<u>Recall:</u> Given a f.o.t lifting diagram

$$\begin{tikzcd} \operatorname{Spec} R/I \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "f"] & \operatorname{Spec} A \end{tikzcd}$$

and a lift f, then

$$\begin{aligned} \mathrm{Lift}(C) &= f + \mathrm{Der}_A(B, I) \\ &= f + \mathrm{Hom}_B(\Omega^1_{B/A}, I) \end{aligned}$$

<u>Proposition:</u> If $f: \operatorname{Spec} B \to \operatorname{Spec} A$ is formally smooth then $\Omega^1_{B/A}$ is a projective B-module. If f is smooth then $\Omega^1_{B/A}$ is finite locally free.

__Proof__ Let $M \to M'$ be a surjection of B-modules we want to show

$$\begin{array}{ccc} \mathrm{Der}_A(B, M) & \longrightarrow & \mathrm{Der}_A(B, M') \\ \| & & \| \\ \mathrm{Hom}_B(\Omega^1_{B/A}, M) & \longrightarrow & \mathrm{Hom}_B(\Omega^1_{B/A}, M') \end{array}$$

is also surjective. An element $\delta_{M'} \in \mathrm{Der}_A(B, M')$ corresponds to a map $\mathrm{Spec}\, A[M'] \to \mathrm{Spec}\, A$

$$a + \delta(a)\varepsilon \longleftarrow a$$

We have

$$\begin{tikzcd} \mathrm{Spec}\, A[M'] \arrow[r, "\mathrm{can}+\delta'"] \arrow[d, hook] & \mathrm{Spec}\, A \arrow[d] \\ \mathrm{Spec}\, A[M] \arrow[r] \arrow[ur, dashed, "f"] & \mathrm{Spec}\, R \end{tikzcd}$$

by smoothness we get an element $f = \mathrm{can} + \delta$ and δ lifts δ'. So $\Omega^1_{B/A}$ is projective.

If $\operatorname{Spec} B \to \operatorname{Spec} A$ is finitely presented $B = A[x_1, \ldots x_n]/I$ then

$$\bigoplus_{i=1}^{n} B \cdot dx_i \to \Omega^1_{B/A} \quad \text{is surjective}$$

So $\Omega^1_{B/A}$ is finite projective.

<u>Proposition</u> Given map of schemes $f: X \to Y$ and $g: Y \to S$ The following hold:

1) If f is formally smooth

then
$$0 \to f^* \Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 \qquad (*)$$
is exact and locally split.

2) If $h = g \circ f$ is formally smooth and the sequence $(*)$ is exact locally split, then f is formally smooth.

Deeper Perspective: There is a cotangent complex $\mathbb{L}_{X/Y}$, $\mathbb{L}_{X/S}$, $f^*\mathbb{L}_{Y/S}$

and a triangle

$$f^*\mathbb{L}_{Y/S} \to \mathbb{L}_{X/S} \to \mathbb{L}_{X/Y} \to$$

this induces on homology

smoothness of f.

$$\begin{tikzcd}
H_1(\mathbb{L}_{X/Y}) \arrow[r] \arrow[d, equal] & H_0(f^*\mathbb{L}_{Y/S}) \arrow[r] \arrow[d, equal] & H_0(\mathbb{L}_{X/S}) \arrow[r] \arrow[d, equal] & H_0(\mathbb{L}_{X/Y}) \arrow[d, equal] \\
0 & f^*\Omega^1_{Y/S} \arrow[r] & \Omega^1_{X/S} \arrow[r] & \Omega^1_{X/Y}
\end{tikzcd}$$

Conversely, if h is formally smooth then $\Omega^1_{X/S}$ "should be projective".

if $0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is locally split, then $\Omega^1_{X/Y}$ is locally a direct summand of $\Omega^1_{X/S}$.

Warning $X = \operatorname{Spec} k$, $Y = \operatorname{Spec} k[\varepsilon]/\varepsilon^2 = S$

then $\Omega^1_{X/S} = \Omega^1_{X/Y} = \Omega^1_{Y/S} = 0$, but $X \to S$ is not formally smooth.

