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x),$$

where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field of the point $x \in X$. Use Nakayama lemma to prove the following statements.

1. The function φ is upper semi-continuous, i.e. for any $n \in \mathbb{Z}$ the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.
2. If $\mathcal{F}$ is locally free and X is connected, then φ is a constant function.
3. Conversely, if X is reduced and φ is constant, then $\mathcal{F}$ is locally free.

Solution. (This is [Hart, Exo.II.5.8])

1. Fix $n \in \mathbb{Z}$. We show that the complement, i.e. any $U = \{x \in X \mid \varphi(x) < n\}$, is open in X. Let $x \in U$ with $\varphi(x) = k < n$ and choose an open affine neighborhood $U = \operatorname{Spec}(A)$ of x such that $\mathcal{F}|_U = \tilde{M}$ for some finitely genertaed A-module. Furthermore, choose generators $m_1, \ldots, m_r$ of M as an A-module. Now (as $\varphi(x) = k$), we have that

$\dim_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} = k$, so we can find a generating set $u_1, \ldots, u_k$ of $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ that consists of elements $u_i \in M_\mathfrak{p}$; morever, by NAK, we have that these u_i also generate $M_\mathfrak{p}$ as $A_\mathfrak{p}$-module. Now we are almost done — comparing the generating sets of $M_\mathfrak{p}$ induced by the generating set of M and of $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ gives us a_{ij}, f_{ij} such that

$$m_j = \sum_i \frac{a_{ij}}{f_{ij}} u_i$$

holds in $M_\mathfrak{p}$. Now for $f = \prod_{i,j} f_{ij}$, we have $\mathfrak{p} \in D(f)$, and for any $q \in D(f)$, we have that $\varphi(q) \leq k < n$ (for any such $\mathfrak{q}$, the images of the m_j in $M_\mathfrak{q}$ generate it as $A_\mathfrak{q}$ module, and so the images of the u_i generate $M_\mathfrak{q}$ as an $A_\mathfrak{q}$-module.) This shows that $U \subseteq X$ is open, as claimed.

2. We claim that if $\mathcal{F}$ is not only coherent but also locally free, then φ is actually continous, i.e. that for any $n \in \mathbb{Z}$, the preimage $U = \varphi^{-1}(n) = \{x \mid \varphi(x) = n\} \subseteq X$ is open: Indeed, let $x \in U$; then there is a neighborhood V of x with $\mathcal{F}|_V$ free of rank n.

   Now, note that the image of φ in $\mathbb{Z}$ is bounded, an in particular, we can find $x \in X$ such that $\varphi(x) = n$ is minimal. Then

   $$V = \{x \in X \mid \varphi(x) > n\} = \bigcup_{k>n} \varphi^{-1}(k) \subseteq X$$

   is open, and by (i), it is also closed. Since we assume X to be connected, either $V = X$ or $V = \emptyset$, and since we started with an n that is actually in the image of φ we necesarilly have $V = X$.

3. Let $x \in X$ with $\varphi(x) = n$ and $U = \operatorname{Spec}(A)$ be an affine neighborhood of x such that $\varphi(x) = n$ with $\mathcal{F}|_U = \tilde{M}$. It suffices to show that $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module for all $\mathfrak{p} \subseteq A$ and argue that $K_\mathfrak{q} = 0$ for all $q \in D(s)$ and hence $K = 0$. By assumption, we can find $m_1, \ldots, m_n \in M$ whose images in $M_\mathfrak{p}/\mathfrak{p}M$ form a $k(x) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$-linear basis. By Nakayama, we have that the $m_1, \ldots, m_n$ still span $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module, and so they also span $M_\mathfrak{q}$ for any $\mathfrak{q} \subseteq \mathfrak{p}$. Now by assumption that φ is constant, we necessarily have that the images of $m_1, \ldots, m_n$ in $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$ are still linearly independent over $A_\mathfrak{q}/\mathfrak{q}$. Thus, for any relation $\sum a_i m_i = 0$ in $M_\mathfrak{p}$ with $a_i \in A_\mathfrak{p}$, we necessarily have $a_i = 0$ in $A_\mathfrak{q}/\mathfrak{q}$ for all $\mathfrak{q}$, i.e. $a_i \in \bigcap_{\mathfrak{q} \subseteq \mathfrak{p}} \mathfrak{q}$. But this intersection vanishes, as $A_\mathfrak{p}$ is reduced (because X is assumed to be reduced). So $M_\mathfrak{p}$ is indeed free.

Compare this to Lemma II.8.9 in Hartshorne.

Exercise 43. *f^* and $\otimes$ are only right exact* (4 points)
Let $(X, \mathcal{O}_X)$ be a ringed space and consider $\mathcal{F}, \mathcal{G} \in \operatorname{Mod}(X, \mathcal{O}_X)$. Show that for all $x \in X$ there exists a natural isomorphism

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

Prove that $\mathcal{F} \otimes_{\mathcal{O}_X} (\ ) \colon \operatorname{Mod}(X, \mathcal{O}_X) \to \operatorname{Mod}(X, \mathcal{O}_X)$ and $f^* \colon \operatorname{Mod}(Y, \mathcal{O}_Y) \to \operatorname{Mod}(X, \mathcal{O}_X)$ for a morphism of ringed spaces $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are both right exact functors. Describe examples showing that in general they are not left exact.

Solution. Not provided.

Exercise 44. *(Non-)Functoriality of* Proj (4 points)
Let $A = \bigoplus_{d=0}^{\infty} A_d$ and $B = \bigoplus_{d=0}^{\infty} B_d$ be graded rings. Recall that a morphism of rings $\varphi : A \to B$ is called *graded* if $\varphi(A_d) \subseteq B_d$.

(i) Let $\varphi : A \to B$ be a graded morphism of rings. Let $U = \{\mathfrak{p} \in \operatorname{Proj} B \mid \varphi(A_+) \not\subseteq \mathfrak{p}\}$. Show that $U \subseteq \operatorname{Proj} B$ is an open subset and show that φ determines a morphism of schemes $U \to \operatorname{Proj} A$.

(ii) Assume that $A_d = B_d$ and let $\varphi : A \to B$ be a graded morphism of rings which induces the identity for $d \gg 0$. Show that $U = \operatorname{Proj} B$ and the induced map $U \to \operatorname{Proj} A$ is an isomorphism.

Solution. (i) As a set, recall that $\operatorname{Proj}(B)$ is the set of homogeneous prime ideals in B not containing $B_+ = \bigoplus_{d\geq 1} B_d$. It is a subset of $\operatorname{Spec}(B)$ and is endowed with the subspace topology. In particular,

$$U = \bigcup_{a\in A_+} \{\mathfrak{p} \in \operatorname{Proj}(B) \mid \varphi(a) \notin \mathfrak{p}\} = \bigcup_{a\in A_+} D_+(\varphi(a))$$

is a union of open subsets, hence it is open. For $a \in A_d \subset A_+$, with $d \geq 1$, the morphism

$$\varphi_{(a)} \colon A_{(a)} \to B_{(\varphi(a))} \colon \frac{f}{a^n} \mapsto \frac{\varphi(f)}{\varphi(a)^n}$$

induces a morphism of schemes

$$f_{(a)} \colon D_+(\varphi(a)) \to D_+(a).$$

Our goal is to define a morphism $U \to \operatorname{Proj}(A)$, hence we need to show that those $f_{(a)}$'s glue together. Let $a \in A_d$ and $a' \in A_{d'}$ with $d, d' \geq 1$. Consider the following diagram:

$$\begin{tikzcd}
D_+(\varphi(a)) \arrow[r, "f_{(a)}"] & D_+(a) \arrow[dr, hook] & \\
D_+(\varphi(aa')) \arrow[u, hook] \arrow[d, hook] \arrow[r, dashed, "f_{(aa')}"] & D_+(aa') \arrow[u, hook] \arrow[d, hook] & \operatorname{Proj}(A) \\
D_+(\varphi(a')) \arrow[r, "f_{(a')}"'] & D_+(a') \arrow[ur, hook] &
\end{tikzcd}$$

where $D_+(\varphi(aa')) = D_+(\varphi(a)) \cap D_+(\varphi(a'))$ *(this is this most important point)* and $D_+(aa') = D_+(a) \cap D_+(a')$. To show that $f_{(a)}$ and $f_{(a')}$ agree on the intersection of their domains, it suffices to show that both restrict to $f_{(aa')}$ on the intersection. This follows directly from the corresponding commutative diagram of rings:

$$\begin{tikzcd}
B_{(\varphi(a))} \arrow[d] & A_{(a)} \arrow[l, "\varphi_{(a)}"'] \arrow[d] \\
B_{(\varphi(aa'))} & A_{(aa')} \arrow[l, "\varphi_{(aa')}"]
\end{tikzcd}$$

As a result, all the scheme morphisms $f_{(a)}$ glue into a morphism of schemes $f \colon U \to \operatorname{Proj}(A)$.

(ii) First, we show that $U = \operatorname{Proj}(B)$. To do this, we claim that the open cover $\operatorname{Proj}(B) = \bigcup_{b\in B_+} D_+(b)$ remains a cover if we restrict to b's of large degree (i.e. greater than some fixed degree). This simply follows from the fact that $D_+(b) = D_+(b^n)$ for any $d, n \geq 1$, $b \in B_d$, and b^n has degree nd (unless it is zero, in which case $D_+(b) = \emptyset = D_+(0)$). By assumption, there exists a $d_0 \geq 0$ such that $\varphi\colon A_d \to B_d$ is the identity for all $d \geq d_0$. Then we get

$$U = \bigcup_{a \in A_+} D_+(\varphi(a)) = \bigcup_{d \geq 1} \bigcup_{a \in A_d} D_+(\varphi(a)) \supset \bigcup_{d \geq d_0} \bigcup_{a \in A_d} D_+(\varphi(a)) = \bigcup_{d \geq d_0} \bigcup_{b \in B_d} D_+(b) = \operatorname{Proj}(B),$$

hence $U = \operatorname{Proj}(B)$.

Now we show that the induced map $f\colon \operatorname{Proj}(B) = U \to \operatorname{Proj}(A)$ is an isomorphism. First, f is locally an isomorphism since it is locally induced by $A_{(a)} \xrightarrow{\varphi(a)} B_{(\varphi(a))}$, for $a \in A_+$ homogeneous. The latter morphism is the identity because elements on the left have the form $\frac{x}{a^n} = \frac{a^{d_0}x}{a^{d_0+n}}$ and such an element is sent to itself since φ induces the identity in degree $\geq d_0$.

Now, observe that $\operatorname{Proj}(A)$ is the gluing of all the $D_+(a)$ for $a \geq d_0$, idem for $\operatorname{Proj}(B)$ with all the $D_+(\varphi(a))$'s. The map f restricts to the identities between the open subset that are glued $D_+(a) \to D_+(\varphi(a))$, hence f is an isomorphism.

Parts (ii) and (iii) of the following exercise use twisting sheaves and the description of rational points of projective space, both of which will be treated in the lecture on Monday.

Exercise 45. *Products of* Proj *and Segre embedding* (4 points)
Let $B = \bigoplus_{d=0}^{\infty} B_d$ and $C = \bigoplus_{d=0}^{\infty} C_d$ be two graded rings with $A := B_0 \cong C_0$. Consider $B \times_A C := \bigoplus_{d=0}^{\infty} B_d \otimes_A C_d$ and the schemes $X := \operatorname{Proj}(B)$ and $Y := \operatorname{Proj}(C)$.

(i) Show that $X \times_{\operatorname{Spec}(A)} Y \cong \operatorname{Proj}(B \times_A C)$.

(ii) Prove that under this isomorphism $\mathcal{O}(1)$ on $\operatorname{Proj}(B \times_A C)$ is isomorphic to $p_1^*\mathcal{O}_X(1) \otimes p_2^*\mathcal{O}_Y(1)$, where p_1 and p_2 are the two projections from $X \times_{\operatorname{Spec}(A)} Y$.

(iii) Now, specialize to $X = \mathbb{P}^{n_1}_{A_0} = \operatorname{Proj} A_0[x_0, \ldots, x_{n_1}]$ and $Y = \mathbb{P}^{n_2}_{A_0} = \operatorname{Proj} A_0[y_0, \ldots, y_{n_2}]$. Show that the surjective graded morphism

$$A_0[z_{00}, \ldots, z_{n_1 n_2}] \to A_0[x_0, \ldots, x_{n_1}] \times_{A_0} A_0[y_0, \ldots, y_{n_2}]$$

that maps z_{ij} to $x_i \otimes y_j$ induces a closed immersion $X \times_{\operatorname{Spec} A} Y \to \mathbb{P}^N_A$. This closed immersion is called *Segre embedding.* What is N? Describe the Segre embedding on rational points if A_0 is a field.

Solution.

1. We show here that the way we glue affines to construct $\operatorname{Proj}(B \times_A C)$ is the same as the way we glue affines to construct $\operatorname{Proj} B \times_{\operatorname{Spec} A} \operatorname{Proj} C$. Take principal opens $D_+(b) = \operatorname{Spec}(B_b)_0 \subset \operatorname{Proj} B, D_+(c) = \operatorname{Spec}(C_c)_0 \subset \operatorname{Proj} C$. The fibre product $D_+(b) \times_{\operatorname{Spec} A} D_+(c)$ is just $\operatorname{Spec}(B_b)_0 \otimes_A (C_c)_0$. *The ring can be identified as $(B \times_A C)[(b^{\deg c} \otimes c^{\deg b})^{-1}]_0$ ¡– the most important input* , by the following maps:

$$\begin{aligned} (B_b)_0 \otimes_A (C_c)_0 &\longleftrightarrow (B \times_A C)[(b^{\deg c} \otimes c^{\deg b})^{-1}]_0 \\ s/b^m \otimes t/c^n &\longmapsto \frac{b^{m(n \deg c - 1)} s \otimes c^{n(m \deg b - 1)} t}{(b^{\deg c} \otimes c^{\deg b})^{mn}} \\ u/b^{k \deg c} \otimes v/c^{k \deg b} &\longleftarrow \frac{u \otimes v}{(b^{\deg c} \otimes c^{\deg b})^k}. \end{aligned}$$

Note that the second map is not well defined if we take the larger graded ring $\bigoplus_n \bigoplus_{i+j=n} B_i \otimes_A C_j$ instead of $B \times_A C$. The isomorphisms is compatible with gluing data and we get the isomorphism.

2. This part follows essentially the same as part 1 by replacing the degree 0 part with degree 1 part.

3. We can show directly if $\phi : A \to B$ is a surjective homomorphism of graded rings (not shifting degrees), then it induces a closed immersion $\operatorname{Proj} B \to \operatorname{Proj} A$. By Exercise 44, the ring homomorphism induces a map $U \to \operatorname{Proj} A$, where $U = \{\mathfrak{p} \in \operatorname{Proj} B | \phi(A_+) \not\subseteq \mathfrak{p}\}$. Since ϕ is surjective, $\phi(A_+) = B_+$ hence U is the whole $\operatorname{Proj} B$. Since localization preserves surjectivity, we have $D_+(\phi(a)) \to D_+(a)$ is a closed immersion. Since closed immersion is LOCT and $D_+(a)$ covers $\operatorname{Proj} A$ when we go over all $a \in A$, we are done.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 46. *Multiplicative group actions and gradings* (+ 4 extra points)
We define the *multiplicative group scheme* $\mathbb{G}_m$ as the scheme $\operatorname{Spec} \mathbb{Z}[x, x^{-1}]$ together with the three morphisms

$$\begin{array}{rcl}
\text{“neutral element” } e: & \operatorname{Spec}\ \mathbb{Z} & \to \mathbb{G}_m \\
 & 1 & \leftarrow\!\shortmid x \\
\text{“multiplication” } m: & \mathbb{G}_m \times_{\operatorname{Spec}\ \mathbb{Z}} \mathbb{G}_m & \to \mathbb{G}_m \\
 & x \otimes x & \leftarrow\!\shortmid x \\
\text{“inverse” } i: & \mathbb{G}_m & \to \mathbb{G}_m \\
 & x^{-1} & \leftarrow\!\shortmid x
\end{array}$$

(i) Rephrase the definition of a group action on a set in terms of commutative diagrams. Use these diagrams to define $\mathbb{G}_m$-actions on schemes.

(ii) Show that giving a $\mathbb{G}_m$-action on an affine scheme $\operatorname{Spec} A$ is the same as giving a $\mathbb{Z}$-grading $A = \bigoplus_{d \in \mathbb{Z}} A_d$.

(iii) Can you find necessary and sufficient conditions on the $\mathbb{G}_m$-action that guarantee $A_d = 0$ for $d < 0$ (i.e. such that A is a graded ring in the sense of the lecture)? Try to find the $\mathbb{G}_m$-action corresponding to the grading on $A_0[x_0, \ldots, x_n]$ that yields projective n-space.

Solution. (i) Let G be a group and X a set. A group action of G on X consists in a map

$$\gamma \colon G \times X \to X \colon (g, x) \mapsto g \cdot x$$

that satisfies: $\gamma(g, \gamma(h, x)) = \gamma(gh, x)$ and $\gamma(e_G, x) = x$ for all $g, h \in G$ and $x \in X$ (with $e_G \in G$ the neutral element).

Writing $e \colon \{1\} \to G \colon 1 \mapsto e_G$ and $m \colon G \times G \to G$ the mutliplication, the above two conditions for γ to define a group action translate into asking that the two following diagrams are commutative:

$$\begin{tikzcd}
G \times G \times X \arrow[r, "m\times\gamma"] \arrow[d, "\mathrm{id}_G \times \gamma"'] & G \times X \arrow[d, "\gamma"] \\
G \times X \arrow[r, "\gamma"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\{1\} \times X \arrow[r, "e\times\mathrm{id}_X"] \arrow[d, "\mathrm{pr}_2"'] & G \times X \arrow[d, "\gamma"] \\
X \arrow[r, "\mathrm{id}_X"'] & X
\end{tikzcd}$$

This motivates the following definition of a $\mathbb{G}_m$-action on a scheme X: it is a scheme morphism

$$\gamma \colon \mathbb{G}_m \times_{\operatorname{Spec}(\mathbb{Z})} X \times X$$

making the exact same two diagram commute, with G replaced by $\mathbb{G}_m$, the cartesian product replaced by the fibre product of (Spec($\mathbb{Z}$)-)schemes and e replaced by the morphism $e\colon \operatorname{Spec}(\mathbb{Z}) \to \mathbb{G}_m$:

$$\begin{tikzcd} \mathbb{G}_m \times \mathbb{G}_m \times X \arrow[r, "m\times\gamma"] \arrow[d, "\mathrm{id}_G \times\gamma"'] & \mathbb{G}_m \times X \arrow[d, "\gamma"] \\ \mathbb{G}_m \times X \arrow[r, "\gamma"'] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}(\mathbb{Z}) \times X \arrow[r, "e\times \mathrm{id}_X"] \arrow[d, "\mathrm{pr}_2"'] & \mathbb{G}_m \times X \arrow[d, "\gamma"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

(ii) Now let $X = \operatorname{Spec}(A)$ be an affine scheme. Consider a morphism $\gamma\colon \mathbb{G}_m \times X \to X$, corresponding to a ring homomorphism $\gamma^*\colon A \to \mathbb{Z}[x, x^{-1}] \times_{\mathbb{Z}} A \cong A[x, x^{-1}]$. The diagrams defining a group action translate into:

$$\begin{tikzcd} A[x^{\pm1}, y^{\pm1}] & A[x^{\pm1}] \arrow[l, "\gamma_y^*\otimes\mathbb{Z}[x^{\pm1}]"'] \\ A[z^{\pm1}] \arrow[u, "A\otimes m^*"] & A \arrow[l, "\gamma_x^*"] \arrow[u, "\gamma_x^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}\otimes_{\mathbb{Z}} A & A[x^{\pm1}] \arrow[l, "e^*\otimes A"'] \\ A \arrow[u, "\cong"] & A \arrow[l, "\mathrm{id}_A"] \arrow[u, "\gamma^*"'] \end{tikzcd}$$

where $m^*(z) = xy$ corresponds to the multiplication m and $e^*(x) = 1$ to the neutral element e. In the square on the left, we denote by γ_x^* the ring homomorphism induced by γ, where x is the name of the chosen coordinate on $\mathbb{G}_m$.

For $a \in A$, we have $\gamma^*(a) = \sum_{n\in\mathbb{Z}} a_n x^n$ for some $a_n \in \mathbb{Z}$ (thought of as the homogeneous terms of a). Commutativity of the above two diagrams then translates into the following equations:

$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} (a_n)_m y^m x^n = \sum_{n\in\mathbb{Z}} a_n x^n y^n \quad \text{and} \quad \sum_{n\in\mathbb{Z}} a_n = a, \tag{9}$$

which are the conditions for γ^* to define a $\mathbb{Z}$-grading on A. Note that the left equation is equivalent

$$(a_n)_m = \begin{cases} a_n & m = n \\ 0 & m \neq m \end{cases} \quad \text{for all } n, m \in \mathbb{Z}. \tag{10}$$

More precisely, we show that the datum of a group scheme action γ of $\mathbb{G}_m$ on X is equivalent to the datum of a $\mathbb{Z}$-grading of the ring A:

- First, assume that γ is a group action, then γ^* satisfies (9). Let

  $$p_n\colon A \to A\colon a \mapsto a_n$$

  be the *projection onto the degree n part.* The left equation of (9) implies that the $p_n{}'s$ are orthogonal projectors, i.e. $p_n \circ p_n = p_n$ and $p_n \circ p_m = 0$ if $n \neq m$, while the right equation of (9) implies that the sum of their images is the whole ring A. In other words, we have a direct sum decomposition (as abelian groups)

  $$A = \bigoplus_{n\in\mathbb{Z}} p_n(A) \subset A[x^{\pm1}]$$

  where the inclusion on the right maps $p_n(a) \in p_n(A)$ to $p_n(a)x^n \in A[x^{\pm1}]$. Thus A is isomorphic to the subring $\bigoplus_{n\in\mathbb{Z}} p_n(A) \subset A[x^{\pm1}]$, and in particular the grading is compatible with the ring multiplication in A, i.e. $p_n(A)p_m(A) \subset p_{nm}(A)$.

