Algebraic geometry 1
Exercise sheet 7

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Exercise 1.

1. Pick any $\mathcal{O}_X$-module $\mathcal{M}$. Then we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X}(f_*\widetilde{N}, \mathcal{M}) &\cong \operatorname{Hom}_A(f_*\widetilde{N}(X), \mathcal{M}(X)) \\ &= \operatorname{Hom}_A(N|_A, \mathcal{M}(X)) \\ &\cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{N|_A}, \mathcal{M}).\end{aligned}$$

   By the Yoneda lemma, this implies that $f_*\widetilde{N} \cong \tilde{N}_{|A}$.

2. For the second part of this exercise, we extend the first part as follows, using that f_* is left-adjoint to f^*

$$\operatorname{Hom}_{\mathcal{O}_Y}(f^*\tilde{\mathcal{M}}, \tilde{N}) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{\mathcal{M}}, f_*\widetilde{\mathcal{N}}) \cong \operatorname{Hom}_A(\widetilde{\mathcal{M}}(B), f_*\widetilde{\mathcal{N}}(B))$$
$$= \operatorname{Hom}_A(M, \widetilde{\mathcal{N}}(A)) \cong \operatorname{Hom}_B(N_{|A} \otimes_A B, \tilde{N}(A)) \cong \operatorname{Hom}_{\mathcal{O}_Y}(\widetilde{\mathcal{M} \otimes_A B}, \widetilde{\mathcal{N}}).$$

   Now, by the Yoneda lemma we again obtain that

$$\widetilde{\mathcal{M} \otimes_A B} \cong f^*\tilde{\mathcal{M}}.$$

   Next, we want to show that we can extend this exercise from affine schemes to schemes.

   Let S_i with $i \in I$ be a cover of S by open affines. Then for each $i \in I$ we get that $g^{-1}(S_i)$ is a subscheme of $Z_i \subset Z$ (unfortunately not necessarilary affine). Now, we cover each of these subschemes Z_i by open affines Z_{ij}. By construction g maps Z_{ij} into S_i. Hence,

$$(g^*\mathcal{M})_{Z_{ij}} = f^*\mathcal{M}_{Z_{ij}} \cong \widetilde{M \otimes_A B},$$

   showing that g^* preserves quasi-coherence.

Exercise 2. For every homogenous polynomial $F(X_0, \ldots, X_n)$ of degree m we attach $\{f_i\}_{i=0,\ldots,n}$, where $f_i(X_{0/i}, \ldots, X_{i-1/i}, X_{i+1/i}, X_{n/i})$ is the unique polynomial such that $\beta_i(f_i) = \frac{F(X_0,\ldots,X_n)}{X_i^m}$, where

$$\beta_i \colon \mathbb{Z}[X_{0/i}, \ldots, X_{n/i}] \to \mathbb{Z}[X_0, \ldots, X_n, X_i^{-1}]$$
$$X_{j/i} \mapsto \frac{X_j}{X_i}$$

First we check that $\alpha_{i,j}^m(f_i) = f_j$. Due to linearity, it is enough to check the case when F is a monomial. Let $F = \prod_{i=0}^n X_i^{m_i}$. Then

$$f_i = \prod_{k=0,k\neq i}^n X_{k/i}^{m_k} \quad \text{and} \quad f_j = \prod_{k=0,k\neq j}^n X_{j/k}^{m_j}.$$

Simply check

$$\alpha_{i,j}^m(f_i) = \prod_{k=0,k\neq i}^n \alpha_{i,j}^m(X_{k/i}^{m_j}) = X_{j/i}^m X_{j/i}^{-1} \prod_{k=0,k\neq i,j}^n (X_{k/j}X_{j/i}^{-1})^{m_k} = f_j$$

where we used $m = \sum_{k=0}^n m_k$ in the last equality. So $\{f_i\}_{i=0,\ldots,n}$ define a global section on $\mathbb{P}_k^n$.

To show injectivity, suppose $f_i = 0$ for all i. Then $\beta_i(f_i) = 0$ for any fixed i and thus $\frac{F(X_0,\ldots,X_n)}{X_i^n} = 0$, so $F(X_0, \ldots, X_n) = 0$.

From this we actually get that if global section $\{f_i\}_{i=0,\ldots,n}$ vanishes on some U_i, then it is 0, which aligns with intuition that a homogenous polynomial will be defined uniquely if we set one variable to be 1 (and remember its degree).