Proof WLOG $X = \operatorname{Spec} C$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} A$.

i) We need to show

$$0 \to C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

is locally split exact.

Now, $\operatorname{Hom}(\Omega^1_{C/A}, C \otimes_B \Omega^1_{B/A})$

$= \operatorname{Der}_A(C, C \otimes_B \Omega^1_{B/A})$.

We have

$$\begin{tikzcd}
\operatorname{Spec} C \arrow[r, equal] \arrow[d] & \operatorname{Spec} C \arrow[d] \\
\operatorname{Spec}(C \oplus [C \otimes \Omega^1_{B/A}]) \arrow[r] & \operatorname{Spec} B \\
(f^\#(b), db) & b \arrow[l, mapsto]
\end{tikzcd}$$

By formal smoothness there is a lift $C \xrightarrow[(\psi, \delta)]{} C \oplus [C \otimes \Omega^1_{B/A}]\varepsilon$

where $\delta : C \to C \otimes \Omega^1_{B/A}$ is a derivation such that $\delta f^\#(b) = db$.

Let $\rho_\delta \in \operatorname{Hom}(\Omega^1_{C/A}, C \otimes_B \Omega^1_{B/A})$

then

$$C \otimes_B \Omega^1_{B/A} \longrightarrow \Omega^1_{C/A} \xrightarrow[s_\delta]{} C \otimes_B \Omega^1_{B/A}$$

$$c \otimes db \longmapsto c \cdot df^\#(b) \longrightarrow c\delta(f^\#(b)) = c \otimes db$$

2) Suppose now $g \circ f$ is formally smooth and the sequence is split exact.

We consider the lifting diagram

$$\begin{tikzcd}
\operatorname{Spec} R/I \arrow[r] \arrow[d] & \operatorname{Spec} C \arrow[r, "f"] & \operatorname{Spec} B \arrow[dl, "g"] \\
\operatorname{Spec} R \arrow[ur, dashed, "\psi"] \arrow[urr, "t_B"] \arrow[r, "t_A"'] & \operatorname{Spec} A &
\end{tikzcd}$$

with $I^2 = 0$

Since $f o g$ is formally smooth there is ψ lifting t_A (but not necessarily lifting t_B).

Consider the space of ψ lifting t_A this is a $\operatorname{Der}_A(C, I)$-torsor.

On rings we want to find an A-linear derivation $\delta : C \to I$ making

$$\begin{tikzcd} C \arrow[r, "\psi^\# + \delta"] & R \\ B \arrow[u, "f^\#"] \arrow[ur, "t_B^\#"'] & \end{tikzcd}$$

commutative.

This is $\psi^\# \circ f^\# - t_B^\# = \delta f^\#$ which we can consider as elements of $\mathrm{Der}_A(B, I)$

But

$$\begin{tikzcd} \mathrm{Der}_A(B, I) \arrow[d, phantom, "\parallel"] & \mathrm{Der}_A(C, I) \arrow[l] \arrow[d, phantom, "\parallel"] \\ \mathrm{Hom}_B(\Omega^1_{B/A}, I) \arrow[d, phantom, "\parallel"] & \mathrm{Hom}_C(\Omega^1_{C/A}, I) \arrow[l] \arrow[dl] \\ \mathrm{Hom}_C(\Omega^1_{B/A} \otimes C, I) & \end{tikzcd}$$

Since $0 \to \Omega^1_{B/A} \otimes C \to \Omega^1_{C/A}$ is split injective, then

$$\mathrm{Hom}_C(\Omega^1_{C/A}, I) \to \mathrm{Hom}_C(\Omega^1_{B/A} \otimes C, I) \to 0$$

is surjective.

Proposition Consider the diagram

$$\begin{tikzcd} Z \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

with i a closed immersion defined by an ideal sheaf $\mathcal{I} = \ker(i : \mathcal{O}_X \to i_*\mathcal{O}_Z)$ and consider the sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0$$

a) If f is formally smooth, then the above sequence is exact. and locally split

b) If g is formally smooth, the sequence is exact and locally split, then f is formally smooth.

