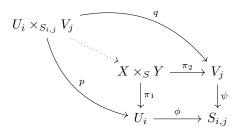
## Algebraic geometry 1 Exercise sheet 6

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## Exercise 1.

1. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram



Therefore, on the level of sets,

$$U_i \times_{S_{i,j}} V_j \subset X \times_S Y$$
,

but in exercise 5.2.1, we showed that this induces an open immersion as locally ringed spaces.

Now observe that

$$\bigcup_{i,j} \left( U_i \times_{S_{i,j}} V_j \right) \longrightarrow Y$$

$$\downarrow \qquad \qquad \downarrow$$

$$X \longrightarrow S$$

commutes, because  $S = \bigcup_{i,j} S_{i,j}$ . Now by uniqueness of the pullback,

$$\bigcup_{i,j} (U_i \times_{S_{i,j}} V_j) \cong U_i \times_{S_{i,j}} V_j.$$

**Exercise 3.** By definition we have to compute a fibred product of  $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$  and  $\operatorname{Spec}(k(p)) \to \operatorname{Spec}(A)$  (where k(p) is the residue field of  $p \in \operatorname{Spec}(A)$  and  $\to$  is the canonical inclusion). Since we are dealing with affine schemes, we can express it concretely as  $\operatorname{Spec}(B \otimes_A k(p))$ . Note that B has the structure of an A-algebra, which is induced by the starting morphism of schemes  $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ . So this exercise reduces to computing these tensor products.

We also observe that k[T] is a PID, which means every non-zero prime ideal is a maximal ideal. This will be handy when computing residue fields, because after quotienting with a non-zero ideal we already get a field (we do not have to further take the quotient field).

- 1. In the first example we do now even have to calculate the tensor product, because we can rewrite  $k[T,U]/(TU-1)=k[T,T^{-1}]$ , so this is just a localization of k[T]. Morphism of spectrums, induced by inclusion into localization, is an open immersion, so fibers will be singletons if  $x \in D(T)$  and empty sets otherwise. And the structure sheaf is also clear, it is just the restriction of structure sheaf  $\mathcal{O}_{\text{Spec}(k[T])}$ .
- 2.
- 3.
- 4.

**Exercise 4.** Take U = D(f) for some  $f \in A$  and let  $U = \bigcup_i D(f_i)$  be some cover. We have to check that

$$M[f^{-1}] \to \operatorname{Eq}\left[\prod_i M[f_i^{-1}] \Longrightarrow \prod_{i,j} M[(f_i f_j)^{-1}]\right]$$

is isomorphism.

This proof is exactly the same as when we proved that  $\mathcal{O}_{\mathrm{Spec}(A)}$  is a sheaf, after we defined it the basis of principal opens.

Then proved that  $A = \text{Eq}\left[\prod_i A[f_i^{-1}] \rightrightarrows \prod_{i,j} A[(f_i f_j)^{-1}]\right]$  where  $\text{Spec}(A) = \bigcup_i D(f_i)$  is a cover.

We can simply tensor the whole diagram and, since tensor product commute with direct limits, we have that

$$M = \operatorname{Eq}\left[\prod_i M \otimes_A A[f_i^{-1}] \Rightarrow \prod_{i,j} M \otimes_A A[(f_i f_j)^{-1}]\right].$$