Introduction to Schemes

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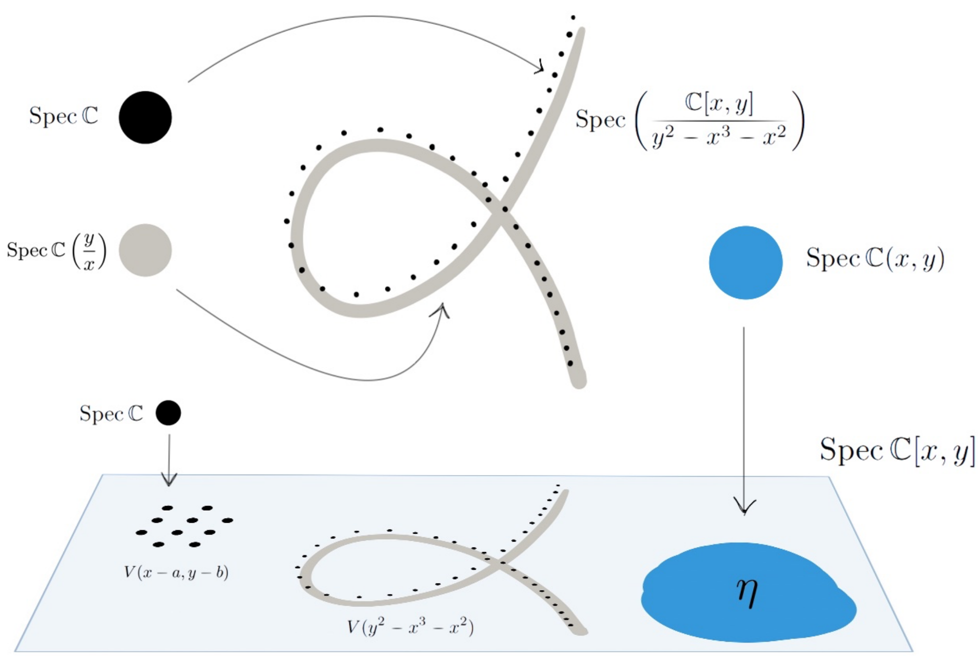

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Contents

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Notation and conventions

All rings are commutative with a unit, denoted 1.
Ring homomorphisms are required to send 1 to 1.
$\subset$ means 'is subset of', i.e., the same thing as $\subseteq$.
The zero ring is not an integral domain (and therefore not a field).
If A is an integral domain, we denote its fraction field by $K(A)$.

Chapter 1
Varieties

We begin with a recollection of the theory of varieties, which will serve as the motivating examples in the theory of schemes. We will contend ourselves to presenting the basic definitions and fundamental properties of the two most important classes of varieties; the affine varieties and the projective varieties. We will later develop the theory of varieties further, as we progress in the book.

Varieties are defined over a fixed algebraically closed ground field, denoted k. It is useful to keep some specific fields in mind, e.g., the field of complex numbers $\mathbb{C}$, the field of algebraic numbers $\bar{\mathbb{Q}}$ or perhaps the algebraic closure $\overline{\mathbb{F}_p}$ of a finite field.

For reasons that will become clear when the 'functor of points' is introduced, we shall write $\mathbb{A}^n(k)$ for the space k^n, and refer to it as the *affine n-space*. The name change from k^n to $\mathbb{A}^n(k)$ is meant to underline that there is more to $\mathbb{A}^n(k)$ than just the set of elements – it will soon be equipped with a topology – and ultimately, it will be a scheme, denoted $\mathbb{A}^n$.

Affine n-space

1.1 Algebraic sets

We begin by defining the *algebraic sets*. These are the subsets of the affine space $\mathbb{A}^n(k)$ whose points are the common solutions of a set of polynomial equations:

algebraic sets

Definition 1.1 (Algebraic sets) *If S is a subset of the polynomial ring $k[x_1, \ldots, x_n]$, we define*

$$Z(S) = \{\, x \in \mathbb{A}^n(k) \mid f(x) = 0 \text{ for all } f \in S \,\},$$

and the algebraic sets *are the subsets of $\mathbb{A}^n(k)$ of this form.*

Note that if $f_1, \ldots, f_r$ are elements of S, then any expression $\sum_{1 \leqslant i \leqslant r} b_i f_i$, with the b_i's being polynomials, also vanishes at points of $Z(S)$. This means that the zero set of the *ideal* $\mathfrak{a}$ generated by S is the same as $Z(S)$, that is, $Z(S) = Z(\mathfrak{a})$. We will therefore almost exclusively work with ideals and tacitly replace a set of polynomials by the ideal it generates. Hilbert's basis theorem tells us that any ideal in $k[x_1, \ldots, x_n]$ is finitely generated, so that an algebraic subset is always described as the set of common zeros of *finitely* many polynomials. Note the two special cases $Z(1) = \varnothing$ and $Z(0) = \mathbb{A}^n(k)$.

The more constraints imposed, the smaller the solution set will be, so if $\mathfrak{b} \subseteq \mathfrak{a}$ are two ideals, one has $Z(\mathfrak{a}) \subseteq Z(\mathfrak{b})$. Sending $\mathfrak{a}$ to $Z(\mathfrak{a})$ is thus an inclusion-reversing map from the partially ordered set of ideals in $k[x_1,\dots,x_n]$ to the partially ordered set of subsets of $\mathbb{A}^n(k)$.

This map is not injective: different ideals can define the same algebraic set*. For instance, the ideals (t) and (t^2) in $k[t]$, both have the origin in the affine line $\mathbb{A}^1(k)$ as their zero set. More generally, all powers $\mathfrak{a}^n$ of an ideal $\mathfrak{a}$ have the same zeros as $\mathfrak{a}$: indeed, as $\mathfrak{a}^n \subseteq \mathfrak{a}$, it holds that $Z(\mathfrak{a}) \subseteq Z(\mathfrak{a}^n)$, and the other inclusion holds as well because $f^n \in \mathfrak{a}^n$ whenever $f \in \mathfrak{a}$. The *radical* $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$ is defined as the ideal

*It is also not surjective: a polynomial only has finitely many zeros, so any proper infinite subset of $\mathbb{A}^1(k)$ is not algebraic.

$$\sqrt{\mathfrak{a}} = \{\, f \mid f^r \in \mathfrak{a} \text{ for some } r > 0 \,\},$$

and the argument above yields that $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$; indeed, the radical is finitely generated, so a power is contained in $\mathfrak{a}$. Two ideals with the same radical therefore have coinciding zero sets, and Hilbert's Nullstellensatz, which we shortly shall see, tells us that the converse is true as well.

The product of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is generated by products $f \cdot g$ with $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$, and hence $Z(\mathfrak{a}\cdot\mathfrak{b}) = Z(\mathfrak{a}) \cap Z(\mathfrak{b})$. For the sum $\mathfrak{a}+\mathfrak{b}$ one easily checks that $Z(\mathfrak{a}+\mathfrak{b}) = Z(\mathfrak{a}) \cap Z(\mathfrak{b})$, which in fact holds true for sums of any cardinality: $Z(\sum_{i\in I}\mathfrak{a}_i) = \bigcap_{i\in I} Z(\mathfrak{a}_i)$.

PROPOSITION 1.2 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals and $\{\mathfrak{a}_i\}_{i\in I}$ a family of ideals in the polynomial ring $k[x_1,\dots,x_n]$.*

i) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $Z(\mathfrak{b}) \subseteq Z(\mathfrak{a})$;*
ii) $Z(\sum_{i\in I}\mathfrak{a}_i) = \bigcap_{i\in I} Z(\mathfrak{a}_i)$
iii) $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a}\cap\mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$;
iv) $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$.

The identities *ii)* and *iii)* tell us that an arbitrary intersection of algebraic sets is algebraic as is the union of two. And as $\mathbb{A}^n(k) = Z(0)$ and $\emptyset = Z(1)$, the algebraic sets constitute the closed sets of a topology on the affine space $\mathbb{A}^n(k)$. It is called the *Zariski topology*.

The Zariski topology

If $X \subset \mathbb{A}^n(k)$ is any subset, there is an induced Zariski topology on X, by declaring that the open sets of X are of the form $X \cap U$, where U is an open set in $\mathbb{A}^n(k)$ (i.e., $U = \mathbb{A}^n(k) - Z(I)$ for some ideal I)).

EXAMPLE 1.3 Let us consider the Zariski topology on affine line $\mathbb{A}^1(k)$. Each non-zero and proper ideal $\mathfrak{a}$ in the polynomial ring $k[t]$ is generated by one element; say $\mathfrak{a} = (f)$, and as the ground field k is algebraically closed, f factors as a product of linear terms: $f = (t-a_1)^{n_1}\dots(t-a_r)^{n_r}$ with $a_i \in k$. Then $Z(f) = \{a_1,\dots,a_r\}$, and apart from the entire line $\mathbb{A}^1(k)$ and the empty set, the closed sets are just the finite sets.

In other words, the Zariski topology on $\mathbb{A}^1(k)$ is the *finite complement topology*, whose proper open sets are exactly those whose complement is finite. Thus the Zariski topology behaves very differently than the usual topology on $\mathbb{A}(\mathbb{C})$, there are much fewer open sets. ☆

The finite complement topology

There is a partial converse to the construction of the zero locus $Z(\mathfrak{a})$ of an ideal. One may consider the set of polynomials vanishing on a given subset of $\mathbb{A}^n(k)$. This is obviously an ideal:

Definition 1.4 *For a subset X of $\mathbb{A}^n(k)$ the ideal $I(X)$ consists of polynomials in $k[x_1, \ldots, x_n]$ that vanish along X; that is,*

$$I(X) = \{ f \in k[x_1, \ldots, x_n] \mid f(t) = 0 \text{ for all } x \in X \}.$$

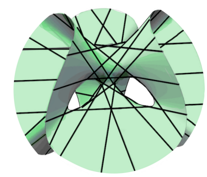
This gives an inclusion-reversing map from the set of subsets of $\mathbb{A}^n(k)$ to the set of ideals in the polynomial ring $k[x_1, \ldots, x_n]$.

Examples

(1.5) Linear subspaces of $\mathbb{A}^n(k)$ are given by linear equations and are algebraic sets.
(1.6) Another classical example is the conic sections: they are the closed algebraic sets in the affine plane $\mathbb{A}^2(k)$ given by quadratic equations of the form $ax^2 + bx + cy^2 = 1$.
(1.7) A more interesting example is the so-called *Clebsch diagonal cubic*; a surface in $\mathbb{A}^3(\mathbb{C})$ with equation

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3.$$

The real points of the surface, i.e., the points in $\mathbb{A}^3(\mathbb{R})$ satisfying the equation, is depicted in the margin. This surface contains 27 lines, all defined over the real numbers.


The Clebsch diagonal cubic

(1.8) Algebraic sets can show a high degree of complexity. The Barth sextic in $\mathbb{A}^2(\mathbb{C})$ is the zero locus of the degree six polynomial

$$4(\phi^2x^2 - y^2)(\phi^2y^2 - z^2)(\phi^2z^2 - x^2) - (1 + 2\phi)(x^2 + y^2 + z^2 - 1)^2$$

where $\phi = (1 + \sqrt{5})/2$. It was found by Wolf Barth in 1964, and has 65 singular points, which is the maximal number for a sextic surface. A plot of the real points is depicted in the margin.

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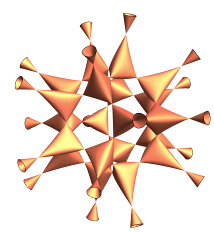
The Barth Sextic with 65 singularities

Hilbert's Nullstellensatz

For algebraic subset X, we just saw that $Z(I(X)) = X$. Hilbert's Nullstellensatz is about the composition of I and Z the other way around, namely about $I(Z(\mathfrak{a}))$. Polynomials in the radical $\sqrt{\mathfrak{a}}$ vanish along $Z(\mathfrak{a})$ (if a power of f vanishes on a set, f vanishes there as well) and therefore $\sqrt{\mathfrak{a}} \subseteq I(Z(\mathfrak{a}))$, and the Nullstellensatz tells us that this inclusion is an equality:

Theorem 1.9 (Hilbert's Nullstellensatz) *Assume that k is an algebraically closed field, and that $\mathfrak{a}$ is an ideal in $k[x_1, \ldots, x_n]$. Then one has*

$$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

The Nullstellensatz has the following fundamental consequences.

Theorem 1.10 (Weak Nullstellensatz) *Let k be an algebraically closed field, and $\mathfrak{a}$ an ideal in the polynomial ring $k[x_1, \ldots, x_n]$.*

i) It holds that $Z(\mathfrak{a})$ is non-empty if and only if $\mathfrak{a}$ is not the unit ideal;

ii) The maximal ideals in $k[x_1, \ldots, x_n]$ are precisely those of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $(a_1, \ldots, a_n) \in \mathbb{A}(k)$.

Proof: *i)*: It is clear that $Z(1) = \emptyset$. If $Z(\mathfrak{a}) = \emptyset$, requiring a polynomial to vanish along $Z(\mathfrak{a})$ imposes no constraint, so $1 \in I(Z(\mathfrak{a}))$ and the Nullstellensatz gives that $1 \in \mathfrak{a}$.

ii): The ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal being the kernel of the evaluation map $k[x_{1n}, \ldots, x] \to k$ which sends f to its value at $(a_1, \ldots, a_n)$. If $\mathfrak{m}$ is a maximal ideal, the Nullstellensatz yields that $Z(\mathfrak{m}) \neq \emptyset$. So take a point $(a_1, \ldots, a_n)$ in $Z(\mathfrak{m})$. Then $(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}$, and as $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal, we must have equality. ❏

Example 1.11 Note that the theorem fails immediately if the ground field is not algebraically closed: The simplest example of an ideal in a polynomial ring with empty zero locus is the ideal $(x^2 + 1)$ in $\mathbb{R}[x]$. ☆

Exercise 1.1 In any ring the radical of an ideal $\mathfrak{a}$ equals the intersection of the prime ideals containing the ideal. Show, using the Nullstellensatz, that in the polynomial ring $k[x_1, \ldots, x_n]$ the radical $\sqrt{\mathfrak{a}}$ equals the intersection of all the *maximal* ideals containing $\mathfrak{a}$. ★

As everywhere in this book, a ring is commutative and has a unit element

Exercise 1.2 Show that the Zariski topology on $\mathbb{A}^2(k)$ is not the product topology on $\mathbb{A}^2(k) = \mathbb{A}^1(k) \times \mathbb{A}^1(k)$. ★

Irreducible sets and varieties

Irreducibility is a notion from point set topology which plays a fundamental role in algebraic geometry. A topological space is *irreducible* if it is not the union of two proper closed subsets; that is, if Y and Y' are closed subsets such that $X = Y \cup Y'$, then either $X = Y$ or $X = Y'$. For an algebraic set $X = Z(\mathfrak{a}) \subseteq \mathbb{A}^n(k)$ being irreducible means that the radical $\sqrt{a}$ of $\mathfrak{a}$ is prime:

Irreducible topological space

Proposition 1.12 *An algebraic set $Z(\mathfrak{a}) \subseteq \mathbb{A}^n(k)$ is irreducible if and only if $\sqrt{\mathfrak{a}}$ is prime.*

Proof: It clearly suffices to treat the case when $\mathfrak{a}$ is radical. Assume that $Z(\mathfrak{a}) = Z(\mathfrak{b}) \cup Z(\mathfrak{b}')$ with radical ideals $\mathfrak{b}$ and $\mathfrak{b}'$ both containing $\mathfrak{a}$. By *iii)* of Proposition 1.2, it holds that $Z(\mathfrak{b}) \cup Z(\mathfrak{b}') = Z(\mathfrak{b} \cap \mathfrak{b}')$. Then by the Nullstellensatz, we get that $\mathfrak{b} \cap \mathfrak{b}' = \mathfrak{a}$. So if $\mathfrak{a}$ is prime, then either $\mathfrak{b} \subseteq \mathfrak{a}$ or $\mathfrak{b}' \subseteq \mathfrak{a}$. That is, either $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{b}' = \mathfrak{a}$.

The implication the other way is easier: if $\mathfrak{a}$ is not prime, it is the intersection of several different prime ideals, so just divide them into two groups and let $\mathfrak{b}$ and $\mathfrak{b}'$ be the corresponding intersections. ❏

Let us give the following preliminary definition of a variety:

DEFINITION 1.13 (TRADITIONAL AFFINE VARIETIES) *A traditional affine variety, is a closed irreducible subset $X \subseteq \mathbb{A}^n(k)$.*

The mappings $X \mapsto I(X)$ and $I \mapsto Z(I)$ give mutually inverse one-to-one inclusion reversing correspondences between the objects in columns of the following table, where $A = k[x_1, \dots, x_n]$:

ALGEBRA	GEOMETRY
maximal ideals of A	points of $\mathbb{A}^n(k)$
prime ideals of A	irreducible closed subsets of $\mathbb{A}^n(k)$
radical ideals of A	closed subsets of $\mathbb{A}^n(k)$
maximal ideals of $A/\mathfrak{a}$	points of $Z(\mathfrak{a})$

EXERCISE 1.3 Show that any open non-empty set in an irreducible topological space is dense. ★

Regular functions and polynomial maps

Consider an algebraic subset $X \subseteq \mathbb{A}^n(k)$. The regular functions on X, or polynomial functions, are simply the restrictions of polynomials in $k[x_1, \dots, x_n]$ to X. Note that two polynomials f and g restrict to the same function on X precisely when there difference $f - g$ vanishes on X, so the set of polynomial functions on X is equal to the quotient ring

$$A(X) = k[x_1, \dots, x_n]/I(X).$$

This ring is called the *affine coordinate ring*, and carries essentially all information about the set X. It has no nilpotent elements since $I(X)$ is a radical ideal, and it is an integral domain if and only if X is irreducible.

Affine coordinate ring of an algebraic set

The correspondence between prime ideals and irreducible closed subsets, shows that the Krull dimension of $A(X)$ equals the combinatorial dimension of the topological space X; that is, the length of the longest chain $X_0 \subset X_1 \subset \cdots \subset X_r = X$ of irreducible closed set in X.

EXAMPLE 1.14 The square root $\sqrt{x}$ is not per se a regular function on $\mathbb{A}^1(\mathbb{C})$, but it defines a regular function on the parabola

$$X = Z(x - y^2)$$

in $\mathbb{A}^2(\mathbb{C})$. Indeed, here the function is simply given by $(x, y) \mapsto y$. Note that the coordinate ring of X is given by

This is related to the notion of a 'Riemann surface', in complex analysis.

$$A(X) = k[x, y]/(x - y^2) \simeq k[y].$$

This is an integral domain of Krull dimension 1, as we expect. ☆

The concept of a ‘polynomial function’ can be extended to ‘polynomial maps’ between algebraic sets; they are maps that under composition carry polynomial functions to polynomial functions:

Definition 1.15 (Polynomial maps) *Let X and Y be two algebraic sets. A map $\phi\colon X \to Y$ is called* a polynomial map *if the composition $g \circ \phi$ is a polynomial function each time g is a polynomial function on Y.*

The composition $\psi \circ \phi$ of two polynomial maps is again a polynomial map, so the algebraic sets form a category with the polynomial maps as morphisms. We say that ϕ is an *isomorphism* when it has an inverse map that is also a polynomial map.

Isomorphism of algebraic sets

When $\phi : X \to Y$ is a polynomial map, and $g \in A(Y)$, the composition $g \circ \phi$ is again a polynomial map $X \to \mathbb{A}^1$. We denote this composition by $\phi^\sharp(g) = g \circ \phi$. This gives us a map

$$\begin{array}{rcl} \phi^\sharp : A(Y) & \to & A(X) \\ g & \mapsto & g \circ \phi \end{array}$$

The map $\phi^\sharp$ is a map of k-algebras: sums and products of polynomial functions on Y are computed pointwise, and constants obviously map to constants. It is a fundamental property of algebraic sets that all k-algebra homomorphisms are realised in his way:

Theorem 1.16 (Main theorem for affine algebraic sets) *Let X and Y be two algebraic sets. The map*

$$\mathrm{Hom}_{\mathsf{alg.sets}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{Alg}/\mathsf{k}}(A(Y), A(X))$$

that sends ϕ to $\phi^\sharp$, is a bijection from the set of polynomial maps to the set of k-algebra homomorphisms.

Proof: The map in the theorem is clearly injective: if for $p \in X$ is a point with $\phi(p) \neq \psi(p)$ there is a polynomial function g on Y with $g(\phi(p)) \neq g(\psi(p))$; that is, $\phi^\sharp(g) \neq \psi^\sharp(g)$.

We begin with treating the case that $Y = \mathbb{A}^m(k)$. In this case, giving a map $\phi\colon X \to Y$ amounts to giving m functions $f_1, \ldots, f_m$ on X, and the map $\phi(x) = (f_1(x), \ldots, f_m(x))$ they define, is a polynomial map precisely when the f_i’s are polynomial functions. Indeed, a polynomial functions on Y is just a polynomial in the coordinates, and composed with ϕ it becomes a polynomial in the f_i’s, which clearly is a polynomial function on X when the f_i’s are. Now, given a homomorphism $F\colon A(Y) = k[u_1, \ldots, u_m] \to A(X)$, we may use the images $f_i = F(u_i)$ to define ϕ as above, and then clearly $\phi^\sharp = F$, since by definition the two agree on the generators u_i.

In the general case, we assume that $Y \subseteq \mathbb{A}^m(k)$. Note that a map $\phi\colon X \to \mathbb{A}^m(k)$ takes values in Y precisely when $\phi^\sharp(g) = g \circ \phi = 0$ for all $g \in I(Y)$. So if $F\colon A(Y) \to A(X)$ is given, the composition $k[u_1, \ldots, u_m] \to A(Y) \to A(X)$ of F with the restriction map is a homomorphism that vanishes $I(Y)$, hence by the first case, it yields a map $X \to \mathbb{A}^m(k)$ that factors through Y. ❑

From a categorical angle, the theorem says that the category of algebraic sets is equivalent to the the category of finitely generated, reduced k-algebras (with the arrows reversed). The subcategory of varieties; that is, irreducible algebraic sets, is then equivalent to the category of integral domains finitely generated over k (with arrows reversed).

Examples

(1.17) Any linear map $\mathbb{A}^n(k) \to \mathbb{A}^m(k)$ is a polynomial map. Indeed, any such map is given by the multiplication by a matrix with entries in k.
(1.18) Consider the algebraic set $X = Z(y^2 - x^3)$ in $\mathbb{A}^2(k)$. The affine coordinate ring of X is given by

$$A(X) = k[x,y]/(y^2 - x^3)$$

This is an integral domain, because the polynomial $y^2 - x^3$ is irreducible.

Consider the polynomial map $\phi\colon \mathbb{A}^1(k) \to \mathbb{A}^2(k)$ given by $t \mapsto (t^2, t^3)$. The image of ϕ is contained in $X \subset \mathbb{A}^2(k)$, and, in fact, ϕ is a bijection between $\mathbb{A}^1(k)$ and X. (Note that $\phi(t) = (0,0)$ only for $t = 0$; and if $(x,y) \neq (0,0)$ lies in X, the assignment $t = y/x$ defines the inverse.)

However, ϕ is not an isomorphism. Indeed, we have $\phi^\sharp(x) = t^2$ and $\phi^\sharp(y) = t^3$. This means that the induced map

$$\phi^\sharp\colon k[x,y]/(y^2 - x^3) \to k[t]$$

has image $k[t^2, t^3]$, so it is not surjective.
(1.19) Note that the “same" X can be embedded into different $\mathbb{A}^n(k)$'s. For instance, the above X can be embedded in $\mathbb{A}^3(k)$ as the zero set $Z(y^2 - x^3, z)$ or $Z(y^2 - x^3, z - xy)$.
(1.20) (*The Frobenius map.*) In this example, we assume that k is of positive characteristic p. The map $k[t] \to k[t]$ given by $t \mapsto t^p$, is in fact a map of k-algebras. It corresponds to the polynomial map $\mathbb{A}^1(k) \to \mathbb{A}^1(k)$ given by $t \mapsto t^p$. This map is bijective, but not an isomorphism.

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Rational functions

Let X be an affine variety. Then $A(X)$ is an integral domain, so it has a fraction field, which we denote by $k(X)$. $k(X)$ is called the ‘fraction field' or ‘field of rational functions' on X. The elements of $k(X)$ can be interpreted as functions on open sets in X; indeed, a fraction $f = a/b$ yields a well defined function on the open set where b dos not vanish.

If $x \in X$ is a point, we say that $f \in k(X)$ is *regular* at x if it can be expressed as a fraction $f = a/b$ with $b(x) \neq 0$. Such a function will automatically also be regular in neighbourhood of x, indeed, it is regular in the complement of the closed set $Z(b)$. Such *Regular functions* sets are called *distinguished open sets* and the standard notion is $D(b)$; that is, $D(b) = \{ x \in X \mid b(x) \neq 0 \}$.

Distinguished open sets

It is easy to check that sums and products of rational functions regular at x, are again regular at x. Thus the functions regular at x form a subring of $k(X)$. We define the *local ring* of X at the point x, denoted $\mathcal{O}_{X,x}$, to be this ring. This is justified by the following:

PROPOSITION 1.21 *The set $\mathcal{O}_{X,x}$ of functions regular at x is a local ring whose maximal ideal $\mathfrak{m}_x$ consists of the functions vanishing at x.*

PROOF: This straightforward: The evaluation map $\mathcal{O}_{X,x} \to k$ has by definition $\mathfrak{m}_x$ as kernel and so is a maximal. If f does not vanish at x, is expressed as $f = a/b$ with both $a(x) \neq 0$ and $b(x) \neq 0$; hence $1/f = b/a$ is regular at x and belongs to $\mathcal{O}_{X,x}$. ❑

Note that a rational function a/b may be regular in larger set than the distinguished open set $D(b)$. The standard example is as follows:

EXAMPLE 1.22 Consider the variety $X \subseteq \mathbb{A}^4_k$ whose equation is $xy - zw = 0$. In the function field, $k(X)$ the equality $x/w = z/y$ holds true. Moreover, the open set $U = D(w) \cup D(y)$ is strictly larger than both $D(w)$ and $D(y)$. For instance, the linear subspace $y = z$ (that is, the set of points $(t, 0, 0, t)$ with $t \in k$) is contained in X. ☆

EXERCISE 1.4 a) Verify that $xy - zw$ is an irreducible polynomial.

b) Verify that x/w as a function on $X = V(xy - zw)$ is not defined along the locus $y = w = 0$. ★

PROPOSITION 1.23 *Let X be a variety. If a rational function $f \in k(X)$ is regular at all points of X, then $f \in A(X)$. In other words,*

$$A(X) = \bigcap_{x \in X} \mathcal{O}_{X,x}.$$

PROOF: Consider the ideal $\mathfrak{a}_f = \{ b \in A \mid bf \in A \}$. It has the property that a rational function f is regular at x if and only if $x \notin Z(\mathfrak{a}_f)$; indeed, $x \notin Z(\mathfrak{a}_f)$ if and only if some $b \in \mathfrak{a}_f$ does not vanish at x, which in turn is equivalent to f being on the form $f = a/b$ for some b with $b(x) \neq 0$. So when f is regular everywhere, it follows that $Z(\mathfrak{a}_f) = \emptyset$, and the Nullstellensatz yields that $1 \in \mathfrak{a}_f$; that is, $f \in A$. ❑

We shall need the following result later on

PROPOSITION 1.24 *Assume that $n \geqslant 2$. A regular function on the open set $\mathbb{A}^n \backslash \{0\}$ is a polynomial function.*

PROOF: Let f be such a rational function. An expression like $f = a/b$ with a and b polynomials, is not valid throughout $\mathbb{A}^n(k) \backslash \{0\}$, since the Hauptidealsatz tells us tat the denominator vanishes in a set larger than $\{0\}$. However, it could *a priori* be that different such expression patch together. So assume $a/b = a'/b'$ with a and b and a' and b' are without common factors. Then $ab' = a'b$, and it follows that $a = ca'$ and $b = cb'$ with c a scalar (as a and b are relatively prime). ❏

EXAMPLE 1.25 Let $X = Z(xy - 1) \subset \mathbb{A}^2(k)$ and consider the map

$$\phi : X \to \mathbb{A}^1(k) - 0$$

given by the first projection. ϕ is actually an isomorphism, with inverse given by $\psi(x) = (x, x^{-1})$ (note that x^{-1} is indeed a regular function on $\mathbb{A}^1(k) - 0$). This means that the regular functions on $\mathbb{A}^1(k) - 0$ is given by polynomials in x and x^{-1}. ☆

1.2 Projective varieties

The projective spaces and the projective varieties are in some sense the algebro-geometric counterparts to the compact topological spaces, with which they share many nice properties.

The projective spaces

We continue working over an algebraically closed ground field k. We first define the projective spaces

DEFINITION 1.26 *The projective n-space $\mathbb{P}^n(k)$ is the quotient of $\mathbb{A}^{n+1}(k)\backslash\{0\}$ by the equivalence relation*

$$(a_0, \ldots, a_n) \sim (ta_0, \ldots, ta_n),$$

where $t \in k$ is non-zero.

Thus in $\mathbb{P}^2(k)$, the tuples $(2,3,4)$ and $(4,6,8)$ represent the same point. In fact, two points are equivalent when they lie on the same line through the origin, so one may think about $\mathbb{P}^n(k)$ as the set of lines in $\mathbb{A}^{n+1}(k)$ through the origin; or if you want, the set of one dimensional linear subspaces of k^{n+1}.

The equivalence class where a point $x \in \mathbb{A}^{n+1}(k)\backslash\{0\}$ lies, will be denoted by $[x]$, but if the point x is specified with coordinates, say $x = (a_0, \ldots, a_n)$, this class will also be denoted by $[x] = (a_0 : \cdots : a_n)$. The a_i's are called the *homogeneous coordinates* of x. Note that they are not coordinates in the usual strict sense of the word, only their ratios are well defined. Note also that no point in $\mathbb{P}^n(k)$ has all homogeneous coordinate equal to 0; the tuple $(0 : \cdots : 0)$ is forbidden.

Homogeneous coordinates of points in $\mathbb{P}^n(k)$

The Zariski topology on $\mathbb{P}^n(k)$

The projective spaces are, like the affine ones, equipped with a natural topology called the *Zariski topology* topology. It is best described by the quotient map

Zariski topology

$$\pi\colon \mathbb{A}^{n+1}(k)\backslash\{0\} \to \mathbb{P}^n(k)$$

which sends $(a_0, \ldots, a_n)$ to $(a_0 : \cdots : a_n)$. This allows us to define a topology on $\mathbb{P}^n$, by declaring a subset $V \subseteq \mathbb{P}^n(k)$ to be closed if and only if the inverse image $\pi^{-1}(V)$ is closed.

There is a construction, similar to $Z(\mathfrak{a})$ in the affine case, that describes all the Zariski closed sets in $\mathbb{P}^n(k)$ in terms of ideals in a polynomial ring. It is slightly more delicate, because polynomials do not give functions on projective space (if f is a polynomial, the value $f(tx_0, \ldots, tx_n)$ of course depends on t). The solution is to use *homogeneous polynomials*; that is, polynomials such that for some natural number d one has

$$f(tx_0, \ldots, tx_n) = t^d f(x_0, \ldots, x_n)$$

for all t. The values of f still depend on the the scalar t, but he point is that whether it is zero or not, *is independent* of t. So we may define the zero set of f in $\mathbb{P}^n(k)$ as

$$Z_+(f) = \{ [x] \in \mathbb{P}^n(k) \mid f(x) = 0 \}.$$

More generally, for each set S of homogeneous polynomials, one may put

$$Z_+(S) = \{ x \in \mathbb{P}^n(k) \mid f(x) = 0 \text{ for all } f \in S \}.$$

These sets are Zariski closed set, because a homogeneous polynomial f vanishes at $[x]$ precisely when it vanishes along the whole fibre of π over $[x]$; in other words, $\pi^{-1}Z_+(S) = Z(S) \cap \mathbb{A}^{n+1}(k) \backslash \{0\}$.

Ideals $\mathfrak{a}$ with a zero-set being equal to a Zariski closed inverse image $\pi^{-1}(X)$, are characterised by the property that if $x \in Z(\mathfrak{a})$, then the entire fibre $\pi^{-1}(\pi(x))$ lies in $Z(\mathfrak{a})$; that is, $\pi^{-1}(\pi(x)) \subseteq Z(\mathfrak{a})$. And these ideals are precisely the *homogenous* ideals. *Homogenous ideals*

Recall that an ideal by definition is *homogenous* if for any $f \in \mathfrak{a}$ each of the homogeneous components of f also lies in $\mathfrak{a}$. Being a homogeneous ideal is equivalent to $\mathfrak{a}$ being generated by homogeneous polynomials. Clearly, if $\mathfrak{a}$ is homogeneous, then $x \in Z(\mathfrak{a})$ implies that $tx \in Z(\mathfrak{a})$ as well, for every $t \in k$. The following lemma shows that the converse holds true as well.

LEMMA 1.27 *Let $x \in \mathbb{A}^{n+1}(k)$ be a point. A polynomial f vanishes at all points on the line through x if and only if all homogeneous components of f do.*

PROOF: Developing f in terms of the homogeneous components f_i, we find

$$f(tx) = t^d f_d(x) + \cdots + t f_1(x) + f_0(x),$$

For $x = (a_0, \ldots, a_n)$ fixed, this is a polynomial in t, so if it zero for infinitely many t, all the coefficients must vanish. That means that all of the f_i must also vanish at x. ❏

We have thus established the desired description of the closed sets in projective space:

PROPOSITION 1.28 *The Zariski closed sets of $\mathbb{P}^n(k)$ are precisely those of the form $Z_+(\mathfrak{a})$ where $\mathfrak{a}$ can be any homogeneous ideal.*

EXAMPLE 1.29 (*The irrelevant ideal.*) The ideal $\mathfrak{m}_+ = (x_0, \ldots, x_n)$ is called the *irrelevant* ideal. It is certainly homogeneous, but its zero locus is empty. (No point has all the homogeneous coordinates equal to zero.) Similarly, any $\mathfrak{m}_+$-primary ideal $\mathfrak{q}$ has empty zero set because $Z(\mathfrak{q}) = Z(\mathfrak{m}_+)$ so that $Z(\mathfrak{q}) \cap \mathbb{A}^{n+1}(k)\backslash\{0\} = \emptyset$. ☆

*

A subset C of an affine space closed under scaling; that is $tx \in C$ when $t \in k$ and $x \in C$ is called a cone. So the zero sets $Z(\mathfrak{a})$ with $\mathfrak{a}$ homogeneous are precisely the cones in $\mathbb{A}^{n+1}k$

EXAMPLE 1.30 (*The complex projective spaces.*) The complex projective spaces $\mathbb{P}^n(\mathbb{C})$ (which topologists usually write as $\mathbb{CP}^n$) are also equipped with a Euclidean topology. It is just the quotient topology inherited from the standard Euclidean topology on $\mathbb{C}^{n+1}$. With this topology they are are compact manifolds. Every one-dimensional subspace of $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$ hits the unit sphere $\mathbb{S}^{2n-1}$ in a unit circle, so the restriction $\pi|_{\mathbb{S}^{2n-1}}$ is continuous surjection $\pi|_{\mathbb{S}^{2n-1}} \colon \mathbb{S}^{2n-1} \to \mathbb{P}^n(\mathbb{C})$. Since the unit sphere $\mathbb{S}^{2n-1}$ is compact, if follows that $\mathbb{P}^n(\mathbb{C})$ is compact as well. It is noteworthy that the $\pi|_{\mathbb{S}^{2n-1}}$ is a fibre bundle with unit circles as fibres. ☆

The projective Nullstellensatz

The usual operations on ideals, like sum, product, intersection and the formation of radicals, yield homogeneous ideals when applied to homogeneous ideals. Moreover, the corresponding equalities between the associated closed sets, as expressed in Proposition 1.2 for the affine case, hold true.

PROPOSITION 1.31 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two homogeneous ideals and $\{\mathfrak{a}_i\}_{i\in I}$ a family of homogeneous ideals in the polynomial ring $k[x_0, \ldots, x_n]$.*

i) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $Z_+(\mathfrak{b}) \subseteq Z_+(\mathfrak{a})$;*
ii) $Z_+(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} Z_+(\mathfrak{a}_i)$
iii) $Z_+(\mathfrak{a}\mathfrak{b}) = Z_+(\mathfrak{a} \cap \mathfrak{b}) = Z_+(\mathfrak{a}) \cup Z_+(\mathfrak{b})$;
iv) $Z_+(\mathfrak{a}) = Z_+(\sqrt{\mathfrak{a}})$.

PROOF: The proposition follows directly from the affine case (Proposition 1.2) by intersecting with $\mathbb{A}^{n+1}(k)\backslash\{0\}$ and pushing down by π. For instance, the last equality in *iii)* follows by the following equalities:

$$\begin{aligned} Z_+(\mathfrak{a}) \cup Z_+(\mathfrak{b}) &= \pi\big(Z(\mathfrak{a}) \cup Z(\mathfrak{b}) \cap \mathbb{A}^{n+1}(k)\backslash\{0\}\big) = \\ &= \pi\big(Z(\mathfrak{a} \cap \mathfrak{b}) \cap \mathbb{A}^{n+1}(k)\backslash\{0\}\big) = Z_+(\mathfrak{a} \cap \mathfrak{b}). \end{aligned}$$

❑

There is also a projective version of the Nullstellensatz. The statement is very similar to that in the affine case, but there are two notable differences. First of all, the irrelevant ideal $\mathfrak{m}_+ = (x_0, \ldots, x_n)$ and all primary ideals with radical equal to $\mathfrak{m}_+$, have empty zero locus. Secondly, one must be careful when defining the vanishing ideal $I(S)$ for a subset $S \subseteq \mathbb{P}^n(k)$; it must be defined as the ideal generated by the *homogeneous* polynomials which vanish along S.

Theorem 1.32 (Projective Nullstellensatz) *Let $\mathfrak{a}$ be a homogeneous ideal in $k[x_0, \ldots, x_n]$.*

i) The zero locus $Z_+(\mathfrak{a})$ is empty if and only if either $1 \in \mathfrak{a}$ or $\sqrt{\mathfrak{a}} = \mathfrak{m}_+$;

ii) If $Z_+(\mathfrak{a}) \neq \emptyset$, it holds true that $I(Z_+(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Proof: *i)*: The set $Z_+(\mathfrak{a})$ is non-empty if and only if $Z(\mathfrak{a}) \cap A^{n+1}(k) \backslash \{0\}$ is non-empty. There are two ways in which the latter can be empty: either $Z(\mathfrak{a}) = \emptyset$ and $1 \in \mathfrak{a}$; or $Z(\mathfrak{a}) = \{0\}$ and $\sqrt{\mathfrak{a}} = \mathfrak{m}_+$.

ii): We see from Lemma 1.27 that $I(Z(\mathfrak{a}))$ is generated by all homogeneous polynomials in $\mathfrak{a}$, but the latter is by definition equal to $I(Z_+(\mathfrak{a}))$. By the affine Nullstellensatz, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ and we are done. ❏

As in the affine case, the mappings I and Z_+ give a way to translate between algebra and geometry.

Proposition 1.33 *The mappings $\mathfrak{a} \mapsto Z_+(\mathfrak{a})$ and $S \mapsto I(S)$ are mutually inverse mappings between the set of proper radical homogenous ideals in $k[x_0, \ldots, x_n]$ and closed subsets of $\mathbb{P}^n(k)$.*

Again one should note that the irrelevant ideal is special: A homogeneous prime ideal corresponds to the empty set if and only if it has radical equal to the irrelevant ideal.

Example 1.34 (*The ideal of a point.*) In the affine case, the maximal ideals in $k[x_1, \ldots, x_n]$ correspond exactly to the points of $\mathbb{A}^n(k)$. In projective space, the points correspond to lines in $\mathbb{A}^{n+1}(k)$, so their ideals are homogeneous, but they are not maximal. A convenient set of generators (certainly not minimal) for the ideal of the point $(a_0, \ldots, a_n)$, are the 2×2-minors of the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ x_0 & x_1 & \ldots & x_n \end{pmatrix} \tag{1.1}$$

Indeed, a variable point $(x_0 : \cdots : x_n)$ lies in in the same one dimensional subspace as $(a_0 : \cdots : a_n)$ precisely when the two corresponding vectors are dependent; *i.e.* precisely when the matrix in (1.1) has rank one. ☆

There is also a projective analogue of Proposition 1.12, whose proof is essentially the same as that in the affine case, relying on *iii)* in Proposition 1.31.

Proposition 1.35 *A closed projective set $Z_+(\mathfrak{a})$ is irreducible if and only if the radical $\sqrt{\mathfrak{a}}$ is prime.*

This leads us to give the following definition.

Definition 1.36 (Tradional projective varieties) *A traditional projective variety is a closed irreducible subset of $\mathbb{P}^n(k)$.*

Distinguished open sets

There are no global coordinates on $\mathbb{P}^n(k)$, but there is a class of standard open subsets were we have good coordinates. These are the so-called distinguished open sets. If $a = (a_0 : \cdots : a_n)$ is a point in $\mathbb{P}^n(k)$, at least one of the homogeneous coordinates is non-zero; say $a_i \neq 0$. Then a belongs to the set

$$D_+(x_i) = \{\, (x_0 : \cdots : x_n) \mid x_i \neq 0 \,\} \subset \mathbb{P}^n(k)$$

and on this set, the ratios x_j/x_i are well defined functions, and can be used as coordinates. One has the pair of maps $\Phi\colon D_+(x_i) \to \mathbb{A}^n(k)$ and $\Psi\colon \mathbb{A}^n(k) \to D_+(x_i)$ given by

$$\Phi\colon (x_0 : \cdots : x_n) \mapsto (x_0/x_i, \ldots, 1, \ldots, x_n/x_i)$$
$$\Psi\colon (t_0, \ldots, 1, \ldots, t_n) \mapsto (t_0 : \cdots : 1 : \cdots : t_n)$$

where the one appears in slot* number i.

To avoid tortuous notation, we here consider $\mathbb{A}^n(k)$ as being the linear subspace $Z(x_i - 1)$ of $\mathbb{A}^{n+1}(k)$ where the i-th coordinate equals one.

Lemma 1.37 *The maps Φ and Ψ are a pair of mutually inverse homeomorphisms between $D_+(x_i)$ with induced topology and $\mathbb{A}^n(k)$.*

Proof: It is easy to see that the two maps are mutually inverse, so the main statement is about them being continuous. This has the algebraic counterpart in homogenisation and dehomogenization of polynomials. To any homogenous polynomial F one associates the dehomogenized polynomial F^d defined by $F(x_0, \ldots, 1, \ldots, x_n)$ simply by setting x_i equal to one. Clearly $\Psi^{-1}(Z_+(\mathfrak{a})) = Z(\mathfrak{a}^d)$ hence Ψ is continuous.

Naturally, there is also a homogenization process: To any polynomial $f(t_0, \ldots, 1, \ldots, t_n)$ of degree d, one associates the homogeneous polynomial* $f^h = x_i^d f(x_0/x_i, \ldots, x_n/x_i)$. Since $x_i \neq 0$ on $D_+(x_i)$, it holds that $\Phi^{-1}(Z(\mathfrak{a})) = Z_+(\mathfrak{a}^h) \cap D_+(x_i)$, and we are through. ❑

The practical recipe is: fill up each term of f with a power of x_i whose exponent is so that the degree of the term becomes d.

Distinguished open sets

For any homogeneous polynomial f, one may define the *distinguished open set* $D_+(f)$ as the set where f does not vanish:

$$D_+(f) = \{\, [x] \in \mathbb{P}^n(k) \mid f(x) \neq 0 \,\}.$$

Examples

(1.38) (*The Serge variety* $\mathbb{P}^1 \times \mathbb{P}^1$*.)* Consider two copies of $\mathbb{P}^1(k)$ one with homogeneous coordinates $(u_0 : u_1)$ and the other with $(t_0 : t_1)$. There is a map $\mathbb{P}^1(k) \times \mathbb{P}^1(k) \to \mathbb{P}^3(k)$ defined by the assignment

$$(t_0 : t_1) \times (u_0 : u_1) \mapsto (t_0u_0 : t_0u_1 : t_1u_0 : t_1u_1).$$

This is well defined since scaling $(t_0 : t_1)$ and $(u_0 : u_1)$ by λ and μ scales $(t_0u_0 : t_0u_1 : t_1u_0 : t_1u_1)$ by $\lambda\mu$, and since one of the t_i's and one of u_i's are non-zero, one of the products t_iu_j's is non-zero as well. The image is closed in $\mathbb{P}^3(k)$. Indeed, it equals the zero locus of $w_0w_3 - w_1w_2$ with w_i's being homogeneous coordinates on $\mathbb{P}^3(k)$.

For instance, in the open affine piece $D_+(w_0)$, it holds that $u_0t_0 = w_0 \neq 0$, so the inverse image equals $D_+(u_0) \times D_+(u_1)$. Normalising (*i.e.* setting $w_0 = t_0 = u_0 = 1$), we bring the map on the form $(1 : t) \times (1 : u) \mapsto (1 : t : u : tu)$, and it becomes clear that the image equals $w_3 = w_1w_2$.

(1.39) (*Rational normal curves.*) Sending $(t_0 : t_1)$ to $(t_0^n : t_0^{n-1}t_1 : \cdots : t_0t_1^{n-1} : t_1^n)$ yields a mapping $\rho \colon \mathbb{P}^1(k) \to \mathbb{P}^n(k)$. It is well defined because when t_0 and t_1 are scald by λ, the product $t_0^{n-1}t_2$ are scald by λ^n, and of course, these product never vanish unless either t_0 or t_1 vanish. The image C_n is called a rational normal curve of degree n.

The map ρ is injective: from points on the image lying in the distinguished open subset $D_+(x_n)$ one recovers the ratio t_0/t_i as x_{n-1}/x_n, and from image points in $D_+(x_0)$ one similarly finds $x_1x_0 = t_1/t_0$, and because the image of ρ is contained in the union $D_+(x_0) \cup D_+(x_1)$, we get injectivity.

The C_n is algebraic subset of $\mathbb{P}^n(k)$ indeed, it equals the common vanishing locus of the 2×2-minors of the matrix.

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \dots & x_{n-1} & x_n \end{pmatrix} \tag{1.2}$$

When $n = 2$, we get back a conic section $x_0x_2 - x_1^2 = 0$ in the plane $\mathbb{P}^2(k)$. The cubic curve C_3 is called the *twisted cubic curve*; 'twisted' because it is does not lie in any plane in $\mathbb{P}^3(k)$.

The twisted cubic curve

(1.40) (*Veronese surface.*) The projective plane $\mathbb{P}^2(k)$ can be embedded in a natural way in projective space $\mathbb{P}^5(k)$ by using all homogeneous quadratic forms as coordinate functions.

$$\begin{array}{rcl} \mathbb{P}^2(k) & \to & \mathbb{P}^5(k) \\ (x_0 : x_1 : x_2) & \mapsto & (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2) \end{array}$$

Giuseppe Veronese (1854–1917) Italian mathematician

The image is the *Veronese surface*. Note that the map makes sense, because a simulations scaling of the x_i's by λ simultaneously scales the w_{ij} by λ^2, and the quadratic monomials cannot all vanish at the same time. If we use coordinates $w_0, \ldots, w_5$ on $\mathbb{P}^5(k)$, the homogeneous ideal of the the surface is given by the 2×2-minors of the matrix

$$\begin{pmatrix} w_0 & w_1 & w_2 \\ w_1 & w_3 & w_4 \\ w_2 & w_4 & w_5 \end{pmatrix}$$

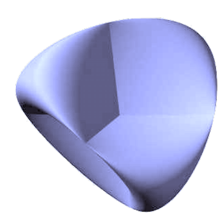
In the margin we have depicted a projection of the real points of the Veronese surface into $\mathbb{R}^3$. Such a projection will always have self-crossings, but is an immersion (i.e., the derivative of the parametrisation vanishes nowhere), and is non-orientable.

☆

EXERCISE 1.5 Show that the induced topology on $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$ is not then product topology. Show that closed sets are given by the vanishing of *bihomogeneous* polynomials.

★

EXERCISE 1.6 Show that the rational normal curve equals the common zero locus of the minors of the matrix 1.2. ★



the roman surface

Rational functions on projective varieties

As we already observed, polynomials do not define functions on projective spaces. However some rational rational functions do. They must be invariant under scaling of the variables, and so are shaped like quotient g/h where g and h are homogeneous polynomials of the same degree. g/h then gives a regular function on the open set $D_+(h) \cap X$.

As in the affine case, a function f is said to be regular throughout an open set U if each point in U has a neighbourhood over which $f = gh^{-1}$ with $h(x) \neq 0$ or $x \in V$, and where a and h are homogeneous polynomials of the same degree. The functions regular at a point x form a ring $\mathcal{O}_{\mathbb{P}^n(k),x}$, which is a local ring whose maximal ideal consists of the regular functions vanishing at x.

Contrary to the affine case, there are not many global regular functions on $\mathbb{P}^n(k)$. In fact, they are all constant. This is true for any projective variety, but we confine ourselves to prove it for projective space itself.

THEOREM 1.41 *The only global regular functions on projective space $\mathbb{P}^n(k)$ are the constants.*

PROOF: Let f be a global regular function on $\mathbb{P}^n(k)$, and consider the composition $\mathbb{A}^n(k)\backslash\{0\} \to \mathbb{P}^n(k) \to k$. It is a global regular function on $\mathbb{A}^n\backslash\{0\}$ which is by Proposition 1.24 on page 16 is a polynomial. However, since this polynomial comes from a function on $\mathbb{P}^n(k)$, it must be constant on lines through the origin. Hence it must have degree 0, that is, it is constant everywhere. ❏

The fact that there are so few global regular functions basically forces us to instead work with regular functions $f : U \to k$ defined on any open $U \subset X$. This is one of the reasons to introduce *sheaves* in the next few chapters.

Chapter 2

The Prime Spectrum

In this chapter, we make the first step towards the notion of a scheme, by defining the spectrum of a ring. The spectrum of a ring A, is a topological space denoted $\operatorname{Spec} A$, which is equipped with a Zariski-like topology with closed sets formed from the ideals of A.

To motivate the definition, assume for a moment that $A = A(X)$ is the coordinate ring of an affine variety X. Thus A is a finitely generated k-algebra over an algebraically closed field k without zerodivisors. The elements of A consists of regular functions $X \to k$.

We know by Hilbert's Nullstellensatz that points of X correspond to the maximal ideals in A: a point x correspond to the ideal $\mathfrak{m}_x$ of regular functions vanishing at x, and conversely, every maximal ideal is the vanishing ideal of a point. The variety X has the Zariski topology in which the closed sets are the zero sets of ideals $\mathfrak{a}$ in A; *i.e.* they have the form $Z(\mathfrak{a}) = \{\, x \in X \mid f(x) = 0 \text{ all } f \in \mathfrak{a} \,\}$. Or bearing the correspondence of points and maximal ideals in mind this may be rephrased as $Z(\mathfrak{a}) = \{\, \mathfrak{m} \mid \mathfrak{m} \subset A \text{ maximal and } \mathfrak{m} \supseteq \mathfrak{a} \,\}$. If X and Y are affine varieties, then any map $\phi : A(Y) \to A(X)$ of k-algebras induces a morphism $f : X \to Y$. Conversely, any such morphism f determines ϕ by $\phi(g) = f \circ g$.

Thus the ideals of the coordinate ring determine both the points of X, as well as the topology.

There is a natural way of generalizing this to all rings, and it involves replacing maximal ideals with prime ideals. Given a ring A, the spectrum of A, denoted $\operatorname{Spec} A$ is simply the set of prime ideals of A. This is then equipped with a Zariski topology where the closed sets are of the form $V(\mathfrak{a}) = \{\, \mathfrak{p} \mid \mathfrak{p} \subset A \text{ prime and } \mathfrak{p} \supseteq \mathfrak{a} \,\}$

The idea of replacing maximal ideals by prime ideals is a fundamental idea in scheme theory. From the categorical perspective, this is a good choice, because ring maps $A \to B$ pull back prime ideals to prime ideals and thus induce maps $\operatorname{Spec} B \to \operatorname{Spec} A$. For affine varieties X and Y, we were lucky that in fact the induced map $A(Y) \to A(X)$ pulls maximal ideals to maximal ideals, but this is not true for general ring maps (e.g., for $\mathbb{Z} \to \mathbb{Q}$).

2.1 *The spectrum of a ring*

Following the Grothendieck maxim of always working at the maximal level of generality, we start out with a commutative ring A with a unit element. We are going to define an *affine scheme*, written $\operatorname{Spec} A$, called the *prime spectrum* of A or simply the *spectrum* of A for short.

In these notes the term 'ring' always refers to a commutative ring with a unit element '1'. Any map of rings must send 1 to 1.

Recall: prime ideals are proper ideals

Definition 2.1 *For a ring A we define its* spectrum *as*

$$\operatorname{Spec} A = \{\, \mathfrak{p} \mid \mathfrak{p} \subset A \text{ is a prime ideal} \,\}.$$

To distinguish between points in $\operatorname{Spec} A$ and ideals in A, we will sometimes write $\mathfrak{p}_x \subseteq A$ for the ideal corresponding to $x \in \operatorname{Spec} A$.

When A is a general ring, we cannot think of elements of A as functions into some fixed field k. However, there still is an analogy between the elements f of A and some sort of functions on $\operatorname{Spec} A$. If x is a point in $\operatorname{Spec} A$ and $\mathfrak{p} = \mathfrak{p}_x$, the localization $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$, and one has the field $k(\mathfrak{p}) = A_\mathfrak{p}/(\mathfrak{p}A_\mathfrak{p})$ (which will also be denoted by $k(x)$). The element f reduced modulo $\mathfrak{p}$ gives an element $f(x) \in k(\mathfrak{p})$, which may be considered as the 'value' of f at x; clearly $f(x) = 0$ if and only if $f \in \mathfrak{p}$.

Definition 2.2 *The field $k(\mathfrak{p})$ is called the* residue field *of* $\operatorname{Spec} A$ *at* $\mathfrak{p}$.

Note in particular that for each $f \in A$ we may speak of the zero set $V(f) = \{x \in \operatorname{Spec} A \,|\, f(x) = 0\}$ in $\operatorname{Spec} A$ of the element f. It is important to note that the 'values' of an element $f \in A$ lie in different fields which might vary with the point. Thus we tweak our notion of a 'regular function' on X: they are not maps into some fixed field, but rather maps into the disjoint union $\coprod_{x \in X} k(x)$.

We can put a topology on the set $\operatorname{Spec} A$ which generalizes the Zariski topology on a variety and which will also be called the *Zariski topology*. The definitions are very similar: the closed sets in $\operatorname{Spec} A$ are defined to be those of the form

The Zariski topology

$$\begin{aligned} V(\mathfrak{a}) &= \{\, x \in \operatorname{Spec} A \mid f(x) = 0 \text{ for all } f \in \mathfrak{a} \,\} \\ &= \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \mathfrak{a} \,\}, \end{aligned}$$

where $\mathfrak{a}$ is any ideal in A. Of course, one has to verify that the axioms for a topology are satisfied. For closed sets the wording of the axioms is that the union of two closed and the intersection of any number (finite or infinite) must be closed. And of course, both the whole space and the empty set must be closed. That the family of subsets of the form $V(\mathfrak{a})$ satisfies these axioms, is the content of the following lemma:

Lemma 2.3 *Let A be a ring and assume that $\{\mathfrak{a}_i\}_{i \in I}$ is a family of ideals in A. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in A. Then the following three statements hold true:*

i) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$;

ii) $V(\sum_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$;

iii) $V(A) = \varnothing$ *and* $V(0) = \operatorname{Spec} A$.

Proof: Prime ideals are by definition proper ideals, so $V(A) = \varnothing$. Also, the zero-ideal is contained in every ideal, so $V(0) = \operatorname{Spec} A$. This proves the last statement. The second follows just as easily, since the sum of a family of ideals is contained in an ideal if and only if each of the ideals is.

The first of the three statements needs an argument. The inclusion $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$ is clear, so we need to show that $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$. If $\mathfrak{b} \nsubseteq \mathfrak{p}$; then there is an element $b \in \mathfrak{b}$ with $b \notin \mathfrak{p}$. But since $ab \in \mathfrak{a} \cap \mathfrak{b}$ for all $a \in \mathfrak{a}$, and $\mathfrak{p}$ contains $\mathfrak{a} \cap \mathfrak{b}$, one has $ab \in \mathfrak{p}$. The ideal $\mathfrak{p}$ is prime, so we may deduce that $a \in \mathfrak{p}$, and consequently one has the inclusion $\mathfrak{a} \subseteq \mathfrak{p}$. ❏

Corollary 2.4 *The collection $\{V(\mathfrak{a})\}$ where $\mathfrak{a}$ runs through all the ideals in A, is the family of closed sets for a topology on* Spec A.

The next lemma is about inclusions between the closed sets of Spec A, and we recognise all the statements from the theory of varieties.

Lemma 2.5 *For two ideals $\mathfrak{a}, \mathfrak{b} \subset A$ we have*

i) $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ *if and only if* $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$. *In particular, one has* $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$;

ii) $V(\mathfrak{a}) = \varnothing$ *if and only if* $\mathfrak{a} = A$;

iii) $V(\mathfrak{a}) = \operatorname{Spec} A$ *if and only if* $\mathfrak{a} \subseteq \sqrt{(0)}$.

Proof: The main point is that the radical of an ideal equals the intersection of all the prime ideals containing it, or expressed with a formula:

See Exercise 2.1

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}. \tag{2.1}$$

In particular, we see that $\mathfrak{a}$ and $\sqrt{\mathfrak{a}}$ are contained in the same prime ideals, so that $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$. To show the first claim in the lemma, we begin with assuming that $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$. From (2.1) we then obtain

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} \supseteq \bigcap_{\mathfrak{p} \in V(\mathfrak{b})} \mathfrak{p} = \sqrt{\mathfrak{b}}.$$

Now, if $\mathfrak{p}$ lies in $V(\mathfrak{a})$ and $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$, the chain of inclusions $\mathfrak{p} \supseteq \sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}} \supseteq \mathfrak{b}$ implies the inclusion $\mathfrak{p} \in V(\mathfrak{b})$ as well. This proves *i)*.

The second claim follows Lemma 2.3, since $\sqrt{\mathfrak{a}} = (1)$ if and only if $\mathfrak{a} = (1)$, and the third holds as $V(0) = \operatorname{Spec} A$. ❏

Corollary 2.6 *The map $\mathfrak{a} \mapsto V(\mathfrak{a})$ gives a one-to-one correspondence between radical ideals of A and closed subsets of* Spec A.

We also get the following corollary:

Corollary 2.7 (Closure of a subset) *The closure of a set $S \subseteq \operatorname{Spec} A$ is given as $\overline{S} = V(\mathfrak{a})$ where $\mathfrak{a} = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$.*

PROOF: Let $\mathfrak{b}$ be the radical ideal with $V(\mathfrak{b}) = \overline{S}$. Then every $\mathfrak{p} \in S$ contains $\mathfrak{b}$ and we may infer that $\mathfrak{b} \subseteq \mathfrak{a}$. On the other hand, $V(\mathfrak{a})$ is closed and $S \subseteq V(\mathfrak{a})$ so that $\overline{S} \subseteq V(\mathfrak{a})$. Hence $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$, and $\mathfrak{a} \subseteq \mathfrak{b}$ by Lemma 2.5. It follows that $\mathfrak{a} = \mathfrak{b}$. ❑

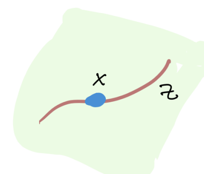
✲ EXERCISE 2.1 Let $\mathfrak{a} \subseteq A$ be an ideal. Show that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p}$. Hint: If $f \notin \sqrt{\mathfrak{a}}$ the ideal $\mathfrak{a}A_f$ is a proper ideal in the localization A_f, hence contained in a maximal ideal. ★

Generic points

The Zariski topology on Spec A is very different from the topology on manifolds that we are used to. For instance, points can fail to be closed. In fact, the next proposition implies that a point $x \in \operatorname{Spec} A$ is closed if and only if the corresponding ideal is maximal. Indeed, applying Corollary 2.7 to $S = \{\mathfrak{p}\}$, we get:

PROPOSITION 2.8 *If $\mathfrak{p}$ is a prime ideal of A, the closure $\overline{\{\mathfrak{p}\}}$ of the one-point set $\{\mathfrak{p}\}$ in* Spec A *equals the closed set $V(\mathfrak{p})$.*

In general, there are of course typically a lot of prime ideals which are not maximal. In fact, the rings having the property that every prime ideal is maximal are quite special; they are the rings of Krull dimension zero, and in the Noetherian case they correspond exactly to the *Artinian rings* (in which case the spectrum is a finite set of points and has the discrete topology).



DEFINITION 2.9 *A point x in a closed subset Z of a topological space X is called a* generic point *for Z if Z is the closure of the singleton $\{x\}$; that is, if $\overline{\{x\}} = Z$,*

In our context, each prime ideal $\mathfrak{p}$ is the generic point of the closed set $V(\mathfrak{p})$. So for example, for an integral domain A, the zero ideal (0) is prime, and it corresponds to the generic point of all of $X = \operatorname{Spec} A = V(0)$. There are simple examples of irreducible topological spaces with more than one generic point. For instance, in spaces with the trivial topology, *i.e.* where the empty set and the entire set are the only open sets, every point is generic. However, we will see shortly that in our context (and in the context of schemes in general) generic points are unique.

Examples

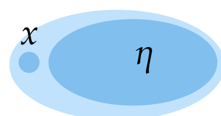
(2.10) (*Fields.*) If K is a field, the prime spectrum Spec K has only one element, which corresponds to the only prime ideal in K, the zero ideal. This also holds true for local rings A with the property that all elements in the maximal ideals are nilpotent, *i.e.* the radical $\sqrt{(0)}$ of the ring is a maximal ideal (see Exercise 2.15). For Noetherian local rings this is equivalent to the ring being an Artinian local ring.

(2.11) (*Artinian rings.*) The ring $A = \mathbb{C}[x]/(x^2)$ is not a field, but it has only one prime ideal (namely the ideal (x)). Note that (0) is not prime, since $x^2 = 0$, but $x \notin (0)$.

More generally, if A is an Artinian ring, then A has only finitely many prime ideals, so $\operatorname{Spec} A$ is a finite set. For a Noetherian ring A, the converse is also true.

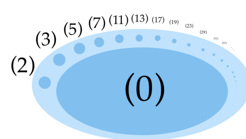
(2.12) (*Discrete valuation rings.*) Consider a discrete valuation ring A, for example the power series ring $\mathbb{C}[[x]]$, or one of the localizations $k[x]_{(x)}$ or $\mathbb{Z}_{(p)}$. (See Appendix A for background on discrete valuation rings). A has only two prime ideals, the maximal ideal $\mathfrak{m}$ and the zero ideal (0). So $\operatorname{Spec} A = \{\eta, x\}$ consists of just two points, with x corresponding to the maximal ideal $\mathfrak{m}$ and η to (0). The point x is closed in $\operatorname{Spec} A$, and therefore $\{\eta\} = X - x$ is open. So η is an open point! The point η is the generic point of $\operatorname{Spec} A$; its closure is the whole $\operatorname{Spec} A$.

The open sets of X are $\emptyset, X, \{\eta\}$. In particular $\operatorname{Spec} A$ is not Hausdorff, as η is contained in the only open set containing x, the whole space.



The spectrum of a DVR

(2.13) (*The spectrum of the integers,* $\operatorname{Spec}\mathbb{Z}$*.*) There are two types of prime ideals in $\mathbb{Z}$: the zero-ideal (0) and the maximal ideals (p), one for each prime number p. The latter give closed points in $\operatorname{Spec}\mathbb{Z}$, but one has $V(0) = \operatorname{Spec}\mathbb{Z}$, so the point corresponding to (0) is the generic point.



The spectrum of the integers

The residue field at a closed point (p) is given by $k(p) = \mathbb{Z}_p/(p)\mathbb{Z}_p = \mathbb{F}_p$, wheras the residue field at (0) is given by $\mathbb{Z}_{(0)} = \mathbb{Q}$.

Each element f of the ring $\mathbb{Z}$ gives rise to a 'regular function' into the various residue fields. Thus for instance $f = 17 \in \mathbb{Z}$ takes the values $f((0)) = 17, f((2)) = \bar{1}$, $f((3)) = \bar{2}$, $f((5)) = \bar{2}$, $f(7) = \bar{3}, \ldots$, in the fields $\mathbb{Q}, \mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7, \ldots$, respectively.

☆

2.2 *Affine spaces*

The most important examples of prime spectra are the affine n-spaces:

Definition 2.14 *We define the* affine n-space *as*

$$\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n].$$

More generally, for a ring R, we define the affine n-space over R *by*

$$\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \ldots, x_n].$$

If k is an algebraically closed field, then $\mathbb{A}^n_k$ is the scheme analogue of the affine n-space $\mathbb{A}^n(k)$ (the variety whose underlying set is k^n). In this setting, Hilbert's Nullstellensatz tells us that points of $\mathbb{A}^n(k)$ are in 1-1 correspondence with maximal ideals in $A = k[x_1, \ldots, x_n]$; they are all of the form $(x_1 - a_1, \ldots, x_n - a_n)$ with $a_i \in k$. Thus $\mathbb{A}^n(k)$ is naturally a subset of the spectrum $\mathbb{A}^n_k$, and the good old Zariski topology on the variety $\mathbb{A}^n(k)$ is the induced topology. However, for $n \geqslant 1$, there are many other prime ideals in A than just the maximal ideals; the zero ideal for instance. So $\mathbb{A}^n_k$ is strictly larger than $\mathbb{A}^n(k)$. The differences between $\mathbb{A}^n_k$ and $\mathbb{A}^n(k)$ become even more apparent if k is not algebraically closed.

Example 2.15 (*The affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$.*) $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ is called the *affine line over k*. In the polynomial ring $k[x]$ all ideals are principal, and all non-zero prime ideals are maximal. In general they are of the form $(f(x))$ where $f(x)$ is an irreducible polynomial, hence of the form $(x - a)$ if we assume that k is algebraically closed. There is only one non-closed point in $\operatorname{Spec} k[x]$, the generic point η, which corresponds to the zero-ideal. The closure $\overline{\{\eta\}}$ is the whole line $\mathbb{A}^1_k$. Thus $\mathbb{A}^1_{\mathbb{C}}$ consists of the generic point η, and the closed points $(x - a)$ for $a \in \mathbb{C}$.

