

# Algebraic geometry 1

## Exercise sheet 8

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### Exercise 1.

1. Let  $0 \neq f \in I$  be a non-zero element. Since  $A$  is a unique factorization domain, we can write

$$f = up_1^{a_1} \dots p_r^{a_r},$$

where  $p_i$  are pairwise non-associated primes. Now,

$$I(p_i) = I[(I \setminus (p_i))^{-1}] = (p_i^{k_i})$$

for some  $k_i \leq a_i$ . Since  $I$  is finite locally free,

$$I = \left( \prod_i p_i^{k_i} \right).$$

2.

**Exercise 2.** Note that for a unique factorization domain  $A$  we get by Gauss that also  $A[x_1, \dots, x_n]$  is a unique factorization domain. This means that by construction of  $\mathbb{P}_A^n$  its local rings are UFD's. Using stacks project, we infer that  $\text{Pic}(\mathbb{P}_A^n) \cong \text{CL}(\mathbb{P}_A^n) = \mathbb{Z}$ .

### Exercise 3.

1. In exercise 2 we showed that all invertible quasicoherent sheaves on  $\mathbb{P}_k^n$  are isomorphic to  $\mathcal{O}_{\mathbb{P}_k^n}(d)$  for some  $d \geq 0$ . So we have to show  $f^* \mathcal{O}_{\mathbb{P}_k^m}(1)$  is an invertible sheaf.

Since invertible  $\mathcal{O}_{\mathbb{P}_k^n}$ -modules are same as line bundles, we have to show that locally  $f^* \mathcal{O}_{\mathbb{P}_k^m}(1)$  is isomorphic to the structure sheaf  $\mathcal{O}_{\mathbb{P}_k^m}$ .

By definition  $f^*\mathcal{O}_{\mathbb{P}_k^m}(1) = f^{-1}\mathcal{O}_{\mathbb{P}_k^m}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}_k^m}} \mathcal{O}_{\mathbb{P}_k^n}$ . Pick some  $x \in \mathbb{P}_k^n$ . Pick small enough affine neighborhood  $f(x) \in U \subseteq \mathbb{P}_k^m$  such that  $\mathcal{O}_{\mathbb{P}_k^m}(1)$  is isomorphic to the structure sheaf  $\mathcal{O}_{\mathbb{P}_k^m}$  on  $U$ . Now pick neighborhood  $x \in W \subseteq \mathbb{P}_k^n$  such that  $f(W) \subseteq U$ .

Then

$$\begin{aligned} f^{-1}\mathcal{O}_{\mathbb{P}_k^m}(1)(W) &= \operatorname{colim}_{f(W) \subseteq V \subseteq U} \mathcal{O}_{\mathbb{P}_k^m}(1)(V) \\ &= \operatorname{colim}_{f(W) \subseteq V \subseteq U} \mathcal{O}_{\mathbb{P}_k^m}(1)(V) \\ &\cong \operatorname{colim}_{f(W) \subseteq V \subseteq U} \mathcal{O}_{\mathbb{P}_k^m}(V) \\ &\cong f^{-1}\mathcal{O}_{\mathbb{P}_k^m}(W). \end{aligned}$$

So locally  $f^{-1}\mathcal{O}_{\mathbb{P}_k^m}(1)$  is isomorphic to  $f^{-1}\mathcal{O}_{\mathbb{P}_k^m}$ , so  $f^{-1}\mathcal{O}_{\mathbb{P}_k^m}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}_k^m}} \mathcal{O}_{\mathbb{P}_k^n}$  is locally isomorphic to  $\mathcal{O}_{\mathbb{P}_k^n}$ , which proves that  $f^*\mathcal{O}_{\mathbb{P}_k^m}(1)$  is an invertible  $\mathcal{O}_{\mathbb{P}_k^n}$ -module and thus isomorphic to  $\mathcal{O}_{\mathbb{P}_k^n}(d)$  for some  $d \geq 0$ .

