Algebraic geometry 1
Exercise sheet 8

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Exercise 1.

1. Let $0 \neq f \in I$ be a non-zero element. Since A is a unique factorization domain, we can write

$$f = up_1^{a_1} \dots p_r^{a_r},$$

where p_i are pairwise nosn-associated primes. Now,

$$I_{(p_i)} = I[(I \backslash (p_i)^{-1})] = (p_i^{k_i})$$

for some $k_i \leq a_i$. Since I is finite locally free,

$$I = (\prod_i p_i^{k_i}).$$

2.

Exercise 2. Note that for a unique factorization domain A we get by Gauss that also $A[x_1, \dots, x_n]$ is a unique factorization domain. This means that by construction of $\mathbb{P}_A^n$ its local rings are UFD's. Using stacks project, we infer that $\mathrm{Pic}(\mathbb{P}_A^n) \cong \mathrm{CL}(\mathbb{P}_A^n) = \mathbb{Z}$.

We now want to give a concrete argument using the given map.

Note that by definion $\mathcal{O}_A^n(0)$ is just the structure sheaf and since maps of groups send 1 to 1, we found the neutral element of this group. One can also check locally that

$$O_{\mathbb{P}_A^n}(m) \otimes_{O_{\mathcal{P}_A^n}} O_{\mathcal{P}_A^n}(n) = O_{\mathcal{P}_A^n}(m+n).$$

This also proves that the given map maps to $\mathrm{Pic}(\mathcal{P}_A^n)$.

It is also quite clear by definition that for $m \neq n$ we have

$$O_{\mathcal{P}_A^n}(m) \ncong O_{\mathcal{P}_A^n}(n). \tag{1}$$

It remains to show surjectivity of this map.

Exercise 3.

1. In exercise 2 we showed that all invertible quasicoherent sheaves on $\mathbb{P}^n_k$ are isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$. So we have to show $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is an invertible sheaf.

   Since invertible $\mathcal{O}_{\mathbb{P}^n_k}$-modules are same as line bundles, we have to show that locally $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^m_k}$.

   By definition $f^*\mathcal{O}_{\mathbb{P}^m_k}(1) = f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^m_k}} \mathcal{O}_{\mathbb{P}^n_k}$. Pick some $x \in \mathbb{P}^n_k$. Pick small enough affine neighborhood $f(x) \in U \subseteq \mathbb{P}^m_k$ such that $\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^m_k}$ on U. Now pick neighborhood $x \in W \subseteq \mathbb{P}^m_k$ such that $f(W) \subseteq U$.

   Then

$$\begin{aligned} f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)(W) &= \operatorname{colim}_{f(W)\subseteq V} \mathcal{O}_{\mathbb{P}^m_k}(1)(V) \\ &= \operatorname{colim}_{f(W)\subseteq V\subseteq U} \mathcal{O}_{\mathbb{P}^m_k}(1)(V) \\ &\cong \operatorname{colim}_{f(W)\subseteq V\subseteq U} \mathcal{O}_{\mathbb{P}^m_k}(V) \\ &\cong f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(W). \end{aligned}$$

   So locally $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)$ is isomorphic to $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}$, so $f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^m_k}} \mathcal{O}_{\mathbb{P}^n_k}$ is locally isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}$, which proves that $f^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is an invertible $\mathcal{O}_{\mathbb{P}^n_k}$-module and thus isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$.

2. At first it was not completely clear to us what the map $f^* \colon \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1)) \to \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ is. So we assumed it is the following:

   For a global section $s \in \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1))$ we first map it with the restriction

$$\Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1)) \to \Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)).$$

   Denote its image with s'. By definition we have

$$\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1)) = \Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)) \otimes_{\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k})} \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})$$

   So include s' into $\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1))$ as $s' \otimes 1$. By part 1 we have an isomorphism $\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1)) \cong \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$. We map $s' \otimes 1$ with this isomorphism to obtain $f^*(s)$.

   The polynomials $y_0, \dots, y_n$ generate $\Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(1))$, which is isomorphic to the module of homogenous polynomials of degree 1. So their restrictions generate $\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1))$. Their images in the tensor product

$$\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)) \otimes_{\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k})} \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})$$

then also stay generators. And finally isomorphism $\Gamma(\mathbb{P}^n_k, f^*\mathcal{O}_{\mathbb{P}^m_k}(1)) \cong \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ also preserves generating set.

So $g_i = f^*(y_i) \in \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ are generators.

If $d \geq 1$, then g_i always vanish at $0 \in \mathbb{A}^{n+1}_k$.