Affine line

An interesting case is when $k = \mathbb{R}$, where $\mathbb{A}^1_{\mathbb{R}}$ is called the *real affine line*. By the Fundamental Theorem of Algebra, a prime ideal $\mathfrak{p}$ of $\mathbb{R}[x]$ is of the form $\mathfrak{p} = (f(x))$ where either $f(x)$ is linear; that is, $f(x) = x - a$ for an $a \in \mathbb{R}$, or f is quadratic with two conjugate complex and non-real roots; that is, $f(x) = (x - a)(x - \bar{a})$ with $a \in \mathbb{C}$ but $a \notin \mathbb{R}$. This shows that the closed points in $\operatorname{Spec} \mathbb{R}[x]$ may be identified with the set of pairs $\{a, \bar{a}\}$ with $a \in \mathbb{C}$. And of course, there is the generic point η corresponding to the zero ideal.

Real affine line

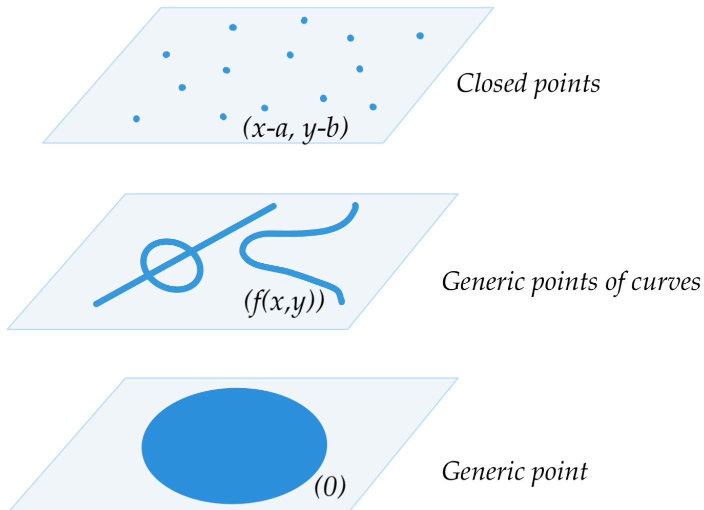
When k is algebraically closed, the residue fields of $\mathbb{A}^1_k$ are of the form

$$k(a) = k[x]_{(x-a)} / (x - a) k[x]_{(x-a)} \simeq k,$$

and $k(\eta) = k[x]_{(0)} = k(x)$. When k is not algebraically closed, we have more interesting residue fields; for instance $\mathfrak{p} = (x^2 + 1)$ defines a point in $\mathbb{A}^1_{\mathbb{R}}$ with residue field $\mathbb{C}$. In general, a maximal ideal $\mathfrak{m}$ in $k[x]$ is generated by an irreducible polynomial, say $f(t)$, and defines a point in $\mathbb{A}^1_k$ whose residue field is the extension of k obtained by adjoining a root of f.

☆

Example 2.16 (*The affine plane $\mathbb{A}^2_k = \operatorname{Spec} k[x_1, x_2]$.*) When k is algebraically closed, the maximal ideals of $k[x_1, x_2]$ are all of the form $(x_1 - a_1, x_2 - a_2)$ and these constitute all the closed points of $\mathbb{A}^2_k$. There are also the prime ideals of the form $\mathfrak{p} = (f)$ where $f(x_1, x_2)$ is an irreducible polynomial. The prime ideal $\mathfrak{p}$ is the generic point of the closed subset $V(f(x_1, x_2))$. In addition to the point $\mathfrak{p}$, the points of $V(f(x_1, x_2))$ are the closed points corresponding to ideals $(x_1 - a_1, x_2 - a_2)$ containing $f(x_1, x_2)$. This condition is equivalent to $f(a_1, a_2) = 0$, so the closed points of $V(f(x_1, x_2))$ correspond to what one in the world of varieties would call *the curve* given by the equation $f(x_1, x_2) = 0$. ☆



2.3 Distinguished open sets

There is no way to describe the open sets in Spec A as simply and elegantly as the closed sets can be described. However there is a natural basis for the topology on Spec A whose sets are easily defined, and which turns out to be very useful. For an element $f \in A$, we let $D(f)$ be the complement of the closed set $V(f)$, that is,

$$D(f) = \{ \mathfrak{p} \mid f \notin \mathfrak{p} \} = X - V(f).$$

These are clearly open sets; we call them the *distinguished open sets*. *Distinguished open sets*

LEMMA 2.17 *One has* $D(f) \cap D(g) = D(fg)$.

PROOF: If $\mathfrak{p}$ is a prime ideal, both $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$ hold true if and only if $fg \notin \mathfrak{p}$. ❑

EXAMPLE 2.18 In $\mathbb{A}^1$ every closed set is of the form $V(f)$ for some polynomial f, so every open set is a distinguished open set $D(f)$. In $\mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x, y]$, the set $U = \mathbb{A}^2 - V(x, y)$ is open, but not of the form $D(f)$. Still, we we have $U = D(x) \cup D(y)$. ☆

Here is the lemma we need:

LEMMA 2.19 *i) The open sets $D(f)$ form a basis for the topology of* Spec A *when f runs through the elements of A.*

ii) A family $\{D(f_i)\}_{i \in I}$ forms an open covering of Spec A *if and only if one may write $1 = \sum_i a_i f_i$ with the a_i's being elements from A only a finite number of which are non-zero.*

PROOF: i) We need to show that any open subset U of Spec A can be written as the union of sets of the form $D(f)$. Observe that, by definition, the complement U^c of U is of the form $U^c = V(\mathfrak{a})$, where $\mathfrak{a} \subset A$ is an ideal, and choose a set $\{f_i\}$ of generators for $\mathfrak{a}$ (not necessarily a finite set). Then we have

$$U = V(\mathfrak{a})^c = V\left(\sum_i (f_i)\right)^c = \left(\bigcap_i V(f_i)\right)^c = \bigcup_i D(f_i). \tag{2.2}$$

ii) Applying the identity (2.2) to $U = \operatorname{Spec} A$, we see that the open sets $D(f_i)$ constitute a covering if and only if $V(\sum_i (f_i)) = \varnothing$, which happens if and only if $\sum_i (f_i) = (1)$. But this happens if and only if 1 is a combination of finitely many of the f_i's. ❑

This lemma tell us that the $D(f)$'s form a very handy basis for the topology: any open set $U \subset \operatorname{Spec} A$ can be written as a union of finitely many $D(f)$'s. Moreover, as the distinguished sets form a basis for the topology, we see that any open cover may be refined to one whose members all are distinguished, and hence it can be reduced to a *finite* covering. A topological space with this property is said to be *quasi-compact*.

The terminology is a little bit unfortunate; spaces in which every open cover has a finite subcover are usually called 'compact'. However, some authors reserve the term 'compact' for quasi-compact and Hausdorff, and this jargon has caught on in the algebraic geometry literature.

LEMMA 2.20 *One has $D(g) \subseteq D(f)$ if and only if $g^n \in (f)$ for a suitable natural number n. In particular, one has $D(f) = D(f^n)$ for all n.*

PROOF: The inclusion $D(g) \subseteq D(f)$ holds if and only if $V(f) \subseteq V(g)$, and by Lemma 2.5 on page 26 this is true if and only if $(g) \subseteq \sqrt{(f)}$, *i.e.* if and only if $g^n \in (f)$ for a suitable n. ❑

In fact, the inclusion $D(g) \subseteq D(f)$ is equivalent to the condition that the localization map $\rho\colon A \to A_g$ extends to a map $\rho_{fg} : A_f \to A_g$. Indeed, ρ extends if and only if $\rho(f)$, *i.e.* f regarded as an element in A_g, is invertible, which in its turn is equivalent to there being an $b \in A$ and an $m \in \mathbb{N}$ such that $g^m(fb - 1) = 0$; or in other words, if and only if $g^m = cf$ for some c and some $m \in \mathbb{N}$. This enables us to define the localization map by

$$\begin{aligned} \rho_{fg} : A_f &\to A_g \\ af^{-n} &\mapsto ac^n g^{-nm}. \end{aligned} \tag{2.3}$$

More generally, for an A-module M, we have *localization maps*

Localization maps

$$\begin{aligned} \rho_{fg} : M_f &\to M_g \\ mf^{-n} &\mapsto mc^n g^{-nm}. \end{aligned} \tag{2.4}$$

where $m \in M$.

EXAMPLE 2.21 (*(The circle).*) Consider $X = \operatorname{Spec} \mathbb{R}[x,y]/(x^2 + y^2 - 1)$. The maximal ideal $\mathfrak{m} = (x, y-1)$ defines the point $(0,1)$ on the circle X. Note that this ideal is not a principal ideal. Nevertheless, the complement $X - \{(0,1)\}$ is a distinguished open set. Indeed, it coincides with $D(y-1)$, because modulo the ideal $(x^2 + y^2 - 1)$, we have

$$(x, y-1)^2 = (x^2, x(y-1), (y-1)^2) = (y-1).$$

☆

Exercise 2.2 Show that $D(f) = \varnothing$ if and only if f is nilpotent. Hint: Use that $\sqrt{(0)} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$. ★

Exercise 2.3 Check that for a nested inclusion $D(h) \subseteq D(g) \subseteq D(f)$, we have $\rho_{fh} = \rho_{gh} \circ \rho_{fg}$. ★

Exercise 2.4 Let A be a ring, let $\mathfrak{a}$ be an ideal in A and let $\{f_i\}_{i \in I}$ be elements from $\mathfrak{a}$. Show that the open distinguished sets $D(f_i)$ cover $\operatorname{Spec} A \backslash V(\mathfrak{a})$ if and only if some power of each element $f \in \mathfrak{a}$ lies in the ideal $(f_i | i \in I)$ generated by the f_i's. ★

2.4 *Irreducible closed subsets*

Irreducible spaces

Recall that a topological space X is said to be *irreducible* if it can not be written as the union two proper closed subsets; that is, if $X = Z \cup Z'$ with Z and Z' closed, then either $Z = X$ or $Z' = X$. Equivalently, the space X is irreducible if and only if any two non-empty open subsets has a non-empty intersection: indeed, to say that $U \cap V = \varnothing$ for two open subsets U and V is to say that $U^c \cup V^c = X$. And so if X is irreducible, either $U^c = X$ or $V^c = X$; that is, either $U = \varnothing$ or $V = \varnothing$. A third way of expressing that X is irreducible, is to say that every non-empty open subset is dense.

Irreducible component

In a topological space X, a maximal, irreducible subset is called an *irreducible component*. Since the closure of an irreducible set is irreducible, the irreducible components are automatically closed, and their union equals the entire space X.

Clearly the closure of any singleton is irreducible: if $\overline{\{x\}}$ is the union of two closed sets, x must lie in one of them, and hence that set equals $\overline{\{x\}}$.

When is Spec A *irreducible?*

From the the theory of varieties we know that the coordinate ring of an affine variety (which by definition is irreducible) is an integral domain, and very simple examples illustrate that reducibility is closely linked to zero divisors in the ring of functions (see Example 2.23 below). In general, we have the following:

Proposition 2.22 *Let A be a ring. Then the following statements hold:*

i) If $\mathfrak{p} \subseteq A$ is a prime ideal, it holds that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$, and $\mathfrak{p}$ is the only generic point of $V(\mathfrak{p})$;

ii) A closed subset $Z \subseteq \operatorname{Spec} A$ is irreducible if and only if Z is of the form $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$;

iii) The space $\operatorname{Spec} A$ itself is irreducible if and only if A has just one minimal prime ideal; in other words, if and only if the nilradical $\sqrt{(0)}$ is prime.

Proof: Statement *i)* is just Proposition 2.8 on page 27. For the uniqueness part, when $V(\mathfrak{p}) = V(\mathfrak{q})$, it holds by Lemma 2.5 on page 26 that both $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{q} \subseteq \mathfrak{p}$.

As the closure of any singleton is irreducible and since we just showed that $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, we know that $V(\mathfrak{p})$ irreducible. For the reverse implication in *ii)*, let $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ be a closed subset. Recall that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a}\subseteq\mathfrak{p}} \mathfrak{p}$, and if $\sqrt{\mathfrak{a}}$ is not prime, there are more than one prime involved in the intersection. We may divide them into two different groups thus representing $\sqrt{\mathfrak{a}}$ as the intersection $\sqrt{\mathfrak{a}} = \mathfrak{b} \cap \mathfrak{b}'$ where $\mathfrak{b}$ and $\mathfrak{b}'$ are ideals whose radicals are different. One concludes that $V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{b}')$, so it is not irreducible.

For the third statement it suffices to observe that $\operatorname{Spec} A = V\big(\sqrt{(0)}\big)$, and again we rely on Lemma 2.5. ❑

A consequence of the lemma is that $\operatorname{Spec} A$ is irreducible whenever A is an integral domain, as in that case (0) is a minimal prime ideal. However, contrary to what holds for coordinate rings of varieties, a ring whose spectrum is irreducible, does not need to be an integral domain, but statement *iii)* tells us that every non-zero divisor is nilpotent – or, in the spirit of the analogy with functions there are non-zero functions that vanish everywhere. The ring $A = k[t]/(t^2)$ is a simple example showing such behaviour. It is not an integral domain, but has only one prime ideal, namely the principal ideal (t).

Note also that the lemma tells us that any non-empty closed and irreducible subset of $\operatorname{Spec} A$ has a unique generic point.

✲ EXERCISE 2.5 Let Z be an irreducible subspace of the topological space X. Show that the closure $\overline{Z}$ is irreducible. If $f\colon X \to Y$ is a continuous map to another topological space Y, show that $f(Z)$ is irreducible. ★

✲ EXERCISE 2.6 Use Zorn's lemma to prove that any irreducible subset of a topological space X is contained in an irreducible component. Show that X is the union of its irreducible components. ★

EXERCISE 2.7 Let X be a topological space and let $Z \subseteq X$ be an irreducible component of X. Let U be an open subset of X and assume that $U \cap Z$ is nonempty. Show that $Z \cap U$ is an irreducible component of U. ★

Connectedness

Connected spaces

Recall that a topological space is *connected* if it cannot be written as a disjoint union of two proper open subsets. Connectedness is a weaker topological notion than irreducibility in the sense that an irreducible space is also connected. However, as the examples below show, the converse does not hold. In the dictionary between algebra and geometry, connectedness of $\operatorname{Spec} A$ translates into absence of non-trivial idempotents in A (see Example 2.24 below).

EXAMPLE 2.23 The prime spectrum $X = \operatorname{Spec} k[x,y]/(xy)$ is a good example of a space which is connected but not irreducible. The coordinate functions x and y are zero-divisors in the ring $k[x,y]/(xy)$, and their zero-sets $V(x)$ and $V(y)$ show that X has two components. Since these two components intersect at the origin, X is connected. ☆

EXAMPLE 2.24 The spectrum $\operatorname{Spec}(A_1 \times A_2)$ of a direct product of two non-trivial rings is not connected. All the ideals are products of the form $\mathfrak{a}_1 \times \mathfrak{a}_2$ of ideals $\mathfrak{a}_i \subseteq A_i$, and they are prime if and only if either $\mathfrak{a}_1$ or $\mathfrak{a}_2$ is prime and the other is the entire ring. Hence $\operatorname{Spec}(A_1 \times A_2)$ is homeomorphic to the disjoint union $\operatorname{Spec} A_1 \cup \operatorname{Spec} A_2$.

See also Exercise ?? for a generalization

With $(1,0) = e_1$ and $(0,1) = e_2$ it holds that $1 = e_1 + e_2$, so that $\mathfrak{a} = \mathfrak{a}e_1 + \mathfrak{a}e_2$, and from $e_i^2 = e_i$, we get that that each $\mathfrak{a}_i = \mathfrak{a}e_i$ is an ideal in A_i. Moreover, as $e_1e_2 = 0$, it holds that $\mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$, in other words, $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2$. It always holds that $e_1e_2 \in \mathfrak{a}$, so for $\mathfrak{a}$ to be prime either e_1 or e_2 must belong to $\mathfrak{a}$; *i.e.* either $\mathfrak{a}_1 = A_1$ or $\mathfrak{a}_2 = A_2$, and obviously the proper one must be prime. Clearly $\mathfrak{a}_i = A_i$ if and only if $e_i \in \mathfrak{a}$, so $\operatorname{Spec} A_i = D(e_j) = V(e_i)$.

Note, by the way, that $\operatorname{Spec} A_i \simeq D(e_j) = V(e_i)$ (with $\{i,j\} = \{1,2\}$), and so both are both open and closed subsets. ☆

EXERCISE 2.8 Assume that X is a topological space that is not connected. Exhibit two non-constant orthogonal idempotents with sum unity in the ring of continuous functions on X. HINT: The characteristic functions of two disjoint open sets whose union equals X, will do. ★

Decomposition into irreducible subsets

From commutative algebra we know that ideals in Noetherian rings have a *primary decomposition*. This is the Lasker–Noether theorem, proved by Emanuel Lasker for ideals in polynomial rings in 1905. Some fifteen years later the general result, as we know it today, was established by Emmy Noether. It states that every ideal $\mathfrak{a}$ in a Noetherian ring can be expressed as an intersection

$$\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$$

where the $\mathfrak{q}_i$'s are primary ideals (See [?]). Recall that primary ideals have radicals that are primes, so the ideals $\sqrt{\mathfrak{q}_i}$ are all prime. They are called the *associated primes* to $\mathfrak{a}$. Such a decomposition is not always unique, but there are partial uniqueness results. The associated prime ideals are unique as are the primary components $\mathfrak{q}_i$ whose associated prime ideals are minimal. However, the so-called *embedded components* are not unique. For instance, one has the equality $(x^2, xy) = (x) \cap (x^2, y)$, but also the equality $(x^2, xy) = (x) \cap (x^2, xy, y^2)$ holds true.

Associated primes

A primary component $\mathfrak{q}_i$ is embedded *if $\sqrt{\mathfrak{q}_i}$ contains the radical $\sqrt{\mathfrak{q}_j}$ of another component $\mathfrak{q}_j$.*

Let A be a ring and consider a primary decomposition of an ideal $\mathfrak{a}$:

$$\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r.$$

Putting $Y_i = V(\sqrt{\mathfrak{q}_i})$, we find $V(\mathfrak{a}) = Y_1 \cup Y_2 \cup \cdots \cup Y_r$, where each Y_i is an irreducible closed set in $\operatorname{Spec} A$. If the prime $\sqrt{\mathfrak{q}_i}$ is not minimal among the associated primes, say $\sqrt{\mathfrak{q}_j} \subseteq \sqrt{\mathfrak{q}_i}$, it holds that $Y_i \subseteq Y_j$, and the component Y_i contributes nothing to union and can be discarded.

In a more general context, a decomposition $Y = Y_1 \cup \cdots \cup Y_r$ of any topological space is said to be *redundant* if one can discard one or more of the Y_i's without changing the union. That a component Y_j can be omitted, is equivalent to Y_j being contained in the union of the rest; that is, $Y_j \subseteq \bigcup_{i \neq j} Y_i$. A decomposition that is not redundant, is said to be *irredudant*. Translating the Lasker–Noether theorem into geometry we arrive at the following:

Redundant decompositions

Irredundant decompositions

PROPOSITION 2.25 *If A is a Noetherian ring, every closed subset $Y \subseteq \operatorname{Spec} A$ can be written as an irredundant union*

$$Y = Y_1 \cup \cdots \cup Y_r,$$

where the Y_i's are irreducible closed subsets. The Y_i's are unique up to order.

Notice, that since embedded components do not show up for radical ideals, we get a clear and clean uniqueness statement.

Examples

(2.26) Consider the closed subset $Y = V(f, g)$ in $\mathbb{A}^3_k = \operatorname{Spec} k[x, y, z]$, with k algebraically closed, where $f = x^2 - yz$ and $g = xz - x$. A primary decomposition of the ideal $I = (f, g)$ is given by

$$I = (x, y) \cap (x, z) \cap (y - x^2, z - 1).$$

This gives a decomposition of Y as

$$Y = V(x, z) \cup V(x, z) \cup V(y - x^2, z - 1),$$

In these situations, it is sometimes easier to use geometric arguments to find the irreducible components of Y. These arguments are legitimate even if they involve only closed points, since closed points are dense in each $V(\mathfrak{p})$ (by the Nullstellensatz).

Since g vanishes at a point in Y, we find that either $x = 0$ or $z = 1$ at the point. If $x = 0$, then the vanishing of f at the point implies that either $y = 0$ or $z = 0$. Thus we find the two components $Y_1 = V(x, y)$ and $Y_2 = V(x, z)$. In the case $z = 1$, then the vanishing of f implies that $y = x^2$, so we obtain a third component $Y_3 = V(y - x^2, z - 1)$. Thus we have find that

$$Y = V(x, z) \cup V(x, z) \cup V(y - x^2, z - 1),$$

and we leave to the reader to verify that the three subsets are irreducible and that the union is irredundant.

(2.27) Consider the closed set $Y = V(\mathfrak{a}) \subset \mathbb{A}^3_k$ given by the ideal

$$\mathfrak{a} = (x^2 - y, xz - y^2, x^3 - xz).$$

Note first that if $x = 0$ at a point in Y, it follows that $y = 0$ there, so $V(x, y) \subset X$. If $x \neq 0$ at the point, the third equation gives $z = x^2$ there, and so by the first and second equations we get $xz - y^2 = x^3 - x^4 = 0$ at the point; and thus $x = 1$, $y = 1$ and $z = 1$. Hence

$$X = V(x, y) \cap V(x - 1, y - 1, z - 1).$$

That is, X is the union of the z-axis, and the point $(1, 1, 1)$. In fact, a primary decomposition of $\mathfrak{a}$ is given by $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3$, where

$$\mathfrak{q}_1 = (x, y), \quad \mathfrak{q}_2 = (x - 1, y - 1, z - 1), \quad \mathfrak{q}_3 = (x^2 - y, xy, y^2, z).$$

Taking radicals, we find that the primes associated to $\mathfrak{a}$ are the following:

$$\mathfrak{p}_1 = (x, y), \quad \mathfrak{p}_2 = (x-1, y-1, z-1), \quad \mathfrak{p}_3 = (x, y, z).$$

Note that $\mathfrak{p}_1 \subset \mathfrak{p}_3$, and $\mathfrak{p}_3$ is thus an embedded component, which does not show up in the decomposition above.

(2.28) Consider $Y = V(yz, xz, y^3, x^2y) \subset \mathbb{A}^3_k$. Computing a primary decomposition, we find that

$$(yz, xz, y^3, x^2y) = (x, y) \cap (y, z) \cap (x^2, y^3, z),$$

and it follows that Y has two irreducible components, the two lines $V(x, y)$ and $V(y, z)$, and that there is an embedded component at the origin.

☆

Noetherian topological spaces

A decomposition result as the one in Proposition 2.25 on the previous page above holds for a much broader class of topological spaces than the closed subsets of spectra. The class in question is the class of the so-called *Noetherian topological spaces*; these satisfy the requirement that every descending chain of closed subsets is eventually stable. That is, if $\{X_i\}$ is a collection of closed subsets forming a chain

Noetherian topological spaces

$$X_1 \supseteq X_2 \supseteq \cdots \supseteq X_i \supseteq X_{i+1} \supseteq \ldots,$$

it holds true that for some index ν one has $X_i = X_\nu$ for $i \geqslant \nu$. It is easy to establish that any closed subset of a Noetherian space endowed with the induced topology is Noetherian.

✲ EXERCISE 2.9 Let X be a topological space. Show that the following three conditions are equivalent:

i) X is Noetherian;

ii) Every open subset of X is quasi-compact;

iii) Every non-empty family of closed subsets of X has a minimal member.

The last statement in the exercise leads to the technique called Noetherian induction *– proving a statement about closed subsets, one can work with 'a minimal counterexample'. It enjoys the property that the statement to be proven holds for every proper closed subset.*

★

The Lasker–Noether decomposition of closed subsets in affine space as a union of irreducibles can be generalized to any Noetherian topological space:

THEOREM 2.29 *Every closed subset Y of a Noetherian topological space X has an irredundant decomposition $Y = Y_1 \cup \cdots \cup Y_r$ where each is Y_i is a closed and irreducible subset of X. Furthermore, the decomposition is unique up to order.*

The Y_i's that appear in the theorem are the *irreducible components* of Y. They are *maximal* among the closed irreducible subsets of Y.

Irreducible components

PROOF: We shall work with the family Σ of those closed subsets of X that cannot be decomposed into a finite union of irreducible closed subsets; or phrased in a different way, the set of counterexamples to the assertion – and of course, we shall prove that it is empty.

Assuming the contrary – that Σ is non-empty – we can find a minimal element Y in Σ because X by assumption is Noetherian. The set Y itself can not be irreducible, so $Y = Y_1 \cup Y_2$ where both the Y_i's are proper subsets of Y and therefore do not belong to Σ. Either is thus a finite union of closed irreducible subsets, and consequently the same is true for their union Y. We have a contradiction, and Σ must be empty.

As to uniqueness, assume that we have a counterexample; that is, two irredundant decomposition such that $Y_1 \cup \cdots \cup Y_r = Z_1 \cup \cdots \cup Z_s$ and such that one of the Y_i's, say Y_1, does not equal any of the Z_k's.

Since Y_1 is irreducible and $Y_1 = \bigcup_j \left(Z_j \cap Y_1\right)$, it follows that $Y_1 \subseteq Z_j$ for some index j. A similarly argument gives $Z_j = \bigcup_i \left(Z_j \cap Y_i\right)$ and Z_j being irreducible, it holds that $Z_j \subseteq Y_i$ for some i, and therefore $Y_1 \subseteq Z_j \subseteq Y_i$. Since the union of the Y_i's is irredundant, we infer that $Y_1 = Y_i$, and hence $Y_1 = Z_j$. Contradiction. ❏

Exercises

(2.10) Show that the Zariski topology on Spec A is Hausdorff if and only if every prime ideal $\mathfrak{p}$ is maximal. Show that the Zariski topology always is T_0.

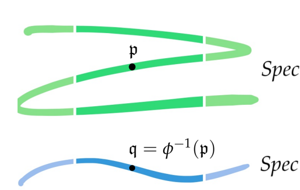
✲ (2.11) Compute a primary decomposition for the following ideals and describe their corresponding closed subsets.

a) $I = (x^2y^2, x^2z, y^2z)$ in $k[x, y, z]$;
b) $I = (x^2y, y^2x)$ in $k[x, y]$;
c) $I = (x^3y, y^4x)$ in $k[x, y]$;
d) $I = (x, y, x - yz)$ in $k[x, y, z]$;
e) $I = (x^2 + (y-1)^2 - 1, y - x^2)$ in $k[x, y]$.

✲ (2.12) Let $\{A_i\}_{i \in I}$ be an infinite sequence of non-trivial rings, and let X be the disjoint union of the spectra Spec A_i. Show that X is not homeomorphic to a spectrum of a ring. ★

2.5 *Morphisms between prime spectra*

Let A and B be two rings and let $\phi\colon A \to B$ be a ring homomorphism. The inverse image $\phi^{-1}(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \subseteq B$ is a prime ideal: that $ab \in \phi^{-1}(\mathfrak{p})$ means that $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, so at least one of $\phi(a)$ or $\phi(b)$ has to lie in $\mathfrak{p}$. Hence sending $\mathfrak{p}$ to $\phi^{-1}(\mathfrak{p})$ gives us a well defined map



$$\Phi : \operatorname{Spec} B \to \operatorname{Spec} A.$$

We will sometimes denote this map by $\operatorname{Spec}(\phi)$. The reason is to emphasize that Spec defines a *contravariant* functor from the category Rings of rings to the category Top of topological spaces. Indeed, in this situation, we have the equality $\operatorname{Spec}\phi \circ \operatorname{Spec}\psi = \operatorname{Spec}(\psi \circ \phi)$, whenever ϕ and ψ are composable ring homomorphisms. This is because of the identity $\phi^{-1}(\psi^{-1}(\mathfrak{p})) = (\psi\phi)^{-1}(\mathfrak{p})$. Also, of course, it holds that $\operatorname{Spec} \mathrm{id}_A = \mathrm{id}_{\operatorname{Spec} A}$.

The basic properties of this map is summarized in the next two propositions.

Proposition 2.30 *Assume that $\phi\colon A \to B$ is a map of rings, and let $\Phi : \operatorname{Spec} B \to \operatorname{Spec} A$ denote the induced map. Then*

i) $\Phi^{-1}(V(\mathfrak{a})) = V(\phi(\mathfrak{a})B)$ *for each ideal $\mathfrak{a} \subset A$. In particular, the map Φ is continuous;*

ii) $\Phi^{-1}(D(f)) = D(\phi(f))$ *for each $f \in A$;*

iii) $\overline{\Phi(V(\mathfrak{b}))} = V(\phi^{-1}(\mathfrak{b}))$ *for each ideal $\mathfrak{b}$ of B.*

Proof: Proof of *i)*: Let $\mathfrak{a} \subseteq A$ be an ideal. Then:

$$\Phi^{-1}(V(\mathfrak{a})) = \{\, \mathfrak{p} \subseteq B \mid \phi^{-1}(\mathfrak{p}) \supseteq \mathfrak{a} \,\} = \{\, \mathfrak{p} \subseteq B \mid \mathfrak{p} \supseteq \phi(\mathfrak{a}) \,\} = V(\phi(\mathfrak{a})B).$$

Indeed, because $\phi^{-1}(\phi(\mathfrak{a})) \supseteq \mathfrak{a}$, it holds true that $\mathfrak{p} \supseteq \phi(\mathfrak{a})$ if and only if it holds true that $\phi^{-1}(\mathfrak{p}) \supseteq \mathfrak{a}$. In particular, the inverse image of any closed subset is again closed, so Φ is continuous.

Proof of *ii)*: Note that for each element $f \in A$ we have the following equalities:

$$\Phi^{-1}(D(f)) = \{\, \mathfrak{p} \subseteq B \mid f \notin \phi^{-1}(\mathfrak{p}) \,\} = \{\, \mathfrak{p} \subseteq B \mid \phi(f) \notin \mathfrak{p} \,\} = D(\phi(f)).$$

Proof of *iii)*: According to Corollary 2.7 on page 26, the closure $\overline{\Phi(V(\mathfrak{b}))}$ equals $V(\mathfrak{a})$ with $\mathfrak{a}$ the ideal given by

$$\mathfrak{a} = \bigcap_{\mathfrak{p} \in \Phi(V(\mathfrak{b}))} \mathfrak{p} = \bigcap_{\mathfrak{b} \subseteq \mathfrak{q}} \phi^{-1}(\mathfrak{q}),$$

$\mathfrak{p} \in \Phi(V(\mathfrak{b}))$ means that $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ for some $\mathfrak{q}$, with $\mathfrak{q} \supseteq \mathfrak{b}$.

and it holds true that

$$\mathfrak{a} = \bigcap_{\mathfrak{b} \subseteq \mathfrak{q}} \phi^{-1}(\mathfrak{q}) = \phi^{-1}\Big(\bigcap_{\mathfrak{b} \subseteq \mathfrak{q}} \mathfrak{q}\Big) = \phi^{-1}(\sqrt{\mathfrak{b}}) = \sqrt{\phi^{-1}(\mathfrak{b})}.$$

Hence $V(\mathfrak{a}) = V(\phi^{-1}(\mathfrak{b}))$, which gives the desired identity. ❏

Proposition 2.31 *With the assumptions of Proposition 2.30, we have:*

i) If ϕ is surjective, then Φ is a homeomorphism from $\operatorname{Spec} A$ onto the closed subset $V(\operatorname{Ker} \phi)$ of $\operatorname{Spec} A$;

ii) If ϕ is injective, then $\Phi(\operatorname{Spec} B)$ is dense in $\operatorname{Spec} A$. In fact, the image $\Phi(\operatorname{Spec} B)$ is dense in $\operatorname{Spec} A$ if and only if $\operatorname{Ker} \phi \subset \sqrt{0}$.

Proof: Proof of *i)*: If $\phi : A \to B$ is surjective, we may assume $B = A/\mathfrak{a}$, where $\mathfrak{a} = \operatorname{Ker} \phi$. There is an inclusion preserving one-to-one correspondence between prime ideals in $A/\mathfrak{a}$, and prime ideals in A containing $\mathfrak{a}$. This shows that Φ is a continuous bijection onto the closed subset $V(\mathfrak{a})$. It also follows that

$$\Phi(V(\mathfrak{b}/\mathfrak{a})) = \{\mathfrak{p} \in \operatorname{Spec} A | \mathfrak{b}/\mathfrak{a} \subseteq \mathfrak{p}/\mathfrak{a} \in \operatorname{Spec}(A/\mathfrak{a})\} = V(\mathfrak{b}),$$

so Φ is closed, and hence it is a homeomorphism.

Proof of *ii)*: Again, by *iii)* of Proposition 2.30, the closure of $\Phi(\operatorname{Spec} B) = \Phi(V(0))$ equals $V(\phi^{-1}(0)) = V(\operatorname{Ker}\phi)$. So $\Phi(\operatorname{Spec} B)$ is dense if and only if $V(\operatorname{Ker}\phi) = \operatorname{Spec} A$. But this happens exactly when $\operatorname{Ker}\phi \subseteq \mathfrak{p}$ for all $\mathfrak{p}$, or equivalently when $\operatorname{Ker}\phi \subseteq \sqrt{0}$. ❏

We include a few prototypical examples:

EXAMPLE 2.32 (*The spectrum* $\operatorname{Spec}(A/\mathfrak{a})$ *of a quotient.*) If $\mathfrak{a} \subseteq A$ is an ideal, the ring homomorphism $A \to A/\mathfrak{a}$ induces a continuous map

$$f\colon \operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A.$$

Note that the prime ideals in $A/\mathfrak{a}$ pull back bijectively to the prime ideals $\mathfrak{p}$ in A containing $\mathfrak{a}$, and this is precisely our closed set $V(\mathfrak{a})$. By the above proposition, the map f is a homeomorphism onto the closed subset $V(\mathfrak{a})$. This is the standard example of a *closed immersion*. We will discuss these in more detail later. ☆

Closed immersions

EXAMPLE 2.33 (*The spectrum* $\operatorname{Spec} A_f$ *of a localization.*) For an element $f \in A$ consider the localization A_f of A in the multiplicative subset $S = \{1, f, f^2, \dots\}$ and the corresponding ring homomorphism $A \to A_f$. The prime ideals in the localized ring A_f are in a natural one-to-one correspondence with the prime ideals $\mathfrak{p}$ of A not containing f; in other words, with the complement $D(f) = \operatorname{Spec} A - V(f)$. Thus the induced map $\operatorname{Spec} A_f \to \operatorname{Spec} A$ is a homeomorphism onto the open set $D(f)$ of $\operatorname{Spec} A$. This is an example of an *open immersion*. ☆

Open immersions

EXAMPLE 2.34 (*Reduction mod p.*) The reduction mod p-map $\mathbb{Z} \to \mathbb{F}_p$ induces a map $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$. The one and only point in $\operatorname{Spec} \mathbb{F}_p$ is sent to the point in $\operatorname{Spec} \mathbb{Z}$ corresponding to the maximal ideal (p). The inclusion $\mathbb{Z} \subseteq \mathbb{Q}$ of the integers in the field of rational numbers induces likewise a map $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$, that sends the unique point in $\operatorname{Spec} \mathbb{Q}$ to the generic point η of $\operatorname{Spec} \mathbb{Z}$.

For every ring A, there is a canonical map $\mathbb{Z} \to A$ which sends 1 to 1. Hence there is a canonical (and in fact, unique) map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$. This map factors through the map $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ described above if and only if A is of characteristic p, which becomes clear if one considers the diagram on the ring level:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{F}_p \arrow[dl, dashed] \\ A & \end{tikzcd}$$

and notes that the canonical map $\mathbb{Z} \to A$ factors via $\mathbb{F}_p$ if and only if A is of characteristic p. ☆

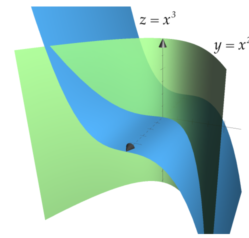
EXAMPLE 2.35 (*(The circle).*) Consider $X = \operatorname{Spec} \mathbb{R}[x,y]/(x^2+y^2-1)$. The ring map

$$\begin{aligned} \phi : \mathbb{R}[x,y]/(u^2+v^2-1) &\to \mathbb{R}[x,y]/(x^2+y^2-1) \\ u &\mapsto x^2-y^2 \\ v &\mapsto 2xy \end{aligned}$$

(originating from the 'squaring map $z \mapsto z^2$) induces a map of schemes $f : X \to X$. Note that ϕ is a map of $\mathbb{R}$-algebras, so f is a morphism of schemes over $\mathbb{R}$. ☆

EXAMPLE 2.36 (*(The twisted cubic).)* Let k be a field. The ring map $\phi \colon k[x,y,z] \to k[t]$ given by $x \mapsto t, y \mapsto t^2, z \mapsto t^3$ defines a morphism of spectra

$$f \colon \mathbb{A}^1_k \to \mathbb{A}^3_k$$



The image of f is the *twisted cubic curve* $V(\mathfrak{a}) \subset \mathbb{A}^3_k$ defined by the ideal $\mathfrak{a} = \operatorname{Ker}\phi = (y - x^2, z - x^3)$.

☆

EXAMPLE 2.37 Let k be a field. The ring map

$$\phi \colon k[x] \to k[x,y]/(xy-1)$$

induces a morphism $\operatorname{Spec} k[x,y]/(xy-1) \to \mathbb{A}^1_k$. On the level of closed points, when k is algebraically closed, this maps (a, a^{-1}) to a. Since $k[x,y]/(xy-1)$ is an integral domain, it has a unique generic point η, and this maps to the generic point of $\mathbb{A}^1_k$. Note that $\operatorname{Spec} k[x,y]/(xy-1) \simeq D(x) \subseteq \mathbb{A}^1_k$ via this morphism. In particular, the image is not closed in $\mathbb{A}^1_k$. ☆

EXAMPLE 2.38 (*The spectrum of the Gaussian integers,* $\operatorname{Spec}(\mathbb{Z}[i])$*.)* The inclusion $\mathbb{Z} \subseteq \mathbb{Z}[i]$ induces a continuous map

$$\phi \colon \operatorname{Spec}(\mathbb{Z}[i]) \to \operatorname{Spec}\mathbb{Z}.$$

We will study $\operatorname{Spec}(\mathbb{Z}[i])$ by studying the fibres of this map. If $p \in \mathbb{Z}$ is a prime, the fibre over $(p)\mathbb{Z}$ consists of those primes that contain $(p)\mathbb{Z}[i]$. These come in three flavours:

i) p stays prime in $\mathbb{Z}[i]$ and the fibre over $(p)\mathbb{Z}$ has one element, namely the prime ideal $(p)\mathbb{Z}[i]$. This happens if and only if $p \equiv 3 \mod 4$;

ii) p splits into a product of two different primes, and the fibre consists of the corresponding two prime ideals. This happens if and only if $p \equiv 1 \mod 4$;

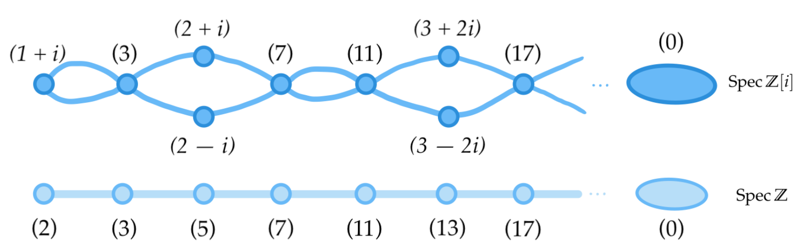
This is related to being able to write p as a sum of squares; if $p = x^2 + y^2$, then $p = (x+iy)(x-iy)$, so it is not prime in $\mathbb{Z}[i]$.

iii) p factors into a product of repeated primes (such a prime is said to 'ramify'). This happens only at the prime (2): note that

$$(2)\mathbb{Z}[i] = (2i)\mathbb{Z}[i] = (1+i)^2\mathbb{Z}[i],$$

which is not radical. So the fibre consists of the single prime $(1+i)\mathbb{Z}[i]$.

The following picture shows $\operatorname{Spec}(\mathbb{Z}[i])$:



The spectrum $\operatorname{Spec}(\mathbb{Z}[i])$

The Galois group $G = \mathrm{Gal}(\mathbb{Q}[i]/\mathbb{Q}) \simeq \mathbb{Z}/2$ acts in this example. This group is generated by the complex conjugation map, which permutes the prime ideals in $\mathrm{Spec}(\mathbb{Z}[i])$ sitting over any (p) in $\mathrm{Spec}(\mathbb{Z})$. So for instance if you look at the primes sitting over say (5), namely $(2+i)$ and $(2-i)$, you see that complex conjugation maps one into the other. Thus we picture $\mathrm{Spec}(\mathbb{Z}[i])$ as some curve lying above $\mathrm{Spec}(\mathbb{Z})$, with G permuting the points in each fibre (though some are fixed by G). ☆

EXAMPLE 2.39 Consider again the real affine line $\mathbb{A}^1_{\mathbb{R}}$ from Example 2.15. Also in this example, there is an action by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$. This group acts on $\mathbb{C}[t]$ via complex conjugation map, that is, the map that sends a polynomial $\sum a_i t^i$ to $\sum_i \bar{a}_i t^i$. The corresponding map from $\mathrm{Spec}\,\mathbb{C}[t]$ to $\mathrm{Spec}\,\mathbb{C}[t]$ defines an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Spec}\,\mathbb{C}[t]$, and the set $\mathrm{Spec}\,\mathbb{R}[t]$ is the quotient space of $\mathrm{Spec}\,\mathbb{C}[t]$ by this action. Indeed, by Example 2.15, this holds for closed points, and clearly the generic point of $\mathrm{Spec}\,\mathbb{C}[t]$ is invariant and corresponds to the generic point of $\mathrm{Spec}\,\mathbb{R}[t]$. The quotient map is the map $\mathrm{Spec}\,\mathbb{C}[t] \to \mathrm{Spec}\,\mathbb{R}[t]$ induced by the inclusion $\mathbb{R}[t] \subseteq \mathbb{C}[t]$. ☆

Exercises

(2.13) In the same vein as Example 2.34, show that a ring A is a $\mathbb{Q}$-algebra (that is, it contains a copy of $\mathbb{Q}$) if and only if the canonical map $\mathrm{Spec}\,A \to \mathrm{Spec}\,\mathbb{Z}$ factors through the generic point $\mathrm{Spec}\,\mathbb{Q} \to \mathrm{Spec}\,\mathbb{Z}$.

✲ (2.14) *Local rings.* Recall that a *local ring* is a ring A with only one maximal ideal.

a) Show that A is local if and only if $\mathrm{Spec}\,A$ has a unique closed point.

b) Give examples of local rings A so that $\mathrm{Spec}\,A$ consists of *i)* one point; *ii)* two points; *iii)* infinitely many points.

c) A map of rings $\phi : A \to B$ is said to be local if $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$. Show that ϕ is local if and only if the induced map $\Phi : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$ maps the unique closed point of $\mathrm{Spec}\,B$ to that of $\mathrm{Spec}\,A$.

d) Give an example of a ring homomorphism $\phi : A \to B$ which is not local. Describe your example in terms of the corresponding map on spectra.

✲ (2.15) Show that the second statement in Example 2.10 is true. That is, show that $\mathrm{Spec}\,A$ has just one element if and only if A is a local ring all whose non-units are nilpotent.

★

Chapter 3

Sheaves

The concept of a sheaf was conceived in the German camp for prisoners of war called Oflag XVII, where French officers taken captive during the fighting in France in the spring 1940 were imprisoned. Among them was the mathematician and lieutenant Jean Leray. In the camp he gave a course in algebraic topology(!!) during which he introduced some version of the theory of sheaves. In modern terms, Leray was aiming to compute the cohomology of a total space of a fibration in terms of invariants of the base and the fibres (and the fibration itself). To achieve this, in addition to the concept of sheaves, he also invented 'spectral sequences'.

Jean Leray
(1906 – 1998)

After the war, the theory of sheaves was developed further by Henri Cartan and Jean-Pierre Serre among others developed the theory of sheaves further, and finally the theory was brought to the state as we know it today by Alexander Grothendieck.

3.1 Sheaves and presheaves

A common theme in mathematics is to study spaces by describing them in terms of their local properties. A manifold is a space which looks locally like Euclidean space; a complex manifold is a space which looks locally like open sets in $\mathbb{C}^n$; an algebraic variety is a space that looks locally like the zero set of a set of polynomials. Here it is clear that point set topology alone is not enough to fully capture the essence of these three notions. However, in each case, the space comes equipped with a distinguished set of functions that does the job: the C^∞-functions, thr holomorphic functions, and the polynomials respectively.

Sheaves provide the general framework for discussing such functions; they are objects that satisfy basic axioms valid in each of the examples above. To explain what these axioms are, let us consider the primary example of a sheaf: the sheaf of continuous maps on a topological space X. By definition, X comes with a collection of 'open sets', and these encode what it means for a map $f\colon X \to Y$ to another topological space Y to be continuous: for every open $U \subseteq Y$, the set $f^{-1}(U)$ should be open in X. For two topological spaces X and Y, we can define for each open $U \subseteq X$, a set of continuous maps

$$C(U, Y) = \{ f\colon U \to Y \mid f \text{ is continuous } \}.$$

Note that if $V \subseteq U$ is another open set, then the restriction $f|_V$ to V of a continuous function f is again continuous, so we obtain a map

$$\rho_{UV}: \quad C(U,Y) \to C(V,Y)$$
$$f \mapsto f|_V.$$

Moreover, note that if $W \subseteq V \subseteq U$, we can restrict to W by first restricting to V, and so $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. The collection of the sets $C(U,Y)$ together with their restriction maps ρ_{UV} constitutes the *sheaf of continuous maps from X to Y*.

An essential feature of continuity is that it is a local property; f is continuous if and only if it is continuous in a neighbourhood of every point, and of course, two continuous maps that are equal in a neighbourhood of every point, are tautologically equal everywhere. A second property is that continuous functions can be glued together: Given an open covering $\{U_i\}_{i\in I}$ of an open set U, and continuous functions $f_i \in C(U_i, Y)$ that agree on the intersections $U_i \cap U_j$ (formally: for each i and j, $f_i(x) = f_j(x)$ for all $x \in U_i \cap U_j$), we can patch the maps f_i together to form a continuous map $f: U \to Y$ which satisfies $f|_{U_i} = f_i$ for each i; we simply define $f(x) = f_i(x)$ for any i such that $x \in U_i$.

Essentially, a *sheaf* on a topological space is a structure that encodes these properties. In each of the examples above, there is a corresponding sheaf of C^∞-functions, holomorphic functions, and regular functions respectively.

One may think of a sheaf as a distinguished set of functions, but they can also be much more general mathematical objects, which in a certain sense behave as sets of functions. The main aspect is that we want the distinguished properties to be preserved under restrictions to open sets, that the objects are determined from their local properties, and that 'glueing' is allowed.

Presheaves

The concept of a sheaf may be defined for any topological space, and the theory is best studied at this level of generality. We begin with the definition of a *presheaf*.

Definition 3.1 *Let X be a topological space. A* presheaf of abelian groups $\mathcal{F}$ *on X consists of the following two sets of data:*

i) for each open $U \subseteq X$, an abelian group $\mathcal{F}(U)$;
ii) for each pair of nested opens $V \subseteq U$ a group homomorphism (called restriction maps*)*

$$\rho_{UV}: \mathcal{F}(U) \to \mathcal{F}(V).$$

The restriction maps must furthermore satisfy the following two conditions:

i) for any open $U \subseteq X$, we have $\rho_{UU} = id_{\mathcal{F}(U)}$;
ii) for any three nested open subsets $W \subseteq V \subseteq U$, one has $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

We will usually write $s|_V$ for $\rho_{UV}(s)$ when $s \in \mathcal{F}(U)$. The elements of $\mathcal{F}(U)$ are usually called *sections* (or *sections over* U). The notation $\Gamma(U,\mathcal{F})$ for the group $\mathcal{F}(U)$ is also common usage; here Γ is the 'global sections'-functor (it is functorial in both U and $\mathcal{F}$). *Sections of a presheaf*

The notion of a presheaf is not confined to presheaves of abelian groups. One may speak about presheaves of sets, rings, vector spaces or whatever you want: indeed, for any category C one may define presheaves with values in C. The definition goes just like for abelian groups, the only difference being that one requires the gadgets $\mathcal{F}(U)$ to be objects from the category C, and of course, the restriction maps are all required to be morphisms in C. We are certainly going to meet sheaves with more structure than the mere structure of abelian groups, *e.g.* sheaves of rings, but they will usually have an underlying structure of abelian group, so we start with these. We will also encounter sheaves of sets; most of the results we establish for sheaves of abelian groups can be proved for sheaves of sets as well, as long as they can be formulated in terms of sets and the proofs are essentially the same.

Sheaves

We are now ready to give the main definition of this chapter:

DEFINITION 3.2 *A presheaf $\mathcal{F}$ is a* sheaf *if it satisfies the two conditions:*

i) *(Locality axiom) Given an open subset $U \subseteq X$ with an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ and sections $s, t \in \mathcal{F}(U)$. If $s|_{U_i} = t|_{U_i}$ for all i, then $s = t \in \mathcal{F}(U)$.*

ii) *(Gluing axiom) If U and $\mathcal{U}$ are as in (i), and if $s_i \in \mathcal{F}(U_i)$ is a collection of sections matching on the overlaps; that is, they satisfy*

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$$

for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ so that $s|_{U_i} = s_i$ for all i.

These two axioms mirror the properties of continuous functions mentioned in the introduction. The Locality axiom says that sections are uniquely determined from their restrictions to smaller open sets. The Gluing axiom says that you are allowed to patch together local sections to a global one, provided they agree on overlaps.

Sub(pre)sheaf

If $\mathcal{F}$ is a presheaf on X, a *subpresheaf* $\mathcal{G}$ is a presheaf such that $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ for every open $U \subseteq X$, and such that the restriction maps of $\mathcal{G}$ are induced by those of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, of course $\mathcal{G}$ is called a *subsheaf*.

There is a nice concise way of formulating the two sheaf axioms at once. For each open cover $\mathcal{U} = \{U_i\}$ of an open set $U \subseteq X$ there is a sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\ \alpha\ } \prod_i \mathcal{F}(U_i) \xrightarrow{\ \beta\ } \prod_{i,j} \mathcal{F}(U_i \cap U_j) \tag{3.1}$$

where the maps α and β are defined by the two assignments $\alpha(s) = (s|_{U_i})_i$, and $\beta(s_i) = (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$. Then $\mathcal{F}$ is a sheaf if and only if these sequences are exact. Indeed, exactness at $\mathcal{F}(U)$ means that α is injective, *i.e.* that $s|_{U_i} = 0$ implies that $s = 0$ (the Locality axiom). Exactness in the middle means that $\operatorname{Ker}\beta = \operatorname{Im}\alpha$, *i.e.* elements s_i satisfying $s_i - s_j = 0$ on $U_i \cap U_j$ must come from an element $s \in \mathcal{F}(U)$ (the Gluing axiom).

This reformulation is sometimes handy when proving that a given presheaf is a sheaf. Moreover, since $\mathcal{F}(U) = \operatorname{Ker}\beta$, we can often use it to compute $\mathcal{F}(U)$ if $\mathcal{F}(U_i)$ and $\mathcal{F}(U_i \cap U_j)$ are known.

Morphisms between (pre)sheaves

A *morphism* (or simply a *map*) *of (pre)sheaves* $\phi\colon \mathcal{F} \to \mathcal{G}$ is a collection of group homomorphisms $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ indexed by the open sets in X and compatible with the restriction maps. In other words, the following diagram commutes for each inclusion $V \subseteq U$ of open sets:

Morphism of presheaves

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\rho_{UV}"] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\phi_V"] & \mathcal{G}(V). \end{tikzcd} \tag{3.2}$$

In this way, the sheaves of abelian groups on X form a category, AbSh_X, whose objects are the sheaves and the morphisms the maps between them. The composition of two maps of sheaves is defined in the obvious way as the composition of the maps on sections. Likewise, we have the category $\mathsf{AbPrSh}_\mathsf{X}$ with the presheaves of abelian groups as objects and morphisms the maps between them.

As usual, a map ϕ between two (pre)sheaves $\mathcal{F}$ and $\mathcal{G}$ is an *isomorphism* if it has a two-sided inverse, *i.e.* a map $\psi\colon \mathcal{G} \to \mathcal{F}$ such that $\phi \circ \psi = \mathrm{id}_{\mathcal{G}}$ and $\psi \circ \phi = \mathrm{id}_{\mathcal{F}}$.

Example 3.3 (*The empty set.*) There is a subtle point about taking U to be the empty set in the definition of a sheaf. If $\mathcal{F}$ is a sheaf, we are forced to define $\mathcal{F}(\varnothing) = 0$. Indeed, the empty set is covered by the empty open covering, and since the empty product equals 0, the sheaf sequence (3.1) takes the form $0 \to \mathcal{F}(\varnothing) \to 0 \to 0$. ☆

Examples

(3.4) (*Continuous functions.*) Take $X = \mathbb{R}^n$ and let $C(X, \mathbb{R})$ be the sheaf whose sections over an open set U is the ring of continuous real valued functions on U, and the restriction maps ρ_{UV} are just the good old restriction of functions. Then $C(X, \mathbb{R})$ is a sheaf of rings (functions can be added and multiplied), and both the sheaf axioms are satisfied. Indeed, any function $f\colon X \to \mathbb{R}$, which restricts to zero on an open covering of X is the zero function. Also, given continuous functions $f_i\colon U_i \to \mathbb{R}$ that agree on the overlaps $U_i \cap U_j$, we can form the continuous function $f\colon U \to \mathbb{R}$ by setting $f(x) = f_i(x)$ for any i such that $x \in U_i$.

In fact, the argument from the beginning of this chapter shows that for any two topological spaces X and Y, the presheaf $\mathcal{F}(U) = C(U, Y)$ of continuous maps $f\colon U \to Y$ forms a sheaf (they are sheaves of sets here, because we cannot in general add or multiply maps).

(3.5) ((*Differential operators*).) Let $X = \mathbb{R}$ and let let $C^r(X, \mathbb{R})$ be the sheaf of functions $f : U \to \mathbb{R}$ which are r times continuously differentiable (note that this is a subheaf of

$C(X, \mathbb{R})$). The differential operator $D = \frac{d}{dx}$ defines a morphism of sheaves $D\colon C^r(X, \mathbb{R}) \to C^{r-1}(X, \mathbb{R})$.

(3.6) (*Holomorphic functions.*) For a second familiar example, let $X \subseteq \mathbb{C}$ be an open set. On X one has the sheaf $\mathscr{A}_X$ of holomorphic functions. That is, for any open $U \subseteq X$ the sections $\mathscr{A}_X(U)$ is the ring of complex differentiable functions on U. Just like in the example above, one checks that $\mathscr{A}_X$ forms a sheaf.

In fact, $\mathscr{A}_X$ is a subsheaf of the sheaf of continuous functions $U \to \mathbb{C}$

One can relax the condition of holomorphy to get a larger sheaf $\mathscr{K}_X$ of meromorphic functions on X. This sheaf contains $\mathscr{A}_X$ as a subsheaf, and the sections over an open U are the meromorphic functions on U. In a similar way, one can get smaller sheaves contained in $\mathscr{A}_X$ by imposing vanishing conditions on the functions. For example if $p \in X$ is any point, one has the sheaf denoted $\mathfrak{m}_p$ of holomorphic functions vanishing at p. Convince yourself that this indeed is a *sheaf of ideals of $\mathscr{A}_X$*.

Meromorphic means: holomorphic on all of U except for a set of isolated points, which are poles of the function.

Our main interest in this book will be the following:

(3.7) (*Algebraic varieties.*) Let X be an algebraic variety (*e.g.* an irreducible algebraic set in $\mathbb{A}^n_k$ or $\mathbb{P}^n_k$) with the Zariski topology. For each open $U \subseteq X$, define the presheaf

$$\mathcal{O}_X(U) = \left\{ f\colon U \to k \,\middle|\, f \text{ is regular} \right\}$$

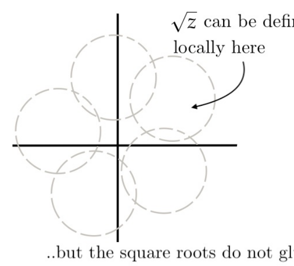
where f is regular if for each point $x \in U$, there is an affine neighbourhood for which f can be represented as a quotient of polynomials g/h with $h(x) \neq 0$.

This is indeed a sheaf: locality holds, because if $f\colon U \to k$ restricts to the zero function on an open covering, it is the zero function. If we are given regular functions $f_i\colon U_i \to k$ on an open overing U_i of U, that agree on the overlaps, they certainly glue to a continuous function $f\colon U \to k$. The function f is also regular, because it restricts to f_i on U_i, and f_i is locally expressible as g/h there.

Define $f\colon U \to k$ by $f(x) = f_i(x)$ whenever $x \in U_i$.

(3.8) (*A presheaf which is not a sheaf.*) Let us continue the set-up in Example 3.6 to exhibit an example of a presheaf which is not a sheaf. Let $X = \mathbb{C} \backslash \{0\}$, and let $\mathscr{A}_X$ denote the sheaf of holomorphic functions. $\mathscr{A}_X$ contains the subpresheaf given by

$$\mathcal{F}(U) = \{\, f \in \mathscr{A}_X(U) \mid f = g^2 \text{ for some } g \in \mathscr{A}_X(U) \,\}.$$



This is not a sheaf, because the Gluing axiom fails: the function $f(z) = z$ is holomorphic, and has a holomorphic square root near any point $x \in X$, but it is not possible to glue these together to a global square root function $\sqrt{z}$ on all of X. Note however, that the Locality axiom holds, because $\mathcal{F}$ is a subpresheaf of the sheaf $\mathscr{A}_X$ (which does satisfy Locality).

(3.9) (*Constant presheaf.*) For any space X and any abelian group A, one has the *constant presheaf* defined by $A(U) = A$ for any nonempty open set U (and $A(\varnothing) = 0$). This is not a sheaf in general.

Constant (pre)sheaves

For instance, if $X = U_1 \cup U_2$ is a disjoint union, and $A = \mathbb{Z}$, then any choice of integers $a_1, a_2 \in \mathbb{Z}$ will give sections of $A(U_1)$ and $A(U_2)$, and they automatically match on the intersection, which is empty. But if $a_1 \neq a_2$, they cannot be glued to an element in $A(X) = \mathbb{Z}$.

In fact, the constant presheaf is a sheaf if and only if any two non-empty open subsets of X have non-empty intersection. Algebraic varieties with the Zariski topology are examples of such spaces.

There is a quick fix for this. We can define the following sheaf A_X by letting

$$A_X(U) = \{\, f\colon U \to A \mid f \text{ is continuous} \,\}$$

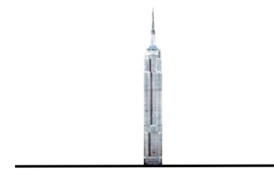
where we give A the discrete topology. For a connected open set U, we then have $A_X(U) = A$. More generally, since f must be constant on each connected component of U, it holds true that

$$A_X(U) \simeq \prod_{\pi_0(U)} A, \tag{3.3}$$

where $\pi_0(U)$ denotes the set of connected components of U. As before, we also must put $A_X(\varnothing) = 0$.

The new presheaf A_X is called the *constant sheaf* on X with value A. It is a sheaf (e.g by the final paragraph of Example 3.4). That being said, the sheaf A_X is not quite worthy of its name as it is not quite constant.

(3.10) *(Skyscraper sheaf.)* Let A be a group. For $x \in X$, we can define a presheaf $A(x)$ by



Skyscraper sheaves

$$A(x)(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

It is straighforward to check that this is a sheaf, usually called a *skyscraper sheaf*.

☆

Exercises

(3.1) Let X be the set with two elements with the discrete topology. Find a presheaf on X which is not a sheaf.

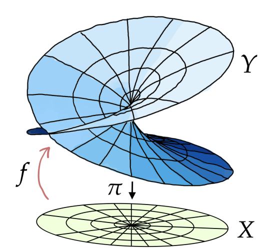
(3.2) In the notation of Example 3.6, the differential operator gives a map of sheaves $D\colon \mathscr{A}_X \to \mathscr{A}_X$, where as previously $X \subseteq \mathbb{C}$ is an open set. Show that the assignment

$$\mathscr{A}(U) = \{\, f \in \mathscr{A}_X(U) \mid Df = 0 \,\}$$

defines a subsheaf $\mathscr{A}$ of $\mathscr{A}_X$. Show that if U is a connected open subset of X, one has $\mathscr{A}(U) = \mathbb{C}$. In general for a not necessarily connected set U, show that $\mathscr{A}(U) = \prod_{\pi_0(U)} \mathbb{C}$ where the product is taken over the set $\pi_0(U)$ of connected components of U.

(3.3) *A Riemann surface.* Let $X = \mathbb{C}\backslash\{0\}$ and let Y denote the complex 'parabola'

$$Y = \{\, (x, y) \mid y^2 = x \,\} \subset \mathbb{C} \times \mathbb{C}$$



Let $\pi\colon Y \to X$ be the projection onto the first factor. Consider the presheaf on X given by

$$\mathcal{G}(U) = \{\, f\colon U \to Y \mid f \text{ is holomorphic, and } \pi \circ f = \mathrm{id}_U \,\}.$$

Show that $\mathcal{G}$ is a sheaf (of sets), and compute $\mathcal{G}(X)$.

(3.4) Let $X \subseteq \mathbb{C}$ be an open set, and assume that $a_1, \ldots, a_r$ are distinct points in X and $n_1, \ldots, n_r$ natural numbers. Define $\mathcal{F}(U)$ to be the set of those functions meromorphic in U, holomorphic away from the a_i's and having a pole order bounded by n_i at a_i. Show that $\mathcal{F}$ is a sheaf of abelian groups. Is it a sheaf of rings?

✲ (3.5) *(The sheaf of homomorphisms).* Given two presheaves $\mathcal{F}$ and $\mathcal{G}$, we may form a presheaf $\mathscr{H}om_X(\mathcal{F},\mathcal{G})$ by letting the sections over an open U be given by

$$\mathscr{H}om_X(\mathcal{F},\mathcal{G})(U) = \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U), \tag{3.4}$$

and letting the restriction maps, well, be the restrictions: if $V \subseteq U$ is another open sets and $\phi\colon \mathcal{F}|_U \to \mathcal{G}|_U$ is a map, the restriction of ϕ to V is simply the restriction $\phi|_V$. Show that $\mathscr{H}om_X(\mathcal{F},\mathcal{G})$ is a sheaf whenever $\mathcal{G}$ is a sheaf.

★

3.2 Stalks

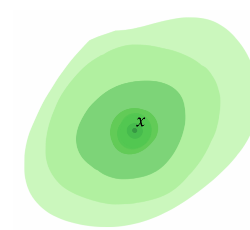
Suppose we are given a presheaf $\mathcal{F}$ of abelian groups on X. With every point $x \in X$ there is an associated abelian group $\mathcal{F}_x$ called the *stalk* of $\mathcal{F}$ at x. The elements of $\mathcal{F}_x$ are called *germs of sections* near x; they are essentially the sections of $\mathcal{F}$ defined in some sufficiently small neighbourhood of x.

Stalk of a (pre)sheaf at a point
Germs of sections

The group $\mathcal{F}_x$ is more formally defined as a *direct limit* of the groups $\mathcal{F}(U)$ as we run through the set of open neighborhoods U containing x (ordered by inclusion):

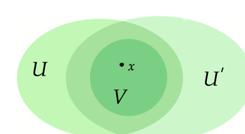
For the reader unfamiliar with direct limits, see Appendix A.

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$$



More concretely, the group $\mathcal{F}_x$ can be defined as follows. We begin with the *disjoint* union $\coprod_{x \in U} \mathcal{F}(U)$ whose elements we index as pairs (s, U) where U is any open neighbourhood of x and s is a section of $\mathcal{F}(U)$. We want to identify sections that coincide near x; that is, we declare (s, U) and (s', U') to be equivalent, and write $(s, U) \sim (s', U')$, if there is an open $V \subseteq U \cap U'$ with $x \in V$ such that s and s' coincide on V; that is, if one has

$$s|_V = s'|_V.$$



This is clearly a reflexive and symmetric relation. It is transitive as well: if $(s, U) \sim (s', U')$ and $(s', U') \sim (s'', U'')$, one may find open neighbourhoods $V \subseteq U \cap U'$ and $V' \subseteq U' \cap U''$ of x over which s and s', respectively s' and s'', coincide. Clearly s and s'' then coincide over the intersection $V' \cap V$. Thus the relation $\sim$ is an equivalence relation.

Definition 3.11 *The* stalk $\mathcal{F}_x$ *at* $x \in X$ *is defined as the set of equivalence classes*

$$\mathcal{F}_x = \coprod_{x \in U} \mathcal{F}(U) / \sim .$$

In case $\mathcal{F}$ is a sheaf of abelian groups, the stalks $\mathcal{F}_x$ are all abelian groups. This is not a priori obvious, because sections over different open sets can not be added. However if (s, U) and (s', U') are given, the restrictions $s|_V$ and $s'|_V$ to any open $V \subseteq U \cap U'$ can be added, and this suffices to define an abelian group structure on the stalks.

The germ of a section

For any neighbourhood U of $x \in X$, there is a natural map $\mathcal{F}(U) \to \mathcal{F}_x$ sending a section s to the equivalence class where the pair (s, U) belongs. This class is called the *germ* of s at x, and a common notation for it is s_x. The map is a homomorphism of abelian groups (rings, modules,..) as one easily verifies. One has $s_x = (s|_V)_x$ for any other open neighbourhood V of x contained in U, or in other words, the following diagram commutes:

Germ of a section

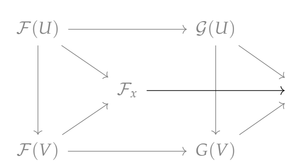
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\rho_{UV}"'] & \mathcal{F}_x \\ \mathcal{F}(V). \arrow[ur] & \end{tikzcd} \tag{3.5}$$

When working with sheaves and stalks, it is important to remember the three following properties; the two first follow right away from the definition, and the third one is easily deduced from the two first.

- The germ s_x of s vanishes if and only if s vanishes on some neighbourhood of x, *i.e.* there is an open neighbourhood U of x with $s|_U = 0$.
- All elements of the stalk $\mathcal{F}_x$ are germs, *i.e.* every one is of the shape s_x for some section s over an open neighbourhood of x.
- The sheaf $\mathcal{F}$ is the zero sheaf if and only if all stalks are zero, *i.e.* $\mathcal{F}_x = 0$ for all $x \in X$.

