Elliptic curves and their moduli spaces
Exercise sheet 2

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Problem 1.

a) From algebraic geometry 1 we know that for every scheme/k T, we have an isomorphism

$$\mathrm{Hom}_{Sch/k}(T, \mathbb{A}^n_k) \to \mathrm{Hom}_k(k[x_1, \dots, x_n], \mathcal{O}_T(T)).$$

which is natural in T. After picking the basis for A, we have a simple identification

$$\begin{aligned} \mathrm{Hom}_k(k[x_1, \dots, x_n], \mathcal{O}_T(T)) &\to \mathcal{O}_T(T) \otimes_k A \\ \varphi &\mapsto \sum_i \varphi(x_i) \otimes x_i, \end{aligned}$$

where we use x_i to denote basis as well. This identification is simply on the level of sets, because although $\mathcal{O}_T(T) \otimes_k A$ has the structure of a (maybe non-commutative) k-algebra, $\mathrm{Hom}_k(k[x_1, \dots, x_n], \mathcal{O}_T(T))$ doesn't seem to have any structure (is there any? Set of maps of k-algebras is nothing more than a set, right? We're assuming it has to map 1 to 1 and be k-linear).

For second part, we want some sort of criterion for when is $\sum_i f_i \otimes x_i$ invertible. We can represent $\mathcal{O}_T(T) \otimes_k A$ by embedding it into algebra of endomorphisms $\mathrm{End}(\mathcal{O}_T(T)^n$ which is isomorphic to matrix algebra $M_{n\times n}(\mathcal{O}_T(T))$. Then we can simply calculate if the element is invertible by evaluating its determinant. In our case determinant is a polynomial in $n = \dim A$ variables (matrix depends on n coefficients, for example first column, all others are implicitly given by that). So for the ring we take $k[x_1, \dots, x_n, \det(x_1, \dots, x_n)^{-1}]$.

Maybe it would be easier to define ring from scratch and not look at open subschemes of $\mathbb{A}^n_k$. In which case we take $k[t_{ij}, i, j \in \{1, \dots, n\}]$, invert $\det(t_{ij})$ and then quotient by relations $t_{ij} = \sum_k a^k_{ij} t_{k1}$ where a^k_{ij} represents k-th coordinate in expansion of the product $(t_{11}x_1 + \dots + t_{n1}x_n)x_j$ as $i \in \{1, \dots, n\}$ and $j \in \{2, \dots, n\}$. In example of $\mathbb{R}$-algebra $\mathbb{C}$ that would mean

$$\mathbb{R}[x_{11}, x_{21}, x_{12}, x_{22}, (x_{11}x_{22} - x_{12}x_{21})^{-1}]/(x_{12} = -x_{21}, x_{22} = x_{11})$$

b)

c) Let $k = \mathbb{R}$, $A = \mathbb{C}$ and $G = \underline{A}^{\times}$. In part a) of problem 1 we argued that $\underline{A}^{\times}$ is the affine open subscheme of $\mathbb{A}_k^n$ we get by inverting the determinant of embedding $A \hookrightarrow \mathrm{End}(k^n) \cong M_{n \times n}(k)$. In this case (one possible) embedding is $a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. So we have to invert $a^2 + b^2$. So now we need to define a scheme morphism. Since both G and $\mathbb{G}_{m,\mathbb{R}}$ are affine, we can give map of rings:

$$\begin{aligned} \mathbb{R}[t, t^{-1}] &\to \mathbb{R}[x, y, (x^2 + y^2)^{-1}] \\ t &\mapsto x^2 + y^2 \end{aligned}$$

So if we have a map $\mathrm{Spec}(\mathbb{R}) \to G$ (i.e. a $\mathbb{R}$-valued point z) and post-compose it with $G \to \mathbb{G}_{m,\mathbb{R}}$, we get $\mathrm{Spec}(\mathbb{R}) \to \mathbb{G}_{m,\mathbb{R}}$ corresponding to $\mathbb{R}$-valued point $z\overline{z}$.

Problem 2. Let k be a field and $\mathbb{G}_{a,k} = \mathrm{Spec}(k[t])$ with $a^* \colon t \mapsto t \otimes 1 + 1 \otimes t$.

a) We want to find maps $\mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$ that will respect operation a. Since $\mathbb{G}_{a,k}$ is affine, this is equivalent to finding all maps $f^* \colon k[t] \to k[t]$ such that

$$\begin{tikzcd} k[t] \otimes_k k[t] & k[t] \arrow[l, "a^*"'] \\ k[t] \otimes_k k[t] \arrow[u, "f^* \otimes f^*"] & k[t] \arrow[l, "a*"'] \arrow[u, "f^*"'] \end{tikzcd}$$

commutes. Since we are working over schemes over k, these are maps of k-algebras, which means that f^* is uniquely defined by its value at t. From commutativity we get the condition

$$f^*(t) \otimes 1 + 1 \otimes f^*(t) = f^*(t \otimes 1 + 1 \otimes t)$$

Writing $f^*(t) = \sum_i a_i t^i$ we get

$$\sum_i a_i t^i \otimes 1 + 1 \otimes \sum_i a_i t^i = \sum_i a_i \sum_j \binom{i}{j} (t^{i-j} \otimes t^j)$$

Since $\mathrm{char}(k) = 0$, none of the elements on right vanish. So by comparing terms we obtain $a_i = 0$ for $i > 1$. So $f^*(t) = a_0 + a_1 t$.

$$(a_0 + a_1 t) \otimes 1 + 1 \otimes (a_0 + a_1 t) = a_0 + a_1 (t \otimes 1 + 1 \otimes t)$$

Compare again and get that $2a_0 = 0$, so $a_0 = 0$ and a_1 can be anything. Therefore endomorphisms $\mathrm{End}(\mathbb{G}_{a,k})$ are parametrized by $a_1 \in k$.

b) Let now $\mathrm{char}(k) = p$. Same as before, but at the step when we have

$$\sum_i a_i t^i \otimes 1 + 1 \otimes \sum_i a_i t^i = \sum_i a_i \sum_j \binom{i}{j} (t^{i-j} \otimes t^j)$$

when i is a power of p, all terms $\binom{i}{j}$ vanish in k, but if i is not a power of p, then there exists j such that $\binom{i}{j}$ does not vanish, meaning a_i has to be 0 for i not a power of p. We also get $a_0 = 0$ by comparing terms. So $f^*(t) = \sum_k a_{p^k} t^{p^k}$. We see that all endomorphisms of $\mathbb{G}_{a,k}$ are of this form and clearly all morphisms of above form in fact define endomorphisms of $\mathbb{G}_{a,k}$