- Now, start with a $\mathbb{Z}$-grading structure on A, i.e. a direct sum decomposition $A = \bigoplus_{n\in\mathbb{Z}} A_n$ where each A_n is an abelian subgroup of A and such that $A_n \cdot A_m \subset A_{nm}$ for all $n, m \in A$. Then the composition

$$\begin{array}{ccccc} A & = & \bigoplus_{n\in\mathbb{Z}} A_n & \hookrightarrow & A[x^{\pm 1}] \\ \\ a & \longmapsto & (a_n)_n & \longmapsto & \sum_{n\in\mathbb{Z}} a_n x^n \end{array}$$

defines a morphism $\gamma^*\colon A \to A[x^{\pm 1}]$. By definition of a graded ring, we automatically have $\sum_{n\in\mathbb{Z}} a_n = a$. Also, since the A_n's are in direct sum, we have that (10) is satisfied. Thus γ^* defines a group scheme action.

(iii) The grading on A has no negative degree part if and only if the morphism $\gamma^*\colon A \to A[x^{\pm 1}]$ factors through $A[x]$. On the level of schemes, this means that $\gamma\colon \mathbb{G}_m \times \operatorname{Spec}(A) \to \operatorname{Spec}(A)$ factors as

$$\mathbb{G}_m \times_{\mathbb{Z}} A = (\mathbb{A}^1_{\mathbb{Z}} - \{0\}) \times_{\mathbb{Z}} A \;\hookrightarrow\; \mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} A = \mathbb{A}^1_A \;\xrightarrow{\tilde{\gamma}}\; \operatorname{Spec}(A)$$

where $\tilde{\gamma}$ is the map induced by the corestriction of γ^* to $A[x]$ (warning: it is not the structure map of $\mathbb{A}^1_A$ as a A-scheme!).

On $A = A_0[x_0, \dots, x_n]$, the grading is just given by the degree of polynomials. Thus the associated $\mathbb{G}_m$-action on $\operatorname{Spec}(A_0[x_0, \dots, x_n])$ corresponds to the ring homomorphism

$$\begin{aligned} \gamma^*\colon A_0[x_0, \dots, x_n] &\longrightarrow A_0[x_0, \dots, x_n, y^{\pm 1}] \\ f(x_0, \dots, x_n) &\longmapsto f(x_0 y, \dots, x_n y) \end{aligned}$$

Exercise 47. *Qcqs lemma* (4 points)

For reference, this is 6.2.8 in [Vakil, Ver.2022], or [Hart, II.5.14], or Görtz-Wedhorn, Theorem 7.22

Let X be a scheme, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. For $s \in \Gamma(X, \mathcal{L})$, define $X_s := \{x \in X \mid s_x \notin \mathfrak{m}_x \mathcal{L}_x\}$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$.

(i) Show that $X_s \subseteq X$ is open.

(ii) Assume that X is quasi-compact and let $t \in \Gamma(X, \mathcal{F})$ such that $t|_{X_s} = 0$. Show that there exists an integer $n > 0$ such that $t \otimes s^{\otimes n} = 0 \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$.

(iii) Assume that X is quasi-compact and quasi-separated. Show that for every section $t' \in \Gamma(X_s, \mathcal{F})$, there exists $n > 0$ and a section $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ such that $t|_{X_s} = t' \otimes s|_{X_s}^{\otimes n}$

(Hint: Reduce to the case where X is an affine scheme and $\mathcal{L}$ is the structure sheaf.)

Solution.

(i) We have that X is covered by affine opens $X = \bigcup U_i$ where $\mathcal{L}_{|U_i} \cong \mathcal{O}_{U_i}$. Since being open is a local property, we can assume X affine and $\mathcal{L} = \mathcal{O}_X$. If $s_x = [s_{|U}, U] \in \mathcal{O}_{X,x} \setminus \mathfrak{m}_x$, then s_x is invertible in $\mathcal{O}_{X,x}$ because $\mathcal{O}_{X,x}$ is local. Thus, there exists $t_x = [g, V] \in \mathcal{O}_{X,x}$ such that $s_x t_x = 1$ in $\mathcal{O}_{X,x}$. It follows that there exists an open neighborhood W of x where we have

$$s_{|W} g_{|W} = 1 \in \mathcal{O}_X(W).$$

Thus, s_y is invertible for all $y \in W$ which proves $W \subset X_s$.

Before we prove (ii) and (iii), we fix some notation. We cover X by a finite number of affine opens $X_1, \dots, X_r$ such that $\mathcal{L}_{|X_i}$ is free for $i = 1, \dots, r$. Let e_i be a generator of $\mathcal{L}(X_i)$ as $\mathcal{O}_X(X_i)$-module and $h_i \in \mathcal{O}_X(X_i)$ such that $s_{|X_i} = h_i e_i$. Note that we have $D(h_i) = X_s \cap X_i$.

(ii) The condition $t_{|X_s} = 0$ implies $t_{|X_s \cap X_i} = 0$, thus there exists $n \geq 1$ such that $h_i^n t_{|X_i} = 0$. We can choose n sufficiently large so that it is independent of i. From this, we directly conclude
$$(t \otimes s^{\otimes n})_{|X_i} = t_{|X_i} \otimes (h_i e_i)^{\otimes n} = h_i^n t_{|X_i} \otimes e_i^{\otimes n} = 0.$$
Thus, $t \otimes s^{\otimes n} = 0 \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$.

(iii) There exists $m \geq 0$ and $f_i \in \mathcal{O}_X(X_i)$ for $i = 1, \dots, r$ such that
$$(h^m)_{|X_i \cap X_s} t'_{|X_i \cap X_s} = f_{i|X_i \cap X_s}, \quad \text{for } i = 1, \dots, r.$$
Set $t_i := f_i \otimes e_i^{\otimes m}$. By construction, $(t_i)_{X_s \cap X_i} = (t' \otimes s^{\otimes m})_{X_s \cap X_i}$. Hence,
$$(t_i)_{|X_s \cap X_i \cap X_j} - (t_j)_{|X_s \cap X_i \cap X_j}) = 0.$$
As X is quasi-separated, all $X_i \cap X_j$ are quasi-compact. Thus, there exists $m_0 \geq 0$ such that
$$((t_i)_{|X_i \cap X_j} - (t_j)_{|X_i \cap X_j})) \otimes s^{m_0} = 0 \in \Gamma(X_i \cap X_j, \mathcal{F} \otimes \mathcal{L}^{\otimes(m+m_0)})$$
So the $t_i \otimes s^{\otimes m_0}$ satisfy the gluing condition. Let $n \geq m_0 + m$ Then also $t_i \otimes s^{\otimes n-m} \in \Gamma(X_i, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ satisfies the gluing condition. It follows that there exists $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ such that
$$t_{|X_s \cap X_i} = (t_i \otimes s^{\otimes n-m})_{|X_s \cap X_i} = (t' \otimes s^{\otimes m} \otimes s^{\otimes n-m})_{|X_s \cap X_i} = (t' \otimes s^{\otimes} n)_{|X_s \cap X_i}.$$
Hence, $t_{|X_s} = (t' \otimes s^n)_{|X_s}$.

Exercise 48. *Veronese embedding* (4 points)
Let $A = \bigoplus_{d \geq 0} A_d$ be a graded ring and $n > 0$. Let $A^{(n)}$ denote the graded ring defined by $A^{(n)} := \bigoplus_{d \geq 0} A_d^{(n)}$ with $A_d^{(n)} := A_{dn}$.

(i) Show that there exists an isomorphism $\varphi \colon \operatorname{Proj}(A) \xrightarrow{\sim} \operatorname{Proj}(A^{(n)})$ with $\varphi^* \mathcal{O}(1) \cong \mathcal{O}(n)$.

(ii) Consider the case $A = A_0[x_0, \dots, x_r]$. Show that the surjection $A_0[y_0, \dots, y_N] \twoheadrightarrow A^{(n)}$ mapping y_i to the i-th monomial of degree n in the variables x_i defines a closed embedding
$$v_n : \mathbb{P}^r_{A_0} \cong \operatorname{Proj}(A) \cong \operatorname{Proj}(A^{(n)}) \hookrightarrow \mathbb{P}^N_{A_0}.$$
Find N. Show that for k a field, on k-rational points the morphism v_n is given by $[\lambda_0 : \dots : \lambda_r] \mapsto [\lambda_0^n : \dots : \lambda^I : \dots : \lambda_r^n]$ with λ^I running through all monomials of degree n.

Solution.

1. Let $D_+(a)$ and $D_+(a^n)$ be principal opens in $\operatorname{Proj} A$ and $\operatorname{Proj} A^{(n)}$. We show that there is an isomorphism of schemes between $D_+(a)$ and $D_+(a^n)$. The isomorphism is given by the ring isomorphism $A_{(a)} \to {A^{(n)}}_{(a^n)}$:
$$\begin{aligned} A_{(a)} &\to {A^{(n)}}_{(a^n)} \\ \frac{r}{a^k} &\mapsto \frac{r \cdot a^{nk-k}}{(a^k)^n} \\ \frac{s}{(a^l)^n} &\leftarrow\!\shortmid \frac{s}{(a^l)^n} \end{aligned}$$

The construction of the isomorphism is compatible with restrictions of principal opens, i.e. the following diagram commute:

$$
\begin{tikzcd}
\operatorname{Proj} A : & & & & \operatorname{Proj} A^{(n)} : & \\
D_+(a) \arrow[dr] \arrow[drrr, dashed] & & D_+(b) \arrow[dl] \arrow[drrr, dashed] & & & \\
& D_+(ab) \arrow[drrr, dashed] & & D_+(a^n) \arrow[dr] & & D_+(b^n) \arrow[dl] \\
& & & & D_+((ab)^n) &
\end{tikzcd}
$$

Hence they define an isomorphism of schemes $\phi : \operatorname{Proj} A \to \operatorname{Proj} A^{(n)}$. To see the second claim, one can look at what happens to $A^{(n)}(1)$ in the isomorphism ϕ. It turns out then there is an isomorphism

$$
\begin{aligned}
A(n)_{(a)} &\to A^{(n)}(1)_{(a^n)} \\
\frac{r}{a^k} &\mapsto \frac{r \cdot a^{nk-k}}{(a^k)^n} \\
\frac{s}{(a^l)^n} &\leftarrow\!\shortmid \frac{s}{(a^l)^n}
\end{aligned}
$$

By Tildifying the isomorphisms we get $\phi^*\mathcal{O}(1) = \mathcal{O}(n)$.

2. By the first part, we have $\operatorname{Proj} A^{(n)} \cong \operatorname{Proj} A = \mathbb{P}^r_{A_0}$. By the third part of Exercise 45, the surjection of rings induces a closed immersion $\mathbb{P}^r \to \mathbb{P}^N$. The dimension of degree n homogeneous polynomials with $r+1$ variables is $\binom{n+r}{n}$, hence $N = \binom{n+r}{n} - 1$. It remains to show that the image of rational points $[\lambda_0 : \cdots : \lambda_r]$ corresponding to the homogeneous prime ideal $(x_i\lambda_j - x_j\lambda_i)_{i,j}$ is mapped to $[\lambda^I]$, corresponding to $(y_J\lambda^I - y_I\lambda^J)$, where I, J are multi-indices of the degree n monomials. For a general ring A and $A^{(n)}$, the constructed ϕ maps a homogeneous prime ideal $\mathfrak{p} = (f_1, \dots, f_r)$ to $\left(\prod_{\sum k_i = n} f_i^{k_i}\right)$. Hence $(x_i\lambda_j - x_j\lambda_i)_{i,j}$ is mapped to $\left(\prod_{\sum k_i = n}(x_i\lambda_j - x_j\lambda_i)^{k_i}\right) = (x^I\lambda^J - x^J\lambda^I)$, which finishes the proof.

Exercise 49. *Weighted projective space* (4 points) Let $n, a_0, \dots, a_n \geq 1$ be integers and A_0 a ring. Let $A = A_0[x_0, \dots, x_n]$. Equip A with the grading such that x_i is homogeneous of degree a_i. We define $\mathbb{P}_{A_0}(a_0, \dots, a_n) \coloneqq \operatorname{Proj} A_0[x_0, \dots, x_n]$ and call it *weighted projective space with weights* $(a_0, \dots, a_n)$ over A_0.

(i) Let $A_0[y_0, \dots, y_n]$ be the polynomial ring with the standard grading. Show that the graded A_0-algebra morphism

$$A_0[x_0, \dots, x_n] \to A_0[y_0, \dots, y_n]; \qquad x_i \mapsto y_i^{a_i}$$

induces a finite surjective morphism $\pi : \mathbb{P}^n_{A_0} \to \mathbb{P}^n(a_0, \dots, a_n)$.

(ii) Show that $\mathbb{P}_{A_0}(a_0, a_1) \cong \mathbb{P}^1_{A_0}$ for all $a_0, a_1 \geq 1$ (not necessarily via π).

(iii) Show that $\mathbb{P}_{A_0}(1,1,2) \cong V_+(y_1^2 - y_0y_2) \subseteq \mathbb{P}^3_{A_0}$. Deduce that $\mathbb{P}_{A_0}(1,1,2)$ is not isomorphic to $\mathbb{P}^2_{A_0}$ over $\operatorname{Spec} A_0$.

(Hint: Reduce to the case where A_0 is a field k and calculate the Zariski tangent space of $V_+(y_1^2 - y_0y_2)$ at the k-rational point $[0:0:0:1]$)

Solution. (i) The morphism $\varphi\colon A_0[x_0,\dots,x_n] \to A_0[y_0,\dots,y_n]\colon x_i \mapsto y_i^{a_i}$ is a graded homomorphism, hence by exercise 44 it induces a morphism of schemes

$$\pi\colon U \to \mathbb{P}^n(a_0,\dots,a_n)$$

where $U = \{\mathfrak{p} \in \operatorname{Proj}(A_0[y_0,\dots,y_n]) \mid \varphi(A_0[x_0,\dots,x_n]_+) \not\subset \mathfrak{p}\}$ is actually the whole projective space $\mathbb{P}^n_{A_0}$. Indeed, if $\mathfrak{p}$ contains $\varphi(A_0[x_0,\dots,x_n]_+)$, then it contains $\varphi(x_i) = y_i^{a_i}$, hence y_i for each i, thus it contains $A_0[y_0,\dots,y_n]_+$, which is not possible by definition of Proj. Therefore, we get a morphism

$$\pi\colon \mathbb{P}^n_{A_0} \to \mathbb{P}^n(a_0,\dots,a_n)$$

and we need to show that it is finite and surjective. These properties are local on the target, hence we can restrict to standard open affines on the target. Let us restrict to $D_+(x_0)$ (for $D_+(x_i)$, this is the same by symmetry). Over this open, the morphism π restricts to a morphism of affine schemes

$$\pi_{(x_0)}\colon D_+(y_0) \to D_+(x_0),$$

corresponding to the morphism of rings

$$\begin{aligned}\varphi_{(x_0)}\colon (A_0[x_0,x_1,\dots,x_n]_{x_0})_0 &\longrightarrow A_0[\tfrac{y_1}{y_0},\dots,\tfrac{y_n}{y_0}]\\ \tfrac{s(x_0,\dots,x_n)}{x_0^k} &\longmapsto \tfrac{s(y_0^{a_0},\dots,y_n^{a_n})}{y_0^{ka_0}}\end{aligned}$$

for $s(x_0,\dots,x_n)$ is homogeneous of degree ka_0, $k>0$. To show that $\pi_{(x_0)}$ is finite and surjective, it suffices to show that $\varphi_{(x_0)}$ is a finite morphism of rings. For $i = 1,\dots,n$, we have

$$(\tfrac{y_i}{y_0})^{a_0a_i} = \varphi_{(x_0)}\left(\tfrac{x_i^{a_0}}{x_0^{a_i}}\right),$$

which concludes.

(ii) Let $a_0, a_1 \geq 1$. We want to show that $\mathbb{P}^1_{A_0}(a_0,a_1)$ and $\mathbb{P}^1_{A_0}$ are isomorphic. First, observe that for any $d \geq 1$, $\mathbb{P}^1_{A_0}(da_0,da_1) \cong \mathbb{P}^1_{A_0}(a_0,a_1)$ under the Veronese isomorphism (exercise 48(i)), hence we can assume that a_0 and a_1 are coprime integers.

Denote the (degree m, one can take $m = a_0a_1$) variables on $\mathbb{P}^1_{A_0}(m,m)$ by z_0, z_1 and the variables on $\mathbb{P}^1_{A_0}(a_0,a_1)$ by x_0, x_1. Consider the morphism of graded rings

$$\begin{aligned}\varphi\colon A_0[z_0,z_1] &\longrightarrow A_0[x_0,x_1]\\ z_0 &\longmapsto x_0^{a_1}\\ z_1 &\longmapsto x_1^{a_0}.\end{aligned}$$

This induces an isomorphism of graded rings $A_0[z_0,z_1] \cong A_0[x_0,x_1]^{(m)}$, hence we conclude that

$$\mathbb{P}^1_{A_0} = \mathbb{P}^1_{A_0}(1,1) \cong \mathbb{P}^1_{A_0}(m,m) \overset{\varphi^*}{\cong} \operatorname{Proj}(A_0[x_0,x_1]^{(m)}) \cong \operatorname{Proj}(A_0[x_0,x_1]) = \mathbb{P}^1_{A_0}(a_0,a_1),$$

where all unlabelled isomorphisms are Veronese isomorphisms.

This isomorphism fails for $n > 1$ by (iii). Convince yourself where the argument above fails when $n > 1$.

(iii) We want to show that $\mathbb{P}^2_{A_0}(1,1,2) \cong V_+(y_1^2 - y_0y_2) \subset \mathbb{P}^3_{A_0}$. First, we have

$$V_+(y_1^2 - y_0y_2) = \mathrm{Proj}(A_0[y_0, y_1, y_2, y_3]/(y_1^2 - y_0y_2)),$$

which can be checked on standard open affines of $\mathbb{P}^3_{A_0}$.

Denote the coordinates on $\mathbb{P}^2_{A_0}(1,1,2)$ by x_0, x_1, x_2 (of respective degrees $1,1,2$). The surjective graded ring morphism

$$\begin{aligned} A_0[y_0, y_1, y_2, y_3] &\longrightarrow A_0[x_0^2, x_0x_1, x_1^2, x_2] = (A_0[x_0, x_1, x_2])^{(2)} \\ (y_0, y_1, y_2, y_3) &\longmapsto (x_0^2, x_0x_1, x_1^2, x_2) \end{aligned}$$

has kernel $(y_1^2 - y_0y_2)$, hence it induces an isomorphism of graded rings

$$A_0[y_0, y_1, y_2, y_3]/(y_1^2 - y_0y_2) \cong (A_0[x_0, x_1, x_2])^{(2)},$$

which in turns induces an isomorphism of schemes $\mathbb{P}^2_{A_0}(1,1,2) \cong V_+(y_1^2 - y_0y_2)$.

It remains to show that $X \coloneqq V_+(y_1^2 - y_0y_2)$ is not isomorphic to $\mathbb{P}^2_{A_0}$ over A_0. First, we assume that $A_0 = k$. Following the hint, we compute the Zariski tangent space of X at the k-rational point $[0:0:0:1]$. This is a local question, hence we can restrict to the open affine $D_+(y_3)$ (which indeed contains the point $[0:0:0:1]$. Over $D_+(y_3)$, we have an isomorphism

$$X \cap D_+(y_3) \cong Y \coloneqq V(y^2 - xz) \subset \mathbb{A}^3_k$$

under which $[0:0:0:1]$ is mapped to the origin $\mathbf{0} = (0,0,0) \in Y \subset \mathbb{A}^3_k(k)$. We have $Y = \mathrm{Spec}(k[x,y,z]/(y^2 - xz) = k[\overline{x}, \overline{y}, \overline{z}])$,

$$\begin{aligned} \mathfrak{m}_{\mathbf{0}} &= (\overline{x}, \overline{y}, \overline{z}) \\ \mathfrak{m}_{\mathbf{0}}^2 &= (\overline{x}^2, \overline{y}^2, \overline{z}^2, \overline{xy}, \overline{yz}, \overline{xz}) \\ \mathfrak{m}_{\mathbf{0}}/\mathfrak{m}_{\mathbf{0}}^2 &= \frac{k.\{\overline{x}, \overline{y}, \overline{z}, \overline{x}^2, \dots\}}{k.\{\overline{x}^2, \overline{y}^2, \overline{z}^2, \overline{xy}, \overline{yz}, \overline{xz}, \dots\}} = k.\{\overline{x}, \overline{y}, \overline{z}\} \cong k^{\oplus 3} \end{aligned}$$

hence the Zariski tangent space of Y at $\mathbf{0}$ has dimension 3, so does the Zariski tangent space of X at $[0:0:0:1]$. Thus X cannot be isomorphic to $\mathbb{A}^2_k$ since the Zariski tangent space of $\mathbb{A}^2_k$ at any k-rational point has dimension 2.