EXAMPLE 3.12 Let $X = \mathbb{C}$, and let $\mathscr{A}_X$ be the sheaf of holomorphic functions in X. What is the stalk $\mathscr{A}_{X,x}$ at a point x? If f and g are two sections of $\mathscr{A}_X$ over a neigbourhood U of the point x having the same germ at p, the fact that f and g admit Taylor series expansions around x, implies that $f = g$ in the connected component containing x of the set where they both are defined. Thus the stalk $\mathscr{A}_{X,x}$ is naturally identified with the ring of power series converging in a neighbourhood of x. ☆

Morphisms of (pre)sheaves induce maps of stalks



A map $\phi \colon \mathcal{F} \to \mathcal{G}$ between two presheaves $\mathcal{F}$ and $\mathcal{G}$ induces for every point $x \in X$ a map

$$\phi_x \colon \mathcal{F}_x \to \mathcal{G}_x$$

between the stalks. Indeed, one may send a pair (s, U) to the pair $(\phi_U(s), U)$, and since ϕ behaves well with respect to restrictions, this assignment is compatible with the equivalence relations; if (s, U) and (s', U') are equivalent and s and s' coincide on an open set $V \subseteq U \cap U'$, the diagram (3.5) gives

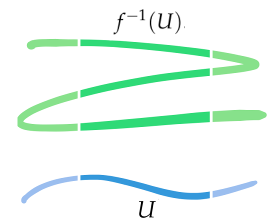
$$\phi_U(s)|_V = \phi_V(s|_V) = \phi_V(s'|_V) = \phi_{U'}(s')|_V.$$

Here we have $(\phi \circ \psi)_x = \phi_x \circ \psi_x$ and $(\mathrm{id}_{\mathcal{F}})_x = \mathrm{id}_{\mathcal{F}_x}$, so the assignments $\mathcal{F} \mapsto \mathcal{F}_x$ and $\phi \mapsto \phi_x$ define a functor from the category of sheaves to the category of abelian groups.

✲ **Exercise 3.6** Let $\phi, \psi : \mathcal{F} \to \mathcal{G}$ be maps of presheaves, with $\mathcal{G}$ a sheaf. Prove that $\phi = \psi$ if and only if ϕ, ψ induce the same maps on stalks, i.e., $\phi_x = \psi_x$ for every $x \in X$. ★

✲ **Exercise 3.7** Let $\phi : \mathcal{F} \to \mathcal{G}$ be maps of sheaves. Prove that ϕ is an isomorphism if and only if ϕ_U is an isomorphism for every open set U. ★

3.3 *The pushforward of a sheaf*



If $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X, we can define a sheaf $f_*\mathcal{F}$ on Y by specifying the sections of $f_*\mathcal{F}$ over the open set $U \subseteq Y$ to be

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U),$$

The restriction maps $\mathcal{F}(f^{-1}U) \to \mathcal{F}(f^{-1}V)$ are those coming from $\mathcal{F}$.

Definition 3.13 *The sheaf $f_*\mathcal{F}$ is called the* pushforward sheaf *or* the direct image *of $\mathcal{F}$.*

It is straightforward to see that $f_*\mathcal{F}$ is a sheaf and not merely a presheaf. Indeed, if $\{U_i\}_i$ is an open covering of U, then $\{f^{-1}(U_i)\}$ is an open covering of $f^{-1}(U)$. A set of gluing data for $f_*\mathcal{F}$ and the given covering consists of sections $s_i \in \Gamma(U_i, f_*\mathcal{F}) = \Gamma(f^{-1}U_i, \mathcal{F})$ matching on the intersections, which means that they coincide in $\Gamma(U_i \cap U_j, f_*\mathcal{F}) = \Gamma(f^{-1}U_i \cap f^{-1}U_j, \mathcal{F})$. These may therefore be glued together to a section in $\Gamma(f^{-1}U, \mathcal{F}) = \Gamma(U, f_*\mathcal{F})$, as $\mathcal{F}$ is a sheaf. The Locality axiom follows for $f_*\mathcal{F}$ because it holds for $\mathcal{F}$.

Example 3.14 Let $\iota\colon \{x\} \to X$ be the inclusion of a closed point in X. If $\mathcal{A}$ is the constant sheaf of a group A on $\{x\}$, then $\iota_*\mathcal{A}$ is the skyscraper sheaf $A(x)$ from Example 3.10 on page 47. More generally, when $\{x\}$ is not necessarily closed, the pushforward sheaf $\iota_*\mathcal{A}$ will be a Godement sheaf with $\iota_*(\mathcal{A})_y = A$ when $y \in \overline{\{x\}}$ and $\iota_*\mathcal{A}_y = 0$ otherwise. ☆

The pushforward also depends functorially on the map f:

Lemma 3.15 *If $g\colon X \to Y$ and $f\colon Y \to Z$ are continuous maps between topological spaces, and $\mathcal{F}$ is a sheaf on X, one has*

$$(f \circ g)_*\mathcal{F} = f_*(g_*\mathcal{F}).$$

(This is indeed an equality, not merely an isomorphism.)

Exercise 3.8 Prove Lemma 3.15. ★

Exercise 3.9 Denote by $\{*\}$ a one point set. Let $f\colon X \to \{*\}$ be the one and only map. Show that $f_*\mathcal{F} = \Gamma(X, \mathcal{F})$ (where strictly speaking $\Gamma(X, \mathcal{F})$ stands for the constant sheaf on $\{*\}$ with value $\Gamma(X, \mathcal{F})$). ★

3.4 *Sheaves defined on a basis*

Recall that a *basis* for a topology on X is a collection of open subsets $\mathscr{B}$ such that any open set of X can be written as a union of members of $\mathscr{B}$. In many situations it turns out to be convenient to define a sheaf by saying what it should be on a specific basis for the topology on X. In this section we will see that there exists a unique way to produce a sheaf, given sections that glue together over open subsets from $\mathscr{B}$.

Let us first make the following definition:

Definition 3.16 *A $\mathscr{B}$-presheaf $\mathcal{F}$ consists of the following data:*

i) For each $U \in \mathscr{B}$, an abelian group $\mathcal{F}(U)$;

ii) For all $U \subseteq V$, with $U, V \in \mathscr{B}$, a restriction map $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$.

As before, these are required to satisfy the relations $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ and $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$. A $\mathscr{B}$-sheaf is a $\mathscr{B}$-presheaf satisfying the Locality and Gluing axioms for open sets in $\mathscr{B}$.

Since the intersections $V \cap V'$ of two sets $V, V' \in \mathscr{B}$ need not lie in $\mathscr{B}$, we need to clarify what we mean in the Gluing axiom. Given a covering of $U \in \mathscr{B}$ by subsets $U_i \in \mathscr{B}$. If $s_i \in \mathcal{F}(U_i)$ are sections such that $s_i|_V = s_j|_V$ for every i, j and every $V \subset U_i \cap U_j$ such that $V \in B$, then the s_i should glue together to an element in $s \in \mathcal{F}(U)$.

The whole point with the notion of $\mathscr{B}$-sheaves is expressed in the following proposition.

We will use this construction when we define the structure sheaf in Chapter 4

Proposition 3.17 *Let X be a topological space and let $\mathscr{B}$ be a basis for the topology on X. Then:*

i) Every $\mathscr{B}$-sheaf $\mathcal{F}_0$ extends to a sheaf $\mathcal{F}$ on X (unique up to unique isomorphism);

ii) If $\phi_0 \colon \mathcal{F}_0 \to \mathcal{G}_0$ is a morphism of $\mathscr{B}$-sheaves, then ϕ_0 extends uniquely to a morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ between the corresponding sheaves;

iii) The stalk of the extended sheaf $\mathcal{F}$ at a point x can be computed as

$$\mathcal{F}_x = \varinjlim_{U \in \mathscr{B},\, U \ni x} \mathcal{F}_0(U).$$

Proof: For clarity, let us denote by $\rho^0_{VV'} \colon \mathcal{F}_0(U) \to \mathcal{F}_0(V)$ the restriction map over subsets $V, V' \in \mathscr{B}$ with $V' \subseteq V$.

Recall Example A.9 on page 415 stating that sections of a sheaf over some open set U may be expressed as an inverse limit of sections over members of a basis contained in U. Bearing this in mind, one is led to defining $\mathcal{F}(U)$ as the inverse limit of the $\mathcal{F}_0(V)$, where V runs over the basis of open subsets contained in the open set U:

$$\begin{aligned}\mathcal{F}(U) &= \varprojlim_{W \subseteq U, W \in \mathscr{B}} \mathcal{F}_0(W) \\ &= \Big\{ (s_W) \in \prod_{W \subseteq U, W \in \mathscr{B}} \mathcal{F}_0(W) \mid \rho^0_{WW'}(s_W) = s'_W \text{ for all } W' \subseteq W, W' \in \mathscr{B} \Big\}.\end{aligned}$$

Note that when $V \subseteq U$ and $V \in \mathscr{B}$, there is a canonical map

$$\pi_{UV} \colon \mathcal{F}(U) \to \mathcal{F}_0(V)$$

induced by the projection $\prod_{W \subset U,\, W \in \mathscr{B}} \mathcal{F}_0(W) \to \mathcal{F}_0(V)$ onto the 'V-th' factor. These maps commute with the restriction maps $\rho^0_{VV'}$, in the sense that $\rho^0_{VV'} \circ \pi_{UV} = \pi_{UV'}$ for all

$V' \subset V \subset U$ with $V, V' \in \mathscr{B}$. In particular, we see that there is a canonical isomorphism $\pi_{VV} \colon \mathcal{F}(V) \to \mathcal{F}_0(V)$ for every $V \in \mathscr{B}$ (well, if $V \in \mathscr{B}$, it will be largest in the set of members of $\mathscr{B}$ contained in V, (by Exercise A.7 on page 416)).

Furthermore, for each pair of open subsets $U' \subseteq U$, the projections

$$\prod_{W \subset U, W \in \mathscr{B}} \mathcal{F}_0(W) \to \prod_{W \subset U', W \in \mathscr{B}} \mathcal{F}_0(W)$$

induce a map

$$\rho_{UU'} \colon \mathcal{F}(U) \to \mathcal{F}(U').$$

By construction this map has the property that $\rho_{UU''} = \rho_{U'U''} \circ \rho_{UU'}$ for each chain of opens $U'' \subseteq U' \subseteq U$, which means that $\mathcal{F}$ is a presheaf on X which extends $\mathcal{F}_0$. It is not hard to check that the two sheaf axioms are satisfied.

We note that *iii)* follows immediately, since we may compute the inverse limit that gives the stalk at a given point using subsets of $\mathscr{B}$, which are cofinal in the set of open neighbourhoods of the point.

Proof of *ii)*: saying $\phi_0 \colon \mathcal{F}_0 \to \mathcal{G}_0$ is a map of $\mathscr{B}$-sheaves amounts to saying that the following diagram commutes for each pair $V' \subseteq V$ of members of $\mathscr{B}$:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "(\phi_0)_V"] \arrow[d] & \mathcal{G}(V) \arrow[d] \\ \mathcal{F}(V') \arrow[r, "(\phi_0)_{V'}"'] & \mathcal{G}(V'). \end{tikzcd}$$

Taking the inverse limit over all open subsets V from $\mathscr{B}$ contained in U, we obtain a natural map

$$\mathcal{F}(U) = \varprojlim_{W \in \mathscr{B}, W \subset U} \mathcal{F}_0(W) \to \varprojlim_{W \in \mathscr{B}, W \subset U} \mathcal{G}_0(W) = \mathcal{G}(U),$$

which extends ϕ_0. Again this must be unique, as it is completely determined by ϕ_0 on stalks. ❑

Chapter 4

Schemes

A scheme has two layers of structures; the bottom one is a topological space on top of which resides a sheaf of rings, called the *structure sheaf*. Without further restrictions, such double structures are called 'ringed spaces', and they form a category which is much larger than the collection of schemes. A scheme is a ringed space that is locally affine; *i.e.* it has an open cover whose members are isomorphic to affine schemes. Before proceeding towards the general definition of scheme, as a warm up, we introduce the structure sheaf on the spectrum of a ring. This makes the spectrum into an affine scheme. Affine schemes then serve as the building blocks of general schemes, and they are cornerstones of the theory. To fully grasp the general definition of a scheme, it is all important to master the mechanics of the affine schemes.

4.1 The structure sheaf on the spectrum of a ring

We have now come to point where we define the structure sheaf on the topological space $\operatorname{Spec} A$. This is a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$ whose stalks all are local rings, so that the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is what one calls a *locally ringed space*.

The two most important properties of the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ are the following:

- Sections over distinguished opens: $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$;
- Stalks: $\mathcal{O}_{\operatorname{Spec} A, x} = A_{\mathfrak{p}_x}$.

These are the two main properties that one uses all the time when working with the structure sheaf on an affine scheme. Moreover, as we shall see, they even characterize the structure sheaf unambiguously.

Motivation

The structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on the prime spectrum $\operatorname{Spec} A$ of a ring A is modelled on the sheaf of regular functions on an affine variety X. Before giving the main definition we recall that situation.

Let $A = A(X)$ be the coordinate ring of X; that is, A is the ring of globally defined regular functions on X. The fraction field K of A is the field of rational functions on X, *i.e.* the functions which are regular in some open subset $U \subset X$. For different open sets U, the set of functions regular in U form different subrings $\mathcal{O}_X(U)$ of K, and if $V \subseteq U$ is another open contained in U, the ring of regular functions $\mathcal{O}_X(V)$ on V of course contains the ring $\mathcal{O}_X(U)$ of those regular on the bigger set U. The restriction map in this case is nothing but the inclusion $\mathcal{O}_X(U) \subseteq \mathcal{O}_X(V)$; it simply considers the functions in $\mathcal{O}_X(U)$ to lie in $\mathcal{O}_X(V)$.

Functions regular on the distinguished open set $D(f) = \{ x \in X \mid f(x) \neq 0 \}$ are allowed to have powers of f in the denominator, and they constitute the subring $A_f \subseteq K$ of elements shaped as af^{-n} with $a \in A$ and n a non-negative integer. As explained in (2.3), if $D(g)$ is another distinguished open set contained in $D(f)$, *i.e.* $D(g) \subseteq D(f)$, then one can write $g^m = cf$ for some $c \in A$ and some suitable $m \in \mathbb{N}$, and hence there is a localization $A_f \subseteq A_g$ (since $f^{-1} = cg^{-m}$).

If one tries to carry out the above construction for a general ring A, one quickly runs into a few obstacles. For instance, there is no natural field K in which the rings $\mathcal{O}_X(U)$ lie as subrings. More dramatically, the localization maps $A_f \to A_g$ may fail to be injective. This happens already in the case $X = \operatorname{Spec} A$, for $A = k[x,y]/(xy)$, which corresponds to the union of x-axis and the y-axis in the affine plane. Since $xy = 0$, the element x maps to 0 via the localization map $A \to A_y$. Geometrically, the regular function x vanishes identically on the open $D(y)$ where $y \neq 0$, and the regular function y vanishes on $D(x)$. So this is by no means a big mystery, it naturally appears once we allow reducible spaces into the mix.

Definition of the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$

One may straight away write down an explicit definition of the structure sheaf (as several texts do), but many readers experience this definition as coming out of the blue. We instead take the approach of defining $\mathcal{O}_{\operatorname{Spec} A}$ as a $\mathscr{B}$-sheaf which has a virtue of being more intuitive. In short, we define $\mathcal{O}_{\operatorname{Spec} A}$ as the sheaf satisfying the two key properties above – they characterize the sheaf uniquely.

Motivated by the above discussion, it makes sense to require the sections of the structure sheaf over $D(f)$ to be the localized ring A_f. There is a subtlety here, because different f's might give identical $D(f)$'s and to avoid choices, we prefer to use a more canonical localization and replace the multiplicative system $\{1, f, f^2, \dots\}$ by its saturation: the multiplicative system

Of course in the end, $\mathcal{O}_X(D(f))$ will be isomorphic to A_f

$$S_{D(f)} = \{ s \in A \mid s \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in D(f) \}.$$

This set does not depend on the particular element f, but only on the set $D(f)$. Furthermore, $S_{D(f)} \subseteq S_{D(g)}$ whenever $D(g) \subseteq D(f)$, so we get naturally induced localization maps $S_{D(f)}^{-1} A \to S_{D(g)}^{-1} A$.

Definition 4.1 *Let $\mathscr{B}$ be the collection of distinguished open subsets $D(f)$. We define the $\mathscr{B}$-presheaf $\mathcal{O}$ by*

$$\mathcal{O}(D(f)) = S_{D(f)}^{-1}A,$$

and for $D(g) \subseteq D(f)$ we let the restriction map be localization map $S_{D(f)}^{-1}A \to S_{D(g)}^{-1}A$.

There is a canonical localization map $\tau\colon A_f \to S_{D(f)}^{-1}A$ since $f \in S_{D(f)}$. The following lemma says that, using τ, we may identify the ring $\mathcal{O}(D(f)) = S_{D(f)}^{-1}A$ with the ring A_f, as we desired.

Lemma 4.2 *The map τ is an isomorphism, permitting us to identify $\mathcal{O}(D(f)) = A_f$.*

Proof: The point is that each element $s \in S_{D(f)}$ does not lie in $\mathfrak{p}$ for any $\mathfrak{p} \in D(f)$; in other words, one has $D(f) \subseteq D(s)$. This is equivalent to $\sqrt{(f)} \subseteq \sqrt{(s)}$, so one may write $f^n = cs$ for some $c \in A$ and $n \in \mathbb{N}$. Suppose that $af^{-m} \in A_f$ maps to zero in $S_{D(f)}^{-1}A$. This means that $sa = 0$ for some $s \in S_{D(f)}$. But then $f^n a = csa = 0$, and therefore $a = 0$ in A_f. This shows that the map τ is injective. To see that is surjective, take any as^{-1} in $S_{D(f)}^{-1}A$ and write this as $as^{-1} = ca(f^n)^{-1} = caf^{-n}$. ❑

The notation $S_{D(f)}^{-1}A$ will only be present in the definition of $\mathcal{O}$; we will usually stick to writing $\mathcal{O}(D(f)) = A_f$ from now on, bearing in mind that it is defined in terms of a canonical localization.

Proposition 4.3 *$\mathcal{O}$ is a $\mathscr{B}$-sheaf of rings.*

Proof: We are given a distinguished set $D(f)$ and an open covering $D(f) = \bigcup_{i \in I} D(f_i)$, where we by quasi-compactness may assume that the index set I is finite. Of course then $D(f_i) \subseteq D(f)$, and we have the 'localize further'-maps $\rho_i\colon A_f \to A_{f_i}$ and $\rho_{ij}\colon A_{f_i} \to A_{f_i f_j}$.

We start by observing that we may assume that $A = A_f$ (in other words, that $f = 1$ and $X = D(f)$.). Indeed, one has $(A_f)_{f_i} = A_{f_i}$ and $(A_f)_{f_i f_j} = A_{f_i f_j}$ since $f_i^{n_i} = h_i f$ for suitable natural numbers n_i.

We will use the following 'partition of unity trick': since $\operatorname{Spec} A$ is covered by the open sets $D(f_i)$, it is covered by the $D(f_i^{n_i})$ as well. Thus we may write $1 = \sum_i b_i f_i^{n_i}$ for some elements $b_i \in A$,

Locality: If $a \in A$ is an element such that $a/1 = 0$ in each localization A_{f_i}, then this means that $f_i^{n_i} \cdot a = 0 \in A$ for some integers n_i. Since there are only finitely many n_i, we may choose an n that works for all f_i, that is, $f_i^n \cdot a = 0$ in A. But this means that a is killed by all elements in the ideal $(f_1^n, \ldots, f_r^n)$. By the above remark the element '1' belongs to this ideal, so $a = 0$ in A.

Gluing: We need to show that given a sequence of elements $a_i \in A_{f_i}$ such that a_i and a_j are mapped to the same element in $A_{f_i f_j}$ for every pair i, j of indices, then there should be an $a \in A$, such that every a_i is the image of a in A_{f_i}, i.e. $\rho_i(a) = a_i$.

Each a_i can be written as $a_i = b_i/f_i^{n_i}$ where $b_i \in A$, and since the indices are finite in number, one may replace n_i with $n = \max_i n_i$. That a_i and a_j induce the same element in the localization $A_{f_i f_j}$ means that we have the equations

$$f_i^N f_j^N (b_i f_j^n - b_j f_i^n) = 0, \tag{4.1}$$

where N a priori depends on i and j, but again due to there being only finitely many indices, it can be chosen to work for all. Equation (4.1) gives

$$b_i f_i^N f_j^m - b_j f_j^N f_i^m = 0 \tag{4.2}$$

where $m = N + n$. Putting $b_i' = b_i f_i^N$ we see that a_i equals b_i'/f_i^m in A_{f_i}, and equation (4.2) takes the form

$$b_i' f_j^m - b_j' f_i^m = 0. \tag{4.3}$$

By the 'partition of unity trick' as above, may write $1 = \sum_i c_i f_i^m$. Defining $a = \sum_i c_i b_i'$, we find

$$a f_j^m = \sum_i c_i b_i' f_j^m = \sum_i c_i b_j' f_i^m = b_j' \sum_i c_i f_i^m = b_j',$$

and hence $a = b_j'/f_j^m$ in A_{f_j}. ❏

Using Proposition 3.17 on page 51 of Chapter 3, we may now make the following definition:

Definition 4.4 *We let $\mathcal{O}_{\operatorname{Spec} A}$ be the unique sheaf extending the $\mathcal{B}$-sheaf $\mathcal{O}$.*

Explicitly, the sections of $\mathcal{O}_{\operatorname{Spec} A}$ over an open set $U \subset X = \operatorname{Spec} A$, are given by the inverse limit of localizations

$$\mathcal{O}_X(U) = \varprojlim_{D(f)\subseteq U} \mathcal{O}(D(f)) = \varprojlim_{D(f)\subseteq U} A_f. \tag{4.4}$$

The distinguished open subsets of U form a directed set when ordered by inclusion, so the first limit is clear. For the second, which relies on the identification in Lemma 4.2 and which is the one mostly used, one tacitly chooses one f for each open $D(f)$. Thus $\mathcal{O}_X(U)$ is an A-module, with universal restriction maps into each of the localizations in the inverse system

$$\begin{tikzcd}
 & A_f \arrow[dr] & & & \\
\cdots \arrow[ur] \arrow[dr] & & A_g \arrow[dr] & & \cdots \\
 & A_{f'} \arrow[ur] \arrow[dr] & & A_h \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & A_{g'} \arrow[ur] \arrow[dr] & & \ddots \\
 & A_{f''} \arrow[ur] & & \ddots &
\end{tikzcd}$$

That being said, we will basically never need to know the group $\mathcal{O}_X(U)$ for U other than a distinguished open set $U = D(f)$. All that matters is that $\mathcal{O}_{\operatorname{Spec} A}$ is the unique sheaf that indeed satisfies the two main properties we want:

Proposition 4.5 *The sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$ as defined above is a sheaf of rings satisfying the two paramount properties, namely*

i) Sections over distinguished opens: $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$;

ii) Stalks: $\mathcal{O}_{\operatorname{Spec} A,x} = A_{\mathfrak{p}_x}$.

In particular, $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$.

Proof: We defined $\mathcal{O}_{\operatorname{Spec} A}$ so that the first property would hold.

For ii), we may compute the stalk using distinguished open sets:

$$\varinjlim_{D(f)\ni x} \mathcal{O}_X(D(f)) = \varinjlim_{f\notin \mathfrak{p}_x} A_f = A_{\mathfrak{p}_x}$$

(See also Example A.5 on page 414.)

The last statement follows from *i)* by taking $f = 1$. ❏

It is worthwhile to consider the special case when A has no zero-divisors. In that case, all the localizations A_f are subrings of the fraction field $K(A)$ of A and the localization maps $A_g \to A_f$ are simply inclusions. Then the intuition from varieties is correct: we can think of elements in $\mathcal{O}_{\operatorname{Spec} A}(U)$ simply as elements g/h in the fraction field of A.

Example 4.6 Consider $X = \operatorname{Spec} \mathbb{Z}$. Then $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(D(p)) = \mathbb{Z}[1/p]$ for each prime number p. The stalks of $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}$ are given by $\mathcal{O}_{\operatorname{Spec} \mathbb{Z},p} = \mathbb{Z}_{(p)}$ in each of the closed points (p) and $\mathcal{O}_{\operatorname{Spec} \mathbb{Z},(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$ at the generic point. ☆

Example 4.7 Let $X = \operatorname{Spec} \mathbb{C}[x]$. Then the stalk of $\mathcal{O}_X$ at the generic point $\eta = \{(0)\}$ is given by $\mathcal{O}_{X,\eta} = \mathbb{C}(x)$. At any closed point $p \in X$, the stalk is given by $\mathcal{O}_{X,p} = \mathbb{C}[x]_{(x-a)}$. ☆

Example 4.8 Let us continue Example 2.12 about spectra of DVR's. In $X = \operatorname{Spec} A$, the spectrum of a DVR, we have three open sets $\varnothing$, η, and X. The structure sheaf takes the following values at these opens:

$$\mathcal{O}_X(\varnothing) = 0, \qquad \mathcal{O}_X(X) = A, \qquad \mathcal{O}_X(\eta) = A_x = K,$$

where K denotes the fraction field of A. The stalks are given by $\mathcal{O}_{X,x} = A_{(x)} = A$ and $\mathcal{O}_{X,\eta} = A_{(0)} = K$. ☆

Example 4.9 While it is straightforward to compute $\mathcal{O}_X$ over an open set $U = D(f)$, one can sometimes use the sheaf exact sequence to compute over other open sets. See Section 6.2 for an explicit computation illustrating this. ☆

Example 4.10 One may use the structure sheaf to give another proof that $X = \operatorname{Spec} A$ being disconnected implies that A is a direct product of rings. Suppose $X = U_1 \cup U_2$, where U_1, U_2 are open and closed subsets and $U_1 \cap U_2 = \varnothing$. The sheaf exact sequence takes the form

$$0 \longrightarrow \mathcal{O}_X(X) = A \longrightarrow \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \longrightarrow \mathcal{O}_X(U_1 \cap U_2) = 0,$$

and we deduce that $A \simeq \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$. ☆

EXERCISE 4.1 *(A-module structure on $\mathcal{O}_{\operatorname{Spec} A}(U)$.)* Let $a \in A$, show that there is a map of sheaves $[a] \colon \mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}_{\operatorname{Spec} A}$, inducing multiplication by a both on $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$ and on the stalks $A_\mathfrak{p}$. HINT: For each distinguished open subset $D(f)$ of $\operatorname{Spec} A$ define $[a] \colon \mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f \to \mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$ as the multiplication by a map; verify that this is a map of $\mathscr{B}$-sheaves. ★

4.2 *The sheaf associated to an A-module*

The construction of the structure sheaf $\mathcal{O}_X$ works more generally: For any A-module M we we can construct a sheaf $\widetilde{M}$ on $\operatorname{Spec} A$. We define it on the distinguished open sets $D(f)$ by

$$\widetilde{M}(D(f)) = M_f,$$

and let the restriction maps be the canonical localization maps: when $D(g) \subseteq D(f)$, it holds true that $g^n = af$ for some a and some n, and there is the canonical localization map $M_f \to M_g$ sending mf^{-r} to $a^r m g^{-nr}$. The same proof as for $\mathcal{O}_X$ (Proposition 4.3 on page 55) shows that this is actually a $\mathscr{B}$-sheaf, and hence gives rise to a unique *sheaf* on X, which we continue to denote by $\widetilde{M}$.

The sheaf $\widetilde{M}$ is a rather special sheaf; it is what's known as a *quasi-coherent sheaf* – these will be studied in more detail in Chapters 13 and 14. This entails in particular, that for each open set $U \subset X$, the group $\widetilde{M}(U)$ has a natural structure of an $\mathcal{O}_X(U)$-module. This is clear for the open sets $U = D(f)$, when the A_f-module is simply M_f, and for general U, it follows by 4.11iii) below.

The tilde-construction is functorial in M. For any A-module homomorphism $\alpha \colon M \to N$, there is an induced map $\widetilde{\alpha} : \widetilde{M} \to \widetilde{N}$. To define $\widetilde{\alpha}$, we use Proposition 3.17 which tells us that it is enough to specify what it should be over each distinguished open set $D(f)$. Given α we simply define $\widetilde{M}(D(f)) \to \widetilde{N}(D(f))$ to be the induced map between the localizations

$$\begin{tikzcd} M_f \arrow[r, "\alpha_f"] \arrow[d] & N_f \arrow[d] \\ M_g \arrow[r, "\alpha_g"] & N_g \end{tikzcd}$$

$$\alpha_f : M_f \to N_f$$

given by $m/f^n \mapsto \alpha(m)/f^n$. This gives a map of $\mathscr{B}$-sheaves by the diagram in the margin. Clearly one has $\widetilde{\phi \circ \psi} = \widetilde{\phi} \circ \widetilde{\psi}$, and the 'tilde-operation' is therefore a covariant functor from A-modules to sheaves on Xs.

The three main properties of the sheaf $\widetilde{M}$ are listed in the proposition that follows.

Proposition 4.11 *Let A be a ring and M and A-module. The sheaf $\widetilde{M}$ on* $\operatorname{Spec} A$ *has the following properties.*

i) Stalks: let $x \in \operatorname{Spec} A$ be a point whose corresponding prime ideal is $\mathfrak{p}$, then the stalk $\widetilde{M}_x$ of M at $x \in X$ is

$$\widetilde{M}_x = M_{\mathfrak{p}}$$

ii) Sections over distinguished open sets: if $f \in A$, one has

$$\Gamma(D(f), \widetilde{M}) = M_f$$

in particular it holds true that $\Gamma(X, \widetilde{M}) = M$;

iii) Sections over arbitrary open sets: for any open subset U of $\operatorname{Spec} A$ *covered by the distinguished sets $\{D(f_i)\}_{i \in I}$, there is an exact sequence*

$$0 \longrightarrow \Gamma(U, \widetilde{M}) \xrightarrow{\alpha} \prod_i M_{f_i} \xrightarrow{\beta} \prod_{i,j} M_{f_i f_j}$$

where α and β are the natural maps as defined in (3.1) *on page* 44*.*

Proof: These properties are completely analogous to the statements in Proposition 4.5 on page 57 about the structure sheaf $\mathcal{O}_X$, and the proofs are exactly the same. The first property follows because the stalks $\widetilde{M}_x$ and the localizations $M_{\mathfrak{p}}$ are direct limits of the same modules over the same inductive system (indexed by the distinguished open subsets $D(f)$ containing x), the second follows from the way we defined $\widetilde{M}$, and the third is just the general exact sequence expressing the space of sections of a sheaf over an open set in terms of the space of sections over members of an open covering and their intersections. ❑

Example 4.12 Let A be a ring and let $I \subset A$ be an ideal. Then $\mathscr{I} = \widetilde{I}$ is an *ideal sheaf* of $\mathcal{O}_X$, i.e., for each $U \subset X$, $\mathscr{I}(U)$ is an ideal of $\mathcal{O}_X(U)$. For $U = D(f)$, $\mathscr{I}(D(f))$ is simply the ideal IA_f of A_f. ☆

Example 4.13 Let $A = k[x,y]/(x^2 + y^2 - 1)$ and $X = \operatorname{Spec} A$. Consider the A-module M given by the cokernel of the map $A^2 \to A^2$ given by the matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, i.e.,

$$M = Ae_1 \oplus Ae_2 / (xe_1 + ye_2).$$

Let $x \in X$ be the point associated to the prime ideal $\mathfrak{p} = (x, y - 1)$. What is the stalk of $\mathcal{F} = \widetilde{M}$ at x? Since y is invertible in $A_{\mathfrak{p}}$, we can use the relation $xe_1 + ye_2 = y(xy^{-1}e_1 + e_2)$ to eliminate e_2, and we find

$$\mathcal{F}_x = M_{\mathfrak{p}} = A_{\mathfrak{p}}e_1 \oplus A_{\mathfrak{p}}e_2 / (xe_1 + ye_2) \simeq A_{\mathfrak{p}} = \mathcal{O}_{X,x}.$$

☆

4.3 *Locally ringed spaces*

We would like to define a scheme to be a space that locally looks like Spec A for some ring A. To be able to make sense of such a definition, we need a suitable category of spaces to work with. We will rely on the two pieces of data we have; the topological space Spec A together with its sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$.

Definition 4.14 (Locally ringed spaces) *A* locally ringed space *is a pair* $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of rings on* X*, such that all the stalks* $\mathcal{O}_{X,x}$ *are local rings.*

To make this into a category, we need to specify what a morphism between two locally ringed spaces is. Reflecting the above definition, a morphism betwen $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ should have two components, one map between the underlying topological spaces X and Y, and one on the level of sheaves. Note that it does not make sense to talk about morphisms $\mathcal{O}_Y \to \mathcal{O}_X$, because the sheaves are defined on different spaces. Instead, once we have specified a continuous map $f : X \to Y$, the sheaf map should be a map $\mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves of rings on Y.

Definition 4.15 (Morphism of locally ringed spaces) *A* morphism of locally ringed spaces *is a pair*

$$(f, \phi) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

where

- $f \colon X \to Y$ *is a continuous map, and*
- $\phi \colon \mathcal{O}_Y \to f_* \mathcal{O}_X$ *is a map of sheaves of rings on* Y*, so that for each* $x \in X$ *and* $y = f(x)$*, the induced map on stalks*

$$\phi_x \colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

  is a map of local rings, that is,

$$\phi_x(\mathfrak{m}_y) \subseteq \mathfrak{m}_x.$$

We usually write $f^\sharp = \phi$.

The second point in the definition requires a few comments. Note first that specifying the sheaf map means that for all open subsets U, we have ring maps

$$f^\sharp(U) \colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U),$$

and we require that they commute with the restriction maps:

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r, "f^\sharp(U)"] \arrow[d, "\rho_{U,V}"] & \mathcal{O}_X(f^{-1}U) \arrow[d, "\rho_{f^{-1}U,f^{-1}V}"] \\ \mathcal{O}_Y(V) \arrow[r, "f^\sharp(V)"] & \mathcal{O}_Y(f^{-1}V) \end{tikzcd} \tag{4.5}$$

The intuition again comes from varieties, where we would like to think of $f^\sharp$ as a way of "pulling back" functions on Y to functions on X. More precisely, if X and Y are affine varieties, we have an induced map $f^\sharp : A(Y) \to A(X)$ given by sending $h : Y \to k$ to $h \circ f : X \to k$. If h is only regular on some open set $U \subset Y$, we may still define a pullback $f^\sharp(h) = h \circ f$, but this is is only regular on $h^{-1}U$. In other words, $f^\sharp(h)$ defines a section of $(f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}U)$.

For a general locally ringed space, however, we do not have the luxury of speaking about functions into some fixed field k, so the ring maps $f^\sharp$ have to be specified as part of the data. We do not allow these to be completely arbitrary ring maps: there is a last condition in the definition involving the induced maps on stalks, which we will now explain.

First of all, we should explain how to define the map

$$f_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \,. \tag{4.6}$$

Starting with $s_y \in \mathcal{O}_{Y,y}$, the germ of a section (s, V) with $V \subset Y$, then $t = f^\sharp(s)$ is a well-defined section of $\mathcal{O}_X(f^{-1}(V))$. Since $f^{-1}(V)$ contains x, we can form its germ $t_x \in \mathcal{O}_{X,x}$. The diagram (4.5) shows that this operation commutes with the restriction maps, so we get an induced map of rings as in (4.6).

Morphisms of locally ringed spaces induce ring maps on stalks.

When X and Y are affine varieties, the corresponding map is simply the map between the localizations

$$f_x^\sharp : A(Y)_{\mathfrak{m}_y} \to A(X)_{\mathfrak{m}_x}$$

which sends a rational function h defined at $y = f(x)$ to $g \circ f$, which is regular at x. Moreover, if h vanishes at $y = f(x)$, the corresponding pullback $f^\sharp(h) = h \circ f$ vanishes at x. This means that the map $f_x^\sharp$ maps the maximal ideal $\mathfrak{m}_y$ into $\mathfrak{m}_x$; or in other words, it is a *map of local rings*. Note that this condition is equivalent to $(f_x^\sharp)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$.

A map of local rings is a ring homomorphism which maps the maximal ideal to the maximal ideal.

In the general setting, thinking about $f^\sharp$ as a way of "pulling back" functions, this condition is very intuitive, but it is in no way automatic when starting from a general map of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$; this is why we need to include it in the defintion. (A second reason to include it will appear in Proposition 4.20 below.)

Here is an example which shows what can go wrong without this requirement:

EXAMPLE 4.16 Let $X = \operatorname{Spec} \mathbb{C}(x)$ and $Y = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$. There is an obvious map $f : X \to Y$ induced by the inclusion $\mathbb{C}[x] \subset \mathbb{C}(x)$. Note that on the level of topological spaces, X consists of a single point η, and f maps η to the generic point ν of Y. The corresponding stalk map $f_\eta^\sharp : \mathcal{O}_{Y,\nu} \to \mathcal{O}_{X,\eta}$ is given by the identity map

$$\mathcal{O}_{Y,\nu} = \mathbb{C}[x]_{(0)} = \mathbb{C}(x) \to \mathbb{C}(x) = \mathcal{O}_{X,\eta}.$$

which is clearly a map of local rings.

On the other hand, we can try to define a strange map $g : X \to Y$ by sending η to some other point $y \in Y$ corresponding to the maximal ideal $(x-a) \subset \mathbb{C}[x]$. The map g is clearly continuous, because X consists of only one point. However, the induced map

$$\mathcal{O}_{Y,y} = \mathbb{C}[x]_{(x-a)} \to \mathbb{C}(x) = \mathcal{O}_{X,\eta}.$$

sends the maximal ideal to the unit ideal in $\mathbb{C}(x)$, so this is not a map of locally ringed spaces. This is of course as it should be, because the function '$x-a$' vanishes at $y \in Y$, but its pullback, e.g., the image of $x-a$ in $\mathcal{O}_{X,\eta}$ does not vanish at $\eta \in X$ (it maps to itself via the evaluation map $\mathcal{O}_{X,\eta} \to k(\eta) = \mathbb{C}(x)$). ☆

If we are given two maps $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp) \colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, we can compose them: the underlying map $X \to Z$ on the topological spaces is naturally the composition $g \circ f$, and since the direct image satisfies $(g \circ f)_* \mathcal{O}_X = g_* f_* \mathcal{O}_X$, we may define $(g \circ f)^\sharp$ as $g^\sharp \circ f^\sharp$. This means that the locally ringed spaces form a category.

Isomorphisms of locally ringed spaces

If X and Y are two locally ringed spaces, an *isomorphism* from X to Y is a morphism $f \colon X \to Y$ that has an inverse morphism; in other words, there is a morphism $g \colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In more concrete terms, this boils down to f being a homeomorphism such that $f^\sharp(U) \colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is a ring isomorphism for every open $U \subset Y$.

The example we are most interested in at the moment is of course the spectrum of a ring:

Proposition 4.17 *For a ring A, the pair $(X, \mathcal{O}_X) = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a locally ringed space.*
For a map of rings $\phi \colon A \to B$, there is an induced map of locally ringed spaces

$$\operatorname{Spec}(\phi) = (h, h^\sharp) \colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

satisfying the two properties

i) *Distinguished open sets: The map $h^\sharp(D(f))$, which fits into the diagram*

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec} A}(D(f)) \arrow[r, "h^\sharp(U)"] \arrow[d, equal] & \mathcal{O}_{\operatorname{Spec} B}(D(\phi(f)) \arrow[d, equal] \\ A_f \arrow[r] & B_{\phi(f)} \end{tikzcd}$$

is the natural localization of the map ϕ; that is, it is given by the assignment $af^{-n} \mapsto \phi(a)\phi(f)^{-n}$.

ii) *Stalks: The map induced by $h^\sharp$ on stalks at $\mathfrak{p} \in \operatorname{Spec} B$ and $\phi^{-1}(\mathfrak{p})$ is the localization map $A_{\phi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ of ϕ.*

Earlier we defined a continuous map $\operatorname{Spec}\phi \colon \operatorname{Spec} A \to \operatorname{Spec} B$ which now will be the topological part, named h, of the new map $\operatorname{Spec}\phi$, and which we shall enrich by a sheafy part $h^\sharp$. Note that the two last assertions of the proposition reflect the two statements in Proposition 4.5. The slogan is: on affine schemes, both $\mathcal{O}_X$ and $h^\sharp$ are just given by the localizations.

Proof: We defined the structure sheaf on $X = \operatorname{Spec} A$ so that $\mathcal{O}_{X,x} = A_{\mathfrak{p}_x}$ at each point for each $x \in X$. In particular, the stalks are local rings, so we have a locally ringed space.

For the second claim, assume that $\phi\colon A \to B$ is a morphism of rings and let $h\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced map given by $h(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. We want to associate to ϕ a map of sheaves of rings

$$h^\sharp\colon \mathcal{O}_{\operatorname{Spec} A} \to h_*\mathcal{O}_{\operatorname{Spec} B}.$$

By Proposition 3.17, it suffices to tell what $\phi^\sharp$ should do to the sections over the distinguished open sets $D(f)$. Here we recall Lemma 2.30, which tells us that

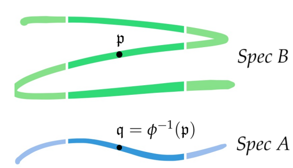
$$h^{-1}(D(f)) = D(\phi(f)).$$

This means that we have the equality $\Gamma(D(f), h_*\mathcal{O}_{\operatorname{Spec} B}) = B_{\phi(f)}$, and we know that $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$. The original map of rings $\phi\colon A \to B$ now localizes to a map $A_f \to B_{\phi(f)}$, sending af^{-n} to $\phi(a)\phi(f)^{-n}$, and this shall be the map $h^\sharp$ on sections over the distinguished open set $D(f)$.

To prove that $h^\sharp$ is well defined, we need to check that it is compatible with the restriction maps among distinguished open sets: indeed, when there is an inclusion $D(g) \subseteq D(f)$, we write as usual $g^m = cf$, and the localization map $A_f \to A_g$ will then send af^{-n} to $ac^n g^{-nm}$. One has $\phi(g)^m = \phi(c)\phi(f)$, which makes the diagram below commutative:

$$\begin{tikzcd}
A_f \arrow[r] \arrow[d] & A_g \arrow[d] \\
B_{\phi(f)} \arrow[r] & B_{\phi(g)}
\end{tikzcd}$$

and this is exactly the required compatibility.



For $\mathfrak{p} \in \operatorname{Spec} B$, with image $\mathfrak{q} = \phi^{-1}(\mathfrak{p}) \in \operatorname{Spec} A$, the map $h^\sharp$ induces a map of stalks

$$h^\sharp_{\mathfrak{p}}\colon \mathcal{O}_{\operatorname{Spec} A,\mathfrak{q}} \to \mathcal{O}_{\operatorname{Spec} B,\mathfrak{p}}$$

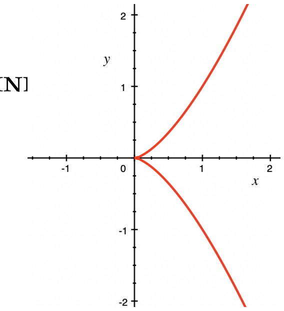
which is just the localization map $A_{\phi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$. Thus the preimage of the maximal ideal of $A_{\phi^{-1}(\mathfrak{p})}$ equals the maximal ideal in $B_{\mathfrak{p}}$, making $h^\sharp_{\mathfrak{p}}$ a map of local rings. Hence $(h, h^\sharp)$ is a morphism of locally ringed spaces. ❑

Example 4.18 (*The Cuspidal cubic.*) Let k be a field, and let

$$A = k[x,y]/(y^2 - x^3)$$

and $B = k[t]$, and let $\phi\colon A \to B$ be the ring map given by $x \mapsto t^2, y \mapsto t^3$. This induces a morphism of locally ringed spaces

$$f\colon \mathbb{A}^1_k \to X = \operatorname{Spec} A$$



This morphism sends $(t-a) \in \operatorname{Spec} k[t]$ to $(x-a^2, y-a^3) \in \operatorname{Spec} A$.

If $p = (t) \in \mathbb{A}^1_k$ denotes the origin, the stalk induced map $f_p^\sharp \colon \mathcal{O}_{\mathbb{A}^1_k,p} \to \mathcal{O}_{X,f(p)}$, is given by map of localizations

$$\phi_{(x,y)} \colon \left(k[x,y]/(y^2-x^3)\right)_{(x,y)} \to k[t]_{(t)} \tag{4.7}$$

☆

Exercise 4.2 Show that that the composition of two morphisms of locally ringed spaces is again a morphism. ★

Exercise 4.3 Show that if $f : X \to Y$ is a morphism of locally ringed spaces, the stalk $f_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ maps induce maps to residue fields $k(y) \to k(x)$; this is, a field extension of $k(y)$. What happens when X and Y are affine varieties? ★

4.4 *Affine schemes*

Definition 4.19 *An* affine scheme *is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.*

Affine schemes form a category AffSch, a subcategory of the category of locally ringed spaces. This category is closely linked to the category of rings, as we will see next.

By Proposition 4.17, the assignment $A \mapsto \operatorname{Spec} A$ gives a contravariant functor from the category Rings of rings to the category AffSch. There is also a contravariant functor going the other way: taking the global sections of the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ gives us the ring A back. Furthermore, a map of affine schemes $f \colon \operatorname{Spec} B \to \operatorname{Spec} A$, comes equipped with a map of sheaves $f^\sharp \colon \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$. Taking global sections gives a ring map

$$A = \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to \Gamma(\operatorname{Spec} A, \phi_* \mathcal{O}_{\operatorname{Spec} B}) = B.$$

We can define a canonical map

$$\Gamma \colon \operatorname{Hom}_{\mathsf{AffSch}}(X,Y) \to \operatorname{Hom}_{\mathsf{Rings}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \tag{4.8}$$

which sends $(f, f^\sharp)$ to the map $f^\sharp(Y) \colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$.

Proposition 4.20 *If X and Y are affine schemes, the map Γ is bijective.*

Proof: Write $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. By construction, we have $A = \mathcal{O}_Y(Y)$ and $B = \mathcal{O}_X(X)$.

If $\phi \colon A \to B$ is a ring homomorphism, it follows from Proposition 4.17 i), that $\Gamma(\operatorname{Spec} \phi) = \phi$. So to establish the bijection, we just need to show that $\operatorname{Spec}(\Gamma(f)) = f$ for a given a morphism $f \colon X \to Y$. We let $\phi = \Gamma(f) = f^\sharp(Y) \colon A \to B$. Let $x \in X$ be a point, corresponding to the prime ideal $\mathfrak{q} \subseteq B$, and let $\mathfrak{p} \subseteq A$ be the prime ideal corresponding to

Note: $B \backslash \mathfrak{q}$ is the subset of elements in B that become invertible in $B_{\mathfrak{q}}$.

$f(x) \in Y$. There is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{\mathfrak{p}} \arrow[r, "f_x^\sharp"'] & B_{\mathfrak{q}} \end{tikzcd}$$

where the vertical maps are the localization maps. By the commutativity of the diagram we have $\phi(A \backslash \mathfrak{p}) \subseteq B \backslash \mathfrak{q}$, so $\phi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$. Now $f_x^\sharp$ is a map of local rings, so in fact $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. This means that $\operatorname{Spec}\phi$ induces the same map as f on the underlying topological spaces. Moreover, we have two morphisms of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$, one induced by f and one induced by $\operatorname{Spec}\phi$. For each x, the induced stalk maps $f_x^\sharp$ and $(\operatorname{Spec}\phi)^\sharp$ coincide with the map $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ above, so also $f^\sharp = (\operatorname{Spec}\phi)^\sharp$ as maps of sheaves (see Exercise 3.6 on page 50). ❑

Here we are using that the stalk maps $f_x^\sharp$ are maps of local rings!

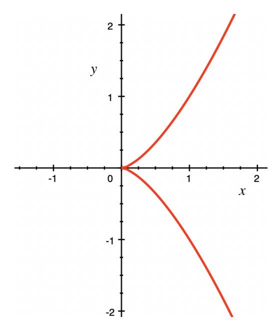
We have established the following important theorem:

Theorem 4.21 *The two functors* Spec *and* Γ *are mutually inverse and define an equivalence between the categories* Rings^{op} *and* AffSch.

In summary, affine schemes X are completely characterized by their rings of global sections $\Gamma(X, \mathcal{O}_X)$, and morphisms between affine schemes $X \to Y$ are in bijective correspondence with ring homomorphisms $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$. In particular, a map f between two affine schemes is an isomorphism if and only if the corresponding ring map $f^\sharp$ is an isomorphism.

Example 4.22 The morphism $f : \mathbb{A}^1 \to \operatorname{Spec} A$ from Example 4.18 is a homeomorphism, but it is not an isomorphism.

Indeed, note that f sends $(t-a) \in \operatorname{Spec} k[t]$ (injectively) to $(x-a^2, y-a^3) \in \operatorname{Spec} A$. By the Nullstellensatz any maximal ideal of A is of that form so it is also surjective. Thus, since f maps the generic point (0) of $\mathbb{A}^1_k$ to the generic point (0) of $\operatorname{Spec} A$, we see that f is a bijection. It is finally a homeomorphism because it is closed; any closed subset of $\mathbb{A}^1_k$ is a finite set of points.



To see it is not an isomorphism, we simply note, that it is induced by $\phi : k[t] \to k[x,y]/(y^2 - x^3)$, which is not an isomorphism (the ring on the right hand side is not even a UFD). In fact, the same argument shows that the stalk map at the origin $f_p^\sharp \colon \mathcal{O}_{\mathbb{A}^1_k, p} \to \mathcal{O}_{X, f(p)}$ given by (4.7) is not even an isomorphism. This confirms our intuition that the cuspidal cubic is not even 'locally isomorphic' to $\mathbb{A}^1_k$ near the origin.

☆

4.5 *Schemes in general*

Finally, we can give the definition of a scheme.

Definition 4.23 *A* scheme *is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme, i.e., there is an open cover U_i of X such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to some affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.*

So as before, a scheme has two components: a topological space X, which is covered by open sets of the form $\operatorname{Spec} A$ and the structure sheaf $\mathcal{O}_X$, which is a sheaf of rings.

The local ring at a point

For a point $x \in X$, we define *the local ring* at x as the stalk $\mathcal{O}_{X,x}$. Note that, the affine open subsets form a basis for the topology of X, so when computing the direct limit that gives the stalk, we may use distinguished affine subsets. So if x is contained in $U = \operatorname{Spec} A$ and corresponds to $\mathfrak{p} \subset A$, we have a natural isomorphism

$$\mathcal{O}_{X,x} = \varinjlim_{D(f)\ni\mathfrak{p}} \mathcal{O}_X(D(f)) = \varinjlim_{f\notin\mathfrak{p}} A_f = A_{\mathfrak{p}}.$$

As before, we think of elements in $\mathcal{O}_{X,x}$ as 'functions defined at x', even if this is strictly only true for well-behaved schemes (see Proposition 4.42).

Residue field

In the local ring $\mathcal{O}_{X,x}$, we also have the *maximal ideal* $\mathfrak{m}_x = \mathfrak{p}A_{\mathfrak{p}}$ and the corresponding *residue field* $k(x) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

Morphism of schemes

A *morphism*, or *map* for short, between two schemes X and Y is simply a map f between X and Y regarded as locally ringed spaces. This also has two components: a continuous map, which we shall denote by f as well, and a map of sheaves of rings

$$f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$$

with the additional requirement that $f^\sharp$ induces a map of local rings on the stalks.

In this way the schemes form a category, which we shall denote by Sch. This contains the category of affine schemes, denoted by AffSch, as a full subcategory.

Relative schemes

Relative schemes

There is also the notion of *relative schemes* where a base scheme S is chosen. A *scheme over S* is a scheme X together with a morphism $f\colon X \to S$, which we call the *structure map* or the *structure morphism*. If two schemes over S are given, say $X \to S$ and $Y \to S$, then a map between them is a map $X \to Y$ compatible with the two structure maps; that is, such that the diagram below is commutative

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

The schemes over S form a category Sch/S, and the set of morphisms as defined above is denoted by $\operatorname{Hom}_S(X, Y)$.

If the base scheme S is affine, say $S = \operatorname{Spec} A$, we say that X is a *scheme over A*, and we write Sch/A for $\mathsf{Sch}/\mathsf{Spec\,A}$ for simplicity. To say that an affine scheme $\operatorname{Spec} B$ is a scheme over $\operatorname{Spec} A$ is the same thing as saying that B is an A-algebra; giving a structure map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is equivalent to giving the map of rings $f^\sharp\colon A \to B$. There is a canonical map from any scheme X to $\operatorname{Spec}\mathbb{Z}$, every scheme is a $\mathbb{Z}$-scheme. On the level of categories one may express this as $\mathsf{Sch} = \mathsf{Sch}/\mathbb{Z}$.

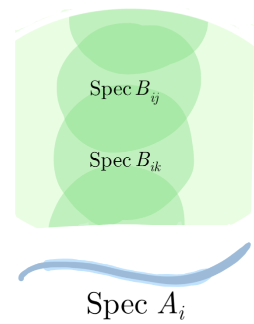
See Exercise **??**

The concept of relative schemes can be thought of as a vast generalization of 'varieties over k'. However, extending this to more general rings or even schemes turns out to be conceptually very fruitful, *e.g.* when discussing properties of morphisms (Chapter **??**) or fibre products (Chapter 8).

Definition 4.24 (Morphisms of finite type) *Let X/S be a scheme over S with structure morphism $f\colon X \to S$. One says that*

i) *X/S is of* locally finite type *if S has a cover consisting of open affine subsets $V_i = \operatorname{Spec} A_i$ such that each $f^{-1}(V_i)$ can be covered by affine subsets of the form $\operatorname{Spec} B_{ij}$, where each B_{ij} is finitely generated as a A_i-algebra;*

ii) *f is of* finite type *if, in i), one can do with a finite number of* $\operatorname{Spec} B_{ij}$.



In case $S = \operatorname{Spec} A$, one says that a scheme over A is of *locally finite type* (respectively of *finite type*) over A if the morphism $X \to \operatorname{Spec} A$ is locally of finite type (respectively of finite type).

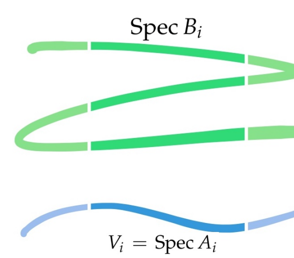
Schemes which are of finite type over a field will be our main concern in this book. This is the class of schemes which are closest to varieties; but we allow non-algebraically closed fields, as well as zero divisors in the structure sheaf.

There is another related, but much stronger finiteness property a morphism can have:

Definition 4.25 (Affine and finite morphisms) *Let $f\colon X \to S$ be a scheme over S. We say that*

i) *f is* affine *if there is a covering $V_i = \operatorname{Spec} A_i$ of S such that each inverse image $f^{-1}(V_i)$ is affine;*

ii) *f is* finite *if it is affine, and in the notation above, if $f^{-1}(V_i) = \operatorname{Spec} B_i$, the A_i-algebra B_i is a finitely generated A_i-module.*



That a scheme X is finite over a field k means that $X = \operatorname{Spec} A$ for A a k-algebra of finite dimension over k. Such an A is Artinian and has only finitely many prime ideals all being maximal. Hence the spectrum $\operatorname{Spec} A$ is a finite set, and the underlying topology is discrete. To underline the enormous difference between the the two finiteness condition, note that X being of finite type simply means it is covered by affine schemes of the form $\operatorname{Spec} k[x_1, \ldots, x_r]/\mathfrak{a}$.

Example 4.26 For $n \geqslant 1$, the structure morphisms $\mathbb{A}^n_k \to \operatorname{Spec} k$ and $\mathbb{P}^1_k \to \operatorname{Spec} k$ are of finite type, but not finite. The morphism $\coprod_{i=1}^{\infty} \mathbb{A}^1_k \to \mathbb{A}^1_k$ (identity on each component) is locally of finite type, but not of finite type. ☆

Example 4.27 Let k be a field, and consider the two k-algebras

$$A = k[x, y]/(y^2 + x^6 - 1) \quad B = k[u, v]/(v^2 + u^3 - 1).$$

The ring map $\phi\colon A \to B$ be the ring map given by $x \mapsto u^2, y \mapsto v$, induces a morphism

$$f\colon X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$$

This is a finite map: Indeed, as an A-module, B decomposes as

$$B = A[u,v]/(u^2 - x, v - y) \simeq A[u]/(u^2 - x) \simeq A \oplus A.$$

The two copies of A reflects the geometric property that the morphism f has two preimages over a typical point in Y. For example, for $\mathfrak{q} = (x-1, y)$, we have

$$f^{-1}V(\mathfrak{q}) = (u^2 - 1, v) = (u-1, v) \cap (u+1, v).$$

so, the preimages are given by $\mathfrak{p}_1 = (u-1, y)$ and $\mathfrak{p}_2 = (u+1, y)$. We will see that this is part of a general phenomenom in Chapter XXX. ☆

The definitions above reference a single open covering of affine schemes $V_i = \operatorname{Spec} A_i$ over which the morphism has the indicated properties. A convenient fact is that it in fact holds for every affine covering, once it holds for one; the proof will be an exercise later in the book (Exercise 19.3 on page 282).

PROPOSITION 4.28 *For each of the properties in Definitions 4.24 and 4.25, if the indicated condition holds for one affine covering, then it holds for every affine covering.*

Open immersions and open subschemes

If X is a scheme and $U \subseteq X$ is an open subset, the restriction $\mathcal{O}_X|_U$ is a sheaf on U making $(U, \mathcal{O}_X|_U)$ into a locally ringed space. This is even a scheme, because if X is covered by affines $V_i = \operatorname{Spec} A_i$, then each $U \cap V_i$ is open in V_i, hence can be covered by affine schemes. It follows that there is a canonical scheme structure on U, and we call $(U, \mathcal{O}_X|_U)$ an *open subscheme* of X and say that U has the *induced scheme structure*. We say that a morphism of schemes $\iota\colon V \to X$ is an *open immersion* if it is an isomorphism onto an open subscheme of X.

Open subschemes

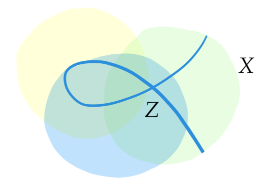
Open immersions

EXAMPLE 4.29 The open set $U = \mathbb{A}^1_k - V(x)$ is an open subscheme of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. Note that there is an isomorphism $U \simeq \operatorname{Spec} k[x, x^{-1}] = \operatorname{Spec} k[x,y]/(xy-1)$, as schemes. ☆

EXAMPLE 4.30 More generally, consider $V = \operatorname{Spec} A_f$ and the map $\iota\colon V \to \operatorname{Spec} A = X$, induced by the localization map $A \to A_f$. This is an open immersion onto the open set $U = D(f) \subset X$. Indeed, we saw in Example 2.33 that ι is a homeomorphism onto U, and it follows from the definition of the sheaf $\mathcal{O}_X$ that the restriction $\mathcal{O}_X|_U$ coincides with the structure sheaf on $\operatorname{Spec} A_f$. ☆

A word of warning: an open subscheme of an affine scheme might not itself be affine (we will see an example of this in Chapter 6).

Closed immersions and closed subschemes



If X is a scheme, we would like to define what it means for a closed subset $Z \subset X$ to be a *closed subscheme* of X. This is a little bit more subtle than the case for open subsets, because for a given closed subset $Z \subset X$, there is no canonical locally ringed space structure on Z.

The prototypical example of a closed subscheme is $\operatorname{Spec}(A/\mathfrak{a})$, which as we have seen, embeds in a natural way as the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec} A$ (Example 2.32). Here the scheme structure is evident. Thus we have a clear intuitive picture of what a closed subscheme should be in general: it is a scheme $(Z, \mathcal{O}_Z)$ with a morphism $\iota \colon Z \to X$, which looks locally like the map $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$.

Definition 4.31 (Closed immersions and subschemes) *Let X and Z be schemes and $\iota \colon Z \to X$ a morphism. The map ι is called a* closed immersion *if there is an affine cover $U_i = \operatorname{Spec} A_i$ of X and ideals $\mathfrak{a}_i$ in A_i such that for each i it holds that $\iota^{-1}(U_i) \simeq \operatorname{Spec}(A_i/\mathfrak{a}_i)$ as schemes over U_i.*

In clear text, the isomorphism $\iota^{-1}U_i \simeq \operatorname{Spec}(A_i/\mathfrak{a}_i)$ being an isomorphism over U_i, means that the diagram

$$\begin{tikzcd}
\iota^{-1}U_i \arrow[r, "\iota"] & U_i = \operatorname{Spec} A_i \\
\operatorname{Spec}(A_i/\mathfrak{a}_i) \arrow[u, "\wr"] \arrow[ur] &
\end{tikzcd}$$

commutes, where the skew map is the canonical closed immersion.

One can refer to Z as a closed subscheme of X, even if this is slightly abuse of language: it is the morphism $Z \to X$ which is a closed immersion, and Z is a priori not assumed to be a closed subset of X.

We will give a more systematic treatment of closed subschemes in Chapter 13, when we discuss quasicoherent ideal sheaves. This has the advantage that we can study the set of closed subscheme of X in terms of data defined on X (ideal sheaves), rather than as maps $Z \to X$ from abstract schemes into X.

In fact, classifying closed subschemes according to the above definition is not so easy even for affine schemes. Of course, for each ideal $\mathfrak{a} \subset A$, we get a closed immersion $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$ and therefore a desired closed subscheme of $\operatorname{Spec} A$ with underlying topological space $V(\mathfrak{a})$. However, because the definition makes reference to a certain affine covering, it is a priori not obvious that all closed subschemes arise from an ideal $\mathfrak{a} \subset A$ in this way, or even that every closed subscheme of $\operatorname{Spec} A$ is an affine scheme. This is nevertheless true, but we will have to postpone the proof until Chapter 13.

Proposition 4.32 *Let $X = \operatorname{Spec} A$ be an affine scheme. The map $\mathfrak{a} \mapsto \operatorname{Spec}(A/\mathfrak{a})$ induces a one-to-one correspondence between the set of ideals of A and the set of closed subschemes of X. In particular, any closed subscheme of an affine scheme is also affine.*

EXAMPLE 4.33 The schemes $\operatorname{Spec} k[x]/(x)$, $\operatorname{Spec} k[x]/(x^2)$, $\operatorname{Spec} k[x]/(x^3), \ldots$ give different subschemes of $\mathbb{A}^1_k$, where k is a field. Still, the underlying topological spaces are the same (a single point). Thus these spectra are homeomorphic, but they are not isomorphic as schemes, because they have non-isomorphic structure sheaves. ☆

EXAMPLE 4.34 Consider the affine 4-space $\mathbb{A}^4_k = \operatorname{Spec} A$, with $A = k[x,y,z,w]$. Then the three ideals

$$I_1 = (x,y), \quad I_2 = (x^2,y) \text{ and } I_3 = (x^2,xy,y^2,xw-yz),$$

have the same radical, and thus give rise to the same closed subset $V(x,y) \subset \mathbb{A}^4_k$, but they give different closed subschemes of $\mathbb{A}^4_k$. ☆

EXERCISE 4.4 Show that being a closed immersion is a property which is local on the target. In clear text: Assume given a morphism $f\colon X \to Y$ and an open cover $\{U_i\}$ of Y. Let $V_i = f^{-1}(U_i)$ and assume that each restriction $f|_{V_i}\colon V_i \to U_i$ is a closed immersion. Prove that then also f is a closed immersion. ★

EXERCISE 4.5 Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of schemes. Prove that if both f and g are closed immersions then $g \circ f$ is one as well. ★

EXERCISE 4.6 Show that a morphism $\iota : Z \to X$ is a closed immerion if and only if it is affine, and the sheaf map $i^\sharp : \mathcal{O}_X \to \iota_*\mathcal{O}_Z$ is surjective. ★

4.6 *Properties of the scheme structure*

In the previous section, we noted that there might be several scheme structures on the same topological space. In this section, we discuss this phenomenon a little bit further.

Recall that a ring A is said to be *reduced* if it has no nilpotent elements. A scheme $(X, \mathcal{O}_X)$ is said to be *reduced* if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced. This condition holds if and only if for every open $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotents: any non-zero nilpotent element in a ring $\mathcal{O}_X(U)$ would have a non-zero germ in at least one local ring $\mathcal{O}_{X,x}$, which would then not be reduced. For the reverse implication, note that any non-zero nilpotent s_x element in $\mathcal{O}_{X,x}$ is induced from some section s of $\mathcal{O}_X(V)$ over some open neighbourhood V of x. Since the support of powers of s are closed, s will be nilpotent in $\mathcal{O}_{X,x}(W)$ for some neighbourhood W of x (possibly smaller than V).

Reduced scheme

If $X = \operatorname{Spec} A$ is an affine scheme, the scheme $X_{\mathrm{red}} = \operatorname{Spec}(A/\sqrt{0})$ is, by definition, a reduced scheme. Moreover, there is a canonical morphism $X_{\mathrm{red}} \to X$, a closed immersion, induced by the quotient map $A \to A/\sqrt{0}$. This map has the following universal property: Any morphism $Y \to X$, where Y is reduced, factors through a map $Y \to X_{\mathrm{red}}$.

This follows from Theorem 4.21 for Y affine, and Theorem 5.6 in general.

In fact, one can for any scheme X, define its *associated reduced scheme* X_{red}, together with a closed immersion $X_{\mathrm{red}} \to X$ satisfying the above universal property. X_{red} has the same underlying topological space as X, but the structure sheaf has been modified to kill all nilpotent elements; it is obtained by "gluing together" each $\operatorname{Spec} A/\sqrt{0}$ for each affine subset $\operatorname{Spec} A$. We will postpone the details of this construction until Chapter 13.

EXAMPLE 4.35 For $A = k[x]/(x^n)$ and $X = \operatorname{Spec} A$, we have $X_{\text{red}} = \operatorname{Spec} A/\sqrt{(0)} = \operatorname{Spec} k$. ☆

EXAMPLE 4.36 For $X = \operatorname{Spec} A$, with $A = k[x,y,z,w]/I$ where

$$I = (x^2, xy, y^2, xw - yz),$$

we have $X_{\text{red}} = \operatorname{Spec} k[x,y,z,w]/(x,y) \simeq \mathbb{A}^2$. ☆

Integral schemes and function fields

Integral scheme

We say that a scheme is *integral* if it is both irreducible and reduced. For an affine scheme, $X = \operatorname{Spec} A$ is integral if and only if A is an integral domain. Indeed, X is reduced if and only if A has no nilpotents – that is, the nilradical vanishes – and X is irreducible if and only if the nilradical $\sqrt{0}$ is prime. These two statements imply that the zero-ideal is prime, and so A is an integral domain.

