

Algebraic geometry 1

Exercise sheet 3

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Exercise 1.

1. Let X be a finite set that is irreducible with respect to some topology \mathcal{F} on X . Then we get $|\mathcal{F}| < \infty$ and since finite unions of closed sets are closed again, we get that

$$X' := \bigcup_{U \subsetneq X \text{ closed}} U$$

is closed in X . Since X is by assumption irreducible, $X \neq X'$, so we can pick $x_0 \in X \setminus X'$, which is by construction generic. For the second part of the exercise we use part 2 of Hochster's Theorem. As a finite set, X is quasicompact and as a basis \mathcal{B} consisting of quasicompact open sets stable under finite intersections take all of the open sets.

It remains to show that X is sober. We need to check that every irreducible subset of X has a unique generic point. The existence of a generic point comes from part of this exercise. Uniqueness of this point is due to the fact that generic points in T_0 spaces are unique if they exist, which follows directly from the definition of T_0 .

1. Let X be a finite irreducible topological space. Since

$$X = \bigcup_{x \in X} \overline{\{x\}}$$

is a finite decomposition in closed sets, we must have $\overline{\{x\}} = X$ for some $x \in X$. This x is a generic point of X .

If we additionally assumed X is T_0 , then this point x would be unique, since in a T_0 space we have $\overline{\{x\}} \neq \overline{\{y\}}$ for $x \neq y$. Also in a finite space the conditions of quasicompactness and the basis being stable under finite intersections are clearly fulfilled. So finite T_0 spaces are spectral.

2. Let us first describe what $\text{Spec}(\mathbb{Z})$ looks like. It is a PID with prime ideals being those (a) for which $a \in \mathbb{Z}$ is a prime number or $a = 0$. So

$$\text{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime}\} \cup \{(0)\}.$$

In this space (0) is a unique generic point.

Closed sets in $\text{Spec}(\mathbb{Z})$ are by definition

$$V((a)) = \{(p) \in \text{Spec}(\mathbb{Z}) \mid (a) \subseteq (p)\} = \{(p) \in \text{Spec}(\mathbb{Z}) \mid p \text{ divides } a\}.$$

So if $a = \prod_i p_i^{k_i}$, then $V((a)) = \{(p_i)\}_i \subseteq \text{Spec}(\mathbb{Z})$. Since any $a \in \mathbb{Z}$ is only divisible by finitely many prime numbers, we get the finite complement topology on $\text{Spec}(\mathbb{Z}) \setminus \{(0)\}$.

Adding generic point (0) to $\text{Spec}(\mathbb{Z}) \setminus \{(0)\}$ is actually the construction $X \rightarrow X^{\text{sob}}$ which we did last week.

Exercise 2. Denote $A = \lim A_i$, $B = \lim B_i$ and $C = \lim C_i$. Also denote maps $A_i \rightarrow A$ with f_i , $B_i \rightarrow B$ with g_i and $C_i \rightarrow C$ with h_i .

By composing α_i and g_i we get $A_i \rightarrow B$ defined as $g_i \circ \alpha_i$. Then by the definition of a colimit we have a unique map $A \rightarrow B$, denoted by α , such that $g_i \circ \alpha_i = \alpha \circ f_i$. In the same way we obtain $\beta: B \rightarrow C$. With these definitions the whole diagram commutes.