Now, for any ring A_0 the same holds, otherwise basechanging along any point $\mathrm{Spec}(k) \to \mathrm{Spec}(A_0)$ would yield an isomorphism in the case of a field.

Exercise 50. *Projection from a point* (4 points)

(i) Let k be a field. Show that the (graded) inclusion $k[x_0, \dots, x_{n-1}] \hookrightarrow k[x_0, \dots, x_n]$ defines a morphism $\pi : \mathbb{P}^n_k \setminus \{[0 : \dots : 0 : 1]\} \to \mathrm{Proj}\, k[x_0, \dots, x_{n-1}] = \mathbb{P}^{n-1}_k$. This is called the *projection from the point* $[0 : \dots : 0 : 1]$ (to the hyperplane $V_+(x_n)$).

(ii) Describe the k-rational points of the fibers of π over k-rational points of $\mathbb{P}^{n-1}_k$.

(iii) Show that for any homogeneous polynomial $f \in k[x_0, \dots, x_n]$ of degree 1 and a point $P \in D_+(f)$, there exists an automorphism $\sigma : \mathbb{P}^n_k \to \mathbb{P}^n_k$ such that $\sigma(V_+(f)) = V_+(x_n)$ and $\sigma(P) = [0 : \dots : 0 : 1]$.
(Hint: first try an explicit point P and a hyperplane $V_+(f)$ of your choice.)

(iv) Let $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$, $\mathbb{P}^3_k = \operatorname{Proj} k[y_0, y_1, y_2, y_3]$, and $\mathbb{P}^2_k = \operatorname{Proj} k[z_0, z_1, z_2]$. Find a homogeneous ideal $I \subseteq k[z_0, z_1, z_2]$ such that $V_+(I)$ is the image of $\pi \circ v_3$, where $v_3 \colon \mathbb{P}^1 \to \mathbb{P}^3$ is the Veronese map of Exercise 48 and π is the projection from $[0 : 0 : 1 : 0]$.

Solution.

(i) *The key is to use Ex. 44 (i)*

Consider the morphism of graded rings $\varphi \colon k[x_0, \ldots, x_{n-1}] \hookrightarrow k[x_0, \ldots, x_n]$ and note that

$$Z \coloneqq \{\mathfrak{p} \in \operatorname{Proj} k[x_0, \ldots, x_n] \mid \varphi((x_0, \ldots, x_{n-1})) \subseteq \mathfrak{p}]\} = (x_0, \ldots, x_{n-1})$$

Hence $Z = [0 : \cdots : 0 : 1]$. Then in the notation of Ex. 44 (i), $U = \mathbb{P}^n \setminus Z$ and we have produced a morphism

$$\pi \colon U \to \mathbb{P}^{n-1}.$$

(ii) Recall that a k-rational point in $\mathbb{P}^{n-1}$ is nothing but a map $\operatorname{Spec} k \to \mathbb{P}^{n-1}$. Recall the description from the class. But for this exercise, WLOG we may assume that the image lands in the affine open $D_+(x_0)$.

Also, maybe it helps to draw a diagram of the form

$$\begin{tikzcd}
\tilde{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D_+(\varphi(x_0)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d, "\pi"] \\
\operatorname{Spec}(k) \arrow[r] & D_+(x_0) \arrow[r] & \mathbf{P}^{n-1}
\end{tikzcd}$$

We want to find the k-rational points on $\operatorname{Spec} k \times_{\mathbb{A}^{n-1}} U$. Cover U by affine charts given by $D_+(x_i)$ and $\operatorname{Spec} k[x_0/x_n, \cdots, x_{n-1}/x_n, x_n/x_i]$ for $i = 0, \cdots, n-1$. Note that

$$\pi^{-1}(D_+(x_0)) \simeq D_+(\varphi(x_0)).$$

Then fibre product $\pi^{-1}(\operatorname{Spec} k)$ is given

$$\operatorname{Spec} k \otimes_{k[x_1/x_0, \cdots, x_{n-1}/x_0]} k[x_1/x_0, \cdots, x_{n-1}/x_0, x_n/x_0] \simeq \operatorname{Spec} k[x_n/x_0]$$

We get that the fibres over k-rational point on $\mathbb{A}^1(k)$.

Something interesting happens here. $X = \mathbb{P}^n \setminus \{[0 : \cdots : 0 : 1]\}$ can be covered by n standard affines. I found it interesting. I thought it is covered by $D_+(x_i)$ for $i = 0, \cdots, n-1$ and $D_+(x_n) \setminus \{[0 : \cdots : 0 : 1]\} \simeq \mathbb{A}^n \setminus \{(0, \cdots, 0)\}$. So one would think that in order to cover X by affine charts we need to consider $D_+(x_i)$ and affine covers of $\mathbb{A}^n \setminus \{(0, \cdots, 0)\}$ by $D_+(x_n x_i)$ for $i = 0, \cdots, n-1$. Note HOWEVER that $D_+(x_n x_i) \subset D_+(x_i)$

(iii) Not provided.

(iv) *Recall that $v_3 \colon \mathbb{P}^1 \to \mathbb{P}^3$ is induced by $k[y_0, y_1, y_2, y_3] \to k[x_0, x_1]$ defined by $y_0 \mapsto x_0^3$, $y_1 \mapsto x_0^2 x_1$, $y_2 \mapsto x_0 x_1^2$ and $y_3 \mapsto x_1^3$.*

Then $\varphi \coloneqq \pi \circ v_3$ is induced by $\varphi' \colon k[z_0, z_1, z_2] \to k[x_0, x_1]$ given by $z_0 \mapsto x_0^3$, $z_1 \mapsto x_0^2 x_1$ and $z_2 \mapsto x_1^3$. We claim that $\ker(\varphi') = I \coloneqq (z_1^3 - z_0^2 z_2)$. Clearly, $I \subset \ker(\varphi')$. To show the reverse inclusion, we restrict to $D_+(z_i)$ for $i = 0, 1, 2$:

– Over $D_+(z_0)$, we get

$$\begin{aligned} \varphi'_{(z_0)} \colon k[z_1, z_2] \to k[x_1] \colon z_1 &\mapsto x_1 \\ z_2 &\mapsto x_1^3 \end{aligned}$$

and thus $\ker(\varphi')_{(z_0)} = (z_1^3 - z_2) = I_{(z_0)}$.

– Over $D_+(z_1)$, we get

$$\begin{aligned} \varphi'_{(z_1)} \colon k[z_0, z_2] \to k[x_0, x_1]_{(x_0^2 x_1)} \colon z_0 &\mapsto \tfrac{x_0^3}{x_0^2 x_1} = \tfrac{x_0}{x_1} \\ z_2 &\mapsto \tfrac{x_1^3}{x_0^2 x_1} = \tfrac{x_1^2}{x_0^2} \end{aligned}$$

and thus $\ker(\varphi')_{(z_1)} = (1 - z_0^2 z_1) = I_{(z_1)}$.

– Over $D_+(z_2)$, we get

$$\begin{aligned} \varphi'_{(z_2)} \colon k[z_0, z_1] \to k[\tfrac{x_0}{x_1}] \colon z_0 &\mapsto \tfrac{x_0^3}{x_1^3} = \left(\tfrac{x_0}{x_1}\right)^3 \\ z_1 &\mapsto \tfrac{x_0^2 x_1}{x_1^3} = \left(\tfrac{x_0}{x_1}\right)^2 \end{aligned}$$

and thus $\ker(\varphi')_{(z_12)} = (z_1^3 - z_0^2) = I_{(z_2)}$.

Therefore $I = \ker(\varphi')$ globally.

Thus $\operatorname{Proj} k[z_0, z_1, z_2]/I \simeq \mathbb{P}^1$ and $V_+(I) = \varphi(\mathbb{P}^1)$.

Exercise 51. *Closed subschemes of products of projective spaces* (4 points)
Let A_0 be a ring, let $n_1, n_2 \geq 1$ be integers and consider

$$X = \mathbb{P}^{n_1}_{A_0} \times_{\operatorname{Spec} A_0} \mathbb{P}^{n_2}_{A_0} = \operatorname{Proj}(A_0[x_0, \dots, x_{n_1}] \times_{A_0} A_0[y, \dots, y_{n_2}]).$$

Let $f_1, \dots, f_m$ be a set of bihomogeneous polynomials, that is, polynomials that are simultaneously homogeneous in the x_i and the y_i. The *closed subscheme Z of X determined by the f_i* is defined as follows: Choose integers m_{xij} and m_{yij} such that all the $x_i^{m_{xij}} f_j$ and $y_i^{m_{yij}} f_j$ have the same (standard) degree, let $I \subseteq A_0[x_0, \dots, x_{n_1}] \times_{A_0} A_0[y, \dots, y_{n_2}]$ be the ideal generated by these new polynomials, and let $Z = V_+(I) \subseteq X$.

(i) Show that the closed subscheme Z does not depend on the chosen m_{xij} and m_{yij}.

(ii) Let $A_0 = k$ be a field and $Z \subseteq \mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k$ a closed subscheme given by the bihomogenous polynomial $x_0 y_1^2 - x_1 y_0^2$. Determine an ideal of image of Z under the Segre embedding $\mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k \to \mathbb{P}^3_k$ of Exercise 45.

Solution.

(i) The main point here is that *in $\mathbb{P}^n$ the locus $V_+(f)$ is the same as the locus of $n+1$ polynomials obtained by multiplying f by the coordinates; namely $V_+(x_0 f, x_1 f, \cdots, x_n f)$.* This is because for any ideal $\mathfrak{a}$, $V(\mathfrak{a} f) = V(\mathfrak{a}) \cup V(f)$. For us $\mathfrak{a}$ is the irrelevant ideal.

Another important point is: $xu^2 - yv^2$ an example of a bihomogenous polynomial in variables (x, y) and (u, v). BUT, $xu - y^2$ is not bihomogenous.

Finally: One needs to know what is the graded ring associated to $\mathbb{P}^n \times \mathbb{P}^m$ (check the previous sheet).

Once these points are completely clear we can move onto solving the exercise: (Lets do it just for a hypersurface, i.e. given by one homogenous polynomial f)

As the exercise suggests we could choose integers m_{xij} and m_{yij} such that all the $x_i^{m_{xij}} f$ and $y_i^{m_{yij}} f$ have the same (standard) degree d. The advantage of doing that is that one can consider the image of $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{m+n+1}$ under the Segre embedding. Then the monomial $x^I y^J$ with $|I| = |J| = d$ can be rewritten in terms of $\prod_{i \in I, j \in J} (x_i y_j)^{s_{ij}}$ such that $\sum_i s_{ij} = J_j$ and $\sum_j s_{ij} = I_i$. Here I_i denotes the degree of x_i in x^I and similarly for J. Let $x_0^{m_0} f = \sum_{I,J} a_{0,I,J} x^I y^J$ of degree d.

Rewriting the monomials this way allows us to consider $Z = V_+(x_i^{m_i} f, y_j^{m_j} f)$ as a subset of $\mathbb{P}^{n+m+1}$ under the Segre embedding and the image is given by vanishing of homogenous polynomials of the form $\sum_{0,I,J} a_{0,I,J} \prod_{i \in I, j \in J} z_{ij}^{s_{ij}}$. (Of course one needs to do this rewriting business for all $x_i^{m_i} f, y_j^{m_j} f$)

The independence of m_{x_i} and m_{y_j} can be argued as follows: Assume f is of bideg (d_1, d_2) with $d_1 > d_2$. Just multiplying f with $y_j^{d_1 - d_2}$ for all j, we already get a collection of bihomogenous degree $d = 2d_2$ polynomials defining Z. As above, lets call the ideal $\mathfrak{a} \subset k[z_{00}, \cdots, z_{mn}]$. Now multiplying by x_i's (all of them) again will give us a bunch of bihomog of degree $(d_2 + 1, d_2)$ polynomials. In order to standardize the degree we need to multiply with y_i's (all of them) again! This means, in terms of the coordinates z_{ij} of $\mathbb{P}^{m+n+1}$, the image of this new set of polynomials generate the ideal $(z_{00}\mathfrak{a}, \cdots, z_{mn}\mathfrak{a})$, which defines the same ideal as $\mathfrak{a}$.

(ii) Let $Z = V_+(I)$ for $I = (x_0 y_1^2 - x_1 y_0^2) \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$. We first need to make them bihomogenous in the same degree. The minimalistic way this can be done is by multiplying the equation by x_0 and x_1. We get

$$V_+(x_0 y_1^2 - x_1 y_0^2) = V_+(x_0^2 y_1^2 - x_0 x_1 y_0^2, x_0 x_1 y_1^2 - x_1^2 y_0^2)$$

Recall that the Segree embedding $\mathbb{P}^1_k \times \mathbb{P}^1_k \hookrightarrow \mathbb{P}^3_k$ is given by

$$([x_0 : x_1], [y_0 : y_1]) \longmapsto [x_0 y_0 : x_1 y_0 : x_0 y_1 : x_1 y_1].$$

The bihomogenous equation of Z in degree (2,2) translates to $z_2^2 - z_0 z_1, z_2 z_3 - z_2^2$ Furthermore the equation defining the image of $\mathbb{P}^1 \times \mathbb{P}^1$ is given by $V_+(z_0 z_3 - z_1 Z_2)$. Thus the image of the curve Z is defined by the ideal

$$(z_2^2 - z_0 z_1, z_2 z_3 - z_2^2, z_0 z_3 - z_1 z_2)$$

This is the famous twisted cubic curve in $\mathbb{P}^3$.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 52. *Grothendieck's classification of Locally free sheaves on $\mathbb{P}^1$* (+ 5 extra points) Let k be a field. Let $\mathcal{F}$ be a locally free sheaf of rank r on $\mathbb{P}^1_k$.

(i) Show that $\mathcal{F}|_{D_+(x_i)} \cong \mathcal{O}^{\oplus r}_{\mathbb{A}^1_k}$.

(ii) Prove the following normal form for matrices over $k[t, t^{-1}]$: Let M be an $r \times r$ matrix over $k[t, t^{-1}]$ with determinant t^n for some $n \in \mathbb{Z}$. Then, there exist matrices $A \in \mathrm{GL}_r(k[t^{-1}])$ and $B \in \mathrm{GL}_r(k[t])$ such that

$$A \cdot M \cdot B = \mathrm{diag}(t^{a_1}, \cdots, t^{a_r}); \quad \text{the diagonal matrix}$$

with $a_1 \geq a_2 \geq \ldots \geq a_r$, $a_i \in \mathbb{Z}$, and the a_i are uniquely determined by M.

(Hint: Assume that M has polynomial entries and argue by induction on r.)

(iii) Conclude that $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_r)$, where the a_i are integers which are uniquely determined by $\mathcal{F}$ up to reordering.

Solution. Not provided.

Exercise 53. *Invertible sheaves* (4 points)
Let $\varphi\colon \mathcal{L} \to \mathcal{M}$ be a homomorphism of invertible sheaves on a scheme $(X, \mathcal{O}_X)$.

(i) Show that φ is an isomorphism if φ is surjective.

(ii) Give an example where φ is injective but not an isomorphism.

Solution.

1. To check that the homomorphism is isomorphism it suffices to check that it induces isomorphisms on stalks. Let $x \in X$ be a point and $\operatorname{Spec} A$ be an affine open containing x, such that $\mathcal{L}$ and $\mathcal{M}$ are trivial on $\operatorname{Spec} A$. Now $\mathcal{L}_x \cong \mathcal{M}_x \cong A_{\mathfrak{m}}$ for some maximal idel $\mathfrak{m} \subset A$. Under this identification, ϕ induces a surjective homomorphism of $A_{\mathfrak{m}}$-modules (Note here it is just a module homomorphism, not a ring hom.) $\alpha : A_{\mathfrak{m}} \to A_{\mathfrak{m}}$. Assume now α is not injective. We take $s \in \ker \alpha$. It follows then $s \cdot \alpha(1) = \alpha(s) = 0$. Hence $\alpha(1)$ annihilates s and therefore $\operatorname{im} \alpha$ is contained in the annihilator of s. If s is not 0, we derive a contradiction to α surjective. Hence s is 0 and α is injective and an isomorphism.

2. There are two kinds of general counterexamples. We can take a line bundle $\mathcal{L}$ with a global section s, then the global section induces an injection $\mathcal{O}_X \to \mathcal{L}, 1 \mapsto s$. In good cases it is an injection but not an isomorphism, for example when X is integral. An explicit example for this case is $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}(1)$. The second kind is the ideal sheaf of an effective Cartier divisor, which gives an injection into the structure sheaf. An explicit example is $\mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^1}$

   Note that these two examples can be considered as the same by tensoring with the line bundle associated to an effective Cartier divisor.

   Also consider $(x) \subseteq k[x]$.

Exercise 54. *Blowing up points in affine space* (4 points)
Let k be a field. Let $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$, $\mathbb{P}^{n-1}_k = \operatorname{Proj} k[y_1, \ldots, y_n]$, and $O = (0, \ldots, 0) \in \mathbb{A}^n_k(k)$. The *blow-up of* $\mathbb{A}^n_k$ *in* O is the closed subscheme $X \subseteq \mathbb{P}^{n-1}_k \times_{\operatorname{Spec} k} \mathbb{A}^n_k = \mathbb{P}^{n-1}_{\mathbb{A}^n_k}$ determined by the homogeneous ideal

$$I = (\{x_i y_j - x_j y_i\}_{i,j=1}^n).$$

Note that X comes with two projections $\pi_1 : X \to \mathbb{A}^n_k$ and $\pi_2 : X \to \mathbb{P}^{n-1}_k$.

(i) Show that π_1 is an isomorphism over $\mathbb{A}^n_k \setminus \{O\}$.

(ii) Show that the fiber $\pi_1^{-1}(O)$ is isomorphic to $\mathbb{P}^{n-1}_k$.

(iii) Show that the fiber of π_2 over a k-rational point of $\mathbb{P}^{n-1}_k$ is isomorphic to $\mathbb{A}^1_k$.

Solution.

(i) *Recall that, (i) homogenous ideals are graded sub-modules. (ii) $X \subset \mathbb{P}^n_R$ be defined I_X then $R[x_0, \cdots, x_n]/I_X$ is a graded ring whose* Proj *is X.*

Fix $i \in \{1, \ldots, n\}$. In the following, we show that π_1 gives an isomorphism of schemes $\pi_1^{-1}(D(x_i)) \xrightarrow{\sim} D(x_i)$. We have $\pi_1^{-1}(D(x_i)) = X \cap \mathbb{P}^{n-1}_{D(x_i)}$ is the closed subscheme corresponding to the homogeneous ideal

$$I_{x_i} = (\{x_j y_l - x_l y_j\}_{j,l=1}^n) = \left(\left\{y_l - \frac{x_l y_i}{x_i}\right\}_{l=1}^n\right) \subset k[y_1, \ldots, y_n, x_1, \ldots, x_n, x_i^{-1}].$$

What happened to the rest of the elements like $x_j y_l - y_j x_l$? Check that, they don't give anything new.

