Algebraic geometry 1
Exercise sheet 5

Solutions by: Eric Rudolph and David Čadež

16. November 2023

Exercise 1.

1. Define

$$X := (U_1 \coprod U_2)/\sim,$$

where $x \sim y$ if $x = \varphi(y)$ and for $i \in \{1, 2\}$

$$\begin{aligned} \pi_i \colon U_i &\to X \\ x &\mapsto \bar{x}. \end{aligned}$$

We can now give X the topology by defining a subset $U \subset X$ to be open if $\pi_i^{-1}(U) \in$ are open in U_i.

We basically take X to be the pushout of $U_1 \leftarrow V_1 \to U_2$, where $V_1 \to U_1$ is the inclusion and $V_1 \to U_2$ and composition of φ and inclusion.

Notice, that π_i are homeomorphic onto open subsets of X. This will become important later. Next we want to define a structure sheaf on X that behaves well with restricting to U_i.

For $U \subset X$ open, let

$$\begin{aligned} \mathcal{O}_X(U) := \ker(\mathcal{O}_{U_1}(\pi_1^{-1}(U)) \oplus \mathcal{O}_{U_2}(\pi_2^{-1}(U)) &\to \mathcal{O}_{U_1}(\pi_1^{-1}(U) \cap V_1) \\ (x, y) &\mapsto x_{|\pi_1^{-1}(U)\cap V_1} - \varphi^\sharp(\pi_2^{-1}(U) \cap V_2)(y_{|\pi_2^{-1}(U)\cap V_2})). \end{aligned}$$

We conclude, that $(X, \mathcal{O}_X)$ is a scheme, because $X = \pi_1(U_1) \cup \pi_2(U_2)$ can be covered by affine schemes using the cover from U_1 and U_2 and since by construction of the structure sheaf $\mathcal{O}_{X|U_1} = \mathcal{O}_{x_i}$. Here we finally used, as promised, that π_i are homeomorphisms onto open subsets of X.

Exercise 2.

1. Take two isomorphic open immersions $(Z, \mathcal{O}_Z)$ and $(W, \mathcal{O}_W)$ as schemes over $(Y, \mathcal{O}_Y)$. So we have a commutative diagram

$$\begin{tikzcd} (Z, \mathcal{O}_Z) \arrow[r, hook] \arrow[d, "\cong"] & (Y, \mathcal{O}_Y) \\ (W, \mathcal{O}_W) \arrow[ur, hook] & \end{tikzcd}$$

from which we get a diagram of topological spaces

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\cong"] & Y \\ W \arrow[ur, hook] & \end{tikzcd}$$

from which it clearly follows that Z and W must be equal as sets.

For the other way, we want to show that for every open $Z \subseteq Y$ there is a unique sheaf $\mathcal{O}_Z$ for which $(\varphi, \varphi^{\#})\colon (Z, \mathcal{O}_Z) \hookrightarrow (Y, \mathcal{O}_Y)$ is an open embedding. Take any two sheaves $\mathcal{O}_Z$ and $\mathcal{O}'_Z$ on Z for which $(\mu, \mu^{\#})\colon (Z, \mathcal{O}'_Z) \hookrightarrow (Y, \mathcal{O}_Y)$ is also open embedding. Then by definition of an open embedding we have isomorphisms $\mu^{-1}\mathcal{O}_Y \to \mathcal{O}_Z$ and $\varphi^{-1}\mathcal{O}_Y \to \mathcal{O}'_Z$. But $\varphi^{-1}\mathcal{O}_Y$ and $\mu^{-1}\mathcal{O}_Y$ are the same, since $\varphi = \mu$, so $\mathcal{O}'_Z \cong \mathcal{O}_Z$. As for the existence: there clearly exists such a sheaf $\mathcal{O}_Z$ simply by taking a restriction $\mathcal{O}_Y \mid_Z$. But (as it says in Davies/Scholze notes) it is not obvious. We have to show that we can cover Z with open subsets, where each of them is isomorphic to an affine scheme. Let $Y = \cup_i Y_i$, where $Y_i \cong \operatorname{Spec} B_i$. Then for every point $x \in Z$ we choose i such that $x \in Y_i \cap Z$. That means there exists some $f \in B_i$ such that $x \in D_{Y_i}(f) \subseteq V_i \cap U$. Since $D_{Y_i}(f) \cong B_[f^{-1}]$, we found a neighborhood of $x \in Z$ that is isomorphic to an affince scheme. We can do that for every $x \in Z$ and thus cover it. So Z is itself a scheme.

Exercise 3.

1. Remember, that the contravariant functor $A \mapsto (Spec(A), \mathcal{O}_{Spec(A)})$ is an equivalence of categories, meaning that $\mathcal{O}^{op}$ is equivalent to $Rings$. Then the affine case follows from the Yoneda embedding.

Exercise 4.

1. Let $F\colon C \to D$ be a functor with adjoints $G, G'\colon D \to C$. By the definition of adjointness, for every arrow $f\colon Fc \to d$ we have unique arrows $\phi f\colon c \to Gd$ and $\mu f\colon c \to G'd$, such that ϕ and μ are natural. In this case take some $d \in D$ and $c = Gd$. Then we have a unique arrow $Gd \to G'd$.

We just have to show this is natural in d, so pick some other $e \in D$ and $FGe \to e$. Same as before we get an arrow $Gb \to G'b$. Using adjointness we have a commutative diagram

$$\begin{tikzcd} FGa \arrow[r] \arrow[d] & a \arrow[d] \\ FGb \arrow[r] & b \end{tikzcd}$$

Then, using the naturality of μ gives that

$$\mu(FGa \to a \to b) = Ga \to G'a \to G'b$$

and

$$\mu(FGa \to FGb \to b) = Ga \to Gb \to G'b$$

Which proves that $a \mapsto (Ga \to G'a)$ is natural. We could easily construct an inverse $a \mapsto (G'a \to Ga)$ which would compose to identity.

2. idk

3. We want to be in the situation of the second part of this exercise. We choose $\mathcal{C} = Psh, \mathcal{D} = \mathcal{E} = Sh$. and $\mathcal{F}$ is sheafification, $\tilde{\mathcal{F}}$ is pullback along f, with $\mathcal{G}$ just being inclusion and $\tilde{\mathcal{G}}$ being the pushforeward along f. So we can conclude that the concatenation of $\mathcal{F} \circ \tilde{\mathcal{F}} \dashv \mathcal{G} \circ \tilde{\mathcal{G}}$. Now the claim follows using the first part of the exercise.

   How to show general case? We want to be in the situation of the second part of this exercise. We choose $\mathcal{C} = Psh, \mathcal{D} = \mathcal{E} = Sh$. and $\mathcal{F}$ is sheafification, $\tilde{\mathcal{F}}$ is pullback along f, with $\mathcal{G}$ just being inclusion and $\tilde{\mathcal{G}}$ being the pushforeward along f. So we can conclude that $\mathcal{F} \circ \tilde{\mathcal{F}} \dashv \mathcal{G} \circ \tilde{\mathcal{G}}$. Now the claim follows using the first part of the exercise.