Algebraic geometry 1
Exercise sheet 10

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18. Dezember 2023

Exercise 1.

1. Since X is closed and irreducible, it is of the form $X = \overline{\{p_0\}}$ for some $p_0 \in \mathbb{A}^n_k$. That means $X \cong \operatorname{Spec}(k[x_1, \dots x_n]/p_0)$. Denote $A = k[x_1, \dots x_n]/p_0$.

   By assumption there is a chain of specializations $p_0 \subset \cdots \subset p_d$ inside X.

   Let $Z \subseteq X \cap V(f_1, \dots, f_r)$ be a irreducible component. Thus it is the closure of a minimal prime ideal in $A/(f_1, \dots, f_r)$.

   By Krull's principal ideal theorem we have $\dim(A/(f_1, \dots, f_r)) \geq d - r$.

   Denote minimal prime ideals in $A/(f_1, \dots, f_r)$ with $q_1, \dots q_l$.

   We argue that

   $$\dim(A/(f_1, \dots, f_r)/q_j) = \dim(A/(f_1, \dots, f_r)).$$

   for any j.

   That follows from A being catenary. If there existed a maximal chain in $A/(f_1, \dots, f_r)$ that starts at q_j we could simply extend it below to get a maximal chain in A. Since all maximal chains in A are of the same length, we get that all maximal chains in $A/(f_1, \dots, f_r)$ are also of the same length.

   Since Z is an irreducible component, we have $Z = \overline{\{q_i\}} \subseteq \operatorname{Spec}(A/(f_1, \dots, f_r))$.

   Therefore any maximal chain in Z is exactly as long as the longest chain in $A/(f_1, \dots, f_r)$. And the longest chain in $A/(f_1, \dots, f_r)$ is at least of length $d - r$.

2. The diagonal $\triangle \subseteq \mathbb{A}^n_k \times \mathbb{A}^n_k$ can be defined as $V(x_i \otimes x_i \mid i = 1, \dots n) \subseteq \operatorname{Spec}(k[x_1, \dots, x_n] \otimes_{\mathbb{Z}} k[x_1, \dots, x_n])$. Using exercise above we get that any irreducible component of $X \cap Y \cong (X \times Y) \cap V(x_i \otimes x_i \mid i = 1, \dots n)$ has dimension at least $d + e - n$.

3. Let $\tilde{X} = f^{-1}(X)$ and $\tilde{Y} = f^{-1}(Y)$ as in the hint.

We have $\dim(\tilde{X}) = d+1$ and $\dim(\tilde{Y}) = e+1$. By the previous exercise we have $\dim(\tilde{X} \cap \tilde{Y}) \geq d+1+e+1-(n+1) = (d+e-n)+1 \geq 1$.

Therefore there there exists $0 \neq x \in \tilde{X} \cap \tilde{Y}$.

Exercise 2.

1. Take $x \in |X|$. If $x \notin f(|Y|)$, we can find an open U_x such that $f^{-1}(U_x) = \emptyset$. So assume $x \in f(|Y|)$. Then look at $f^{-1}(x)$. Take an open affine $V_x \subseteq |Y|$ with $f^{-1}(x) \in V_x$. Since f is homeomorphism on its image, we have can take $U_x = f(V_x)$ an affine neighborhood of x such that $f^{-1}(U_x) = V_x$ is affine.

2.

Exercise 3.

1. We have a map $k \to \Gamma(X, \mathcal{O}_X)$. For any $f \in \Gamma(X, \mathcal{O}_X)$ we can define $k[x] \to \Gamma(X, \mathcal{O}_X)$ by $x \mapsto f$.

   Showing that $g(X)$ does not contain the generic point of $\mathbb{A}^1_k$ is equivalent to showing that $k[x] \to \Gamma(X, \mathcal{O}_X)$ is not injective.

   We have a composition $k \to k[x] \to \Gamma(X, \mathcal{O}_X)$. So also $X \to \mathbb{A}^1_k \to \operatorname{Spec}(k)$.

   Map $X \to \operatorname{Spec}(k)$ is proper.

   Map $\mathbb{A}^1_k \to \operatorname{Spec}(k)$ is separated, since it is a map of affine schemes. (Follows from the fact that $k[x] \otimes_k k[x] \to k[x]$ is surjective, and thus $\mathbb{A}^1_k \to \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ a closed immersion.)

   Thus by the proposition from the lectures, the map $g\colon X \to \mathbb{A}^1_k$ is proper. In particular it is closed. Since X is connected, the image $g(X)$ must be connected as well.

   Using the hint, we can postcompose to obtain $X \to \mathbb{A}^1_k \to \mathbb{P}^1_k$. Now the conclusion should be that the image of X in $\mathbb{P}^1_k$ is also closed. Since $\mathbb{A}^1_k \subseteq \mathbb{P}^1_k$ is not closed, the image of X in $\mathbb{A}^1_k$ can also not be closed. Therefore it must be a single point.

   Since we did not exactly understand why should $X \to \mathbb{P}^1_k$ be closed, we decided to rather show that $X \to \mathbb{A}^1_k$ cannot be surjective, as that would imply $\mathbb{A}^1_k$ being universally closed over $\operatorname{Spec}(k)$ (which we've shown during the lectures to be false).

   Instead of doing it abstractly, we can show that $X \to \mathbb{A}^1_k$ being surjective would imply $\mathbb{A}^2_k \to \mathbb{A}^1_k$ being closed.

By the universal property of $\mathbb{A}^2_k$ we get a map $X \to \mathbb{A}^2_k$, induced by $X \to \mathbb{A}^1_k$. So we have a map $X \to \mathbb{A}^2_k \to \mathbb{A}^1_k$. Denote $\alpha\colon X \to \mathbb{A}^2_k$ and $\beta\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$. If α would be surjective, then for any $U \subseteq \mathbb{A}^2_k$ we would have $(\beta \circ \alpha)(\alpha^{-1}(U)) = \beta(U)$. Since $\beta \circ \alpha$ is closed by assumption, this would prove that β is closed. That is not true, so $\beta \circ \alpha$ is not surjective.

We've shown that the image of $X \to \mathbb{A}^1_k$ is a single point. Since this point is closed, it is not the generic point. This shows that $k[x] \to \Gamma(X, \mathcal{O}_X)$ induced by $f \in \Gamma(X, \mathcal{O}_X)$ is not injective.

2. We have a map $k \to \Gamma(X, \mathcal{O}_X)$. It cannot be 0, since X is locally finite type over $\operatorname{Spec}(k)$. So it is injective.

   It is also surjective, since for any $f \in \Gamma(X, \mathcal{O}_X)$ the map $k[x] \to \Gamma(X, \mathcal{O}_X)$ defined by $x \mapsto f$ is not injective. Therefore $k \cong \Gamma(X, \mathcal{O}_X)$.