From here,

$$X \simeq \operatorname{Proj}\left(\frac{k[x_1, \cdots, x_n, x_i^{-1}][y_1, \cdots, y_n]}{(y_l - \frac{x_l y_i}{x_i})_{l=1}^n}\right) \simeq \operatorname{Proj} k[x_1, \cdots, x_n, x_i^{-1}][y_i] \simeq D_+(x_i)$$

(ii)

$$k[x_1, \cdots, x_n]/(x_1, \cdots, x_n) \otimes_{k[x_1, \cdots, x_n]} k[x_1, \cdots, x_n, y_1, \cdots, y_n, y_i^{-1}]_0 \simeq k[y_1, \cdots y_n, y_i^{-1}]_0.$$

Hence $\pi^{-1}(0, \cdots, 0) \simeq \mathbb{P}^n_k$.. Fix $1 \leq i \leq n$ and consider the basic open $D_+(y_i) \subset \mathbb{P}^{n-1}_k$. By construction,

$$\begin{aligned} &\pi_1^{-1}(O) \cap D_+(y_i) \times_{\operatorname{Spec}(k)} \mathbb{A}^n_k \\ = \operatorname{Spec} &\Bigg(\underbrace{k\left[\frac{y_1}{y_i}, \ldots, \frac{y_n}{y_i}, x_1, \ldots, x_n\right] \Big/ I_{(y_i)} \otimes_{k[x_1, \ldots, x_n]} k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)}_{A_i :=}\Bigg), \end{aligned}$$

where

$$I_{(y_i)} = \left(\left\{\frac{y_l}{y_i} x_j - \frac{y_j}{y_i} x_l\right\}_{j,l=1}^n\right) = \left(\left\{x_j - \frac{y_j}{y_i} x_i\right\}_{j=1}^n\right).$$

Since

$$I_{(y_i)} \subset (x_1, \ldots, x_n) k\left[\frac{y_1}{y_i}, \ldots, \frac{y_n}{y_i}, x_1, \ldots, x_n\right],$$

we obtain a canonical ring isomorphism

$$\varphi_i' : A_i \cong k\left[\frac{y_1}{y_i}, \ldots, \frac{y_n}{y_i}, x_1, \ldots, x_n\right] / (x_1, \ldots, x_n) = k\left[\frac{y_1}{y_i}, \ldots, \frac{y_n}{y_i}\right].$$

Hence, we obtain a morphism of schemes

$$\varphi_i : \pi_1^{-1}(O) \cap D_+(y_i) \times_{\operatorname{Spec}(k)} \mathbb{A}^n_k \to D_+(y_i) \subset \mathbb{P}^{n-1}_k.$$

Similarly, for fixed $1 \leq i, j \leq n$, holds

$$\begin{aligned} &\pi_1^{-1}(O) \cap D_+(y_i y_j) \times_{\operatorname{Spec}(k)} \mathbb{A}^n_k \\ = \operatorname{Spec} &\Bigg(\underbrace{k\left[\frac{y_1}{y_i}, \ldots, \frac{y_n}{y_i}, \frac{y_i}{y_j}, x_1, \ldots, x_n\right] \Big/ I_{(y_i y_j)} \otimes_{k[x_1, \ldots, x_n]} k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)}_{A_{ij} :=}\Bigg), \end{aligned}$$

where

$$I_{(y_i y_j)} = (I_{(y_i)})_{(y_j)} = \left(\left\{x_j - \frac{y_l y_j}{y_i y_j} x_i\right\}_{j=1}^n\right).$$

The same argument as above, gives an isomorphism of rings

$$\varphi_{ij} : A_{ij} \xrightarrow{\sim} k\Big[\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}, \frac{y_i}{y_j}\Big].$$

By construction, the following diagram commutes for all i, j:

$$\begin{tikzcd}
A_i \arrow[dr] \arrow[drr, dashed, "\varphi_i"] & & A_j \arrow[dl] \arrow[drrr, dashed, "\varphi_j"] & & & \\
& A_{ij} \arrow[drr, dashed, "\varphi_{ij}"] & k\Big[\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}\Big] \arrow[dr] & & & k\Big[\frac{y_1}{y_j}, \dots, \frac{y_n}{y_j}\Big] \arrow[dll] \\
& & & k\Big[\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}, \frac{y_i}{y_j}\Big] & &
\end{tikzcd}$$

Thus, the morphisms φ_i glue to an isomorphism of schemes $\pi_1^{-1}(O) \cong \mathbb{P}_k^{n-1}$.

(iii) Fix $1 \le i \le n$ and consider the basic open $D_+(y_i) \subset \mathbb{P}_k^{n-1}$. We have

$$\pi_2^{-1}(D_+(y_i)) = X \cap D_+(y_i) \times_{\mathrm{Spec}(k)} \mathbb{A}_k^n = \mathrm{Spec}\left(k\Big[\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}, x_1, \dots, x_n\Big] \Big/ I_{(y_i)}\right),$$

where $I_{(y_i)}$ is as in (ii). We have that π_2 corresponds to the natural map

$$k\Big[\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}\Big] \to k\Big[\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}, x_1, \dots, x_n\Big] \Big/ I_{(y_i)}.$$

Note that the natural map

$$k\Big[x_i, \frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}\Big] \to k\Big[\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}, x_1, \dots, x_n\Big] \Big/ I_{(y_i)}$$

is an isomorphism of rings compatible with π_2. Hence, we have a commuting diagram:

$$\begin{tikzcd}
\pi_2^{-1}(D_+(y_i)) \arrow[rr, "\cong"] \arrow[dr, "\pi_2"'] & & D_+(y_i) \times_{\mathrm{Spec}(k)} \mathbb{A}_k^1 \arrow[dl, "\mathrm{pr}_1"] \\
& D_+(y_i) &
\end{tikzcd}$$

It follows that the fiber of each rational point in $\mathbb{P}_k^{n-1}$ is isomorphic to $\mathbb{A}_k^1$.

Exercise 55. *The standard Cremona involution* (4 points)
Let k be a field and let

$$\begin{aligned}
\sigma : \mathbb{P}_k^2 \setminus \{[0:0:1], [0:1:0], [1:0:0]\} &\to \mathbb{P}_k^2 \\
[a_0 : a_1 : a_2] &\mapsto [a_1 a_2 : a_0 a_2 : a_0 a_1]
\end{aligned}$$

be the morphism determined by the global sections $x_1x_2, x_0x_2, x_0x_1 \in \Gamma(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(2))$.

(i) Show that $\sigma(V_+(x_i))$ is a point for all i.

(ii) Show that σ^2 is well-defined on $\mathbb{P}^2_k \setminus (\cup_{i=1}^3 V_+(x_i))$ and that it extends to the identity on $\mathbb{P}^2_k$. Conclude that there is no morphism $\mathbb{P}^2 \to \mathbb{P}^2$ that restricts to σ on $\mathbb{P}^2_k \setminus \{[0:0:1],[0:1:0],[1:0:0]\}$.

(iii) *(+1 extra point)* Show that σ induces a morphism $X \to X$, where X is the blow-up of $\mathbb{P}^2$ at $[1:0:0],[0:1:0]$, and $[0:0:1]$, i.e., the scheme obtained by glueing the blow-ups of the three affine planes $\mathbb{A}^2_k \cong D_+(x_i) \subseteq \mathbb{P}^2_k$ in $(0,0)$. Conclude that σ defines an automorphism of X.

Solution. (i) Consider a point $(1:a_1:a_2) \in V_+(x_0)$. By definition

$$\sigma(s_0(1:a_1:a_2), s_1(0:a_1:a_2), s_2(1:a_1:a_2))$$

for $s_0, s_1, s_2 \in \Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(2))$ three sections given explicitly by $s_0 = x_1x_2$, $s_1 = x_0x_2$ and $s_2 = x_0x_1$. Hence, we have

$$(0:a_1:a_2) \mapsto (a_1a_2:0:0) = (1:0:0).$$

Thus $\sigma(V_+(x_0)) = [1:0:0]$, $\sigma(V_+(x_1)) = [0:1:0]$ and $\sigma(V_+(x_2)) = [0:0:1]$.
Note that σ is not defined everywhere on $V_+(x_i)$.

(ii) By (i) $\sigma(V_+(x_i))$ map to $[1:0:0],[0:1:0],[0:0:1]$ respectively for $i=0,1,2$. Since σ is well-defined outside these image points, σ^2 is well-defined outside the preimages of these theree points. Hence σ^2 is well-defined on $\mathbb{P}^2_k \setminus (\cup_{i=1}^3 V_+(x_i))$.
Given $[a_0:a_1:a_2] \in \mathbb{P}^2_k \setminus (\cup_{i=1}^3 V_+(x_i))$, note that $a_0, a_1, a_2 \neq 0$. Hence we have

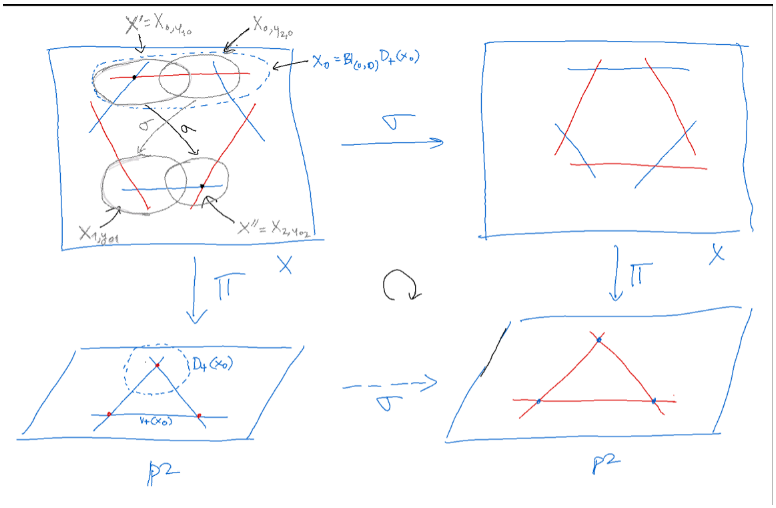
$$\sigma^2([a_0:a_1:a_2]) = \sigma([a_1a_2:a_0a_2:a_0a_1]) = [a_0^2a_1a_2 : a_1^2a_0a_2 : a_2^2a_0a_1] = [a_0:a_1:a_2].$$

Therefore $\sigma^2|_{\mathbb{P}^2_k \setminus (\cup_{i=1}^3 V_+(x_i))} = \mathrm{Id}$ and hence $\mathrm{Id}\colon \mathbb{P}^2 \to \mathbb{P}^2$ extends σ^2.
For the second assertion, assume to the contrary that there exists $\widetilde{\sigma}\colon \mathbb{P}^2 \to \mathbb{P}^2$ extending σ. Since $\widetilde{\sigma}^2|_{\mathbb{P}^2_k \setminus (\cup_{i=1}^3 V_+(x_i))} = \sigma^2 = \mathrm{Id}_{\mathbb{P}^2_k \setminus (\cup_{i=1}^3 V_+(x_i))}$ and $\mathbb{P}^2$ is integral and closed, we obtain *from the equalizer exercise* that $\widetilde{\sigma}^2 = \mathrm{Id}$. So $\widetilde{\sigma}$ is an isomorphism. However, by (i) we know that

$$\sigma(V_+(x_i)) = \widetilde{\sigma}(V_+(x_i)) = \mathrm{pt}\,.$$

An isomorphism cannot contract any line.

(iii)First note that we only need to to define the maps over open charts. Gluing is not an issue since outside the six coloured lines below σ does remains the same and hence by the equalizer exercise things glue. How we choose the charts is the tricky part.
Solution. A.



First, note that on the intersection $D_+(x_0) \cap D_+(x_1) \cap D_+(x_2) = \operatorname{Spec} k[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_0}{x_1}, \frac{x_0}{x_2}]$, the standard Cremona involution sends $\frac{x_j}{x_i}$ to $\frac{x_i}{x_j}$.

We cover $\mathbb{P}^2$ by the $D_+(x_i)$. The blow-up of $(0,0) \in \mathbb{A}^2_k = D_+(x_i)$ is given as a subscheme X_i of $\mathbb{A}^2_k \times \mathbb{P}^1_k$ by the ideal

$$\begin{aligned}
&(\frac{x_1}{x_0}y_{10} - \frac{x_2}{x_0}y_{20}) \text{ if } i = 0\\
&(\frac{x_0}{x_1}y_{01} - \frac{x_2}{x_1}y_{21}) \text{ if } i = 1\\
&(\frac{x_0}{x_2}y_{02} - \frac{x_1}{x_2}y_{12}) \text{ if } i = 2.
\end{aligned}$$

Consider $X' = X_{0,y_{20}}$ and $X'' = X_{2,y_{02}}$. Then, $X' = \operatorname{Spec} k[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{y_{10}}{y_{20}}]/(\frac{x_1}{x_0}\frac{y_{10}}{y_{20}} - \frac{x_2}{x_0})$ and $X'' = \operatorname{Spec} k[\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{y_{12}}{y_{02}}]/(\frac{x_0}{x_2} - \frac{x_1}{x_2}\frac{y_{12}}{y_{02}})$. Define a morphism $X'' \to X'$ by

$$\begin{aligned}
\frac{y_{12}}{y_{02}} &\mapsto \frac{x_1}{x_0}\\
\frac{x_1}{x_2} &\mapsto \frac{y_{10}}{y_{20}}.
\end{aligned}$$

Then, on the locus where all the x_i are invertible, we have

$$\begin{aligned}
\frac{y_{10}}{y_{20}} &= \frac{x_2}{x_0} / \frac{x_1}{x_0} = \frac{x_2}{x_1}\\
\frac{y_{12}}{y_{02}} &= \frac{x_0}{x_2} / \frac{x_1}{x_2} = \frac{x_0}{x_1},
\end{aligned}$$

so the morphism defined above extends the Cremona involution. Since we can extend the involution one chart at a time (by the equalizer exercise) and the S_3-action given by permutation of coordinates permutes the six affine charts of X transitively, we are done.

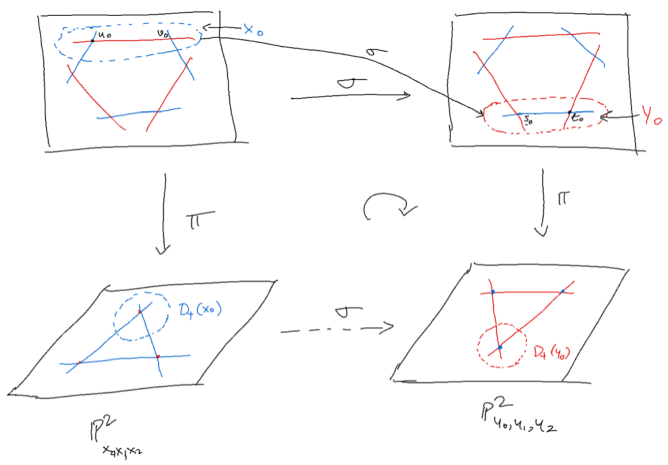
Solution. B. *This solution is very similar to the above. The difference is that we treat the target $\mathbb{P}^2$ as a different space and σ as a birational map, as opposed to birational self-map.*

Recall that Blow-up of $D_+(x_i)$ at $[1:0:0]$ is given by $X_0 \subset \mathbb{P}^1_{D_+(x_0)}$ defined by the homogenous ideal

$$(x_1 v_0 / x_0 - x_2 u_0 / x_0) \subset k[x_1/x_0, x_2/x_0][u_0, v_0].$$

One could write the other blow-ups similarly and X is obtained by gluing X_0, X_1 and X_2. We let $\pi_i \colon X_i \to D_+(x_i)$ the blow-up maps. Let $\pi \colon X \to \mathbb{P}^2$ denote the blow-up map. Note that $X_i \cap X_j \simeq D_+(x_i x_j)$ for all $i, j = 0, 1, 2$.

Let y_0, y_1, y_2 be the coordinates of the $\mathbb{P}^2$ in the image. Similarly X can also be obtained by gluing Y_0 given by Blowing up $D_+(y_0)$ at $[1:0:0]$ and Y_1 and Y_2.



Define $\sigma \colon X_0 \to Y_0$ via the induced map of graded rings

$$\frac{k[y_1/y_0, y_2/y_0, s_0, t_0]}{(y_1 t_0/y_0 - y_2 s_0/y_0)} \to \frac{k[x_1/x_0, x_2/x_0, u_0, v_0]}{(x_1 v_0/x_0 - x_2 u_0/x_0)}$$

By sending

$$y_i/y_0 \mapsto x_i/x_0 \text{ and } s_0 \mapsto v_0, t_0 \mapsto u_0.$$

The maps glue by the equalizer exercise.

The map above looks so simple since on $\mathbb{P}^2$ the map "roughly speaking" looks like $(x_0 : x_1 : x_2) \mapsto (1/x_0 : 1/x_1 : 1/x_2)$ and locally that's exactly what's happening $y_1/y_0 = x_0 x_2/x_1 x_2 = x_0/x_1$ and similarly $y_2/y_0 = x_0/x_2$. The exceptional $\mathbb{P}^1 \subset X_0$ then simple gets mapped to the expectional in Y_0 (switches direction). So we are hiding the hard work, that's explicit in Solution A.

Exercise 56. *Some examples of curves and surfaces* (4 points)
Let k be an algebraically closed field.

(i) Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(2)$. Show that the morphism $\mathbb{P}^2 \to \mathbb{P}^5_k$ induced by the global sections $x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2 \in \Gamma(X, \mathcal{L})$ is the Veronese map of Exercise 48. In general, for integers $n, k \geq 0$ think about what kind of map $\mathcal{O}_{\mathbb{P}^n}(k)$ induces on $\mathbb{P}^n$.

(ii) Show that the sections $x_0^2, x_1^2, x_2^2, x_1(x_0 - x_2), (x_0 - x_1)x_2 \in \Gamma(X, \mathcal{L})$ induce a closed immersion $X \hookrightarrow \mathbb{P}^4_k$.

(iii) Consider the subspace $V \subseteq \Gamma(X, \mathcal{L})$ in (i) spanned by those of the chosen basis above that vanish at the point $[0:0:1]$ and consider the morphism $\mathbb{P}^2 \setminus [0:0:1] \to \mathbb{P}^4$ induced by V. Describe the closure of the image in $\mathbb{P}^4$ of the cuspidal cubic curve given by $x_1^2 x_2 = x_0^3$ in $\mathbb{P}^2$ under this morphism.

Solution.

(i) By construction, the map $X = \mathbb{P}^2_k \to \mathbb{P}^5_k$ defined by $\mathcal{L}$ and the listed sections locally on target (over $D_+(y_i) \subset \mathbb{P}^5_k$ for $i \in \{0, \dots, 5\}$, where y_i are the homogeneous coordinates on $\mathbb{P}^5_k$) looks like the map $D_+(s_i) \to D_+(x_i)$ corresponding to the ring map

$$k[\frac{y_0}{y_i}, \dots, \frac{y_5}{y_i}] \to \Gamma(X_{s_i}, \mathcal{O}_{X_{s_i}}) = k[x_0, x_1, x_2]_{(s_i)}$$

$$\frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}$$

where s_i denotes the i-th monomial of degree 2 in variables x_0, x_1, x_2.

But this description coincides with how the Veronese embedding $\mathbb{P}^2_k \to \mathbb{P}^5_k$ looks locally on target. Hence the two are equal.

Analogously, the sections of $\mathcal{O}_{\mathbb{P}^n_k}(k)$ given by all monomials of degree k in variables $x_0, \dots, x_n$ will give the k-th Veronese embedding $\mathbb{P}^n_k \to \mathbb{P}^N_k$.

(ii) First, note that the sections $s_0 = x_0^2, s_1 = x_1^2, s_2 = x_2^2, s_3 = x_1(x_0 - x_2), s_4 = (x_0 - x_1)x_2 \in \Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(2))$ indeed globally generate $\mathcal{O}_{\mathbb{P}^2_k}(2))$, because already x_0^2, x_1^2, x_2^2 do, as $X_{x_i^2} = D_+(x_i^2) = D_+(x_i)$. Thus, we do get a morphism $f : \mathbb{P}^2_k \to \mathbb{P}^4_k$.

Recall Rem.14.7 from the lectures, namely that *any* (finite) collection $(s_1, \dots, s_n)$ of sections of a line bundle $\mathcal{L}$ on X give rise to a morphism

$$X \supseteq \bigcup_{i=1}^{n} X_{s_i} \longrightarrow \mathbb{P}^n_k.$$

Next, we will use Proposition 14.11 to show that f is a closed immersion. First, we note that each of the X_{s_i} is affine because $X_{s_i} = D_+(s_i) \subset \mathbb{P}^2_k$. It is left to see that for each $i \in \{0, \dots, 4\}$, the ring map

$$k[\frac{y_0}{y_i}, \frac{y_1}{y_i}, \frac{y_2}{y_i}, \frac{y_3}{y_i}, \frac{y_4}{y_i}] \to k[x_0, x_1, x_2]_{(s_i)},$$

$$\frac{y_j}{y_i} \mapsto \frac{s_j}{s_i}$$

is surjective.

For $i = 0$, the map is

$$k[\frac{y_1}{y_0}, \frac{y_2}{y_0}, \frac{y_3}{y_0}, \frac{y_4}{y_0}] \to k[\frac{x_1}{x_0}, \frac{x_2}{x_0}],$$

$$(\frac{y_1}{y_0}, \frac{y_2}{y_0}, \frac{y_3}{y_0}, \frac{y_4}{y_0}) \mapsto (\frac{x_1^2}{x_0^2}, \frac{x_2^2}{x_0^2}, \frac{x_1(x_0 - x_2)}{x_0^2}, \frac{(x_0 - x_1)x_2}{x_0^2}).$$

We have that $\frac{x_1(x_0-x_2)}{x_0^2} - \frac{(x_0-x_1)x_2}{x_0^2} = \frac{x_1-x_2}{x_0}$ lies in the image, hence so does $\frac{(x_1-x_2)^2}{x_0^2}$, and therefore $\frac{2x_1x_2}{x_0^2} = \frac{x_1^2}{x_0^2} + \frac{x_2^2}{x_0^2} - \frac{(x_1-x_2)^2}{x_0^2}$ lies in the image, and since $\mathrm{char}(k) \neq 2$, $\frac{x_1x_2}{x_0^2}$ is then also contained in the image. This implies that $\frac{x_1}{x_0} = \frac{x_1(x_0-x_2)}{x_0^2} + \frac{x_1x_2}{x_0^2}$ and $\frac{x_2}{x_0} = \frac{(x_0-x_1)x_2}{x_0^2} + \frac{x_1x_2}{x_0^2}$ lie in the image, which shows surjectivity.

