

# Algebraic geometry 1

## Exercise sheet 9

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### Exercise 2.

1. Suppose we have a vector bundle of rank  $n$  on  $\mathbb{P}_k^1$ . How do we construct a matrix  $\in \mathrm{GL}_n(k[T^{\pm 1}])$ ?

Take a rank  $n$  vector bundle  $\mathcal{E}$ . Since Picard group of  $U_0$  and  $U_1$  are trivial, we have isomorphisms  $\alpha, \beta$ . So on  $\mathrm{Spec}(k[T^{\pm 1}]) \subseteq U_0$  we have an isomorphism  $\Gamma(U_0 \cap U_1, \mathcal{O}_{U_0}^n) = (k[T^{\pm 1}])^n \cong \mathcal{E}(U_0 \cap U_1)$ .

Combining this with an isomorphism  $\mathcal{E}(U_0 \cup U_1) \cong (k[T^{\pm 1}])^n = \Gamma(U_0 \cap U_1, \mathcal{O}_{U_1}^n)$ , we get an isomorphism  $(k[T^{\pm 1}])^n \cong (k[T^{\pm 1}])^n$ .

Let  $\mathcal{D}$  be another rank  $n$  vector bundle on  $\mathbb{P}_k^1$ , and let  $\varphi: \mathcal{E} \rightarrow \mathcal{D}$  be an isomorphism between them. On  $U_0$  and  $U_1$  we get induced isomorphisms

$$(k[T])^n = \mathcal{E}(U_0) \rightarrow \mathcal{D}(U_0) = (k[T])^n$$

and

$$(k[T^{-1}])^n = \mathcal{E}(U_1) \rightarrow \mathcal{D}(U_1) = (k[T^{-1}])^n$$

...

On the right side, we are given transition maps.

We have that

$$\alpha_{|U_0 \cap U_1}^{-1} \circ \beta_{|U_0 \cap U_1}$$

is invertible, because by assumption  $\alpha$  and  $\beta$  are isomorphisms. To see injectivity, remember that given sheaves on a cover and transition maps, we can uniquely (up to isomorphism) glue them to get a sheaf on the whole space.

Well definedness of this map comes from the fact that if two vector bundles  $V_1 \cong V_2$  are isomorphic, then the transition map is the same.

It remains to show surjectivity.

2. By left multiplication we get a diagonal matrix we get that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathrm{GL}_2(k[x]),$$

so assume this without loss of generality.

Next, observe that the determinant of  $A$  in this case is in  $k[x, x^{-1}] \cap k[x]$ , so we can write

$$\det(A) = cx^n.$$

Since  $k[x]$  is an euclidean domain, we can now assume that  $a_{12} = 0$ . Using our observation regarding the derterminant, we conclude that  $a_{11}$  and  $a_{22}$  are monomials (with non-negative degree).

We think that we can also assume w.l.o.g that  $\deg(a_{11}) \geq \deg(a_{22})$ .

Now we can eliminate  $a_{21}$  by adding  $k[x]$  multiples of the second column to the first column, to eliminate all the terms in  $a_{21}$  with degree greater  $\deg(a_{22})$  and all terms in  $a_{12}$  with degree smaller  $\deg(a_{21})$  by adding  $k[x^{-1}]$  multiples of the first row to the second row (to eliminate all terms in  $a_{12}$  with degree smaller  $\deg(a_{11})$ ).

3. Claim: Every line bundle on  $\mathbb{P}^1$  can be written as

$$\mathcal{O}_{\mathbb{P}^1}(d_n) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n).$$

Proof of claim:

By part 1 of this exercise, we can characterize the isomorphism classes of rank  $n$  vector bundles by looking at the transition functions.

In the second part of this exercise, we showed that (for  $n = 2$ , but actually inductively for all  $n$ ) these transition functions can be writte as  $T^d$ . The claim now follows from observing that the transition matrix of

$$\mathcal{O}_{\mathbb{P}^1}(d_n) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n)$$

is given by

$$T^{(d_1, \dots, d_n)}.$$

### Exercise 3.

1. By exercise 1 on sheet 3 we get that

$$X \rightarrow \mathbb{A}_R^{n+1} \times_{\mathrm{Spec}(R)} \mathbb{P}_R^n$$

is a closed immersion, so it remains to check that

$$\mathcal{O}_X \rightarrow f_* \mathcal{O}_{\mathbb{A}_R^{n+1} \times_{\mathrm{Spec}(R)} \mathbb{P}_R^n}$$

is a surjection.