

# Algebraic geometry 1

## Exercise sheet 11

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**Exercise 1.** We claim that  $\mathcal{O}_{X,x}$  is a normal, local noetherian domain of dimension at most one. Normality is by definition of normality of  $X$ . Stalks are of course local rings by definition of locally ringed space. Noetherian comes from the assumption that  $X$  is of finite type over  $k$ . Also, by The Stacks Project

$$\dim(X) = \sup_{x \in X} \dim(\mathcal{O}_{X,x}).$$

Hence,  $\dim(\mathcal{O}_{X,x}) \leq 1$ .

If  $\dim(\mathcal{O}_{X,x}) = 0$ , then  $\mathcal{O}_{X,x}$  is of course regular as the only prime ideal is  $(0)$ .

If  $\dim(\mathcal{O}_{X,x}) = 1$ , then we know from the lecture that the maximal ideal  $\mathfrak{m} \subset \mathcal{O}_{X,x}$  is principal (and not zero) and hence  $\mathcal{O}_{X,x}$  is regular.

**Exercise 3.**

1. Since  $k$  is algebraically closed, the only irreducible polynomials  $f \in k[x, y]$  are of degree 1.

Hence, we can write

$$f_r = l_1 \dots l_r,$$

where  $l_i \in k[x, y]$  is of degree 1. From the assumption that  $f_r$  is homogeneous it follows that the  $l_i$  are homogeneous.

Therefore, we can write

$$Z = V(f_r) = V(l_1 \dots l_r) = \cup_i V(l_i)$$

and since  $V(l_i)$  is a line through the origin,  $Z$  can be written as the finite union of lines through the origin.

2. We first want to prove that  $\dim(\mathcal{O}_{X,(x,y)}) = 1$  for all  $r$ . The prime ideals  $p$  in this ring fulfil  $(f) \subset p \subset (x, y)$ . Remember that we can write down these prime ideals explicitly as in "What do primes of  $k[x, y]$  look like". From this the claim follows.

We know that  $\dim_k(m_{\mathcal{O}_{X,(x,y)}}/m_{\mathcal{O}_{X,(x,y)}}^2)$  is the number of generators of  $m_{\mathcal{O}_{X,(x,y)}}$ .

Now if  $r = 1$ , then we can write  $f = g(x, y)x + h(x, y)y$  and w.l.o.g. we have  $g(0, 0) = 1$ , meaning that it is invertible (after localizing). Therefore  $f = x + h(x, y)y$ , so  $y \mid x$  meaning  $(x, y) = (y)$ . On the other hand, if  $r > 1$ , then  $x \nmid y$  and  $y \nmid x$  meaning that  $m$  is no principal ideal showing that  $X$  is singular at zero in this case. (This can be seen by writing  $f$  as  $f = x^2h_1(x, y) + xyh_2(x, y) + y^2h_3(x, y)$ ).

3. By part two of this exercise, all the schemes have a singular point at the origin. I don't know why they do not have singular points anywhere else.

**Exercise 4.** We will show that the restriction map on global sections is an isomorphism, i.e. that

$$\Gamma(X, \mathcal{O}_X) \cong \Gamma(U, \mathcal{O}_X)$$

This immediately implies the claim of the exercise by definition of vector bundle (if rings are isomorphic then so is their finite sum).

By definition,  $\text{codim}(Z) \geq 2$ . Take  $Y \subset X$  an irreducible component of codimension 1. By construction,  $Y$  and  $U$  intersect nontrivially (either  $Z \cap Y = \emptyset$  or  $Z \subsetneq Y$ ). In particular,  $U$  contains the generic point  $\mu$  of  $Y$ . This means that  $\Gamma(U, \mathcal{O}_X) \subset \mathcal{O}_{X, \mu}$ .

The hint tells us that  $A$  is the intersection of all localizations of  $A$  at prime ideals  $p$  of height 1. Those prime ideals of height 1 correspond to irreducible components of codimension 1. Hence, we have just shown that

$$\Gamma(U, \mathcal{O}_X) \subset A = \Gamma(X, \mathcal{O}_X).$$

The other inclusion is immediate.