For $i = 1$, the map is

$$k[\frac{y_0}{y_1}, \frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{y_4}{y_1}] \to k[\frac{x_0}{x_1}, \frac{x_2}{x_1}],$$

$$(\frac{y_0}{y_1}, \frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{y_4}{y_1}) \mapsto (\frac{x_0^2}{x_1^2}, \frac{x_2^2}{x_1^2}, \frac{x_0 - x_2}{x_1}, \frac{x_0 x_2}{x_1^2} - \frac{x_2}{x_1}).$$

We have that $\frac{(x_0-x_2)^2}{x_1^2}$ lies in the image, and hence also $\frac{x_0x_2}{x_1^2} = \frac{1}{2}(\frac{x_0^2}{x_1^2} + \frac{x_2^2}{x_1^2} - \frac{(x_0-x_2)^2}{x_1^2})$ lies in the image. Thus, also the generators $\frac{x_2}{x_1} = \frac{x_0x_2}{x_1^2} - (\frac{x_0x_2}{x_1^2} - \frac{x_2}{x_1})$ and $\frac{x_0}{x_1} = \frac{x_0-x_2}{x_1} + \frac{x_2}{x_1}$ are contained in the image.

In a similar manner, one can show the surjectivity for the other i's.

(iii) As described in the question, we consider the sections $x_0^2, x_1^2, x_0x_1, x_0x_2, x_1x_2 \in \Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(2))$ and the induced morphism $f : \mathbb{P}^2_k \setminus [0 : 0 : 1] \to \mathbb{P}^4_k$. Similarly to part (i), we recognize that this morphism is actually the one that corresponds by Exercise 44(i)[4] to the morphism of graded rings

$$k[y_0, y_1, y_2, y_3, y_4] \xrightarrow{\varphi} k[x_0, x_1, x_2]^{(2)},$$

$$(y_0, y_1, y_2, y_3, y_4) \mapsto (x_0^2, x_1^2, x_0x_1, x_0x_2, x_1x_2),$$

bearing in mind the isomorphism $\mathbb{P}^2_k \cong \operatorname{Proj} k[x_0, x_1, x_2]^{(2)}$.

Consider the homogeneous ideal $I = (x_0x_1^2x_2 - x_0^4, x_1^3x_2 - x_0^3x_1, x_1^2x_2^2 - x_0^3x_2)$ (technically speaking, this gives two different ideals depending on whether we view it as an ideal in $k[x_0, x_1, x_2]$ or $k[x_0, x_1, x_2]^{(2)}$, but it should be clear from the context which of the two possibilities we mean each time). We have $C \coloneqq V_+(x_1^2x_2 - x_0^3) = V_+(I) \subset \mathbb{P}^2_k$. Consider now the ideal $J = (y_3y_1 - y_0^2, y_4y_1 - y_2y_0, y_4^2 - y_3y_0) \subseteq k[y_0, y_1, y_2, y_3, y_4]$.

We observe that $\varphi(J) \subset I$, which shows that $f(C \setminus [0 : 0 : 1]) \subset V_+(J) =: C'$, and hence for dimension reasons we must have $\overline{f(C \setminus [0 : 0 : 1])} = C'$.

Exercise 57. *Ramified coverings* (4 points)
Let k be an algebraically closed field of characteristic $\operatorname{char}(k) = p \geq 0$. Consider a homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ of degree d and the two closed subschemes $X = V_+(f) \subseteq \mathbb{P}^n_k$ and $Y = V_+(f - x_{n+1}^d) \subseteq \mathbb{P}^{n+1}_k$. Show that the projection $g \colon Y \to \mathbb{P}^n_k$ from $[0 : \ldots : 0 : 1]$ satisfies the following properties:

(i) Restricted to the intersection $Y \cap V_+(x_{n+1}) = V_+(f - x_{n+1}^d, x_{n+1}) \subseteq \mathbb{P}^{n+1}_k$, the morphism g yields an isomorphism with X.

(ii) If $p \nmid d$, then for every closed point in the complement of $X \subseteq \mathbb{P}^n_k$ the fiber of g is a reduced scheme consisting of exactly d k-rational points.

(iii) If $p \mid d$, then the fiber of g over every closed point of $\mathbb{P}^n_k$ is non-reduced.

Solution. (i) The projection $g \colon Y \to \mathbb{P}^n_k$ from $[0 : \cdots : 0 : 1]$ is the morphism induced by the inclusion of graded rings $k[x_0, \ldots, x_n] \hookrightarrow k[x_0, \ldots, x_{n+1}]$. In particular, the restriction of g to $Y \cap V_+(x_{n+1})$ is induced by the homomorphism

$$\gamma \colon k[x_0, \ldots, x_n] \longrightarrow \frac{k[x_0, \ldots, x_{n+1}]}{(f - x_{n+1}^d, x_{n+1})} = \frac{k[x_0, \ldots, x_n]}{(f)},$$

which induces a closed embedding identifying $Y \cap V_+(x_{n+1})$ with $X \subset \mathbb{P}^n_k$.

(ii) Let $P \in \mathbb{P}^n_k - X$ be a closed point. Let us assume it is in $D_+(x_0)$, then we are in the following situation:

[4] (Non-)functoriality of Proj

$$\begin{array}{ccc}
Y & \xrightarrow{\quad g \quad} & \mathbb{P}^n_k \\
\cup & & \cup \\
Y - g^{-1}(X) & \longrightarrow & \mathbb{P}^n_k - X \\
\Cup & & \Cup \\
g^{-1}(P) & \longrightarrow & P
\end{array}$$

For simplicity, let us assume that $P \in D_+(x_0)$, then this diagram becomes

$$\begin{array}{ccc}
Y_{(x_0)} & \xrightarrow{\quad g_{(x_0)} \quad} & \mathbb{A}^n_k = D_+(x_0) \\
\cup & & \cup \\
Y_{(x_0)} - g^{-1}(X_{(x_0)}) & \longrightarrow & \mathbb{A}^n_k - X_{(x_0)} \\
\Cup & & \Cup \\
g^{-1}(P) & \longrightarrow & P
\end{array}$$

where $X_{(x_0)} = V(f(1, x_1, \dots, x_n)) \subset \mathbb{A}^n_k$ and $Y_{(x_0)} = V(f(1, x_1, \dots, x_n) - x_{n+1}^d) \subset \mathbb{A}^{n+1}_k$. Now the morphism $g_{(x_0)}$ corresponds to the ring homomorphism

$$g^*_{(x_0)} \colon k[x_1, \dots, x_n] \to \frac{k[x_1, \dots, x_{n+1}]}{(f(1, x_1, \dots, x_n) - x_{n+1}^d)} : x_i \mapsto x_i.$$

Since k is algebraically closed and $P \in \mathbb{A}^n_k$ is a closed point, it can be given written in homogeneous coordinates $[1 : a_1 : \dots : a_n]$ where we can put 1 as 0-th entry since $P \in D_+(x_0)$. The fact that $P \notin X_{(x_0)}$ means that $c \neq f(1, a_1, \dots, a_n) \neq 0$. Then we can compute the fibre $g^{-1}(P)$ as the spectrum of the ring

$$\begin{aligned}
\frac{k[x_1, \dots, x_{n+1}]}{(f(1, x_1, \dots, x_n) - x_{n+1}^d)} \bigotimes_{k[x_1, \dots, x_n]} \frac{k[x_1, \dots, x_n]}{(x_1 - a_1, \dots, x_n - a_n)} &\cong \frac{k[x_{n+1}]}{(f(1, a_1, \dots, a_n) - x_{n+1}^d)} \\
&= \frac{k[y]}{(c - y^d)} \cong k^d
\end{aligned}$$

where the last isomorphism uses the chinese remainder theorem and the fact that $c - y^d$ has d distinct roots, since $p \nmid d$. Thus $g^{-1}(P) \cong \operatorname{Spec}(k[y]/(c - y^d))$ is a reduced scheme with d k-rational points.

Geometrically, this exercise means that $Y - g^{-1}(X) \to \mathbb{P}^n_k - X$ is a d-fold covering. However, the covering is ramified (*branched*) in X, since the fibre over a point $P \in X$ is a non-reduced point of the form $\operatorname{Spec}(k[y]/(y^d))$.

(iii) Now let us assume that $p \mid d$. The fibre over a point $P = [a_0 : \dots : a_n] \notin X$ has the form

$$\operatorname{Spec}(k[y]/(c - y^d)) = \operatorname{Spec}(k[y]/(\zeta - y^{d/p})^p)$$

where ζ is a d-th root of $c \coloneqq f(a_0, \dots, a_n)$. This is clearly non-reduced. Similarly, the fibre over a point $P \in X$ has the form

$$\operatorname{Spec}(k[y]/(y^d))$$

which is again non-reduced.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 58. *Closed points exist on quasi-compact schemes* (+ 2 extra points)
Let X be a quasi-compact scheme. Show that for every $x \in X$, there exists a closed point in $\overline{\{x\}} \subseteq X$.

Solution. Not provided.

Exercise 59. *Tensor products with ample invertible sheaves* (4 points)
Let X be a scheme of finite type over a Noetherian ring A. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on X. Show the following:

(i) If $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

(ii) If $\mathcal{L}$ is ample, then there exists $n_0 \geq 0$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample for all $n \geq n_0$.

(iii) if $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$.

(iv) If $\mathcal{L}$ is very ample (over A) and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample.

Solution. Recall that a sheaf an invertible sheaf $\mathcal{L}$ on a Noetherian scheme X is ample if for all $\mathcal{F} \in \mathrm{Coh}(X)$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. Recall also that a (quasi-coherent) sheaf $\mathcal{F}$ is globally generated if there is a surjection $\bigoplus_{i \in I} \mathcal{O}_X \twoheadrightarrow \mathcal{F}$.

First, we show that the tensor product of globally generated sheaves is globally generated. Let $\mathcal{F}$ and $\mathcal{G}$ be globally generated. Then we have two surjections $f\colon \bigoplus_{i \in I} \mathcal{O}_X \twoheadrightarrow \mathcal{F}$ and $g\colon \bigoplus_{i \in J} \mathcal{O}_X \twoheadrightarrow \mathcal{G}$. Then using (twice) that the tensor product is right-exact, hence preserves surjections, we obtain the surjection

$$\bigoplus_{(i,j) \in I \times J} \mathcal{O}_X \cong \bigoplus_{i \in I} \mathcal{O}_X \otimes \bigoplus_{i \in J} \mathcal{O}_X \overset{f \otimes \mathrm{id}}{\twoheadrightarrow} \mathcal{F} \otimes \bigoplus_{i \in J} \mathcal{O}_X \overset{\mathrm{id} \otimes g}{\twoheadrightarrow} \mathcal{F} \otimes \mathcal{G}$$

which shows that $\mathcal{F} \otimes \mathcal{G}$ is globally generated. The same proof works for *finitely* globally generated sheaves.

(i) Assume that $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated. We want to show that $\mathcal{L} \otimes \mathcal{M}$ is ample. Let $\mathcal{F} \in \mathrm{Coh}(X)$. Since $\mathcal{L}$ is ample, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. Thus, for all $n \geq n_0$, $\mathcal{F} \otimes (\mathcal{L} \otimes \mathcal{M})^{\otimes n} \cong \mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{M}^{\otimes n}$ is globally generated (using that products of globally generated sheaves are globally generated, and that $\mathcal{M}$ is already gobally generated).

(ii) Since $\mathcal{L}$ is ample, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is globally generated for all $n \geq n_0$. Then, for all $n \geq n_0 + 1$, the sheaf

$$\mathcal{L}^{\otimes n} \otimes \mathcal{M} = \mathcal{L} \otimes (\mathcal{L}^{\otimes n-1} \otimes \mathcal{M})$$

is the product of an ample sheaf with a globally generated sheaf, hence it is ample by (i).

(iii) First, we show that: Let $\mathcal{N}$ be an invertible sheaf on X. If $\mathcal{N}^{\otimes n}$ is ample for some $n \geq 1$, then $\mathcal{N}$ itself is ample. (Actually this was done in class)

Indeed, assume that $\mathcal{N}^{\otimes n}$ is ample and let $\mathcal{F} \in \mathrm{Coh}(X)$. Then applying the definition of ampleness to the (finitely many) coherent sheaves $\mathcal{F}, \mathcal{N} \otimes \mathcal{F}, \ldots, \mathcal{N}^{\otimes n-1} \otimes \mathcal{F}$, then taking the maximum of the obtained ranks, we obtain that there exists $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, all the sheaves

$$(\mathcal{N}^{\otimes n})^{\otimes m} \otimes \mathcal{F}, (\mathcal{N}^{\otimes n})^{\otimes m} \otimes \mathcal{N} \otimes \mathcal{F}, \ldots, (\mathcal{N}^{\otimes n})^{\otimes m} \otimes \mathcal{N}^{\otimes n-1} \otimes \mathcal{F}$$

are globally generated. This exactly means that for all $l \geq m_0 n$, $\mathcal{N}^{\otimes l} \otimes \mathcal{F}$ is globally generated (using Euclidean division of l by n). Thus $\mathcal{N}$ is ample.

Now let us focus on the exercise. Let $\mathcal{L}$ and $\mathcal{M}$ be ample invertible sheaves. We need to show that $\mathcal{L} \otimes \mathcal{M}$ is ample. By ampleness, there exists n_0 such that $\mathcal{L}^{\otimes n}$ and $\mathcal{M}^{\otimes n}$ are globally generated for all $n \geq n_0$. In particular, $\mathcal{L}^{\otimes n_0+1} = \mathcal{L} \otimes \mathcal{L}^{\otimes n_0}$ is ample by (i), and again by (i) we get that

$$(\mathcal{L} \otimes \mathcal{M})^{\otimes n_0+1} = \mathcal{L}^{\otimes n_0+1} \otimes \mathcal{M}^{\otimes n_0+1}$$

is ample since $\mathcal{L}^{\otimes n_0+1}$ is ample and $\mathcal{M}^{\otimes n_0+1}$ is globally generated. Using what we proved right above (ampleness can be checked after taking a tensor power), this implies that $\mathcal{L} \otimes \mathcal{M}$ is ample, as desired.

(iv) We are in the setting of Theorem 15.6 from the notes, thus $\mathcal{L}$ if ample if and only if $\mathcal{L}^{\otimes n}$ is very ample (over A) for some $n \geq 1$.

Assume that $\mathcal{L}$ is very ample and that $\mathcal{M}$ is globally generated. Then by Theorem 15.6, $\mathcal{L}$ is ample, hence $\mathcal{L} \otimes \mathcal{M}$ is ample by (i). Then by Theorem 15.6 again, we get that $(\mathcal{L} \otimes \mathcal{M})^{\otimes n}$ is very ample (over A) for some n. How to get very ampleness of $\mathcal{L} \otimes \mathcal{M}$? Actually this is not clear at all. We need to try a different approach.

Let $f \colon X \to \operatorname{Spec}(A)$ be the structure morphism. By assumption, f is of finite type and A is a Noetherian ring. Assume moreover that $\mathcal{L}$ is very ample relative to f. Then there exists an immersion $i \colon X \hookrightarrow \mathbb{P}^n_A$ such that $i^*\mathcal{O}(1) \cong \mathcal{L}$.

Since $\mathcal{M}$ is a globally generated and X is quasi-compact (because it is finite type over A), $\mathcal{M}$ is actually finitely globally generated[5]. By Theorem 14.3, a choice of finitely many (say $m+1$) sections globally generating $\mathcal{M}$ induces a morphism $g \colon X \to \mathbb{P}^m_A$ such that $g^*\mathcal{O}(1) \cong \mathcal{M}$.

In order to combine these two morphisms into projective spaces, we use the Segre embedding (exercise 45) $\sigma \colon \mathbb{P}^n_A \times_A \mathbb{P}^m_A \hookrightarrow \mathbb{P}^N_A$ which satisfies $\sigma^*(\mathcal{O}(1)) = \operatorname{pr}_1^* \mathcal{O}(1) \otimes \operatorname{pr}_2^* \mathcal{O}(1)$. Consider the composition

$$f \colon X \xrightarrow{\Delta} X \times_A X \times_A X \xrightarrow{i \times g} \mathbb{P}^n_A \times_A \mathbb{P}^m_A \xrightarrow{\sigma} \mathbb{P}^N_A.$$

We claim that f is an immersion and that $f^*\mathcal{O}(1) \cong \mathcal{L} \otimes \mathcal{M}$.

Denote by $\pi \colon \mathbb{P}^m_A \to \operatorname{Spec}(A)$ the structure map. The composition

$$X \xrightarrow{\Delta} X \times_A X \times_A X \xrightarrow{i \times g} \mathbb{P}^n_A \times_A \mathbb{P}^m_A \xrightarrow{\operatorname{id} \times \pi} \mathbb{P}^n_A \times_A \operatorname{Spec}(A) = \mathbb{P}^n_A$$

is nothing but the immersion $i \colon X \hookrightarrow \mathbb{P}^n_A$. We want to show that $(i \times g) \circ \Delta$ is also an immersion. By the Cancellation theorem (Theorem 9.9), it suffices to show that the diagonal of $\operatorname{id} \times \pi$ is an immersion, which is true by exercise 32(ii). Therefore, $(i \times g) \circ \Delta$ is an immersion, so is the composition $f = \sigma \circ (i \times g) \circ \Delta$.

On the other hand, we have

$$f^*\mathcal{O}(1) = \Delta^*(i \times g)^*\sigma^*\mathcal{O}(1) = \Delta^*(i \times g)^*(\operatorname{pr}_1^* \mathcal{O}(1) \otimes \operatorname{pr}_2^* \mathcal{O}(1)) \cong \Delta^*(\operatorname{pr}_1^* \mathcal{L} \otimes \operatorname{pr}_2^* \mathcal{M}) = \mathcal{L} \otimes \mathcal{M},$$

which concludes.

Exercise 60. *Extending coherent sheaves* (4 points)
The goal of this exercise is to extend a coherent sheaf defined on an open subscheme; in particular the following statement.

$(*)$ If X is a Noetherian scheme, $i : U \hookrightarrow X$ is an open subscheme of X, $\mathcal{F}$ is a coherent sheaf on U, and $\mathcal{G}$ is a quasi-coherent sheaf on X such that $\mathcal{F} \subseteq \mathcal{G}|_U$, then there exists a coherent subsheaf $\mathcal{F}' \subseteq \mathcal{G}$ such that $\mathcal{F}'|_U = \mathcal{F}$.

[5] The fact that $\mathcal{M}$ is globally generated means that $X = \bigcup_{s \in \Gamma(X, \mathcal{M})} X_s$, hence finitely many opens suffice by quasi-compactness.

(i) Prove that every quasi-coherent sheaf on a Noetherian affine scheme is the union of its coherent subsheaves.

(Here, we say that a sheaf of abelian groups $\mathcal{F}$ on a topological space X is a union of subsheaves of abelian groups $\mathcal{F}_\alpha$ if for every open $U \subset X$ the group $\mathcal{F}(U)$ is the union of its subgroups $\mathcal{F}_\alpha(U)$.)

(ii) Show that $(*)$ holds if X is affine.

(iii) Show that $(*)$ holds.

Solution.

1. The statement is equivalent to that the natural map

$$\bigoplus_{\mathcal{H} \subseteq \mathcal{F} \text{ coh}} \mathcal{H} \to \mathcal{F}$$

is surjective, which by the equivalence of the category of quasi-coherent sheaves on affine schemes and the category of modules is equivalent to that the natural map

$$\bigoplus_{N \subseteq M \text{ fin. gen.}} N \to M$$

is surjective. The latter statement is simple commutative algebra.

2. Note first that the inclusion $i : U \hookrightarrow X$ is quasi-compact and quasi-separated. Therefore, $i_*\mathcal{F}$ is quasi-coherent. By replacing $\mathcal{G}$ with $i_*\mathcal{F}$ we may assume that $\mathcal{G}|_U = \mathcal{F}$. Note that $i_*\mathcal{F}$ is quasi-coherent since i is quasi-compact and quasi-separated. Now by part 1, there exists a direct sum of coherent sheaves such that $\bigoplus_{i \in I} \mathcal{H}_i \to \mathcal{G}$ is surjective. Hence $\bigoplus_{i \in I} \mathcal{H}_i|_U \to \mathcal{F}$ is surjective. Taking a point $x \in U$, we see that the stalk $\mathcal{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module and hence there exists a finite sub direct sum $\bigoplus_{i \in I_x} \mathcal{H}_{i,x}$ of $\bigoplus_{i \in I} \mathcal{H}_{i,x}$ which already surjects on $\mathcal{F}_x$. Here I_x is a finite sub index set of I. Since everything is coherent, this surjection extends to a surjection $\bigoplus_{i \in I_x} \mathcal{H}_i|_{U_x} \to \mathcal{F}|_{U_x}$ on a small open neighbourhood U_x of x. Taking now all open neighbourhoods U_x for all points $x \in U$, we get an open cover of U with a finite sub cover $U_{x_1}, \ldots, U_{x_n}$. Taking the direct sum $\bigoplus_{i \in \bigcup_{j=1}^n I_{x_j}} \mathcal{H}_i$, we see that it is a coherent sheaf on X such that its restriction to U surjects onto $\mathcal{F}$.

