Algebraic geometry 1
Exercise sheet 6

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Exercise 1.

1. By the universal property of the fiber product of locally ringed spaces, we have the following commutative diagram

$$\begin{tikzcd}
U_i \times_{S_{i,j}} V_j \arrow[dr, dotted] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & V_j \arrow[d, "\psi"] \\
& U_i \arrow[r, "\phi"] & S_{i,j}
\end{tikzcd}$$

Therefore, on the level of sets,

$$U_i \times_{S_{i,j}} V_j \subset X \times_S Y,$$

but in exercise 5.2.1, we showed that this induces an open immersion as locally ringed spaces.

Now observe that

$$\begin{tikzcd}
\bigcup_{i,j} \left(U_i \times_{S_{i,j}} V_j\right) \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

commutes, because $S = \bigcup_{i,j} S_{i,j}$. Now by uniqueness of the pullback,

$$\bigcup_{i,j} \left(U_i \times_{S_{i,j}} V_j\right) \cong U_i \times_{S_{i,j}} V_j.$$

I guess this is a good step in the direction of understanding why the pullback in the category of sheaves exists, right? If we assume X, Y, S to be sheaves and $U_i, V_j, S_{i,j}$ to be affine schemes, then by the above argument we found a cover of $X \times_S Y$ by affine schemes.

2. Surjectivity follows, because a pullback of schemes in partiular makes

$$\begin{tikzcd} \mid X \times_S Y \mid \arrow[r] \arrow[d] & \mid X \mid \arrow[d, "\psi"] \\ \mid Y \mid \arrow[r, "\phi"] & \mid S \mid \end{tikzcd}$$

commute for all ψ, ϕ.

Exercise 3. By definition we have to compute a fibred product of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and $\mathrm{Spec}(k(p)) \to \mathrm{Spec}(A)$ (where $k(p)$ is the residue field of $p \in \mathrm{Spec}(A)$ and $\to$ is the canonical inclusion). Since we are dealing with affine schemes, we can express it concretely as $\mathrm{Spec}(B \otimes_A k(p))$. Note that B has the structure of an A-algebra, which is induced by the starting morphism of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. So this exercise reduces to computing these tensor products.

We also observe that $k[T]$ is a PID, which means every non-zero prime ideal is a maximal ideal. This will be handy when computing residue fields, because after quotienting with a non-zero ideal we already get a field (we do not have to further take the quotient field).

1. In the first example we do now even have to calculate the tensor product, because we can rewrite $k[T, U]/(TU - 1) = k[T, T^{-1}]$, so this is just a localization of $k[T]$. Morphism of spectrums, induced by inclusion into localization, is an open immersion, so fibers will be singletons if $x \in D(T)$ and empty sets otherwise. And the structure sheaf is also clear, it is just the restriction of structure sheaf $\mathcal{O}_{\mathrm{Spec}(k[T])}$.

2.

3.

4.

Exercise 4. Take $U = D(f)$ for some $f \in A$ and let $U = \cup_i D(f_i)$ be some cover. We have to check that

$$M[f^{-1}] \to \mathrm{Eq}\left[\prod_i M[f_i^{-1}] \rightrightarrows \prod_{i,j} M[(f_i f_j)^{-1}]\right]$$

is isomorphism.

This proof is exactly the same as when we proved that $\mathcal{O}_{\mathrm{Spec}(A)}$ is a sheaf, after we defined it the basis of principal opens.

Then proved that $A = \mathrm{Eq}\left[\prod_i A[f_i^{-1}] \rightrightarrows \prod_{i,j} A[(f_i f_j)^{-1}]\right]$ where $\mathrm{Spec}(A) = \cup_i D(f_i)$ is a cover.

We can simply tensor the whole diagram and, since tensor product commute with direct limits, we have that

$$M = \mathrm{Eq}\left[\prod_i M \otimes_A A[f_i^{-1}] \rightrightarrows \prod_{i,j} M \otimes_A A[(f_i f_j)^{-1}]\right].$$