Algebraic geometry 1
Exercise sheet 2

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Exercise 1. Let $I = (f_1, \dots, f_r) \subseteq k[x_1, \dots, x_n]$ be an ideal and $X = V(I) \subseteq \mathbb{A}^n(k)$ be its vanishing locus.

1. We have to show
$$\overline{X} = \bigcap \{V^+(h) \mid h \text{ homogenous}, h(X) = 0\} = V(\{\tilde{g} \mid g \in I\}).$$
Pick any homogenous h that vanishes on $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$. By letting $x_{n+1} = 1$ we see that $h(x_1, \dots, x_n, 1) \in \sqrt{I}$, since it vanishes on X. Therefore we have $l \in \mathbb{N}$ such that $h(x_1, \dots, x_n, 1)^l = \sum_{i=1}^r \alpha_i f_i$ for some $\alpha_i \in k[x_1, \dots, x_n]$.
Homogenization is almost a bijection between homogenous polynomials in $n+1$ variables and all polynomials in n variables. It is bijective between homogenous polynomials in $n+1$ variables that are not divisible by x_{n+1} and polynomials in n variables. The "inverse" to homogenization would be the letting $x_{n+1} = 1$.
Adding these two together we get $x_{n+1}^{ml} \widetilde{\sum_{i=1}^r \alpha_i f_i} = h^l$. Therefore
$$h = \sum_{i=1}^r \beta_i \widetilde{f_i}$$
and thus $\bigcap \{V^+(h) \mid h \text{ homogenous}, h(X) = 0\} \supseteq V(\{\tilde{g} \mid g \in I\})$.

2. We calculate $\overline{X}$ using the first part of the exercise and take the numbering of the varieties from the first exercise sheet:
$$\begin{aligned}
\overline{X_1} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(x^2 - xz) \cap V(z) = (0:1:0), \\
\overline{X_2} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(z^2 - xy) \cap V(z) = (1:0:0) \cup (0:1:0), \\
\overline{X_3} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(xy) \cap V(z) = (1:0:0) \cup (0:1:0), \\
\overline{X_4} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(x^2 - xy) \cap V(z) = (0:1:0), \\
\overline{X_5} \cap \mathbb{P}^2_k \backslash \mathbb{A}^2_k &= V(x) \cap V(z) = (0:1:0).
\end{aligned}$$

3.

Define $f(x,y) := xy - 1$ and $g(x,y) := xy$. Then we have

$$X = V(f,g) = \emptyset,$$

and since $\emptyset$ is closed in projective space as the vanishing set of a non-zero constant, we have $\overline{X} = \emptyset$. However,

$$(0:1:0) \in V^+(xy - z, xy) = V^+(\tilde{f}, \tilde{g}),$$

proving that in this case $\overline{X} \neq V(\tilde{f}, \tilde{g})$.

Exercise 2. Let $F \in k[x,y,z]$ be a non-zero homogenous polynomial of degree 2.

1. Let us first prove the statement in the hint. So let

   $$T(x,y) = (ax + by + u, cx + dy + v)$$

   with $ad - cb \neq 0$, be an affine linear transformation of $\mathbb{A}^2(k)$. We can extend it to an automorphism of $\mathbb{P}^2(k)$ by defining

   $$\widetilde{T}(x:y:z) = (ax + by + uz : cx + dy + vz : z)$$

   It is an extension of T, because we have $\widetilde{T} \circ i = i \circ T$ where $i \colon \mathbb{A}^2(k) \to \mathbb{P}^2(k)$ is given by $(x,y) \mapsto (x:y:1)$. Mapping $\widetilde{T}$ is bijective with inverse

   $$\widetilde{T}^{-1}(x:y:z) = (dx - by - (du - bv)z : -cx + ay + (cu - av)z : (ad - bc)z).$$

   This proves that $\widetilde{T}$ is an automorphism of $\mathbb{P}^2(k)$.

   Lets consider the polynomial F now. We looks at its zero-set on $\mathbb{A}^2(k)$, which is exactly $V(F) \cap V(z-1) = V(F, z-1) = V(F(x,y,1))$.

   This set could be empty, in which case $F(x,y,1)$ must be a non-zero constant (because k is algebraically closed). A general form of F would be $F(x,y,z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$. If we set $z = 1$ and F becomes a non-zero constant, we get that a_3 is a non-zero constant and other a_i are 0. So $F(x,y,z) = az^2$, which means $V(F)$ is isomorphic to the projective line $V(x) = V(x^2)$.