Moreover, it is not so hard to prove the following:

PROPOSITION 4.37 *A scheme X is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for each open $U \subseteq X$.*

Any irreducible subset of a scheme has a unique generic point. In particular, an integral scheme X has a unique generic point η. It is the only point which is dense in X and it belongs to every open non-empty subset of X. In particular, if $U = \operatorname{Spec} A \subseteq X$ is an open affine, $\eta \in U$ and corresponds to the zero ideal in A (which is prime). The local ring $\mathcal{O}_{X,\eta}$ is thus equal to the field of fractions of A. We define the *function field* $k(X)$ of X to be the stalk $\mathcal{O}_{X,\eta}$ at the generic point.

Function field

EXAMPLE 4.38 The function field of $\operatorname{Spec} \mathbb{Z}$ is $\mathcal{O}_{\operatorname{Spec} \mathbb{Z},(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$. ☆

EXAMPLE 4.39 The function field of $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ is $k(x_1, \ldots, x_n)$. ☆

EXAMPLE 4.40 (*The quadratic cone.*) The quadric cone $Q = \operatorname{Spec} k[x,y,z]/(x^2 - yz)$ is integral, as it is the spectrum of an integral domain. The function field of Q is given by

$$K(k[x,y,z]/(x^2 - yz)) \simeq k(x,y)$$

. ☆

Since y is invertible, we can use $z = y^{-1}x^2$ to eliminate z.

For an integral scheme X, it is sometimes fruitful to consider sections of $\mathcal{O}_X$ as elements in $k(X)$; one may heuristically think about them as 'rational functions' on X, thus pushing the analogy with function just before Definition 2.2 in Section 2.1 further. This is legitimate in view of the following. For any non-empty open U the generic point η belongs to U, and there is a canonical 'stalk map' $\Gamma(U, \mathcal{O}_X) \to \mathcal{O}_{X,\eta} = k(X)$ which is easily seen to be injective. These maps are compatible with restrictions, *i.e.* all diagrams

$$\begin{tikzcd}
\Gamma(U, \mathcal{O}_X) \arrow[r, hook] \arrow[d, "\rho_{UV}"'] & k(X) \\
\Gamma(V, \mathcal{O}_X) \arrow[ur, hook]
\end{tikzcd}$$

where $V \subseteq U$ are two open subsets, commute. Identifying the $\Gamma(U, \mathcal{O}_X)$ with their images in $k(X)$, the restriction maps just become inclusions. The same reasoning applies to the other stalks $\mathcal{O}_{X,x}$: they all lie in $k(X)$. Moreover, it holds true that $x \in U$ if and only if $\Gamma(U, \mathcal{O}_X) \subseteq \mathcal{O}_{X,x}$. We say that an element $f \in k(X)$ is *defined* in the point x if $f \in \mathcal{O}_{X,x}$.

LEMMA 4.41 *Let X be an integral scheme and let $f \in k(X)$. The set $U_f = \{ x \in X \mid f \in \mathcal{O}_{X,x} \}$ where f is defined, is open.*

PROOF: Let $x \in U_f$ and let $\operatorname{Spec} A$ be an affine neighbourhood of x. Consider the ideal $\mathfrak{a}_f = \{ b \in A \mid bf \in A \}$. If $\mathfrak{p}$ is a prime in A, then $f \in A_\mathfrak{p}$ if and only if $\mathfrak{a}_f \nsubseteq \mathfrak{p}$; that is, $V(\mathfrak{a}_f)$ is the complement of $U_f \cap \operatorname{Spec} A$. ❑

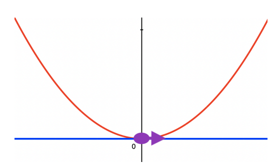
PROPOSITION 4.42 *Let X be an integral scheme with function field K. Then*

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} = \left\{ f \in K \;\middle|\; \begin{array}{l} \text{for each point } x \in U,\ f \text{ can be} \\ \text{represented as } g/h \text{ where } h(x) \neq 0 \end{array} \right\} \subset K.$$

PROOF: There are two equalities to prove here. To prove the first, assume first that U is affine, say $U = \operatorname{Spec} A$. Then $A = \bigcap A_\mathfrak{p}$ where the intersection extends over all prime ideals in A; indeed, if the ideal $\mathfrak{a}_f$ is proper, it will be contained in a maximal ideal $\mathfrak{m}$, hence $f \notin A_\mathfrak{m}$. If U is general, the statement follows since $\Gamma(U, \mathcal{O}_X) = \bigcap \Gamma(V, \mathcal{O}_X)$, where the intersection extends over all non-empty open affine subsets $V \subseteq U$. Indeed, in general $\Gamma(U, \mathcal{O}_X)$ equals the inverse limit $\Gamma(U, \mathcal{O}_X) = \varprojlim \Gamma(V, \mathcal{O}_X)$, and this inverse limit becomes the intersection when all rings are identified with subrings of K.

To prove the second, let $x \in X$ be a point, and let $\operatorname{Spec} A \ni x$ be an affine open containing x. Then an element $f \in K = K(A)$ lies in $\mathcal{O}_{X,x} = A_\mathfrak{p} \subset K$ if and only if it can be expressed as $f = a/s$ where $s \notin \mathfrak{p}$. ❑

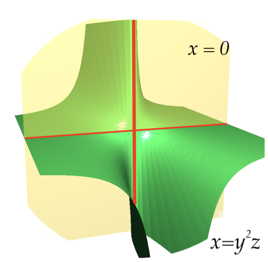
EXAMPLE 4.43 Non-reduced schemes appear frequently when two schemes X and Y intersect. For instance, consider the parabola $X = \operatorname{Spec} k[x,y]/(y - x^2)$ and the line $Y = \operatorname{Spec} k[x,y]/(y)$. The intersection of these is given by the ideal $I = (y - x^2, y) = (x^2, y)$, which is a non-radical ideal. Thus the nilpotent elements of $k[x,y]/(x^2, y)$ in some sense accounts for the "tangency" of the intersection $X \cap Y$. ☆



EXAMPLE 4.44 Here is a similar example in $\mathbb{A}^3_k$. Consider

$$X = \operatorname{Spec} k[x,y,z]/(x - y^2 z)$$

which defines a closed subscheme of $\mathbb{A}^3$ (a cubic surface). The intersection of X with the plane given by $x = 0$ is given by the ideal $I = (x, y^2 z) = (x, y^2) \cap (x, z)$. Thus the intersection $Z = \operatorname{Spec} k[x,y,z]/I$ is neither irreducible nor reduced: It has two irreducible components corresponding to the lines $x = y = 0$ and $x = z = 0$; the former has "multiplicity 2", since the plane is tangent to X along that line. ☆



EXAMPLE 4.45 (*Schemes of matrices.*) Let k be a field and consider the k-scheme

$$\mathbb{M}_n = \mathbb{A}_k^{n^2} = \operatorname{Spec} k[x_{ij}]_{1 \leqslant i,j \leqslant n}.$$

As the notation suggests, this is a scheme whose points over k parameterize $n \times n$-matrices with entries in k. $\mathbb{M}_n$ contains several interesting subschemes:

The *general linear group*, $GL_n(k) \subset \mathbb{M}_n$ defined as the open subscheme of invertible matrices; this is the distinguished open set $D(\det M)$, where $\det M$ is the determinant of the matrix of variables $M = (x_{ij})$. $GL_n(k)$ is an example of a *group scheme*, i.e., it is a scheme G equipped with scheme morphisms $m : G \times G \to G$ and $i : G \to G$ satisfying the usual axioms of being a group. As such, $GL_n(k)$ also contains many other closed subschemes which are also group schemes: the *special linear group* $SL_n(k)$, which is defined by $V(\det M - 1) \subset \mathbb{M}_n$; and the orthogonal group $O(n)$ (defined in $\mathbb{M}_n$ by the ideal generated by the relations $M^t M = I$, which are polynomial in the x_{ij}); and the *special orthogonal group* $SO(n)$, which is defined by $M^t M = I$ and $\det M = 1$. ☆

EXAMPLE 4.46 (*Nilpotent matrices.*) A particularly interesting example is given by the set of nilpotent matrices, i.e., matrices A such that $A^k = 0$ for some $k > 0$. We can put a scheme structure on this set by noting that an $n \times n$-matrix A is nilpotent if and only if $A^n = 0$. The equation $M^n = 0$ gives n^2 degree n relations in the variables x_{ij}, and the ideal J they generate define a closed subscheme $N = \operatorname{Spec}(k[x_{ij}]/J)$ of $\mathbb{M}_n$. Interestingly, the subscheme N is typically non-reduced. Indeed, note that the trace of a nilpotent matrix is always zero, so the equation $TrM = 0$ gives a linear polynomial in the x_{ij} which vanishes on all the closed points of N. So if k is algebraically closed, the Nullstellensatz implies that $\operatorname{Tr} M$ lies in $\sqrt{J}$, and so J is not radical.

We can put a different scheme structure on the set of nilpotent matrices, using the fact that a matrix A is nilpotent if and only if it has characteristic polynomial equal to x^n. Note that the coefficients of the characteristic polynomial

$$\det(tI - A) = x^n - c_1(A)x^{n-1} + \cdots + (-1)^n c_n(A)$$

are polynomials in the entries of A, so we see that we get n equations $c_1(M) = \cdots = c_n(M) = 0$, that define a subscheme in $\mathbb{M}_n$ with the same underlying topological space as N. In fact, it is not too hard to check that the ideal I generated by the $c_i(M)$ is radical, so that $N_{\text{red}} = \operatorname{Spec}\left(k[x_{ij}]/I\right)$. ☆

4.7 *Affine varieties and integral schemes*

Let X be an affine variety over k and let $A = A(X)$ denote its affine coordinate ring. To X we can consider the associated affine scheme $X^s = \operatorname{Spec} A$. The ring A is a finitely generated k-algebra with no zerodivisors. This translates into X being an integral scheme, of finite type over k.

If k is algebraically closed, then Hilbert's nullstellensatz implies that each maximal ideal $\mathfrak{m} \subset A$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some point $a \in X$, or more intrisically, $\mathfrak{m}$ is the kernel of the evaluation map $A \to k$ at $a \in X$. In particular, there is a natural injection $X \subset X^s$. Note that $V(I) \cap X = Z(I)$, so the classical Zariski topology on X is simply the restriction of the Zariski topology on $X^s = \operatorname{Spec} A$. Thus we can think of X^s as X plus added points $\mathfrak{p} = I(Y)$, one for each subvariety $Y \subset X$.

The structure sheaf $\mathcal{O}_{X^s}$ on X^s is constructed via the ring A. If $U = D(f) \subset X$ is a distinguished open set, we know that U has the structure of an affine variety as well (defined by the extra equation $tf - 1 = 0$ in $X \times \mathbb{A}^1$). Moreover, $\mathcal{O}_{X^s}(U) = A_f$, so the structure sheaf also knows about the regular function on U. The same in fact holds for any open set U by covering U using distinguished opens.

Exercises

(4.7) Show that any irreducible subset of a scheme has a unique generic point.

✲ (4.8) Which of the topologies on a set with three points is the underlying topology of a scheme?

✲ (4.9) Let X be a scheme.

a) Show that any irreducible and closed subset $Z \subseteq X$ has a unique generic point. Such topological spaces are called *sober*. HINT: Reduce to the affine case.

b) Show that in general schemes are not Hausdorff (they are not T_1 in the topologists's jargon). What are the possible underlying topologies of affine schemes that are Hausdorff?

c) Show that X satisfies the zeroth separation axiom (they are T_0); that is, given two points x and y in X, there is an open subset of X containing one of them but not the other.

d) Show that every quasi-compact and sober topological space has a closed point. (However, there are schemes that are not quasi-compact without closed points, see Proposition **??** on page ??).

(4.10) *The sheaf of units.* Let X be a scheme with structure sheaf $\mathcal{O}_X$. We say that $s \in \mathcal{O}_X(U)$ is a *unit* if there exists a multiplicative inverse $s^{-1} \in \mathcal{O}_X(U)$.

a) Show that $s \in \mathcal{O}_X(U)$ is a unit if and only if for all $x \in X$, the germ s_x is a unit in the ring $\mathcal{O}_{X,x}$; that is, if and only if s_x does not lie in the maximal ideal of $\mathcal{O}_{X,x}$.

b) We let $\mathcal{O}_X^\times(U)$ denote the subgroup of units in $\mathcal{O}_X(U)$. Show that $\mathcal{O}_X^\times(U)$ is a sub*sheaf* of $\mathcal{O}_X$.

(4.11) *The Frobenius morphism.* Let p be a prime number and let A be a ring of characteristic p. The ring map $F_A : A \to A$ given by $a \mapsto a^p$ is called the *Frobenius map* on A.

a) Show that F_A induces the identity map on $\operatorname{Spec} A$;

b) Show that if A is local, then F_A is a local homomorphism;

c) For a scheme X over $\mathbb{F}_p$, define the *Frobenius morphism* $F : X \to X$ by the identity on the underlying topological space and with $F^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ given by $g \mapsto g^p$. Show that F_X is a morphism of schemes;

d) Show that F_X is natural in the sense that if $f : X \to Y$ is a morphism of schemes over $\mathbb{F}_p$, we have $f \circ F_X = F_Y \circ f$.

In particular, this exercise shows that for a morphism of schemes $f : X \to Y$, in order to check that f is an isomorphism, is not enough to check that f is a homeomorphism; also the map $f^\sharp$ must be an isomorphism.

(4.12) Let X an integral scheme over a ring A, and let $f \in k(X)$. Show that there is a morphism $\phi\colon U_f \to \mathbb{A}^1_A$ such that $\phi^\sharp\colon A[t] \to \Gamma(U_f, \mathcal{O}_X)$ is given by $t \mapsto f$.

(4.13) Prove Proposition 4.37. That is, prove that a scheme X is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for each open $U \subseteq X$.

(4.14) For each of the following rings A, decide whether the corresponding morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ is finite or finite type:

a) $\mathbb{Z}[i]$
b) $\mathbb{Z}[\frac{1}{p}]$
c) $\mathbb{Z}_{(p)}$
d) $\mathbb{Z} \times \mathbb{Z}$
e) $\mathbb{Z}[x]$

(4.15) Let X be a scheme and let $x \in X$ be a point. Show that x is a closed point if and only if the corresponding morphism $\operatorname{Spec} k(x) \to X$ is finite.

Chapter 5

Gluing and first results on schemes

It is sometimes said that 'algebraic geometry is the study of the geometry of zero sets of polynomials'. After Grothendieck, perhaps a more precise slogan is that 'algebraic geometry is the geometry of rings'.

While this is true, the theory of schemes is much richer than just the spectra of rings. This is essentially due to the enormous flexibility we have in gluing: we are allowed to glue together new schemes out of old ones, as well as sheaves on them, and also morphisms between these. The aim of this chapter is to explain the conditions under which this can be done. We begin with gluing together sheaves (which is the easiest case and which works over any topological space), and then move on to schemes and morphisms. In the final part of the chapter we outline some applications of these constructions to the study of schemes.

5.1 *Gluing maps of sheaves*

This is the easiest gluing situation we encounter in this chapter. The setting is as follows. We are given two sheaves $\mathcal{F}$ and $\mathcal{G}$ on the topological space X and an open covering $\{U_i\}_{i\in I}$ of X. On each open set U_i we are given a map of sheaves $\phi_i\colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$, and we assume that the following gluing condition holds on the overlaps:

- $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}$

for all $i, j \in I$ where U_{ij} denotes the intersection $U_{ij} = U_i \cap U_j$. Then we have

> **Proposition 5.1 (Gluing morphisms of sheaves)** *Under the assumptions above, there exists a unique map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ such that $\phi|_{U_i} = \phi_i$.*

Proof: Take a section $s \in \mathcal{F}(V)$ where $V \subseteq X$ is open, and let $V_i = U_i \cap V$. Then $\phi_i(s|_{V_i})$ is a well defined element in $\mathcal{G}(V_i)$, and it holds true that $\phi_i(s|_{V_{ij}}) = \phi_j(s|_{V_{ij}})$ by the gluing condition. Hence the sections $\phi_i(s|_{V_i})$'s of the $\mathcal{G}|_{V_i}$'s glue together to a section of $\mathcal{G}$ over V, which we define to be $\phi(s)$. This gives the desired map of sheaves.

The uniqueness also follows: if ϕ and ψ are two morphisms of sheaves so that $\phi(s)|_{U_i} = \psi(s)|_{U_i}$ for all $i \in I$ then $\phi(s) = \psi(s)$, by the Locality axiom for $\mathcal{G}$, and hence $\phi = \psi$. ❑

5.2 *Gluing sheaves*

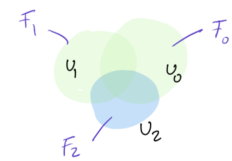
The setting in this section is a topological space X and an open covering $\{U_i\}_{i\in I}$ of X with a sheaf $\mathcal{F}_i$ on each open subset U_i. We want to "glue" the $\mathcal{F}_i$ together; that is, we search for a global sheaf $\mathcal{F}$ restricting to $\mathcal{F}_i$ on each U_i. The intersections $U_i \cap U_j$ are denoted by U_{ij}, and triple intersections $U_i \cap U_j \cap U_k$ are written as U_{ijk}.

As usual, the sheaves can take values in any category, but the main situation we have in mind is when the sheaves are sheaves of abelian groups.

The gluing data consists of isomorphisms $\tau_{ji}\colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$. The idea is to identify sections of $\mathcal{F}_i|_{U_{ij}}$ with $\mathcal{F}_j|_{U_{ij}}$ using the isomorphisms τ_{ij}. For the gluing process to be possible, the τ_{ij}'s must satisfy the three conditions

i) $\tau_{ii} = \mathrm{id}_{\mathcal{F}_i}$
ii) $\tau_{ji} = \tau_{ij}^{-1}$
iii) $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$

where the last identity takes place where it makes sense: on the triple intersection U_{ijk}. Observe that the three conditions parallel the three requirements for a relation being an equivalence relation; the first reflects reflectivity, the second symmetry and the third transitivity.



The third requirement is obviously necessary in order to glue together sections: a section s_i of $\mathcal{F}_i|_{U_{ijk}}$ will be identified with its image $s_j = \tau_{ji}(s_i)$ in $\mathcal{F}_j|_{U_{ijk}}$, and in its turn, s_j is going to be equal to $s_k = \tau_{kj}(s_j)$. Then, of course, s_i and s_k are also identified, which means that $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$.

Proposition 5.2 (Gluing sheaves) *In the setting as above there exists a sheaf $\mathcal{F}$ on X, unique up to isomorphism, such that there are isomorphisms $\nu_i\colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$ satisfying $\nu_j = \tau_{ji} \circ \nu_i$ over the intersections U_{ij}.*

Proof: If $V \subseteq X$ is an open set, we write $V_i = U_i \cap V$ and $V_{ij} = U_{ij} \cap V$. We are going to define the sections of $\mathcal{F}$ over V, and they are of course obtained by gluing togehter sections of the $\mathcal{F}_i$'s over V_i's along the V_{ij}'s using the isomorphisms τ_{ji}. We define

$$\mathcal{F}(V) = \{\, (s_i)_{i\in I} \mid \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}} \,\} \subseteq \prod_{i\in I} \mathcal{F}_i(V_i). \tag{5.1}$$

The τ_{ji}'s are maps of sheaves and therefore are compatible with all restriction maps, so if $W \subseteq V$ is another open set, we have $\tau_{ji}(s_i|_{W_{ij}}) = s_j|_{W_{ij}}$ if $\tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}}$. Because of this, the defining condition (5.1) is compatible with componentwise restrictions, and they can therefore be used as the restriction maps in $\mathcal{F}$. We have thus defined a presheaf on X.

The first step in what remains of the proof, is to establish the isomorphisms $\nu_i\colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$. To avoid getting confused by the names of the indices, we shall work with a fixed index $\alpha \in I$. Suppose $V \subseteq U_\alpha$ is an open set. Then naturally one has $V = V_\alpha$, and projecting from the product $\prod_i \mathcal{F}_i(V_i)$ onto the component $\mathcal{F}_\alpha(V) = \mathcal{F}_\alpha(V_\alpha)$ gives us[1] a map $\nu_\alpha\colon \mathcal{F}|_{V_\alpha} \to \mathcal{F}_\alpha$. This map sends the section $(s_i)_{i\in I}$ to s_α. The situation is summarized

[1]Since restrictions operate componentwise, it is straightforward to verify that this map is compatible with restrictions.

in the following commutative diagram

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r] \arrow[dr, "\nu_\alpha"'] & \prod_{i\in I} \mathcal{F}_i(V_i) \arrow[d, "p_\alpha"] \\ & \mathcal{F}_\alpha(V). \end{tikzcd}$$

We proceed to show that the ν_α's give the desired isomorphisms.

To begin with, we note that on the intersections $V_{\alpha\beta}$ the requirement in the proposition, that $\nu_\beta = \tau_{\beta\alpha} \circ \nu_\alpha$, is fulfilled. This follows directly from the definition in (5.1) that $s_\beta|_{V_{\alpha\beta}} = \tau_{\beta\alpha}(s_\alpha|_{V_{\alpha\beta}})$.

The map ν_α is surjective: take a section $\sigma \in F_\alpha(V)$ over some $V \subseteq U_\alpha$ and define $s = (\tau_{i\alpha}(\sigma|_{V_{i\alpha}}))_{i\in I}$. Then s satisfies the condition in (5.1) and is an honest element of $\mathcal{F}(V)$. Indeed, by the third gluing condition we obtain

$$\tau_{ji}(\tau_{i\alpha}(\sigma|_{V_{ji\alpha}})) = \tau_{j\alpha}(\sigma|_{V_{ji\alpha}})$$

for every $i, j \in I$, and that is just the condition in (5.1). As $\tau_{\alpha\alpha}(\sigma|_{V_{\alpha\alpha}}) = \sigma$ by the first gluing request, the element s projects to the section σ of $\mathcal{F}_\alpha$.

The map ν_α is injective: this is clear, since if $s_\alpha = 0$ it follows that $s_i|_{V_{i\alpha}} = \tau_{i\alpha}(s_\alpha) = 0$ for all $i \in I$. Now $\mathcal{F}_\alpha$ is a sheaf and the $V_{i\alpha}$ constitute an open covering of V_α, so we may conclude that $s = 0$ by the Locality axiom for sheaves.

The final step is to show that $\mathcal{F}$ is a sheaf, and we start with the Gluing axiom: so suppose that $\{V_\alpha\}$ is an open covering of $V \subseteq X$ and that $s_\alpha \in \mathcal{F}(V_\alpha)$ is a bunch of sections matching on the intersections $V_{\alpha\beta}$. Since $\mathcal{F}|_{U_i \cap V}$ is a sheaf (we just checked that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{F}_i$) the sections $s_\alpha|_{V_\alpha \cap U_i}$ patch together to give sections s_i in $\mathcal{F}(U_i \cap V)$ matching on the overlaps $U_{ij} \cap V$. This last condition means that $\tau_{ij}(s_i) = s_j$. By definition, $(s_i)_{i\in I}$ defines a section in $\mathcal{F}(V)$ restricting to s_i, and we are done.

The Locality axiom is easier: given a section $s = (s_i)_{i\in I}$ in $\mathcal{F}(V)$ and a covering $\mathcal{W} = \{W_j\}_{j\in J}$ of V such that $s|_{W_j} = 0$ for all $j \in J$, then also $s|_{W_j \cap V_i} = 0$, and since $\{W_j \cap V_i\}_{j\in J}$ forms a covering of V_i, we must have $s|_{V_i} = 0$ as well, since $\mathcal{F}|_{V_i} = \mathcal{F}_i$ is a sheaf. But then from the description (5.1), we thus see that $s = 0$. ❑

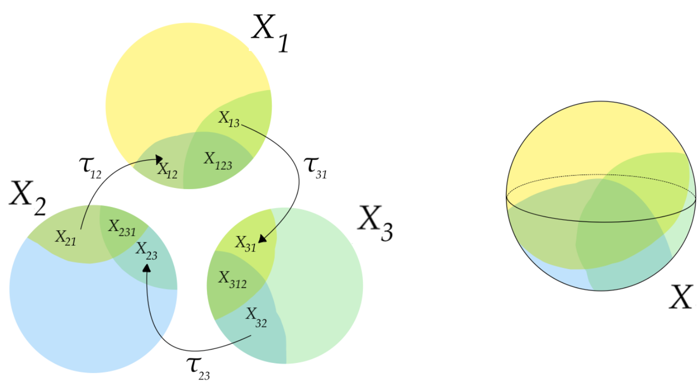
Exercise 5.1 Show the uniqueness statement in the proposition. ★

Exercise 5.2 Let $\{U_i\}_{i\in I}$ be an open cover of X. Let $\mathscr{B}$ be the collection of open sets V so that $V \subset U_i$ for some i. Show that $\mathscr{B}$ is a basis for the topology, and use this to give another proof of Proposition 5.2 on the previous page. ★

5.3 Gluing schemes

The possibility of gluing different schemes together along open subschemes is a fundamental property of schemes. It gives a plethora of new examples; the most prominent ones being the projective spaces. The gluing process is also an important part in many general existence proofs, like in the construction of the fibre product, which as we are going to show, exists without any restrictions in the category of schemes. This is in stark contrast with the theory of varieties, where two varieties that are glued together, easily can fail to be a variety.

In the present setting of 'scheme gluing' we are given a family $\{X_i\}_{i \in I}$ of schemes indexed by a set I. In each of the schemes X_i we are given a collection of open subschemes X_{ij}, where the indices i and j run through I. Their role is to form the glue lines in the process, *i.e.* the contacting surfaces that are to be glued together: in the glued scheme they will be identified and will be equal to the intersections of X_i and X_j. The identifications of the different pairs of the X_{ij}'s are encoded by a family of scheme isomorphisms $\tau_{ji} \colon X_{ij} \to X_{ji}$. Furthermore, we let $X_{ijk} = X_{ik} \cap X_{ij}$ (these are the various triple intersections before the gluing has been done), and we have to assume that $\tau_{ji}(X_{ijk}) = X_{jik}$. Notice that X_{ijk} is an open subscheme of X_i.



The three following gluing conditions, very much alike the ones we saw for sheaves, must be satisfied for the gluing to be possible:

i) $\tau_{ii} = \mathrm{id}_{X_i}$;
ii) $\tau_{ij} = \tau_{ji}^{-1}$;
iii) The isomorphism τ_{ij} takes X_{ijk} into X_{jik} and one has $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$ over X_{ijk}.

PROPOSITION 5.3 (GLUING SCHEMES) *Given gluing data X_i, τ_{ij} as above, there exists a scheme X with open immersions $\psi_i \colon X_i \to X$ such that $\psi_i|_{X_{ij}} = \psi_j|_{X_{ji}} \circ \tau_{ji}$, and such that the images $\psi_i(X_i)$ form an open covering of X. Furthermore, one has $\psi_i(X_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$. The scheme X is uniquely characterized by these properties up to a unique isomorphism.*

PROOF: To build the scheme X, we first build the underlying topological space X and subsequently equip it with a sheaf of rings. For the latter, we rely on the gluing technique for sheaves presented in Proposition 5.2. And finally, we need to show that X is locally affine; this follows however immediately once the immersions ψ_i are in place – the X_i's are schemes and therefore locally affine.

On the level of topological spaces, we start out with the disjoint union $\coprod_i X_i$ and proceed by introducing an equivalence relation on it. We declare two points $x \in X_{ij}$ and

$x' \in X_{ji}$ to be equivalent when $x' = \tau_{ji}(x)$. Observe that if the point x does not lie in any X_{ij} with $i \neq j$, we leave it alone, and it will not be equivalent to any other point.

The three gluing conditions imply readily that we obtain an equivalence relation in this way. The first requirement entails that the relation is reflexive, the second that it is symmetric, and the third ensures it is transitive. The topological space X is then defined to be the quotient of $\coprod_i X_i$ by this relation equipped with the quotient topology: if $\pi : \coprod_i X_i \to X$ denotes the quotient map, a subset U of X is open if and only if $\pi^{-1}(U)$ is open.

Topologically, the maps $\psi_i \colon X_i \to X$ are just the maps induced by the open inclusions of the X_i's in the disjoint union $\coprod_i X_i$. They are clearly injective since a point $x \in X_i$ is never equivalent to another point in X_i. Now, X has the quotient topology so a subset U of X is open if and only if $\pi^{-1}(U)$ is open, and this holds if and only if $\psi_i^{-1}(U) = X_i \cap \pi^{-1}(U)$ is open for all i. In view of the formula

$$\pi^{-1}(\psi_i(U)) = \bigcup_j \tau_{ji}(U \cap X_{ij})$$

we conclude that each ψ_i is an open map, hence a homeomorphism onto its image.

To simplify notation, we now write X_i for $\psi_i(X_i)$, which is in accordance with our intuitive picture of X as being the union of the X_i's with points in the X_{ij}'s identified according to the τ_{ij}'s. Then X_{ij} becomes $X_i \cap X_j$ and X_{ijk} becomes the triple intersection $X_i \cap X_j \cap X_k$.

On X_{ij}, we have the isomorphisms $\tau_{ji}^\sharp \colon \mathcal{O}_{X_j}|_{X_{ij}} \to \mathcal{O}_{X_i}|_{X_{ij}}$; the sheaf maps of the scheme isomorphisms $\tau_{ji} \colon X_{ij} \to X_{ji}$. In view of the third gluing condition $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$, valid on X_{ijk}, we obviously have $\tau_{ki}^\sharp = \tau_{ji}^\sharp \circ \tau_{kj}^\sharp$. The two first gluing conditions translate into $\tau_{ii}^\sharp = \mathrm{id}$ and $\tau_{ji}^\sharp = (\tau_{ij}^\sharp)^{-1}$. The conclusion is that the gluing properties needed to apply Proposition 5.2 are satisfied, and we are allowed to glue the different $\mathcal{O}_{X_i}$'s together and thus to equip X with a sheaf of rings. This sheaf of rings restricts to $\mathcal{O}_{X_i}$ on each of the open subsets X_i, and therefore its stalks are local rings. So $(X, \mathcal{O}_X)$ is a locally ringed space that is locally affine, hence a scheme.

We leave it to the reader to prove the uniqueness statement in the proposition. ❑

EXERCISE 5.3 Prove the uniqueness part in the above proposition. ★

Global sections of glued schemes

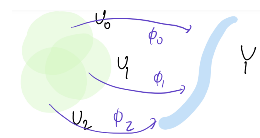
The standard exact sequence for computing global sections from an open covering is a valuable tool in the setting of glued schemes. If X is obtained by gluing the open subschemes X_i along isomorphisms $\tau_{ji} \colon X_{ij} \to X_{ji}$, it reads:

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \xrightarrow{\ \alpha\ } \bigoplus_i \Gamma(X_i, \mathcal{O}_{X_i}) \xrightarrow{\ \rho\ } \bigoplus_{i,j} \Gamma(X_{ij}, \mathcal{O}_{X_{ij}}) \tag{5.2}$$

where $\rho\big((s_i)_{i\in I}\big) = \big(s_i|_{X_{ij}} - \tau_{ij}^\sharp(s_j|_{X_{ji}})\big)_{i\in I}$ and $\alpha(s) = (\psi_i^\sharp(s))_{i,j\in I}$.

5.4 Gluing morphisms of schemes

Suppose we are given schemes X and Y and an open covering $\{U_i\}_{i\in I}$ of X. Assume further that there is given a family of morphisms $\phi_i\colon U_i \to Y$ which match on the intersections $U_{ij} = U_i \cap U_j$. The aim of this paragraph is to show that the ϕ_i's can be glued together to give a morphism $X \to Y$:



Proposition 5.4 (Gluing morphisms of schemes) *Assume given gluing data ϕ_i as above, there exists a unique map of schemes $\phi\colon X \to Y$ such that $\phi|_{U_i} = \phi_i$.*

Proof: Clearly the map on topological spaces is well defined and continuous, so if $U \subseteq Y$ is an open set, we have to define $\phi^\sharp\colon \Gamma(U, \mathcal{O}_Y) \to \Gamma(U, \phi_*\mathcal{O}_X) = \Gamma(\phi^{-1}U, \mathcal{O}_X)$. So take any section $s \in \mathcal{O}_Y(U)$ over U. This gives sections $t_i = \phi_i^\sharp(s)$ of $\mathcal{O}_X(U_i)$. But since $\phi_i^\sharp$ and $\phi_j^\sharp$ restrict to the same map on U_{ij}, we have $t_i|_{U_{ij}} = t_j|_{U_{ij}}$. The t_i therefore patch together to a section $t \in \mathcal{O}_X(\phi^{-1}U)$, which is the section we are aiming at: we may define $\phi^\sharp(s)$ to be t. Proving the uniqueness statement is again left to the student. ❑

Exercise 5.4 Let X and Y be schemes and let $\mathscr{B}$ be a base for the topology on X. Suppose that there is a collection of morphisms $\phi_U\colon U \to Y$, one for each $U \in \mathscr{B}$, such that if $V \in \mathscr{B}$ satisfies $V \subset U$, we have

$$\phi_U|_V = \phi_V.$$

Show that there exists a unique morphism of schemes $\phi\colon X \to Y$ such that $\phi|_U = \phi_U$. ★

5.5 Universal properties of maps into affine schemes

For a general scheme X, one may very well consider the associated affine scheme $\operatorname{Spec}\Gamma(X, \mathcal{O}_X)$. This is however in general very different from X (for instance, for the projective line $\mathbb{P}^1_k$ introduced in the next chapter, $\operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ will be just a point). There is however still a canonically defined morphism $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$, which satisfies the following universal property:

Proposition 5.5 *Let X be any scheme. Then there is a canonical map of schemes $\psi\colon X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ inducing the identity on global sections of the structure sheaves.*

There is in fact an even stronger relationship between maps of affine schemes and ring homomorphisms:

Theorem 5.6 (Maps into affine schemes) *For any scheme X, the canonical map*

$$\Phi_X\colon \operatorname{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\mathsf{Rings}}(A, \Gamma(X, \mathcal{O}_X))$$

given by $(f, f^\sharp) \mapsto f^\#(X)$ is bijective.

Proof: Let $\{U_i\}$ be an affine covering of X. By the affine schemes case (Theorem 4.21), we know that each Φ_{U_i} is bijective. This also gives that Φ_X is injective: If we are given two morphisms $\phi, \psi : X \to \operatorname{Spec} A$, that map to the same ring map $\beta : A \to \mathcal{O}_X(X)$, we get morphisms $\phi_i : U_i \to \operatorname{Spec} A$ and $\psi_i : U_i \to \operatorname{Spec} A$. For each i these both correspond to the ring map given by composing β with the restriction $\beta_i : A \to \mathcal{O}_X(X) \to \mathcal{O}_X(U_i)$, thus $\phi_i = \psi_i$, by the bijectivity of Φ_{U_i}. Then $\phi = \psi$ by the uniqueness part of Proposition 5.4, so Φ_X is injective.

To show that Φ_X is surjective let $\beta\colon A \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism. The maps induced by restriction, $\beta_i\colon A \to \Gamma(X, \mathcal{O}_X) \to \Gamma(U_i, \mathcal{O}_X)$, and induce also morphisms of schemes $f_i\colon U_i \to \operatorname{Spec} A$. We claim that the f_i's may be glued together to a map $f\colon X \to \operatorname{Spec} A$. This is a consequence of the following diagram being commutative where $V \subseteq U_i \cap U_j$ is an open affine:

$$\begin{tikzcd}
 & \Gamma(U_i, \mathcal{O}_{U_i}) \arrow[dr] & & \\
A \arrow[ur, "\beta_i"] \arrow[r, "\beta"] \arrow[dr, "\beta_j"'] & \Gamma(X, \mathcal{O}_X) \arrow[u] \arrow[d] \arrow[r] & \Gamma(U_i \cap U_j, \mathcal{O}_X) \arrow[r] & \Gamma(V, \mathcal{O}_X). \\
 & \Gamma(U_j, \mathcal{O}_{U_j}) \arrow[ur] & &
\end{tikzcd}$$

Indeed, note that for $V \subseteq U_i \cap U_j$ affine, the diagram implies that the restrictions $f_i|_V$ and $f_j|_V$ induce the same element in $\operatorname{Hom}_{\mathsf{Rings}}(A, \Gamma(V, \mathcal{O}_X))$, and so they are equal on V (according to Theorem 4.21). Since this is true for any V, the f_i are equal on all of $U_i \cap U_j$. So by gluing the f_i's, we obtain a morphism $f\colon X \to \operatorname{Spec} A$. It must hold that $\Phi_X(f) = \beta$, since Φ_X is injective (which we just proved) and since $f|_{U_i}$ maps to β_i via Φ_{U_i} for each i. This completes the proof. ❑

Proposition 5.5 above follows immediately by applying the theorem to $A = \Gamma(X, \mathcal{O}_X)$.

Corollary 5.7 *The canonical map $\psi\colon X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ is universal among the maps from X to affine schemes; i.e. any map $\phi\colon X \to \operatorname{Spec} A$ factors as $\phi = \eta \circ \psi$ for a unique map $\eta\colon \operatorname{Spec} \Gamma(X, \mathcal{O}_X) \to \operatorname{Spec} A$.*

Proof: In the theorem above, ψ corresponds to the identity map $\mathrm{id}_{\Gamma(X, \mathcal{O}_X)}$ on the right hand side. The morphism η is the map of Spec's induced by the ring map $\phi^\sharp\colon A \to \Gamma(X, \mathcal{O}_X)$. We check that it factors ϕ: the morphism $(\eta \circ \psi)\colon X \to \operatorname{Spec} A$ satisfies $(\eta \circ \psi)^\# = \psi^\# \circ \eta^\# = \phi^\#$ and hence it coincides with ϕ by the above theorem. ❑

As a special case, we note that there is a bijection

$$\operatorname{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} \mathbb{Z}) \simeq \operatorname{Hom}_{\mathsf{Rings}}(\mathbb{Z}, \Gamma(X, \mathcal{O}_X)).$$

Since ring maps always preserve the unit element, the set on the right is clearly a one-point set. So there exist precisely one morphism of schemes $X \to \operatorname{Spec} \mathbb{Z}$. In categorical terms this means that $\operatorname{Spec}\mathbb{Z}$ is a *final object* in the category of schemes Sch.

The category Sch also has an *initial object*, the empty scheme; it equals the spectrum of the zero ring, $\operatorname{Spec} 0$, which has the empty set as underlying topological space. Given any scheme X there is clearly a unique morphism $\operatorname{Spec} 0 \to X$, which on the level of sheaves sends every section of $\mathcal{O}_X$ to zero.

R-valued points

For a scheme X, it makes sense to study morphisms $\operatorname{Spec} R \to X$ from affine schemes into it. We call such morphisms *R-valued points*, and the set of all such will be denoted by $X(R)$; that is, $X(R) = \operatorname{Hom}_{\mathsf{Sch}}(\operatorname{Spec} R, X)$. The jargon here is justified from the following:

Example 5.8 Recall the 'absolute' affine space $\mathbb{A}^n = \operatorname{Spec}\mathbb{Z}[x_1, \ldots, x_n]$. An R-valued point of $\mathbb{A}^n$ is a morphism $g\colon \operatorname{Spec} R \to \operatorname{Spec}\mathbb{Z}[x_1, \ldots, x_n]$, which determines and is determined by the n-tuple $(a_i) = (g^*(x_i))$ of elements in R. Hence,

$$\mathbb{A}^n(R) = R^n.$$

Now, let $X = \operatorname{Spec}\mathbb{Z}[x_1, \ldots, x_n]/\mathfrak{a}$ where $\mathfrak{a} = (f_1, \ldots, f_r)$ is an ideal. The set of R-points of X can be found similarly: any morphism

$$g\colon \operatorname{Spec} R \to \operatorname{Spec}\mathbb{Z}[x_1, \ldots, x_n]/\mathfrak{a}$$

is determined by the n-tuple $(a_i) = (g^*(x_i))$, and the n-tuples that occur are exactly those such that $f \mapsto f(a_1, \ldots, a_n)$ defines a homomorphism

$$\mathbb{Z}[x_1, \ldots, x_n]/\mathfrak{a} \to R.$$

In other words, the n-tuples (a_i) in R^n which are solutions of the equations $f_1 = \cdots = f_r = 0$. ☆

Example 5.9 More generally, one may replace $\mathbb{Z}$ in the example above by any ring A and R by any A-algebra: if $\mathfrak{a} = (f_1, \ldots, f_r)$ is an ideal in $A[x_1, \ldots, x_n]$ and $X = \operatorname{Spec} A[x_1, \ldots, x_n]/\mathfrak{a}$ the set $\operatorname{Hom}_{\mathsf{Sch}/A}(\operatorname{Spec} R, X)$ consists of tuples (a_i) in R^n that are solutions of the equations $f_1 = \cdots = f_r = 0$. ☆

Example 5.10 *(A conic with no real points.)* Let $X = \operatorname{Spec} A$, where A is the real algebra $A = \mathbb{R}[x, y]/(x^2 + y^2 + 1)$. Note that the conic $x^2 + y^2 + 1 = 0$ has no real points, so $X(\mathbb{R}) = \varnothing$. However, A has infinitely many maximal ideals, so that X as well as $X(\mathbb{C})$ are infinite. ☆

Example 5.11 *(Pythagorean triples.)* Consider $X = \operatorname{Spec}\mathbb{Z}[x, y]/(x^2 + y^2 - 1)$. Then $X(\mathbb{R})$ is the unit circle in $\mathbb{R}^2$ while $X(\mathbb{Q})$ consists of pairs $(m/r, n/r)$ in $\mathbb{Q}^2$ corresponding to the 'Pythagorean triples' of integers; *i.e.* triples m, n and r so that $m^2 + n^2 = r^2$. ☆

The sets $X(R)$ of points over R are obviously important in number theory, as they naturally generalize the solution set of the polynomials $f_1 = \cdots = f_r = 0$. Of course, even when R is a field, it can be very difficult to describe the set $X(K)$ of K-valued points $\operatorname{Spec} K \to X$, or even determining whether $X(K) \neq \varnothing$ (a most spectacular example is $\operatorname{Spec}\mathbb{Z}[x, y]/(x^n + y^n - 1)$, $n \geqslant 3$ for $K = \mathbb{Q}$).

When K is a field, the underlying topological $\operatorname{Spec} K$ is very simple; it is just a singleton since the only prime ideal in K is the zero ideal. However, the structure sheaf $\mathcal{O}_{\operatorname{Spec} K}$ on $\operatorname{Spec} K$ carries more information: morphisms $\operatorname{Spec} L \to \operatorname{Spec} K$ (*i.e.* the elements of $X(L)$) correspond exactly to field extensions $L \supseteq K$. In particular, $\operatorname{Spec} K$ and $\operatorname{Spec} L$ are isomorphic if and only if $K \simeq L$.

The residue fields play an important role here. The following result says that they satisfy a universal property with respect to morphisms from spectra of fields to X.

PROPOSITION 5.12 *Let X be a scheme and let K be a field. Then to give a morphism of schemes* $\operatorname{Spec} K \to X$ *is equivalent to giving a point $x \in X$ plus an embedding $k(x) \to K$.*

More generally, one may for a fixed scheme S define $X(S)$ to be the set of all morphisms $S \to X$; the so-called *S-valued points* of X. In the example above, we have for any scheme S,

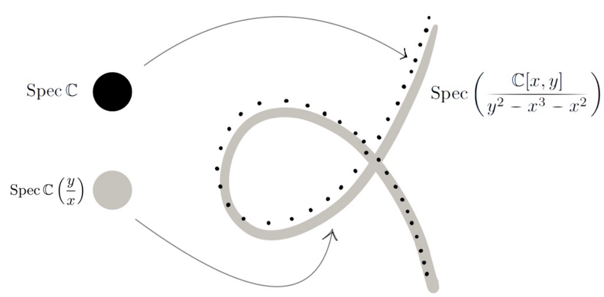
$$\mathbb{A}^n(S) = \operatorname{Hom}_{\mathsf{Sch}}(S, \mathbb{A}^n) = \Gamma(S, \mathcal{O}_S)^n.$$

In fancy terms, this says that $\mathbb{A}^n$ represents the functor taking a scheme to n-tuples of elements of $\Gamma(S, \mathcal{O}_S)$. We shall see a similar functorial characterization of projective space $\mathbb{P}^n$ later in the book.

EXAMPLE 5.13 Let $R = \mathbb{C}[x,y]/(y^2 - x^3 - x^2)$ and $X = \operatorname{Spec} R$. Then X is the *nodal cubic curve* over $\mathbb{C}$. X contains two types of points:

- Closed points $p \in X$. These correspond to maximal ideals $\mathfrak{m} = (x-a, y-b)$ where a, b satisfy $b^2 = a^3 + a^2$. The residue field equals $k(p) = R/\mathfrak{m} \simeq \mathbb{C}$. Thus these are the $\mathbb{C}$-valued points of X.

- The generic point η. This corresponds to the zero ideal. The residue field here equals the fraction field of R, which is isomorphic to $\mathbb{C}(t)$ (via the substitution $x = t^2 - 1, y = t^3 - t$). Thus the generic point is a $\mathbb{C}(t)$-valued point of X.



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✲ EXERCISE 5.5 Prove Proposition 5.12 ★

EXERCISE 5.6 Beware that there may be a huge difference between abolute scheme isomorphisms and relative ones. Show that:

a) $\mathrm{Hom}_{\mathsf{Sch}/\mathbb{C}}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathbb{C}) = \{\mathrm{id}\}$
b) $\mathrm{Hom}_{\mathsf{Sch}/\mathbb{R}}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathbb{C}) \simeq \mathbb{Z}/2\mathbb{Z}$.
c) $(\operatorname{Spec}\mathbb{C})(\mathbb{C}) = \mathrm{Hom}_{\mathsf{Sch}}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathbb{C})$ is unountable.

★

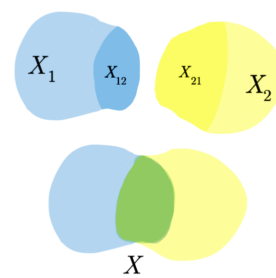
Chapter 6

Examples constructed by gluing

6.1 Gluing two schemes together

To concretify the gluing techniques introduced in Chapter 5, we will in more detail study the simple case of schemes obtained by gluing together just two schemes.

We start out with two schemes X_1 and X_2 with respective open subsets $X_{12} \subseteq X_1$ and $X_{21} \subseteq X_2$; these are open subschemes equipped with their canonical induced scheme structures obtained by restricting the structure sheaves. Furthermore, we assume given an isomorphism $\tau\colon X_{21} \to X_{12}$. The gluing conditions are trivially fulfilled, and these data allow us to glue together X_1 and X_2 along X_{12} and X_{21}, and we thus construct a new scheme X.



On the level of topological spaces X is obtained from the disjoint union $X_1 \coprod X_2$ by forming the quotient modulo the equivalence relation with $x \sim \tau(x)$ for $x \in X_{21} \subseteq X_2$ and giving X the quotient topology. Moreover, each morphism $\iota_j\colon X_j \to X$ is an open immersion, allowing us to view each X_j as an open subset of X.

The sections of the sheaf $\mathcal{O}_X$ over an open $U \subseteq X$ is determined by the exact sequence (5.2), which in the present context takes the shape below. We insists on being precise and write $U_j = \iota_j^{-1}U$ (which is identified with $X_j \cap U$) and $U_{ji} = \iota_j^{-1}(U_1 \cap U_2)$ (which is identified with $X_1 \cap X_2 \cap U$; this can be done in two ways related by the 'glue' τ).

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(U, \mathcal{O}_X) \arrow[r] & \Gamma(U \cap X_1, \mathcal{O}_X) \times \Gamma(U \cap X_2, \mathcal{O}_X) \arrow[r] \arrow[d, "\wr"] & \Gamma(U \cap X_1 \cap X_2, \mathcal{O}_X) \arrow[d, "\wr"] \\
 & & \Gamma(U_1, \mathcal{O}_{X_1}) \times \Gamma(U_2, \mathcal{O}_{X_2}) \arrow[r, "\rho"'] & \Gamma(U_{21}, \mathcal{O}_{X_2})
\end{tikzcd}$$

The components of the left map are just the restrictions, *i.e.* it maps s to the pair $(\iota_1^\sharp s|_{U \cap X_1}, \iota_2^\sharp s|_{U \cap X_2})$ (which after identification is nothing but $(s|_{U \cap X_1}, s|_{U \cap X_2})$), and the second sends a pair (s, t) to $\rho(s, t) = t|_{U_{21}} - \tau^\sharp(s|_{U_{12}})$.

The main example to keep in mind is when X_1 and X_2 are both affine, say $X_1 = \operatorname{Spec} R$ and $X_2 = \operatorname{Spec} S$, and they are glued together along two distinguished open subsets $D(u)$ and $D(v)$ for some $u \in R$ and $v \in S$. The 'glue' τ is induced from a ring isomorphism between localizations

$$\phi : R_u \to S_v.$$

The geometric picture is as in the following diagram of schemes

$$\begin{tikzcd}
& \operatorname{Spec} R_u = D(u) \arrow[dl, hook] & D(v) = \operatorname{Spec} S_v \arrow[l, "\tau", "\simeq"'] \arrow[dr, hook] & \\
\operatorname{Spec} R & & & \operatorname{Spec} S.
\end{tikzcd}$$

The global sections of $\mathcal{O}_X$ is computed by the standard sequence on the previous page, which in the present staging takes the form

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow R \times S \xrightarrow{\ \rho\ } S_v. \tag{6.1}$$

Here $\rho(r,s) = s/1 - \phi(r/1)$ with $s/1$ and $r/1$ denoting the images of s and r respectively in S_v and R_u. In other words, elements in $\mathcal{O}_X(X)$ correspond to pairs $(r,s) \in R \times S$ such that $s/1 = \phi(r/1)$ in the localized ring S_v.

We can also study sheaves on the glued scheme X. Proposition 5.2 tells us that giving a sheaf $\mathcal{F}$ on X is equivalent to specifying (i) a sheaf $\mathcal{F}_1$ on X_1; (ii) a sheaf $\mathcal{F}_2$ on X_2; (iii) a sheaf isomorphism

$$\nu_{12} : \mathcal{F}_2|_{D(v)} \to \mathcal{F}_1|_{D(u)}$$

where we use the isomorphism τ to identify $D(u)$ and $D(v)$. In the special case that $\mathcal{F}_1 = \widetilde{M}$ and $\mathcal{F}_2 = \widetilde{N}$ for modules M and N (over S and R respectively), we can specify this by giving an isomorphism of R_u-modules

$$v_{12} : N_u \to M_v$$

Many important examples arise from this basic construction. We will now survey a few of these.

6.2 *A scheme that is not affine*

The first application is to see that the affine plane minus the origin is not an affine scheme.

Let k be a field, let $\mathbb{A}^2_k = \operatorname{Spec} A$ where $A = k[u,v]$, and consider the open subset $U = \mathbb{A}^2_k - V(u,v)$; this is precisely the affine plane with the closed point corresponding to the origin removed. Since U is an open set of $\mathbb{A}^2_k$, there is a canonical scheme structure on U as described in Section 4.5 on page 68. We contend that U can not be isomorphic to an affine scheme, the key point being that the restriction map $\Gamma(\mathbb{A}^2_k, \mathcal{O}_{\mathbb{A}^2_k}) \to \Gamma(U, \mathcal{O}_U)$ is an isomorphism. Indeed, if U were an affine scheme, Theorem 5.6 on page 81 would then imply that the inclusion map $U \to \mathbb{A}^2_k$ were an isomorphism, which obviously is not true as it *e.g.* is not surjective.

Let us check that the restriction map really is an isomorphism. The two distinguished open sets $D(u) = \operatorname{Spec} A_u$ and $D(v) = \operatorname{Spec} A_v$ form an open affine covering of U, and the exact sequence (6.1) takes the following form:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(U, \mathcal{O}_U) \arrow[r] & A_u \times A_v \arrow[r, "\rho"] & A_{uv}, \\
& A \arrow[u, "i^\sharp"] & &
\end{tikzcd}$$

where ρ is the difference between the two localization maps; that is, it maps a pair (au^{-m}, bv^{-n}) to $au^{-m} - bv^{-n}$ considered as an element in A_{uv}. We have included the restriction map $i^\sharp$ in the diagram, which is the just the map coming from the inclusion map $i\colon U \to \mathbb{A}^2_k$. It sends an element $a \in A$ to the pair $(a/1, a/1)$ in $A_u \times A_v$.

Elements of $\Gamma(U, \mathcal{O}_U)$ correspond to a pairs (au^{-m}, bv^{-n}) in the kernel of ρ, and for such a pair the relation

$$av^n = bu^m$$

holds in $A = k[u, v]$. Since A is a UFD, it follows that there is an element $c \in A$ with $a = cu^m$ and $b = cv^n$; that is, $au^{-m} = bv^{-n}$, and this shows that $i^\sharp$ is surjective. Hence it is an isomorphism since it obviously is injective.

EXERCISE 6.1 Show that $\mathbb{A}^1_k$ minus the origin is an affine scheme. ★

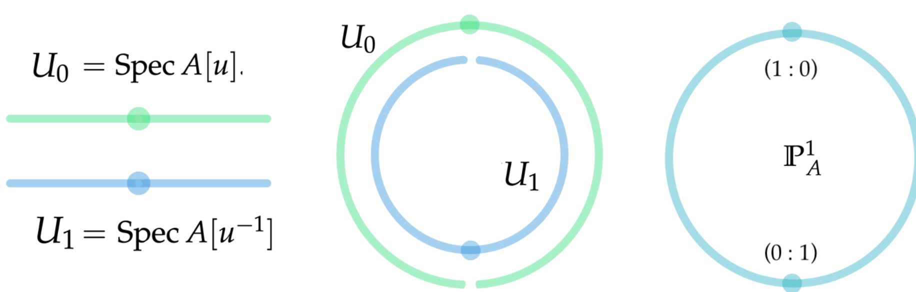
EXERCISE 6.2 Let $X = \operatorname{Spec} k[x, y, z, w]/(xw - yz)$ and consider the open set $U = X - V(x, y)$. Use the above strategy to compute $\mathcal{O}_X(U)$. Conclude that U is not affine. ★

6.3 *The projective line*

In elementary courses on complex analysis one encounters the Riemann sphere. This is the complex plane $\mathbb{C}$ with one point added, the point at infinity. If z is the complex coordinate centered at the origin, the inverse $w = z^{-1}$ is the coordinate centered at infinity. Another name for the Riemann sphere is the complex projective line, denoted $\mathbb{CP}^1$.

The construction of $\mathbb{CP}^1$ can be vastly generalized and works in fact over any ring A. Let u be a variable ('the coordinate function at the origin') and let $U_0 = \operatorname{Spec} A[u]$. The inverse u^{-1} is a variable as good as u ('the coordinate at infinity'), and we let $U_1 = \operatorname{Spec} A[u^{-1}]$. Both are copies of the affine line $\mathbb{A}^1_A$ over A.

Inside U_0 we have the distinguished open set $U_{01} = D(u)$, which is canonically isomorphic to the prime spectrum $\operatorname{Spec} A[u, u^{-1}]$, the isomorphism coming from the inclusion $A[u] \subseteq A[u, u^{-1}]$. In the same way, inside U_1 there is the distinguished open set $U_{10} = D(u^{-1})$. This is also canonically isomorphic to the spectrum $\operatorname{Spec} A[u^{-1}, u]$, the isomorphism being induced by the inclusion $A[u^{-1}] \subseteq A[u^{-1}, u]$. Hence U_{01} and U_{10} are isomorphic schemes (even canonically), and we may glue U_0 to U_1 along U_{01}. The result is called the *projective line over* A and is denoted by $\mathbb{P}^1_A$.



Gluing two affine lines to get $\mathbb{P}^1_A$

Note that the complement of U_1 equals $V(u) \subseteq U_0 = \operatorname{Spec} A[u]$, which is isomorphic to $\operatorname{Spec} A$, so when $A = k$ is a field, $\mathbb{P}^1_k$ is a 'one-point 'compactification' of U_0. Of course, a similar statement holds true for the complement of U_0.

The following computation is very important.

PROPOSITION 6.1 *We have* $\Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}) = A$.

PROOF: The projective line $\mathbb{P}^1_A$ is covered by the two open affines U_0 and U_1, and the standard exact sequence (6.1) above takes the form

$$\begin{tikzcd} \Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}) \arrow[r] & \Gamma(U_0, \mathcal{O}_{\mathbb{P}^1_A}) \times \Gamma(U_1, \mathcal{O}_{\mathbb{P}^1_A}) \arrow[r] \arrow[d, "\wr"] & \Gamma(U_{01}, \mathcal{O}_{\mathbb{P}^1_A}) \arrow[d, "\wr"] \\ & A[u] \times A[u^{-1}] \arrow[r, "\rho"'] & A[u, u^{-1}], \end{tikzcd}$$

where the map ρ sends a pair $(f(u), g(u^{-1}))$ of polynomials with coefficients in A, one in the variable u and one in u^{-1}, to the difference $g(u^{-1}) - f(u)$. We claim that the kernel of ρ equals A; *i.e.* the polynomials f and g must both be constants.

So assume that $f(u) = g(u^{-1})$. Write $f(u) = au^n + \textit{lower terms in } u$, and in a similar way, write $g(u^{-1}) = bu^{-m} + \textit{lower terms in } u^{-1}$, where both $a \neq 0$ and $b \neq 0$. Without loss of generality we may assume that $m \geqslant n$. Now, suppose that $m \geqslant 1$. Upon multiplication by u^m we obtain $b + uh(u) = u^m f(u)$ for some polynomial $h(u)$, and putting $u = 0$ we get $b = 0$, which is a contradiction. Hence $m = n = 0$ and we are done. ❑

In particular, the global sections of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$ is just $\mathbb{C}$, which is a special case of Liouville's theorem that the only global holomorphic functions are the constants. We note that we also have got yet another example of a scheme which is not affine: if $\mathbb{P}^1_A$ were affine, it would have to be isomorphic to $\operatorname{Spec} \mathbb{C}$ according to Theorem 4.21 on page 65. But this is clearly not the case, as $\mathbb{P}^1_{\mathbb{C}}$ contains infinitely many closed points (*e.g.* it contains $\mathbb{A}^1_{\mathbb{C}}$ as an open subset).

Another morale to extract from this is that the group $\Gamma(X, \mathcal{O}_X)$ does not give much information about X for general schemes, in contrast to the case when $X = \operatorname{Spec} A$ is affine.

✲ **EXERCISE 6.3** Let $X = \operatorname{Spec} A$ be an affine scheme over a field k. Show that every morphism $\mathbb{P}^1_A \to X$ is constant. ★

The projective line $\mathbb{P}^1$ as a quotient

The two examples we have constructed are in fact closely related. In particular, there is a natural morphism between them:

$$\pi : \mathbb{A}^2_A - V(u, v) \to \mathbb{P}^1_A,$$

which we shall construct by gluing together the two morphisms $\phi_1 \colon D(u) = \operatorname{Spec} A[u, v]_u \to \operatorname{Spec} A[u^{-1}v]$ and $\phi_2 \colon D(v) = \operatorname{Spec} A[u, v]_v \to \operatorname{Spec} A[uv^{-1}]$ induced by the inclusions

$A[u^{-1}v] \subset A[u,v]_u$ and $A[uv^{-1}] \subset A[u,v]_v$. With $t = uv^{-1}$ the two targets yields $\mathbb{P}^1_A$ when glued together along $\operatorname{Spec} A[t,t^{-1}] = \operatorname{Spec} A[uv^{-1}, vu^{-1}]$; the union of the sources equals $D(u) \cup D(v) = V(u,v)^c$ and $D(u) \cap D(v) = D(uv) = \operatorname{Spec} A[u,v]_{uv}$. The gluing condition is satisfied as we see by applying Spec to the following commutative diagram (which, in fact, is a diagram of inclusion between subrings of $A[u,v]_{uv} = A[u,u^{-1},v,v^{-1}]$):

$$\begin{tikzcd}
A[u,v]_u \arrow[dr, hook] & A[u^{-1}v] \arrow[l, hook'] \arrow[dr, hook] & \\
& A[u,v]_{uv} & A[u^{-1}v, uv^{-1}] \arrow[l, hook'] \\
A[u,v]_v \arrow[ur, hook] & A[uv^{-1}] \arrow[l, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

On the level of closed points, when $A = k$ is an algebraically closed field, the morphism π is exactly the *quotient morphism* used in the construction of the projective line as a quotient space.

A family of sheaves on $\mathbb{P}^1$

The projective spaces, in particular $\mathbb{P}^1_A$, carry a family of sheaves, there is one for each integer, which play a foremost role in algebraic geometry. We shall construct these sheaves on the projective line $\mathbb{P}^1_A$ by the gluing techniques so far explained; they will be denoted by $\mathcal{O}_{\mathbb{P}^1_A}(m)$ with $m \in \mathbb{Z}$.

Let $X = \mathbb{P}^1_A$ and let $U_0 = \operatorname{Spec} A[u]$ and $U_1 = \operatorname{Spec} A[u^{-1}]$ be the usual covering. Consider the intersection $U_0 \cap U_1 = \operatorname{Spec} R$, where $R = A[u,u^{-1}]$. Multiplication by u^m gives an isomorphism

$$R \xrightarrow{[u^m]} R,$$

and by Exercise 4.1 this induces an isomorphism of sheaves

$$\tau\colon \mathcal{O}_{U_1}|_{U_0 \cap U_1} \to \mathcal{O}_{U_0}|_{U_0 \cap U_1}.$$

Now, we define a sheaf $\mathcal{O}_{\mathbb{P}^1_A}(m)$ by gluing $\mathcal{O}_{U_1}$ to $\mathcal{O}_{U_0}$ along $U_0 \cap U_1$ via this isomorphism. Note that the direction of τ is very important; we could of course have used the multiplication map the other way round, but would then had obtained another sheaf, namely $\mathcal{O}_{\mathbb{P}^1_A}(-m)$.

By construction, the sheaf $\mathcal{O}_{\mathbb{P}^1_A}(m)$ has the property that $\mathcal{O}_{\mathbb{P}^1_A}(m)|_{U_0} \simeq \mathcal{O}_{U_0}$ and $\mathcal{O}(m)_{\mathbb{P}^1_A}|_{U_1} \simeq \mathcal{O}_{U_1}$. In the jargon of Chapter 16 it is a 'locally free sheaf'. Let us, however, show that when $m \neq 0$, the sheaf $\mathcal{O}_{\mathbb{P}^1_A}(m)$ is not isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^1_A}$. In particular, we see that a sheaf is not determined by its stalks alone.

To prove this claim, we compute global sections using the standard sequence:

$$0 \longrightarrow \Gamma(\mathcal{O}_{\mathbb{P}^1_A}, \mathcal{O}_{\mathbb{P}^1_A}(m)) \longrightarrow A[t] \oplus A[t^{-1}] \xrightarrow{\rho} A[t,t^{-1}]$$

where $\rho(p(u), q(u^{-1})) = u^m q(u^{-1}) - p(u)$. If $m < 0$, clearly there are no non-trivial polynomials p and q satisfying $u^m q(u^{-1}) = p(u)$, and we infer that $\Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}(m)) = \operatorname{Ker}\rho = 0$; hence $\mathcal{O}_{\mathbb{P}^1_A}(m)$ cannot be isomorphic to $\mathcal{O}_{\mathbb{P}^1_A}$. For $m \geqslant 0$ there are, however, such polynomials: every polynomial $p(u)$ of degree at most m is on the form $u^m q(u^{-1})$, and q is uniquely determined by p. We have shown the following:

PROPOSITION 6.2 *For $m \geqslant 0$ we have*

$$\Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}(m)) = A \oplus Au \oplus \cdots \oplus Au^m.$$

Closed subschemes of $\mathbb{P}^1$

Let us have a closer look at the sheaf $\mathcal{O}_{\mathbb{P}^1_A}(-1)$, which is usually called the *tautological sheaf* on $\mathbb{P}^1$. We claim that there is a map of sheaves

$$\phi\colon \mathcal{O}_{\mathbb{P}^1_A}(-1) \to \mathcal{O}_{\mathbb{P}^1},$$

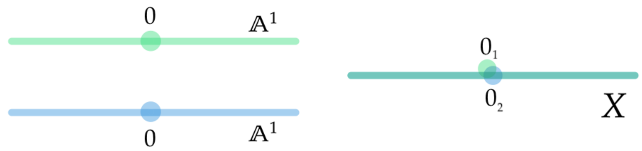
which makes $\mathcal{O}_{\mathbb{P}^1_A}(-1)$ into a subsheaf of $\mathcal{O}_{\mathbb{P}^1}$. We shall define ϕ locally on each of the open sets U_0 and U_1 and subsequently apply the gluing lemma for morphisms of sheaves. On the open set $U_0 = \operatorname{Spec} A[u]$ we use the multiplication by u as the $\phi_0\colon \mathcal{O}_{U_0} \to \mathcal{O}_{U_0}$. Likewise, we define $\phi_1\colon \mathcal{O}_{U_1} \to \mathcal{O}_{U_1}$ by the identity map. To be able to glue them together, we need to verify that the two agree on the intersection $U_0 \cap U_1 = \operatorname{Spec} A[u, u^{-1}]$, but this follows directly from the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{U_0}|_{U_0 \cap U_1} & \xrightarrow{u} & \mathcal{O}_{U_0}|_{U_0 \cap U_1} \\
{\scriptstyle u^{-1}}\uparrow & & \uparrow\| \\
\mathcal{O}_{U_1}|_{U_0 \cap U_1} & \xrightarrow{=} & \mathcal{O}_{U_1}|_{U_0 \cap U_1}
\end{array}$$

Here all four sheaves are equal to $\mathcal{O}_{U_0 \cap U_1}$. The right vertical map is the gluing-map for the sheaf $\mathcal{O}_{\mathbb{P}^1_A}$ and the left one that for $\mathcal{O}_{\mathbb{P}^1_A}(-1)$, whilst the horizontal maps are the restrictions $\phi_0|_{U_0 \cap U_1}$ and $\phi_1|_{U_0 \cap U_1}$. Thus we have the desired map $\phi : \mathcal{O}_{\mathbb{P}^1_A}(-1) \to \mathcal{O}_{\mathbb{P}^1}$. This map is injective, because it is injective over U_0 and U_1 (see Lemma 12.2).

6.4 *The affine line with a doubled origin*

We intend to glue together two copies X_1 and X_2 of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[u]$ over a field k along their common open subset $X_{12} = \operatorname{Spec} k[u, u^{-1}]$ with the identity morphism $\phi : k[u, u^{-1}] \to k[u, u^{-1}]$ as glue. The resulting scheme contains two $\mathbb{A}^1_k$'s which overlap outside the origin. But since the the gluing process does nothing over the origins of each $\mathbb{A}^1_k$, there are now *two* points in X that replace the origin; X is sometimes called the *affine line with two origins*.



