Elliptic curves and their moduli spaces
Exercise sheet 7

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Problem 1.

1. Cover X with $D(x)$ and $D(y)$. On these the scheme is $\operatorname{Spec}(k[y,z]/z^2)$ and $\operatorname{Spec}(k[x,z]/z^2)$ respectively.

   First we show that the ideal of nilpotent elements in both of them is the principal ideal generated by z. I think we've shown during algebra 1 that a polynomial is nilpotent if and only if all of its coefficients are nilpotent. It is clear that nilpotent elements in $k[z]/z^2$ are (z). Write $k[y,z]/z^2 = (k[z]/z^2)[y]$, we see that nilpotent elements of $(k[z]/z^2)[y]$ are polynomials where all coefficients are divisible by z. That shows $\mathcal{N}$ on $\operatorname{Spec}(k[y,z]/z^2)$ is given by (z). The situation on $\operatorname{Spec}(k[x,z]/z^2)$ is exactly the same.

   Now $\mathcal{N}$ clearly has a natural structure of $\mathcal{O}_{X_{red}}$-module, because $\mathcal{N}^2 = 0$. So multiplication with $\mathcal{O}_X$ "factors through" multiplication with $\mathcal{O}_{X_{red}}$.

2.

Problem 2.

1. For the start assume all schemes involved are affine.

   Let $X = \operatorname{Spec} A, S = \operatorname{Spec} R, T = \operatorname{Spec} R', Z = \operatorname{Spec}(A/I)$.

   Then we have a diagram

$$\begin{tikzcd}
\operatorname{Spec}(A/I \otimes_R R') \arrow[r] \arrow[d] & \operatorname{Spec}(A/I) \arrow[d] \\
\operatorname{Spec}(A \otimes_R R') \arrow[r] \arrow[d] & \operatorname{Spec}(A) \arrow[d] \\
\operatorname{Spec}(R') \arrow[r] & \operatorname{Spec}(R)
\end{tikzcd}$$

In this case the pullback is

$$f^*\mathcal{I} = (I \otimes_R R')\tilde{}$$

and the base change is the kernel of surjection $A \otimes_R R' \to A/I \otimes_R R'$, i.e.

$$\mathcal{I}_{R'} = (\ker(A \otimes_R R' \to A/I \otimes_R R'))\tilde{}.$$

If $R \to R'$ is flat, then

$$0 \to I \otimes_R R' \to A \otimes_R R' \to A/I \otimes_R R' \to 0$$

is exact, which shows that in this case the map

$$I \otimes_R R' \to \ker(A \otimes_R R' \to A/I \otimes_R R')$$

is an isomorphism.

Because we already have a natural map $f^*\mathcal{I} \to \mathcal{I}_T$, it is enough to check that is is an isomorphism on an affine open cover, which reduces it to affine case.

2. If a non zero-divisor $a \in A$, we have an isomorphism $A \cong (a) = I$ as A-modules.

   Assume $\mathcal{I}$ is line bundle, given by I on $\mathrm{Spec}(A) \subseteq X$. If there is an isomorphism $A \to I$, then I is a principal ideal generated by the image of 1 under isomorphism above.

3. Let Z be an effective Cartier divisor and flat over S.

   Pick affine local covers as in part a).

   Suppose $\mathcal{I}$ is, locally on $\mathrm{Spec}(A) \subseteq X$, defined by $a \in A$. By flatness we have an isomorphism $f^*\mathcal{I} \to \mathcal{I}_T$, so $\mathcal{I}_T$ is isomorphic to $(a) \otimes_R R'$, which is a principal ideal of $A \otimes_R R'$ defined by $(a) \otimes 1$.