   Suppose now $V := V(F(x,y,1)) \subseteq \mathbb{A}^2(k)$ is not empty. Note that $F(x,y,1)$ is of the form we treated in exercise 3 of sheet 1. Then we can use an affine linear transformation T on $\mathbb{A}^2(k)$ so that $V \subseteq \mathbb{A}^2(k)$ assumes the forms of one of the 5 curves we treated in exercise 3 sheet 1, namely $V(x), V(y - x^2), V(xy), V(xy - 1), V(x(x-1))$. We showed earlier that his transformation can be extended to an isomorphism $\widetilde{T}$ of $\mathbb{P}^2(k)$.

Loosely speaking, we might have "lost some information" when we focused on $\mathbb{A}^2(k) \cap V(F)$, but it turns out that we didnt, if we take into account that the curve we started with was non-zero homogenous polynomial of degree 2. We are able to "retrieve all of it back" by taking the closure and remembering that original curve was of degree 2.

Since the curve $V(F)$ is by definition of topology closed, the curve $\widetilde{T}(V(F)) \subseteq \mathbb{P}^2(k)$ will also be closed. By construction we have on of the following cases

- $V(x) \cap \mathbb{A}^2(k) = V(x^2) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$
- $V(xy) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$
- $V(xy-1) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$
- $V(y-x^2) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$
- $V(x(x-1)) \cap \mathbb{A}^2(k) \subseteq \widetilde{T}(V(F))$

In each case we can take the closure of the left side and, since right side is closed, the inclusion will still hold. We use 1st exercise of this sheet to calculate the closure by homogenizing the polynomial. Namely we get one of the following cases

- $V(x) = V(x^2) \subseteq \widetilde{T}(V(F))$
- $V(xy) \subseteq \widetilde{T}(V(F))$
- $V(xy-z^2) \subseteq \widetilde{T}(V(F))$
- $V(yz-x^2) \subseteq \widetilde{T}(V(F))$
- $V(x^2-xz) \subseteq \widetilde{T}(V(F))$

These inclusions are in fact equalities,

Exercise 3.

1. We define a closed subset $V(f-g) \subseteq X$, which contains the open set $\mathcal{U} \subseteq V(f-g)$. By definition the completent $\mathcal{U}^C$ is closed and $V(f-g) \cup \mathcal{U}^C = X$. Since X is irreducible and $\mathcal{U}$ is non-empty, we have $f = g$.

2. Lets show first that $\chi\colon A \mapsto \chi_A(A)$ vanishes on diagonalizable matrices with pairwise different eigenvalues: For $A = TDT^{-1}$ we have

$$\chi_A(A) = \sum_{i=0}^{n} \alpha_i A^i = \sum_{i=0}^{n} \alpha_i (TDT^{-1})^i = T\left(\sum_{i=0}^{n} \alpha_i D^i\right) T^{-1}.$$

Denote $D = \operatorname{diag}(d_1, \ldots, d_n)$ and notice that

$$\sum_{i=0}^{n} \alpha_i D^i = \sum_{i=0}^{n} \operatorname{diag}(\alpha_i d_1^i, \ldots, \alpha_i d_n^i) = \operatorname{diag}(\chi_A(d_1), \ldots, \chi_A(d_n)) = 0$$

because characteristic polynomial vanishes on eigenvalues d_i.

Since $\mathbb{A}^{n\times n}(L)$ is irreducible and χ vanishes on open subset of it, namely on diagonalizable matrices with pairwise different eigenvalues, χ must be 0 on whole $\mathbb{A}^{n\times n}(L)$.

Exercise 4.

1. First, we note that points in projective space are closed. To see this, take some point $a = (a_0 : ... : a_n) \in \mathbb{P}^n_k$. Then $a = V(\{f \in k[x_0, ..., x_n] : f = x_i a_j - x_j a_i\})$. Now, we let $X \subset \mathbb{P}^n_k$ be some quasi-projective variety and $x, y \in X$ with $x \neq y$. Define $U := \{x\}^c$, which is open as the complement of a closed set and by definition fulfills the properties $x \notin U$ and $y \in U$.