

Algebraic geometry 1

Exercise sheet 10

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Exercise 1.

1. Since X is closed and irreducible, it is of the form $X = \overline{\{p_0\}}$ for some $p_0 \in \mathbb{A}_k^n$. That means $X \cong \text{Spec}(k[x_1, \dots, x_n]/p_0)$. Denote $A = k[x_1, \dots, x_n]/p_0$. By assumption there is a chain of specializations $p_0 \subset \dots \subset p_d$ inside X . Let $Z \subseteq X \cap V(f_1, \dots, f_r)$ be a irreducible component. Thus it is the closure of a minimal prime ideal in $A/(f_1, \dots, f_r)$. By Krull's principal ideal theorem we have $\dim(A/(f_1, \dots, f_r)) \geq d - r$. Denote minimal prime ideals in $A/(f_1, \dots, f_r)$ with q_1, \dots, q_l . We argue that

$$\dim(A/(f_1, \dots, f_r)/q_j) = \dim(A/(f_1, \dots, f_r)).$$

for any j .

That follows from A being catenary. If there existed a maximal chain in $A/(f_1, \dots, f_r)$ that starts at q_j we could simply extend it below to get a maximal chain in A . Since all maximal chains in A are of the same length, we get that all maximal chains in $A/(f_1, \dots, f_r)$ are also of the same length.

Since Z is an irreducible component, we have $Z = \overline{\{q_i\}} \subseteq \text{Spec}(A/(f_1, \dots, f_r))$.

Therefore any maximal chain in Z is exactly as long as the longest chain in $A/(f_1, \dots, f_r)$. And the longest chain in $A/(f_1, \dots, f_r)$ is at least of length $d - r$.

2. The diagonal $\Delta \subseteq \mathbb{A}_k^n \times \mathbb{A}_k^n$ can be defined as $V(x_i \otimes x_i \mid i = 1, \dots, n) \subseteq \text{Spec}(k[x_1, \dots, x_n] \otimes_{\mathbb{Z}} k[x_1, \dots, x_n])$. Using exercise above we get that any irreducible component of $X \cap Y \cong (X \times Y) \cap V(x_i \otimes x_i \mid i = 1, \dots, n)$ has dimension at least $d + e - n$.
3. Let $\tilde{X} = f^{-1}(X)$ and $\tilde{Y} = f^{-1}(Y)$ as in the hint.

We have $\dim(\tilde{X}) = d + 1$ and $\dim(\tilde{Y}) = e + 1$. By the previous exercise we have $\dim(\tilde{X} \cap \tilde{Y}) \geq d + 1 + e + 1 - (n + 1) = (d + e - n) + 1 \geq 1$. Therefore there exists $0 \neq x \in \tilde{X} \cap \tilde{Y}$.

Exercise 2.

- 1.
- 2.