Algebraic geometry 1
Exercise sheet 11

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15. Januar 2024

Exercise 3.

1. Since k is algebraically closed, the only irreducible polynomials $f \in k[x, y]$ are of degree 1.

   Hence, we can write

   $$f_r = l_1 \dots l_r,$$

   where $l_i \in k[x, y]$ is of degree 1. From the assumption that f_r is homogenous it follows that the l_i are homogenous.

   Therefore, we can write

   $$Z = V(f_r) = V(l_1 \dots l_r) = \cup_i V(l_i)$$

   and since $V(l_i)$ is a line through the origin, Z can be written as the finite union of lines through the origin.

2. We first want to prove that $\dim(\mathcal{O}_{X,(x,y)}) = 1$ for all r. The prime ideals p in this ring fulfil $(f) \subset p \subset (x, y)$. Remember that we can write down these prime ideals explicitely as in "What do primes of k[x,y] look like". From this the claim follows.

   We know that $\dim_k(m_{\mathcal{O}_{X,(x,y)}}/m^2_{\mathcal{O}_{X,(x,y)}})$ is the number of generators of $m_{\mathcal{O}_{X,(x,y)}}$.

   Now if $r = 1$, then we can write $f = g(x, y)x + h(x, y)y$ and w.l.o.g. we have $g(0, 0) = 1$, meaning that it is invertible (after localizing). Therefore $f = x + h(x, y)y$, so $y \mid x$ meaning $(x, y) = (y)$ On the other hand, if $r > 1$, then $x \nmid y$ and $y \nmid x$ meaning that m is no principal ideal showing that X is singular at zero in this case. (This can be seen by writing f as $f = x^2 h_1(x, y) + xyh_2(x, y) + y^2 h_3(x, y)$).

3. By part two of this exercise, all the schemes have a singular point at the origin. I don't know why they do not have singular points anywhere else.

Exercise 4. We will show that the restriction map on global sections is an isomorphism, i.e. that

$$\Gamma(X, \mathcal{O}_X) \cong \Gamma(U, \mathcal{O}_X)$$

This immediately implies the claim of the exercise by definition of vector bundle (if rings are isomorphic then so is their finite sum).

By definition, $\operatorname{codim}(Z) \geq 2$. Take $Y \subset X$ an irreducible component of codimension 1. By construction, Y and U intersect nontrivially (either $Z \cap Y = \emptyset$ or $Z \subsetneq Y$). In particular, U contains the generic point μ of Y. This means that $\Gamma(U, \mathcal{O}_X) \subset \mathcal{O}_{X,\mu}$.

The hint tells us that A is the intersection of all localizations of A at prime ideals p of height 1. Those prime ideals of height 1 correspond to irreducible components of codimension 1. Hence, we have just shown that

$$\Gamma(U, \mathcal{O}_X) \subset A = \Gamma(X, \mathcal{O}_X).$$

The other inclusion is immediate.