Take some $(a_0, \ldots, a_n) \in V(g_0, \ldots, g_m) \subseteq \mathbb{A}^{n+1}_k$. If $a_i \neq 0$ for some i, then the line going through $(a_0, \ldots, a_n)$ and 0 would lie in $V(g_0, \ldots, g_m)$. Then $(g_0, \ldots, g_m)$ would be contained in the set of equations parametrizing this line. Therefore it wouldn't be generating the whole module.

3. If $m < n$, then $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$, which can be identified as a k-module of homogenous polynomials of degree d, cannot be generated by m elements. It is a vector space of dimension $\binom{n+d}{n}$ and $\binom{n+d}{n} > m$ for $d > 0$. Therefore $d = 0$.

   Now we show that f must be constant. Suppose $f(\mathbb{P}^n_k)$ has two points. Then we can separate these two point with two independent polynomials $s, t \in k[y_0, \ldots, y_m]_1$. Then $s \otimes 1$ and $t \otimes 1$ are independent elements of $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})$-module

$$\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k}(1)) \otimes_{\Gamma(\mathbb{P}^n_k, f^{-1}\mathcal{O}_{\mathbb{P}^m_k})} \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}).$$

   But $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(0))$ is the trivial line bundle, so it cannot contain two elements which are independent over global sections of line bundle itself.

Exercise 4.

1. Let $U_i = \mathrm{Spec}(A_i)$.

   Take a point $x \in U_1 \cap U_2$.

   Take a principal open $x \in D(f) \subseteq U_1$ ($f \in U_1$). Then find a smaller principal open $x \in D(g) \subseteq D(f) \subseteq U_2$ ($g \in U_2$).

   Now we show that $D(g)$ is also a principal open in U_1.

   Since $D(f) \subseteq U_2$ open, we have a map $\mathcal{O}(U_2) \to \mathcal{O}(D(f))$, which induces $A_2 \to (A_1)_f$. Denote by $g' = g|_{\mathrm{Spec}((A_1)_f)}$ the image of g under this map. Since $g' \in (A_1)_f$, we can write it as $g' = \frac{h}{f^n}$. Then $D(g) = D(g) \cap D(f) = D(g') \cap D(f) = D(h) \cap D(f) = D(hf)$, where $h, f \in A_1$. This shows that $D(g)$ is also principal open in U_1.

2. We have to show that the property of being of finite presentation is a local property and that f as defined above is locally of finite presentation.

   Let $\mathrm{Spec}(B) \subseteq X$ and $\mathrm{Spec}(A) \subseteq S$ open affines. Pick a point $x \in \mathrm{Spec}(B)$. Then $x \in \mathrm{Spec}(B) \cap \mathrm{Spec}(B_i)$ for some i. Pick some neighborhood $x \in U \subseteq \mathrm{Spec}(B) \cap \mathrm{Spec}(B_i)$ such that U is principal open in $\mathrm{Spec}(B)$ and in $\mathrm{Spec}(B_i)$.

Now take a neighborhood $f(x) \in V \subseteq f(U)$ so that V is principal open in $\mathrm{Spec}(A)$ and in $\mathrm{Spec}(A_i)$. Now take another smaller neighborhood $x \in U' \subseteq f^{-1}(V)$ such that U' is principal open in $\mathrm{Spec}(B)$ and in $\mathrm{Spec}(B_i)$.

So we have $U' \to V$, where both U' and V are principal opens of $\mathrm{Spec}(B_i)$ and $\mathrm{Spec}(A_i)$ respectively. Since $A_i \to B_i$ is of finite presentation, then localizations $(A_i)_f \to (B_i)_g$ (for some $f \in A_i$ and $g \in B_i$) are as well.

So for every point $x \in \mathrm{Spec}(B)$ we can find a principal open neighborhood in $x \in D(f_x)$ and a principal open neighborhood $f(x) \in D(g_x)$ such that $A_{g_x} \to B_{f_x}$.

Since $\mathrm{Spec}(B)$ is quasi-compact, we have $\mathrm{Spec}(B) = D(f_1) \cup \cdots \cup D(f_n)$. Denote $g_1, \ldots, g_n \in A$ be the respective elements in A.

We have composition $\mathrm{Spec}(B_{f_i}) \to \mathrm{Spec}(A_{g_i}) \hookrightarrow \mathrm{Spec}(A)$, which induces a map of rings $A \to A_{g_i} \to B_{f_i}$. Since $A_{g_i} \cong A[X]/(Xg_i - 1)$ and $A_{g_i} \to B_{f_i}$ are of finite presentation by assumption, and being of finite presentation is stable under compositions, we have that $A \to B_{f_i}$ are of finite presentation for every i.

Now its just commutative algebra to show that $A \to B$ is of finite presentation as well, so I hope its okay to assume this part. Otherwise we could just rewrite something like Lemma 00EP.