Algebraic geometry 1
Exercise sheet 10

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Exercise 1.

1. Since X is closed and irreducible, it is of the form $X = \overline{\{p_0\}}$ for some (Eric thinks unique) $p_0 \in \mathbb{A}^n_k$. That means $X \cong \mathrm{Spec}(k[x_1, \dots x_n]/p_0)$. Denote $A = k[x_1, \dots x_n]/p_0$.

   By assumption there is a chain of specializations $p_0 \subset \cdots \subset p_d$ inside X.

   Let $Z \subseteq X \cap V(f_1, \dots, f_r)$ be a irreducible component. Thus it is the closure of a minimal prime ideal in $A/(f_1, \dots, f_r)$.

   By Krull's principal ideal theorem we have $\dim(A/(f_1, \dots, f_r)) \geq d - r$.

   Denote minimal prime ideals in $A/(f_1, \dots, f_r)$ with $q_1, \dots q_l$.

   (Eric thinks that there is a unique generic point here again, since X is sober, so there should only be one of these prime ideals, right?)

   We argue that

   $$\dim(A/(f_1, \dots, f_r)/q_j) = \dim(A/(f_1, \dots, f_r)).$$

   for any j.

   That follows from A being catenary. If there existed a maximal chain in $A/(f_1, \dots, f_r)$ that starts at q_j we could simply extend it below to get a maximal chain in A. Since all maximal chains in A are of the same length, we get that all maximal chains in $A/(f_1, \dots, f_r)$ are also of the same length.

   Since Z is an irreducible component, we have $Z = \overline{\{q_i\}} \subseteq \mathrm{Spec}(A/(f_1, \dots, f_r))$.

   Therefore any maximal chain in Z is exactly as long as the longest chain in $A/(f_1, \dots, f_r)$. And the longest chain in $A/(f_1, \dots, f_r)$ is at least of length $d - r$.

2. The diagonal $\triangle \subseteq \mathbb{A}^n_k \times \mathbb{A}^n_k$ can be defined as $V(x_i \otimes x_i \mid i = 1, \dots n) \subseteq \mathrm{Spec}(k[x_1, \dots, x_n] \otimes_{\mathbb{Z}} k[x_1, \dots, x_n])$.

   (Should there not be a minus instead of $\otimes$ in the above expression?)

Using exercise above we get that any irreducible component of $X \cap Y \cong (X \times Y) \cap V(x_i \otimes x_i \mid i = 1, \ldots n)$ has dimension at least $d + e - n$.

3. Let $\tilde{X} = \overline{f^{-1}(X)}$ and $\tilde{Y} = \overline{f^{-1}(Y)}$ as in the hint.

   We have $\dim(\tilde{X}) = d + 1$ and $\dim(\tilde{Y}) = e + 1$. By the previous exercise we have $\dim(\tilde{X} \cap \tilde{Y}) \geq d + 1 + e + 1 - (n + 1) = (d + e - n) + 1 \geq 1$.

   Therefore there there exists $0 \neq x \in \tilde{X} \cap \tilde{Y}$.

Exercise 2.

1.

2.