3. We take inductions on the number of affine opens that cover the scheme X.
If there exists one affine open that covers X, i.e. X is affine, the statement follows from part 2. We then assume that the statement $(*)$ holds for all schemes admitting a cover of $n-1$ affine opens. Take now a scheme X with a cover of n affine opens. By replacing $\mathcal{G}$ with $i_*\mathcal{F}$ we assume that $\mathcal{G}|_U = \mathcal{F}$. We may wrtie X as $Y \cup Z$, where Y is a scheme with a cover of $n-1$ affine opens and Z is affine. There exists a coherent subsheaf $\mathcal{H}_Y$ of $\mathcal{G}|_Y$ on Y, such that $\mathcal{H}_Y|_{Y \cap U} = \mathcal{F}|_{Y \cap U}$. Since $\mathcal{H}_Y|_{Y \cap Z}$ and $\mathcal{F}|_{Z \cap U}$ agree on $U \cap Y \cap Z$, they glue to a coherent sheaf $\mathcal{H}'$ on $Z \cap (U \cup Y)$. Now applying the induction hypothesis again, we find a coherent subsheaf $\mathcal{H}_Z$ of $\mathcal{G}|_Z$ on Z, such that $\mathcal{H}_Z|_{Z \cap (U \cup Y)} = \mathcal{H}'|_{Z \cap (U \cup Y)}$. In particular, $\mathcal{H}_Z|_{Z \cap U} = \mathcal{F}|_{Z \cap U}$, and $\mathcal{H}_Z|_{Z \cap Y} = \mathcal{H}_Y|_{Z \cap Y}$. Hence we can glue $\mathcal{H}_Z$ and $\mathcal{H}_Y$ and get a coherent sheaf we desire.

Exercise 61. *Morphisms from projective spaces* (4 points)
Let k be a field and let $f : \mathbb{P}^n_k \to X$ be a morphism of k-schemes. Show the following:

(i) If X is affine, then $f(\mathbb{P}^n_k)$ is a (k-rational) point of X.

(ii) If $X = \mathbb{P}^m_k$, then f is the composition of a d-fold Veronese morphism for a unique $d \geq 0$ with iterated projections from points, inclusion of linear subspaces and an automorphism of $\mathbb{P}^m_k$.

(iii) If X is quasi-projective over k, then either $f(\mathbb{P}^n_k)$ is a (k-rational) point of X or the fibres of f over the k-rational points are finite.

(iv) (+1 extra point) Furthermore if $f(\mathbb{P}^n_k)$ is not a k-rational point then f is finite.

Solution.

(i) Every map of k-schemes $f\colon \mathbf{P}^n_k \to \mathrm{Spec}(A)$ is uniquely determined by its behaviour on global section, i.e. recall that we have

$$\hom_{k\text{-Sch}}(\mathbf{P}^n_k, \mathrm{Spec}(A)) = \hom_{k\text{-Alg}}(A, k).$$

Now every non-trivial k-algebra morphism $f^{\#} A \to k$ is necessarily surjective (its image is a k-linear subspace of k after all), and so its kernel gives a closed point of $\mathrm{Spec}(A)$.

(ii) We assume that $f(\mathbb{P}^n)$ does not lie in any hyperplane sections. This is equivalent to saying that the sections defining the map to the projective space do not satisfy any linear relations.

We first get two easy cases out of the way, when $d = 0$, $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d) = k$ and hence f maps to a point in $\mathbb{P}^m_k$. If $d = 1$, the sections of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ are the coodinates of $\mathbb{P}^n$ and hence either $m \geq n$ and $f =$ an inclusion of a linear subspace of $\mathbb{P}^m$ (i.e. a composition of inclusion of linear hyperplanes) or $m \leq n$ and then f is defined (up-to an isomorphism of $\mathbb{P}^m$) by finitely many of these coordinates. This amounts to iterated projection from the leftover coordinate points.

Now assume $d > 1$. Let $s_0, \cdots, s_m \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ that defines f. Up-to an automorphism of $\mathbb{P}^m$ we may assume that $s_i = x^{I_i}$ where x^{I_i} is a monomial of degree d in the coordinates of $\mathbb{P}^n$. Let $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ be the standard Veronese map defined by the monomial x^I of degree d. Consider the projection from the point $[0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \mathbb{P}^{\binom{n+d}{d}-1}$ where 1 is in some J'th position such that $J \neq I_i$ for any $i = 0, \cdots, m$. The composition of the Veronese and the projection from this point is defined by all sections of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)$ except x^J. Iterating this process removes all sections of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)$ except for x^{I_i}'s recovering our original f.

(iii), (iv) Not provided.

Exercise 62. *Calculating divisor class groups* (4 points)
Let k be a field. Calculate the divisor class group of the affine cone over the Veronese image of $\mathbb{P}^1_k$ in $\mathbb{P}^2_k$.

Solution. Recall that the affine cone over a projective variety $X \subset \mathbb{P}^n$ defined by homogenous polynomials $(f_1, \cdots, f_k)$ is given by $\mathrm{Spec}\, \frac{k[x_0,\cdots,x_n]}{(f_1,\cdots,f_k)}$. Let $X = \mathrm{Spec}\, k[x,y,z]/(xy - z^2) \subset \mathbb{A}^3$, i.e. the affine cone over the Veronese line.

First note that $k[x,y,z]/(xy - z^2)$ is not an UFD. Indeed, $xy = z^2$ in the ring. Therefore it is possible that the class group is non-trivial.

In order to calculate the class group, the best strategy is to throw out a prime divisor D such that the complement $X \setminus D$ is the Spec of an UFD. To this end, let $D = (y = z = 0)$ be the x-axis in $\mathbb{A}^3$. Note that $D \subset X$ since $(x,0,0) \in V(xy - z^2)$. Therefore, D is a prime divisor on X. Let η_D be the generic point of D. Since x is invertible in $k[x,y,z]_{(y,z)}$, in the local ring we obtain the isomorphism

$$\mathcal{O}_{X,\eta_D} \simeq \operatorname{Spec} k[x,y,z]_{(y,z)}/(xy - z^2) \simeq \operatorname{Spec} k[x,z]_{(z)}.$$

This is a discrete valuation ring, where the valutaion of y with respect to z gives $v_D(y) = 2$ since $y = x^{-1}z^2$ in the DVR above. Hence formally one can write $V(y) \cap X = 2D$. Since it is defined by a single polynomial, we have $V(y) \cap X = 2D \sim 0$.

Consider now the exact sequence

$$\mathbb{Z} \cdot D \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus D) \to 0$$

Note that

$$X \setminus D \simeq \operatorname{Spec} k[x,y,y^{-1},z]/(xy - z^2) \simeq \operatorname{Spec} k[y,y^{-1},z].$$

and hence it is the affine scheme associated to an UFD. From *Prop. 16.28* we have $\mathrm{Cl}(X \setminus D) = 0$ and hence from the exact sequence it follows that $\mathrm{Cl}(X) \simeq \mathrm{Im}(\mathbb{Z} \cdot D) \simeq \mathbb{Z}/2\mathbb{Z}$. The last isomorphism follows from $2D = 0 \in \mathrm{Cl}(X)$ as discussed above.

*We use $V(y) = 2D$ for **two** reasons: first and the most obvious one is to conclude that $2D \sim 0$; second and not so obvious: $X \setminus D = X \setminus 2D$ as schemes and we understand the latter by localizing at y.*

Exercise 63. *Birational isomorphisms in small dimension* (4 points)
Two integral schemes X and Y are called *birational* (over a scheme S) if there exist non-empty open subschemes $U \subseteq X$ and $V \subseteq Y$ and an isomorphism $U \cong V$ (over S).

(i) Let X and Y be integral normal schemes of dimension 1 which are proper over a field k. Show that if X and Y are birational over k, then they are isomorphic over k.

(Hint: Use Exercise 36 (ii))

(ii) Let k be a field. Calculate $\mathrm{Pic}(\mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k)$. Deduce that $\mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k$ and $\mathbb{P}^2_k$ are not isomorphic over k (even though they are birational over k).

Solution.

(i) Let $U \subset X$, $V \subset Y$ be nonempty open and $f : U \xrightarrow{\cong} V$ an isomorphism over k. As X is an irreducible Noetherian 1-dimensional scheme, we have $X \setminus U = \{x_1, \dots, x_n\}$ with all $x_i \in X$ closed points. By assumption, each $\mathcal{O}_{X,x_i}$ is a normal Noetherian local domain of dimension 1, hence a DVR. Since also Y is proper over k, by Exercise 36(ii) there exists an open subset $U_1 \subset X$ containing U and x_1, and a unique morphism $f_1 : U_1 \to Y$ with $f_1|_U = f$. Repeating this procedure at most n times, we find a morphism $\widetilde{f} : X \to Y$ extending f. Similarly, we find a unique extension $\widetilde{(f^{-1})} : Y \to X$ of $f^{-1} : V \to U$, and we have that $(\widetilde{(f^{-1})} \circ \widetilde{f})|_U = \mathrm{id}_U$ and $(\widetilde{f} \circ \widetilde{(f^{-1})})|_V = \mathrm{id}_V$ and hence, by the equalizer exercise (since X is reduced, Y/k is separated and U, V are dense), $\widetilde{(f^{-1})} \circ \widetilde{f} = \mathrm{id}_X$ and $\widetilde{f} \circ \widetilde{(f^{-1})} = \mathrm{id}_Y$, so $\widetilde{f}$ is an isomorphism *by the equalizer exercise*.

(ii) Denote $X = \mathbb{P}^1_k \times_k \mathbb{P}^1_k$, $Z = \mathbb{P}^1_k \times_k \{[0:1]\}$, $U = X \setminus Z$. We have the following exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0, \tag{11}$$

and here $\mathrm{Cl}(X) = \mathrm{Cl}(\mathbb{P}^1_k \times_k \mathbb{P}^1_k)$, $\mathrm{Cl}(U) = \mathrm{Cl}(\mathbb{P}^1_k \times_k \mathbb{A}^1_k) = \mathrm{Cl}(\mathbb{P}^1_k) = \mathbb{Z}$. Now, we claim that the leftmost map in (11) (which sends m to mZ) is injective and that $\mathrm{Cl}(\mathbb{P}^1_k \times_k \mathbb{P}^1_k) \to \mathrm{Cl}(\mathbb{P}^1_k)$ admits a section.

For the injectivity, suppose that mZ (for some $m \in \mathbb{Z}$) is a principal divisor, i.e. $mZ = \operatorname{div}(f)$ for some $f \in k(\mathbb{P}^1_k \times_k \mathbb{P}^1_k)^\times$. Then $mZ \cap \mathbb{P}^1_{k(T)}$ is also a principal divisor (where $\mathbb{P}^1_{k(T)} = \{\eta\} \times_k \mathbb{P}^1_k$, where η is the generic point of $\mathbb{P}^1_k$) corresponding to the same rational function $f \in k(\mathbb{P}^1_{k(T)})^\times = k(\mathbb{P}^1_k \times_k \mathbb{P}^1_k)^\times$.

Write $\mathbb{P}^1_{k(T)} = \operatorname{Proj} k(T)[T_0, T_1]$. Looking at $\mathbb{A}^1_{k(T)} = D_+(T_0) \subset \mathbb{P}^1_{k(T)}$, we see that since the valuation of f is zero at every codimension 1 (= closed) point of $\mathbb{A}^1_{k(T)}$ (as $\operatorname{div}(f) = mZ \cap \mathbb{P}^1_{k(T)}$), we must have that both f and $\frac{1}{f}$ lie in any localization of $k(T)[\frac{T_1}{T_0}]$ at any height 1 (= maximal) prime ideal $\mathfrak{p} \subset k(T)[\frac{T_1}{T_0}]$. I.e. $f \in \bigcap_{\mathfrak{p} \text{ of height } 1} k(T)[\frac{T_1}{T_0}]_{\mathfrak{p}} = k(T)[\frac{T_1}{T_0}]$ and f is a unit in this ring, i.e $f \in k(T)^\times$. Hence $\operatorname{div}(f) = 0$, and so $m = 0$, proving the injectivity.

Since $\mathbb{Z}$ is projective, the sequence (11) splits, showing that $\operatorname{Cl}(\mathbb{P}^1_k \times_k \mathbb{P}^1_k) \cong \mathbb{Z} \times \mathbb{Z}$ and in particular, $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ is not isomorphic to $\mathbb{P}^2_k$.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 64. *The Grothendieck group of a scheme* (+ 5 extra points)
Let X be a Noetherian scheme. The *Grothendieck group* $K_0(X)$ of X is defined as the quotient of the free abelian group generated by all coherent sheaves on X by the subgroup generated by the expressions $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$, whenever there is an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent sheaves on X.

(i) If X is integral and $\mathcal{F}$ is a coherent sheaf on X, we define the *rank* of $\mathcal{F}$ as $\operatorname{rank}(\mathcal{F}) := \dim_{\mathcal{O}_{X,\eta}}(\mathcal{F}_\eta)$, where η is the generic point of X. Show that $\operatorname{rank}(-)$ defines a surjective homomorphism from $K_0(X)$ to $\mathbb{Z}$.

(ii) Let $Y \subseteq X$ be a closed subscheme and let $\mathcal{F}$ be a coherent sheaf on X with support on Y. Show that $\mathcal{F}$ admits a finite filtration by coherent sheaves

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_n = \mathcal{F}$$

such that each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is the pushforward of a coherent sheaf on Y.

(iii) Let $\iota : Y \hookrightarrow X$ be a closed immersion. Show that there is an exact sequence

$$K_0(Y) \xrightarrow{\alpha} K_0(X) \xrightarrow{\beta} K_0(X - Y) \to 0,$$

where α is induced by ι_* and β is induced by $(-)|_{X-Y}$.
(Hints: First, note that $\beta \circ \alpha = 0$, so that β induces a homomorphism

$$\overline{\beta} : K_0(X)/\alpha(K_0(Y)) \to K_0(X - Y).$$

Then, use Exercise 54 and Part (ii) of the current exercise to construct an inverse to $\overline{\beta}$.)

(iv) Let k be a field. Calculate $K_0(\operatorname{Spec} k), K_0(\mathbb{A}^1_k)$, and $K_0(\mathbb{P}^1_k)$.

Exercise 65. *Geometric reducedness* (4 points)
Let k be a field, let A and B be k-algebras, and let $k \subseteq K$ be a field extension.

(i) Assume that A is non-reduced. Show that $A \otimes_k K$ is non-reduced.

(ii) Show that if $A \otimes_k B$ is non-reduced, then there exist finitely generated k-algebras $A' \subseteq A$ and $B' \subseteq B$ such that $A' \otimes_k B'$ is non-reduced.

(iii) Show that if K is finitely generated and separable over k, then A is reduced if and only if $A \otimes_k K$ is reduced.

(iv) Conclude that a scheme X over a perfect field is reduced if and only if it is geometrically reduced.

Solution.

(i) Assume that A is non-reduced. Let $a \in A$ be a nilpotent element of A, i.e. for some integer n we have $a^n = 0$. Then we have $(a \otimes 1)^n = 0 \in A \otimes_k K$.

(ii) If $A \otimes_k B$ is non-reduced, then we let $x \in A \otimes_k B$ such that x is nilpotent. Since $A \otimes_k B$ is generated as an algebra by the pure tensor products of the form $a \otimes b$, we write $x = \sum_{i=1}^{\ell} a_i \otimes b_i$ for finitely many $a_i \in A$ and $b_i \in B$. Let $A' \coloneqq k[a_1, \cdots, a_\ell]$ and $B' \coloneqq k[b_1, \cdots, b_k]$. Then $z \in A' \otimes_k B'$ is also nilpotent.

(iii) If K is finitely generated and separable over k, we may assusme that $K = k(x_1, \ldots, x_k, y)$ where $x_1, \ldots, x_k$ is a transcedental basis and the minimal polynomial of y over $k(x_1, \ldots, x_k)$ is separable.

We already know by (i) that if A is non-reduced then $A \otimes_k K$ is also non-reduced. Conversely if $A \otimes_k K$ is non-reduced, by (ii) there exists a finitely generated k-subalgebra A' of A such that $A' \otimes_k K$ is non-reduced. If A were reduced, so would be A'. Furthermore, A' has finitely many minimal primes and hence embeds into finite product of fields $F_1 \times \cdots \times F_\ell$ given by localizing at the minimal primes. Since tensoring with a field extension is exact, we obtain $A' \otimes_k K$ also embeds in $\prod_{j=1}^{\ell} F_i \otimes_k K$. It is thus sufficient to show that for a field F, $F \otimes_k K$ contains no nilpotent. To this end, let $P(T)$ be the minimal polynomial of y. Since $P(T)$ is a minimal separable polynomial, it has distinct roots. Assume that over the field $F \otimes_k k(x_1, \ldots, x_n) \simeq F(x_1, \ldots, x_n)$ we can factor $P(T) = P_1(T) \cdots P_s(T)$. Then by Chinese remainder theorem we have the ring isomorphism $F \otimes_k K \simeq F(x_1, \ldots, x_n)[T]/(P(T)) \simeq E_1 \times \cdots \times E_s$, i.e. the k-algebra given by product of field extensions $F \subseteq E_i$. Hence it cannot contain any nilpotent.

(iv) Let X be a schme over a perfect field k. Let $k \subset K$ denote any finitely generated field extension. If X is non-reduced, by (i) $X \otimes_k K$ is also non-reduced. For the converse, note that by (iii) above X is reduced if and only if for all finitely generated sub-extension $k \subset K' \subset K$, the scheme $X_{K'}$ is reduced. But by the converse of (ii), if for all finitely generated sub-extension $k \subset K' \subset K$, the scheme $X_{K'}$ is reduced, then $X \otimes_k K$ is also reduced.

Exercise 66. *An alternative definition of the cotangent sheaf* (4 points)
Let A be a ring and B an A-algebra. Let $\rho : B \otimes_A B \to B$ be the multiplication map. Let $I = \operatorname{Ker}(\rho)$. We consider $B \otimes_A B$ as a B-module via multiplication on the right, so that I/I^2 becomes a B-module. Consider the map

$$\begin{aligned} d : B &\to I/I^2 \\ b &\mapsto b \otimes 1 - 1 \otimes b. \end{aligned}$$

(i) Show that d is an A-linear derivation.

(ii) Show that the pair $(I/I^2, d)$ satisfies the universal property of the module of relative differentials of B over A.

(iii) Let $f : X \to Y$ be a morphism of schemes, let $\Delta : X \to X \times_Y X$ be the diagonal, and let $\mathcal{I}$ be the kernel of $\Delta^\sharp : \mathcal{O}_{X\times_Y X} \to \Delta_* \mathcal{O}_X$. Show that $\Omega_{X/Y} \cong \Delta^*(\mathcal{I}/\mathcal{I}^2)$.

Solution.

(i) We directly check the axioms:

(1) Let $b, b' \in B$. Then, we have

$$d(b+b') = (b+b') \otimes 1 - 1 \otimes (b+b') = b \otimes 1 - 1 \otimes b + b' \otimes 1 - 1 \otimes b' = d(b) + d(b').$$

(2) Let $b, b' \in B$. We have

$$d(bb') = bb' \otimes 1 - 1 \otimes bb' = b(b' \otimes 1 - 1 \otimes b') + b'(b \otimes 1 - 1 \otimes b) = bd(b') + b'd(b).$$

(3) For given $a \in A$ holds

$$d(a) = a \otimes 1 - 1 \otimes a = a \otimes 1 - a \otimes 1 = 0.$$

Thus, d is a derivation.

(ii) Given a derivation $d_M : B \to M$. We define a morphism of A-modules

$$\varphi : B \otimes_A B \to B \oplus M, \quad x \otimes y \mapsto (xy, xd_M(y)).$$

We obtain a morphism of A-modules

$$f : I \to M, \quad \sum_i x_i \otimes y_i \mapsto \sum_i x_i d_M(y_i).$$

Given $\sum_i x_i \otimes y_i, \sum_j z_j \otimes w_j \in I$, then we have

$$\begin{aligned} f\Big(\Big(\sum_i x_i \otimes y_i\Big)\Big(\sum_j z_j \otimes w_j\Big)\Big) &= \sum_{i,j} x_i z_j d_M(y_i w_j) \\ &= \sum_{i,j} x_i z_j w_j d_M(y_i) + x_i z_j y_i d_M(w_j) \\ &= \sum_i x_i \Big(\sum_j z_j w_j\Big) d_M(y_i) + \sum_j z_j \Big(\sum_i x_i y_i\Big) d_M(w_j) \\ &= 0. \end{aligned}$$

Thus, f induces a map on the quotient $f : I/I^2 \to M$. For $b \in B$ holds

$$f(d(b)) = f(b \otimes 1 - 1 \otimes b) = -d_M(b).$$

On I/I^2, we have the B-module structure

$$b\Big(\sum_i x_i \otimes y_i\Big) = \sum_i bx_i \otimes y_i = \sum_i x_i \otimes by_i.$$

It follows directly that f is B-linear. Moreover, note that we have the general identity

$$x \otimes y = (1 \otimes y)(x \otimes 1 - 1 \otimes x) + 1 \otimes xy$$

for all $x, y \in B$. It follows that $d(B)$ generates I/I^2 as B-module. Indeed, suppose we are given $\sum_i x_i \otimes y_i \in I$, then modulo I^2 we have

$$\sum_i x_i \otimes y_i = \sum_i (1 \otimes y_i) d(x_i).$$

Altogether, the above statements imply the universal property of relative differentials.

