Algebraic geometry 1
Exercise sheet 5

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Exercise 1.

1. Define

$$X := U_1 \coprod U_2/\sim,$$

where $x \sim y$ if $x = \varphi(y)$ and for $i \in \{1, 2\}$

$$\pi_i : U_i \to X$$
$$x \mapsto \bar{x}.$$

We can now give X the structure of a topological space by defining a subset $U \subset X$ to be open if $\pi^{-1}(U) \in$ are open in U_1.
Notice, that π_i are homeomorphic onto open subsets of X. This will become important later. Next we want to define a structure sheaf on X that behaves well with restricting to U_i.
For $U \subset X$ open, let

$$\mathcal{O}_X(U) := ker(\mathcal{O}_{U_1}(\pi^{-1}(U)) \oplus \mathcal{O}_{U_2}(\pi^{-1}(U)) \to \mathcal{O}_{U_1}(\pi^{-1}(U) \cap U_1)$$
$$(x, y) \mapsto x_{|\pi^{-1}(U) \cap V_1} - \varphi^\sharp(\pi_2^{-1}(U) \cap V_2)(y_{|\pi_2^{-1}(U) \cap V_2})),$$

where the substraction in the above term comes from the group structure of $\mathcal{O}_{U_1}(\pi^{-1}(U) \cap V_1)$. This is of course a group again, as the kernel of a ring map.
We conclude, that $(X, \mathcal{O}_X)$ is a scheme, because $X = \pi_1(U_1) \cup \pi_2(U_2)$ can be covered by affine schemes using the cover from U_1 and U_2 and since by construction of the structure sheaf $\mathcal{O}_{X|U_1} = \mathcal{O}_{x_i}$. Here we finally used, as promised, that π_i are homeomorphisms onto open subsets of X.

Exercise 3.

1. Remember, that the contravariant functor $A \mapsto (Spec(A), \mathcal{O}_{Spec(A)})$ is an equivalence of categories, meaning that $\mathcal{O}^{op}$ is equivalent to $Rings$. Then the affine case follows from the Yoneda embedding. How to show general case?