Recall: we have $\mathbb{P}^n_A$ with its standard cover $D^+(x_i)$ $i \in \{0, \dots, n\}$.

The $\mathcal{O}_{\mathbb{P}^n_A}(d)[D^+(x_i)] = \left(A[x_0, \dots, x_n, \frac{1}{x_i}]\right)_d$

$$\Gamma\left(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)\right) = \left(A[x_0, \dots, x_n]\right)_d$$

Or better

$$\bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)[D^+(x_i)] = A[x_0, \dots, x_n, \tfrac{1}{x_i}]$$

and

$$\Gamma\left(\mathbb{P}^n_A, \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)\right) = A[x_0, \dots, x_n]$$

$$\operatorname{Ker}\left(\prod_{i=0}^{n} \Gamma(D^+(x_i), \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow \prod_{0 \le i < j \le n} \Gamma\left(D^+(x_i x_j), \mathcal{O}_{\mathbb{P}^n}(d)\right)\right)$$

$$d(s)_{i_0, i_1} = s_{i_0} - s_{i_1}$$

Theorem Let A be a ring and $n \geq 0$. Then

$$H^q(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) = \begin{cases} A[x_0, \dots x_n]_d & \text{if } q=0 \\ 0 & \text{if } q \neq 0, n \\ \left(\frac{1}{x_0 x_1 \dots x_n} A\left[\frac{1}{x_0}, \dots, \frac{1}{x_n}\right]\right)_d & q=n \end{cases}$$

Proof we use Čech cohomology with respect to the cover

$$\mathcal{U} = (D_+(x_i))_{i=0}^n ,$$

$$C^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(d)) = \bigoplus_{i_0 < i_1 < \dots i_q} \left(A[x_0, \dots, x_n, \frac{1}{x_{i_0} x_{i_1} \cdots x_{i_q}}]\right)_d$$

we can give a $\mathbb{Z}^{n+1}$-grading to $C^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(d))$ by declaring that for each $\underline{a} = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ $x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}$ has multidegree $\underline{a}$.

Let $$C^q(\underline{a}) = \bigoplus_{i_0 < \dots < i_q} \left(A\left[x_0, \dots x_n, \frac{1}{x_{i_0} \cdots x_{i_q}}\right] \right)_{\underline{a}}$$

and

$$C^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(d)) = \bigoplus_{\substack{i_0 < \dots < i_q \\ \sum_{i=0}^n a_i = d}} \left(A\left[x_0, \dots x_n, \frac{1}{x_{i_0} \cdots x_{i_q}}\right] \right)_{\underline{a}}$$

We have $$\bigoplus_{d \in \mathbb{Z}} H^q(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) \simeq \bigoplus_{\underline{a} \in \mathbb{Z}^{n+1}} H^q(C^\bullet(\underline{a}))$$

We calculate $H^q(C^\bullet(\underline{a}))$ instead.

<u>Case 1</u>:

Suppose $a_i < 0$ $\forall i$, Then

$$C^q(\underline{a}) = \begin{cases} 0 & \text{if } q \neq n \\ A \cdot \dfrac{1}{x_0^{|a_0|} \cdots x_n^{|a_n|}} & \text{if } q = n \end{cases}$$

So $$\bigoplus_{\underline{a} < 0} H^q(C^\bullet(\underline{a})) = \begin{cases} 0 & \text{if } q \neq n \\ \dfrac{1}{x_0 \cdots x_n} A\left[\dfrac{1}{x_0}, \dots \dfrac{1}{x_n}\right] & q = n \end{cases}$$

Case 2: By symmetry WLOG $a_0 \geq 0$

For $q \geq 0$ we have a morphism

$$h : C^{q+1}(\underline{a}) \longrightarrow C^{q}(\underline{a})$$

$$h(s)_{i_0,\dots,i_q} = \begin{cases} 0 & \text{if } i_0 = 0 \\ s_{0,i_0,\dots,i_q} & \text{if } i_0 > 0 \end{cases}$$

For $q > 0$ we claim that

$$0 \overset{h}{\sim} \mathrm{id}$$

$$\begin{tikzcd}
C^{q-1} \arrow[r] & C^{q} \arrow[r, "d"] \arrow[dl] & C^{q+1} \arrow[dl, "h"'] \\
C^{q-1} \arrow[r] & C^{q} \arrow[r] & C^{q+1}
\end{tikzcd}$$

h is homotopy $0 \overset{h}{\sim} \mathrm{id}$ i.e.