2. The polynomials  $y_0, \dots, y_n$  generate the module of homogenous polynomials of degree 1.

#### Exercise 4.

1. Let  $U_i = \operatorname{Spec}(A_i)$ .

Take a point  $x \in U_1 \cap U_2$ .

Take a principal open  $x \in D(f) \subseteq U_1$  ( $f \in U_1$ ). Then find a smaller principal open  $x \in D(g) \subseteq U \subseteq U_2$  ( $g \in U_2$ ).

Now we show that  $D(g)$  is also a principal open in  $U_1$ .

Since  $D(f) \subseteq U_2$ , we have a map  $\mathcal{O}(U_2) \rightarrow \mathcal{O}(D(f))$ , which induces  $A_2 \rightarrow (A_1)_f$ . Denote by  $g' = g|_{\operatorname{Spec}((A_1)_f)}$  the image of  $g$  under this map. Since  $g' \in (A_1)_f$ , we can write it as  $g' = \frac{h}{f^n}$ . Then  $D(g) = D(g) \cap D(f) = D(g') \cap D(f) = D(h) \cap D(f) = D(hf)$ , where  $h, f \in A_1$ . This shows that  $D(g)$  is also principal open in  $U_1$ .

2. We have to show that the property of being of finite presentation is a local property and that  $f$  as defined above is locally of finite presentation.

Let  $\operatorname{Spec}(B) \subseteq X$  and  $\operatorname{Spec}(A) \subseteq S$  open affines. Pick a point  $x \in \operatorname{Spec}(B)$ . Then  $x \in \operatorname{Spec}(B) \cap \operatorname{Spec}(B_i)$  for some  $i$ . Pick some neighborhood  $x \in U \subseteq \operatorname{Spec}(B) \cap \operatorname{Spec}(B_i)$  such that  $U$  is principal open in  $\operatorname{Spec}(B)$  and in  $\operatorname{Spec}(B_i)$ .

Now take a neighborhood  $f(x) \in V \subseteq f(U)$  so that  $V$  is principal open in  $\operatorname{Spec}(A)$  and in  $\operatorname{Spec}(A_i)$ . Now take another smaller neighborhood  $x \in U' \subseteq f^{-1}(V)$  such that  $U'$  is principal open in  $\operatorname{Spec}(B)$  and in  $\operatorname{Spec}(B_i)$ .

So we have  $U' \rightarrow V$ , where both  $U'$  and  $V$  are principal opens of  $\text{Spec}(B_i)$  and  $\text{Spec}(A_i)$  respectively. Since  $A_i \rightarrow B_i$  is of finite presentation, then localizations  $(A_i)_f \rightarrow (B_i)_g$  (for some  $f \in A_i$  and  $g \in B_i$ ) are as well.

So for every point  $x \in \text{Spec}(B)$  we can find a principal open neighborhood in  $x \in D(f_x)$  and a principal open neighborhood  $f(x) \in D(g_x)$  such that  $A_{g_x} \rightarrow B_{f_x}$ .

Since  $\text{Spec}(B)$  is quasi-compact, we have  $\text{Spec}(B) = D(f_1) \cup \dots \cup D(f_n)$ . Denote  $g_1, \dots, g_n \in A$  be the respective elements in  $A$ .

We have composition  $\text{Spec}(B_{f_i}) \rightarrow \text{Spec}(A_{g_i}) \hookrightarrow \text{Spec}(A)$ , which induces a map of rings  $A \rightarrow A_{g_i} \rightarrow B_{f_i}$ . Since  $A_{g_i} \cong A[X]/(Xg_i - 1)$  and  $A_{g_i} \rightarrow B_{f_i}$  are of finite presentation by assumption, and being of finite presentation is stable under compositions, we have that  $A \rightarrow B_{f_i}$  are of finite presentation for every  $i$ .

Now its just commutative algebra to show that  $A \rightarrow B$  is of finite presentation as well, so I hope its okay to assume this part. Otherwise we could just rewrite something like Lemma 00EP.