Algebraic geometry 1
Exercise sheet 8

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Exercise 1.

Exercise 2.

Exercise 3.

1. In exercise 2 we showed that all invertible quasicoherent sheaves on $\mathbb{P}^n_k$ are isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$. So we have to show $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is an invertible sheaf.

   Since invertible $\mathcal{O}_{\mathbb{P}^n_k}$-modules are same as line bundles, we have to show that locally $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^m_k}$.

   By definition $f^*\mathcal{O}_{\mathbb{P}^m_k}(1) = f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^m_k}} \mathcal{O}_{\mathbb{P}^n_k}$. Pick some $x \in \mathbb{P}^n_k$. Pick small enough affine neighborhood $f(x) \in U \subseteq \mathbb{P}^m_k$ such that $\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^m_k}$ on U. Now pick neighborhood $x \in W \subseteq \mathbb{P}^m_k$ such that $f(W) \subseteq U$.

   Then

$$\begin{aligned} f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)(W) &= \operatorname{colim}_{f(W)\subseteq V} \mathcal{O}_{\mathbb{P}^m_k}(1)(V) \\ &= \operatorname{colim}_{f(W)\subseteq V\subseteq U} \mathcal{O}_{\mathbb{P}^m_k}(1)(V) \\ &\cong \operatorname{colim}_{f(W)\subseteq V\subseteq U} \mathcal{O}_{\mathbb{P}^m_k}(V) \\ &\cong f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(W). \end{aligned}$$

   So locally $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}$, so $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^m_k}} \mathcal{O}_{\mathbb{P}^n_k}$ is locally isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}$, which proves that $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is an invertible $\mathcal{O}_{\mathbb{P}^n_k}$-module and thus isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$.

2. At first it was not completely clear to us what the map $f^*\colon \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1)) \to \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ is. So we assumed it is the following: For a global section

$f \in \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1))$ we first map it with the restriction

$$\Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1)) \to \Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)).$$

Denote its image with f'.

By definition we have

$$\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1)) = \Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)) \otimes_{\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k})} \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})$$

So include f' into $\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1))$ as $f' \otimes 1$.

The polynomials $y_0, \ldots, y_n$ generate the module of homogenous polynomials of degree 1.

Exercise 4.

1. Let $U_i = \mathrm{Spec}(A_i)$.

   Take a point $x \in U_1 \cap U_2$.

   Take a principal open $x \in D(f) \subseteq U_1$ ($f \in U_1$). Then find a smaller principal open $x \in D(g) \subseteq D(f) \subseteq U_2$ ($g \in U_2$).

   Now we show that $D(g)$ is also a principal open in U_1.

   Since $D(f) \subseteq U_2$ open, we have a map $\mathcal{O}(U_2) \to \mathcal{O}(D(f))$, which induces $A_2 \to (A_1)_f$. Denote by $g' = g|_{\mathrm{Spec}((A_1)_f)}$ the image of g under this map. Since $g' \in (A_1)_f$, we can write it as $g' = \frac{h}{f^n}$. Then $D(g) = D(g) \cap D(f) = D(g') \cap D(f) = D(h) \cap D(f) = D(hf)$, where $h, f \in A_1$. This shows that $D(g)$ is also principal open in U_1.

2. We have to show that the property of being of finite presentation is a local property and that f as defined above is locally of finite presentation.

   Let $\mathrm{Spec}(B) \subseteq X$ and $\mathrm{Spec}(A) \subseteq S$ open affines. Pick a point $x \in \mathrm{Spec}(B)$. Then $x \in \mathrm{Spec}(B) \cap \mathrm{Spec}(B_i)$ for some i. Pick some neighborhood $x \in U \subseteq \mathrm{Spec}(B) \cap \mathrm{Spec}(B_i)$ such that U is principal open in $\mathrm{Spec}(B)$ and in $\mathrm{Spec}(B_i)$.

   Now take a neighborhood $f(x) \in V \subseteq f(U)$ so that V is principal open in $\mathrm{Spec}(A)$ and in $\mathrm{Spec}(A_i)$. Now take another smaller neighborhood $x \in U' \subseteq f^{-1}(V)$ such that U' is principal open in $\mathrm{Spec}(B)$ and in $\mathrm{Spec}(B_i)$.

   So we have $U' \to V$, where both U' and V are principal opens of $\mathrm{Spec}(B_i)$ and $\mathrm{Spec}(A_i)$ respectively. Since $A_i \to B_i$ is of finite presentation, then localizations $(A_i)_f \to (B_i)_g$ (for some $f \in A_i$ and $g \in B_i$) are as well.

   So for every point $x \in \mathrm{Spec}(B)$ we can find a principal open neighborhood in $x \in D(f_x)$ and a principal open neighborhood $f(x) \in D(g_x)$ such that $A_{g_x} \to B_{f_x}$.

Since $\mathrm{Spec}(B)$ is quasi-compact, we have $\mathrm{Spec}(B) = D(f_1) \cup \cdots \cup D(f_n)$. Denote $g_1, \ldots, g_n \in A$ be the respective elements in A.

We have composition $\mathrm{Spec}(B_{f_i}) \to \mathrm{Spec}(A_{g_i}) \hookrightarrow \mathrm{Spec}(A)$, which induces a map of rings $A \to A_{g_i} \to B_{f_i}$. Since $A_{g_i} \cong A[X]/(Xg_i - 1)$ and $A_{g_i} \to B_{f_i}$ are of finite presentation by assumption, and being of finite presentation is stable under compositions, we have that $A \to B_{f_i}$ are of finite presentation for every i.

Now its just commutative algebra to show that $A \to B$ is of finite presentation as well, so I hope its okay to assume this part. Otherwise we could just rewrite something like Lemma 00EP.