This scheme is not affine: the sheaf sequence from before takes the form

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X,\mathcal{O}_X) \arrow[r] & \Gamma(\mathbb{A}^1_k,\mathcal{O}_{\mathbb{A}^1_k})\oplus\Gamma(\mathbb{A}^1_k,\mathcal{O}_{\mathbb{A}^1_k}) \arrow[r] \arrow[d, "\shortparallel"] & \Gamma(X_{12},\mathcal{O}_{X_{12}}) \arrow[d, "\shortparallel"] \\ & & k[u]\oplus k[u] \arrow[r, "\rho"] & k[u,u^{-1}] \end{tikzcd}$$

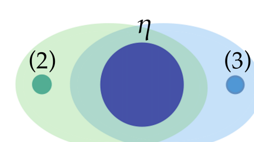
where now $\rho(a_1,a_2) = a_1 - a_2$, and it follows that either open inclusion $\iota\colon \mathbb{A}^1_k \to X$ induces an isomorphism $\Gamma(X,\mathcal{O}_X) \simeq \Gamma(\mathbb{A}^1_k,\mathcal{O}_{\mathbb{A}^1_k}) = k[u]$. However, the open inclusion $\iota\colon \mathbb{A}^1_k = \operatorname{Spec} k[u] \to X$ is not an isomorphism (it is not surjective, since the image misses one of the two origins).

Exercise 6.4 Imitate the construction of the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ on $\mathbb{P}^1_k$ to form a family of sheaves on X. ✮

6.5 *Semi-local rings*

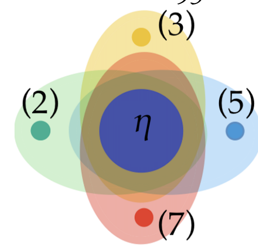
Example 6.3 (*Semi-local rings.*) The rings $\mathbb{Z}_{(2)}$ and $\mathbb{Z}_{(3)}$ are both discrete valuation rings whose maximal ideals are (2) and (3) respectively. Their fraction fields are both equal to $\mathbb{Q}$. Let $X_1 = \operatorname{Spec}\mathbb{Z}_{(2)}$ and $X_2 = \operatorname{Spec}\mathbb{Z}_{(3)}$. Both have a generic point that is open, so there is a canonical open immersion $\operatorname{Spec}\mathbb{Q} \to X_i$ for $i = 1,2$. Hence we can glue the two along their generic points and thus obtain a scheme X with one open point η and two closed points. Let us compute the global sections of $\mathcal{O}_X$ using the now classical sequence for the open covering $\{X_1, X_2\}$:

$$\begin{tikzcd} \Gamma(X,\mathcal{O}_X) \arrow[r] & \Gamma(X_1,\mathcal{O}_X)\times\Gamma(X_2,\mathcal{O}_X) \arrow[r] \arrow[d, "\shortparallel"'] & \Gamma(X_1\cap X_2,\mathcal{O}_X) \arrow[d, "\shortparallel"] \\ & \mathbb{Z}_{(2)}\times\mathbb{Z}_{(3)} \arrow[r, "\rho"'] & \mathbb{Q}. \end{tikzcd}$$



The map ρ sends a pair (an^{-1}, bm^{-1}) to the difference $an^{-1} - bm^{-1}$, hence the kernel consists of the diagonal, so to speak, in $\mathbb{Z}_{(2)}\times\mathbb{Z}_{(3)}$, which is isomorphic to the intersection $\mathbb{Z}_{(2)}\cap\mathbb{Z}_{(3)}$. This is a semi-local ring with the two maximal ideals (2) and (3). Hence there is a map $X \to \operatorname{Spec}\mathbb{Z}_{(2)}\cap\mathbb{Z}_{(3)}$ and it is left as an exercise to show that this is an isomorphism. ☆

Example 6.4 (*More semi-local rings.*) More generally, if $P = \{p_1,\ldots,p_r\}$ is a finite set of distinct prime numbers, one may let $X_p = \operatorname{Spec}\mathbb{Z}_{(p)}$ for $p \in P$. There is, as in the previous case, a canonical open embedding $\operatorname{Spec}\mathbb{Q} \to X_p$. Let the image be $\{\eta_p\}$. Obviously conditions for gluing the η_p's together are all satisfied (the transition maps are all equal



to $\mathrm{id}_{\operatorname{Spec}\mathbb{Q}}$ and $X_{pq} = \{\eta_p\}$ for all p). We do the gluing and obtain a scheme X. Again, to compute the global sections of the structure sheaf, we use the standard sequence

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X) \arrow[r] & \prod_{p\in P} \Gamma(X_p, \mathcal{O}_X) \arrow[r] \arrow[d, "\simeq"'] & \prod_{p,q\in P} \Gamma(X_p \cap X_q, \mathcal{O}_X) \arrow[d, "\simeq"] \\ & \prod_{p\in P} \mathbb{Z}_{(p)} \arrow[r, "\rho"'] & \prod_{p,q\in P} \mathbb{Q}. \end{tikzcd}$$

The map ρ sends a sequence $(a_p)_{p\in P}$ to the sequence $(a_p - a_q)_{p,q\in P}$, and it follows that the kernel of ρ equals the intersection $A_P = \bigcap_{p\in P} \mathbb{Z}_{(p)}$. This is a semi-local ring whose maximal ideals are the $(p)A_p$'s for $p \in P$. There is a canonical morphism $X \to \operatorname{Spec} A_P$, and again we leave it to the industrious student to verify that this is an isomorphism. ☆

Exercise 6.5 Verify the claims in Examples 6.3 and 6.4 above that X is isomorphic respectively to $\operatorname{Spec}\mathbb{Z}_2 \cap \mathbb{Z}_3$ and to $\operatorname{Spec} A_P$. Hint: Use the unicity statement in Proposition 5.3 on page 79. ★

Exercise 6.6 Glue $\operatorname{Spec}\mathbb{Z}_{(2)}$ to itself along the generic point to obtain a scheme X. Show that X is not affine. Hint: Show that $\Gamma(X, \mathcal{O}_X) = \mathbb{Z}_{(2)}$. ★

6.6 *The blow-up of the affine plane*

In this section, we will construct the *blow-up of* $\mathbb{A}^2$ *at the origin*, by gluing together two affine schemes. We begin by recalling the classical construction for varieties. To be precise, we write $\mathbb{A}^2(k)$ for the variety, and $\mathbb{A}^2_k$ for the scheme, etc.

The blow-up as a variety

Let k be an algebraically closed field, and consider the affine plane $\mathbb{A}^2(k)$. There is a rational map $f : \mathbb{A}^2(k) \dashrightarrow \mathbb{P}^1(k)$ that sends a point (x, y) to the point $(x : y)$ (in homogeneous coordinates on $\mathbb{P}^1(k)$). This map is not defined at the origin $(0,0)$, but we can still associate with it the closure X in $\mathbb{A}^2(k) \times \mathbb{P}^1(k)$ of its graph, which lies in $\big(\mathbb{A}^2(k) - (0,0)\big) \times \mathbb{P}^1(k)$.

To describe the graph in more detail it is better to rename the homogenous coordinates on $\mathbb{P}^1(k)$ to $(s : t)$. If the coordinate t is nonzero, it holds that $(s : t) = (st^{-1} : 1)$, so the part of the graph where $y \neq 0$, is given by the equation $xy^{-1} = st^{-1}$; or in other words, by $xt - ys = 0$. And similarly, the same equation gives the part where $x \neq 0$ since there $yx^{-1} = ts^{-1}$. Hence X is defined in $\mathbb{A}(k)^2 \times \mathbb{P}^1(k)$ by the single equation

$$X = Z(xt - ys) \subset \mathbb{A}^2(k) \times \mathbb{P}^1(k).$$

We also have two projection maps $p\colon X \to \mathbb{A}^2(k)$ and $q\colon X \to \mathbb{P}^1(k)$. Let us analyze the fibres of these two maps. The fibres of p are easy to describe. If $(x, y) \in \mathbb{A}^2(k)$ is not the origin, then $p^{-1}(x, y)$ consists of a single point: the equation $xt = ys$ allows us to determine the point $(s : t)$ uniquely since either $x \neq 0$ or $y \neq 0$. However, when $(x, y) = (0,0)$, any choices of s and t satisfy the equation, so $p^{-1}(0,0) = (0,0) \times \mathbb{P}^1(k)$. In

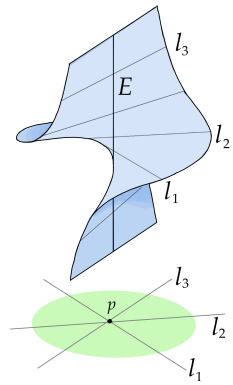


Figure 6.1: *The blow-up of the plane at a point*

particular, this inverse image is one-dimensional; it is called the *exceptional divisor* of X, and is frequently denoted by E.

Similarly, if $(s : t) \in \mathbb{P}^1(k)$ is a point, the fibre

$$q^{-1}(s : t) = \{(x, y) \times (s : t) \mid xt = ys\} \subset \mathbb{A}(k)^2 \times (s : t)$$

is the line in $\mathbb{A}^2(k)$ with $sx - ty = 0$ as equation, s and t being the coefficients. The map q is an example of a *line bundle*; all of its fibres are affine lines; that is, $\mathbb{A}^1(k)$'s. We will see these again later on in the book.

The standard covering of $\mathbb{P}^1(k)$ as a union of two $\mathbb{A}^1(k)$'s gives an affine cover of X: If $U \subset \mathbb{P}^1(k)$ is the open set where $s \neq 0$, we can normalize by setting $s = 1$, and the equation $xt = sy$ becomes $y = tx$. Hence x and t may serve as affine coordinates on $q^{-1}(U)$, and $q^{-1}(U) \simeq \mathbb{A}^2(k)$. In these coordinates, the morphism $p : X \to \mathbb{A}^2_k$ restricts to the map $\mathbb{A}^2(k) \to \mathbb{A}^2(k)$ given by $(x, t) \mapsto (x, xt)$. Similarly, if V denotes the open set where $t \neq 0$, it holds that $q^{-1}(V) = \mathbb{A}^2(k)$ with affine coordinates y and s, and the map p is given here as $(y, s) \mapsto (sy, y)$.

The blow-up as a scheme

Inspired by the above discussion we proceed to define the scheme-analogue of the blow-up of $\mathbb{A}^2_k$ at a point. It will be defined as a scheme over $\mathbb{Z}$ rather than over a field k (we get a blow-up of $\mathbb{A}^2_A$ for any ring A by tensorizing everything below by A). Also, in addition to the scheme X, we also want a morphisms of schemes $p \colon X \to \mathbb{A}^2$ and $q \colon X \to \mathbb{P}^1$ having similar properties to the morphisms in the example above.

Consider the affine plane $\mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x, y]$. The prime ideal $\mathfrak{p} = (x, y) \subset \mathbb{Z}[x, y]$ corresponds to the origin of $\mathbb{A}^2(k)$ in the analogy with the situation above. Consider the diagram

$$\begin{tikzcd}
 & \mathbb{Z}[x,y] \arrow[dl] \arrow[dr] & \\
\mathbb{Z}[x,t] \arrow[dr] & & \mathbb{Z}[y,s] \arrow[dl] \\
 & R = \mathbb{Z}[x,y,s,t]/(xt-y, st-1) &
\end{tikzcd}$$

Here the two diagonal maps in the upper part are given by $x \mapsto x$, $y \mapsto xt$ and $y \mapsto y$, $x \mapsto ys$ respectively, and the two others are induced by obvious inclusions.

Note that the ring R is isomorphic to $\mathbb{Z}[x,s,t]/(st-1) = \mathbb{Z}[x,t,t^{-1}]$ as well as to $\mathbb{Z}[y,s,t]/(st-1) = \mathbb{Z}[y,s,s^{-1}]$. Since this ring is a localization of both $\mathbb{Z}[x,t]$ and $\mathbb{Z}[y,s]$, we can identify its spectrum both as an open subset of $\operatorname{Spec}\mathbb{Z}[x,t]$ and as an open subset of $\operatorname{Spec}\mathbb{Z}[y,s]$. From this we get a diagram

$$\begin{tikzcd}
 & \operatorname{Spec}\mathbb{Z}[x,y] & \\
U = \operatorname{Spec}\mathbb{Z}[x,t] \arrow[ur] & & \operatorname{Spec}\mathbb{Z}[y,s] = V \arrow[ul] \\
 & \operatorname{Spec} R \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

where the bottom diagonal maps are the two open immersions. Hence we can glue these two affine spaces together along $\operatorname{Spec} R$ to obtain a new scheme X. By construction, the restriction of the maps $\operatorname{Spec}\mathbb{Z}[x,t] \to \operatorname{Spec}\mathbb{Z}[x,y]$ and $\operatorname{Spec}\mathbb{Z}[y,s] \to \operatorname{Spec}\mathbb{Z}[x,y]$ to $\operatorname{Spec} R$ coincide with the map $\operatorname{Spec} R \to \operatorname{Spec}\mathbb{Z}[x,y]$ which is induced by $\mathbb{Z}[x,y] \to R$. Therefore they may be glued together to a morphism (the 'blow-up morphism')

$$p\colon X \to \mathbb{A}^2 = \operatorname{Spec}\mathbb{Z}[x,y].$$

To complete the discussion, we should define the corresponding morphism $q\colon X \to \mathbb{P}^1$. Again we work locally. On the affine open $U = \operatorname{Spec}\mathbb{Z}[x,t]$ we have a map $U \to \mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[t]$ induced by the inclusion $\mathbb{Z}[t] \subset \mathbb{Z}[x,t]$. Similarly, on $V = \operatorname{Spec}\mathbb{Z}[y,s]$ we have a map $V \to \mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[s]$. Checking if they can be glued together, amounts to seeing what happens on the overlap $U \cap V = \operatorname{Spec} R$. However, on $\operatorname{Spec} R$ it holds that $t = s^{-1}$, so using the standard description of $\mathbb{P}^1$ as being glued together of two affine lines, we see that the maps $\mathbb{Z}[t] \to R$ and $\mathbb{Z}[s] \to R$ induce the desired morphism $q\colon X \to \mathbb{P}^1$.

✲ **Exercise 6.7** Compute the space $\Gamma(X, \mathcal{O}_X)$ of global sections and describe the canonical map $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$. ★

Exercise 6.8 Imitate the construction above to define the blow-up of $\mathbb{A}^n_{\mathbb{Z}}$ along a codimension 2 linear space $V(x,y)$. ★

6.7 A small resolution of a quadric

We consider the following two copies of $\mathbb{A}^3_k$:

$$X_1 = \operatorname{Spec} k[x, y, t], X_2 = \operatorname{Spec} k[z, w, s]$$

The maps $s \mapsto t^{-1}$, $z \mapsto xt$, $w \mapsto yt$ defines an isomorphism

$$\tau : k[z, w, s]_s \simeq k[x, y, t]_t.$$

This allows us to glue X_1 and X_2 along $D(t)$ and $D(s)$. The corresponding scheme will be denoted by X.

The scheme X contains a closed subscheme C which is isomorphic to $\mathbb{P}^1_k$. Indeed, $L_1 = \operatorname{Spec} k[x, yt]/(x, y)$ and $L_2 = \operatorname{Spec} k[z, w, s]/(z, w)$ define closed subschemes of X_1 and X_2 respectively. As abstract schemes, these are isomorphic to $\mathbb{A}^1_k$, and it is easy to see that the gluing above induce the standard gluing of X_1 and X_2 into a $\mathbb{P}^1_k$.

We claim that there is a morphism

$$\phi : X \to \mathbb{A}^4.$$

To define it, we work on X_1 and X_2. Define $\phi_1 : X_1 \to \mathbb{A}^4$ by the ring map

$$\sigma_1 : k[u_1, \ldots, u_4] \to k[x, y, t]$$

sending $(u_1, \ldots, u_4) \mapsto (x, y, xt, yt)$. Likewise, define $\phi_2 : X_2 \to \mathbb{A}^4$ by the ring map

$$\sigma_2 : k[u_1, \ldots, u_4] \to k[z, w, s]$$

sending $(u_1, \ldots, u_4) \mapsto (zs, ws, z, w)$. These morphisms are compatible via the isomorphism τ; this follows by the commutative diagram

$$\begin{tikzcd} k[u_1, \ldots, u_4] \arrow[r, "\sigma_1"] \arrow[dr, "\sigma_2"'] & k[x, y, t] \arrow[r] & k[x, y, t]_t \\ & k[z, w, s] \arrow[r] & k[z, w, s]_s \arrow[u, "\tau"'] \end{tikzcd}$$

Note that in both cases, the image is contained in the quadric hypersurface $Q = V(u_1 u_4 - u_2 u_3)$, so that ϕ really is a map $\phi : X \to Q$.

EXERCISE 6.9 In the above example:

a) Compute $\Gamma(X, \mathcal{O}_X)$ and describe the morphism $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$.

b) Show that ϕ maps C to the origin $V(u_1, \ldots, u_4) \subset Q$.

c) Show that ϕ restricts to an isomorphism on the open set $X - C$.

★

6.8 Projective spaces

We now give examples of some more involved gluings where more than two parts are joined and we shall construct the all important projective spaces over a ring A which are omnipresent in algebraic geometry. So let A be a ring, and consider the subrings of $A[x_0, x_0^{-1}, \ldots x_n, x_n^{-1}]$ given by

$$R_i = A[x_0 x_i^{-1}, \ldots, x_n x_i^{-1}]$$

for $i = 0, \ldots, n$. Each R_i is isomorphic to a polynomial ring in n variables over A, and $U_i = \operatorname{Spec} R_i$ is an affine space $\mathbb{A}^n_A$. Note that we have equalities

$$R_i[x_i x_j^{-1}] = R_j[x_j x_i^{-1}] \tag{6.2}$$

for each i and j; indeed, this follows from the identities $x_l x_i^{-1} = x_l x_j^{-1} \cdot x_j x_i^{-1}$ valid for all i, j and l. Each $U_{ij} = \operatorname{Spec} R_i[x_i x_j^{-1}]$ is the standard open $D(x_j x_i^{-1})$ in U_i, and using the equalities from (6.2) as transition functions, we can glue together the affine spaces $U_i = \operatorname{Spec} R_i \simeq \mathbb{A}^n_A$ along the U_{ij}'s (the transition-function are identities, and the gluing condition are trivially fulfilled, and note that the triple intersections are $U_{ijl} = \operatorname{Spec} R_i[x_i x_j^{-1}, x_i x_l^{-1}]$). In this way we construct a scheme which we shall denote by $\mathbb{P}^n_A$, and this is the *projective n-space* over A.

The projective n-space

Note that each $\operatorname{Spec} R_i$ comes with a canonical map $\operatorname{Spec} R_i \to \operatorname{Spec} A$, induced by the inclusion $A \subset R_i$. Moreover, the isomorphisms used as glue above are all 'over A'; that is, they are A-algebra homomorphisms, and thus they are compatible with the inclusions $A \subset R_i$. Hence they may be glued together to form a morphism $\mathbb{P}^n_A \to \operatorname{Spec} A$.

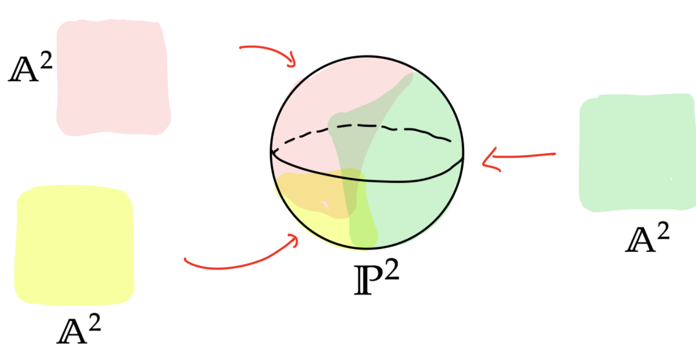
Note in particular, that for $n = 1$ we obtain the projective line $\mathbb{P}^1_A$ constructed earlier. An argument similar to that in Proposition 6.1 gives

Proposition 6.5 $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$.

Example 6.6 (*The projective plane.*) The *projective plane* $\mathbb{P}^2_k$ is formed by gluing together the three copies of the affine plane $\mathbb{A}^2_k$. The notation quickly becomes cluttered when working with the fractions as in the previous paragraph, and one way to avoid this is to single out one of the variables, say x_0, and 'set it equal to one', or expressed more seriously, renaming $x_1 x_0^{-1}$ to x and $x_2 x_0^{-1}$ to y. The other fractions follow suit; for instance, $x_1 x_2^{-1} = xy^{-1}$. With this convention the three affine pieces become:

$$U_0 = \operatorname{Spec} k[x, y], \quad U_1 = \operatorname{Spec} k[x^{-1}, yx^{-1}], \quad U_2 = \operatorname{Spec} k[y^{-1}, xy^{-1}].$$

☆



Gluing three affine planes to $\mathbb{P}^2$

We proceed by giving some examples.

EXAMPLE 6.7 (*A triangle of reference.*) In the standard affine open $U_0 = \operatorname{Spec} k[x,y]$ the subset $V(x) = \operatorname{Spec} k[y]$ is an affine line $\mathbb{A}^1_k$, and the same is true for the subset $V(xy^{-1}) = \operatorname{Spec} k[y^{-1}]$ in $U_2 = \operatorname{Spec} k[y^{-1}, xy^{-1}]$. The two affine lines match in the overlap $U_0 \cap U_2 = \operatorname{Spec} k[x,y,y^{-1}]$ because the equality $(x) = (xy^{-1})$ of principal ideals holds in $k[x,y,y^{-1}]$. Note that the lines meet $U_0 \cap U_2$ in the closed subset equal to $\operatorname{Spec} k[y,y^{-1}]$. Hence when the U_i's are glued together to form $\mathbb{P}^2_k$, the two affine lines are glued together to a projective line; indeed, the gluing setup is given by the inclusions $k[y] \subset k[y,y^{-1}] \supset k[y^{-1}]$, which is precisely the recipe for $\mathbb{P}^1_k$.

Returning to the coordinates x_0, x_1 and x_2, the projective line just constructed is denoted $V(x_1)$. In a completely symmetric way we find two other lines in $\mathbb{P}^2$, one is $V(x_0)$ and the other $V(x_2)$. They constitute what one calls a 'triangle of reference'. ☆

EXAMPLE 6.8 Consider the three ideals

$$\begin{aligned} I_0 &= (y^2 - x^3) \subset k[x,y]; \\ I_1 &= (x^{-1}(yx^{-1})^2 - 1) \subset k[x^{-1}, yx^{-1}]; \\ I_2 &= (y^{-1} - (xy^{-1})^3) \subset k[y^{-1}, xy^{-1}]. \end{aligned}$$

Each ideal I_i defines a closed subscheme of the corresponding affine piece $U_i = \mathbb{A}^2_k$, and it is readily checked that they agree on the overlaps $U_i \cap U_j$. For instance, in $U_0 \cap U_1 = \operatorname{Spec} k[x, x^{-1}, y]$, we have

$$\big((x^{-1})(yx^{-1})^2 - 1\big) = \big(x^{-3}(y^2 - x^3)\big) = (y^2 - x^3),$$

since x is invertible in $k[x,x^{-1},y]$. Thus the three subschemes glue together to a closed subscheme $Z \subset \mathbb{P}^2_k$.

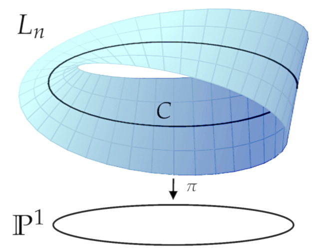
In Chapter 10, we will see that there is a much more economic way of specifying subschemes of $\mathbb{P}^n$ using graded ideals. In fact, the above subscheme is defined by a single homogeneous polynomial, $F = x_0x_2^2 - x_1^3$. ☆

EXERCISE 6.10 Prove Proposition 6.5. (A more general result will be proved in Chapter 18). ★

6.9 *Line bundles on $\mathbb{P}^1$*

The sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ which we constructed in Example 6.3 have a geometric alter ego, the so-called line bundles L_n. (Here we work with $A = k$, a field). The two concepts are closely related and the connection between them will be further explored in Chapter 16. Here we contend ourselves to give the construction and a closer description of a few of the L_n's.

The sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ were obtained by gluing $\mathcal{O}_{U_0}$ and $\mathcal{O}_{U_1}$ via the multiplication by t^n map on $\mathcal{O}_{U_0 \cap U_1}$, where $U_0 = \operatorname{Spec} k[u]$, $U_1 = \operatorname{Spec} k[u^{-1}]$ form the standard affine cover of $\mathbb{P}^1_k$, and their intersection equals $U_0 \cap U_1 = \operatorname{Spec} k[u, u^{-1}]$. The new schemes L_n will be constructed essentially by the same gluing process, but schemes and not sheaves will be



joined together; two copies of the affine plane, $\operatorname{Spec} k[u,s]$ and $\operatorname{Spec} k[u^{-1},t]$ will be glued together with glue being 'multiplication of one coordinate by a power of the other'. The schemes L_n come equipped with a canonical morphism $\pi\colon L_n \to \mathbb{P}^1_k$.

To be precise, consider the following commutative diagram of ring maps

$$\begin{tikzcd}
k[u,s] \arrow[r, hook] & k[u,u^{-1},s] \arrow[rr, "\rho", "\simeq"'] & & k[u,u^{-1},t] & k[u^{-1},t] \arrow[l, hook'] \\
k[u] \arrow[u, hook] \arrow[rr, hook] & & k[u,u^{-1}] \arrow[ul, hook] \arrow[ur, hook] & & k[u^{-1}] \arrow[u, hook] \arrow[ll, hook']
\end{tikzcd}$$

where the map $\rho\colon k[u,u^{-1},s] \to k[u,u^{-1},t]$ is defined by sending s to $u^n t$ and u to u, and the others are the obvious inclusions*. Applying Spec to this diagram results in the gluing diagram

*The attentive student will observe a change of sign compared to the sheaf case; this is not merely an annoying convention, but is dictated by a deeper but p.t. mysterious principle.

$$\begin{tikzcd}
\mathbb{A}^2 = \operatorname{Spec} k[u,s] \arrow[d] & D(u) \simeq D(u^{-1}) \arrow[l, hook'] \arrow[r, hook] \arrow[d] & \mathbb{A}^2 = \operatorname{Spec} k[u^{-1},t] \arrow[d] \\
U_0 & U_0 \cap U_1 \arrow[l, hook'] \arrow[r, hook] & U_1.
\end{tikzcd}$$

The gluing conditions are fulfilled, and hence we obtain scheme L_n admitting a morphism $\pi\colon L_n \to \mathbb{P}^1$. Note that if $x \in \mathbb{P}^1$ is a closed point, say $x \in U_0$, then the fibre $\pi^{-1}(x)$ is isomorphic to the affine line $\mathbb{A}^1_{k(x)}$. This explains the term 'line bundle': intuitively L_n is a family of affine lines parameterized by the base space $\mathbb{P}^1_k$.

The zero section

There is a copy of $\mathbb{P}^1_k$ embedded in L_n, called the *zero section* of L_n; that is, there is a closed immersion $\iota\colon \mathbb{P}^1_k \to L_n$ with image a closed subscheme $C \subset L_n$, having the property that $C \cap \pi^{-1}(x)$ is the origin in each fibre $\mathbb{A}^1_{k(x)}$ of π. Intuitively, it is defined by the equation $s = 0$ or $t = 0$ in each fibre. The proper definition is as follows: Inside each $\mathbb{A}^2_k$ we have the closed subschemes $\mathbb{A}^1_k \simeq V(s) \subset \operatorname{Spec} k[u,s]$ and $\mathbb{A}^1_k \simeq V(t) \subset \operatorname{Spec} k[u^{-1},t]$, and the salient point is that when the two $\mathbb{A}^2_k$'s are glued together, the subschemes $V(s)$ and $V(t)$ match up and are glued together to form a $\mathbb{P}^1_k$. Indeed, the gluing data on the algebraic level are expressed in the diagram

$$\begin{tikzcd}
k[u,s] \arrow[r, hook] \arrow[d, two heads] & k[u,u^{-1},s] \arrow[r, "\rho", "\simeq"'] \arrow[d, two heads] & k[u,u^{-1},t] \arrow[d, two heads] & k[u^{-1},t] \arrow[l, hook'] \arrow[d, two heads] \\
k[u,s]/(s) \arrow[r, hook] \arrow[d, "\wr"'] & k[u,u^{-1},s]/(s) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & k[u,u^{-1},t]/(t) \arrow[dl, "\simeq"] & k[u^{-1},t]/(t) \arrow[l, hook'] \arrow[d, "\wr"] \\
k[u] \arrow[r, hook] & k[u,u^{-1}]) & & k[u,u^{-1}] \arrow[ll, hook']
\end{tikzcd}$$

where we note that the bottom row precisely expresses the recipe for $\mathbb{P}^1_k$, so Spec of the vertical maps may be glued together to a map $\iota\colon \mathbb{P}^1_k \to L_n$.

On each affine open, ι is given by factoring out some ideal, hence is a closed immersion. Moreover, each morphism $k[u] \to k[u,s] \to k[u,s]/(s) \simeq k[u]$ and $k[u^{-1}] \to k[u^{-1},t] \to k[u^{-1},t]/(t) \simeq k[u^{-1}]$ is the identity. It follows that $\pi \circ \iota = \mathrm{id}_{\mathbb{P}^1_k}$, and ι is a 'section' of π.

A few particular cases

The schemes L_n give a rich source of examples in algebraic geometry, and we will come back to them frequently in the book. For now let us compute and study a few explicit examples.

L_0

The scheme L_0 is glued together of two copies of $\mathbb{A}^2_k$ with the help of the inclusions $k[u,t] \to k[u,u^{-1},u] \leftarrow k[u^{-1},t]$. In addition to π, the bundle L_0 admits a morphism $L_0 \to \mathbb{A}^1_k$ obtained by gluing together the two maps $\operatorname{Spec} k[t,u] \to \operatorname{Spec} k[t]$ and $\operatorname{Spec} k[u^{-1},t] \to \operatorname{Spec} k[t]$. Anticipating the notion of a fibre product (which we will study in detail in Chapter 8), this makes, in fact, L_0 isomorphic to the *fiber product* $\mathbb{P}^1 \times_k \mathbb{A}^1_k$. It is the scheme associated to the product variety $\mathbb{P}^1(k) \times \mathbb{A}^1(k)$.

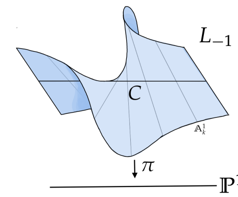
L_1

We claim that the scheme L_1 is isomorphic to the complement of a closed point P in $\mathbb{P}^2$, *i.e.* $Y = \mathbb{P}^2 \backslash \{P\}$. Indeed, choose coordinates x_0, x_1 and x_2 in the projective plane and consider the two distinguished open sets $D_+(x_0) = \operatorname{Spec} k[x_1x_0^{-1}, x_2x_0^{-1}]$ and $D_+(x_1) = \operatorname{Spec} k[x_0x_1^{-1}, x_2x_1^{-1}]$. Their union in $\mathbb{P}^2_k$ equals the complement of the closed point $P = (0:0:1)$. Renaming the variables $u = x_0x_1^{-1}$, $s = x_2x_1^{-1}$ and $t = x_2x_0^{-1}$ we find that $D_+(x_0) = \operatorname{Spec} k[u,s]$ and $D_+(x_1) = k[u^{-1},t]$ and the identity $x_2x_1^{-1} = x_0x_1^{-1} \cdot x_2x_0^{-1}$ turns into the equality $s = ut$, which is precisely the gluing data for L_1.

Geometrically the morphism $\mathbb{P}^2_k \backslash \{P\} \to \mathbb{P}^1_k$ is given by 'projection from the point P'. We will make this more precise in Chapter **??**. The fibres of π are the lines in $\mathbb{P}^2_k$ through P (with the point P removed) and the zero section equals the line 'at infinity'; *i.e.* the line $V(x_2)$.

L_{-1}

We have in fact seen the scheme L_{-1} before: it is isomorphic to the blow-up of $\mathbb{A}^2_k$ at the origin. To see this, one needs only check that the gluing maps are exactly the same. In



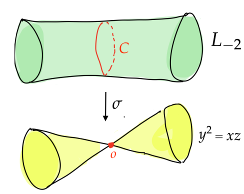
particular, the map $\pi : L_1 \to \mathbb{P}^1_k$, is exactly the map constructed in Section 6.6, and the zero-section C corresponds to the exceptional divisor E. See Exercise 6.11 below.

L_{-2}

The scheme L_{-2} is quite interesting. It is the so-called *resolution of a quadratic cone*. There is a surjective morphism $\sigma : L_{-2} \to \operatorname{Spec} R$, where $R = k[x,y,z]/(y^2 - xz)$, and $\operatorname{Spec} R$ is singular whereas L_{-2} is not. To construct σ, we need to construct the maps with source R in the diagram

$$\begin{tikzcd}
 & & & R \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
k[u,s] \arrow[rr, hook] & & k[u,u^{-1},s] \arrow[rr, "\rho"] & & k[u,u^{-1},t] & & k[u^{-1},t] \arrow[ll, hook'] \\
k[u] \arrow[u] \arrow[rrr, hook] & & & k[u,u^{-1}] \arrow[ul] \arrow[ur] & & & k[u^{-1}] \arrow[u] \arrow[lll, hook']
\end{tikzcd}$$

where the rest of the maps form the gluing data for the bundle L_{-2}; in particular, the gluing map $\rho\colon k[u,u^{-1},s] \to k[u,u^{-1},t]$ will be the isomorphism so that $s \mapsto u^{-2}t$. We define the homomorphism $R \to k[u,s]$ by the assignments $x \mapsto s$, $y \mapsto us$ and $z \mapsto u^2 s$, and $R \to k[u^{-1},t]$ by the assignments $x \mapsto u^{-2}t$, $y \mapsto u^{-1}t$ and $z \mapsto t$. This makes all three upper triangles commute. Applying Spec to the diagram we get a morphism $\sigma\colon L_{-2} \to \operatorname{Spec} R$.



Let us study the fibers of this morphism, and we begin by figuring out what happens over the open set $U = \operatorname{Spec} k[u,s]$, where σ is given by the map

$$\sigma_U\colon \operatorname{Spec} k[u,s] \to \operatorname{Spec} R$$

induced by the upper left map in the diagram. Consider the prime ideal $\mathfrak{p} = (x,y,z) \subset R$ with zero-set the origin. We have $\sigma^{-1}(V(\mathfrak{p})) = V(s, su, u^2 s) = V(s)$, which means that the whole 'u-axis' $V(s)$ in $\mathbb{A}^2 = \operatorname{Spec} k[u,s]$ is collapsed onto the origin P in $\operatorname{Spec} R$. Likewise, the 'u^{-1}-axis' in $\mathbb{A}^2 = \operatorname{Spec} k[u^{-1},t]$ is collapsed to the origin. It follows that the whole zero-section C in L_{-2} is mapped to the origin. In fact, the zero-section C is the only subscheme of L_{-2} which is contracted by this map; σ is an isomorphism outside C.

This is consistent with the fact that any morphism from $\mathbb{P}^1_k$ to an affine scheme is constant.

Proposition 6.9 *σ restricts to an isomorphism $L_{-2} - C \to \operatorname{Spec} R \backslash \{P\}$.*

Proof: The complement $\operatorname{Spec} R\backslash\{P\}$ of the origin is covered by the two distinguished open sets $D(x), D(z)$ (note that $D(y) = D(y^2) = D(xz)$ by the quadratic relation defining R). Likewise, the complement $L_{-2}\backslash C$ of the zero-section is covered by the distinguished open subsets $D(s) \subset \operatorname{Spec} k[u,s]$ and $D(t) \subset \operatorname{Spec} k[u^{-1},t]$. It holds that $\sigma_U^{-1}(V(x)) = V(s) \subset \operatorname{Spec} k[u,s]$, and this means that the restriction $\sigma|_U = \sigma_U$ maps $D(s)$ onto $D(x)$. In fact, using the identification $D(x) = \operatorname{Spec} R_x$, and the identity $R_x = (k[x,y,z]/(y^2 - xz))_x \simeq$

$k[x,y]_x$ we see that $\sigma|_U$ is the map

$$\operatorname{Spec} k[u,s]_s \to \operatorname{Spec} k[x,y]_x$$

induced by the ring map which is defined by $x \mapsto s$ and $y \mapsto us$; this is an isomorphism since we have inverted s. Hence $\sigma|_U$ is an isomorphism over $D(x)$. A symmetric argument shows that $\sigma|_V$ is an isomorphism over $D(z)$; and all together, σ is an isomorphism outside C. ❑

Exercises

✲ (6.11) Check that L_{-1} is indeed the blow-up constructed in Section 6.6.

(6.12) Show that for $n \geqslant 0$, the scheme L_{-n} admits a morphism $\sigma : L_{-n} \to Y$ contracting the zero-section C to a point.

(6.13) For the morphism $\pi : L_n \to \mathbb{P}^1_k$, show that

$$\pi_* \mathcal{O}_{L_n} = \mathcal{O}_{\mathbb{P}^1_k}.$$

★

6.10 *Double covers of projective space*

Algebraic geometry is not only about polynomials, but algebraic functions as well. A simple example of this is the square root of a polynomial; for instance, of $f(x_1,\ldots,x_n) \in R = A[x_1,\ldots,x_n]$. Due to the sign ambiguity, this does not give a function on $\mathbb{A}^n_A$, and this leads to the construction of a new scheme which is a double cover of $\mathbb{A}^n_A = \operatorname{Spec} R$: the closed subscheme $X = V(t^2 - f) \subseteq \operatorname{Spec} R[t]$ maps in a two-to-one-fashion onto $\operatorname{Spec} R$ (with a liberal interpretation of two-to-one when A is of characteristic 2). By definition, the coordinate t is a square root of f defined on the double cover X.

We intend to generalize this to double covers of projective spaces by gluing together the double covers we just construced. We begin with the case of $\mathbb{P}^1$; over a field, this gives the so-called 'hyperelliptic curves'.

Hyperelliptic curves

Let k be a field and let g be a non-negative integer. We will consider the two affine schemes $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B$, where

$$A = \frac{k[x,y]}{(y^2 - a_{2g+1}x^{2g+1} - \cdots - a_1 x)} \text{ and } B = \frac{k[u,v]}{(v^2 - a_{2g+1}u - \cdots - a_1 u^{2g+1})}$$

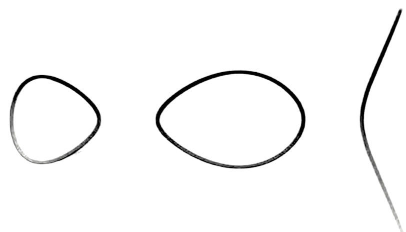
with scalars $a_1,\ldots,a_{2g+1} \in k$. The two distinguished open sets $D(x) = \operatorname{Spec} A_x$ and $D(u) = \operatorname{Spec} B_u$ are isomorphic: the assignments $\phi(u) = x^{-1}$ and $\phi(v) = x^{-g-1}y$ give an isomorphism $\phi\colon B_u \to A_x$. It is well-defined as the little calculation

$$\begin{aligned}\phi(v^2 - a_{2g+1}u - \cdots - a_1 u^{2g+1}) &= y^2 x^{-2g-2} - a_{2g+1}x^{-1} - \cdots - a_1 x^{-2g+1} \\ &= x^{-2g-2}(y^2 - a_{2g+1}x^{2g+1} \cdots - a_1 x)\end{aligned}$$

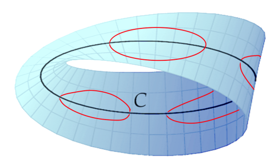
shows that the defining ideal for B_u maps into the one defining A_x, and one verifies effortlessly that the inverse homomorphism is given as $x \mapsto u^{-1}$ and $y \mapsto vu^{-g-1}$. We can thus glue X_1 and X_2 together along the open subsets $D(x)$ and $D(u)$.

Hyperelliptic curves

The resulting scheme X is what is called a *hyperelliptic curve* or a *double cover of* $\mathbb{P}^1_k$. In the case $g = 1$, X is an example of an *elliptic curve*. Here is an illustration of the real points of one of the affine charts for $g = 2$:



Notice that the gluing is very similar to the schemes L_n introduced in Section 6.9. In fact, X is naturally a closed subscheme of L_{-g-1}, since L_{-g-1} is glued together by $U = \operatorname{Spec} k[x, y]$ and $V = \operatorname{Spec}[u, v]$ using the same gluing maps, and X is locally given by the quotient maps $k[x, y] \to A$ and $k[u, v] \to B$ on the two affine open sets.

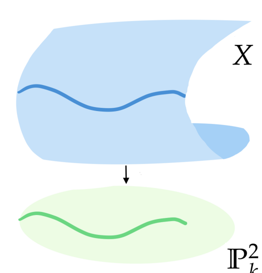


In particular, X admits a morphism $f\colon X \to \mathbb{P}^1_k$. In more detail, we can see the map as the map glued together by the two maps $\operatorname{Spec} A \to \operatorname{Spec} k[x]$ and $\operatorname{Spec} B \to \operatorname{Spec} k[u]$, induced by the two natural inclusions $k[x] \subset A$ and $k[u] \subset B$. The 'gluing diagram'

$$\begin{tikzcd} k[u] \arrow[r, "u\mapsto x^{-1}"] \arrow[d, hook] & k[x] \arrow[d, hook] \\ B_u \arrow[r, "\phi"] & A_x \end{tikzcd}$$

commutes, so by the Gluing Lemma for morphisms (Proposition 5.4 on page 81) the inclusions match on the overlap and patch together to the desired map $X \to \mathbb{P}^1_k$. Note that the correspondence $u \mapsto x^{-1}$ gives the standard construction of $\mathbb{P}^1_k$ by gluing together the two affine lines $\operatorname{Spec} k[x]$ and $\operatorname{Spec} k[u]$.

It is the morphism $f\colon X \to \mathbb{P}^1_k$ that lies behind the name 'double cover': the general fibre $f^{-1}(q)$ consists of two points, when k is algebraically closed and of characteristic different from two.



Higher-dimensional double covers

The above construction generalizes in a straightforward manner to higher-dimensional projective spaces. We will even consider projective spaces over any ring A.

Let A be a ring and let $R = A[x_0, \ldots, x_n]$ with the usual grading. Let $f \in R$ be a homogeneous polynomial of degree $2d$, and for each $0 \leqslant i \leqslant n$ let

$$S_i = A[x_0 x_i^{-1}, \ldots, x_n x_i^{-1}, y x_i^{-d}] / \big((y x_i^{-d})^2 - f(x_0 x_i^{-1}, \ldots, x_n x_i^{-1})\big)$$

For each pair i, j letting $S_{ij} = S_i[x_i x_j^{-1}]$, one checks that $S_{ij} = S_{ji}$; indeed, this reduces to the identity

$$x_i^{2d} x_j^{-2d} \big((y x_i^{-d})^2 - f(x_0 x_i^{-1}, \ldots, x_n x_i^{-1})\big) = (y x_j^{-d})^2 - f(x_0 x_j^{-1}, \ldots, x_n x_j^{-1}).$$

It is then straightforward to verify that the $\operatorname{Spec} S_i$'s glue together along the open subschemes $\operatorname{Spec} S_{ij}$'s to a scheme X. Moreover, keeping the notation R_i from the previous section, the morphisms $\operatorname{Spec} S_i \to \operatorname{Spec} R_i$, induced by the inclusions $R_i \to S_i$, glue together to a morphism $\pi \colon X \to \mathbb{P}^n_A$.

EXAMPLE 6.10 *(A Del Pezzo surface.)* Let us consider the case $f(x_0, x_1, x_2) = x_1^4 + x_0^3 x_1 + x_2^2 (x_2 - x_0)^2$. Note that

$$S_0 \simeq k[u, v, y] / \big(y^2 - u^3 - u + v^2(v^2 - 1)\big)$$

via the identifications $u = x_1 x_0^{-1}$ and $v = x_2 x_0^{-1}$. So the scheme X is a surface glued together of three open sets, each isomorphic to a quartic surface in $\mathbb{A}^3_k$. The 'double cover' morphism is given by $\pi \colon \operatorname{Spec} S_0 \to \operatorname{Spec} k[u, v]$.

The closed subset $V(u)$ is interesting: Note that

$$(y^2 - u^4 - u + v^2(v-1)^2, u) = (y + v(v-1), u) \cap (y - v(v-1), u)$$

So the preimage $\pi^{-1}(V(u))$ consists of two components, each mapping isomorphically to $V(u)$. ☆

EXERCISE 6.14 Assume that k is algebraically closed. Let $a_{2g+1} = 1$ and $a_1 = -1$ and $a_i = 0$ for the other indices. Determine the image of $D(x)$ and $D(u)$ in $\mathbb{P}^1_k$. Find all points in $\mathbb{P}^1_k$ where the fibre of the double cover f does not consist of exactly two points. How many are there? ★

EXERCISE 6.15 *(Adapted from New Zealand Mathematical Olympiad 2019 Problem 5.)* Consider the hyperelliptic curve $X = X_1 \cup X_2$, where

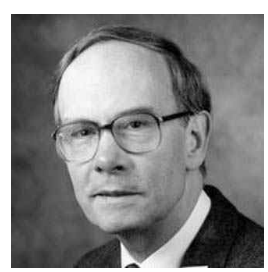
$$X_1 = \operatorname{Spec} \mathbb{Z}[x, y] / (x^4 - x^3 + 3x^2 + 5 - y^2)$$

and

$$X_2 = \operatorname{Spec} \mathbb{Z}[u, v] / (1 - u + 3u^2 + 5u^4 - v^2)$$

and $u = x^{-1}$, $v = x^2 y^{-1}$ on the overlap. Determine the set of $\mathbb{Z}$-points, $X(\mathbb{Z})$. ★

6.11 *Hirzebruch surfaces*



Friedrich Hirzebruch (1927–2012)

Let $r \geqslant 0$ be an integer and consider the scheme X which is glued together by the four affine scheme charts

$$\begin{aligned} U_{00} &= \operatorname{Spec} k[x, y] & U_{01} &= \operatorname{Spec} k[x, y^{-1}] \\ U_{10} &= \operatorname{Spec} k[x^{-1}, x^r y] & U_{11} &= \operatorname{Spec} k[x^{-1}, x^{-r} y^{-1}] \end{aligned} \tag{6.3}$$

When $k = \mathbb{C}$, these are the so-called *Hirzebruch surfaces*. In many ways, these surfaces behave as the 'Möbius strips' in algebraic geometry. We will study these surfaces in several contexts in the book. For now, we will explain that X admits a morphism to $\mathbb{P}^1_k$.

The inclusions

$$\begin{array}{cc} k[x] \subset k[x,y] & k[x] \subset k[x,y^{-1}] \\ k[x^{-1}] \subset k[x^{-1}, x^r y] & k[x^{-1}] \subset k[x^{-1}, x^{-r}y^{-1}] \end{array} \tag{6.4}$$

induce morphisms $U_{ij} \to \mathbb{A}^1_k$. Moreover, these agree over the various intersections $U_{ij} \cap U_{jl}$, and so we obtain a morphism $X \to \mathbb{P}^1_k$.

EXERCISE 6.16 Show that when $r = 1$, the surface above is isomorphic to the blow-up of $\mathbb{P}^2$ at a point. (Hint: Show that the latter can be described using four affine charts). ★

Chapter 7

Geometric properties of schemes

In this chapter we survey some of the main geometric properties of schemes. We have already seen a few such properties in Chapter 4: these were properties that could be formulated just in terms of the underlying topological spaces. For instance, a scheme $(X, \mathcal{O}_X)$ was said to be *connected* if the topological space X is connected. Recall that this means that X can not be written as a disjoint union of two proper, open subsets. In particular, for $X = \operatorname{Spec} A$, being connected was reflected in the algebraic condition that A was not the direct product of two non-zero rings B and C. Likewise, we said that a scheme X is *irreducible* if its underlying topological space is irreducible; that is, it is not the union of two different proper, closed subsets. For $X = \operatorname{Spec} A$ this amounts to saying that A has a single minimal prime.

In this chapter we will survey more subtle geometric properties of schemes. These properties reflect both the underlying topological space, as well as the structure sheaf $\mathcal{O}_X$.

7.1 Noetherian schemes

By the correspondence between irreducible subsets of $\operatorname{Spec} A$ and prime ideals of A, we immediately see that if A is a Noetherian ring, the prime spectrum $\operatorname{Spec} A$ is a Noetherian topological space, *i.e.* all descending chains of irreducible closed subsets stabilize. Indeed, such a chain is of the form $V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq \cdots$, where we may assume that the ideals $\mathfrak{a}_n$ are prime. Then the condition that $V(\mathfrak{a}_n)$ is decreasing, corresponds to the sequence $(\mathfrak{a}_n)$ being increasing, and so it has to be stationary because A is Noetherian.

The converse however, is not true. A simple example is the following:

Example 7.1 Consider the polynomial ring $A = k[t_1, t_2, \ldots]$ in countably many variables t_i and mod out by the square $\mathfrak{m}^2$ of the maximal ideal generated by the variables, $\mathfrak{m} = (t_1, t_2, \ldots)$. The resulting ring A has just one prime ideal, the one generated by the t_i's. So $\operatorname{Spec} A$ has just one point, and hence is a Noetherian space. The ring A, however, is clearly not Noetherian; the sole prime ideal requires infinitely many generators, namely all the t_i's. ☆

In light of this example, we take a different route to define Noetherian schemes:

Definition 7.2 *i) A scheme is* locally Noetherian *if it can be covered by open affine subsets* $\operatorname{Spec} A_i$ *where each* A_i *is a Noetherian ring.*

ii) A scheme is Noetherian *if it is both locally Noetherian and quasi-compact.*

Recall from Chapter 4 that a scheme X is *quasi-compact* if every open cover of X has a finite subcover. We also showed that affine schemes were quasi-compact: Any open covering can be refined to a covering by distinguished open sets $D(f_i)$, and when $\operatorname{Spec} A = \bigcup_i D(f_i)$, the ideal generated by the f_i's contains 1, and the finitely many $D(f_i)$'s with f_i occurring in an expansion of 1, will do.

From the definition, it follows that a general scheme is Noetherian if and only if it can be covered by finitely many open affines $\operatorname{Spec} A_i$ where each A_i is Noetherian.

In fact, with the new definition, we now have

Proposition 7.3 $\operatorname{Spec} A$ *is Noetherian (as a scheme) if and only if* A *is Noetherian.*

Proof: We can see this as a purely algebraic fact: Refining the cover, we may assume that each $A_i = A_{f_i}$, and we need to show that A is Noetherian provided that each localization A_{f_i} is Noetherian and $1 \in (f_1, \ldots, f_r)$.

Let $I \subset A$ be an ideal. We need to show that I is finitely generated. By assumption, each ideal I_{f_i} is finitely generated, say by generators $g_{ij}) = a_{ij} f_i^N \in A_{f_i}$. That means that $f_i^N \cdot g_{ij} = a_{ij}/1$ also generate I_{f_i} (since f_i is invertible in A_{f_i}). Consider the map $\phi : \bigoplus_{i,j} A \to I$ sending the (i,j)-th basis vector e_{ij} to a_{ij}. By assumption, $\phi_{\mathfrak{p}}$ is surjective for all prime ideals $\mathfrak{p}$ (since the $D(f_i)$ cover $\operatorname{Spec} A$). Hence ϕ is also surjective, and I is finitely generated. ❑

Proposition 7.4 *If* X *is a Noetherian scheme, then its underlying topological space is Noetherian.*

Proof: Since X is quasi-compact it may be covered by a finite number of open affine subsets, and since a descending chain stabilizes if the intersection with each of those open sets stabilizes, we reduce to showing the proposition for $X = \operatorname{Spec} A$ with A a Noetherian ring, which is clear. ❑

Proposition 7.5 *Let* X *be a (locally) Noetherian scheme. Then any closed or open subscheme of* X *is also (locally) Noetherian.*

Proof: Without loss of generality, we may assume that X is Noetherian. Let $\{X_i\}_{i \in I}$ be a *finite* open cover with $X_i = \operatorname{Spec} A_i$ and assume that each A_i is Noetherian. Let $Y \subset X$ be an open or closed subscheme. We will show that each $Y \cap X_i$ is Noetherian. In particular, since $Y \cap X_i$ is a closed subscheme of an affine scheme, we reduce to considering the case where $X = \operatorname{Spec} A$ where A is Noetherian.

First case, Y is open. Then there are elements $f_1, \ldots, f_n \in A$ such that $Y = \bigcup_{i=1}^n D(f_i) = \bigcup_{i=1}^n \operatorname{Spec}(A_{f_i})$. If A is Noetherian, then so is each of the localizations A_{f_i}, so Y is also Noetherian.

Second case, Y is closed. Then $Y = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$. If A is Noetherian, then so is $\operatorname{Spec}(A/\mathfrak{a})$, and again Y is Noetherian. ❑

EXAMPLE 7.6 All of the examples from Chapter 6 are Noetherian. This follows because they are glued together by finitely many schemes of the form $\operatorname{Spec} R$ where R is a Noetherian ring. ☆

EXAMPLE 7.7 Let k be a field. The following schemes are not Noetherian:

i) $\coprod_{i=1}^{\infty} \mathbb{A}_k^1$;

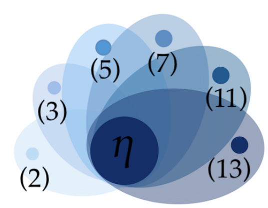
ii) $\operatorname{Spec} \bigoplus_{i=1}^{\infty} k[x]$;

iii) $\operatorname{Spec} \prod_{i=1}^{\infty} k[x]$,

where the union is the disjoint union. Indeed, the disjoint union $\coprod_{i=1}^{\infty} \mathbb{A}_k^1$ is not quasi-compact (thus not affine). The latter two are affine (thus quasi-compact), but non-isomorphic, because their rings of global sections $\mathcal{O}_X(X)$ are non-isomorphic.

The ideal structure of infinite products of rings can in fact be very complicated. For instance $\operatorname{Spec} \prod_{i \in \mathbb{N}} k$ is described by all *ultrafilters* on $\mathbb{N}$ (if you know what such exotic creatures are); anyhow, it is a compact, totally disconnected and Hausdorff(!) topological space. It is called the *Stone–Čech compactification of* $\mathbb{N}$. ☆

EXAMPLE 7.8 In Example 6.4 on page 92 we worked with a finite set of primes, but the hypothesises of the gluing theorem impose no restrictions on the number of schemes to be glued together, and we are free to take $\mathcal{P}$ infinite, for example we can use the set $\mathcal{P}$ of all primes! The glued scheme $X_{\mathcal{P}}$ is a peculiar animal: it is neither affine nor Noetherian, but it is locally Noetherian. There is a map $\phi \colon X_{\mathcal{P}} \to \operatorname{Spec} \mathbb{Z}$ which is bijective and continuous, but not a homeomorphism, and it has the property that for all open subsets $U \subseteq \operatorname{Spec} \mathbb{Z}$ the map induced on sections $\phi^\sharp \colon \Gamma(U, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}}) \to \Gamma(\phi^{-1}U, \mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism, in other words, $\phi^\sharp \colon \mathcal{O}_{\operatorname{Spec} \mathbb{Z}} \to \phi_*(\mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism!



As before we construct the scheme $X_{\mathcal{P}}$ by gluing the different $\operatorname{Spec} \mathbb{Z}_{(p)}$'s together along the generic points. However, when computing the global sections, we see things changing. The kernel of ρ is still $\bigcap_{p \in \mathcal{P}} \mathbb{Z}_{(p)}$, but now this intersection equals $\mathbb{Z}$: indeed, a rational number $\alpha = a/b$ lies in $\mathbb{Z}_{(p)}$ precisely when the denominator b does not have p as factor, so lying in all $\mathbb{Z}_{(p)}$, means that b has no non-trivial prime-factor. That is, $b = \pm 1$, and hence $\alpha \in \mathbb{Z}$.

There is a morphism $X_{\mathcal{P}} \to \operatorname{Spec} \mathbb{Z}$ which one may think about as follows. Each of the schemes $\operatorname{Spec} \mathbb{Z}_{(p)}$ maps in a natural way into $\operatorname{Spec} \mathbb{Z}$, the mapping being induced by the inclusions $\mathbb{Z} \subseteq \mathbb{Z}_{(p)}$. The generic points of the $\operatorname{Spec} \mathbb{Z}_p$'s are all being mapped to the generic point of $\operatorname{Spec} \mathbb{Z}$. Hence they patch together to give a map $X_{\mathcal{P}} \to \operatorname{Spec} \mathbb{Z}$. This is a continuous bijection by construction, but it is not a homeomorphism: indeed, the subsets $\operatorname{Spec} \mathbb{Z}_{(p)}$ are open in $X_{\mathcal{P}}$ by the gluing construction, but they are not open in $\operatorname{Spec} \mathbb{Z}$, since their complements are infinite, and the closed sets in $\operatorname{Spec} \mathbb{Z}$ are just the finite sets of maximal ideals.

The topology of the scheme $X_{\mathcal{P}}$ is not Noetherian since the subschemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ form an open cover that obviously can not be reduced to a finite cover. However, it is locally Noetherian, as the open subschemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ are Noetherian. The sets $U_p = X_{\mathcal{P}}\backslash\{(p)\}$ map bijectively to $D(p) \subseteq \operatorname{Spec}\mathbb{Z}$ and $\Gamma(U_p, \mathcal{O}_{X_{\mathcal{P}}}) = \mathbb{Z}_p$, but U_p and $D(p)$ are not isomorphic. ☆

7.2 *The dimension of a scheme*

Recall that the *Krull dimension* of a ring A is defined as the supremum of the length of all chains of prime ideals in A. For a scheme, we make the following similar definition, which works for any topological space.

Definition 7.9 (Dimension) *Let X be a topological space. The dimension of X is the supremum of all integers n such that there exists a chain*

$$Y_0 \subset Y_1 \subset \cdots \subset Y_n$$

of distinct irreducible closed subsets of X.

Note that this supremum might not be a finite number, in which case we say that $\dim X = \infty$. If X is a scheme, we define the dimension of X as the dimension of the underlying topological space. In particular, it holds true that $\dim X = \dim X_{\text{red}}$.

Here are a few basic properties of the dimension:

Proposition 7.10 *i) If $Y \subset X$ is any subset, then $\dim Y \leqslant \dim X$.*

ii) If X is covered by open sets U_i, then $\dim X = \sup_i \dim U_i$.

iii) If X is covered by closed sets Z_i, then $\dim X = \sup_i \dim Z_i$.

iv) If $\dim X$ is finite, and $Y \subset X$ is a closed irreducible subset such that $\dim Y = \dim X$, then $Y = X$.

In the case where $X = \operatorname{Spec} A$ is affine, the closed irreducible subsets of X are of the form $V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal. Using this observation we find

Proposition 7.11 *The dimension of $X = \operatorname{Spec} A$ equals the Krull dimension of A.*

Example 7.12

i) $\dim \operatorname{Spec}\mathbb{Z}$ equals one. All maximal chains have the form $V(p) \subset V(0) = \operatorname{Spec}\mathbb{Z}$ for prime numbers p.

ii) $\dim \operatorname{Spec}(k[\epsilon]/\epsilon^2) = \dim \operatorname{Spec} k = 0$.

iii) The dimension of $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \ldots, x_n]$ is equal to $n + \dim A$ when A is a Noetherian ring (for general rings $\dim \mathbb{A}^n_A$ takes values between $\dim A + n$ and $\dim A + 2n$, and all values are possible). In particular, when $A = k$ is a field, $\mathbb{A}^n_k$ has dimension n. A maximal chain is $V(x_1) \supset V(x_1, x_2) \supset \cdots \supset V(x_1, \ldots, x_n)$.

For $\mathbb{A}^1_{\mathbb{Z}}$, the dimension is two, and a maximal chain of prime ideals in $\mathbb{Z}[x]$ is given by $(f(x), p) \supset (p) \supset (0)$, where $f(x)$ is an irreducible polynomial, and p is a prime number.

☆

Having finite dimension does not guarantee that a scheme is Noetherian (see Example 7.1). More seriously, there are even Noetherian rings whose Krull dimension is infinite. The first example was constructed by Masayoshi Nagata, the great master of counterexamples in algebra. Although each maximal chain of prime ideals in a Noetherian ring will be of finite length (prime ideals satisfy the descending chain condition) there can be arbitrary long ones (see Problem 7.4).

Example 7.13 (*Zero-dimensional schemes.*) The schemes $\operatorname{Spec}\mathbb{Z}/p\mathbb{Z}$, $\operatorname{Spec}\mathbb{C}[x]/x^n$ and $\operatorname{Spec}\mathbb{C}[x,y]/(x^2, xy, y^3)$ have dimension zero. More generally, the spectrum of an Artinian ring has dimension 0 (and in the case where A is Noetherian, $\operatorname{Spec} A$ has dimension 0 if and only if A is Artinian). However, there are non-Noetherian rings, *e.g.* $\prod_{i=1}^{\infty} \mathbb{F}_2$, which have dimension 0 and even infinitely many points. ☆

Exercise 7.1 Show that $A = \prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$ has dimension 0, but it is not Noetherian. ★

Exercise 7.2 The ring $A = \prod_{n=1}^{\infty} \mathbb{Z}/2^n\mathbb{Z}$ has infinite Krull dimension[1]. Show that $\operatorname{Spec} A$ is still Noetherian as a topological space. ★

Codimension

For a closed subset $Y \subset X$, we can define $\dim Y$ and $\dim X$ using closed subsets contained in Y and X respectively. There is also a relative notion, the *codimension* of Y inside X, defined in terms of closed subsets of X containing Y. These three numbers will in many important cases be related by the relationship $\dim Y + \operatorname{codim} Y = \dim X$ (which jutifies the name 'codimension'), although this formula does not hold in general, it is not even true for all spectra of Noetherian integral domains*.

**For instance, there are Noetherian domains A of dimension two with a principal maximal ideal $\mathfrak{m}$ (see Section 26.3 or Section ?? in CA); then $\mathfrak{m}$ is of height one by Krull's Principal Ideal Theorem, but* $\dim A - \dim A/\mathfrak{m} = 2$.

Definition 7.14 (Codimension) *Let $Y \subseteq X$ be an irreducible closed subset of X. We define the* codimension *of Y as the supremum of all integers n such that there exists a chain*

$$Y = Y_0 \subset Y_1 \subset \cdots \subset Y_n$$

of distinct irreducible closed subsets of X.

By the correspondence between irreducible closed subsets and prime ideals, the codimension of the closed subset $V(\mathfrak{p})$ in $\operatorname{Spec} A$ equals the height of the prime $\mathfrak{p}$ in A, that is, the maximal length of a chain of distinct prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r = \mathfrak{p}$, or equivalently, the Krull dimension $\dim A_{\mathfrak{p}}$ of the localized ring $A_{\mathfrak{p}}$.

[1]This was shown in R. Gilmer, W. Heinzer, *Products of commutative rings and zero-dimensionality*. Trans. Amer. Math. Soc. 331 (1992), 663–680.

Proposition 7.15 *Let X be a scheme. Let $x \in X$ be a point and set $Z = \overline{\{x\}}$. Then* $\dim \mathcal{O}_{X,x} = \operatorname{codim}(Z, X)$.

Proof: Take a chain $Z \subset Z_1 \subset \ldots Z_n$ of distinct irreducible closed subsets . Then for any open neighborhood U of x the generic points $\eta_1, \ldots, \eta_n$ of the Z_i's are contained in U. Thus if $U = \operatorname{Spec} A$ is an affine open neigborhood of x, then the generic points correspoind to prime ideals $\mathfrak{p}_x \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$ in A. Taking the supremum gives the claim. ❏

Dimension theory for schemes of finite type over a field

One should have in mind that codimension can be counterintuitive even for Noetherian schemes; for instance, there are Noetherian affine schemes of any dimension with closed points being of codimension one; we shall see a two-dimensional one in Proposition 26.13.

For integral schemes of finite type over fields however, the theory is simpler, and we can study the dimension in terms of the function field:

Recall that the transcendence degree *of a field extension K/k is the maximal number of elements of K which are algebraically independent.*

Theorem 7.16 *Let X be an integral scheme of finite type over a field k, with function field K. Then*

i) The dimension $\dim X$ *equals the trancendence degree of K over k (in particular,* $\dim X < \infty$*);*

ii) For each $U \subseteq X$ open, $\dim U = \dim X$*;*

iii) If $Y \subset X$ is a closed subset, then $\operatorname{codim} Y = \inf\{\dim \mathcal{O}_{X,p} | p \in Y\}$ *and*

$$\dim Y + \operatorname{codim} Y = \dim X.$$

In particular, for a closed point $p \in X$, $\dim X = \dim \mathcal{O}_{X,p}$.

Proof: To prove *i)* we may assume that $X = \operatorname{Spec} A$ is affine. The hypothesis on X gives that A is a finitely generated k-algebra with quotient field K. In this case, *i)* is a consequence of the Noether normalization lemma (see Chapter 11 of Atiyah–MacDonald).

Note that statement *ii)* follows from *i)* since X and an open subset U have the same function field.

To prove *iii)*, we may again assume that $X = \operatorname{Spec} A$, and use the formula

$$\dim A/\mathfrak{p} + \operatorname{ht} \mathfrak{p} = \dim A$$

which holds for prime ideals in finitely generated k-algebras. ❏

Example 7.17 The scheme $\mathbb{P}^n_k$ satisfies the conditions of the theorem. It's dimension is n, which follows because $\mathbb{P}^n_k$ contains $\mathbb{A}^n_k$ as an open dense subset, and $\mathbb{A}^n_k$ has dimension n. ☆

Example 7.18 The quadric cone $Q = \operatorname{Spec} k[x, y, z]/(x^2 - yz)$ of Example 4.40 on page 71 has dimension 2. This follows because the function field $K = k(Q)$ is isomorphic to $k(x, y)$, which has transcendence degree 2 over k. Alternatively, we can use the morphism $\mathbb{A}^2_k \to \mathcal{Q}$ which are isomorphisms over an open set $U \subset \mathbb{A}^2_k$ (which thus also has dimension 2). ☆

EXAMPLE 7.19 It's important to note that the formula $\dim Y + \operatorname{codim} Y = \dim X$ does not always hold, even if X is the spectrum of a very nice ring. Indeed, let $X = \operatorname{Spec} A$ where $A = R[t]$ and R is any DVR with generator t of the maximal ideal (for instance, the localization $R = k[t]_{(t)}$). The prime $\mathfrak{p} = (tu - 1)$ has height one, but $A/\mathfrak{p} \simeq R[1/t]$ is a field, hence of dimension zero. However, $\dim A = \dim R + 1 = 2$. ☆

For schemes which aren't integral but still of finite type, we still have a good control over the dimension. First of all, the dimension of X is the same as of X_{red}, so we may assume that X is reduced. Then, if $X = \bigcup X_i$ is the decomposition into irreducible components, we have that X_i is integral, and $\dim X$ is the supremum of all $\dim X_i$.

