Algebraic geometry 2
Exercise sheet 1

Solutions by: Esteban Castillo Vargas and David Čadež

18. April 2024

Exercise 1. Lets first prove that if M is not torsion free it cant be flat. Take $r \in R$ and $m \in M$ such that $rm = 0$ and $r \neq 0 \neq m$. We have an exact sequence of R-modules

$$0 \to (r) \to R \to R/(r) \to 0$$

but when tensoring with M we get

$$0 \to (r) \otimes_R M \to R \otimes_R M \to R/(r) \otimes_R M \to 0$$

which is not exact, because $(r) \otimes_R M \to R \otimes_R M \cong M$ is not injective (it maps $r \otimes m \mapsto 0$).

For the other direction, take m a maximal ideal of R. Since R is a Dedeking domain, R_m is also normal and thus a PID (we proved that last year during the lectures). We've shown a module over a PID is torsion-free exactly when it is flat in Algebra 1. We did it by showing that flatness can be checked on all finitely generated submodules and that a finitely generated module over a PID is flat if and only if it is free.

Exercise 2. Map $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ sends generic point to generic point if and only if $R \to A$ injective.

And clearly A is a torsion free R module if and only if $R \to A$ is injective.

Using first exercise we get that $R \to A$ is flat if and only if $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ send generic point to generic point.

Exercise 3.

i) The derivative of $z \mapsto zg(z)$ at z is $g(z) + z\frac{dg}{dz}(z)$, which is $g(0)$ at 0, therefore non-zero. So by theorem from complex analysis there exists a holomorphic inverse on some neighborhood of 0.

Second part: from complex analysis we know that if g is a holomorphic function on a simply connected open Ω with $g \neq 0$ on Ω, then there exists $\tilde{g}$ on Ω with $e^{\tilde{g}} = g$. So for h we can take $e^{\frac{1}{n}\tilde{g}}$.

ii) Pick $y \in Y$ and $V \subseteq X$ a neighborhood of $f(y) \in X$ with $V \cong \mathbb{D}$ (WLOG with $f(y)$ corresponding to 0). Take $U \subseteq f^{-1}(V)$ with $y \in U \cong \mathbb{D}$ (WLOG with y corresponding to 0). Because zero set of a non-zero holomorphic map is discrete, we can pick U such that y is the only zero of the function $U \to V \to \mathbb{D}$. So now we have holomorphic $h\colon \mathbb{D} \cong U \to V \cong \mathbb{D}$, for which $0 \mapsto 0$. Let n_y be the degree of this root. Therefore we can write $h(z) = z^{n_y} g(z)$ for some holomorphic $g\colon \mathbb{D} \to \mathbb{D}$. Observe that since 0 is the only root, we have $g(z) \neq 0$ for all $z \in \mathbb{D}$. By part i) we have that there exists n-th root of g, i.e. a holomorphic function p with $p^{n_y} = g$ on $\mathbb{D}$. Note that since $g \neq 0$ on $\mathbb{D}$, same is true for p. We can write $h(z) = z^{n_y} p^{n_y}(z)$. By part i), the function $z \mapsto zp(z)$ is biholomorphic, so it has a holomorphic inverse. Precomposing h with this inverse yields a function $\tilde{h}\colon \mathbb{D} \to \mathbb{D}$ with $z \mapsto z^{n_y}$.

iii) Suppose we have two local descriptions with U_1 and U_2, which have non-empty open intersecton in Y. We can assume both map to same $V \cong \mathbb{D}$. We obtain a local neighborhood of 0 in $U_1 \cap U_2$ that is biholomorphic to its image in $U_1 \cap U_2 \rightharpoonup U_2$. Since this change of coordinates is biholomorphic, degree of the root has to be 1, so n_1 and n_2 corresponding with descriptions with U_1 and U_2 are the same as well.

For every point $y \in Y$ we found a neighborhood U on which it identifies with $z \to z^{n_y}$. From this local identification it follows that for every other point $z \in U$ with $z \neq y$ there exists a neighborhood $\tilde{U} \subseteq U$ for which $\tilde{U} \xrightarrow{\sim} f(\tilde{U})$. So for every point in $U \setminus \{y\}$ the map f is locally biholomorphic, which means $n_z = 1$. Therefore the set of points y with $n_y > 1$ is discrete. Because manifold Y is compact, that set must be finite.

Exercise 4.

i) Functor $\mathrm{Hom}_A(-, I)$ is always left-exact, so we only have to check right-exactness.

Let
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
be exact. We want to show
$$0 \to \mathrm{Hom}_A(M_3, I) \to \mathrm{Hom}_A(M_2, I) \to \mathrm{Hom}_A(M_1, I) \to 0$$
is exact.

There is a natural isomorphism of abelian groups
$$\mathrm{Hom}_A(M_i, I) \cong \mathrm{Hom}_{\mathbb{Z}}(M_i \otimes_A A, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(M_i, \mathbb{Q}/\mathbb{Z}).$$

(here by natural we mean functorial, i.e. that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(M_3, I) & \longrightarrow & \mathrm{Hom}_A(M_2, I) & \longrightarrow & \mathrm{Hom}_A(M_1, I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(M_3, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(M_2, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(M_1, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

commutes. We had some confusion around the meaning of naturality, functoriality and something being canonical.)

In the hint it says that $\mathbb{Q}/\mathbb{Z}$ is injective $\mathbb{Z}$-module, so we get that

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(M_3, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(M_2, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(M_1, \mathbb{Q}/\mathbb{Z}) \to 0$$

is exact. Therefore

$$0 \to \mathrm{Hom}_A(M_3, I) \to \mathrm{Hom}_A(M_2, I) \to \mathrm{Hom}_A(M_1, I) \to 0$$

is also exact.

ii) There is a (forgetful) faithful functor from category of A-modules to category of abelian groups, which preserves monomorphisms and has a right adjoint. Then it is true that if the latter category has enough injectives than also former has enough injectives.