To show surjectivity, take $\{f_i\}_{i=0,\ldots,n}$ for which $\alpha_{i,j}^m(f_i) = f_j$ holds for every i, j. Then simply choose some i and let $F(X_0, \ldots X_n) = X_i^m \beta_i(f_i)$. So we found F that maps to $\{f_i\}_{i=0,\ldots,n}$.

Exercise 3.

1. We have a cover $\mathbb{P}_{\mathbb{Z}}^n = \cup_i U_i$, where $U_i = \mathrm{Spec}(\mathbb{Z}[X_{j/i}, j \neq i])$. We defined $\mathbb{P}_k^n$ to be simply the fibered product $\mathbb{P}_{\mathbb{Z}}^n \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k)$. We can use 1st exercise from sheet 6, to get a cover

$$\begin{aligned}
\mathbb{P}_k^n &= \bigcup_i U_i \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k) \\
&= \bigcup_i \mathrm{Spec}(\mathbb{Z}[X_{j/i}, j \neq i]) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k) \\
&= \bigcup_i \mathrm{Spec}(\mathbb{Z}[X_{j/i}, j \neq i] \otimes_{\mathbb{Z}} k) \\
&= \bigcup_i \mathrm{Spec}(k[X_{j/i}, j \neq i]).
\end{aligned}$$

Define morphism $\mathbb{P}^n_k \to (\mathbb{P}^n_k(k))^{\text{sob}}$ on the cover.

Lemma 06N9 We have that for a space X and a covering $X = \bigcup_i X_i$, the space X is sober if and only if X_i is sober for every i.

We showed on sheet 3 that soberification of an $\mathbb{A}^n_k(k)$ is $\operatorname{Spec}(k[X_1, \ldots, X_n])$.

So we have $(\mathbb{P}^n_k(k))^{\text{sob}} = \bigcup_i (\mathbb{A}^n_k(k))^{\text{sob}} = \bigcup_i \operatorname{Spec}(k[X_1, \ldots, X_n])$.

Define morphism

$$\mathbb{P}^n_k = \bigcup_i \operatorname{Spec}(k[X_{j/i}, j \neq i]) \to (\mathbb{P}^n_k(k))^{\text{sob}} = \bigcup_i \operatorname{Spec}(k[X_1, \ldots, X_n])$$

with the obvious isomorphism for every i.

2. We defined $V(s) \subseteq \mathbb{P}^n_k$ locally on affine subschemes. Our definition assumed we have a line bundle $\mathcal{L}$ on $(X, \mathcal{O}_X)$.

   In our case $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_k}(d)$

   Locally on $U_i = \operatorname{Spec}(k[X_{j/i}, j \neq i])$ we have equality $\mathcal{O}_{\mathbb{P}^n_k}(d)|_{U_i} = \mathcal{O}_{U_i}$ by definition.

   We show that restrictions

   So we have $V(s)|_{U_i} = \operatorname{Spec}(k[X_{j/i}, j \neq i]/(g))$, where $g \in k[x_{0/i}, \ldots, x_{n/i}]$ is the unique polynomial with $\beta_i(g) = \frac{F(X_0, \ldots, X_n)}{X_i^m}$ as in exercise 2.

   On the other hand we had $V^+(f) \cap U_i = V(f(X)$

Exercise 4. We don't really want to do all the explicit calculations, so we only show what we think is maybe the main takeaway of this exercise.

For some polynomial $f \in \mathbb{R}[x, y]$ we have that

$$\begin{aligned}
&V(f) \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C}) \\
&\cong \operatorname{Spec}(\mathbb{R}[x, y]/(f)) \otimes_{Spec(\mathbb{R})} \operatorname{Spec}(\mathbb{C}) \\
&\cong \operatorname{Spec}(\mathbb{R}[x, y]/(f) \times_{\mathbb{R}} \mathbb{C}) \\
&\cong \operatorname{Spec}(\mathbb{C}[x, y]/(f)).
\end{aligned}$$

In the following, we take $f(x, y) := xy - 1$ and $g(x, y) := x^2 + y^2 - 1$. We know from the first sheet, that

$$\mathbb{C}[x, y]/(f) \cong \mathbb{C}[x, y]/(g),$$

but one can easily check that

$$\mathbb{R}[x, y]/(f) \ncong \mathbb{R}[x, y]/(g),$$

since the left side has strictly more units than the right side.

Therefore, this is an example showing that varieties, so in particular schemes being isomorphic is not stable under base change.