(iii) Let $W = \bigcup_{U,V} U \times_V U \subset X \times_Y X$, where $V \subset Y$ runs over all open affines and U runs over all open affines contained in $f^{-1}(V)$. Then, we have a factorization

$$\Delta : X \overset{\Delta_W}{\to} W \hookrightarrow X \times_Y X$$

where Δ_W is a closed immersion. Let $\mathcal{J}$ be the kernel of $\mathcal{O}_W \to \Delta_{W,*}\mathcal{O}_X$. Then,

$$\Delta_W^*(\mathcal{J}/\mathcal{J}^2) \cong \Delta^*(\mathcal{I}/\mathcal{I}^2).$$

Let $V \subset Y, U \subset f^{-1}(V)$ be affine. By (ii) we have an isomorphism of $\mathcal{O}_X(U)$-modules $\Delta_W^*(\mathcal{J}/\mathcal{J}^2)(U) \cong \Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)}$ which is compatible with localization. Thus, we can glue these local isomorphisms to a global isomorphism of $\mathcal{O}_X$-modules.

Exercise 67. *Projective Jacobian criterion* (4 points)
Let k be a field and let $X \subseteq \mathbb{P}_k^n$ be a closed subscheme given by the saturated ideal $(f_1, \dots, f_m)$ with f_i homogeneous of degree d_i. We define the *Jacobian matrix* J_X of X as

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_0} & \cdots & \frac{\partial f_m}{\partial x_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

(i) Show the Euler identity: For every homogeneous polynomial f of degree d, one has $\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = df$.

(ii) Show that the rank of J_X at a point $x \in X$ is well-defined and equals the rank of the Jacobian matrix of the affine scheme $X \cap D_+(x_i)$ for any $D_+(x_i)$ containing x. Conclude that if $k(x)$ is separable over k, then X is smooth at $x \in X$ if and only if $J_X(x)$ has co-rank $\dim \mathcal{O}_{X,x} + \text{tr.deg.}(k(x)/k) + 1$. *Said differently, we have* corank $J_X(x) - 1 =$ Rank $\Omega_{X/k}(x) = n -$ Rank $J_X(x)$.

Solution.

1. It now suffices to show the result for monomials, let $f = \prod_{i=0}^n x_i^{\alpha_i}$. Then we have

$$\begin{aligned} x_i \frac{\partial f}{\partial x_i} &= \alpha_i x_i x_i^{\alpha_i - 1} \prod_{j \neq i} x_j^{\alpha_j} \\ &= \alpha_i \prod_{j=0}^n x_j^{\alpha_j} \\ &= \alpha_i f. \end{aligned}$$

Summing over i then yields

$$\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = \sum_{i=0}^n \alpha_i f = df.$$

2. If $x \in \mathbb{P}^n_k$, we make sense of the value $\frac{\partial f_j}{\partial x_i}(x)$ in the following way. We have $\frac{\partial f_j}{\partial x_i} \in \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d_j - 1))$, where d_j is the degree of f_j. Now, pick any open subset $x \in U \subset \mathbb{P}^n_k$ such that $\mathcal{O}_{\mathbb{P}^n_k}(d_j - 1)|_U \cong \mathcal{O}_U$ and fix any such isomorphism $\varphi_j : \mathcal{O}_{\mathbb{P}^n_k}(d_j - 1)|_U \xrightarrow{\cong} \mathcal{O}_U$. Then $\varphi_U(\frac{\partial f_j}{\partial x_i}|_U)_x \in \mathcal{O}_{\mathbb{P}^n_k, x}$ and we define $\frac{\partial f_j}{\partial x_i}(x)$ to be its residue class in $k(x) = \mathcal{O}_{\mathbb{P}^n_k, x}/\mathfrak{m}_x$. Using this same φ_j for all $\frac{\partial f_j}{\partial x_i}$, we can make sense of the value of the Jacobian $J_X(x)$ at the point x. Of course, the matrix $J_X(x)$ depends on the choice of U and φ, but any other choice will give rise to a matrix whose columns will differ by multiplication by some nonzero elements in $k(x)$. In particular, its rank stays the same.

   Now smoothness is a local condition, so it suffices to show the claim on an affine open, and after reordering, it suffices to show that in the affine subscheme $U_0 = \{z \in \mathbb{P}^n \mid z_0 \neq 0\}$ the intersection $X \cap U_0$ is regular at $y = (1, y_1, \dots, y_n)$, where $y_i = x_i/x_0$. This can be using the Jacobi criterion (in the affine setting) from the lecture, as soon as we know $\mathrm{rk} J_y = \mathrm{rk} J_x$. Up to change of variables, J_y is the right $(n-1) \times m$ submatrix of J_x, and the left most column of J_x is given by $\frac{\partial f_i}{\partial x_0}(P)$. Now, the Euler lemma tells us that

$$\begin{aligned} \frac{\partial f_i}{\partial x_0} &= d\frac{f_i}{x_0} - \frac{1}{x_0}\sum_{j=1}^n x_j \frac{\partial f_i}{\partial x_j} \\ &= -\sum_{j=1}^n y_j \frac{\partial f_i}{\partial y_j}, \end{aligned}$$

   so the first column of J_x is a linear combination of the columns on J_y, and their rank is the same.

Exercise 68. *Some explicit computations* (4 points)
Let k be an algebraically closed field. Describe the non-smooth points of the following schemes over k:

(i) $X = V_+(y^2 z - x^3 + xz^2) \subseteq \mathbb{P}^2_k$.

(ii) $X = V_+(\sum_{i=0}^r x_i^2) \subseteq \mathbb{P}^n_k$ for $1 \leq r \leq n$.

(iii) $X = V_+(f - x_{n+1}^d) \subseteq \mathbb{P}^{n+1}_k$ for $f \in k[x_0, \dots, x_n]_d$. Can X be regular if $\mathrm{char}(k) \mid d$?

Solution.
Use the Jacobian criterion.

(i) Assume $\mathrm{char}(k) \neq 2, 3$. On $D_+(z) \subset \mathbb{P}^2_k$, we have $X \cap D_+(z) = V(f(x,y)) \subset \mathbb{A}^2_k$, where $f(x,y) = y^2 - x^3 + x$.

$$\begin{cases} \frac{\partial f}{\partial x} = -3x^2 + 1 = 0 \\ \frac{\partial f}{\partial y} = 2y = 0, \end{cases} \tag{12}$$

   which has the solution $(x,y) = (\pm\frac{1}{\sqrt{3}}, 0)$. As we see, these two points do not lie on $X \cap D_+(z)$.

   On $D_+(y) \subset \mathbb{P}^2_k$, we have $X \cap D_+(y) = V(f(x,z)) \subset \mathbb{A}^2_k$, where $f(x,z) = x^3 - xz^2 - z$.

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - z^2 = 0 \\ \frac{\partial f}{\partial y} = -2xz - 1 = 0, \end{cases} \tag{13}$$

Note that $(X \cap D_+(y)) \setminus (X \cap D_+(z)) = [0 : 1 : 0]$, which corresponds to the point $(0,0)$ under our identification $D_+(y) \cong \mathbb{A}^2_k$. But $(0,0)$ is not a solution of (13).

Finally, $(X \cap D_+(x)) \setminus (X \cap D_+(z)) = \emptyset$. Therefore, X is smooth over k.

If $\mathrm{char}(k) = 3$, then both (12) and (13) have no solutions, hence in this case X is also smooth over k.

If $\mathrm{char}(k) = 2$, then (12) gives a singular point $(x,y) = (1,0) \in X \cap D_+(z)$, which corresponds to $[1 : 0 : 1] \in X \subset \mathbb{P}^2_k$.

(ii) Assume $\mathrm{char}(k) \neq 2$. Using the projective Jacobian criterion, we get

$$\frac{\partial f}{\partial x_i} = 2x_i = 0 \text{ for } i = 0, \ldots, r, \tag{14}$$

which implies $x_0 = \cdots = x_r = 0$. Hence, if $r = n$, then X is smooth over k, and if $r < n$, the singular closed points of X are of the form $[0 : \cdots : 0 : x_{r+1}, \ldots, x_n] \in \mathbb{P}^n_k(k)$ for which $\sum_{i=r+1}^n x_i^2 = 0$.

If char(k) $= 2$, then every closed point of X is singular.

(iii) Assume $\mathrm{char}(k) \nmid d$. Use the projective Jacobian criterion. We have

$$\begin{cases} \frac{\partial f}{\partial x_i} = 0 \text{ for } i = 0, \ldots, n \\ dx_{n+1}^{d-1} = 0, \end{cases} \tag{15}$$

and hence, in particular $x_{n+1} = 0$. By the Euler identity (see Exercise 67), any point satisfying the system (15) also satisfies $f(x_0, \ldots, x_n) = 0$. Hence, singular closed points of X are the points $[a_0 : \cdots : a_n : 0] \in \mathbb{P}^{n+1}_k(k)$ such that $\frac{\partial f}{\partial x_i}(a_0, \ldots, a_n) = 0$ for $i = 0, \ldots, n$.

If $\mathrm{char}(k) \mid d$, then $V_+(f - x^d)$ may or may not be smooth. Let $\mathrm{Char}(k) = 2$. For example if $f = x_0x_1 + x_1x_2 + x_2x_0$, then $V_+(f - x_3^2)$ is singular at $[1 : 1 : 1 : 1]$. However if $f = x_0x_1$ Then $V_+(f - x_2^2)$ is smooth everywhere.

In characteristic 0, $V_+(f - x_{n+1}^d)$ is smooth everywhere if and only f is smooth.

Exercise 69. *Exterior powers of sheaves* (4 points)
Recall that if A is a ring and M is an A-module, the *tensor algebra* of M is the graded (non-commutative) A-algebra $T(M) = \bigoplus_{n=0}^{\infty} T^n(M)$ where $T^n(M) = M^{\otimes n}$. The *exterior algebra* $\Lambda(M)$ is the quotient of $T(M)$ by the homogeneous ideal generated by the $x \otimes x$ with $x \in M$ and the n-th exterior power of M is $\Lambda(M)_n$.

Now, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$, we define the corresponding operations for $\mathcal{F}$ by applying them to each $\mathcal{F}(U)$ and passing to the sheafification.

(i) Assume that $\mathcal{F}$ is locally free of rank n. Show that $\Lambda^r(\mathcal{F})$ is locally free of rank $\binom{n}{r}$. In particular, $\det(\mathcal{F}) := \Lambda^n\mathcal{F}$ is an invertible sheaf which we call the *determinant* of $\mathcal{F}$.

(ii) Assume that $\mathcal{F}$ is locally free of rank n. Show that the natural map $\Lambda^r(\mathcal{F}) \otimes_{\mathcal{O}_X} \Lambda^{n-r}(\mathcal{F}) \to \Lambda^n(\mathcal{F})$ induces an isomorphism $\Lambda^r(\mathcal{F}) \cong \mathcal{H}om_{\mathcal{O}_X}(\Lambda^{n-r}(\mathcal{F}), \Lambda^n(\mathcal{F}))$

(iii) Let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a short exact sequence of locally free sheaves of finite rank. Show that, for any r, there exists a sequence of subsheaves

$$0 = F^{r+1} \subseteq F^r \subseteq \ldots \subseteq F^1 \subseteq F^0 = \Lambda^r(\mathcal{F})$$

such that $F^p/F^{p-1} \cong \Lambda^p(\mathcal{F}') \otimes \Lambda^{n-p}(\mathcal{F}'')$. Deduce that $\det(\mathcal{F}) \cong \det(\mathcal{F}') \otimes_{\mathcal{O}_X} \det(\mathcal{F}'')$. (Analogously, one can define the symmetric algebra and the symmetric power of sheaves of $\mathcal{O}_X$-modules and then an analogous filtration exists for symmetric powers with respect to short exact sequences. We will come back to this in AG2.)

Solution.

This is Hartshorne, Exercise II.5.16.
The first important observation is that the constructions of $T(M)$ and $\bigwedge(M)$ are compatible with the localization at one element, i.e. $T(M[f^{-1}] \cong T(M)[f^{-1}]$ canonically. In particular, if $\mathcal{F}$ is a quasi-coherent sheaf, say $\mathcal{F}|_U \cong \widetilde{M}$ for some affine open U, then $T(\mathcal{F})|_U \cong \widetilde{T(M)}$ and $\bigwedge(\mathcal{F})|_U \cong \widetilde{\bigwedge M}$.

1. If $\mathcal{F}|_U \cong A^n$ for an affine open $U = \operatorname{Spec} A$, then by the observation above, $\bigwedge^r(\mathcal{F})|_U \cong \widetilde{\bigwedge^r A^n}$, which is free of rank $\binom{n}{r}$.

2. It suffices to check everything on stalks, hence the question is reduced to the commutative algebra computation $\bigwedge^r A^n \cong \operatorname{Hom}\left(\bigwedge^{n-r} A^n, \bigwedge^n A^n\right)$. Let $e_1, \ldots, e_n$ be a basis, and let $I \subseteq 1, \cdots, n$ be a sub index set with cardinality r. We write e_I for the element $\bigwedge_{i\in I} e_i \in \bigwedge^r A^n$, where the wedge is taken in the order of integers in I. The isomorphism is then given by:

$$e_I \mapsto (e_J \mapsto e_{I \coprod J}),$$

where $I \coprod J$ denotes the union of I, J counting multiplicities. In particular, $e_{I \coprod J}$ is nonzero if and only if I and J are disjoint.

3. Take an affine open U such that $\mathcal{F}'|_U, \mathcal{F}|_U, \mathcal{F}''|_U$ are free. Since $\mathcal{F}''|_U$ is projective, there is a unique isomorphism $\mathcal{F}|_U \cong \mathcal{F}'|_U \oplus \mathcal{F}''|_U$ making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{F}'|_U \arrow[r] \arrow[d, "id"] & \mathcal{F}|_U \arrow[r] \arrow[d] & \mathcal{F}''|_U \arrow[r] \arrow[d, "id"] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{F}'|_U \arrow[r] & \mathcal{F}'|_U \oplus \mathcal{F}''|_U \arrow[r] & \mathcal{F}''|_U \arrow[r] & 0
\end{tikzcd}$$

We then write $\mathcal{F}'|_U \cong A^s$, $\mathcal{F}''|_U \cong A^t$, $\mathcal{F}|_U \cong A^{s+t}$. From concrete computations in commutative algebra we deduce that there exists a canonical isomorphism

$$\bigwedge^r A^{s+t} \cong \bigoplus_{i=0}^{r} \left(\bigwedge^i A^s\right) \otimes \left(\bigwedge^{r-i} A^t\right)$$

By defining $F^p A := \bigoplus_{i=0}^{p} \left(\bigwedge^i A^s\right) \otimes \left(\bigwedge^{r-i} A^t\right)$ to be the first p components, we see that this filtration satisfies the property asked in the exercise (perhaps we have to reverse the indices of the filtration). Moreover since everything is canonical, i.e. everything commutes with taking localizations, this construction glues well to a filtration of global quasi-coherent sheaves.
In particular, when taking $r = s + t$, there is only one nonzero component in the direct sum, namely $\det \mathcal{F}' \otimes \det \mathcal{F}''$.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 70. *Smoothness in characteristic* 0 (+ 4 extra points)
The goal of this exercise is to show that a scheme X of finite type over a field k of characteristic 0 is smooth if and only if $\Omega_{X/k}$ is locally free. For this, we have to show that if $\Omega_{X/k}$ is locally free, then it automatically has the correct rank.

(i) Show that $\Omega_{(\mathbb{F}_p[x]/x^p)/\mathbb{F}_p}$ is free of positive rank. In particular, the statement we want to prove is false in characteristic $p > 0$, even over perfect fields.

(ii) Let A be a k-algebra. Let $a \in A$ be an element such that Ada is a direct summand of $\Omega_{A/k}$. Show that a is not nilpotent.

(iii) Conclude that X is smooth if $\Omega_{X/k}$ is locally free.

(Remark: The Zariski–Lipman conjecture asks whether it is enough to assume that the dual of $\Omega_{X/k}$ is locally free to guarantee that X is smooth over k. This conjecture is known under certain assumptions on the singularities of X)

Solution. Not provided.

Exercise 71. *Examples of smooth hypersurfaces* (3 points)
Let k be any field and d a natural number. Give an explicit example of a smooth hypersurface $V_+(f_d)$, where f_d is homogeneous of degree d.

(Remark: Next term, we will prove a theorem of Bertini, which implies that if k is infinite, then a "general choice" of f_d will yield a smooth hypersurface.)

Solution. Let $X = V_+(f_d)$ and $n = \dim V_+(f_d)$. Recall that for any $x \in X$ we have $\operatorname{Rank} \Omega_{X/k} \otimes \kappa(x) = n - \operatorname{rank}(J_{A,x})$. Thus X is smooth at a closed point x if and only if $\operatorname{Rank}(J_{A,x}) = 1$.

If $\operatorname{char}(k) \nmid d$: set $f_d = x_0^d + \ldots + x_n^d$. Given $x \in V_+(f_d)$. We have

$$J_{A,x} = \left(dx_0^{d-1}, \ldots, dx_n^{d-1}\right)$$

Then $\operatorname{Rank}(J_{A,x}) = 0$ if and only if $x_i = 0$ for all $i = 0$. But $(0 : \cdots : 0) \notin \mathbb{P}^{n+1}$ and hence X is smooth.

Note that one could also check smoothness at every point using the same Jacobian criterion. Let $\mathfrak{p}$ denote the homogeneous prime ideal corresponding to a point $x \in X$. If x is not smooth, we have $dx_0^{d-1}, \ldots, dx_n^{d-1} \in \mathfrak{p} \in \mathfrak{p}$. Hence,

$$(x_0, \ldots, x_n) \subset \mathfrak{p}$$

which gives a contradiction.

If $\operatorname{Char} k \mid d$, then we let $f = x_0^d + \sum_i x_i x_{i+1}^{d-1}$. The projective Jacobian matrix for f looks like

$$\left(x_1^{d-1}, x_0(d-1)x_1^{d-2} + x_2^d, \ldots, x_{n-2}(d-1)x_{n-1}^{d-2} + x_n^d, x_{n-1}(d-1)x_n^{d-2}\right)$$

Note that if the Jacobian does not have maximal rank, its rank is 0, i.e. each term is zero. This leads to $x_1 = 0$ and hence for all $i > 0$ $x_i = 0$. So the only point where the Jacobian could be zero is $(1 : 0 : \cdots : 0)$. But this point does not lie on $V_+(f)$. Hence f is smooth everywhere.

Exercise 72. *Projective tangent spaces* (4 points)
Let k be a field, let $X = V_+(f_1, \dots, f_m) \subseteq \mathbb{P}^n_k$ and let $x = [a_0 : \dots : a_n] \in X$ be a k-rational point. The *projective tangent space of X at x* is the closed subscheme $T_x X \subseteq \mathbb{P}^n_k$ determined by the linear polynomials

$$\sum_{j=0}^{n} \frac{\partial f_i}{\partial x_j}(x) x_j.$$

(i) Prove that $T_x X$ does not depend on the choice of the f_i.

(ii) Show that $x \in X$ is a smooth point if and only if $\dim(T_x X) = \dim \mathcal{O}_{X,x}$.

(iii) Calculate the tangent line of the curve $C = V_+(y^2 z - x^3 + xz^2) \subseteq \mathbb{P}^2_k$ at $[0 : 1 : 0]$.

Solution.

(i) Let $I_f = (f_1, \dots, f_m), I_g = (g_1, \dots, g_s)$. Assume $(I_f)^{\mathrm{sat}} = (I_g)^{\mathrm{sat}}$. Denote $\partial I_f(x) = (\sum_{j=0}^n \frac{\partial f_i}{\partial x_j}(x) x_j \mid i = 1, \dots, m)$, $\partial I_g(x) = (\sum_{j=0}^n \frac{\partial g_i}{\partial x_j}(x) x_j \mid i = 1, \dots, s)$. We want to see that $\partial I_f(x) = \partial I_g(x)$.

First, note that we know that the ranks of the projective Jacobian matrices $J_f(x)$ and $J_g(x)$ are equal. Therefore, without loss of generality, we can assume that the columns of $J_f(x)$ and $J_g(x)$ are linearly independent (since passing to a linearly independent subsystem of columns which spans all the other columns will not change the ideals $\partial I_f(x)$ and $\partial I_g(x)$). Hence, $\mathrm{rk} J_f(x) = \mathrm{rk} J_g(x) = m = s \leq n$.

For each $i = 1, \dots, m$, we can write $g_i = \sum_{k=1}^m h_{ki} f_k$ for some $h_{ki} \in k[x_0, \dots, x_n]$, and these h_{ki} can be chosen to be homogeneous. Then

$$\frac{\partial g_i}{\partial x_j}(x) = \sum_{k=1}^{m} \left(\frac{\partial h_{ki}}{\partial x_j}(x) f_k(x) + h_{ki}(x) \frac{\partial f_k}{x_j}(x) \right) = \sum_{k=1}^{m} h_{ki}(x) \frac{\partial f_k}{x_j}(x),$$

where the second equality uses that $x \in X$.

