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SS 2024

Algebraic Geometry II

7. Exercise sheet

Exercise 1 (4 points):

Let X be a scheme. For a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ we define $\underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{L})$ as the sheaf of sets

$$V \subset X \longmapsto \mathrm{Isom}(\mathcal{O}_V, \mathcal{L}_{|V})$$

1) Prove that $\underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{L})$ is a $\mathcal{O}_X^\times$-torsor, where $\mathcal{O}_X^\times \subset \mathcal{O}_X$ denotes the sheaf of units.
2) Prove that sending $\mathcal{L}$ to $\underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{L})$ defines a bijection $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$.

Exercise 2 (4 points):

Let X be a topological space and let $1 \to N \to G \to Q \to 1$ be a short exact sequence of sheaves of groups on X (i.e., for every $x \in X$ the morphism $G_x \to Q_x$ is surjective with kernel N_x). Prove that there is a natural short exact sequence of pointed sets

$$1 \to N(X) \to G(X) \to Q(X) \to H^1(X, N) \to H^1(X, G) \to H^1(X, Q),$$

i.e., the image of each morphism is exactly the preimage of the distinguished point under the next morphism.

Exercise 3 (4 points):

Let k be a field and let X be a smooth curve over k. Let $x_1, \ldots, x_n \in X$ be closed points and set $U := X \setminus \{x_1, \ldots, x_n\}$.
1) Show that there is a short exact sequence $\mathbb{Z}^n \to \mathrm{Pic}(X) \to \mathrm{Pic}(U) \to 0$.
2) If $\mathrm{Pic}(X)$ is an infinitely generated abelian group, show that a product $\prod_{i \in I} \mathcal{L}_i$ of $\mathcal{O}_X^\times$-torsors is in general not a $\prod_{i \in I} \mathcal{O}_X^\times$-torsor.
Hint/Remark: Use the exact sequence $k(U)^\times \to \mathrm{Div}_U^1 \to \mathrm{Pic}(U) \to 0$ and its analog for X. If k is algebraically closed and X separated, then $\mathrm{Pic}(X)$ is infinitely generated unless X is isomorphic to an open subset of $\mathbb{P}_k^1$.

Exercise 4 (4 points):

Let k be a field and let X be a smooth, separated curve over k.
1) Assume that $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, which is torsion, i.e., for $U = \mathrm{Spec}(A) \subseteq X$ affine and open, $\mathcal{M}(U)$ is a torsion A-module. Show that $H^1(X, \mathcal{M}) = 0$.
2) Assume that $X = \mathbb{P}_k^1$ and that $x, y \in X$ are two different closed points. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be the ideal sheaf of the (reduced) closed subscheme $\{x, y\} \subseteq X$. Show that $H^1(X, \mathcal{I}) \neq 0$.
Hint: For 1) pick some open affine U containing the support of $\mathcal{M}$ and argue that it is sufficient to split a short exact sequence $0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{O}_X \to 0$ over U.

To be handed in on: Thursday, 06.06.2024 (during the lecture or via eCampus).