$s = (hd + dh)[s]$.

This shows $H^q(C^q(\underline{a})) \longrightarrow H^q(C^q(\underline{a}))$ the identity is 0 so $H^q(C^q(\underline{a})) = 0$.

$$(hd(s))_{i_0,\dots,i_q} = \begin{cases} 0 & \text{if } i_0 = 0 \\[2ex] d(s)_{0,i_0,\dots,i_q} & \text{if } i_0 > 0 \end{cases}$$

but

$$d(s)_{0,i_0,\dots,i_q} = s_{i_0,\dots,i_q} + \sum_{j=0}^{q} (-1)^{j+1} s_{0,i_0,\dots,\underbrace{\hat{i}_j}_{\text{omit } i_j},\dots,i_q}$$

On the other hand

$$(dh(s))_{i_0,\dots,i_q} = \sum_{j=0}^{q} (-1)^j (h(s))_{i_0,\dots,\hat{i}_j,\dots,i_q}$$

$$= \begin{cases} s_{0,i_1,\dots,i_q} = s_{i_0,i_1,\dots,i_q} & \text{if } i_0 = 0 \\[2ex] \displaystyle\sum_{j=0}^{q} (-1)^j s_{0,i_0,\dots,\underbrace{\hat{i}_j}_{\text{omit } i_j},\dots,i_q} & \text{if } i_0 \neq 0. \end{cases}$$

We see $\left(h\partial(s)+\partial h(s)\right)_{i_0,\dots,i_q} = s_{i_0,\dots,i_q}$

So $h\partial(s)+\partial h(s) = s.$

Since $\mathrm{Id} \overset{h}{\sim} 0$ when $q>0$

This shows $H^q(C^\bullet(\underline{a}))=0$ if $q\geq 1$ and $a_i \geq 0$ for some i.

<u>Theorem</u>: Let A be a Noetherian ring and $\mathcal{F}$ a coherent sheaf on $X=\mathbb{P}^n_A$. The following hold:

1) For all $i\geq 0$ $H^i(X,\mathcal{F})$ is a finitely generated A-module

2) There is $N>>0$ s.t. $H^i(X,\mathcal{F}(d))=0$ $\forall\, i>0.$ $\forall\, d\geq N$

Proof i) We show finiteness by induction on i and $H^{n-i}(X, \mathcal{F})$.

Base case $i=-1$ | Grothendieck vanishing.

Recall that $\mathcal{O}(1)$ is ample.
In particular there is $d \gg 0$ s.t. $\mathcal{F}(d) := \mathcal{F} \otimes \mathcal{O}(1)^{\otimes d}$ is globally generated.

Since $\mathcal{F}$ is coherent, there is

$$0 \to K \to \bigoplus_{i=0}^{N} \mathcal{O}_X \to \mathcal{F}(d) \to 0$$

and

$$0 \to K' \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{E} = \bigoplus_{i=0}^{N} \mathcal{O}_X(-d)$.

we set

$$\underbrace{H^{n-i}(X, \mathcal{E})}_{\substack{\text{coherent} \\ \text{by computation.}}} \longrightarrow H^{n-i}(X, \mathcal{F}) \to \underbrace{H^{n-(i-1)}(X, K')}_{\substack{\text{coherent} \\ \text{by induction}}}$$

2) By Grothendieck vanishing only finitely many $H^i(X, -) \neq 0$.