EXAMPLE 7.20 Consider $X = \mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$ and $Y = V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal

$$\mathfrak{a} = \left(xy - x, x^2, y^2z - z, y^3 - y, xy^2 - xy\right) = (z,y,x) \cap \left(y-1, x^2\right) \cap (y+1, x)$$

The associated primes of $\mathfrak{a}$ are $\mathfrak{p}_1 = (x, y+1)$, $\mathfrak{p}_2 = (x, y-1)$ and $\mathfrak{p}_3 = (x,y,z)$. So Y has three components: $L = V(x, y+1), M = V(x, y-1)$ (two lines), and $p = V(x,y,z)$ (the origin). The dimension of Y equals the largest of the dimension of each component, and $\dim L = 1, \dim M = 1, \dim p = 0$, so $\dim Y = 1$. The codimension of Y in X equals the maximum of the heights of the associated primes of $\mathfrak{a}$, *i.e.* $\operatorname{ht}(\mathfrak{p}_1) = 2$. So the codimension of Y equals 2. ☆

EXERCISE 7.3 *(A polynomial ring of excess dimension.)* The aim of this exercise is to exhibit a one-dimensional local ring R such that the Krull dimension of the polynomial ring $R[T]$ equals three; that is $\dim R[T] = \dim R + 2$.

The ring R is no more exotic than the ring of rational functions $f(x,y)$ in two variables over an algebraically closed field k that are defined and constant on the y-axis. The elements of R, when written in lowest terms, have a denominator not divisible by x, and $f(0,y)$, which is then meaningful, lies in k. The example was originally constructed by Krull as an example of a non-Noetherian domain with just one non-zero prime ideal.

a) Show that the ideals $\mathfrak{a}_r = (x, xy^{-1}, \ldots, xy^{-r})$ with $r \in \mathbb{N}$ form an acsending chain that does not stabilize. Conclude that R is not Noetherian.
b) Show that R is local with the set $\mathfrak{m}$ of elements $f \in R$ that vanish along the y-axis as the maximal ideal.
c) * Prove that there are no other primes than $\mathfrak{m}$ and (0) in R. HINT: Show first that any element in R is of the form $x^i y^j \alpha$ where $i \geqslant 0$, $j \in \mathbb{Z}$ and α is a unit in R.
d) Prove that $\mathfrak{q} = \{\, F(T) \in R[T] \mid F(y) = 0 \,\}$ is a non-zero prime ideal strictly contained in $\mathfrak{m}R[T]$. Conclude that $\dim R[T] \geqslant 3$. HINT: Consider the chain $(0) \subset \mathfrak{q} \subset \mathfrak{m}R[T] \subset \mathfrak{m}R[T] + (T)$.

★

EXERCISE 7.4 *(Nagata's example.)* Let $B = k[x_1, x_2, \ldots]$ be the polynomial ring in countably many variables, and decompose the set of natural numbers into a union $\mathbb{N} = \bigcup_i J_i$ of disjoint finite sets J_i whose cardinality tends to infinity with i. Any such partition will do, but a specific example can be the following. The first set J_1 has 1 as its sole element, J_2 concists of the next two integers, J_3 of the next three etc.

Masayoshi Nagata (1927–2008)

Let $\mathfrak{n}_i$ be the ideal in B generated by the x_j's for which $j \in J_i$, and let S be the multiplicative closed subset $\bigcap_i B \backslash \mathfrak{n}_i$ of B; *i.e.* the set of elements from B not lying in any of the $\mathfrak{n}_i$'s. Nagata's example is the localized ring $A = S^{-1}B$. The aim of the exercise is to prove that A is Noetherian, but of infinite Krull dimension. We let $\mathfrak{m}_i$ denote $\mathfrak{n}_i A$; the ideal in A generated by the x_j's with $j \in J_i$.

We shall need the rational function field $K_i = k(x_j | j \notin J_i)$ in the variables x_j whose index does not lie in J_i, and the polynomial ring $K_i[x_j | j \in J_i]$ over K_i in the remaining variables; that is, those x_j for which $j \in J_i$. Moreover, the ideal $\mathfrak{a}_i$ will be the ideal in $K_i[x_j | j \in I_i]$ generated by the latter; that is, $\mathfrak{a}_i = (x_j | j \in J_i)$.

a) Show that $B_{\mathfrak{n}_i} \simeq K_i[x_j | j \in I_i]_{\mathfrak{a}_i}$.
b) Prove that $A_{\mathfrak{m}_i} = B_{\mathfrak{n}_i}$ and conclude that each local ring $A_{\mathfrak{m}_i}$ is Noetherian with $\dim A_{\mathfrak{m}_i} = \#J_i$ and hence that $\dim A = \infty$.
c) * Show that A is Noetherian. Hint: Any ideal is contained in finitely many of the of the $\mathfrak{m}_i$'s, and therefore finitely generated.

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Chapter 8
Fibre products

8.1 Introduction

From the theory of varieties we know that we can construct the *Cartesian product* $X \times Y$ of two varieties X and Y. If $X = Z(f_1, \ldots, f_s) \subset \mathbb{A}^n(k)$ and $Y = Z(g_1, \ldots, g_t) \subset \mathbb{A}^m(k)$ are two affine varieties, then their product $X \times Y$ is the affine variety $Z(f_1, \ldots, f_s, g_1, \ldots, g_t) \subset \mathbb{A}^{m+n}(k)$, and departing from this, the general case is handled by a gluing process.

In this chapter, we will consider a vast generalization of this construction. For any scheme S and any two S-schemes $X \to S$ and $Y \to S$ we will construct a new scheme, denoted $X \times_S Y$, equipped with projection morphisms $\pi_X \colon X \times_S Y \to X$ and $\pi_Y \colon X \times_S Y \to Y$ satisfying a certain universal property. The aim of this chapter is to prove the fundamental theorem (which certainly is 'Cartesian' in spirit): Fibre products of schemes exist.

The fact that all fibre products exist is one of the most important properties of the category of schemes, and one can argue that this is the definitive reason for transitioning from varieties to schemes: the fibre product of two varieties is in general not a variety, but it is a scheme.

The general fibre product is moreover extremely useful in many situations and takes on astonishingly versatile roles. At the end of the chapter we shall explain some of the various contexts where fibre products appear, including *base change* and *scheme theoretic fibres*. We begin the chapter by recalling the definition of the fibre product of sets, then transition into a very general situation to discuss fibre products in general categories, and then finally, return to the context of schemes. The strategy for proving existence will be similar to what one does for varieties; one constructs the fibre product first when X, Y and S are affine schemes, and subsequently by using several gluing constructions shows it exists in general. The majority of the chapter will be devoted to going through the steps of this gluing procedure. Towards the end, we will treat the main applications and see a series of examples.

Fibre products of sets.

As a warming up, we recall the fibre product in the category of sets, Sets. The points of departure is two sets X and Y both equipped with a map to a third set S; *i.e.* we are given a diagram

$$\begin{tikzcd}
X \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\
& S &
\end{tikzcd}$$

The fibre product $X \times_S Y$ is the subset of the cartesian product $X \times Y$ consisting of the pairs whose two components have the same image in S; that is,

$$X \times_S Y = \{\, (x, y) \mid \phi_X(x) = \phi_Y(y) \,\}.$$

Clearly the diagram below where π_X and π_Y denote the restrictions of the two projections to the fibre product (in other words, $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$), is commutative.

$$\begin{tikzcd}
& X \times_S Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\
& S &
\end{tikzcd} \tag{8.1}$$

And more is true: the fibre product enjoys a universal property. Given any two maps $\psi_X \colon Z \to X$ and $\psi_Y \colon Z \to Y$ such that $\phi_X \circ \psi_X = \phi_Y \circ \psi_Y$ there is a unique map $\psi \colon Z \to X \times_S Y$ satisfying $\pi_X \circ \psi = \psi_X$ and $\pi_Y \circ \psi = \psi_Y$. Defining such a ψ is easy, just use the map whose two components are ψ_X and ψ_Y and observe that it takes values in $X \times_S Y$ since the relation $\phi_X \circ \psi_X = \phi_Y \circ \psi_Y$ holds.

$$\begin{tikzcd}
& X \times_S Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[dr, "\phi_X"'] & Z \arrow[l, "\psi_X"'] \arrow[r, "\psi_Y"] \arrow[u, dashed] & Y \arrow[dl, "\phi_Y"] \\
& S &
\end{tikzcd} \tag{8.2}$$

Giving the two ψ's is to give a commutative diagram like (8.1) above with Z replacing the product $X \times_S Y$ (as in the lower part of (8.2)), and the universal property is to say that 8.1 is universal, or in categorical terms, *final* among such diagrams. One also says that (8.1) is a *Cartesian diagram* or a *Cartesian square*. Speaking of names, the reason for the name 'fibre product' is that the fibres of the map $X \times_S Y \to S$ are the product of the fibres of the two maps $X \to S$ and $Y \to S$.

Cartesian diagram

EXERCISE 8.1 Show that if Y is a subset of S, then the fibre product $X \times_S Y$ equals the preimage $\phi_X^{-1}(Y)$. More strikingly: assume that also X is a subset of S, show that the fibre product, $X \times_S Y$, then will be equal to the intersection $X \cap Y$. If S is reduced to a singleton, show that $X \times_S Y$ is just the good old Cartesian product $X \times Y$. ★

The fibre product in general categories

The notion of a fibre product formulated as the solution to a universal problem as above, is *mutatis mutandis* meaningful in any category C. Given any two arrows $\phi_X \colon X \to S$ and $\phi_Y \colon Y \to S$ in the category C, an object, which we shall denote $X \times_S Y$, is said to be a *fibre product* of the objects X and Y, or more precisely of the two arrows ϕ_X and ϕ_Y, if the following two conditions are fulfilled:

Fibre product

- There are two arrows $\pi_X \colon X \times_S Y \to X$ and $\pi_Y \colon X \times_S Y \to Y$ in C such that $\phi_X \circ \pi_X = \phi_Y \circ \pi_Y$ (called the projections);
- For any two arrows $\psi_X \colon Z \to X$ and $\psi_Y \colon Z \to Y$ in C such that $\phi_X \circ \psi_X = \phi_Y \circ \psi_Y$, there is a *unique* arrow $\psi \colon Z \to X \times_S Y$ satisfying $\pi_X \circ \psi = \psi_X$ and $\pi_Y \circ \psi = \psi_Y$.

These properties may naturally be expressed through commutative diagrams, identical to the ones used for sets in the previous section, and the notions of a *Cartesian diagram* and *Cartesian squares* are carried over to any category.

The two arrows $\pi_X \circ \psi$ and $\pi_Y \circ \psi$ that determine the arrow $\psi \colon Z \to X \times_S Y$, are called *the components* of ϕ, and the notation $\psi = (\psi_X, \psi_Y)$ is sometimes used. If $\psi_X \colon X' \to X$ and $\psi_Y \colon Y' \to Y$ are two arrows over S, there is a unique arrow denoted $\psi_X \times \psi_Y$ from $X' \times_S Y'$ to $X \times_S Y$ whose components are $\psi_X \circ \pi_{X'}$ and $\psi_Y \circ \pi_{Y'}$.

Components

If the fibre product exists, it is unique up to a unique isomorphism, as is true for solutions to any universal problem. However, it is a good exercise to check this in detail in this specific situation.

Exercise 8.2 Show that if the fibre product exists in the category C, it is unique up to a unique isomorphism. ★

It is not so hard to come up with examples of categories where fibre products do *not* exist. For instance, consider the funny category C where the objects are subsets X of the integers with an even number of elements, and the morphisms given by inclusions $Y \subset X$. In this category, the fibre product of $Y \subset X$ and $Z \subset X$ over X would be $Y \cap Z \subset X$. However, $Y \cap Z$ does not necessarily have an even number of elements!

What is of course much more disappointing, is that fibre products fail to exist in our good old categories like manifolds or affine varieties. This shows yet another reason why we need to make the transition from varieties to schemes.

Exercise 8.3

a) Give an example showing that the fibre product does not always exist in the category of manifolds;
b) Give an example showing that the fibre product does not always exist in the category of affine varieties.

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Exercise 8.4 Let C be a category and X, Y and S three object from S. Convince yourself that the simple product of two objects in C$/S$ equals their fibre product in C. Show that one formally may add a final object $*$ to any category C and that the fibre product $X \times_* Y$

exists in the extended category if and only if the simple product $X \times Y$ exists in C, and in case they exist, they are equal. ★

8.2 Fibre products of schemes

It is a fundamental property of schemes that their fibre products exist unconditionally, and most of this chapter is devoted to a proof of this. It consists of a rather long sequence of reductions basically to the affine case — where the spectrum of the tensor product provides a product — and the glueing techniques developed in Chapter 5. We shall prove

Theorem 8.1 (Existence of fibre products) *Let $X \to S$ and $Y \to S$ be two schemes over the scheme S. Then their fibre product $X \times_S Y$ exists.*

The projections onto X and Y will frequently be denoted by respectively π_X and π_Y. We will see several examples later which show that the underlying set of a product can be very different from the product of the underlying sets of X and Y. However, the 'scheme-valued points' behave well; that is, for any S-scheme $T \to S$, there is a canonical isomorphism of sets of T-points

$$\mathrm{Hom}_{\mathsf{Sch}/S}(T, X \times_S Y) \simeq \mathrm{Hom}_{\mathsf{Sch}/S}(T, X) \times \mathrm{Hom}_{\mathsf{Sch}/S}(T, Y).$$

This is just another way of formulating the universal property of a product.

Products of affine schemes

The category AffSch of affine schemes is, more or less by definition, equivalent to the category of rings, and in the category of rings we have the tensor product. The tensor product enjoys a universal property *dual* to the one of the fibre product. To be precise, assume A_1 and A_2 are B-algebras, *i.e.* we have two maps of rings α_i

$$\begin{tikzcd} A_1 & & A_2 \\ & B \arrow[ul, "\alpha_1"] \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

There are two maps $\beta_i \colon A_i \to A_1 \otimes_B A_2$ sending $a_1 \in A_1$ to $a_1 \otimes 1$ and a_2 to $1 \otimes a_2$, respectively. These are both ring homomorphisms, since $aa' \otimes 1 = (a \otimes 1)(a' \otimes 1)$ respectively $1 \otimes aa' = (1 \otimes a)(1 \otimes a')$, and they fit into the commutative diagram

$$\begin{tikzcd} & A_1 \otimes_B A_2 & \\ A_1 \arrow[ur, "\beta_1"] & & A_2 \arrow[ul, "\beta_2"'] \\ & B. \arrow[ul, "\alpha_1"] \arrow[ur, "\alpha_2"'] & \end{tikzcd} \tag{8.3}$$

because $\alpha_1(b) \otimes 1 = 1 \otimes \alpha_2(b)$ by the definition of the tensor product $A_1 \otimes_B A_2$ (this is the significance of the tensor product being taken over B; one can move elements in B from one side of the $\otimes$-glyph to the other).

Moreover, the tensor product is *universal* in this respect. Indeed, assume that $\gamma_i\colon A_i \to C$ are B-algebra homomorphisms, *i.e.* $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$; or phrased differently, they fit into the commutative diagram analogous to (8.3) with the β_i's replaced by the γ_i's. The association $a_1 \otimes a_2 \to \gamma_1(a_1)\gamma(a_2)$ is B-bilinear and hence extends to a B-algebra homomorphism $\gamma\colon A_1 \otimes_B A_2 \to C$, which obviously has the property $\gamma \circ \beta_i = \gamma_i$. Expressed diagrammatically, the universal property appears as

$$\begin{tikzcd} & A_1 \otimes_B A_2 \arrow[d, dashed] & \\ A_1 \arrow[ur, "\beta_1"] \arrow[r, "\gamma_1"] & C & A_2 \arrow[ul, "\beta_2"'] \arrow[l, "\gamma_2"'] \\ & B \arrow[ul, "\alpha_1"] \arrow[ur, "\alpha_2"'] & \end{tikzcd} \tag{8.4}$$

Transforming all this into geometry by applying the Spec-functor, we arrive a the diagram

$$\begin{tikzcd} & \operatorname{Spec}(A_1 \otimes_B A_2) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \operatorname{Spec} A_1 \arrow[dr] & & \operatorname{Spec} A_2 \arrow[dl] \\ & \operatorname{Spec} B, & \end{tikzcd} \tag{8.5}$$

and the affine scheme $\operatorname{Spec}(A_1 \otimes_B A_2)$ enjoys the property of being universal among affine schemes sitting in a diagram like (8.5). Hence $\operatorname{Spec}(A_1 \otimes_B A_2)$ equipped with the two projections π_1 and π_2 serves as the fibre product in the category AffSch of affine schemes. One even has the stronger statement; it is the fibre product in the larger category Sch of schemes.

Proposition 8.2 *Given morphisms $\phi_i\colon \operatorname{Spec} A_i \to \operatorname{Spec} B$ for $i = 1, 2$. Then the spectrum $\operatorname{Spec}(A_1 \otimes_B A_2)$ with the two projection π_1 and π_2 defined as above, is the fibre product of the $\operatorname{Spec} A_i$'s in the category of schemes.*

Unravelled, the conclusion of the proposition reads: if Z is any scheme and $\psi_i\colon Z \to \operatorname{Spec} A_i$ are morphisms with $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$, there exists a unique morphism $\psi\colon Z \to \operatorname{Spec}(A_1 \otimes_B A_2)$ such that $\pi_i \circ \psi = \psi_i$ for $i = 1, 2$.

Proof: We know that the proposition is true whenever Z is an affine scheme; so the subtle point is that Z may not necessarily be affine. For short, we let $X = \operatorname{Spec}(A_1 \otimes_B A_2)$.

The proof is just an application of the gluing lemma for morphisms. One covers Z by open affines U_α and covers the intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by open affine subsets $U_{\alpha\beta\gamma}$ as well. By the affine case of the proposition, for each U_α we get a map $\psi_\alpha\colon U_\alpha \to X$, such that $\pi_i \circ \psi_\alpha = \psi_i|_{U_\alpha}$. By the uniqueness part of the affine case, these maps must coincide on the open affines $U_{\alpha\beta\gamma}$, and therefore on the intersections $U_{\alpha\beta}$. They can thus be patched together to a map $\psi\colon Z \to X$, which is unique since the ψ_α's are unique. ❑

A preliminary lemma

Recall that any open subset U of a scheme Y has a canonically defined scheme structure as an open subscheme (the structure sheaf is defined as $\mathcal{O}_Y|_U$). Hence, if f is any morphism $f\colon X \to Y$, the inverse image $f^{-1}(U)$ is in a natural way an open subscheme of X. The following lemma will turn out to be useful:

Lemma 8.3 *If $X \times_S Y$ exists and $U \subseteq X$ is an open subscheme, then $U \times_S Y$ exists and is canonically isomorphic to the open subscheme $\pi_X^{-1}(U)$ with the two restrictions $\pi_Y|_{\pi_X^{-1}(U)}$ and $\pi_X|_{\pi_X^{-1}(U)}$ as projections.*

Proof: The situation is displayed in the following diagram

$$\begin{tikzcd}
 & & Y \\
Z \arrow[rru, bend left, "\psi_Y"] \arrow[r, dashed] \arrow[rd, "\psi_U"'] & \pi_X^{-1}(U) \arrow[r, hook] \arrow[ru] \arrow[d] & X \times_S Y \arrow[u, "\pi_Y"'] \arrow[d, "\pi_X"] \\
 & U \arrow[r] & X,
\end{tikzcd}$$

and we are to verify that $\pi_X^{-1}(U)$ together with the restriction of the two projections satisfy the universal property. If Z is a scheme and $\psi_U\colon Z \to U$ and $\psi_Y\colon Z \to Y$ are two morphisms over S, we may consider ψ_U as a map into X, and therefore they induce a map of schemes $\psi\colon Z \to X \times_S Y$ with $\psi_X = \pi_X \circ \psi$ and $\psi_Y = \pi_Y \circ \psi$. Clearly $\pi_X \circ \psi = \psi_U$ takes values in U and therefore ψ takes values in $\pi_X^{-1}(U)$. It follows immediately that ψ is unique (see the Exercise below), and we are through. ❑

When identifying $\pi_X^{-1}(U)$ with $U \times_S Y$, the inclusion map $\pi_X^{-1}(U) \subseteq X \times_S Y$ will correspond to the map $\iota \times \mathrm{id}_Y$ where $\iota\colon U \to X$ is the inclusion, so anticipating the notion of *base change*, one may reformulate the lemma by saying that open immersions stay open immersions under *base change* (see Proposition 8.12 on page 126).
Exercise 8.5 Assume that $U \subseteq X$ is an open subscheme and let $\iota\colon U \to X$ be the inclusion map. Let ϕ and ψ be two maps of schemes from a scheme Z to U and assume that $\iota \circ \phi = \iota \circ \psi$. Then $\phi = \psi$. ★

The basic patching

The following proposition will lay at the ground for all the gluing necessary for the construction of a fibre product:

Proposition 8.4 *Let $\phi_X \colon X \to S$ and $\phi_Y \colon Y \to S$ be two maps of schemes, and assume that there is an open covering $\{U_i\}_{i\in I}$ of X such that $U_i \times_S Y$ exist for all $i \in I$. Then $X \times_S Y$ exists. The products $U_i \times_S Y$ form an open covering of $X \times_S Y$ and projections restrict to projections.*

Proof: The tactics will be to patch together the different schemes $U_i \times_S Y$, and see that the result, indeed, is a product $X \times_S Y$. We begin with some useful notation: let $U_{ij} = U_i \cap U_j$ be the intersections of the U_i and U_j, and let $\pi_i \colon U_i \times_S Y \to U_i$ denote the projections. By Lemma 8.3 there are isomorphisms $\theta_{ji} \colon \pi_i^{-1}(U_{ij}) \to U_{ij} \times_S Y$, and the gluing functions we shall use are $\tau_{ji} = \theta_{ij}^{-1} \circ \theta_{ji}$, which identify $\pi_i^{-1}(U_{ij})$ with $\pi_j^{-1}(U_{ij})$. Here is the picture:

$$U_i \times_S Y \supseteq \pi_i^{-1}(U_{ij}) \xrightarrow[\simeq]{\theta_{ji}} U_{ij} \times_S Y \xrightarrow[\simeq]{\theta_{ij}^{-1}} \pi_j^{-1}(U_{ji}) \subseteq U_j \times_S Y.$$

The gluing maps τ_{ij} satisfy the gluing conditions because they are compositions of maps that satisfy them, and we shall see that the scheme resulting from the gluing process will serve as the product $X \times_S Y$.

The two projections are essential parts of the product and must not be forgotten: The projection onto Y stays invariable there since Y is never touched during the construction. The projection onto X is obtained by gluing the projections π_i along the $\pi_i^{-1}(U_{ij})$. Under the identification of $\pi_i^{-1}(U_{ij})$ with the product $U_{ij} \times_S Y$ the projection π_{ij} onto U_{ij} corresponds to the restriction $\pi_i|_{\pi_i^{-1}(U_{ij})}$ by Lemma 8.3, and this means that the equalities $\pi_i|_{\pi_i^{-1}(U_{ij})} = \pi_{ij} \circ \theta_{ji}$ hold true. Saying that the two restrictions $\pi_i|_{\pi_i^{-1}(U_{ij})}$ and $\pi_j|_{\pi_j^{-1}(U_{ij})}$ become equal after gluing, is to say that the equality

$$\pi_i|_{\pi_i^{-1}(U_{ij})} = \pi_j|_{\pi_j^{-1}(U_{ij})} \circ \tau_{ji}$$

holds (remember that in the gluing process points x and $\tau_{ji}(x)$ are identified), but this is the case since

$$\pi_j|_{\pi_j^{-1}(U_{ij})} \circ \tau_{ji} = \pi_{ij} \circ \theta_{ij} \circ \tau_{ji} = \pi_{ij} \circ \theta_{ij} \circ \theta_{ij}^{-1} \circ \theta_{ji} = \pi_{ij} \circ \theta_{ji} = \pi_i|_{\pi_i^{-1}(U_{ij})}.$$

Consequently we may glue the π_i's together to obtain the projection π_X.

It is a matter of easy verifications that the glued scheme with the two projections has the required universal property. ❑

It is worth while commenting that the product $X \times_S Y$ is not defined as a particular scheme, it is just an isomorphism class of schemes — products are only defined up to isomorphism (but surely a unique one). So speaking about *the* product is an abuse of language (justified by there being a *unique* isomorphism respecting projections between any two 'products'). In the proof above, while both $\pi_i^{-1}(U_{ij})$ and $\pi_j^{-1}(U_{ij})$ represent products $U_{ij} \times_S Y$, they are however not equal, merely canonically isomorphic. In the construction we could have used any of them or any non-specified representative of the isomorphism class (as we in fact did, since this makes the situation appear more symmetric in i and j).

An immediate consequence of the Proposition 8.4 is that fibre products exist over an affine base S.

LEMMA 8.5 *Assume that S is affine, then $X \times_S Y$ exists.*

PROOF: First if Y as well is affine, we are done. Indeed, cover X by open affine sets U_i. Then $U_i \times_S Y$ exists by the affine case, and we are in the position to apply Proposition 8.4 above. We then cover Y by affine open sets V_i. As we just verified, the products $X \times_S V_i$ all exist, and applying Proposition 8.4 once more we may conclude that $X \times_S Y$ exists. ❑

The final reduction

Let $\{S_i\}$ be an open affine covering of S and let $U_i = \phi_X^{-1}(S_i)$ and $V_i = \phi_Y^{-1}(S_i)$. By Lemma 8.5 the products $U_i \times_{S_i} V_i$ all exist. Using the following lemma and, for the third time, the gluing statement in Proposition 8.4, we are through with a proof of the existence of fibre products (Theorem 8.1 on page 117).

LEMMA 8.6 *With current notation, we have the equality $U_i \times_{S_i} V_i = U_i \times_S Y$. That is, $U_i \times_S Y$ exists and the projections are π_{U_i} and $\pi_Y|_{V_i}$.*

PROOF: We contend that $U_i \times_{S_i} V_i$ satisfies the universal product property of $U_i \times_S Y$. The key diagram is

$$\begin{tikzcd}
 & Z \arrow[dl, "\psi"'] \arrow[dr, "\psi'"] & & \\
U_i \arrow[dr, "\phi_X|_{U_i}"'] & & Y \arrow[dl, "\phi_Y"] & V_i \arrow[l, hook'] \\
 & S & &
\end{tikzcd}$$

where ψ and ψ' are two given maps. If one follows the left path in the diagram, one ends up in S_i, and hence the same must hold following the right path. But then, V_i being equal to the inverse image $\phi_Y^{-1}(S_i)$, it follows that ψ' factors through V_i, and by the universal property of $U_i \times_{S_i} V_i$ there is a morphism $Z \to U_i \times_{S_i} V_i$ with the requested properties. ❑

Notation

If $S = \operatorname{Spec} A$ one often writes $X \times_A Y$ in short for $X \times_{\operatorname{Spec} A} Y$. If $S = \operatorname{Spec} \mathbb{Z}$, one writes $X \times Y$. In case $Y = \operatorname{Spec} B$ the shorthand notation $X \otimes_A B$ is frequently seen as well; it avoids writing Spec twice.

As in any category, diagrams arising from fibre products are frequently called *Cartesian diagram*; that is, the diagram

Cartesian diagrams

$$\begin{tikzcd} Z \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\phi_X"] \\ Y \arrow[r, "\phi_Y"'] & S \end{tikzcd}$$

is said to be a Cartesian diagram if there is an isomorphism $Z \simeq X \times_S Y$ with π_X and π_Y corresponding to the two projections.

✲ EXERCISE 8.6 *(Basic properties of the fibre product.)* Let X, Y and Z be three schemes over the scheme S. Show the following basic properties of the fibre product; there are canonical isomorphisms over S all compatible with all actual projections and all unique:

a) *(Reflectivity)* $X \times_S S \simeq X$;
b) *(Symmetry)* $X \times_S Y \simeq Y \times_S X$;
c) *(Associativity)* $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$.

Assume moreover that T is a scheme over S and that Y is as well a scheme over T; show that there is a unique S-isomorphism compatible with all projections

d) *(Transitivity)* $X \times_S T \times_T Y \simeq X \times_S Y$,

where $X \times_S T$ is a scheme over T via the projection onto T and Y is a scheme over S via the map $T \to S$. ★

EXERCISE 8.7 *(Functoriality.)* Show by using the universal property that given two S-morphisms $\phi\colon X' \to X$ and $\psi\colon Y' \to Y$ there is a unique morphism $\phi \times \psi\colon X' \times_S Y' \to X \times_S Y$ such that the two squares in the diagram commute.

$$\begin{tikzcd} X' \arrow[d, "\phi"'] & X' \times_S Y' \arrow[l, "\pi_{X'}"'] \arrow[r, "\pi_{Y'}"] \arrow[d, "\phi\times\psi"] & Y' \arrow[d, "\psi"] \\ X & X \times_S Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y. \end{tikzcd}$$

Be precise about what uniqueness means. If $\phi'\colon X'' \to X'$ and $\psi'\colon Y'' \to Y'$ is another pair of morphisms, show that $(\phi \times \psi) \circ (\phi' \times \psi') = (\phi \circ \phi') \times (\psi \circ \psi')$, with *equality* unregarded of which representatives of the products are used (recall that the product is merely defined up to (unique) isomorphism). ★

8.3 Examples

Varieties versus Schemes

In the important case that X and Y are integral schemes of finite type over the algebraically closed field k the product of the two as varieties coincides with their product as schemes over k, with the usual interpretation that the variety associated to the scheme X is the set of closed points $X(k)$ with induced topology.

The product $X \times_k Y$ will be a variety (*i.e.* an integral scheme of finite type over k) and the closed points of the product $X \times_k Y$ will be the direct product of the closed points in X and Y.

It is not obvious that $A \otimes_k B$ is an integral domain when A and B are, and in fact, in general, even if k is a field, it is by no means true. But it holds true whenever A and B are of finite type over k and k is an algebraically closed field. The standard reference for this is Zariski and Samuel's book *Commutative algebra I* which is the Old Covenant for algebraists. It is also implicit in Hartshorne's book ([**?**]), exercise **3.15** b) on page 22.

However that the tensor product $A \otimes_k B$ is of finite type over k when A and B are, is straightforward. If $u_1, \ldots, u_m$ generate A over k and $v_1, \ldots, v_m$ generate B over k the products $u_i \otimes 1$ and $1 \otimes v_j$ generate $A \otimes_k B$.

Non-algebraically closed fields

The situation is more subtle when one works over fields that are not algebraically closed. To illustrate some of the phenomena that can occur, we study a few basic examples. They all show different aspects of the product of spectra of two fields; *i.e.* the spectrum of a tensor product of fields.

Examples

(8.7) A simple but illustrative example is the product $\operatorname{Spec}\mathbb{C} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$. This scheme has *two* distinct closed points, and it is not integral — it is not even connected!

The example also shows that the underlying set of the fibre product is not necessarily equal to the fibre product of the underlying sets, although this was true for varieties over an algebraically closed field. In the present case the three schemes involved all have just one element and their fibre product has just one point. So we issue warnings: The product of integral schemes is in general not necessarily integral! The underlying set of the fibre product is not always the fibre product of the underlying sets.

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is in fact isomorphic to the direct product $\mathbb{C} \times \mathbb{C}$ of two copies of the complex field $\mathbb{C}$; indeed, we compute using that $\mathbb{C} = \mathbb{R}[t]/(t^2+1)$ and find

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}[t]/(t^2+1) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[t]/(t^2+1) = \mathbb{C}[t]/(t-i)(t+i) = \mathbb{C} \times \mathbb{C},$$

where for the last equation we use the Chinese remainder theorem and that the rings $\mathbb{C}[t]/(t \pm i)$ both are isomorphic to $\mathbb{C}$.

(8.8) This previous little example can easily be generalized: assume that L is a simple, separable field extension of K of degree d; that is, $L = K(\alpha)$ where the minimal polynomial $f(t)$ of α over K is separable and of degree d. Let Ω be a field extension of K in which the polynomial $f(t)$ splits completely, *e.g.* a normal extension of L (*e.g.* any algebraically closed field containing K) – then by an argument completely analogous to the one above, one finds the equality

$$L \otimes_K \Omega = \Omega \times \ldots \times \Omega,$$

where the product has d factors. Consequently $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec}\Omega$ has an underlying set with d points, even if the three sets of departure all are prime spectra of fields and thus singletons.

One may push this further and construct examples where the fibre product $\operatorname{Spec} K \times_{\operatorname{Spec} K} \operatorname{Spec} \Omega$ is not even Noetherian and has infinitly many points!

(8.9) Let k be a field and x and y two variables. Consider the tensor product $A = k(x) \otimes_k k(y)$. We can regard this as a localization of $k[x, y]$ where we invert everything in the multiplicative set $S = \{p(x)q(y) | p(x), q(y) \neq 0\}$. Let us show that A has infinitely many maximal ideals. Suppose that $\mathfrak{m} \subset A$ is a maximal ideal; it has the form $S^{-1}\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset k[x, y]$ which is maximal among the primes that do not intersect S. In this case we must have $\mathfrak{p} \cap k[x] = 0$, since otherwise there would be a non-zero $p(x) \in \mathfrak{p} \cap S$. Similarly $\mathfrak{p} \cap k[y] = 0$, which implies that $\mathfrak{p}$ has height at most 1. Hence, either $\mathfrak{p} = (0)$, or $\mathfrak{p} = (f)$ for some irreducible polynomial $f \in k[x, y]$ not a product of a polynomial from $k[x]$ and one from $k[y]$. If follows that A has dimension 1, and A has infinitely many maximal ideals – in fact uncountably many if *e.g.* $k = \mathbb{C}$.

This example shows how strange the fibre product really is – $\operatorname{Spec} A$ is an infinite set, even though it is the fibre product of two schemes with one-point underlying sets. We will see more examples like this in the end of this chapter.

(8.10) In this example we let $L \in \mathbb{C}[x, y]$ be a linear polynomial that is not real, for example $L = x + iy + 1$, and we introduce the real algebra $A = \mathbb{R}[x, y]/(L\bar{L})$. The product $L\bar{L}$ of L and its complex conjugate is a real irreducible quadric; which in the concrete example is $(x + 1)^2 + y^2$. The prime spectrum $\operatorname{Spec} A$ is therefore an integral scheme. However, the fibre product $\operatorname{Spec} A \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ is not irreducible being the union of the two conjugate lines $L = 0$ and $\bar{L} = 0$ in $\operatorname{Spec} \mathbb{C}[x, y]$.

The scheme $\operatorname{Spec} A$ has just one real point, namely the point $(-1, 0)$ (*i.e.* corresponding to the maximal ideal $(x + 1, y)$). The $\mathbb{C}$-points however, are plentiful. They are $\mathbb{C}$-points of $\mathbb{A}^2_{\mathbb{R}}(\mathbb{C})$, which all are the orbits $\{(a, b), (\bar{a}, \bar{b})\}$ of the complex conjugation with (a, b) non-real, and form the subset of those $\{(a, b), (\bar{a}, \bar{b})\}$ such that $L(a, b) = 0$.

(8.11) Another example along same lines as Example 8.8 shows that the fibre product $X \times_S Y$ is not necessarily reduced even if both X and Y are; the point being to use an inseparable polynomial $f(t)$ instead of a separable one in 8.8. Let k be a non-perfect field whose characteristic is p, which means there is an element $a \in k$ not being a p-th power. Let L be the field extension $L = k(b)$ where $b^p = a$. That is, $L = k[t]/(t^p - a)$; this is a field since $t^p - a$ is an irreducible polynomial over k. However, upon being tenzored by itself over k, it takes the shape

$$L \otimes_k L = L[t]/(t^p - a) = L[t]/(t^p - b^p) = L[t]/\left((t - b)^p\right),$$

which is not reduced, the non-zero element $t - b$ being nilpotent. So we issue a third warning: the fibre product of integral schemes is not in general reduced!

☆

Exercises

(8.8) With the assumptions of the example above, check the statement that $L \otimes_K \Omega \simeq \Omega \times \ldots \times \Omega$, the product having d factors.

(8.9) Assume that A is an algebra over the field k having a countable set $\{e_1, e_2, \ldots, e_i, \ldots\}$ of mutually orthogonal idempotents, *i.e.* $e_i e_j = 0$ if $i \neq j$ and $e_i e_i = 1$, and assume that $e_i A \simeq k$. Assume also that every element is a finite linear combination of the e_i's. Show that the ideal I_j generated by the e_i's with $i \neq j$ is a maximal ideal.
(8.10) Let k be a field and let X and Y be schemes over k. Show that the k-points of $X \times_k Y$ are in bijection with the usual Cartesian product of sets $X(k) \times Y(k)$.
(8.11) Let $A = \mathbb{R}[x, y]/(x^2 + xy + 1)$. Show that $\operatorname{Spec} A$ is irreducible and that $\operatorname{Spec} A \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ is isomorphic to the disjoint union of two affine lines $\mathbb{A}^1_{\mathbb{C}}$.
(8.12) Let $a \in \mathbb{Q}$ be a rational number that is not a square and for each $i \in \mathcal{N}$ let $L_i = \mathbb{Q}(\eta_i)$ where $\eta_i^{2i} = a$. Then the L_i's form a tower of fields

$$L_1 \subset \cdots \subset L_i \subset L_{i+1} \subset \cdots \subset \mathbb{C},$$

and each L_{i+1} is a quadratic extension of L_i. Let L be the union of the L_i's. Show that the algebras $L_i \otimes_{\mathbb{Q}} \mathbb{C}$ form a tower

$$L_1 \otimes_{\mathbb{Q}} \mathbb{C} \subset \cdots \subset L_i \otimes_{\mathbb{Q}} \mathbb{C} \subset L_{i+1} \otimes_{\mathbb{Q}} \mathbb{C} \subset \cdots \subset \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$$

and that $L \otimes_{\mathbb{Q}} \mathbb{C} = \bigcup_i L_i \otimes_{\mathbb{Q}} \mathbb{C}$. Show that each $L_{i+1} \otimes_{\mathbb{Q}} \mathbb{C}$ is isomorphic to $L_i \otimes_{\mathbb{Q}} \mathbb{C} \times L_i \otimes_{\mathbb{Q}} \mathbb{C}$ as a $\mathbb{C}$-algebra and conclude that $L \otimes_{\mathbb{Q}} \mathbb{C}$ as an $\mathbb{C}$-algebra is isomorphic to direct sum of countably many copies of $\mathbb{C}$'s. Conclude that $\operatorname{Spec} L \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{C}$ is not even a Noetherian scheme.
(8.13) *The dimension of a product.* For two integral schemes of finite type over k, show that

$$\dim X \times_k Y = \dim X + \dim Y.$$

★

8.4 Base change

The fibre product is in constant use in algebraic geometry, and it is an astonishingly versatile and flexible instrument. We shall comment on some of the most frequently encountered applications, and we begin with the notion of base change.

In its simplest and earliest appearances base change is just extending the field over which one works; *e.g.* in Galois theory, or even in the theory of real polynomials, when studying an equation with coefficients in a field k one often finds it fruitful to study the equation over a bigger field K. Generalizing this to extensions of algebras over which one works, and then to schemes, one arrives naturally at the fibre product.

Let X be a scheme over S and $T \to S$ be a map. Considering $T \to S$ as a change of base schemes one frequently writes $X_T = X \times_S T$ and says that X_T is obtained from X by *base change* or frequently that X_T is the *pullback* of X along $T \to S$. This is a functorial construction since if $\phi\colon X \to Y$ is a morphism over S, there is induced a morphism $\phi_T = \phi \times \mathrm{id}_T$ from X_T to Y_T over T, and one easily checks that $\phi_T \circ \pi_T$ coincides with the natural projection map $X_T \to T$ (or in other words, the outer rectangle in the diagram

Base change

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "\phi_T"] & X \arrow[d, "\phi"] \\ Y_T \arrow[r, "\pi_Y"] \arrow[d, "\pi_T"] & Y \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

in the margin is Cartesian). If $\mathcal{P}$ is a property of morphisms, one says that $\mathcal{P}$ is stable under base change if for any T over S, the map ϕ_T has the property $\mathcal{P}$ whenever ϕ has it. Examples 8.10 and 8.11 showed that neither being irreducible nor being reduced are properties stable under base change.

On the other hand, one way of phrasing Lemma 8.3 on page 119 is to say that being an open immersion is stable under base change. The same applies to closed immersions:

Proposition 8.12 (Pullbacks of open and closed immersions) *Assume given a Cartesian diagram of schemes*

$$\begin{tikzcd} Z_Y \arrow[r] \arrow[d, "\phi_Y"'] & Z \arrow[d, "\phi"] \\ Y \arrow[r] & X \end{tikzcd}$$

If the morphism $\phi\colon Z \to X$ is a closed immersion, then the morphism $\phi_Y\colon Z_Y \to Y$ is one as well. If ϕ is an open immersion, then ϕ_Y is also an open immersion.

Proof: Only the statement about closed immersions needs a proof. Assume first that X and Y are both affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. From $X = \operatorname{Spec} A$ it follows that $Z = \operatorname{Spec} A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ (Proposition 4.32 on page 69), and therefore $Z_Y = Z \times_X Y = \operatorname{Spec} A/\mathfrak{a} \otimes_A B = \operatorname{Spec} B/\mathfrak{a}B$. Hence Z_Y is a closed subscheme of Y.

The issue is local on Y (Exercise 4.4 on page 70), so assume that $U \subseteq Y$ is an open affine that maps into an open affine $V \subseteq X$ (one may cover Y by such by first covering X by affine opens and subsequently covering each of their inverse images in Y by affine opens). Then by Lemma 8.6 on page 121 one has $\phi^{-1}(V) \times_X Y = \phi^{-1}(V) \times_V U$, and by the affine case this is a closed subscheme of U. ❑

Exercises

(8.14) *Finite type and base change.* Show that being of finite type (respectively being finite or being locally of finite type) is a property stable under base change. Show that the product of two morphisms of finite type (respectively of finite or locally of finite type) is of finite type (respectively of finite or locally of finite type).

* (8.15) *Flat base change.* In a base change staging, with X a scheme over S and $T \to S$ a morphism, it is often crucial to be able to compare functions on the schemes X and X_T, and for so-called flat base changes the situation is optimal. Assume X is a scheme over $\operatorname{Spec} A$ and let B be an A-algebra. Show that there is a natural B-algebra homomorphism $\Gamma(X, \mathcal{O}_X) \otimes_A B \to \Gamma(X_B, \mathcal{O}_{X_B})$ which is an isomorphism whenever B is a flat A-algebra.

(8.16) Let p and q be two different primes. Show the following identities:

a) $\operatorname{Spec} \mathbb{F}_p \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_q = \emptyset$;
b) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(p)} = \operatorname{Spec} \mathbb{Z}_{(p)}$;
c) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(q)} = \operatorname{Spec} \mathbb{Q}$.

8.5 Scheme theoretic fibres

In most parts of mathematics, when one studies a map of some sort, it is very useful to understand the fibres of the map. This is also true in the theory of schemes.

Suppose that $\phi\colon X \to Y$ is a morphism of schemes and that $y \in Y$ is a point. On the level of topological spaces, we are interested in the preimage $\phi^{-1}(y)$, and we aim at giving a scheme theoretic definition of the fibre $\phi^{-1}(y)$. Having the fibre product at our disposal, nothing is more natural than defining the fibre to be the fibre product $\operatorname{Spec} k(y) \times_Y X$, where $\operatorname{Spec} k(y) \to Y$ is the map corresponding to the point y; recall that the field $k(y)$ is defined as $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ and that the 'point-map' $\operatorname{Spec} k(y) \to Y$ is the composition $\operatorname{Spec} k(y) \to \operatorname{Spec} \mathcal{O}_{Y,y} \to Y$. It is common to write X_y for the scheme theoretic fibre and reserve the notation $\phi^{-1}(y)$ for the topological fibre. Thus the scheme theoretic fibre of ϕ over y fits into the Cartesian diagram

$$\begin{tikzcd} X_y = X \times_Y \operatorname{Spec} k(y) \arrow[r] \arrow[d] & X \arrow[d, "\phi"] \\ \operatorname{Spec} k(y) \arrow[r] & Y. \end{tikzcd}$$

Note that the fibre X_y enjoys the property that for every morphism $\psi\colon Z \to X$ the composition $\phi \circ \psi$ factors through $\operatorname{Spec} k(y) \to Y$ if and only if ψ itself 'takes values' (read factors through) the fibre X_y.

As the next lemma will show, the underlying topological space of X_y is the topological fibre $\phi^{-1}(y)$, but it will be endowed with an additional and canonical scheme structure. In many cases, the fibre will not be reduced, and this is mostly a good thing since it makes certain continuity results true.

Proposition 8.13 *Let X and Y be schemes and $\phi\colon X \to Y$ a morphism. Let $y \in Y$ be a point. Then*

i) The inclusion $X_y \to X$ of the scheme theoretic fibre is a homeomorphism onto the topological fibre $\phi^{-1}(y)$;

ii) If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, it holds that $X_y = \operatorname{Spec}(B/\mathfrak{p}_y B)_{\mathfrak{p}_y}$;

iii) If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ and y is a closed point, one has $X_y = \operatorname{Spec} B/\mathfrak{p}_y B$.

Proof: We begin with the two affine cases *ii)* and *iii)*, but even in the general case we may obviously assume that Y is affine, say $Y = \operatorname{Spec} A$.

The map ϕ between two affine schemes is induced by a map of rings $\alpha\colon A \to B$. Let $\mathfrak{p} = \mathfrak{p}_y \subseteq A$ be the prime ideal corresponding to $y \in Y$. We have the equalities

$$\{\, \mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{p} \subseteq \alpha^{-1}(\mathfrak{q}) \,\} = \{\, \mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{p}B \subseteq \mathfrak{q} \,\} = V(\mathfrak{p}B).$$

In the particular case that $\mathfrak{p}$ is a maximal ideal, the inclusion $\mathfrak{p} \subseteq \alpha^{-1}(\mathfrak{q})$ must be an equality, and the sets above describe the fibre set-theoretically:

$$\phi^{-1}(\mathfrak{p}) = \{\, \mathfrak{q} \subseteq B \mid \mathfrak{q} \supseteq \mathfrak{p}B \,\} = V(\mathfrak{p}B).$$

The closed subset $V(\mathfrak{p}B)$ of $\operatorname{Spec} B$ with induced topology from $\operatorname{Spec} B$ is canonically homeomorphic to $\operatorname{Spec}(B/\mathfrak{p}B)$. Thus we have a homeomorphism between $\operatorname{Spec} B/\mathfrak{p}B$ and the topological fibre $\phi^{-1}(\mathfrak{p})$. On the other hand, by standard equalities between tensor products one has

$$B/\mathfrak{p}B = M \otimes_A B/\mathfrak{p}A = B \otimes_A k(y),$$

and so the scheme theoretical fibre $\phi^{-1}(y)$, which is given by $X_y = X \times_Y \operatorname{Spec} k(y) = \operatorname{Spec} B \otimes_A k(y)$, is in a canonical way homeomorphic to the topological fibre. This proves *iii)*.

If $\mathfrak{p}$ is not a maximal ideal, the set $\operatorname{Spec} B/\mathfrak{p}B$ can certainly be bigger than the fibre, the extra prime ideals being those $\mathfrak{q}$ for which $\alpha^{-1}(\mathfrak{q})$ is strictly bigger than $\mathfrak{p}$. When we localize in the multiplicative system $S = A \backslash \mathfrak{p} \subseteq A$, these superfluous prime ideals become non-proper since they all contain elements of the form $\alpha(s)$ with $s \in S$. It follows that the points in the fibre $\phi^{-1}(y)$ correspond exactly to the primes in the localized ring $(B/\mathfrak{p}B)_\mathfrak{p}$. Standard formulas for tensor products give on the other hand the equality

$M \otimes_A A/I \simeq M/IM$

$$(B/\mathfrak{p}B)_\mathfrak{p} = B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = B \otimes_A k(y).$$

The Zariski topology on the spectrum $\operatorname{Spec}(B/\mathfrak{p}B)_\mathfrak{p}$ (which is the topology it has as a scheme-fibre) coincides with the one induced from the Zariski topology of $\operatorname{Spec}(B/\mathfrak{p}B)$ (which is the topology as topological fibre), and hence *ii)* is proven.

In the general case, one considers the diagram (8.6) below where U is open and affine in X. The two small squares are Cartesian, the lower by the definition of the fibre and the upper one after Lemma 8.3 on page 119, and because the fibre product is transitive (Exercise 8.6 on page 122), it follows that the larger square is Cartesian as well; and so $X_y \cap U$ is the fibre over y of the map $U \to Y$.

$$\begin{tikzcd} X_y \cap U \arrow[r] \arrow[d] & U \arrow[d] \\ X_y \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k(y) \arrow[r] & Y \end{tikzcd} \tag{8.6}$$

By the affine case, the two topologies we examine agree on $X_y \cap U$, and as U can be any open affine, they share a basis and must be equal. ❑

Example 8.14 We take a look at a simple but classic example: the map

$$\phi \colon X = \operatorname{Spec} k[x,y]/(x - y^2) \to \operatorname{Spec} k[x]$$

induced by the injection $B = k[x] \to k[x,y]/(x - y^2) = A$. Geometrically one would say this is just the projection of the 'horizontal' parabola onto the x-axis.

If $a \in k$, computing the fibre over $\mathfrak{m}_a = (x - a)$ yields that the fibre X_a equals the spectrum of the ring

$$k[x,y]/(x - y^2) \otimes_{k[x]} k(a) \simeq k[y]/(y^2 - a).$$

where $k(a)$ denotes the field $k(a) = k[x]/(x-a)$ (which of course is just a copy of k), and where we use the isomorphism $A/\mathfrak{a} \otimes_A M \simeq M/\mathfrak{a}M$ for an ideal $\mathfrak{a}$ in A and an A-module M.

Several cases can occur, apart from the characteristic two case, which is special.

i) In case a has a square root in k, say $b^2 = a$, the polynomial $y^2 - a$ factors as $(y-b)(y+b)$, and the fibre becomes the product

$$\operatorname{Spec} k[y]/(y-b) \times \operatorname{Spec} k[y]/(y+b),$$

which is the disjoint union of two copies of $\operatorname{Spec} k$.

ii) If a does not have a square root in k, the fibre equals $\operatorname{Spec} k(\sqrt{a})$ where $k(\sqrt{a})$ is a quadratic field extension of k. The fibre is a singleton, but with multiplicity two, the multiplicity materializing as the degree of the field extension $k \subset k(\sqrt{a})$.

iii) The final case appears when $a = 0$. The fibre is not reduced, but equals $\operatorname{Spec} k[y]/(y^2)$. The fibre is again a singleton with multiplicity two, the multiplicity this time materializes as the length of $\mathcal{O}_{X_a}$.

We also notice that the generic fibre of ϕ is the quadratic extension $k(x)(\sqrt{x})$ of the function field $k(x)$. In all cases the 'number of points' in the fibre when counted with the proper multiplicity, equals two, and this illustrates one of the permanence properties alluded to above.

When k is not algebraically closed there are also points P in $\operatorname{Spec} k[x]$ whose maximal ideals $\mathfrak{m}_P = (f(x))$ are generated by a non-linear polynomial irreducible over k. The analyses above goes through word by word with the sole change that k is replaced by the extension $K = k(P) = k[x]/(f(x))$ of k. When a has a square root in K, the fibre X_P becomes the disjoint union of two copies of $\operatorname{Spec} K$, and when not, it will equal $\operatorname{Spec} K(\sqrt{a})$.

Over perfect fields k of characteristic two the picture is completely different. Then a is a square, say $a = b^2$ and as $(y^2 - b^2) = (y-b)^2$ none of the fibres are reduced (they equal $\operatorname{Spec} k[y]/(y-b)^2$), except the generic fibre which is $k(x)(\sqrt{x})$. ☆

EXAMPLE 8.15 A similar example can be obtained from the map

$$f : \operatorname{Spec} B \to \operatorname{Spec} A,$$

where $B = \operatorname{Spec} k[x,y,z]/(xy-z)$ and $A = k[z]$ and f is induced from the inclusion $k[z] \to k[x,y,z]/(xy-z)$. As before, we assume k algebraically closed, pick a closed point $a \in \operatorname{Spec} A$, and consider the fibre

$$X_a = \operatorname{Spec}(B \otimes_A k(a)) = \operatorname{Spec} k[x,y]/(xy-a)$$

Again there are two cases. If $a \neq 0$, then $xy - a$ is an irreducible polynomial, and so X_a is an integral scheme. This is intuitive, since it corresponds to the hyperbola $V(xy-a)$ in $\mathbb{A}^2_k$. If $a = 0$, we are left with $X_0 = \operatorname{Spec} k[x,y]/(xy)$, which is not irreducible; it has two components corresponding to $V(x)$ and $V(y)$. X_0 is reduced however.

For good measure, we also consider the fibre over the generic point η of $\operatorname{Spec} A$. This corresponds to

$$k[x,y,z]/(xy-z) \otimes_{k[z]} k(z) = k(z)[x,y]/(xy-z),$$

which is an integral domain. Hence X_η is integral. ☆

Exercises

✲ (8.17) Compute the fibre product $\operatorname{Spec}(\mathbb{Z}/2) \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}(\mathbb{Z}/3)$. Explain your answer geometrically.

✲ (8.18) With reference to Example 2.38 on page 40:

a) Show that the fibre of ϕ over a prime ideal (p) is homeomorphic to

$$\operatorname{Spec}\mathbb{F}_p[x]/(x^2+1)$$

and that $\dim_{\mathbb{F}_p}\mathbb{F}_p[x]/(x^2+1)$ =2. Hint: Use that $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1)$.

b) Show that $\mathbb{F}_p[x]/(x^2+1)$ is a field if and only if x^2+1 does not have a root in $\mathbb{F}_p$.

c) Show that $\mathbb{F}_p[x]/(x^2+1)$ is a field if and only if $(p)\mathbb{Z}[i]$ is a prime ideal.

✲ (8.19) The map $\operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$ from Example 2.38 illustrates a general pattern for scheme-theoretical fibres of so-called 'finite' morphisms:

a) Assume that A is an algebra over a field k of dimension n as a vector space. Show that $\operatorname{Spec} A$ has at most n points. Show that $\operatorname{Spec} A$ has exactly n points if and only if A is the direct product of n copies of k.

b) Let $A \subseteq B$ be two rings and assume that B is free of rank n as an A-module. Show that the fibres of the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ has at most n points.

(8.20) Consider the ring map

$$\phi\colon \mathbb{C}[x,y] \to \mathbb{C}[x,y,z]/(xz-y)$$

which induces $\Phi :: \operatorname{Spec}\mathbb{C}[x,y,z]/(xz-y) \to \mathbb{A}^2_{\mathbb{C}}$. Describe the map Φ On the level of closed points, this maps (a,ab,b) to (a,ab), and the generic point maps to the generic point. In this example, the image is neither open nor closed: it equals $D(x) \cup V(x,y)$.

(8.21) Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let $\phi\colon X \to Y$ be a map such that $\phi^\sharp\colon A \to B$ makes B into a free A-module of rank n. Prove that for each point $y \in Y$ it holds that

$$\sum_{x\in\phi^{-1}(y)} [k(x) : k(y)] \operatorname{length}\mathcal{O}_{X_y,x} = n.$$

(8.22) Let p and q be two different prime numbers and consider the morphism $\phi\colon \mathbb{A}^1_k \to \mathbb{A}^2_k$ induced from the map $k[x,y] \to k[t]$ which is defined by the assignments $x \mapsto t^p$ and $y \mapsto t^q$. Determine all scheme theoretic fibres of ϕ.

(8.23) Let k an algebraically closed field. Consider the k-algebra A given as $A = k[x,y,z]/(xy,xz,yz)$ and let $X = \operatorname{Spec} A$. Consider the map $\phi\colon X \to \mathbb{A}^1$ dual to the k-algebra homomorphism $k[t] \to A$ that sends t to $x+y+z$. Determine all scheme theoretic fibres of ϕ. Hint: Heuristics: $X(\mathbb{C})$ is the union of the axes in $\mathbb{C}^3$ and the map sends points on $X(\mathbb{C})$ to the sum of their coordinates.

(8.24) *A du Val singularity.* Let k be an algebraically closed field. Let $n \geqslant 2$ be an integer and consider the two rings $A = k[x,y,z]/(xy-z^{n+1})$ and $B = k[u,v,z]/(uv-z^{n-1})$. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Show that the assignments $x \mapsto uz$, $y \mapsto vz$ and $z \mapsto z$ induce a morphism $\phi\colon Y \to X$. Prove that ϕ is birational and determine its scheme theoretic fibres over closed points.

(8.25) *Another du Val singularity.* Let k be an algebraically closed field. Let $A = k[x,y,z]/(x^2 + y^3 + z^5)$ and let $X = \operatorname{Spec} A$ (this is the famous du Val singularity E_8). Furthermore, let $B = k[u,v,z]/(u^2 + u^3 z + z^3)$ and $Y = \operatorname{Spec} B$. Show that the assignments $x \mapsto uz$, $y \mapsto vz$ and $z \mapsto z$ induce a morphism $\phi\colon Y \to X$. Prove that ϕ is birational and determine its scheme theoretic fibres over closed points.

(8.26) Describe the scheme theoretic fibres in all points of the following morphisms.

a) $f : \operatorname{Spec} \mathbb{C}[x,y]/(xy-1) \to \operatorname{Spec} \mathbb{C}[x]$;
b) $f : \operatorname{Spec} \mathbb{C}[x,y]/(x^2-y^2) \to \operatorname{Spec} \mathbb{C}[x]$;
c) $f : \operatorname{Spec} \mathbb{C}[x,y]/(xy) \to \operatorname{Spec} \mathbb{C}[x]$;
d) $f : \operatorname{Spec} \mathbb{Z}[x,y]/(xy^2-m) \to \operatorname{Spec} \mathbb{Z}$, where m is a non-zero integer.

(8.27) Determine all the scheme theoretic fibres of the morphism $\operatorname{Spec} \mathbb{Z}[(1+\sqrt{5})/2] \to \operatorname{Spec} \mathbb{Z}[\sqrt{5}]$ induced by the natural inclusion $\mathbb{Z}[\sqrt{5}] \subseteq \mathbb{Z}[(1+\sqrt{5})/2]$.

★

Chapter 9

Separated schemes

The topology on schemes behaves very differently from the usual Euclidean topology. In particular, schemes are not Hausdorff, except in trivial cases – the open sets in the Zariski topology are simply too large. Still we would like to find an analogous property that can serve as a satisfactory substitute for this property, so that we for instance have "uniqueness of limits". The route we take is to impose that the diagonal should be closed (closed in the Zariski topology of the product, of course).

By the immense freedom we have for gluing schemes together there are lots of non-separated schemes in the world of schemes. On the other hand, the non-separated schemes are a bit strange, and one doesn't frequently encounter non-separated schemes in practice. In fact, the first edition of EGA reserved the word *preschema* for what we today call schemes and *schema* for a separated scheme.

There are counterexamples: 'moduli spaces' and 'quotient schemes' are classes of schemes which are sometimes non-separated.

More importantly, some very nice and advantageous properties hold only for separated schemes, and this legitimates the notion. For instance, in a separated scheme, the intersection of two affine subsets is again affine (this is a property which will be important later on).

Of course, one needs good criteria to be sure we have a large class of separated schemes. We will soon see that all affine schemes are separated, and we will see in Chapter 10 that the same is true for projective schemes also.

We begin with defining the diagonal and giving some of its properties, the proofs are of a quite formal nature and also work in any category where the statements are meaningful.

9.1 The diagonal

Let X/S be a scheme over S. There is a canonical map $\Delta_{X/S}\colon X \to X \times_S X$ of schemes over S called the *diagonal map* or the *diagonal morphism*. The two component maps of $\Delta_{X/S}$ are both equal to the identity id_X; that is, the defining properties of $\Delta_{X/S}$ are $\pi_i \circ \Delta_{X/S} = \mathrm{id}_X$ for $i = 1, 2$ where the π_i's denote the two projections.

In the case that X and S are affine schemes, the diagonal has a simple and natural interpretation in terms of algebras; it corresponds to the most natural map, namely the multiplication map:

$$\mu\colon A \otimes_B A \to A.$$

The multiplication map sends $a \otimes a'$ to the product aa', and then extends to $A \otimes_B A$ by linearity. The projections correspond to the two algebra homomorphisms $\iota_i\colon A \to A \otimes_B A$

that send a to $a \otimes 1$ respectively to $1 \otimes a$. Clearly it holds that $\mu \circ \iota_i = \mathrm{id}_A$, and on the level of schemes this translates into the defining relations for the diagonal map. Moreover, μ is clearly surjective, so we have established the following:

PROPOSITION 9.1 *If X is an affine scheme over the affine scheme S, then the diagonal $\Delta_{X/S} \colon X \to X \times_S X$ is a closed immersion.*

The conclusion here is not generally true for schemes, and shortly we shall give counterexamples. However, from the proposition we just proved, it follows readily that the image $\Delta_{X/S}(X)$ is always *locally closed*, *i.e.* the diagonal is locally a closed immersion:

PROPOSITION 9.2 *The diagonal $\Delta_{X/S}$ is locally a closed immersion.*

PROOF: Begin with covering S by open affine subsets and subsequently cover each of their inverse images in X by open affines as well. In this way one obtains a covering of X by affine open subsets U_i whose images in S are contained in affine open subsets S_i. The products $U_i \times_{S_i} U_i = U_i \times_S U_i$ are open and affine, and their union is an open subset containing the image of the diagonal. By Proposition 9.1 above the diagonal restricts to a closed immersion of U_i in $U_i \times_{S_i} U_i$. ❑

9.2 *Separated schemes*

We have now come to the definition of the property that will play the role of the Hausdorff property for schemes.

DEFINITION 9.3 *One says that the scheme X/S is* separated *over S, or that the structure map $X \to S$ is separated, if the diagonal map $\Delta_{X/S} : X \to X \times_S X$ is a closed immersion. One says for short that X is separated if it is separated over* $\operatorname{Spec} \mathbb{Z}$.

Recall that being a closed immersion is a local property on the target. Translating this for Δ, we see that $f : X \to S$ is separated if and only if there exists an open covering $\{S_i\}$ of S such that $f^{-1}S_i \to S_i$ is separated.

In fact, since Δ is a locally closed immersion, it suffices to check that the image of the diagonal $\Delta_{X/S}$ is a closed subset of $X \times_S X$. In particular, this means that being separated is a condition that only involves the underlying topological spaces of the map $f : X \to S$.

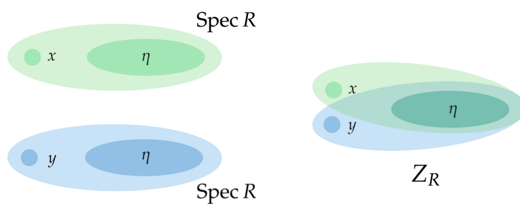
Examples

(9.4) Any morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes is separated (by Proposition 9.1).
(9.5) If $X \to S$ is separated, and $i : Z \to X$ is a closed immersion, then $Z \to S$ is also separated. This is because $\Delta_{Z/S}$ factors as $Z \to X \to X \times_S X$, which is a composition of two closed immersions.
(9.6) If $X \to S$ is separated, and $U \to X$ is an open set, then $U \to S$ is also separated.

(9.7) The simplest example of a scheme that is not separated is obtained by gluing the prime spectrum of a discrete valuation ring to itself along the generic point.

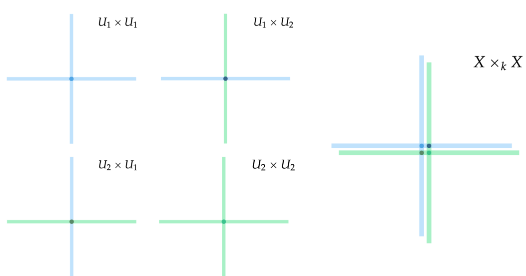
To give more details let R be a DVR with fraction field K. Then $\operatorname{Spec} R = \{x, \eta\}$ where x is the closed point corresponding to the maximal ideal, and η is the generic point corresponding to the zero ideal. The generic point η is an open point (the complement of $\{\eta\}$ is the closed point x) and corresponding to the open immersion $\operatorname{Spec} K \to \operatorname{Spec} R$. By the gluing lemma for schemes (Proposition 5.3 on page 79), we may glue one copy of $\operatorname{Spec} R$ to another copy of $\operatorname{Spec} R$ by identifying the generic points; that is, the open subschemes $\operatorname{Spec} K$ in the two copies.

In this manner we construct a scheme Z_R together with two open immersions $\iota_i \colon \operatorname{Spec} R \to Z_R$. They send the generic point η to the same point, which is an open point in Z_R, but they differ on the closed point x.



It follows that the diagonal is not closed. If $\pi \colon \operatorname{Spec} R \to X \times X$ is the morphism induced by ι_1 and ι_2, the preimage $\pi^{-1}(\Delta_X(X))$ of the diagonal is exactly the set of points in $\operatorname{Spec} R$, where the morphisms ι_i agree. But this set has exactly one point, the open point, which is not closed. (See also Proposition 9.14 below).

(9.8) The affine line X with two origins constructed on page 91 in Chapter 6 is not separated over $S = \operatorname{Spec} k$. It was constructed as the union of two affine lines $U_i = \operatorname{Spec} k[u]$ glued together along their common open subset $U_{12} = \operatorname{Spec} k[u, u^{-1}]$. Hence there are two open immersions $\operatorname{Spec} k[u] \to X$ which agree on U_{12}, which is not closed.



It is also instructive to examine the diagonal in detail. Denote the two origins by O_1 and O_2. Then the scheme $X \times_k X$ is glued together by four affine charts $U_1 \times U_1$, $U_1 \times U_2$, $U_2 \times U_1$, and $U_2 \times U_2$, all isomorphic to $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$. Here the image $\Delta_X(X)$ intersects $U_1 \times U_1$ along the diagonal $V(x - y) \subset \mathbb{A}^2_k \simeq U_1 \times U_1$. On the other hand, in $U_1 \times U_2$, the intersection $U_1 \times U_2 \cap \Delta(X)$ does not contain the origin (O_1, O_2). It follows that $\Delta_X(X)$ is not closed. In fact, the fibre product $X \times_k X$ contains four origins

$$(O_1, O_1), (O_1, O_2), (O_2, O_1), (O_2, O_2).$$

The image of the diagonal morphism only contains the two origins (O_1, O_1) and (O_2, O_2), while the closure of $\Delta_X(X)$ contains all four.

☆

We mostly care about separatedness for schemes because it disallows pathological examples like the affine line with two origins as above. The schemes we are most interested in are $\mathbb{A}^n$, $\mathbb{P}^n$ and their closed subschemes, and as we will see, these are all separated.

