Algebraic geometry 2
Exercise sheet 4

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Exercise 1. First of all we can use that finite projective modules are locally finite free. So since we are searching for an open neighbourhood of a point, we can first localize to some neighbourhood where N is finite free. (So assume $N = A^n$ is finite free.)

Since $M \otimes_A k(x)$ is finite dimensional $k(x)$-vsp, we can pick a basis $\{b_i \otimes 1\}_{i=1,\dots,m}$. Let $g\colon F := A^n \to M$ be defined by $e_i \mapsto b_i$. At x we obtain an isomorphism $F \otimes_A k(x) \xrightarrow{\sim} M \otimes_A k(x)$.

The composition $F \to M \to N$ is a map of free A-modules, so it can be represented by a matrix $J \in M_{n\times m}(A)$. At x this matrix has rank $m = \dim_{k(x)}(M \otimes_A k(x))$. So there is a neighbourhood U on which it has rank at least m (here we use argument from the previous sheet: U is taken to be the non-vanishing locus of determinant of some appropriate minor). On U, the composition $F \xrightarrow{J} N$ has left inverse $N \xrightarrow{I} F$ (i.e. it is injective).

On U, the section of the map $M \to N$ is given by composition $N \xrightarrow{I} F \xrightarrow{g} M$, which is what we wanted to show.

Exercise 3. By the definition of formally étale, the exercise reduces to show that in a diagram

$$\begin{array}{ccc} \mathbb{F}_p & \longrightarrow & R \\ \downarrow & & \downarrow \\ A & \xrightarrow{g} & R/I, \end{array}$$

where $I^2 = 0$, there exists a unique lift $A \to R$.

We can define a lift very explicitly:

Define $(-)^p\colon R \to R$ with $x \mapsto x^p$. Since R has characteristic p, this is a homomorphism. Ideal I is clearly contained in the kernel, so it factors through the quotient: $R \to R/I \to R$. Denote $u\colon R/I \to R$.

By assumption A is a perfect $\mathbb{F}_p$-algebra, so Frobenius endomorphism is an automorphism. We claim that a composition

$$A \xrightarrow{\mathrm{Fr}_A^{-1}} A \xrightarrow{g} R/I \xrightarrow{u} R$$

lifts g. Indeed, for any $x = y^p \in A$, we have $(u \circ g \circ \mathrm{Fr}_A^{-1})(x) = g(y)^p = g(x)$.

Now we prove uniqueness: Let φ, ψ be two lifts. Take any $x = y^p \in A$. Since they are lifts, we have $\varphi(y) - \psi(y) \in I$. But then $(\varphi(y) - \psi(y))^p = 0$ and thus also $\varphi(y^p) - \psi(y^p) = \varphi(x) - \psi(x) = 0$, so $\varphi = \psi$.