Disclaimer This is a copy TEXed from memory rather than the original exam. The exact phrasing does not match the real exam, and the exercises may contain errors.

Remark According to a tutor this exam is similar to the 16/17 Scholze exams, which are not available at the Fachschaft.

Exercise 1

1. Define what it means for a morphism of schemes to be: of finite type, separated, proper.
2. Let $f : X \to Y$ and $g : Y \to S$ morphisms of schemes such that $g \circ f$ is separated. Show that f is also a separated morphism.

   Note: Obviously you are not allowed to just quote this result from the lecture

Exercise 2

1. Define what it means for an $\mathcal{O}_X$-module on a scheme X to be quasi-coherent and what it means for an $\mathcal{O}_X$-module on a noetherian scheme X to be coherent.
2. Let $f : X \to Y$ be a finite morphism of noetherian schemes and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Show that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module.
3. Show by a counterexample that (2) is no longer true if "finite morphism" is replaced by "morphism of finite type".

Exercise 3 Let k be a field and $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ defined by (in homogeneous coordinates)

$$[x : y] \to [x^2 : y^2]$$

1. Show that $f_*\mathcal{O}_{\mathbb{P}^1_k}$ is a locally free of rank 2.
2. Show that $f_*\mathcal{O}_{\mathbb{P}^1_k} \cong \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}(-1)$.

Exercise 4 Let k be an algebraically closed field with characteristic neither 2 nor 3. Define $C = \operatorname{Spec} k[x, y]/(f)$ with $f(x, y) = xy^2 - x - y$.

1. Show that C is smooth at all its k-rational points.

   Remark: The original exam may have asked to show that C is normal instead.

2. Show that the vanishing locus of $xy^2 - xz^2 - yz^2 \in \Gamma(\mathcal{O}_{\mathbb{P}^2_k}(3), \mathbb{P}^2_k)$ is the schematic closure $\overline{C}$ of the image of C under the immersion

$$\mathbb{A}^2_k \hookrightarrow \mathbb{P}^2_k, (x, y) \to [x : y : 1]$$

3. Compute the normal compactification $\overline{C}_{norm}$ of C as the normalization of $\overline{C}$. Determine $|\overline{C}_{norm} \backslash \overline{C}|$.

Exercise 5 Let k be an algebraically closed field with characteristic neither 2 nor 3. Define $C = \operatorname{Spec} k[x, y, t]/(f)$ with $f(x, y, t) = y^2 - x^4 - (t^2 - 3t + 2)x$. Let $h : C \to \operatorname{Spec} k[t]$ given by $(x, y, t) \to t$.

1. Show that for $a \in \{1, 2\}$ the fiber $h^{-1}((t - a))$ is reducible and has its only non-smooth k-rational point at $(0, 0)$.

2. Show that for $a \in k \backslash \{1, 2\}$ the fiber $h^{-1}((t - a))$ is normal.