Algebraic geometry 1
Exercise sheet 1

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Exercise 1.

1. Closed subsets of $\mathbb{A}^1(k)$ are $\emptyset$, $\mathbb{A}^1(k)$ and all finite subsets.
2. That is true because there exists no product of two closed subsets of $\mathbb{A}^1(k)$ that would cover closed subspace $V(x-y) \subseteq \mathbb{A}^2(k)$ and not cover the point $(x-1, y)$. Basis for closed sets of the product topology are the whole space and finite union of lines parallel to one of the coordinates.

Exercise 2.

1. The polynomials from the definition exist.
2. Discriminant is a polynomial function on coefficients of a polynomial.
3. A matrix has pairwise different eigenvalues if and only if its characteristic polynomial has only simple roots. That is exactly when discriminant of its characteristic polynomial is non-zero. We can compose maps from 1) and 2) to get a map $h\colon V \to \mathbb{A}^1(k)$ that vanishes on a matrix M if and only if M is not diagonalizable with pairwise different eigenvalues. Since $\{0\}$ is a closed set in $\mathbb{A}^1(k)$ and h continuous, we get that $\{M \in V \mid h(M) \neq 0\}$ is open in V.

Exercise 3. Using the hint we look at linear transformations that would simplify polynomial. Write $x \mapsto ux + vy$ and $y \mapsto wx + zy$. We get that if $a_2^2 - 4a_1a_3 = 0$, then we can pick $u = \sqrt{a_1}$ and $v = \sqrt{a_3}$ and $x^2 \mapsto a_1x^2 + a_2xy + a_3y^2$. Otherwise we can define $u = 1$, $w = a_1$, v is the solution to $v(a_2 - va_1) = a_3$ and $z = a_2 - va_1$.

From now on a_4, a_5 and a_6 are not the same as in the original polynomial.
So if $f(x, y) = x^2 + a_4x + a_5y + a_6$, then V is either

- if $a_5 \neq 0$; V is isomorphic to a parabola
- if $a_5 = 0$ and $a_4^2 - 4a_6 \neq 0$; V is isomorphic to disjoint union of two lines
- if $a_5 = 0$ and $a_4^2 - 4a_6 = 0$; V is isomorphic to a single line

If $f(x, y) = xy + a_4x + a_5y + a_6$, then V is either isomorphic to a hyperbola or a union of coordinate lines. We can write f in the form $(x+u)(y+v) - z$ for some suitable $u, v, z \in k$. Explicitly we get $u = a_4$, $v = a_5$ and $z = uv - a_6$. So if $a_4a_5 = a_6$, then $z = 0$ and V is isomorphic to the union of coordinate lines, otherwise V is isomorphic to a hyperbola. But again note that these a_4, a_5, a_6 are not the ones in original polynomial, because we used a linear transormation earlier.

Exercise 4. Lets look at coordinate rings:

1. $\mathcal{O}_{\{y-x^2=0\}} = k[x, y]/(y - x^2) = k[x]$
2. $\mathcal{O}_{\{xy-1=0\}} = k[x, y]/(xy - 1) = k[x, x^{-1}]$
3. $\mathcal{O}_{\{xy=0\}} = k[x, y]/(xy)$
4. $\mathcal{O}_{\{x(x-1)=0\}} = k[x, y]/(x(x - 1))$
5. $\mathcal{O}_{\{x=0\}} = k[x, y]/(x) = k[y]$

The 1st and 5th are clearly isomorphic. Number 2 is not isomorphic to any else, because it has strictly more invertible elements than just the field k. Number 3 and 4 are only ones with zero-divisors, so they could only be isomorphic to each other. But they are not, because number 4 has an idempotent element (other than 0 and 1) and 3 doesn't.