Hence $J_g(x) = J_f(x) H(x)$, where $H = (h_{ki})_{k,i=1,\dots,m} \in \mathrm{Mat}_{m \times m}(k)$. Since $\mathrm{rk} J_f(x) = \mathrm{rk} J_g(x) = m$, we must have that $H(x)$ is invertible. Therefore,

$$\partial I_g(x) = ((x_0, \dots, x_n) \cdot J_g(x)) = ((x_0, \dots, x_n) \cdot J_f(x) H(x)) = ((x_0, \dots, x_n) \cdot J_f(x)) = \partial I_f(x).$$

(ii) It is clear that $\dim T_x X = n - \mathrm{rk}(J_X(x))$, where J_X is the projective Jacobian matrix. Now, by the projective Jacobian criterion (bearing in mind that the point x is rational), we have that X is smooth at x if and only if $n - \mathrm{rk}(J_X(x)) = \dim \mathcal{O}_{X,x}$, which proves our claim.

(iii) We have $f = y^2 z - x^3 + xz^2$, $x = [0 : 1 : 0]$ and $\frac{\partial f}{\partial x}(x) = 0$, $\frac{\partial f}{\partial y}(x) = 0$. $\frac{\partial f}{\partial z}(x) = 1$. Thus, $T_x C = V_+(z) \subset \mathbb{P}^2_k$.

Calculate the intersection and see that it intersects only at one point, strange? No, it's just a flex point!

Exercise 73. *Slicing criterion for regularity* (4 points)
The goal of this exercise is to show that an effective Cartier divisor that passes through a non-regular point cannot be regular.

(i) Let A be a ring and let $x \in \operatorname{Spec} A$ be a closed point with maximal ideal $\mathfrak{m}$. Let $f \in \mathfrak{m}$. Show that the Zariski tangent space (from Exercise 17) of $\operatorname{Spec} A/(f)$ at x is isomorphic to the subspace of the Zariski tangent space of $\operatorname{Spec} A$ at x that is annihilated by f.

(ii) Let X be a Noetherian scheme and $D \subseteq X$ an effective Cartier divisor. Assume that $x \in D$ is regular. Show that $x \in X$ is regular.

Solution. (i) First, Zariski tangent spaces are local, thus we can assume that A is a local ring, with unique maximal ideal $\mathfrak{m}$ and $f \in \mathfrak{m}$. Second observe that x has the same residue field in $\operatorname{Spec}(A)$ as in $\operatorname{Spec}(A/(f))$, since

$$A/\mathfrak{m} \cong (A/(f))/(\mathfrak{m}/(f)).$$

The projection $\mathfrak{m} \to \mathfrak{m}/(f)$ induces the dashed arrow ϕ in the following commutative square:

$$\begin{tikzcd}
\mathfrak{m} \arrow[r, two heads] \arrow[d, two heads] & \mathfrak{m}/(f) \arrow[d, two heads] \\
\mathfrak{m}/\mathfrak{m}^2 \arrow[r, dashed, "\phi"] & (\mathfrak{m}/(f))/(\mathfrak{m}/(f))^2
\end{tikzcd}$$

Moreover, ϕ is automatically surjective. Taking duals, we get an injective map

$$\begin{aligned}
\phi^\vee \colon ((\mathfrak{m}/(f))/(\mathfrak{m}/(f))^2)^\vee &\longrightarrow (\mathfrak{m}/\mathfrak{m}^2)^\vee \\
\overline{v} &\longmapsto (g \mapsto \overline{v}(\overline{g}))
\end{aligned}$$

where $\overline{g} = \phi(g)$. Observe that $\phi^\vee$ is a map from the Zariski tangent space $T_x(\operatorname{Spec}(A/(f)))$ into the Zariski tangent space $T_x(\operatorname{Spec}(A))$, thus it remains to identify its image. We claim that

$$\operatorname{Im}(\phi^\vee) = \{v \in (\mathfrak{m}/\mathfrak{m}^2)^\vee \mid v(f) = 0\}. \tag{16}$$

- The inclusion $\subset$ is direct, since $\phi(f) = 0$.
- For the other inclusion, let v be an element of the right-hand-side, i.e. $v(f) = 0$. We need to show that v factors through ϕ. In the following diagram,

$$\begin{tikzcd}
\mathfrak{m} \arrow[r, two heads] \arrow[d, two heads, "\pi"'] & \mathfrak{m}/(f) \arrow[d, two heads, "\pi'"] \arrow[dr, dashed, "v'"] & \\
\mathfrak{m}/\mathfrak{m}^2 \arrow[r, "\phi"] \arrow[rr, bend right, "v"'] & (\mathfrak{m}/(f))/(\mathfrak{m}/(f))^2 \arrow[r, dashed, "w"] & k(x)
\end{tikzcd}$$

  the composition $v \circ \pi$ sends (f) to zero, since $v(f) = 0$, thus we get the dashed arrow v'. Now $v'(\overline{g}\overline{h}) = v(gh) = 0$ for any two elements $g, h \in \mathfrak{m}$, thus we get the dashed arrow w, making everything commute. We have $\phi^\vee(w) = v$, hence this concludes the proof.

(ii) Let X be a Noetherian scheme and $D \subset X$ an effective Cartier divisor. Assume that $x \in D$ is regular. This means, $f \notin \mathfrak{m}^2$. We want to show that $x \in X$ is regular as well.

Since D is effective Cartier, it is locally given by the vanishing locus of a single function f (which is a non-zero divisor). Being regular is local, thus we can restrict to such an (affine) open subset and assume that $D = V(f) \subset \operatorname{Spec}(A) = X$. We can even assume that A is a local ring.

By definition, that $x \in D$ is regular means that

$$\dim_{k(x)}(\mathfrak{m}/(f))/(\mathfrak{m}/(f))^2 = \dim(A_{\mathfrak{m}}/(f)).$$

But the equality (16) in part (i) tells us that

$$\dim_{k(x)}(\mathfrak{m}/(f))/(\mathfrak{m}/(f))^2 = \dim_{k(x)} \mathfrak{m}/\mathfrak{m}^2 - 1,$$

since $\operatorname{Im}(\phi^\vee)$ is the kernel of the nonzero functional $v \mapsto v(f)$ (hence it has codimension 1). Observe that $v \mapsto v(f)$ is nonzero since $f \in \mathfrak{m} - \mathfrak{m}^2$, because if f was in $\mathfrak{m}^2$ then D would be non-regular at x.

On the other hand, $\dim(A_{\mathfrak{m}}/(f)) = \dim(A_{\mathfrak{m}}) - 1$ since minimal primes of (f) have height 1 (by Krull's principal ideal theorem). Therefore, we get

$$\dim_{k(x)} \mathfrak{m}/\mathfrak{m}^2 = \dim(A_{\mathfrak{m}})$$

and $x \in X$ is regular.

See `https://metaphor.ethz.ch/x/2017/hs/401-3132-00L/dimension.pdf` for more results about the relation between height and dimension. In particular we used Theorem 19 from those notes.

Exercise 74. *Left-exactness of conormal sequence* (4 points)
Let k be a perfect field and X a smooth k-scheme of finite type. Let $Z \subseteq X$ be a reduced closed subscheme with ideal sheaf $\mathcal{I}_Z$. Assume that $\mathcal{I}_Z/\mathcal{I}_Z^2$ is locally free. Show that the conormal sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{\delta} \Omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \Omega_{Z/k} \to 0$$

is exact.

Solution. We may assume that Z is irreducible. Recall that there exists an open subset $U \subset X$ such that $Z \cap U$ is smooth or equivalently $\Omega_Z|_U$ is locally free of rank $\dim Z$. Using Theorem 18.26 we have a short exact sequence

$$0 \to \mathcal{I}_{Z\cap U}/\mathcal{I}_{Z\cap U}^2 \to \Omega_{U/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \Omega_{Z\cap U/k} \to 0$$

Therefore $\operatorname{Supp}(\ker(\delta)) \subset X \setminus U$. Since $\mathcal{I}_Z/\mathcal{I}_Z^2$ is locally free, it cannot have a subsheaf supported at a closed subscheme. To see the last claim, note that the support of a section of a sheaf is contained in the support of the sheaf. Therefore, for any open $V \subset X$, such that $V \cap Z \neq \emptyset$, we can choose a section $s \in \Gamma(V, \ker(\delta))$. But $\Gamma(V, \ker(\delta)) \subset \Gamma(V, \mathcal{I}_Z/\mathcal{I}_Z^2)$. Since the latter is locally free (assume free on V), its section s must be supported everywhere on V. Hence $\ker(\delta) = 0$.

Exercise 75. *Frobenius and differentials* (5 points)
All schemes considered in this exercise are schemes over $\mathbb{F}_p$. For a scheme X, the *absolute Frobenius* $F_X : X \to X$ is defined as $(\mathrm{id}_X, F^\sharp)$, where $F^\sharp(U)$ is the p-th power map (which is a ring homomorphism, since $\mathcal{O}_X(U)$ has characteristic p).

(i) Show that, for any $\mathbb{F}_p$-scheme S, base change along F_S determines a functor $(-)^{(p/S)}$: $(\mathrm{Sch}/S) \to (\mathrm{Sch}/S)$. For any S-scheme X, the morphism $F_{X/S} = (F_X, \pi_X) : X \to X^{(p/S)}$ is called *S-linear (or relative) Frobenius*. If S is clear from the context, we write $(-)^{(p)}$ instead of $(-)^{(p/S)}$.

(ii) Show that for any morphism $f : X \to Y$ of S-schemes, we have $f \circ F_X = F_Y \circ f$ and $f^{(p)} \circ F_{X/S} = F_{Y/S} \circ f$.

(iii) Show that $F_{X/S}$ is a universal homeomorphism.

(iv) If $S = \operatorname{Spec} A$ for some ring A and $X = \operatorname{Spec} A[x_1, \dots, x_n]/I$ for an ideal I, show that $X^{(p/S)} \cong A[x_1, \dots, x_n]/I^{(p)}$, where $I^{(p)} = (\{f^{(p)}\}_{f \in I})$, where $f^{(p)}$ denotes raising the coefficients of a polynomial f to the p-th power. Show that, under this identification, the map $F_{X/S}$ is induced by the A-linear map determined by $x_i \mapsto x_i^p$. *Observe that, absolute Frobenius raise everything to p-th power but relative Frobenius raises just the variables to p-th power.*

(v) Conclude that, for every morphism of $\mathbb{F}_p$-schemes $X \to Y$, there is an isomorphism $\Omega_{X/Y} \cong \Omega_{X/X^{(p/Y)}}$.

Solution. (i) and (ii) Lets do the first part of (ii) first To see that

$$\begin{tikzcd} X' \arrow[r, "F_{X'}"] \arrow[d, "f"'] & X' \arrow[d, "f"] \\ X \arrow[r, "F_X"'] & X \end{tikzcd}$$

commutes, we just observe that the same is true for affine schemes, i.e. if $\varphi\colon A \to B$ is a ring map, then $\varphi(a^p) = \varphi(a)^p$. We can then define the relative Frobenius for an S-scheme $f\colon X \to S$ as the unique map $F_{X/S}$ making the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "F_{X/S}"'] \arrow[rr, bend left, "F_X"] \arrow[dr, bend right, "f"'] & X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ & S \arrow[r, "F_S"'] & S \end{tikzcd}$$

commutative. To see that $(-)^{(p)}$ is functorial, we note that we can write

$$f^{(p)} = f \times \mathrm{id}\colon X \times_{S,F_S} S \to Y \times_{S,F_S} S,$$

which by a diagram chase should then also give the identities $f^{(p)} \circ F_{X/S} = F_{Y/S} \circ f$.

(iii) Not provided.

(iv), we first note that in an affine setting, the fiber product

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ S \arrow[r, "F_S"'] & S \end{tikzcd}$$

with $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ and the structure map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ corresponding to a ring map $A \to B$, can be described as

$$X^{(p)} = B \otimes_{F_A, A} A,$$

i.e. the first B-factor is an A-module via the composition $A \xrightarrow{F_A} A \to B$. This gives the desired description of $X^{(p/S)}$.

(v) we view the Frobenius relative Y as a morphism of Y-schemes:

$$\begin{tikzcd} X \arrow[rr, "F_{X/Y}"] \arrow[dr, "f"'] & & X^{(p/Y)} \arrow[dl] \\ & Y & \end{tikzcd}$$

This then gives the exact sequence

$$F_{X/Y}^*\Omega_{X^{(p)}/Y} \longrightarrow \Omega_{X/Y} \longrightarrow \Omega_{X/X^{(p/Y)}} \longrightarrow 0$$

(Note that it is not a priori left-exact, because $F_{X/Y}$ is not smooth). So to show the desired claim, it is enough to see that the first map is zero. Since the first map can be described via pulling back the differential forms on $X^{(p/Y)}$ one can see (at least when $X = \operatorname{Spec} A[x_1, \dots, x_n]/I$ as in (iv)) that the differential of the map $x_i \mapsto x_i^p$ is zero.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 76. *Universally injective morphisms* (+ 4 extra points)
Recall that a field extension $k \subseteq K$ is called *purely inseparable* if $\operatorname{char}(k) = p \neq 0$ and for every $a \in K$, there exists $n > 0$ such that $a^{p^n} \in k$.

Let $f : X \to Y$ be a morphism of schemes. Show that the following properties are equivalent:

(i) f is universally injective.

(ii) f is *radicial*, i.e., it is injective and for all points $x \in X$, $k(x)$ is a trivial or purely inseparable extension of $k(f(x))$.

(iii) For every field k, the morphism $\operatorname{Hom}(\operatorname{Spec} k, X) \to \operatorname{Hom}(\operatorname{Spec} k, Y)$ is injective.

Conclude that, for every field k, the inclusion $k[x^2, x^3] \to k[x]$ induces a universal homeomorphism of affine schemes.

Solution. Not provided.

Except in the bonus exercise below, we assume that k is a perfect field and every curve is geometrically integral, smooth, and complete over k.

Exercise 77. *Curves of genus* 0
Let X be a curve over k.

(i) Show that $X \cong \mathbb{P}^1_k$ if and only if there exists a divisor D on X with $\deg(D) = 1$ and $h^0(X, \mathcal{O}_X(D)) \geq 2$.

(ii) Show that $X \cong \mathbb{P}^1_k$ if and only if $g(X) = 0$ and $X(k) \neq \emptyset$.

Solution. (i) One direction is easy: if there exists an isomorphism $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$, we let D be an effective divisor associated to a section of $\varphi^*\mathcal{O}(1)$. To see the other use these two sections to produce a rational map $\varphi\colon \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$. Use Ex. 36 to extend φ. Use Prop. 20.10 to conclude that φ is an isomorphism.

(ii) Take a point $x \in X(k)$. Then, considered as a divisor, x has degree 1. By Riemann–Roch, we have $h^0(X, \mathcal{O}_X(x)) = 2$, so we conclude by (1).

Exercise 78. *Existence of curves of given genus*
Let $Q := \mathbb{P}^1_k \times \mathbb{P}^1_k$ and let $g \geq 0$ be any non-negative integer.

(i) Find a bihomogeneous polynomial $f(x_0, x_1, y_0, y_1)$ which has degree $g+1$ in the x_i and degree 2 in the y_i such that $V_+(f) \subseteq Q$ is smooth.

(ii) Conclude that curves of genus g exist.

Solution. (i) Not provided.

(ii) Sketch: Let $X = V_+(f)$ and let p_1, p_2 be the two projections. The polynomial f determines a section of $p_1^*\mathcal{O}(g+1) \otimes p_2^*\mathcal{O}(2)$. We have $\omega_{\mathbb{P}^1\times\mathbb{P}^1} \cong p_1^*\mathcal{O}(-2) \otimes p_2^*\mathcal{O}(-2)$. Hence, by adjunction, $\omega_X \cong p_1^*\mathcal{O}(g-1)|_X$. So, $\deg(\omega_X) = (g-1)\cdot \deg(p_1^*\mathcal{O}(1)|_X)$. The zero locus of a section of $p_1^*\mathcal{O}(1)$ is a fiber F of p_1 over a k-rational point of $\mathbb{P}^1$. The intersection of F with X yields a divisor of degree 2, hence $\deg(\omega_X) = 2g-2$, so X has genus g.

Exercise 79. *Genus degree formula*
Let $X = V_+(f_d) \subseteq \mathbb{P}^2_k$ be a smooth hypersurface with f_d homogeneous of degree d.

(i) Show that $g(X) = \frac{1}{2}(d-1)(d-2)$.

(ii) Conclude that curves of genus 2 never admit a closed immersion to $\mathbb{P}^2_k$.

Solution. (i) Note that by the adjunction formula $\omega_X \simeq \mathcal{O}_X(d-3)$. Hence, $\deg(\omega_X) = (d-3)\deg(\mathcal{O}_X(1))$. The degree of $\mathcal{O}_X(1)$ can be calculated by taking a general line ℓ in $\mathbb{P}^2_k$ and taking the length of the intersection $\ell \cap X$. Since f_d is homogeneous of degree d, the latter is d. Hence,

$$\deg(\omega_X) = (d-3)\deg(\mathcal{O}_X(1)) = d(d-3).$$

On the other hand, we know that $2g(X) - 2 = \deg(\omega_X)$. Solving for $g(X)$ yields the formula.

(ii) If $\frac{1}{2}(d-1)(d-2) = 2$, then we obtain $d(d-3) = 2$. Hence either $d = 2$ and $d-3 = 1$, or $d = 1$ and $d-3 = 2$. Both sets of equations are absurd.

Exercise 80. *Hyperelliptic curves*
A curve X over k is called *hyperelliptic* if $g(X) \geq 2$ and X admits a finite morphism of degree 2 to $\mathbb{P}^1_k$.

(i) Show that every curve of genus 2 is hyperelliptic.

(ii) Show that all the curves constructed in Exercise 78 are hyperelliptic.

Solution. (i) By the definition of genus $\Gamma(C, \omega_C) = 2$ and $\deg \omega_C = 2.2 - 2 = 2$, the linear system $|D|$ associated to ω_C is of dimension 2 and has degree 2. It remains to show that $|D|$ is base-point free. Assume $|D|$ has a base-point P and write $|D| = P + |D'|$. Then, $\deg(D') = 1$, so, by Riemann–Roch, we have

$$h^0(X, \mathcal{O}_X(D')) \leq h^0(X, \mathcal{O}_X(P)) = 1,$$

a contradiction. Hence, $|D|$ determines the desired morphism $X \to \mathbb{P}^1_k$ of degree 2.

(ii) Hint: Let $X = V_+(f) \subset Q$ be the smooth curve of genus g as in Ex. 78. Let $p\colon Q \to \mathbb{P}^1$ be the first projection. Argue that this is a degree 2 map.

Exercise 81. *Plane quartics*
Let X be a curve over k. Assume that $g(X) = 3$ and X admits a closed immersion to $\mathbb{P}^2_k$.

(i) Give an explicit example of such an X.

(ii) Show that the morphism $X \to \mathbb{P}^2_k$ is given by a basis of $H^0(X, \omega_X)$. In other words, the effective divisors D on X given by global sections of $H^0(X, \omega_X)$ are exactly the intersections of X with lines in $\mathbb{P}^2_k$.

(iii) Show that if D is an effective divisor of degree 2 on X, then $|D| = \{D\}$. Conclude that X is not hyperelliptic.

(Hint: Reduce to the case where k is algebraically closed and apply the criterion for very ampleness that will be discussed in Monday's lecture)

Solution. (i) Assume $\operatorname{Char} k \neq 2$. Let $X = V_+(x_0^4 + x_1^4 + x_2^4) \subset \mathbb{P}^2_k$. By the genus degree formula in Ex. 79 $g(X) = 3$.

(ii) This follows from the fact that $\omega_X \simeq \mathcal{O}_X(1)$ by adjunction and $\Gamma(X, \omega_X) = 3$.

(iii) By Riemann–Roch, we observe that

$$\dim \Gamma(X, \mathcal{O}_X(D)) - \dim \Gamma(X, \omega_X(-D)) = 2 - 3 + 1 = 0.$$

If $\dim \Gamma(X, \mathcal{O}_X(D)) > 1$, then by the equation above $\dim \Gamma(X, \omega_X(-D)) > 1$. On the other hand, since ω_X defines the embedding $X \hookrightarrow \mathbb{P}^2_k$, by the very ampleness criterion we have for any points $p, q \in X$ $\dim \Gamma(X, \omega_X(-p-q)) = 3 - 2 = 1$. Hence we obtain that $|D| = \mathbb{P}(\Gamma(X, \mathcal{O}_X(D)) \simeq \{D\}$.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 82. *Curves are quasi-projective*
In this exercise, k is an arbitrary field. In the lecture, we have seen that regular complete curves over a field k are projective. The goal of this exercise is to see that, more generally, all curves over a field k are quasi-projective. So, let X be a curve over k.

(i) Let $U \subseteq X$ be a non-empty open affine subset. Show that there exists a globally generated invertible sheaf $\mathcal{L}$ and a global section s of $\mathcal{L}$ such that $X_s = U$.

(ii) Conclude that there exists a globally generated invertible sheaf $\mathcal{L}$ on X such that for every $x \in X$, there exists a global section s of $\mathcal{L}$ such that $x \in X_s$.

(iii) Show that the invertible sheaf $\mathcal{L}$ in (ii) is very ample. Conclude that X is quasi-projective over k. In particular, if X is complete, then it is projective over k, and we do not need to assume that X is regular for this.

Solution. Not provided.