Proof WLOG $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$,

$Z = \operatorname{Spec} B/J$.

a) We want

$$\begin{tikzcd} i^*\Omega^1_{X/S} \arrow[r, bend left, "r"] & J/J^2 \arrow[l, bend left, "d"] \end{tikzcd}$$

with $r \circ d = \mathrm{id}_{J/J^2}$. We interpret $r \in \operatorname{Der}_A(B, J/J^2)$. Consider the lifting problem:

$$\begin{tikzcd}
B/J & B/J \arrow[l] \arrow[dl, dashed, "u"'] & B \arrow[l, "\pi_J"'] \\
B/J^2 \arrow[u] & A \arrow[l] \arrow[u, "f^\#"'] \arrow[ur]
\end{tikzcd}$$

Since f is smooth u exists.

This induces two A-algebra maps

$$\begin{tikzcd} B \arrow[r, shift left, "\pi_{J^2}"] \arrow[r, shift right, "u \circ \pi_J"'] & B/J^2 \arrow[r] & B/J. \end{tikzcd}$$

then $\delta = \pi_{J^2} - u \circ \pi_J \in \operatorname{Der}_A(B, J/J^2)$

the induced map is

$$\rho : B/J \otimes_B \Omega^1_{B/A} \longrightarrow J/J^2$$
$$v \otimes db \longmapsto v\,\delta(b)$$

If $b \in J$ then $\rho(db) = \delta(b)$

$$= \pi_{J^2}(b) - \underbrace{u \circ \pi_J(b)}_{=0}$$

$$= b \in J/J^2 .$$

b) Formal smoothness is about lifting

$$C = \begin{tikzcd} R/I & B/J^2 \arrow[l] \\ R \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd} \qquad \mathrm{Lift}(C) \neq \emptyset.$$

$$C = \begin{tikzcd} R/I & B/J^2 \arrow[l] & B \arrow[l] \arrow[dll, dashed, "N"'] \\ R \arrow[u] & A \arrow[l] \arrow[u] \arrow[ur] & \end{tikzcd}$$

by smoothness of g we can lift to N.

If $N(J) = 0$, we win.
Note that $N(J) \subseteq I$ so $N(J^2) = 0$.

We consider $N + \mathrm{Der}_A(B, I)$ the set of all possible lifts.

It suffices to find $\delta \in \mathrm{Der}_A(B, I)$ with $\delta(j) = -N(j)$ $\forall j \in J$.

$$\mathrm{Der}_A(B, I) \simeq \mathrm{Hom}(\Omega^1_{B/A}, I) \xrightarrow{\text{surjective}} \mathrm{Hom}_B(J/J^2, I)$$

$$\mathrm{Hom}_B(J/J^2, I) \ni N|_J$$

(since the sequence is exact locally split).

Example: $S = \mathrm{Spec}\, R$, $X = \mathbb{A}^n_S = \mathrm{Spec}\, R[x_1 \dots x_n] = B$

$Z = V(I) \subseteq \mathbb{A}^n_S$, $I \subseteq R[x_1 \dots x_n]$, R Noetherian, with $I = (f_1, \dots, f_m)$. $Z = \mathrm{Spec}\, A = \mathrm{Spec}\, B/I$

then we set

$$\begin{tikzcd}
\bigoplus_{j=1}^m A \cdot e_j \arrow[r, "\sigma"] \arrow[d, two heads] & \bigoplus_{i=1}^n A \cdot \partial x_i \arrow[d, "\cong"] & & \\
I/I^2 \arrow[r, "d"] & A \otimes_B \Omega^1_{B/R} \arrow[r] & \Omega^1_{A/R} \arrow[r] & 0
\end{tikzcd}$$

where $\sigma(e_j) = df_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} dx_i$

we can think of σ as a matrix with $\sigma_{ij} = \left(\frac{\partial f_j}{\partial x_i}\right)$

By previous proposition, Z is finally smooth over R if and only if $I/I^2 \xrightarrow{d} A \otimes_B \Omega^1_{B/R}$ is injective and locally split.