Some of the most basic properties of separated morphisms are listed in the next proposition:

Proposition 9.9 *The following hold true:*

i) (Immersions) Locally closed immersions are separated, in particular open and closed immersions are;

ii) (Composition) Let $f\colon T \to S$ and $g\colon X \to T$ be morphisms. If both f and g are separated, the composition $g \circ f$ is separated as well. If X is separated over S, it is separated over T;

iii) (Base change) Being separated is a property stable under base change: if $f\colon X \to S$ is separated and $T \to S$ is any morphism, then $f_T\colon X \times_S T \to T$ is separated:

Proof: The first statement was proven in XXX. To prove statement *ii)*, let the two separated morphisms be $f\colon X \to T$ and $g\colon T \to S$. The crux of the proof is that the following diagram is Cartesian (see Exersice **??**):

$$\begin{tikzcd} X \times_T X \arrow[r, "\iota"] \arrow[d] & X \times_S X \arrow[d, "f \times f"] \\ T \arrow[r, "\Delta_{T/S}"] & T \times_S T. \end{tikzcd} \tag{9.1}$$

Note that $\Delta_{X/S} = \iota \circ \Delta_{X/T}$. Assume first that $T \to S$ is separated, then $\Delta_{T/S}$ is a closed immersion, hence ι will be one as well (Proposition 8.12 on page 126). It follows that $\Delta_{X/S} = \iota \circ \Delta_{X/T}$ is a closed immersion (composition of closed immersions are closed immersions), and so X is separated over S. For the converse statement, assume that X is separated over S. Then the composition $\iota \circ \Delta_{X/T}$ being equal to $\Delta_{X/S}$ is a closed immersion, hence $\Delta_{X/T}$ is closed.

When proving statement *iii)*, it suffices to cite Exercise **??** on page ??, that diagonals pull back to diagonals, and again Proposition 8.12, that pullbacks of closed immersions are closed immersions. ❑

We introduce separatedness mostly because they give good formal properties. In some sense the schemes category is still a little bit "too large", and separated schemes have properties that make them closer to varieties. Here is one of these properties:

Proposition 9.10 *Assume that X is a separated scheme over an affine scheme $S = \operatorname{Spec} A$, and assume that U and V are two affine open subsets of X. Then the intersection $U \cap V$ is also affine, and the natural product map $\Gamma(U, \mathcal{O}_U) \otimes_A \Gamma(V, \mathcal{O}_V) \to \Gamma(U \cap V, \mathcal{O}_X)$ is surjective.*

Proof: The product $U \times_S V$ is an open and affine subset of $X \times_S X$, and $U \cap V = \Delta_{X/S}(X) \cap (U \times_S V)$. So if the diagonal is closed, $U \cap V$ is a closed subscheme of the affine scheme $U \times_S V$, hence affine (Proposition 4.32). By the construction of the fibre product of affine schemes one has

$$\Gamma(U \times_S V, \mathcal{O}_{U \times_S V}) = \Gamma(U, \mathcal{O}_U) \otimes_A \Gamma(V, \mathcal{O}_V),$$

and as $U \cap V$ is a closed subscheme of $U \times_S V$, the restriction map

$$\Gamma(U \times_S V, \mathcal{O}_{U \times_S V}) \to \Gamma(U \cap V, \mathcal{O}_{U \cap V})$$

is surjective, as we wanted to show. ❑

Conversely, we have

Proposition 9.11 *Let X be a scheme over $\operatorname{Spec} A$, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine cover of X such that*

i) all intersections $U_i \cap U_j$ are affine;
ii) $\Gamma(U_i, \mathcal{O}_X) \otimes_A \Gamma(U_j, \mathcal{O}_X) \to \Gamma(U_i \cap U_j, \mathcal{O}_X)$ is surjective for each $i, j \in I$.

Then X is separated over S.

Proof: Let $\pi_1, \pi_2 \colon X \times_S X \to X$ be the two projections and let $\Delta \colon X \to X \times_S X$ denote the diagonal morphism $\Delta_{X/S}$. Let $U_i = \operatorname{Spec} B_i$ and $U_j = \operatorname{Spec} B_j$ be two open sets in the covering $\mathcal{U}$. We have

$$\Delta^{-1}(\pi_1^{-1}(U_i) \cap \pi_2^{-1}(U_j)) = \Delta^{-1}(\pi_1^{-1}(U_i)) \cap \Delta^{-1}(\pi_2^{-1}(U_j)) = U_i \cap U_j, \tag{9.2}$$

Also, from the universal property of the fibre product it ensues that $\pi_1^{-1}(U_i) \cap \pi^{-1}(U_j) = U_i \times_S U_j \subset X \times_S X$, and from this we deduce that Δ is a closed immersion if each restriction

$$\Delta_{ij} \colon U_i \cap U_j \to U_i \times_S U_j$$

of Δ is a closed immersion. But this follows from the assumptions: by *i)* the intersection $U_i \cap U_j$ is affine, say $U_i \cap U_j = \operatorname{Spec} C_{ij}$, and by *ii)* the ring homomorphism $B_i \otimes_A B_j \to C_{ij}$ is surjective. Hence Δ_{ij} is a closed immersion for each i, j, and the proof is complete. ❑

Example 9.12 The above provides us with a convenient criterion to check that a scheme is separated, given an affine covering. For instance, let us show that the projective line $\mathbb{P}^1_k$ is separated. $\mathbb{P}^1_k$ is covered by the two affine subsets $U_1 = \operatorname{Spec} k[x]$ and $U_2 = \operatorname{Spec} k[x^{-1}]$, which have affine intersection $\operatorname{Spec} k[x, x^{-1}]$. To conclude, we need only check that the map

$$k[x] \otimes_k k[x^{-1}] \to k[x, x^{-1}]$$

is surjective, and it is. ☆

EXAMPLE 9.13 Here is a non-separated scheme where two affine open sets have non-affine intersection. We glue two copies of the affine plane $\mathbb{A}^2_k$ together along the complement $U_{12} = \mathbb{A}^2_k - V(x,y)$ of the origin. If U_1 and U_2 denote the two open immersions of the affine plane, then $U_1 \cap U_2 = U_{12}$, but the open set U_{12} is not affine (see the example in Section 6.2 on page 87). In this example, the multiplication map in the proposition coincides with $k[x,y] \otimes k[x,y] \to \Gamma(U_{12}, \mathcal{O}_{U_{12}})$, which *is* surjective. ☆

Another useful property is that morphisms into separated schemes are determined on open dense sets, at least when the source is reduced:

PROPOSITION 9.14 *Let X and Y be two schemes over S and $f_1, f_2\colon Y \to X$ two morphisms over S. Assume that Y is a reduced scheme and X is separated over S. Moreover, assume there is an open immersion $\iota\colon U \to Y$ with dense image such that $f_1 \circ \iota = f_2 \circ \iota$. Then $f_1 = f_2$.*

PROOF: We may assume that Y is affine, say $Y = \operatorname{Spec} A$. The morphism $f\colon Y \to X \times_S X$ whose two components are f_1 and f_2 enters in the diagram

$$\begin{tikzcd} U \arrow[r, "\iota"] \arrow[dr, "j"'] & Y \arrow[r, "f"] & X \times_S X \\ & E \arrow[u, "\eta"'] \arrow[r] & X, \arrow[u, "\Delta_{X/S}"'] \end{tikzcd}$$

where the right square is Cartesian. We have assumed that $f_1 \circ \iota = f_2 \circ \iota$, so $f \circ \iota$ factors through $\Delta_{X/S}$ and, according to the universal property of pullbacks, the map ι factors through E. Now, pullbacks of closed immersions are closed immersions, so that the image $\eta(E)$ is closed, and by Proposition 4.32 on page 69 it is shaped like $\operatorname{Spec} A/\mathfrak{a}$ for some ideal $\mathfrak{a}$. The image $\eta(E)$ contains the dense set $\iota(U)$ and therefore is equal to Y. Thus $\sqrt{\mathfrak{a}} = (0)$, and I is contained in the nilradical of A which is zero as A is reduced. Hence η is an isomorphism. Consequently, f factors through the diagonal and $f_1 = f_2$. ❑

EXAMPLE 9.15 The above proposition fails when X is not separated. For instance, if X is the affine line with two origins, then there are two morphisms $\iota_j\colon \mathbb{A}^1_k \to X$ for $j = 1,2$ which agree on a dense open set, but they are not equal. ☆

EXAMPLE 9.16 Likewise, it may fail when the scheme Y is not reduced: let $Y = \operatorname{Spec} k[x,y]/(y^2, xy)$ and consider the two maps $f_j\colon Y \to \operatorname{Spec} k[u]$, $j = 1,2$ defined by $u \mapsto x$ and $u \mapsto x + y$ respectively. These agree over the distinguished open set $D(x)$, but they are different. ☆

Exercises

(9.1) Show that $X \to S$ is separated if and only if the image of the diagonal map $\Delta_{X/S}$ is a closed subset of $X \times_S X$.

(9.2) Show that $Z_R \times Z_R$ with Z_R as in Example 9.7 above is obtained by gluing *four* copies of $\operatorname{Spec} R$ together along their generic points. Show that the diagonal is open and not closed.

(9.3) *The graph of a morphism.* A morphism $\phi\colon X \to Y$ over S has a *graph* $\Gamma_\phi\colon X \to X \times_S Y$; it is the pullback of the diagonal $\Delta_{Y/S}$ under the morphism $\phi \times \mathrm{id}_Y\colon X \times Y \to Y \times_S Y$. Show that the graph is a closed immersion when Y is separated.

(9.4) *Closed immersions.* Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes.

a) Assume that g is separated. Show that if the composition $g \circ f$ is a closed immersion, then f is a closed immersion. HINT: Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ & X \arrow[r, "g\circ f"'] & Z \end{tikzcd}$$

where the square is Cartesian and Γ_f is the graph of f.

b) Show by an example that in general f is not necessarily a closed immersion even if $g \circ f$ is. HINT: For one of the copies of $\mathbb{A}^1$, say U_1, in the affine line X with two origins constructed on page 91 in Chapter 6, exhibit a morphism $X \to \mathbb{A}^1$ that restricts to the identity on U_1.

✲ (9.5) Let T be a scheme and X and Y two schemes over T with structure maps $\phi_X\colon X \to T$ and $\phi_Y\colon Y \to T$. Let $T \to S$ be a morphism.

a) Show there is a Cartesian diagram

$$\begin{tikzcd} X \times_T Y \arrow[r, "\iota"] \arrow[d] & X \times_S Y \arrow[d, "\phi_X \times \phi_Y"] \\ T \arrow[r, "\Delta_{T/S}"] & T \times_S T \end{tikzcd}$$

where ι is the natural inclusion as in Exercises **??**.

b) Show that ι is separated and that ι is a closed immersion if T is separated over S.

(9.6) Recall that a morphism $\phi\colon X \to Y$ is said to be *affine* if for some cover $\{U_i\}$ of Y of open affine sets, the inverse images $\phi^{-1}(U_i)$ are affine (Definition 4.25 on page 67). Show that affine morphisms are separated.

(9.7) Let R and S be two DVR's with the same fraction field, and denote by $\mathfrak{m}_R$ and $\mathfrak{m}_S$ the two maximal ideals. Assume that R and S different in the sense that $\mathfrak{m}_R \cap S \nsubseteq \mathfrak{m}_S$ and $\mathfrak{m}_S \cap R \nsubseteq \mathfrak{m}_R$. Let Z be the scheme obtained by gluing $\operatorname{Spec} R$ and $\operatorname{Spec} S$ together along the generic points. Show that Z is affine, more precisely, show that Z is isomorphic to $\operatorname{Spec}(R \cap S)$.

(9.8) *The Hausdorff axiom.* Let Y be a separated scheme over S and let $f, g\colon X \to Y$ be two S-morphisms from X to Y. Show that the set $Z \subset X$ of points $x \in X$ so that $f(x) = g(x)$, is closed in X.

(9.9) *Equalizers.* Let X and Y be schemes over S and f_1 and f_2 two morphisms from Y to X. Let $f\colon Y \to X \times_S X$ be the morphism whose components are the f_i's; that is, $f_i = \pi_i \circ f$ (as usual, the π_i's are the two projections). The pullback $f^{-1}\Delta_{X/S}$ is called the *equalizer* of the f_i's, and we shall denote it by $\eta\colon E \to Y$. In other words, the diagram below is Cartesian:

Equalizer

$$\begin{tikzcd} E \arrow[r, "\eta"] \arrow[d] & Y \arrow[d, "f"] \\ X \arrow[r, "\Delta_{X/S}"] & X \times_S X. \end{tikzcd}$$

a) Show that a morphism $g\colon Z \to Y$ satisfies $f_1 \circ g = f_2 \circ g$ if and only if g factors via η;
b) Show that X is separated if and only if all equalizers of maps into X are closed.

(9.10) Let $X = \operatorname{Spec}\mathbb{C}$ and $S = \operatorname{Spec}\mathbb{R}$. Recall that the product $X \times_S X$ consists of two (closed) points. Which one is the diagonal? Can you find another $\mathbb{R}$-algebra A so that if $Y = \operatorname{Spec} A$ it holds that $Y \times_S Y \simeq X \times_S X$ and the diagonal is the other point?

(9.11) Let A be a B-algebra. Show that the kernel of the multiplication map $\mu\colon A \otimes_B A \to A$ is generated by the elements of shape $a \otimes 1 - 1 \otimes a$. Hint: It holds true that $\sum_i a_i \otimes b_i = \sum_i (a_i \otimes 1 - 1 \otimes a_i) \cdot 1 \otimes b_i + \sum_i 1 \otimes a_i b_i$.

✲ (9.12) Prove a partial converse to statement *ii)* in Proposition 9.9:

a) if f and g are two composable morphisms of schemes and $f \circ g$ and f are separated, prove that then g is separable. Hint: Show that ι in the diagram (9.1) always is separated, then resort to Exercise 9.4 above.
b) Show by way of an example that the third alternative is not true; *i.e.* exhibit f and g so that $f \circ g$ and g are separated but f is not.

✲ (9.13) Show that if a scheme X is separated (over $\mathbb{Z}$), then for every scheme Y and every morphism $f\colon X \to Y$, the morphism f is separated.

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Chapter 10

Projective schemes

The projective varieties are fundamental in the theory of varieties, not just because they are interesting objects of study, but also because they in many aspects are easier to handle than non-projective ones. In the scheme world, there is a construction extending the notion of projective varieties; from any positively graded ring R one constructs a scheme $\operatorname{Proj} R$ called the *projective spectrum*. The construction is somewhat parallel to that of the prime spectrum of a ring, but there are several key differences between the two. For instance, and perhaps most strikingly, $\operatorname{Proj} R$ does not depend functorially on R in the sense that maps between graded rings do not always give maps between the projective schemes. Moreover, different R's may yield isomorphic projective spectra.

Before we begin the construction, we include a motivating section recalling some features of the projective spaces over the complex numbers.

10.1 Motivation

Let us recall the usual construction of the complex projective spaces: as a topological space, $\mathbb{CP}^n$ is the quotient space

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} - 0\right) / \mathbb{C}^*$$

by the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ that scales the coordinates. The orbits of $\mathbb{C}^*$ in $\mathbb{C}^{n+1} \backslash \{0\}$ are just the lines through the origin with the origin removed, so this is the traditional 'variety way' of thinking about projective spaces as the set of lines through the origin of $\mathbb{CP}^n$.

We can translate this into algebra as follows: for each polynomial function f on $\mathbb{C}^{n+1}$ and each $\lambda \in \mathbb{C}^*$, we get a new polynomial function f^λ by defining $f^\lambda(x) = f(\lambda x)$, and this gives an action of $\mathbb{C}^*$ on the polynomial ring $R = \mathbb{C}[x_0, \ldots, x_n]$. From this, we can decompose R as a $\mathbb{C}$-vector space into a direct sum

$$R = \sum_{d \in \mathbb{Z}} R_d$$

of eigenspaces; that is, each R_d is the subspace of elements $f \in R$ with $\mathbb{C}^*$ acting as $f^\lambda = \lambda^d \cdot f$. In fact, $\mathbb{C}^*$ acts through algebra homomorphisms, which means that $(fg)^\lambda = f^\lambda g^\lambda$, the algebra R decomposes as $R = \bigoplus_{d \in \mathbb{Z}} R_d$ as a vector space over $\mathbb{C}$, which implies that $R_d \cdot R_{d'} \subseteq R_{d+d'}$. In other words, R is a *graded algebra*.

Leaving the realm of complex manifolds and entering the world of schemes, we want to take the quotient of $\mathbb{A}^{n+1}_{\mathbb{C}} - 0 = \operatorname{Spec} \mathbb{C}[x_0, \ldots, x_n] - V(x_0, \ldots, x_n)$ by this action. We write $\mathbb{P}^n_{\mathbb{C}}$, rather than $\mathbb{C}\mathbb{P}^n$, for the corresponding quotient space equipped with the quotient topology; this notation is to emphasize that the quotient is taken with respect to the Zariski topology, and not the usual topology.

One may try to put a scheme structure on $\mathbb{P}^n_{\mathbb{C}}$ by looking for reasonable open covers. Note that the open subsets of $\mathbb{P}^n_{\mathbb{C}}$ correspond to $\mathbb{C}^*$-invariant open subsets of $\mathbb{A}^{n+1}_{\mathbb{C}} - 0$. It is not too hard to see that $D(f) \subset \mathbb{A}^{n+1}_{\mathbb{C}}$ is $\mathbb{C}^*$-invariant if and only if f is a homogeneous polynomial, and in case it is, we shall write $D_+(f) \subset \mathbb{P}^n_{\mathbb{C}}$ for the open subset corresponding to $D(f) \subset \mathbb{A}^{n+1}_{\mathbb{C}} - 0$.

To define a structure sheaf on $\mathbb{P}^n_{\mathbb{C}}$ we must figure out what the spaces of sections $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(D_+(f))$ should be. While it is true that $D(f)$, being an affine scheme, has a structure sheaf whose global sections is $\mathbb{C}[x_0, \ldots, x_n, f^{-1}]$, we have to take more care in deciding which sections to take, to make things compatible with the $\mathbb{C}^*$-action: a function on $D_+(f)$ should be a function on $D(f)$ that is invariant under the action of $\mathbb{C}^*$. That is, we should have $g^\lambda = g$, which means precisely that g has degree zero. Thus we define

$$\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(D_+(f)) = \mathbb{C}[x_0, \ldots, x_n, f^{-1}]_0,$$

where the subscript means that we take the degree 0 part.

We can generalize the above construction for any affine $\mathbb{C}$-scheme with an action of $\mathbb{C}^*$. Such a scheme corresponds to a graded $\mathbb{C}$-algebra R. To make a reasonably good quotient it is necessary to remove the locus in $\operatorname{Spec} R$ that is fixed by $\mathbb{C}^*$, and it is not too hard to prove the following:

Lemma 10.1 *The fixed locus of $\mathbb{C}^*$ acting on $\operatorname{Spec} R$ is $V(R_+)$, where R_+ denotes the ideal generated by elements of positive degree.*

We then proceed to consider the quotient space P of $\operatorname{Spec} R - V(R_+)$ by $\mathbb{C}^*$. Again, the $\mathbb{C}^*$-invariant distinguished open subsets in $\operatorname{Spec} R$ of the form $D(f)$ where f is homogeneous, constitute a basis for the topology on $\operatorname{Spec} R - V(R_+)$. These descend to open subsets $D_+(f) \subset P = (\operatorname{Spec} R - V(R_+))\,/\mathbb{C}^*$, which form a basis for the quotient topology. Finally, whenever f is homogeneous, $\mathcal{O}_{\operatorname{Spec} R}(D(f))$ has a natural grading, and we may define a $\mathscr{B}$-sheaf on P by setting $\mathcal{O}_P(D_+(f)) = \mathcal{O}_{\operatorname{Spec} R}(D(f))_0$, and of course, we must check that we get a scheme P.

Beside of inducing a grading on R, the action of $\mathbb{C}^*$ plays very little role here. Realizing this, we can in fact build a scheme P from any graded ring R: We construct the topological space of P from the set of *homogeneous* prime ideals of R (with the induced Zariski topology), and define a structure sheaf on it by the formula like the one above. This is essentially the 'Proj'-construction.

10.2 *Basic remarks on graded rings*

In the literature one finds various definitions of what a graded ring should be; some are more general than others. Here we shall consider rings graded by the natural numbers $\mathbb{N}$:

A *graded ring* R is a ring with a decomposition

$$R = \bigoplus_{n \in \mathbb{N}} R_n = R_0 \oplus R_1 \oplus \cdots$$

as an abelian group such that $R_m \cdot R_n \subset R_{m+n}$ for each $m, n \geqslant 0$. Note that R_0 is a subring of R and that each of the R_n's is an R_0-module. The elements in R_n are said to be *homogeneous* of degree n, and one writes $\deg x = n$ when $x \in R_n$. (Note that 0 has no well-defined degree, but is considered homogeneous of any degree.)

Every non-zero element $x \in R$ can be expressed uniquely as a sum $x = \sum_{n \in \mathbb{N}_0} x_n$ with $x_n \in R_n$. The non-zero terms in the sum are called the *homogeneous components* of x.

A homomorphism $\phi \colon R \to S$ between two graded rings R and S is homogeneous of degree d if $\phi(R_n) \subseteq S_{dn}$. In fact, one may allow d to be a rational number, but except in a few rather exotic cases, it will be a natural number. The graded rings together with the homogeneous maps form a category GrRings. That two graded rings are isomorphic means that they are isomorphic as rings and that the gradings are the same except for a possible scaling of the degrees.

An R-module M is *graded* if it has a similar decomposition $M = \bigoplus_{n \in \mathbb{Z}} M_n$ as an abelian group such that $R_m \cdot M_n \subset M_{m+n}$ for all. A map of graded R-modules is an R-linear map $\phi : M \to N$ such that $\phi(M_n) \subset N_n$ for all $n \in \mathbb{Z}$. Note that contrary to what we required for maps between graded rings, degrees are preserved.

As usual, a non-zero element $x \in M$ is *homogeneous* of degree n if it lies in M_n. Just like ring elements, any member $x \in M$ may be expressed in a unique way as $x = \sum_{n \in \mathbb{N}_0} x_n$ with each x_n in M_n, and the non-zero terms are called the *homogeneous components* of x.

An ideal $\mathfrak{a} \subset R$ is *homogeneous* if the homogeneous components of each element in $\mathfrak{a}$ belongs to $\mathfrak{a}$. This is the case if and only if $\mathfrak{a}$ is generated by homogeneous elements. It is readily verified that intersections, sums and products of homogeneous ideals are homogeneous.

We will write R_+ for the sum $\bigoplus_{n>0} R_n$; this is naturally a homogeneous ideal of R, which we call the *irrelevant ideal*. If $R = k[x_0, \ldots, x_n]$ with the standard grading, then $R_+ = (x_0, \ldots, x_n)$.

EXAMPLE 10.2 (*Veronese rings.*) Common examples of graded rings are the so-called *Veronese rings* rings associated to a graded ring R. For any natural number d, we let let $R^{(d)}$ denote the subring of R given by $\bigoplus_{n \geqslant 0} R_{nd}$. ☆

Veronese rings

Localization

Occasionally we will meet graded rings having elements of negative degree; they are defined as above except that they decompose as

$$R = \bigoplus_{n \in \mathbb{Z}} R_n.$$

These are sometimes called $\mathbb{Z}$-graded rings. One way such rings appear is as localizations of graded rings. Indeed, if $T \subseteq R$ is a multiplicative system all whose elements are homogeneous, one may define a grading on $T^{-1}R$ by letting $\deg g/t = \deg g - \deg t$ for $t \in T$ and g a homogeneous element from R. In other words, one puts

$$(T^{-1}R)_n = \{\, f/t \in T^{-1}R \mid f \in R_n, t \in T \text{ and } \deg f - \deg t = n \,\}.$$

Then, as is easily verified, the localized ring $T^{-1}R$ decomposes as the direct sum as $T^{-1}R = \bigoplus_{n\in\mathbb{Z}}(T^{-1}R)_n$, which makes it a $\mathbb{Z}$-graded ring. The same construction also works very well for graded modules, so that $T^{-1}M$ is a graded module whose homogeneous elements are of shape xt^{-1} with x homogeneous and $\deg xt^{-1} = \deg x - \deg t$.

One example of multiplicative sets of the graded sort, is the sets $T(\mathfrak{p})$ consisting of all homogeneous elements in R not lying in a given homogeneous prime ideal $\mathfrak{p}$. Another example is the set S of non-negative powers of a homogeneous element f.

Some algebraic geometry-texts use the notations $M_{(\mathfrak{p})}$ and $M_{(f)}$ for the degree zero part of the localizations $M_\mathfrak{p}$ and M_f respectively. We will however not adopt this notation (*e.g.* writing things like '$k[x]_{(x)}$' gets a little bit confusing and one ends up with monstrous notation like '$k[x]_{((x))}$').

Example 10.3 For the polynomial ring $R = A[x_0, \ldots, x_n]$ equipped with the standard grading, the degree 0 part of the localization R_{x_j} is generated by the monomials $x_0x_j^{-1}, \ldots, x_nx_j^{-1}$, so

$$(R_{x_j})_0 = A[x_0x_j^{-1}, \ldots, x_nx_j^{-1}].$$

☆

Exercises

(10.1) Let R be a graded ring and $\mathfrak{p}$ a homogeneous prime ideal. Show that $(R_\mathfrak{p})_0$ is a local ring whose maximal ideal is given as $\mathfrak{q} = \{\, fg^{-1} \mid f \in \mathfrak{p}, g \in T(\mathfrak{p}), \deg f = \deg g \,\}$.

(10.2) Let R be a graded ring and $\mathfrak{p}$ a homogeneous ideal in R. Show that $\mathfrak{p}$ is prime if and only if $x \cdot y \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ for all homogeneous elements x and y.

(10.3) Let R and S be graded rings and $\phi\colon R \to S$ a homomorphism of graded rings. Show that the inverse image $\phi^{-1}(\mathfrak{p})$ of an ideal $\mathfrak{p} \subseteq S$ is homogeneous whenever $\mathfrak{p}$ is.

★

10.3 *The Proj construction*

Motivated by the discussion in the introduction, we make the following definition:

> Definition 10.4 *Let R be a graded ring. We denote by* Proj R *the set of homogeneous prime ideals of R that do not contain the irrelevant ideal R_+. It is called the* projective spectrum *of R.*

One endows $\operatorname{Proj} R$ with a topology by letting for each homogeneous ideal $\mathfrak{a}$,

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Proj} R \mid \mathfrak{p} \supseteq \mathfrak{a} \,\},$$

and just like in the case of $\operatorname{Spec} R$, these sets comply with the axioms for closed sets of a topology. This topology is called the *Zariski topology* on $\operatorname{Proj} R$. Indeed, the following identities hold true; the conditions that the primes are homogeneous and not contain the irrelevant ideal, do not disturb the proofs which remain *mutatis mutandis* the same as for the closed sets in $\operatorname{Spec} R$:

i) $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$;
ii) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$;
iii) $V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})$,

where $\mathfrak{a}$, $\mathfrak{b}$ and the $\mathfrak{a}_i$'s are homogeneous ideals. The key point is that sums, products and radicals persist being homogeneous when the involved ideals are.

The reason behind the name 'the irrelevant ideal' is that R_+ does not play any role when it comes to forming closed sets in Proj R, neither do ideals whose radical equals R_+. This is made clear by the following lemma. Note that by definition we have $V(R_+) = \varnothing$.

Lemma 10.5 *For any homogeneous ideal $\mathfrak{a}$ it holds that $V(\mathfrak{a}) = V(\mathfrak{a} \cap R_+)$. In fact, if I is an ideal such that $\sqrt{I} = R_+$, it holds that $V(\mathfrak{a}) = V(\mathfrak{a} \cap I)$.*

Proof: Since $V(R_+) = \varnothing$, condition *iii)* above implies that $V(I) = \varnothing$, and condition *ii)* then gives $V(\mathfrak{a} \cap I) = V(\mathfrak{a}) \cup V(I) = V(\mathfrak{a})$. ❑

Thus, when constructing the closed sets $V(\mathfrak{a})$, it suffices to work with ideals contained in the irrelevant ideal. In fact, we can take $\mathfrak{a}$ lying in any prescribed power of the irrelevant ideal.

Incidentally, we will not get more closed sets if we allow all ideals $\mathfrak{a}$ and not just the homogeneous ones: any ideal $\mathfrak{a}$ has a 'homogenization' associated with it, which is the ideal generated by all homogeneous components of members of $\mathfrak{a}$, and which gives the same closed subset $V(\mathfrak{a})$ of Proj R — a homogenous prime ideal contains $\mathfrak{a}$ if and only if all homogenous components of elements in $\mathfrak{a}$ lie in it. In other words, the Zariski topology on Proj R is nothing but the topology induced on Proj R from the Zariski topology on Spec R by means of the inclusion Proj $R \subset$ Spec R.

Distinguished open subsets

As in the affine case, there are some distinguished open sets. For each $f \in R$ which is homogeneous of positive degree, we let $D_+(f)$ be the collection of homogeneous ideals in (not containing the irrelevant ideal R_+) that do not contain f, or in other words, $D_+(f) = D(f) \cap$ Proj R. These are open sets with respect to the Zariski topology on Proj R; the complement of $D_+(f)$ equals the closed set $V(f)$.

The next result is important in understanding the local structure of Proj R. In particular, it will be essential when we define the scheme structure on Proj R.

Proposition 10.6 *We have $D_+(f) \cap D_+(g) = D_+(fg)$. Also, the $D_+(f)$ form a basis for the topology on* Proj R *when f runs through the homogeneous elements of R of positive degree.*

Proof: The first part is evident by the definition of a prime ideal. The second follows as in the affine case: $V(\mathfrak{a})$ is the intersection of the $V((f))$'s for the homogeneous $f \in \mathfrak{a} \cap R_+$, so $\operatorname{Proj} R - V(\mathfrak{a})$ is the union of these $D_+(f)$. Hence every open set is a union of sets of the form $D_+(f)$. ❑

EXERCISE 10.4 Let R be a graded ring and f and $\{f_i\}_{i\in I}$ homogenous elements from R with $\deg f > 0$. Show that the distinguished open sets $D_+(f_i)$ cover $D_+(f)$ if and only if a power of f lies in the ideal $(f_i | i \in I)$. ★

Dehomogenization and homogenization

In the affine case, there is a canonical homeomorphism between $D(f)$ and $\operatorname{Spec} R_f$ which associates a prime $\mathfrak{p} \in D(f)$ with the prime ideal $\mathfrak{p}R_f$. In perfect analogy with this, associating the degree zero part of $\mathfrak{p}R_f$ with $\mathfrak{p} \in D_+(f)$ gives a homeomorphism between $D_+(f)$ and $\operatorname{Spec}(R_f)_0$.

EXAMPLE 10.7 To illustrate this correspondence in a simple example, which will hopefully clarify what's going on in the general case, let us consider the ring $R = k[x,y,z]$, and the distinguished open set $D_+(z)$. The monomials of degree zero in R_z are products of powers of xz^{-1} and yz^{-1}, so we have $(R_z)_0 = k[xz^{-1}, yz^{-1}]$. Consider then a principal ideal $\mathfrak{a} = (f)$ in R generated by a homogeneous polynomial f of degree d. Because z is invertible in R_z, and because of the identity

$$f(xz^{-1}, yz^{-1}, 1) = z^{-d} f(x,y,z),$$

the ideal $\mathfrak{a}R_z$ becomes $\mathfrak{a}R_z = (z^{-d}f)R_z$; moreover, since $z^{-d}f$ is of degree zero it lies in $(R_z)_0$, and consequently it holds true that $\left(\mathfrak{a}R_z\right)_0 = \mathfrak{a}R_z \cap R_0 = (z^{-d}f)R_0$. So when we pass to $(R_z)_0$, the generator f is replaced by the *dehomogenized* polynomial $z^{-d}f$.

There is also a straightforward way of making a polynomial g in the ring $k[xz^{-1}, yz^{-1}]$ homogeneous: one simply gives g a factor z^d with d being the degree of g. This will almost all the time be an inverse to the dehomogenization process above; there is just one fallacy: any factor of f which is a power of z, disappears when f is dehomogenized, and there is no means of recovering it knowing only $z^{-d}f$. ☆

The general set up of the isomorphism $D_+(f) \simeq \operatorname{Spec}(R_f)_0$ follows the pattern in the example. Basically one dehomogenizes elements of the ideals with respect to f (and homogenizes to get them back), but expressed in a necessarily general formalism.

PROPOSITION 10.8 *Let R be a graded ring and let $f \in R$ be homogeneous of degree d. The canonical map $\phi\colon D_+(f) \to \operatorname{Spec}(R_f)_0$ that is defined by*

$$\phi(\mathfrak{p}) = \mathfrak{p}R_f \cap (R_f)_0,$$

has the following three properties:

i) ϕ is a homeomorphism;

ii) For any homogeneous element $g \in R$ such that $D_+(g) \subset D_+(f)$, letting $u = g^d f^{-\deg g} \in (R_f)_0$, we have $\phi(D_+(g)) = D(u)$;

iii) If $\mathfrak{a} \subset R$ is a homogeneous ideal, then $\phi(V(\mathfrak{a}) \cap D_+(f)) = V(\mathfrak{a}R_f \cap (R_f)_0)$.

PROOF: Note that ϕ is just the restriction to $\operatorname{Proj} R \cap \operatorname{Spec} R_f$ of the canonical map $\operatorname{Spec} R_f \to \operatorname{Spec}(R_f)_0$ induced by the inclusion map $(R_f)_0 \subset R_f$. Therefore it is continuous.

To prove *i)*, that ϕ is a homeomorphism, we begin by construct the inverse map $\psi\colon \operatorname{Spec}(R_f)_0 \to D_+(f)$. Given a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R_f)_0$, let us define $\psi(\mathfrak{p})$ to be the direct sum $\psi(\mathfrak{p}) = \bigoplus_{n \geqslant 0} \psi(\mathfrak{p})_n$ where

$$\psi(\mathfrak{p})_n = \{x \in R_n | x^d \cdot f^{-n} \in \mathfrak{p}\}.$$

One may think of $\psi(\mathfrak{p})_n$ as consisting of the homogenous elements of degree n that end up in $\mathfrak{p}$ when dehomogenized (recall that $d = \deg f$).

The first thing to check is that $\psi(\mathfrak{p})$ is a homogeneous prime ideal. Once we know it is an ideal, it will be homogeneous by definition, and it is nearly trivial to see it is closed under multiplication by elements from R. The tricky part is actually to prove it is an additive subgroup. To that end, assume we are given two elements $x, y \in \psi(\mathfrak{p})_n$. By definition we have $x^d \cdot f^{-n} \in \mathfrak{p}$ and $y^d \cdot f^{-n} \in \mathfrak{p}$, so by the Binomial theorem $(x+y)^{2d} \cdot f^{-2n} \in \mathfrak{p}$; indeed, each term in the expanded sum contains either $x^d \cdot f^{-n}$ or $y^d \cdot f^{-n}$. Therefore it holds that $(x+y)^d \cdot f^{-n} \in \mathfrak{p}$, and consequently $\psi(\mathfrak{p})$ is closed under addition.

To show that $\psi(\mathfrak{p})$ is prime, it suffices to verify the defining property for pairs of homogeneous elements, so we assume that $x \in R_m$ and $y \in R_n$ are two elements such that $(xy)^d f^{-(m+n)} \in \mathfrak{p}$. Since $f \notin \mathfrak{p}$, it holds that $(xy)^d \in \mathfrak{p}$, and hence either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Therefore we have either $x^d f^{-m} \in \psi(\mathfrak{p})_m$ or $y^d f^{-n} \in \psi(\mathfrak{p})_n$; in other words, $\psi(\mathfrak{p})$ is prime.

That the maps ϕ and ψ are mutually inverse follows almost by construction. Indeed, to verify $\mathfrak{p} \subseteq \psi(\mathfrak{p}R_f \cap (R_f)_0)$, note that elements $x \in \mathfrak{p}_n = \mathfrak{p} \cap R_n$ satisfy $x^d f^{-n} \in \mathfrak{p}R_f \cap (R_f)_0$. Conversely, if $x \in \psi(\mathfrak{p}R_f \cap (R_f)_0)_n$, then $x^d f^{-n} \in \mathfrak{p}R_f \cap (R_f)_0$, and there is an $N > 0$ so that $f^N x^d \in \mathfrak{p}$. As $\mathfrak{p}$ is prime, and $f \notin \mathfrak{p}$, we have $x^d \in \mathfrak{p}$, and hence $x \in \mathfrak{p}$. This proves that $\psi \circ \phi = 0$ and proving $\phi \circ \psi = \mathrm{id}$ is done similarly. Thus ϕ is bijective. That it is a homeomorphism follows from it being continuous and open — openness follows from statement *ii)*, which we now proceed to prove.

Proof of *ii)*: Let $g \in R$ be an element with $D_+(g) \subset D_+(f)$. Then for $\mathfrak{p} \in D_+(f)$, the following series of equivalences holds true since $\mathfrak{p}R_f$ is a prime ideal:

$$\begin{aligned} \mathfrak{p} \in D_+(g) &\Leftrightarrow g^r f^{-s} \notin \mathfrak{p}R_f \text{ for some } r, s > 0 \\ &\Leftrightarrow g^r f^{-s} \notin \mathfrak{p}R_f \text{ for all } r, s > 0 \\ &\Leftrightarrow g^{\deg f} f^{-\deg g} \notin \mathfrak{p}R_f \cap (R_f)_0 = \phi(\mathfrak{p}) \end{aligned}$$

Hence $\phi(D_+(g)) = D(u)$.

Proof of *iii)*: This follows essentially by the definition of ϕ. Let $\mathfrak{p} \in V(\mathfrak{a}) \cap D_+(f)$, so that $\mathfrak{a} \subseteq \mathfrak{p}$ and $f \notin \mathfrak{p}$. Then $\mathfrak{a}R_f \cap (R_f)_0 \subseteq \mathfrak{p}R_f \cap (R_f)_0$, which gives the inclusion '$\subset$'. Conversely, given a prime ideal $\mathfrak{p} \subset (R_f)_0$ such that $\mathfrak{a}R_f \cap (R_f)_0 \subseteq \mathfrak{p}$, its preimage $\mathfrak{p}' = \mathfrak{p} \cap R$ will be a homogeneous prime ideal in R not containing f, and so $\phi(\mathfrak{p}') = \mathfrak{p}'R_f \cap (R_f)_0 \supseteq \mathfrak{a}R_f \cap (R_f)_0$. This completes the proof. ❑

There is an analogue homogenization-dehomogenization process for modules, which we shall need later on when dealing with coherent sheaves on $\operatorname{Proj} R$:

Proposition 10.9 *Let R be a graded ring and M a graded R-module. Let f and g be two homogeneous elements such that $D_+(g) \subseteq D_+(f)$. If we let $u = g^d f^{-\deg g} \in (R_f)_0$, there is a canonical homomorphism $(M_f)_0 \to (M_g)_0$ which induces an isomorphism $((M_f)_0)_u \simeq (M_g)_0$.*

Proof: Write $g^k = af$. The localization map $M_f \to M_g$ is given by $xf^{-m} \mapsto a^m x g^{-mk}$, where $x \in M$. This induces a map $(M_f)_0 \to (M_g)_0$ because $\deg x + m(\deg a - k \deg g) = \deg x - m \deg f$. The element u acts as an invertible element on $(R_g)_0$, so the map $(M_f)_0 \to (M_g)_0$ factors via a map

$$\rho : ((M_f)_0)_u \to (M_g)_0.$$

We claim that this is an isomorphism.

ρ surjective: Explicitly, we have

$$\rho\left(xf^{-n}u^{-m}\right) = f^{tm-n} x g^{-dm},$$

where $t = \deg g$. Take any element $y \cdot g^{-l} \in (M_g)_0$ where $\deg y = tl$. Choose m large so that $dm \geqslant l$. Define $x = g^{dm-l} y \cdot f^{-(tm-n)} \in (M_f)_0$. We have $\deg x = nd$, and hence $xf^{-n} \cdot u^{-m} \in ((M_f)_0)_u$ is an element that maps to the given $y \cdot g^{-l}$.

ρ is injective: If $xf^{-n} \in (M_f)_0$ maps to 0 in $(M_g)_0$, then there is an $l > 0$ so that $g^{ld} a^n x = 0 \in M$. Multiplying up by powers of a and f, we get a relation of the form $g^{(l+n)d} x = 0 \in M$, and hence $u^{(l+n)d} x = 0 \in (M_f)_0$. But then $xf^{-n} = 0 \in ((M_f)_0)_u$. ❑

Proj R *as a scheme*

We shall now give Proj R a scheme structure, and the first step will be to make it a locally ringed space. In other words, we need to define the structure sheaf — a sheaf of rings $\mathcal{O}_{\operatorname{Proj} R}$— and while doing this, relying on Proposition 10.8, we shall see that the locally ringed space is locally affine. So $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R})$ will be a scheme. The order of the day is: restrict $\mathcal{O}_{\operatorname{Spec} R}$ to Proj R and 'take degree zero parts'.

To carry out the order of the day, we let $\mathscr{B}$ be the basis for the topology on Proj R consisting of the distinguished open subsets. For each $D_+(f)$, we set

$$\mathcal{O}(D_+(f)) = (R_f)_0.$$

The localization maps $R_f \to R_g$ are all homogenous of degree zero so that $(R_f)_0$ maps into $(R_g)_0$, and we may use the maps $(R_f)_0 \to (R_g)_0$ as restriction maps. This gives us a $\mathscr{B}$-presheaf $\mathcal{O}$. We proceed to establish that $\mathcal{O}$ is a $\mathscr{B}$-sheaf: If $\{D_+(f_i)\}$ is a covering of $D_+(f)$ (with the f_i's homogeneous), the distinguished open subsets $D(f_i)$ of Spec R will cover $D(f)$, and consequently the standard sequence

$$0 \longrightarrow R_f \xrightarrow{\alpha} \textstyle\prod_i R_{f_i} \xrightarrow{\beta} \prod_{i,j} R_{f_i f_j} \tag{10.1}$$

will be exact simply because $\mathcal{O}_{\operatorname{Spec} R}$ is a sheaf. Singling out pieces of degree zero is an exact operation and applied to (10.1) yields the exact sequence

$$0 \longrightarrow (R_f)_0 \xrightarrow{\alpha} \prod_i (R_{f_i})_0 \xrightarrow{\beta} \prod_{i,j} (R_{f_i f_j})_0$$

which exactly says that $\mathcal{O}$ is a $\mathscr{B}$-sheaf. We denote the unique sheaf extension of $\mathcal{O}$ by $\mathcal{O}_{\operatorname{Proj} R}$. Notice that the formula $\mathcal{O}_{\operatorname{Proj} R}(D_+(f)) = (R_f)_0$ is still valid.

According to Proposition 10.8 on page 145 there is a canonical homeomorphism $D_+(f) \simeq \operatorname{Spec}(R_f)_0$ which sends $D_+(g) \subset D_+(f)$ to the subset $D(u) \subset \operatorname{Spec}(R_f)_0$ with $u = f^{\deg g} g^{-\deg f}$. Because u is of degree zero, it holds that $(R_g)_0 \simeq \big((R_f)_0\big)_u$. This means that $\mathcal{O}$ restricts to the $\mathscr{B}$-sheaf induced by the structure sheaf on $\operatorname{Spec}(R_f)_0$, and so $\mathcal{O}_{\operatorname{Proj} R}$ restricts to $\mathcal{O}_{\operatorname{Spec}(R_f)_0}$. The locally ringed space $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R})$ is therefore locally affine; in other words it is a scheme.

Definition 10.10 *For a graded ring R, we call the scheme* $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R})$ *the* projective spectrum *of R.*

The projective spectrum $\operatorname{Proj} R$ is in a natural way a scheme over $\operatorname{Spec} R_0$: the structure map $\pi\colon \operatorname{Spec} R \to \operatorname{Spec} R_0$ restricts to a continuous map on $\operatorname{Proj} R$, which turns out to be a morphism. For this to be true, it suffices that its restriction to $D_+(f)$ be a morphism for each homogeneous f. But under the identification $\phi\colon D_+(f) \simeq \operatorname{Spec}(R_f)_0$ from Proposition 10.8 this restriction turns into the composition $\pi|_{D_+(f)} \circ \phi^{-1}$, and one check that this coincides with the structure map $\operatorname{Spec}(R_f)_0 \to \operatorname{Spec} R_0$ (that comes from the map $R_0 \to (R_f)_0$); in other words, it holds true that

$$\phi(\mathfrak{p}) \cap R_0 = \big(\mathfrak{p} R_f \cap (R_f)_0\big) \cap R_0 = \mathfrak{p} \cap R_0.$$

Indeed, one inclusion is obvious, and if for some $x \in \mathfrak{p}$ it holds that $y = f^{-n} x \in R_0$, we find $y \in \mathfrak{p}$ since $x = f^n y$ lies there, but f does not.

Among the most prominent varieties are the projective spaces, and they have analogues over any ring, in fact over any base scheme.

Definition 10.11 *We define the* projective n-space *to be the scheme*

$$\mathbb{P}^n = \operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_n].$$

More generally, for a ring A, the projective n-space over A *is the scheme*

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n].$$

There is even a projective n-space $\mathbb{P}^n_S$ over any scheme S. It is defined as $\mathbb{P}^n_S = \mathbb{P}^n \times_{\mathbb{Z}} S$; it may be obtained by gluing together the projective schemes $\mathbb{P}^n_A$ (which are open subsets in $\mathbb{P}^n_S$), one for each affine open set $U = \operatorname{Spec} A \subset S$.

Examples

(10.12) Let A be a ring and let $R = A[t]$ with the grading given by $\deg t = 1$ and $\deg a = 0$ for all $a \in A$. Then the structure map gives an isomorphism $\operatorname{Proj} R \simeq \operatorname{Spec} A$.

(10.13) (*The projective line $\mathbb{P}^1_A$ once more.*) Let us study the case of a polynomial ring in $R = A[s, t]$ where s and t have degree one and see that $\mathbb{P}^1_A = \operatorname{Proj} R$ ties up with the version of $\mathbb{P}^1_A$ as defined in Chapter 6 (in Section 6.3 on page 88); indeed, we shall see that the new $\mathbb{P}^1_A$ is glued together from affine schemes in precisely the same manner as is the old $\mathbb{P}^1_A$.

Note that $\operatorname{Proj} R$ is covered by the distinguished open sets $D_+(s)$ and $D_+(t)$ (since s and t generate the irrelevant ideal). Write for simplicity $U = D_+(s) \simeq \operatorname{Spec}(R_s)_0$ and $V = D_+(t) \simeq \operatorname{Spec}(R_t)_0$. Then $\operatorname{Proj} R$ is glued together from U, V along $U \cap V = D_+(st) \simeq \operatorname{Spec}(R_{st})_0$.

Note first that the degree zero part of $R_s \simeq A[s, s^{-1}, t]$ equals $A[s^{-1}t]$, and symmetrically it holds that $(R_t)_0 = A[st^{-1}]$. The intersection $D_+(st)$ is the degree zero part of R_{st} which is given as $(R_{st})_0 = A[s^{-1}t, st^{-1}]$. In other words, if we write $u = s^{-1}t$, it holds true that $(R_s)_0 = A[u]$, $(R_t)_0 = A[u^{-1}]$ and that $(R_{st})_0 = A[u, u^{-1}] = A[u]_u$. Hence $U \simeq \operatorname{Spec} A[u] = \mathbb{A}^1_A$ and $V \simeq \operatorname{Spec} A[u^{-1}] \simeq \mathbb{A}^1_A$, and they are patch together along $\operatorname{Spec} A[u, u_{-1}]$, exactly as in the gluing scheme used to construct the old $\mathbb{P}^1_A$ in Section 6.3.

(10.14) (*Projective n-space.*) In the same vein, we can show that $\mathbb{P}^n_A = \operatorname{Proj} R$ where $R = A[x_0, \ldots, x_n]$ coincides with the previous construction of $\mathbb{P}^n_A$ via gluing.

The case when $A = k$ is a field is the most interesting. In this case $\mathbb{P}^n_k$ is a scheme whose closed k-points $\mathbb{P}^n(k)$ coincides with the *variety* of projective n-space.

Since $\mathbb{P}^n_k$ is covered by $n+1$ copies of $\mathbb{A}^n_k$, $\mathbb{P}^n_k$ is reduced and it is also irreducible, since $\operatorname{Spec} R - V(R_+)$ is. Thus $\mathbb{P}^n_k$ is an integral scheme. Then, since $\mathbb{A}^n_k$ is a dense open subset, we have $k(\mathbb{P}^n_k) = k(\mathbb{A}^n_k) = k(X_1, \ldots, X_n)$. In particular, this has transcendence degree n over k, so that $\mathbb{P}^n_k$ has dimension n by Theorem 7.16. More intrinsically, we may also write

$$k(\mathbb{P}^n) = \left\{ \frac{g(x_0, \ldots, x_n)}{h(x_0. \ldots, x_n)} \,\middle|\, g, h \text{ homogeneous of the same degree} \right\}.$$

(10.15) (Proj *of a one dimensional ring.*) Let $R = k[x, y]/(xy)$ with the natural grading. Speaking geometrically, $\operatorname{Spec} R$ is the union of the x- and y-axes, and so $\operatorname{Spec} R - V(x, y)$ is the union of the axes with the origin excluded, and we expect $\operatorname{Proj} R$ to consist of two points. On the other hand, R has, apart from the irrelevant ideal $R_+ = (x, y)$, just two homogeneous prime ideals, (x) and (y), so $\operatorname{Proj} R$ indeed has just two points.

☆

Some basic properties of Proj R

A few fundamental properties are as follows:

PROPOSITION 10.16 (PROPERTIES OF Proj R) *Let R be a graded ring.*

i) Proj R *is separated;*

ii) *If R is Noetherian, then* Proj R *is Noetherian; in particular,* Proj R *is quasi-compact;*

iii) *If R is finitely generated over R_0, then* Proj R *is of finite type over* Spec R_0*;*

iv) *If R is an integral domain, then* Proj R *is integral.*

PROOF: We use the fact that X is covered by the affine open sets $D_+(f)$ where f runs over the homogeneous elements of R^+. These open sets are clearly affine (Proposition 10.8), and so is their intersection: $D_+(f) \cap D_+(g) = D_+(fg)$. Thus to prove that Proj R is separated, we need only check condition *ii)*, namely that $(R_f)_0 \otimes (R_g)_0 \to (R_{fg})_0$ is surjective for any $f, g \in R^+$, which it is.

The remaining properties can all be checked on an affine covering, although in *ii)* and *iii)* it must be finite. In our case Proj R is covered by the affines Spec$(R_f)_0$, which are Noetherian (respectively of finite type or integral) provided R is Noetherian (respectively finitely generated or an integral domain) and in both cases that R is Noetherian or of finite type over R_0, the irrelevant ideal is finitely generated, and so Proj R is covered by finitely many $D_+(f)$'s; (let f run through a set of generators of R_+). ❏

EXAMPLE 10.17 When the ring R is not Noetherian, it may very well happen that Proj R is not quasi-compact (!), which is in stark contrast with the case of affine schemes: the prime spectrum Spec A is always quasi-compact whatever the ring A is. For an explicit example we may take $R = k[x_1, x_2, \dots]$ to be a polynomial ring in infinitely many variables. Then $R_+ = (x_1, x_2, \dots)$, and Proj R is covered by the open sets $D_+(x_i)$, but there is clearly no finite sub-cover: if I is any finite subset of $\mathbb{N}$, the family $\{ D_+(x_i) \mid i \in I \}$ does not cover Proj R as its union does not contain the prime ideal $(x_i | i \in I)$. Compare this with the quasi-affine scheme Spec $R - \{R_+\}$, which neither is quasi-compact.

This situation is somewhat counterintuitive, given the usual heuristic that complex projective varieties (*i.e.* closed subsets of the compact space $\mathbb{CP}^n$) are compact, whereas affine varieties (*e.g.* $\mathbb{A}^n$ or $\mathbb{A}^1 - 0$) are not. The explanation is that the usual notions of 'compactness' do not behave so well in the Zariski topology; there are other notions like 'properness' which better capture the properties we want. ☆

Exercises

(10.5) Prove that for a graded ring R, and homogeneous elements $f, g \in R$, the natural map $(R_f)_0 \otimes (R_g)_0 \to (R_{fg})_0$ is surjective.

✲ (10.6) If R is an integral domain, show that the function field of $X = \operatorname{Proj} R$ is given as

$$k(X) = \left\{ \frac{g}{h} \,|\, g, h \text{ have the same degree} \right\} \subset K(R) \tag{10.2}$$

.

(10.7) Show that Proj R is empty if and only if every element in R_+ is nilpotent.

✲ (10.8) Give examples of a non-noetherian graded ring R such that $\operatorname{Proj} R$ is Noetherian, of R that is not of finite type over a field k, but $\operatorname{Proj} R$ is, and R which is not an integral domain, but whose projective spectrum $\operatorname{Proj} R$ is integral. HINT: The irrelevant ideal is irrelevant.

★

10.4 *Functoriality*

Unlike the case of affine schemes, the proj-construction is not entirly functorial. A graded ring homomorphism $\phi\colon R \to S$ does not induce a morphism between the projective spectra $\operatorname{Proj} S$ and $\operatorname{Proj} R$. The reason is that some primes in $\operatorname{Proj} S$ may pullback to R to contain the irrelevant ideal R_+. However, as we will see shortly, this is the only obstruction to defining a morphism, and discarding the badly behaved primes, we find and open set where a morphism can be defined.

Given a homomorphism $\phi\colon R \to S$ of graded rings we introduce the set $G(\phi) \subset \operatorname{Proj} S$ of homogeneous prime ideals $\mathfrak{p}$ in S that do not contain $\phi(R_+)$; or equivalently, they do not contain $\phi(R_+)S = R_+S$. In particular, these prime ideals have their inverse images $\phi^{-1}(\mathfrak{p})$ in $\operatorname{Proj} R$, and the assignment $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ sets up a map

$$F\colon G(\phi) \to \operatorname{Proj} R.$$

The set $G(\phi)$ is an open subset of $\operatorname{Proj} S$ being the complement of $V(R_+S)$ and has the canonical induced scheme structure as an open subscheme of $\operatorname{Proj} S$. Giving it that structure, we have:

PROPOSITION 10.18 *Let $\phi\colon R \to S$ be a homomorphism of graded rings. Then the map $F\colon G(\phi) \to \operatorname{Proj} R$ is a morphism of schemes.*

PROOF: First of all, the map F is continuous because the Zariski topologies on $\operatorname{Proj} R$ and $\operatorname{Proj} S$ are induced from those of $\operatorname{Spec} S$ and $\operatorname{Spec} R$, and F is the restriction of the map between the two Spec's induced by ϕ.

As usual, it suffices to check that the restriction of F will be a morphism on each distinguished open subset, and that these agree on intersections. To be precise, we consider distinguished open subsets $D_+(f)$ and $D_+(\phi(f))$; when f runs through R_+ the former cover $\operatorname{Proj} R$ and the latter $G(\phi)$. Note that $F^{-1}(D_+(f))$ equals $D_+(\phi(f))$ — which is contained in the set $G(\phi)$ because $G(\phi) = V(R_+S)^c$.

We rely on the canonical isomorphisms between $D_+(f)$ and $\operatorname{Spec}(R_f)_0$ and between $D_+(\phi(f))$ and $\operatorname{Spec}(S_{\phi(f)})_0$ established in Proposition 10.8. The natural map $(R_f)_0 \to (S_{\phi(f)})_0$ induced by ϕ gives a morphism $D_+(\phi(f)) \to D_+(f)$, whose underlying topological map clearly equals $F|_{D_+(\phi(f))}$, just follow a homogeneous prime ideal in S the two ways to $(R_f)_0$ around the diagram

$$\begin{tikzcd} (S_{\phi(f)}) \arrow[r, hook] & S_f \arrow[r, hook] & S \\ (R_f)_0 \arrow[u] \arrow[r, hook] & R_f \arrow[u] \arrow[r, hook] & R. \arrow[u] \end{tikzcd}$$

That these morphisms match up over intersections, is a matter of easy verification: indeed, the map $f\colon (R_{fg}) \to S_{\phi(fg)}$ induced by ϕ, equals the localization of both the maps $R_f \to S_{\phi(f)}$ and $R_g \to S_{\phi(g)}$ induced by ϕ. ❑

Example 10.19 (*Projection from a point.*) To illustrate why restriction to the open set $G(\phi)$ is necessary, we consider the case where $R = k[x_0, x_1]$, $S = k[x_0, x_1, x_2]$ and ϕ is the inclusion map. Note that the prime ideal $\mathfrak{a} = (x_0, x_1)$ defines an element in $\operatorname{Proj} S$, but its restriction to R is the whole irrelevant ideal of R. In fact, $G(\phi) = \operatorname{Proj} S - V(\mathfrak{a})$, and the map

$$\psi : \mathbb{P}^2_k - V(\mathfrak{a}) \to \mathbb{P}^1_k$$

is nothing but the projection from the point $(0 : 0 : 1)$ which sends a point with homogeneous coordinates $(x_0 : x_1 : x_2)$ to the one with coordinates $(x_0 : x_1)$. It is a good exercise to prove that there can be no morphisms $\mathbb{P}^m_k \to \mathbb{P}^n_k$ for $m > n$ in general. (See Section **??**). ☆

Example 10.20 Consider the map $\phi\colon k[u, v] \to k[x, y]$ of graded k-algebras defined by the two assignments $u \mapsto x^2$ and $v \mapsto y^2$. The exceptional set $G(\phi)$ is empty, because if a prime ideal $\mathfrak{p} \subseteq k[x, y]$ contain s $(u, v)k[x, y] = (x^2, y^2)$, it will certainly contain (x, y). Hence the map ϕ gives rise to a morphism $\mathbb{P}^1_k \to \mathbb{P}^1_k$. Its action on k-points is $(a_0 : a_1) \mapsto (a_0^2 : a_1^2)$. ☆

✲ Exercise 10.9 Let A be a ring and let $R = A[x_0, \dots, x_n]$. Define a morphism of *schemes*

$$\pi : \mathbb{A}^{n+1}_A - V(R_+) \to \mathbb{P}^n_A$$

that generalizes the usual quotient construction of $\mathbb{CP}^n$. ★

Exercise 10.10 In this exercise A denotes a ring. Consider the homomorphism of graded rings $\phi\colon A[x_0, x_1, x_2] \to A[x_0, x_1, x_2]$ defined by the three assignments $x_i \to x_j x_k$ where the indices satisfy $\{i, j, k\} = \{1, 2, 3\}$. Determine the open set $G(\phi)$ in the two cases

a) $A = k$ is a field;

b) A is the ring of integers.

★

Closed immersions

A large number of examples of morphisms between projective spectra as constructed above are the ones associated with graded quotient homomorphism $\phi\colon R \to R/\mathfrak{a}$, where $\mathfrak{a} \subset R$ is a homogeneous ideal. In this case $\phi(R_+) = (R/\mathfrak{a})_+$ so $G(\phi)$ is the entire spectrum $\operatorname{Proj} R/\mathfrak{a}$, and the corresponding map ι is defined everywhere. Thus we obtain a morphism

$$\iota\colon \operatorname{Proj} R/\mathfrak{a} \to \operatorname{Proj} R,$$

whose image is $V(\mathfrak{a})$. We contend that ι is a closed immersion, and notice that one may verify this on an open cover of Proj R. So let $f \in R$ be a homogeneous element. We know that $\iota^{-1}(D_+(f)) = D_+(\phi(f))$, and that the restriction of ι to $\iota^{-1}(D_+(f))$ may be identified with the morphism

$$\operatorname{Spec}((R/\mathfrak{a})_{\phi(f)})_0 \to \operatorname{Spec}(R_f)_0$$

induced by the degree zero part of the localization $R_f \to (R/\mathfrak{a})_f$ of ϕ. But this is obviously surjective, hence $\iota|_{\iota^{-1}(D_+(f))}$ is a closed immersion.

We will prove in Chapter 15 that, in fact, any closed immersion arises in this way, under some mild assumptions on R.

Example 10.21 (*Homogeneous coordinates.)* The simplest conceivable closed immersion is that of a closed point in $\mathbb{P}^n_k$. At least if k is algebraically closed, such points a are given by their *homogeneous coordinates* $a = (a_0 : \cdots : a_n)$, the submaximal ideal corresponding to a is generated by the minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_n \\ a_0 & a_1 & \dots & a_n \end{pmatrix}. \tag{10.3}$$

Indeed, the vanishing of those minors describes vectors in k^{n+1} dependent on $(a_0, \dots, a_n)$; or in other words, points lying on the line through $(a_0, \dots, a_n)$.

There is an analogue of this for projective spaces $\mathbb{P}^n_A$ over an arbitrary ring A that to an n-tuple $a = (a_0, \dots, a_n)$ of elements from A gives an A-point of $\mathbb{P}^n_A$; that is, a section of the structure map $\pi\colon \mathbb{P}^n_A \to \operatorname{Spec} A$, which we denote $= (a_0 : \cdots : a_n)$. The appropriate necessary condition on the a_i's (generalizing the condition that not all a_i be zero) is that the a_i's generate the unit ideal in A. Moreover, two such tuples give the same section if and only if they are proportional by a unit from A.

Let $\mathfrak{a}$ be the ideal in $A[x_0, \dots, x_n]$ generated by the minors of the matrix (10.3); in other words

$$\mathfrak{a} = (a_i x_j - a_j x_i \,|0 \leqslant i, j \leqslant n).$$

We claim that π induces an isomorphism between $V(\mathfrak{a})$ and Spec A; its inverse will then be a closed embedding $\iota_a\colon \operatorname{Spec} A \to \mathbb{P}^n_A$. The open distinguished sets $D(a_i)$ cover Spec A, and it will suffice to see that the restriction $\pi|_{\pi^{-1}(D(a_i))}\colon V(\mathfrak{a}) \cap \pi^{-1}(D(a_i)) \to D(a_i)$ is an isomorphism for each i. So replacing Spec A by $D(a_i)$, we may well assume that one of the a_i's, say a_0, is invertible. Since $a_0 x_i - a_i x_0$ belongs to $\mathfrak{a}$, we deduce that $x_i - a_i a_0^{-1} x_0 \in \mathfrak{a}$, and hence $A[x_0, \dots, x_n]/\mathfrak{a} = A[x_0]$. By Example 10.12, it follows that the structure map restricts to an isomorphism on $V(\mathfrak{a})$. Clearly a simultaneous scaling does not change $a_i a_0^{-1}$, and if $a_i a_0^{-1} = a_i' a_0'^{-1}$, it holds that $a_i' = a_0' a_0^{-1} a_i$.

It is not true in general that all maps Spec A to $\mathbb{P}^n$ are of the 'homogeneous coordinate form' $(a_1 : \cdots : a_n)$, but locally near each point they are, so in particular if A is a local ring (*e.g.* a field) it holds true. Later we shall give a general description of morphisms into projective spaces.

Lemma 10.22 *Let A be a ring and $x \in \operatorname{Spec} A$ a closed point. Assume that $\sigma\colon \operatorname{Spec} A \to \mathbb{P}^n_A$ is a section of the structure map.*

i) *There is an open affine* $\operatorname{Spec} A' \subseteq \operatorname{Spec} A$ *and elements* $a_i \in A'$ *with at least one* a_i *a unit such that* $\sigma|_{\operatorname{Spec} A'} = (a_0 : \cdots : a_n)$. *Another such tuple* $(a'_0 : \cdots : a'_n)$ *is the same map if and only if* $a'_i = \alpha a_i$ *for a unit* $\alpha \in A'$.

ii) *In particular, if A is a local ring every section* σ *of the structure map is of the form* $(a_0 : \cdots : a_n)$ *for some tuple* $a = (a_0, \ldots, a_n)$.

One must remember that the lemma is relative to a *fixed* sequence of variables $x_0, \ldots, x_n$.

PROOF: Assume that a section $\sigma\colon \operatorname{Spec} A \to \mathbb{P}^n_A$ of the structure map is given. Then the image of the point x lies in $D_+(x_\nu)$ for some ν. We let $\operatorname{Spec} A'$ be an open affine neighbourhood of x contained in $\sigma^{-1}(D_+(x_\nu))$. Then the restriction $\sigma|_{\operatorname{Spec} A'}$ factors through $D_+(x_\nu)$.

This means that $\sigma^\sharp$ is a map from $A[x_\nu x_\nu^{-1}, \ldots, x_n x_\nu^{-1}]$ to A'; the images $a_i = \sigma^\sharp(x_i x_\nu^{-1})$ are elements in A' and $(a_0 : \cdots : 1 : \cdots : a_n)$ will be the appropriate homogeneous coordinates giving the map $\sigma|_{\operatorname{Spec} A'}$ (where the 'one' is in the ν-th slot); indeed, with $\mathfrak{a}$ as above, the section σ factors through $V(\mathfrak{a})$, and as the structure map of $V(\mathfrak{a})$ is an isomorphism, f will be an isomorphism onto $V(\mathfrak{a})$

Strictly speaking, the section $(a_0 : \cdots : a_n)$ *will be a map into* $\mathbb{P}^n_{A'}$*, but this is an open subset of* $\mathbb{P}^n_A$

The second statement follows, since in that case already σ factors through $D_+(x_\nu)$. ❑

☆

10.5 Projective schemes

Let S be a scheme and let X be a scheme over S. We say that X is *projective* over S, or that the structure morphism $f\colon X \to S$ is *projective*, if $f\colon X \to S$ factors as $f = \pi \circ \iota$ where $\iota\colon X \to \mathbb{P}^n_S$ is a closed immersion and $\pi\colon \mathbb{P}^n_S \to S$ is the projection. X is *quasi-projective* over S if $X \to S$ factors via an open immersion $X \to \overline{X}$ and a projective S-morphism $\overline{X} \to S$.[1]

Projective morphisms

Quasi-projective morphisms

The primary examples is of course $X = \mathbb{P}^n_A \to \operatorname{Spec} A$ for a ring A. More generally, if $X = \operatorname{Proj} R$ where R is a graded R_0-algebra generated in degree one by finitely many elements and $S = \operatorname{Spec} R_0$, then X is projective over S, since in this case, we can define the projective immersion ι by taking a surjection $R_0[x_0, \ldots, x_n] \to R$, which upon taking Proj, gives a closed immersion $X \to \mathbb{P}^n_{R_0}$.

Note that projectivity is a relative notion: it is the morphism $X \to S$ which is projective, not X itself. For instance, $\mathbb{P}^1_{k[t]}$ is projective over $\operatorname{Spec} k[t]$, but it is not over $\operatorname{Spec} k$. Intuitively, it is the fibres of $X \to S$ which are projective; in the example, the (scheme-theoretic) fibre over $s \in S = \operatorname{Spec} k[t]$ equals the projective line $\mathbb{P}^1_{k(s)}$ over $k(s)$. Still, if we are working in the category of schemes over, say, a field k or $\mathbb{Z}$, we still refer to a scheme X being 'projective' when it is projective over the base scheme.