It suffices to show for fixed i and fixed $\mathcal{F}$ there is d_i with $H^i(X, \mathcal{F}(N)) = 0 \quad \forall N \geq d_i$.

We do this by reverse induction in i. We have

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

$$\mathcal{E} = \bigoplus_{i=1}^{n} \mathcal{O}(-d_{\mathcal{F}})$$

$$H^i(X, \mathcal{E}(N)) \to H^i(X, \mathcal{F}(N)) \to H^{i+1}(X, \mathcal{K}(N))$$

explicitly $N - d_{\mathcal{F}} \geq 0$

by induction.

Theorem If $f: X \to Y$ is a projective map of Noetherian schemes, and $\mathcal{F} \in \mathrm{Coh}(X)$ then each $R^i f_* \mathcal{F}$ is coherent.

Proof We have

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_Y \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

where i is a closed immersion.

Then $R^i f_* \mathcal{F} \simeq R^i \pi_* i_* \mathcal{F}$

since i_* is exact.

WLOG $X = \mathbb{P}^n_Y$.

From last lecture $R^i f_* \mathcal{F}$ is $\mathrm{QCoh}(Y)$ and coherence can be checked on affines $\mathrm{Spec}\, A \underset{\text{open}}{\subseteq} Y$.

Then $R^i f_* \mathcal{F}|_{\mathrm{Spec}\, A} = \overbrace{H^i(\mathbb{P}^n_A, \mathcal{F}|_{\mathbb{P}^n_A})}$.

By previous theorem this is a coherent A-module.

<u>Theorem</u> (Chow's Lemma) Let $f: X \to Y = \operatorname{Spec} A$ be a proper map of Noetherian schemes. There exists a commutative diagram

$$\begin{tikzcd} X' \arrow[d, "g"'] \arrow[dr, "f'"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with f' projective and g proper and birational.

<u>Theorem</u>: Let $f: X \to Y$ be proper, and $\mathcal{F} \in \operatorname{Coh}(X)$. Then $R^i f_* \mathcal{F}$ is coherent.

<u>proof</u> Take diagram

$$\begin{tikzcd} X' \arrow[d, "g"'] \arrow[dr, "f'"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

as in Chow's lemma.

We may assume the theorem holds for all $\mathcal{F} \in Coh(X)$ s.t. $\dim(\operatorname{supp} \mathcal{F}) < \dim(X)$.

We have $R^i g_*(g^* \mathcal{F})$ are all coherent and if $i \geq 1$ then $\operatorname{supp} R^i g_*(g^*\mathcal{F}) < \dim X$.

We have a spectral sequence

$$E_2^{p,q} = R^p f_*(R^q g_*(g^*\mathcal{F})) \Rightarrow R^{p+q} f'_*(g^*\mathcal{F})$$

Coh by dimension induction

$$\begin{array}{ccc} R^0 f R^2 g & R^1 f R^2 g & R^2 f R^2 g \\ R^0 f R^1 g & R^1 f R^1 g & R^2 f R^1 g \\ R^0 f R^0 g & R^1 f R^0 g & R^2 f R^0 g \end{array}$$

The terms of $E_\infty^{p,q}$ are all coherent since $R^{p+q} f'_* g^* \mathcal{F}$ is so every sub-quotient is.

After a finite number of pases the term $E_r^{0,q} = E_\infty^{0,q}$ so this term is coherent.

Since $E_r^{0,q} = \mathrm{Coker}(S \to E_{r-1}^{0,q})$ with S coherent this shows inductively that $E_{r-i}^{0,q}$ is coherent for all i.

This shows that

$$R^i f_* g_* g^* \mathcal{F} \quad \text{is coherent.}$$

Finally,

$$0 \to K \to \mathcal{F} \to g_* g^* \mathcal{F} \to G \to 0$$

since g is birational

$$\dim \mathrm{Supp}(K),\ \dim \mathrm{Supp}\, G < \dim(X)$$

This shows $R^i f_* F$ is coherent.