Given $z \in Z$ corresponding to a prime ideal $p \subseteq A$. If $I/I^2 \otimes A_p \xrightarrow{d} A_p \otimes_B \Omega^1_{B/R}$ is split injective then this splitting spreads to a neighborhood. Since I/I^2 is finitely presented A-module.

So the locus $Z^{sm} \subseteq Z$ where Z is smooth over S is open.

We wish to rephrase this in terms of $\mathcal{J}$.

<u>Lemma</u> Let (A, m) be a local ring, $\mu: F_1 \to F_2$ a map of finite free modules. Then μ is injective split $\iff$ $\mu \otimes_A A/m$ is injective split.

<u>proof</u> $\Rightarrow$) Easy.

$\Leftarrow$) Let $k = A/m$,

then $F_1 \otimes_A k \subseteq F_2 \otimes_A k$

we can choose a basis $\{\bar{e}_1 \ldots \bar{e}_r\}$ of $F_1 \otimes_A k$ with $r = \mathrm{rank}(F_1 \otimes_A k)$, and a basis $\{\overline{\mu(e_1)}, \ldots, \overline{\mu(e_r)}, \bar{d}_{r+1}, \ldots, \bar{d}_n\}$ for $(F_2 \otimes_A k)$. Since A is local these basis lift to basis $\{e_1, \ldots, e_r\}$ and $\{\mu(e_1), \ldots, \mu(e_r), d_{r+1}, \ldots, d_n\}$

the map $\mu(e_i) \longmapsto e_i$

$d_j \longmapsto 0$

defines a splitting.

Theorem (Jacobian Criterion) Fix the following diagram of schemes

$$\begin{tikzcd} \operatorname{Spec} A = Z \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & \mathbb{A}^n_R = \operatorname{Spec} B \arrow[dl, "j"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

$\operatorname{Spec} A = \operatorname{Spec} B/I$

with i closed and locally of finite presentation,

fix a point $z \in Z$ f is smooth at z iff there is $z \in U \underset{\text{open}}{\subseteq} \mathbb{A}^n_R$ and polynomials $f_1, \dots, f_m \in R[x_1, \dots x_n]$ s.t.

$Z \cap U = V(f_1, \dots, f_m) \cap U$ and that $J(z) = \left(\frac{\partial f_j}{\partial x_i}\right)(z)$ has rank m

Proof Since $J(z): \bigoplus_{j=1}^{m} k(z)e_j \to \bigoplus_{i=1}^{n} k(z)\,dx_i$ is injective split we set

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{j=1}^{m} A\cdot e_j \arrow[r, "J"'] \arrow[d] & \bigoplus_{i=1}^{n} A\cdot dx_i \arrow[d, "\cong"] & & \\
& I/I^2 \arrow[r, "d"'] & A\otimes_B \Omega^1_{B/R} \arrow[r] & \Omega^1_{A/R} \arrow[r] & 0
\end{tikzcd}$$

on a neighborhood.
(Away from the locus where the m-minors of $J(z)$ vanish).

This shows $\bigoplus_{j=1}^{m} A\cdot e_j \xrightarrow{\sim} I/I^2$

and that $I/I^2 \xrightarrow{d} A\otimes_B \Omega^1_{B/R}$ is injective locally split.

Conversely, if f is smooth at z and

$\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m \in I/I^2 \otimes k(z) = I \otimes_B k(z)$ are a basis then locally I is generated by lifts $f_1, \dots, f_m \in I$.

$$\begin{tikzcd}
\bigoplus_{j=1}^{m} A \cdot e_j \arrow[r, "J"'] \arrow[d] & \bigoplus_{i=1}^{n} A \cdot dx_i \arrow[d, "\cong"] & \\
I/I^2 \arrow[r, "d"'] & A \otimes_B \Omega^1_{B/R} \arrow[r] & \Omega^1_{A/R} \arrow[r] & 0
\end{tikzcd}$$

smoothness implies d injective which shows $\operatorname{rank} J(z) = m$.