EXAMPLE 10.23 For $A = \mathbb{C}[t]$, the scheme $X = \operatorname{Proj} A[x,y,z]/(zy^2 - x^3 - txz^2)$ is projective over $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} A$. The fiber of $X \to \mathbb{A}^1_{\mathbb{C}}$ over any closed point $a \in \mathbb{A}^1_{\mathbb{C}}$ is an integral projective subscheme of dimension one: $V(zy^2 - x^3 - axz^2) \subset \mathbb{P}^2_{\mathbb{C}}$. ☆

[1] There exist slightly different definitions in the literature, see `https://stacks.math.columbia.edu/tag/01VW`

10.6 The Veronese embedding

Let R be a graded ring and let d be a positive integer. In Example 10.2 we defined the Veronese ring $R^{(d)}$ associated to a graded ring R as $\bigoplus_n R_{dn}$. I this section we aim at showing that the inclusion $\phi\colon R^{(d)} \to R$ induces an isomorphism

$$v_d\colon \operatorname{Proj} R \to \operatorname{Proj} R^{(d)}.$$

The first step will naturally be to show that v_d is a morphism. This is true because in this case $G(\phi) = \operatorname{Proj} R$ since any prime $\mathfrak{p}$ such that $\mathfrak{p} \supseteq R_+ \cap R^{(d)}$ must also contain all of R_+; indeed, if $a \in R_+$, note that $a^d \in R_+ \cap R^{(d)}$ and so $a \in \mathfrak{p}$ as well! The map v_d is called the *Veronese embedding*, or the *d-uple embedding* of $\operatorname{Proj} R$.

The Veronese embedding

Proposition 10.24 *The Veronese embedding v_d is an isomorphism.*

Proof: There are many things to check here, so we will sketch the proof, and leave the remaining verifications for the reader.

First we note that v_d is injective: if $\mathfrak{p}, \mathfrak{q} \in \operatorname{Proj} R$ are two prime ideals such that $\mathfrak{p} \cap R^{(d)} = \mathfrak{q} \cap R^{(d)}$, then for a homogeneous element $x \in R$ it holds true that

$$x \in \mathfrak{p} \Leftrightarrow x^d \in \mathfrak{p} \Leftrightarrow x^d \in \mathfrak{q}, \Leftrightarrow x \in \mathfrak{q}$$

and hence $\mathfrak{p} = \mathfrak{q}$.

To show that v_d is surjective, let $\mathfrak{q} \in \operatorname{Proj} R^{(d)}$ and define the homogeneous ideal in R by

$$\mathfrak{p} = \bigoplus_{n=0}^{\infty} \left\{ x \in R_n | x^d \in \mathfrak{q} \right\}.$$

It is easy to see that multiplication by homogenous elements from R leaves $\mathfrak{p}$ invariant, and that if a product of homogeneous elements lies in $\mathfrak{p}$, one of the factors does. Using the little trick from Proposition 10.8 considering $(x+y)^{n+m}$ with x and y homogeneous of degree n and m, one infers it is additively closed as well. So $\mathfrak{p}$ is a prime ideal. That $\mathfrak{p} \cap R^{(d)} = \mathfrak{q}$ follows immediately from $\mathfrak{p}$ being prime. So we conclude that v_d is bijective.

The proof then proceeds to show that the maps v_d and its inverse are open, hence they are homeomorphisms. Now, $f^d \in R^{(d)}$ for each homogeneous $f \in R_+$, so for each prime $\mathfrak{p} \in \operatorname{Proj} R$ it holds true that $f \in \mathfrak{p}$ if and only if $f^d \in \mathfrak{p} \cap R^{(d)}$. This shows that v_d maps $D_+(f)$ bijectively onto $D_+(f^d)$. For the different f, the distinguished open subsets $D_+(f)$ cover $\operatorname{Proj} R$, and the $D_+(f^d)$ cover $\operatorname{Proj} R^{(d)}$, and it follows that v_d is a homeomorphism.

Finally, one checks that v_d induces isomorphisms between the two distinguished open sets $D_+(f)$ and $D_+(f^d)$; this is a consequence of the identification in Proposition 10.8 on page 145 and the undemanding equality $(R^{(d)}_{f^d})_0 = (R_f)_0$. ❑

Example 10.25 (*Classic Veronese varieties.*) These varieties are named after the Italian Mathematician Guiseppe Veronese, one of the founders of the algebraic geometry of higher dimensional varieties. He considered maps $\mathbb{P}^n \to \mathbb{P}^N$ given by a basis for the homogenous part of the polynomial ring $R = k[x_0, \ldots, x_n]$ of degree d (so N is the dimension of that space). For instance, the map $\mathbb{P}^2 \to \mathbb{P}^5$ that sends a point with homogeneous coordinates $(x_0 : \cdots : x_2)$ to $(x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2)$ is one of the sort whose image is the famous Veronese surface.

In Proj-terminology a Veronese embedding corresponds to the ring homomorphism $k[t_0, \ldots, t_N] \to k[x_0, \ldots, x_n] = R$ that sends each variable t_i to a corresponding basis elements. The image of this map is precisely the Veronese ring $R^{(d)}$, and thus it induces, according to Section 10.4 on page 152, a closed immersion of $\operatorname{Proj} R^d$ into $\operatorname{Proj} R$. As a footnote, let us note that this explains the *a priori* mysterious qualifier 'embedding' in the name 'Veronese embedding' above. ☆

REMARK ON RINGS GENERATED IN DEGREE ONE We will frequently assume that the ring R is *generated in degree one*, that is, R is generated as an R_0-algebra by R_1. The reason for this will become clear in the next section. Intuitively, it is because we want $\operatorname{Proj} R$ to be covered by the 'affine coordinate charts' $D_+(x)$ where x should have degree 1.

We remark that this assumption is in fact not very restrictive: Any projective spectrum of a finitely generated ring is isomorphic to the Proj of a ring generated in degree 1. This is because of the basic algebraic fact that if R is finitely generated, then some subring $R^{(d)}$ will have all of its generators in one degree, and since $\operatorname{Proj} R^{(d)} \simeq \operatorname{Proj} R$, we don't change the Proj by replacing R with $R^{(d)}$.

EXERCISE 10.11 Let x and y be two points in $\mathbb{P}^n_k$. Prove there is an open affine $U \subseteq \mathbb{P}^n_k$ containing both x and y. ★

10.7 *The Segre embedding*

Recall that for affine schemes $X = \operatorname{Spec} B, Y = \operatorname{Spec} C$ over $S = \operatorname{Spec} A$, the fibre product $X \times_S Y$ was defined as $\operatorname{Spec}(B \otimes_A C)$. There is a similar statement for Proj:

THEOREM 10.26 *Let R, R' be graded rings with $R_0 = R'_0 = A$. Let $S = \bigoplus_{n \geqslant 0}(R_n \otimes R'_n)$. Then*

$$\operatorname{Proj} S \simeq \operatorname{Proj} R \times_A \operatorname{Proj} R'.$$

EXERCISE 10.12 Prove Theorem 10.26. Hint: Prove that $S_{f \otimes g} \simeq (R_f)_0 \otimes_A (R'_g)_0$ for $f \in R$ and $g \in R'$. Then compare both sides by gluing together fibre products over distinguished open sets. ★

COROLLARY 10.27 *Let A be a ring and let $m, n \geqslant 1$ be integers. Then there is a closed immersion*

$$\sigma_{m,n} : \mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}$$

PROOF: Consider the A-algebra $S = \bigoplus_{n \geqslant 0}(R_n \otimes R'_n)$ above, where $R = A[x_0, \ldots, x_m]$ and $R' = A[y_0, \ldots, y_n]$ are the polynomial rings. Consider the following morphism of graded

A-algebras.

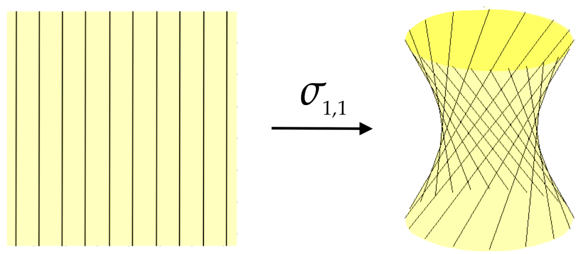
$$A[z_{ij}]_{0\leqslant i\leqslant m, 0\leqslant j\leqslant n} \to A[x_0,\ldots,x_m]\otimes A[y_0,\ldots,y_n]$$
$$z_{ij} \mapsto x_i \otimes y_j.$$

It is clear that S is generated as an $R_0\otimes R'_0$-algebra by the products $x_i\otimes y_j$, so the map is surjective and thus we get the desired closed immersion. ❏

Example 10.28 Let $R = k[x_0, x_1]$, $R' = k[y_0, y_1]$. Then $u_{ij} = x_i y_i$ defines an isomorphism

$$S = \bigoplus_{n\geqslant 0}(R_n\otimes R'_n) \to k[u_{00}, u_{01}, u_{10}, u_{11}]/(u_{00}u_{11} - u_{01}u_{10}).$$

This recovers the usual embedding of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ as a quadric surface in $\mathbb{P}^3_k$.



A smooth quadric surface

☆

10.8 More intricate examples

The construction of the projective spectra is definitely more subtle than the construction of the prime spectra. As we allude to in the introductory remark, one of the vast differences is that unlike the harmonious relation between rings and prime spectra, which is one–to–one, a huge number of different graded rings give rise to isomorphic Proj's. Already the Veronese embeddings furnish an infinity of examples: for any natural number d, the two schemes $\operatorname{Proj} R$ and $\operatorname{Proj} R^{(d)}$ are isomorphic (see Section 10.6). The first following example gives examples of a different kind. Our second example illustrates the versatility of the Proj-construction and its usefulness to making 'quotient by k^*-actions' other than 'schemy-versions' of the classical one from projective geometry, a taste of the so-called weighted projective spaces.

Example 10.29 (*The weigthed projective spaces $\mathbb{P}(p,q)$.)* Let k be a field and p and q two relatively prime natural numbers and let $d = pq$. Consider the polynomial ring $R = k[x, y]$, but endow it with the non-standard grading giving x degree p and y degree q. We claim that $\operatorname{Proj} R \simeq \mathbb{P}^1_k$, or more specifically that $R^{(d)}$ is isomorphic to the polynomial ring $A = k[u, v]$ graded in the non-standard but innocuous way that $\deg u = \deg v = d$. Clearly $\operatorname{Proj} A \simeq \mathbb{P}^1_k$.

Observe that a homogeneous element in $R^{(d)}$ is a linear combination of monomials $x^\alpha y^\beta$ with $p\alpha + q\beta = dn$; hence q divides α and p divides β and so $\alpha' + \beta' = n$ with $\alpha = q\alpha'$

and $\beta = p\beta'$. There is a homomorphism of graded k-algebras $A = k[u, v] \to R^{(d)}$ that sends $u \to x^q$ and $v \to y^p$. This is injective since x^q and y^p are algebraically independent when x and y are, so to see it is an isomorphism, it will suffice to check it is surjective on each homogeneous component: now, as we just saw, $(R^{(d)})_{dn}$ has a basis consisting of the monomials $x^{q\alpha}y^{p\beta}$ with $\alpha + \beta = n$; and for the same α's and β's the monomials $u^\alpha v^\beta$ form a basis for A_n. ☆

EXAMPLE 10.30 (*The weighted projective space* $\mathbb{P}(1, 1, p)$.) This is another example along the same lines. Again we begin with a polynomial ring $R = k[x, y, z]$ endowed with a slightly exotic grading; we put $\deg x = \deg y = 1$ and $\deg z = p$ for some natural number p. Then $\operatorname{Proj} k[x, y, z]$ is a so-called *weighted projective space* and one often sees it denoted by $\mathbb{P}(1, 1, p)$.

As the reader might guess, both this example and the previous one are special cases of the general construct $\mathbb{P}(p_1, \ldots, p_r) = \operatorname{Proj} k[x_1, \ldots, x_r]$ where we give $k[x_1, \ldots, x_r]$ a grading by setting $\deg x_i = p_i$.

The scheme $X = \operatorname{Proj} R$ has a covering of the three open affines $D_+(x)$, $D_+(y)$ and $D_+(z)$. Both $D_+(x)$ and $D_+(y)$ are isomorphic to $\mathbb{A}^2_k$; it is a straightforward exercise to verify that $(R_x)_0 = k[yx^{-1}, zx^{-p}]$ and $(R_y)_0 = k[xy^{-1}, zy^{-p}]$, and that these are polynomial rings. However, the third distinguished open affine $D_+(z)$ is not isomorphic to $\mathbb{A}^2_k$. In fact, it has a singularity! Clearly $x^{p-i}y^i z^{-1}$, for $0 \leqslant i \leqslant p$, are homogeneous elements of degree zero in $(R_z)_0$, and it is almost trivial that they generate $(R_z)_0$, so that $(R_z)_0 = k[x^p z^{-1}, \ldots, y^p z^{-1}]$. One recognizes this ring as an isomorphic copy of the p^{th} Veronese ring $A^{(p)}$ of the polynomial ring $A = k[u, v]$. And anticipating parts of the story, this is the cone over a so-called *projective normal curve* of degree p, whose apex is a singular point. ☆

EXAMPLE 10.31 (*The Blow-up as a Proj.*) Consider the ring $A = k[x, y]$ and the ideal $I = (x, y)$. We can form a new graded ring by introducing a new formal variable t and setting

$$R = \bigoplus_{k \geqslant 0} I^k t^k$$

where $I^0 = A$. In R, the new variable t has degree 1, and the other variables x and y have degree 0. One may think about R as the subring of $A[t]$ of polynomials shaped like $\sum_\nu a_\nu t^\nu$ where the coefficient a_ν belongs to I^ν.

The map $\mathfrak{p} \mapsto \mathfrak{p} \cap A$, induces a morphism

$$\pi : \operatorname{Proj} R \to \operatorname{Spec} A = \mathbb{A}^2_k.$$

The irrelevant ideal R_+ is generated by xt and yt so that $\operatorname{Proj} R$ is glued together by the two open affine subschemes $\operatorname{Spec}(R_{xt})_0$ and $\operatorname{Spec}(R_{yt})_0$.

Note that there is a map of graded rings $\phi \colon A[u, v] \to R$, where both u and v are of degree one, which is given by the assignments

$$\begin{aligned} u &\mapsto xt \\ v &\mapsto yt. \end{aligned}$$

This is surjective since I is generated by x and y. Note also that the kernel contains the element $xv - yu$. In fact, by Exercise 10.13 below, we have

LEMMA 10.32 $R \simeq A[u,v]/(xv - yu)$.

From this description we see that $\operatorname{Proj} R$ is covered by the two distinguished open sets $D_+(u) = \operatorname{Spec}(R_v)_0$ and $D_+(v) = \operatorname{Spec}(R_v)_0$. Here

$$(R_u)_0 \simeq (A[u,v]_u/(xv - yu))_0 = k[x, vu^{-1}]$$

and

$$(R_v)_0 \simeq (A[u,v]_v/(xv - yu))_0 = k[y, uv^{-1}].$$

These are glued along $\operatorname{Spec}(R_{uv})_0 \simeq (A[u,v]_{uv}/(xv - yu))_0$, and one finds

$$(A[u,v]_{uv}/(xv - yu))_0 = k[x, y, uv^{-1}, vu^{-1}]/(x \cdot vu^{-1} - y) \simeq k[x, uv^{-1}, vu^{-1}]$$

In particular, we see that $\operatorname{Proj} R$ coincides with the previous blow-up construction. ☆

✲ EXERCISE 10.13 Prove Lemma 10.32. ★

✲ EXERCISE 10.14 *(The weighted projective space $\mathbb{P}(1,1,p)$.)* Let R be as in the Example 10.30 above, and let $A = k[x,y,w]$ with the usual grading. Furthermore, let $\alpha \colon R \to A$ be the homomorphism that sends z to w^p and neither touches x nor y. Show that α is homogeneous of degree zero and induces a morphism $\pi \colon \mathbb{P}^2_k \to X$. Describe the fibres of π over closed points in case k is algebraically closed.

★

EXERCISE 10.15 Let $R = k[x,y,z]$ be the polynomial ring given the grading $\deg x = 1$, $\deg y = 2$ and $\deg z = 3$, and let $X = \operatorname{Proj} R$ (also denoted $\mathbb{P}(1,2,3)$). The aim of the exercise is to describe the three covering distinguished subschemes $D_+(x)$, $D_+(y)$ and $D_+(z)$.

a) Show that $(R_x)_0 = k[yx^{-2}, zx^{-3}]$ and that $D_+(x) \simeq \mathbb{A}^2_k$.
b) Show that $(R_y)_0 \simeq k[x^2y^{-1}, z^2y^{-6}, xzy^{-2}]$. Show that the graded algebra homomorphism $k[u,v,w] \to (R_y)_0$ given by the assignments $x \mapsto yx^{-2}$, $v \mapsto z^2y^{-6}$ and $w \mapsto xzy^{-2}$ induces an isomorphism $k[u,v,w]/(w^2 - uv) \simeq (R_y)_0$. Hence $D_+(y)$ is a hypersurface in $\mathbb{A}^3_k$; the so-called 'cone over a quadric'. Show it is not isomorphic to $\mathbb{A}^2_k$ (check the local ring at the origin).
c) Show that $R_z = k[x^3z^{-1}, y^3z^{-2}, xyz^{-1}]$ and that the map $k[u,v,w] \to (R_z)_0$ defined by the assignments $x \mapsto x^3z^{-1}$, $v \mapsto y^3z^{-2}$ and $w \mapsto xyz^{-1}$ induces an isomorphism $k[u,v,w]/(w^3 - uv) \simeq (R_z)_0$. Show that it is not isomorphic to $\mathbb{A}^2$.
d) Show that the map $R_+ \to k[U,V,W]$ sending $x \mapsto U$, $y \to V^2$ and $z \mapsto W^3$ induces a map $\mathbb{P}^2_k \to \operatorname{Proj} R$, and describe the fibres over closed points.

★

Chapter 11

Schemes of finite type over a field

With the notion of separatedness we can finally state the definition of a variety: *affine variety*

Affine varieties

Definition 11.1 *Let k be a field. An affine variety over k is a scheme isomorphic to* Spec A, *where A is an integral domain of finite type over k.*

Example 11.2 The schemes

$$\mathbb{A}^1_{\overline{\mathbb{Q}}} = \operatorname{Spec} \overline{\mathbb{Q}}[t], \qquad \operatorname{Spec} \mathbb{C}[x,y]/(x^2 - y^3), \qquad \operatorname{Spec} \overline{\mathbb{F}}_p[x,y,z]/(x^2 - yz)$$

are affine varieties, whereas the following schemes are not:

$$\operatorname{Spec} \overline{\mathbb{Q}}[t]/t^2, \qquad \operatorname{Spec} \mathbb{C}[x,y]/(xy), \qquad \operatorname{Spec} \mathbb{Z}.$$

☆

Definition 11.3 *Let k be a field.*
A prevariety *over k is an irreducible scheme of finite type over k which has a finite affine covering consisting of affine varieties.*
A variety *X over k (or k-variety) is a k-scheme which is separated over k. That is, a k-scheme such that*

i) *X is integral*
ii) *$X \to \operatorname{Spec} k$ is separated*
iii) *$X \to \operatorname{Spec} k$ is of finite type.*

Prevarieties

The terminology one finds in the literature at this point, is varying; some authors do not require varieties to be irreducible (but they are always reduced), and many require the base field k to be algebraically closed.

11.1 Abstract varieties

We have already said that schemes are generalizations of algebraic varieties, but on the other hand, we have also seen that even the simplest schemes, *e.g.* $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, behave differently than varieties in the sense that they usually have many non-closed points. Thus for this generalization to make sense, we should expect there to be a canonical way to 'add non-closed points' to an algebraic variety so that the resulting topological space has the structure of a scheme. Let us explain what this means more precisely.

Let k be an algebraically closed field and let V be a variety over k. We first consider the case where V is affine. Each affine variety has a coordinate ring $A = A(V)$; it is canonically attached to V being the ring of regular functions on V. From $A(V)$, we can build $V^s = \operatorname{Spec} A$, which is an affine scheme whose closed points are in bijection with the points of V (that is, $V^s(k) = V$) according to the Nullstellensatz. Thus the 'new points' correspond to the non-maximal ideals of A; in fact, one way of thinking about this is that the scheme V^s in some sense is the collection of *all subvarieties* of the variety V.

Moreover, the fundamental theorem of affine varieties tells us that maps $\phi\colon V \to W$ between two affine varieties are in one-one-correspondence with k-algebra homomorphisms $\phi^\sharp\colon A(W) \to A(V)$, which exactly parallels our Theorem 5.6. Hence putting $\phi^s = \operatorname{Spec} \phi^\sharp$, we obtain a morphism $\phi^s\colon V^s \to W^s$ which extends ϕ. As $\phi^\sharp$ is a map of k-algebras, the morphism ϕ^s is a morphism of schemes over $\operatorname{Spec} k$.

Summing up, we have thus defined a functor $s\colon \mathsf{AffVar/k} \to \mathsf{Sch/k}$, where $\mathsf{AffVar/k}$ denotes the category of affine varieties over k. As morphisms of k-varieties $V \to W$ and affine k-schemes $V^s \to W^s$ are both in canonical bijection with k-algebra homomorphisms $A(W) \to A(V)$, the functor s is therefore *fully faithful*, in the sense that the assignment $\phi \mapsto \phi^s$ is a bijection

$$\operatorname{Hom}_{\mathsf{AffVar}/k}(V, W) \simeq \operatorname{Hom}_{\mathsf{Sch}/k}(V^s, W^s).$$

In the general case, a variety V has an open cover by affine varieties V_i, and gluing can be performed in both the category of varieties as well as in the categories of schemes, and it is a matter of straightforward checking that this gives a well-defined scheme V^s containing each V_i^s as an open subscheme. The gluing works equally well for morphisms, so we again obtain a functor, which we denote

$$s\colon \mathsf{Var}/k \to \mathsf{Sch}/k,$$

where now Var/k is the category of all varieties over k. Once again this functor is fully faithful, in the sense that the induced maps between $\operatorname{Hom}_{\mathsf{Var}/k}(V, W)$ and $\operatorname{Hom}_{\mathsf{Sch}/k}(V^s, W^s)$ are bijective. So two varieties give rise to isomorphic schemes over k if and only if they are isomorphic as varieties, and each scheme isomorphism is unambiguously determined by the variety isomorphism. In particular, this tells us that the category of varieties Var/k is equivalent to a full subcategory of Sch/k. We have already seen that s is far from being surjective, *e.g.* $\operatorname{Spec} k[x]/(x^2)$ does not come from a variety.

This definition should be compared with the definition from Chapter 5. There we defined a variety to be a scheme in the image of the functor

$$\mathsf{Var}/k \to \mathsf{Sch}/k,$$

which associates a k-variety V to a scheme V^s over k. As varieties satisfy the Hausdorff axiom, it is immediate that the corresponding scheme V^s is separated. Thus the two notions agree.

From now on a 'variety' will always refer to a scheme satisfying the conditions in Definition 11.3. Basically, any theorem from the 'classical setting' regarding varieties carries over to varieties in the new sense. This is justified by the following theorem:

Theorem 11.4 *The functor $V \to V^s$ is* fully faithful *and gives and equivalence between the category of varieties* Var/k *and the subcategory of* Sch/k *of schemes satisfying Definition 11.3.*

Dominant maps

We say that a morphism of schemes $f\colon X \to Y$ is *dominant* if the image of f is dense in Y. When X and Y is affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ the algebraic counterpart is that map $f^\sharp\colon B \to A$ is injective, at least when B is reduced:

Dominant morphism

That B is reduced is a necessary condition. The natural map $B \to B/\sqrt{0}$ induces a dominant map between the spectra (even a homeomorphism).

Lemma 11.5 *Let A and B be rings with B reduced. Then a morphism $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ has a dense image if and only if $f^\sharp\colon B \to A$ is injective.*

Proof: This is just *ii)* of Proposition 2.31 on page 38 bearing in mind that $\sqrt{0} = 0$ in B. ❑

Lemma 11.6 *Let $f\colon X \to Y$ be a morphism of schemes. Then the following are equivalent.*

i) *f is dominant;*

ii) *$f(U)$ is dense for one dense subset $U \subseteq X$;*

iii) *$f(U)$ is dense for all dense subsets $U \subseteq X$;*

If X is irreducible, they are also equivalent to

iv) *Y is irreducible and the generic point of X maps to the generic point of Y.*

Proof: The key point is that for any subset $U \subseteq X$ it holds true that $f(\overline{U}) \subseteq \overline{f(U)}$. For a dense subset U, if $f(X)$ is dense, $f(U)$ will be dense. Hence *i)* implies *ii)* and *iii)*. Assuming that $f(U)$ is dense, but $f(X)$ is not leads to the flagrant contradiction that the complement of $\overline{f(X)}$ which is a non-empty open subset, meets $f(U)$ (since $f(U)$ is dense). Thus *ii)* yields *i)*.

Finally, if X is irreducible, the generic point η is dense, so to prove *iv)* we just cite the equivalence of *i)* and *ii)* with $U = \{\eta\}$. ❑

The lemma is purely topological, the algebraic counterpart is the following

Proposition 11.7 *Let $: X \to Y$ be a morphism of schemes and assume that X is irreducible and Y is reduced.*

i) *f is dominant;*

ii) *For all affine open subsets $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, the ring map $f^\sharp\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective;*

iii) *For all pairs of affine open subsets $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, the ring map $f^\sharp\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective.*

Note that if X additionally is integral, then Y will be integral, and f induces a homomorphism $f^\sharp\colon k(Y) \to k(X)$ between the function fields.

11.2 Birational vs biregular geometry

Two varieties are said to be *birationally equivalent* if they they have isomorphic open subsets. This is a much weaker relation than being isomorphic; for instance, blowing up a point in $\mathbb{P}^2_k$ yields a variety which is birationally equivalent with but not isomorphic to $\mathbb{P}^2_k$.

Birationally equivalent varieties

Let us be precise about what a rational map from X to Y is. Heuristically, just like rational functions, it is a morphism $U \to Y$ where U is an open non-empty subset of X. To avoid the ambiguity in the domain of definition U, one introduces an equivalence relation between such pairs (U, f), and says that two pairs (U, f) and (U', f') are equivalent if $f|_{U\cap U'} = f'|_{U\cap U'}$. A *rational map* is then an equivalence class of such pairs. However, in each class there is a preferred member for which the open set U is maximal, and this is another way of resolving the ambiguity. A rational map is denoted with a dashed arrow $f\colon X \dashrightarrow Y$ (with the set of definition tacitly understood).

Rational map between varieties

EXERCISE 11.1 Show that there is a maximal open set where a rational map is defined. HINT: The map can be extended on the union of all the U's appearing in the pairs (U, f) of the equivalence class. ✯

One says that a rational map f is *dominant* if $f(U)$ is dense in Y where U is some open set where f is defined (if true for one U, it holds for all). Let $g\colon Y \dashrightarrow Z$ be another rational map say defined on $V \subseteq Y$. The open set $f^{-1}(V)$ is non-empty since $f(U)$ being dense entails that $f(U) \cap V \neq \emptyset$, and on $f^{-1}(V)$ the composition $g \circ f$ is defined. We conclude that dominant rational maps can be composed, and so the varieties over k together with the dominant rational maps form a category Rat_k.

Dominant rational map

A map dominant rational map $f\colon X \dashrightarrow Y$ is *birational* if it is an isomorphism in Rat_k; or in clear text, if there is dominant rational map $g\colon Y \dashrightarrow X$ so that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. One says that X and Y are *birationally equivalent* if there is birational map between them.

Birational maps

Birationally equivalent varieties

EXAMPLE 11.8 Sending $(u_0 : u_1 : u_2)$ to $(u_1u_2 : u_0u_2 : u_0u_1)$ is a rational map from $\mathbb{P}^2_k$ to $\mathbb{P}^2_k$ defined away from the three coordinate points $(0:1:0)$, $(1:0:1)$ and $(1:1:0)$. It is birational with itself as inverse. ☆

EXAMPLE 11.9 Sending $(u_0 : u_1) \times (v_0 : v_1)$ to $(u_0v_0 : u_1v_0 : u_1v_1)$ is a rational map $\mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$. Defined away from $(1:0) \times (0:1)$. It is also birational with $(t_0 : t_1 : t_2) \mapsto (t_0 : t_1) \times (t_1 : t_2)$ as inverse; this map is defined away from $(0:0:1)$ and $(1:0:0)$. ☆

A fundamental truth is that the study of of dominant rational maps, basically is reduced to the study of extensions of function fields:

THEOREM 11.10 *Let X and Y be two varieties over k. Then there is a one-to-one correspondence between rational dominant maps $X \dashrightarrow Y$ and k-algebra homomorphisms $k(Y) \subseteq k(X)$. In particular, two varieties are birationally equivalent if and only if their function fields are isomorphic as k-algebra.*

We need a little lemma.

LEMMA 11.11 *Let A and B be two domains of finite type over a field k and denote their fraction fields by K and L respectively. Assume that $\phi\colon L \to K$ is a k-algebra homomorphism. Then there is some element $d \in A$ so that $\phi(B) \subseteq A_d$.*

PROOF: Let $b_1, \ldots, b_r$ generate B over k. Each $\phi(b_i)$ is of the form $\phi(b_i) = a_i/c_i$ with $a_i, c_i \in A$. Then $d = c_1 \ldots c_r$ does the job. ❑

PROOF OF THE THEOREM: Let $U = \operatorname{Spec} A \subseteq X$ and $V = \operatorname{Spec} B \subseteq Y$ be open affine subsets. Given a dominant rational map $f\colon X \dashrightarrow Y$, we may chose U and V so that U maps into V. In view of the comment after Proposition 11.7, we have the stalk map $f^\sharp\colon k(Y) \to k(X)$. Note on $B \subseteq k(Y)$ it induces the map $f^\sharp\colon B \to A$.

For the converse, if a k-homomorphism $\phi\colon L \to K$ is given, there is according to Lemma 11.11 an element $d \in A$ so that $\phi(B) \subseteq A_d$; then ϕ induces a morphism $\operatorname{Spec} A_d \to V \subseteq Y$ hence a rational map $X \dashrightarrow Y$. Evidently, B maps injectively into A_d so the morphism is dominant.

One leisurely verifies that the two assignments are mutually inverses (the key comment is that all maps between coordinate rings of affines are restrictions of $f^\sharp\colon k(Y) \to k(X)$) ❑

Associating X to the function field $k(X)$ defines a functor from the category Rat_k of varieties over k and dominant rational maps to the category of fields of finite type over k and k-homomorphism. Theorem 11.10 tells us that it is fully faithful; that is $\operatorname{Hom}_{\mathsf{Rat}_k}(X, Y) \simeq \operatorname{Hom}_{\mathsf{Alg}_k}(k(X), k(Y))$. In fact, as we shortly will see, it is also essentially surjective: every field K of finite type over k is of the form $k(X)$ for some variety X. So it makes the two categories 'essentially equivalent', but there is no natural functor that serves as the inverse functor—there is no good, systematic way to pick out one particular model for each field K.

A variety X so that $k(X) \simeq K$ is called a model for the field K.

THEOREM 11.12 (MAIN THEOREM OF BIRATIONAL GEOMTRY) *The assignment $X \to k(X)$ is fully faithful and essentially surjective functor between the following categories:*

i) The category of projective varieties and dominant rational maps;

ii) The category of finitely generated field extensions of k and k-algebra homomorphisms.

PROOF: Given a field K of finite type over k whose transcendence degree is d. Assume that $K = k(t_1, \ldots, t_r)$ and let $A = k[t_1, \ldots, t_r]$. Then $X = \operatorname{Spec} A$ is of dimension d since the Krull dimension of A equals the transcendence degree of its fraction field (xxx). To get a projective projective variety, embed X in some affine space $\mathbb{A}^r_k$ and close it up in $\mathbb{P}^n_k$. ❑

Note, to obtain a non singular model X for each field X is highly desirable, but extremely difficult. An illustrious result of Hironaka's is that it is true in characteristic zero, but in positive characteristic it is still un-known, except in low dimensions .

11.3 *Tangent spaces*

The tangent spaces of general schemes will necessarily be of a rather abstract nature, and as a motivation we first recall the concrete situation for affine varieties embedded in some affine space $\mathbb{A}_k^n$ over a field k.

Consider such an affine variety $X \subseteq \mathbb{A}_k^n$, say $X = V(I)$ where $I = (f_1, \dots, f_r)$. For a k-point $p \in X$, we define the *(embedded) tangent space* of X at p as the subspace of vectors $v = (v_1, \dots, v_n) \in k^n$ satisfying the linear equations

$$\sum_{i=1}^{n} \partial f_j / \partial x_i(p) \cdot v_i = 0 \text{ for } j = 1, \dots, r. \tag{11.1}$$

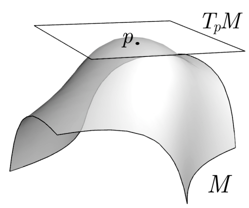
An equivalent definition, without reference to a specific generating set of I, is the following:

$$\sum_{i=1}^{n} \partial f / \partial x_i(p) \cdot v_i = 0 \text{ for all } f \in I. \tag{11.2}$$

The two definitions are equivalent, for if $f = \sum_j g_j f_j$, the chain rule gives

$$\partial f / \partial x_i(p) = \textstyle\sum_j f_j(p) \partial g_j / \partial x_i(p) + g_j(p) \partial f_j / \partial x_i(p) = \sum_i g_j(p) \partial f_j / \partial x_i(p),$$

where the last equality holds since the f_i's vanish at p. Vectors satisfying (11.1) therefore also satisfy (11.2), and the reverse implication is trivial. In particular the tangent space as defined in (11.1), is independent of the chosen set of generators for the ideal.



Note that T_pX by definition is a sub-k-vector space of k^n; it is the null space of the *Jacobian matrix*

Jacobian matrix

$$J(f_1, \dots, f_r) = \left(\partial f_i / \partial x_j(p)\right), \tag{11.3}$$

where $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n$. The dimension of T_pX is therefore given by

$$\dim T_pX = n - \operatorname{rank} J(f_1, \dots, f_r). \tag{11.4}$$

There is an intrinsic description of the tangent space T_pX, which does not rely on any specific embedding of X, and which will be the inspiration for the general definition.

Suppose for simplicity that $p = (0, \dots, 0)$ is the origin (we may always arrange this by a linear change of coordinates), and write $\mathfrak{M} = (x_1, \dots, x_n) \subset k[x_1, \dots, x_n]$ for the maximal ideal at p. For a polynomial $f \in k[x_1, \dots, x_n]$, we consider its *linearization at p*, given by

$$Df = \sum_{i=1}^{n} \partial f / \partial x_i(0) x_i.$$

This is just the linear part of the Taylor expansion at p. Note that the coordinates $x_1, \ldots, x_n$ give a basis for the dual space $(k^n)^\vee = \mathrm{Hom}_k(k^n, k)$. Hence we may view Df as a linear functional on k^n, and in this way we get a k-linear map

$$D \colon \mathfrak{M} \to (k^n)^\vee.$$

It is clear that D is surjective, since $D(x_i) = x_i$. A polynomial f lies in kernel of D precisely when all terms are of degree at least two, or phrased differently, the kernel of f equals $\mathfrak{M}^2$. Hence D induces an isomorphism of k-vector spaces

$$\mathfrak{M}/\mathfrak{M}^2 \simeq (k^n)^\vee.$$

Returning to the variety X and the tangent space T_pX, we take the dual of the inclusion $T_pX \subset k^n$, to obtain a surjection

$$(k^n)^\vee \to (T_pX)^\vee.$$

Concretely, this map is given by restricting a linear functional on k^n to the subspace T_pX. The composition

$$\theta : \mathfrak{M}/\mathfrak{M}^2 \to (k^n)^\vee \to (T_pX)^\vee$$

is also surjective.

We claim that $\mathrm{Ker}\,\theta = \mathfrak{M}^2 + I$. Indeed, note that $f \in \mathrm{Ker}\,\theta$ if and only if Df restricts to 0 on T_pX. This happens if and only if $Df = Dg$ for some $g \in I$ (since T_pX is the zero locus of Dg for all $g \in I$); that is, if and only if $f - g \in \mathrm{Ker}\, D = \mathfrak{M}^2$, or equivalently, $f \in \mathfrak{M}^2 + I$.

It follows that we have isomorphisms of k-vector spaces

$$(T_pX)^\vee \simeq \mathfrak{M}/(\mathfrak{M}^2 + I) \simeq \mathfrak{m}/\mathfrak{m}^2. \tag{11.5}$$

where $\mathfrak{m} \subset \mathcal{O}_{X,p}$ is the maximal ideal. Taking duals, we now have:

Proposition 11.13 *There is a natural isomorphism*

$$T_pX \simeq \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k). \tag{11.6}$$

the Zariski tangent space

Motivated by the above discussion for embedded affine varieties, we proceed to give the general definition of tangent spaces for schemes. It was introduced for varieties by Oscar Zariski in a fundamental paper ([?]) in 1947, and bears the name the *Zariski tangent space*.

Let X be a scheme and let $x \in X$ be a point. We consider the local ring $A = \mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m}_x$. The quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$ is in a natural way a vector space over the residue class field $k(x) = A/\mathfrak{m}_x$.

Definition 11.14 *The* Zariski tangent space *T_xX to X at x is the dual vector space of $\mathfrak{m}_x/\mathfrak{m}_x^2$. That is,*

$$T_xX = \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

The space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is called the Zariski cotangent space *of X at x. An element of T_xX is called a* tangent vector*; it is a linear functional $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k(x)$.*

The cotangent space is functorial in the following sense. Let $f\colon X \to Y$ be a morphism and let $y = f(x)$. Part of the structure of the morphism is a ring homomorphism $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. It localizes to a homomorphism of local rings $f^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ which takes the maximal ideal into the maximal ideal, and being a ring map, sends $\mathfrak{m}_y^2$ into $\mathfrak{m}_x^2$. Therefore it induces an additive map

$$f_x^\sharp\colon \mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2.$$

Moreover, for each morphism g composable with f one has

$$(g\circ f)_x^\sharp = f_x^\sharp \circ g_{f(x)}^\sharp$$

since $(g\circ f)^\sharp = f^\sharp \circ g^\sharp$. The map $f_x^\sharp$ is, however, just a map of $k(y)$-vector spaces. In general, there is no way to make $\mathfrak{m}_y/\mathfrak{m}_y^2$ a $k(x)$-vector space, and for this reason the tangent spaces are not functorial in general; the required duals will be with respect to different fields.

One exception is when X and Y are varieties over some field k, and x and y both are k-points. Then $k(x) = k(y) = k$, and we are permitted to take duals to get a map $df\colon T_xX \to T_yY$. Once the tangent-maps are defined, they behave functorially:

$$d(g\circ f)_x = dg_y \circ df_x$$

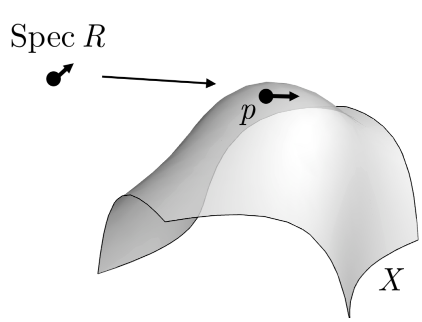
when $g\colon Y \to Z$ is a map of k-schemes and $g(y)$ is a k-point.*

* *Note that this only works if* $\dim_{k(y)} T_yY$ *is finite; this subtle point is another reason why the cotangent space* $\mathfrak{m}/\mathfrak{m}^2$ *is preferable to the tangent space.*

Zariski tangent spaces and the ring of dual numbers

The ring of dual numbers

Let k be a field. The ring $k[\epsilon]/(\epsilon^2)$ is called the *ring of dual numbers* over k. We shall with a slightly abusive notation write it as $k[\epsilon]$ tacitly understanding that $\epsilon^2 = 0$. The spectrum of $k[\epsilon]$ is a very simple scheme: its underlying topological space is a single point. However, the non-reduced structure on $\operatorname{Spec} k[\epsilon]$ shows that it is more interesting than $\operatorname{Spec} k$. We picture it as a point p with a vector 'sticking out of it'.



There are other interesting tiny algebras related to $k[\epsilon]$. If V any vector space over k, one may form the 'infinitesimal' k-algebra $D_V = k \oplus V$ where V is as a maximal ideal with square zero; that is, the multiplication is $(a+w)\cdot(b+v) = ab + (aw+bv)$. The pertinent property of D_V is that k-algebra homomorphisms $D_V \to k[\epsilon]$ correspond bijectively to linear functionals on V; in other words, there is an isomorphism

$$\mathrm{Hom}_{\mathsf{Alg}_k}(D_V, k[\epsilon]) \simeq \mathrm{Hom}_k(V, k).$$

Indeed, if $\alpha\colon D_V \to k[\epsilon]$ is given, the restriction $\alpha|_V$ is k-linear and takes values in $(\epsilon) = k$. For the inverse map, if $\alpha\colon V \to k$ is a given functional, the assignment $a + v \mapsto a + \alpha(v)\epsilon$ defines a k-algebra map.

PROPOSITION 11.15 *Let X be a scheme over k. To give a k-morphism* $\mathrm{Spec}(k[\epsilon]) \to X$ *is equivalent to giving a k-rational point $x \in X$ (meaning that $k(x) = k$), and an element of $T_x X$.*

PROOF: Fix a k-point x of X. Every map $\mathrm{Spec}\, k[\epsilon] \to X$ that sends p to x, must factor through each open affine neighbourhood of x, and we may as well assume that X is affine, say $X = \mathrm{Spec}\, A$. Let $\mathfrak{m} = \mathfrak{m}_x$. A homomorphism $\alpha\colon A \to k[\epsilon]$ corresponds to a morphism $\mathrm{Spec}\, k[\epsilon] \to X$ that sends p to x, precisely when the diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "x"'] & k[\epsilon] \arrow[dl, "p"] \\ k & \end{tikzcd}$$

*where by abuse of language x and p denote the maps corresponding to the k-points x and p.

commutes. Such maps factor in a unique manner through the canonical map $A \to A/\mathfrak{m}^2$ (since $\alpha(\mathfrak{m}) \subseteq (\epsilon)$ and $\epsilon^2 = 0$). Now, the reduction map $A/\mathfrak{m}^2 \to A/\mathfrak{m} = k$ splits as an algebra homomorphism, the structure map $k \to A/\mathfrak{m}^2$ being a section, and $A/\mathfrak{m}^2$ decomposes as an k-algebra into $A/\mathfrak{m}^2 = k \oplus (\mathfrak{m}/\mathfrak{m}^2)$; in other words, $A/\mathfrak{m}^2 = D_{\mathfrak{m}/\mathfrak{m}^2}$ in the terminology above. It follows that

$$\mathrm{Hom}_{\mathsf{Alg}_k}(A, k[\epsilon]) \simeq \mathrm{Hom}_{\mathsf{Alg}_k}(A/\mathfrak{m}^2, k[\epsilon]) \simeq \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k),$$

and we are through. ❑

EXERCISE 11.2 *(Tangent space of a functor.)* It is a remarkable fact that one may even define tangent spaces for many (contravariant) functors on the category Sch/k that are not (*a priori*) representable. This holds true for most moduli functors, and one may *e.g.* compute dimensions and check regularity without knowing the functor is representable or without having a grasp on the representing object. The actual condition is that the functor commute with fibered products.

So let $F\colon \mathsf{Sch}_k \to \mathsf{Sets}$ be such a contravariant functor, and fix a k-point of F; that is, an element $p \in F(\mathrm{Spec}\, k)$. Denote by $F_0(\mathrm{Spec}\, k[\epsilon])$ the subset of $F(\mathrm{Spec}\, k[\epsilon])$ of elements that maps to p under the map $F(\eta)\colon F(\mathrm{Spec}\, k[\epsilon]) \to F(\mathrm{Spec}\, k)$ induced by the structure map $\eta\colon k \to k[\epsilon]$. The aim is to equip $F_0(\mathrm{Spec}\, k[\epsilon])$ with a natural structure of a vector space over k.

a) Show that diagonal of $\mathrm{Spec}\, k[\epsilon]$ induce a map

$$F_0(\mathrm{Spec}\, k[\epsilon]) \times F_0(\mathrm{Spec}\, k[\epsilon]) \to F_0(\mathrm{Spec}\, k[\epsilon])$$

that is an abelian group law.

b) For each $\alpha \in k$ let $\phi_\alpha \colon k[\epsilon] \to k[\epsilon]$ be given as $\phi_\alpha(a + b\epsilon) = a + \alpha b\epsilon$. Show that ϕ_α is an k-algebra homomorphism. Let α act on $F_0(\operatorname{Spec} k[\epsilon])$ through $F(\phi_\alpha)$. Show that this action together with the addition from *a*) gives a vector space structure on F_0.

★

Exercise 11.3 Let V and W be two vector spaces over k. Show that there is a functorial isomorphism $\operatorname{Hom}_{\mathsf{Alg}_k}(D_V, D_W) \simeq \operatorname{Hom}_k(V, W)$. ★

11.4 *Normal schemes and normalization*

Recall that a ring is called *normal* if all it's localizations $A_\mathfrak{p}$ are domains integrally closed in their fraction field. We will be mostly interested in the case when A is an integral domain; in this case the condition is equivalent to A itself being integrally closed in its fraction field K. Motivated by all the desirable algebraic properties of such rings, we make the following definition:

Definition 11.16 *Let X be an integral scheme with fraction field K. We say that X is* normal *at a point $x \in X$ if the ring $\mathcal{O}_{X,x}$ is integrally closed (viewed as a subring of K).*

Example 11.17 $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are normal schemes, because the local rings is isomorphic to $k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ which is a localization of an UFD, hence normal. ☆

Example 11.18 More generally, a scheme which is *locally factorial* (meaning that all stalks $\mathcal{O}_{X,x}$ are UFD's), is also normal. [CA notes chapter 7]. In particular, all regular schemes are normal. (This follows from the algebraic fact that local regular rings are UFDs (Atiyah–MacDonald)). ☆

For an integral scheme X, we will define a new scheme $\overline{X}$ which is a normal scheme, and a morphism $\pi \colon \overline{X} \to X$. There are many schemes with this property (take $\operatorname{Spec} K \to X$ for instance), so to get something more canonical, we want $\overline{X}$ and π to have a certain universal property. The scheme $\overline{X}$ is called the *normalization* of X.

$$\begin{tikzcd} & \overline{X} \arrow[d, "\pi"] \\ Y \arrow[ur, dashed, "h"] \arrow[r, "g"'] & X \end{tikzcd}$$

Theorem 11.19 *Let X be an integral scheme, then there is a normal scheme $\overline{X}$, and a morphism $\pi \colon \overline{X} \to X$ satisfying the following universal property: For any dominant morphism $g \colon Y \to X$ from a normal scheme Y, there is a unique morphism $h \colon Y \to \overline{X}$ such that $g = \pi \circ h$.*

Proof: The uniqueness part follows from the universal property. We therefore only need to check the existence.

Suppose first that $X = \operatorname{Spec} A$ is affine. We denote by A' be the normalization of A in the fraction field K.

Let Y be a normal scheme and let $B = \mathcal{O}_Y(Y)$. For a dominant morphism $g \colon Y \to X$, the corresponding map $g^{\sharp(X)} \colon A \to B$ is injective, so it extends to a unique morphism $A' \to B$, by the universal property of normalization of rings. Hence g factors via a unique morphism $g' \colon Y \to \operatorname{Spec} A'$. In particular, the canonical map $\pi\colon \operatorname{Spec} A' \to \operatorname{Spec} A$ satisfies the universal property in the theorem.

Now, let X be an arbitrary integral scheme, and let $U_i = \operatorname{Spec} A_i$ be an affine cover. Note that there are normalization morphisms $\pi_i \colon \bar{U}_i \to U_i$ defined by the inclusions $A_i \subset A_i'$. Consider the open set $U_{ij} = U_i \cap U_j$, which is an open set in both U_i and U_j. As $\pi_i|_{\pi_i^{-1}(U_{ij})} \colon \pi_i^{-1}(U_{ij}) \to U_{ij}$ and $\pi_j|_{\pi_j^{-1}(U_{ij})} \colon \pi_j^{-1}(U_{ij}) \to U_{ij}$ are both normalizations of U_{ij}, they coincide. Hence by the Gluing Lemma for morphisms, the morphisms π_i glue together to a scheme $\bar{X}$ and a morphism $\pi\colon \bar{X} \to X$. ❑

Theorem 11.20 *If X is a variety, the normalization $\overline{X}$ has the following properties:*

i) $\pi\colon \overline{X} \to X$ is surjective;

ii) $\overline{X}$ and X have the same dimension;

iii) There is an open subset $U \subset X$ so that π restricted to $\pi^{-1}(U)$ is an isomorphism;

iv) If X is of finite type over a field or over $\mathbb{Z}$, then $\pi\colon \overline{X} \to X$ is a finite morphism.

Proof: The proof relies on some of the basic properties of the integral closure.

Both statements *i)* and *ii)* follow from the Going-Up theorem.

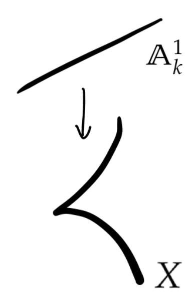
Statement *iii)* holds true because being normal is a generic property; that is, for a finitely generated integral domain A, the localization $A_{\mathfrak{p}}$ is normal for all $\mathfrak{p} \in U$ in a non-empty open subset U.

Finally, statement *iv)* follows from the fact that if A is an integral domain which is finitely generated over a field, then the normalization $\widetilde{A}$ in the fraction field K of A is a finite A-module. (This statement is essentially a consequence of Noether's normalization lemma.) ❑

In general, the normalization map $\pi : \overline{X} \to X$ need not be finite in the sense of Definition 4.25 on page 67. The first examples of Noetherian integral domains A whose integral closure is not finite over A were found by Yasuo Akizuki and Friedrich Karl Schmidt in 1935. See also Exercise 12.10 in [CA].

We perform normalization because normal schemes have better properties than arbitrary ones. For example in normal varieties regular functions defined outside a closed subvariety of codimension $\geqslant 2$ can be extended to regular functions defined everywhere ("Algebraic Hartogs theorem") . For instance, a curve is non-singular if and only if it is normal, so that normalization is the same as desingularization for curves.

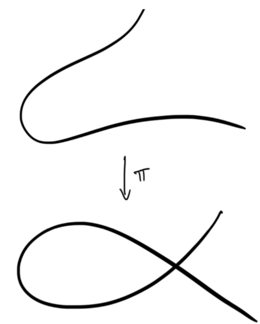
Examples



(11.21) *(Cuspidal cubic.)* Let k be a field, and let $X = \operatorname{Spec} A$ where $A = k[x,y]/(y^2 - x^3)$. This is the *cuspidal cubic curve* in $\mathbb{A}^2_k$.

There is an isomorphism of k-algebras $A \xrightarrow{\simeq} k[t^2, t^3]$ given by sending $x \mapsto t^2$ and $y \mapsto t^3$. It is clear that $k[t^2, t^3]$ is an integral domain with fraction field $K = k(t)$. Moreover, the normalization of A equals $\overline{A} = k[t]$. The inclusion $A \subset \overline{A}$ induces the normalization morphism $\pi : \mathbb{A}^1_k \to X$, and this is an isomorphism over the open set $D(t) \subset \mathbb{A}^1_k$ where t is inverible.

(11.22) *(Nodal cubic.)* Let now $X = \operatorname{Spec} A$ with A being the ring $A = k[x,y]/(y^2 - x^3 - x^2)$, where k now is a field whose characteristic is not two (if the characteristic is two, we are back in previous cuspidal case). This is the *nodal cubic curve* in $\mathbb{A}^2_k$. Here it is a little bit tricker to find the normalization, but it helps to think about it geometrically.



If we think of the corresponding affine variety $\{ (x,y) \mid y^2 = x^3 + x^2 \} \subset \mathbb{A}^2(k)$, we see that the origin $(0,0)$ is a special point: a line $l \subset \mathbb{A}^2_k$ through the closed point $(0,0) \in X$ (with equation $y = tx$) will intersect X at $(0,0)$ and at one more point (with $x = t^2 - 1$), and this gives a parameterization of the curve, which is generically one-to-one.

Back in the scheme world, we imitate this by introducing the parameter $t = yx^{-1}$ in the function field K of X, the equation $y^2 = x^3 - x^2$ then reduces to $t^2 = 1 + x$ after being divided by x^2. Moreover, the element t is integral, since it satisfies the monic equation $T^2 - x - 1 = 0$ (which has coefficients in A). Since $x = t^2 - 1$ and $y = x \cdot y/x = t^3 - t$, we see that

$$A = k[t^2 - 1, t^3 - t] \subseteq k[t] \subseteq K = k(t),$$

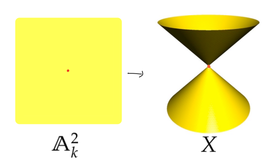
and since $k[t]$ is integrally closed, any element in K which is integral over A, can be written as a polynomial in t. So $\overline{A} = k[t]$ is the integral closure of A in $k(t)$. The normalization map $\pi : \operatorname{Spec} \overline{A} \to \operatorname{Spec} A$ is an isomorphism outside the origin $(0,0) \in X$. Geometrically the map π identifies two points $(t+1)$ and $(t-1)$ in $\mathbb{A}^1_k$ to the origin in X.

(11.23) *(The quadratic cone.)* Consider the affine scheme $X = \operatorname{Spec} A$ where $A = \mathbb{C}[x,y,z]/(xy - z^2)$. Note that this is not a factorial scheme as $xy = z^2$ and one easily checks that x, y and z all are irreducible elements, so we cannot immediately conclude that A is normal. However, there are a few ways to see that it is in fact so:

i) There is an isomorphism of rings

$$\phi\colon A \to \mathbb{C}[u^2, uv, v^2],$$

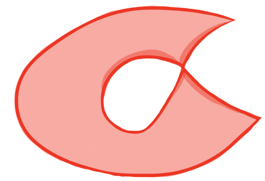
and the latter algebra is normal in $k(u,v)$.



ii) Let $B = \mathbb{C}[x,y]$, so that $A = B[z]/(z^2 - xy)$. Then $B \subset A$ is a ring extension making A into a finite B-module, in fact, it is a free module of rank two with basis 1, z. We get an inclusion of fields $K(B) = \mathbb{C}(x,y) \subset K(A)$ obtained by adjoining the element z, which equals the square root $\sqrt{xy}$, to $\mathbb{C}(x,y)$. Each element of $K(A)$ can be written as $w = u + vz$ where $u, v \in \mathbb{C}(x,y)$. If this element is integral over A, it will also be integral over B. In fact, w satisfies the minimal polynomial

$$T^2 - 2uT + (u^2 - xyv^2) = 0,$$

and if this is a polynomial integral over B, we must have $2u \in \mathbb{C}[x,y]$, and hence $u \in \mathbb{C}[x,y]$. Moreover, it ensues that the element $(u^2 - xyv^2)$ belongs to $\mathbb{C}[x,y]$, so also that $xyv^2 \in \mathbb{C}[x,y]$, from which it readily follows that $v \in \mathbb{C}[x,y]$ since a potential denominator of v^2 will be a square and can therefore not be neutralized by the non-square element xy.



(11.24) Here is an example of a non-normal surface with an isolated singularity. We let X be the scheme obtained by identifying two points in $\mathbb{A}^2_k$; X is the affine variety given by the k-algebra

$$A = \{f \in k[x,y] \mid f(0,0) = f(0,1)\}.$$

Then the normalization $\overline{X}$ is the affine plane.

☆

EXERCISE 11.4 Let k be a field and consider the k-algebra

$$A = k[x, xy, y^2, y^3] \subset k[x,y]$$

Show that $X = \operatorname{Spec}(A)$ is a surface with the maximal ideal $\mathfrak{m} = \langle x, xy, y^2, y^3 \rangle \subset A$ the unique point $x \in X$ so that $\mathcal{O}_{X,x}$ is not normal. ★

✲ EXERCISE 11.5 *(A consequence of Noether's normalization lemma.)* Let $X = \operatorname{Spec} A$ be an affine scheme of dimension n, of finite type over a field k.

(i) Show that there is a *finite* morphism

$$X \to \mathbb{A}^n_k$$

Such a morphism is called a *Noether normalization of* X.

(ii) Find a Noether normalization for the scheme

$$X = \operatorname{Spec} \mathbb{C}[x,y,z]/I$$

where $I = (xy - z^2, x^2y - xy^3 + z^4 - 1)$. ★

Chapter 12

More on sheaves

When working with sheaves on a space the toolbox include some traditional tools, well-known from abelian groups: the possibility to form kernels, cokernels and images of maps, and the notion of exact sequences. We may also form arbitrary direct sums and products of sheaves. In short, the category AbSh_X of sheaves on X is an abelian category with arbitrary products and direct sums.

Kernels

Let X be a topological space and let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves on X.

Definition 12.1 *The* kernel $\operatorname{Ker}\phi$ *of* ϕ *is the subsheaf of* $\mathcal{F}$ *defined by*

$$(\operatorname{Ker}\phi)(U) = \operatorname{Ker}\phi_U.$$

In other words, $(\operatorname{Ker}\phi)(U)$ *consists of the sections in* $\mathcal{F}(U)$ *that map to zero under* $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$.

The requirement in the definition is compatible with the restriction maps, because $\phi_V(s|_V) = \phi_U(s)|_V$ for any section s over the open set U and any open $V \subseteq U$. Thus we have defined a subpresheaf of $\mathcal{F}$.

For the two sheaf axioms: the Locality axiom holds for $\operatorname{Ker}\phi$ because $\mathcal{F}$ is a sheaf. Moreover, if $\{U_i\}$ is an open covering of U and $\{s_i\}$ a family of sections with s_i belonging to $\operatorname{Ker}\phi(U_i)$ that agree on overlaps, one may glue together the s_i's to a section s of $\mathcal{F}$ over U. One has $\phi(s)|_{U_i} = \phi(s|_{U_i}) = \phi(s_i) = 0$, and from the Locality axiom for $\mathcal{G}$ it follows that $\phi(s) = 0$.

Lemma 12.2 *For each* $x \in X$, $(\operatorname{Ker}\phi)_x = \operatorname{Ker}\phi_x$.

A map $\phi\colon \mathcal{F} \to \mathcal{G}$ is said to be *injective* if $\operatorname{Ker}\phi = 0$; this is equivalent to ϕ_U being injective for each open U. In light of the previous lemma, it is also equivalent to the condition that $\operatorname{Ker}\phi_x = 0$ for all x; *i.e.* that all stalk maps ϕ_x are injective. One often expresses this by saying that 'ϕ is injective on stalks'.

Images

When it comes to images the situation is not as nice as for kernels. If we are given a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ we can define the *image presheaf*, contained in $\mathcal{G}$, by letting $(\operatorname{Im}\phi)'(U) = \operatorname{Im}\phi_U$. However, this is not necessarily a sheaf: if $s_i = \phi_{U_i}(t_i)$ are gluing data for the image presheaf with respect to a cover $\{U_i\}$ of U, there is no reason for the t_i's to match on the intersections $U_{ij} = U_i \cap U_j$, even if the s_i's do.

Image presheaf

However in the case when $\mathcal{G}$ is a sheaf, we can define the *image sheaf* $\operatorname{Im}\phi$, by taking the sections of $\mathcal{G}$ which are 'locally images of ϕ'. This is a subsheaf of $\mathcal{G}$; it is the smallest subsheaf of $\mathcal{G}$ containing the images of ϕ.

Definition 12.3 *For a map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, we define the* image sheaf $\operatorname{Im}\phi$ *by*

$$(\operatorname{Im}\phi)(U) = \left\{ t \in \mathcal{G}(U) \;\middle|\; \begin{array}{l} \textit{there exists a covering } U_i \textit{ of } U \textit{ and sections} \\ s_i \in \mathcal{F}(U_i) \textit{ such that } t|_{U_i} = \phi(s_i) \end{array} \right\}.$$

Note that this is a presheaf precisely because the diagram (3.2) commutes.

Let us check that $\operatorname{Im}\phi$ is indeed a sheaf. The Locality axiom holds free, because $\operatorname{Im}\phi$ is a subpresheaf of $\mathcal{G}$ and $\mathcal{G}$ is a sheaf. As for the Gluing axiom, suppose we are given a covering U_i of an open set U, plus sections $t_i \in \operatorname{Im}\phi$ such that the t_i agree on the overlaps $U_i \cap U_j$. Since $\mathcal{G}$ is a sheaf, the t_i glue to a section $t \in \mathcal{G}(U)$. The section t however, is by construction locally an image $\phi(s)$: each U_i has an open cover V_{ij} such that $t_i|_V = \phi(s_{ij})$, and so $t \in \operatorname{Im}\phi(U)$.

Unlike the situation for kernels, forming the image of a map of sheaves does *not* always commute with taking sections. However, any section of the form $t = \phi(s)$ in $\mathcal{G}$ clearly lies in $\operatorname{Im}\phi$, so at least we have $\operatorname{Im}\phi_U \subseteq \Gamma(U, \operatorname{Im}\phi)$. Luckily the situation is better locally: forming images commutes with forming stalks:

Lemma 12.4 *For each $x \in X$ we have* $(\operatorname{Im}\phi)_x = \operatorname{Im}\phi_x$.

Proof: Let $t_x \in \operatorname{Im}\phi_x$ and pick an $s_x \in \mathcal{F}_x$ with $\phi_x(s_x) = t_x$. We may extend these elements to sections s and t over some open neighbourhood V, so that $\phi_V(s) = t$, and t is a section of $\operatorname{Im}\phi$ over V. This shows that $\operatorname{Im}\phi_x \subseteq (\operatorname{Im}\phi)_x$. Conversely, if t is a section of $\mathcal{G}$ over an open U containing x locally lying in the image presheaf, the restriction $t \subseteq V$ lies in $\operatorname{Im}\phi_V$ for some smaller neighbourhood V of x, hence the germ t_x lies in $\operatorname{Im}\phi_x$. ❑

The map $\phi\colon \mathcal{F} \to \mathcal{G}$ is said to be *surjective* if the *image sheaf* $\operatorname{Im}\phi = \mathcal{G}$. This is equivalent to all the stalk maps ϕ_x being surjective (one says that 'ϕ is surjective on stalks'). However, we underline that this condition does not imply that the maps ϕ_U are surjective for every open U. (This is an important observation; we will discuss it in detail later on.).

Example 12.5 Consider the subscheme $Z = \operatorname{Spec} k[x,y]/(y)$ of $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ and let $i : Z \to \mathbb{A}^2_k$ denote the inclusion. Consider the morphism of sheaves

$$i^\sharp : \mathcal{O}_Z \to i_*\mathcal{O}_{\mathbb{A}^2_k}$$

We claim that the naive image presheaf $\mathcal{G}$ given by $\mathcal{G}(U) = \operatorname{Im} i^\sharp(U)$ is not a sheaf. To see why, let $U = D(x)$ and $V = D(y)$. Then we have

$$\mathcal{G}(U) = \mathcal{O}_Z(i^{-1}U) = \mathcal{O}_Z(U \cap Z) = k[x]_x$$

and

$$\mathcal{G}(V) = \mathcal{O}_Z(\varnothing) = 0 \quad \text{and} \quad \mathcal{G}(U \cap V) = \mathcal{O}_Z(\varnothing) = 0.$$

Note that the sections $x^{-1} \in \mathcal{G}(V)$ and 0 both restrict to 0 in $\mathcal{G}(U \cap V) = \mathcal{O}_Z(\varnothing) = 0$. However, they do not glue to a section of $\mathcal{G}(U \cup V)$, because $\mathcal{G}(U \cap V) = k[x]$. ☆

However, for the map ϕ to be an *isomorphism*, one has the following

Proposition 12.6 *Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves. Then the following four conditions are equivalent*

i) The map ϕ is an isomorphism;

ii) For every $x \in X$ the map on stalks $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism;

iii) One has $\operatorname{Ker}\phi = 0$ and $\operatorname{Im}\phi = \mathcal{G}$;

iv) For all open subsets $U \subseteq X$ the map on sections $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism.

Proof: Most of the implications are straightforward from what we have done so far and are left to the reader. We comment just on the two main salient points.

Firstly, assume that all the stalk maps $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ are isomorphism, and let us deduce that all maps $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ on sections are isomorphisms. It is clear that ϕ_U is injective, since forming kernels commute with taking sections as in Lemma 12.2 on page 173.

For surjectivity, take an element $t \in \mathcal{G}(U)$; we need to show that $t = \phi_U(s)$ for some $s \in \mathcal{F}(U)$. For each $x \in U$ there is a germ s_x induced by a section $s_{(x)}$ of $\mathcal{F}$ over some open neighbourhood U_x of x that satisfies $\phi_x(s_x) = t_x$. Let $t_{(x)} = t|_{U_x}$. After shrinking the neighbourhood U_x, we may assume that $\phi_{U_x}(s_{(x)}) = t_{(x)}$. Note that the $t_{(x)}$'s match on the intersections $U_x \cap U_{x'}$, all being restrictions of the section t, and therefore the $s_{(x)}$'s match as well because ϕ_{U_x} is injective (as we just observed above). Hence, the sections $s_{(x)}$ patch together to a section s of $\mathcal{F}$ over U that must map to t since it does so locally.

Secondly, if all the ϕ_U's are isomorphisms, we have all the inverse maps ϕ_U^{-1} at our disposal. They commute with restrictions since the maps ϕ_U do. Indeed, from $\phi_V \circ \rho_{UV} = \rho_{UV} \circ \phi_U$ one obtains $\rho_{UV} \circ \phi_U^{-1} = \phi_V^{-1} \circ \rho_{UV}$. Thus the ϕ_U^{-1}'s define a map $\phi^{-1}\colon \mathcal{G} \to \mathcal{F}$ of sheaves, which of course, is inverse to ϕ. We conclude that ϕ is an isomorphism. ❑

Exercise 12.1 Check that $(\operatorname{Ker}\phi)_x = \operatorname{Ker}(\phi_x)$ for a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$. ★

12.1 *Godement sheaves*

Godement sheaves

We will consider a class of rather peculiar sheaves, called *Godement sheaves*, to demonstrate the versatility and the generality of the notion of sheaves. They will also be important later, when we define the sheafification of a presheaf, and they open the path to a convenient definition of sheaf cohomology.

Suppose that we are given, for each point $x \in X$, an abelian group A_x. The groups A_x can be chosen in a completely arbitrary way, at random if you will. The choice of these groups gives rise to a *sheaf* $\mathscr{A}$ on X, whose set of sections over an open set $U \subseteq X$ is given by

$$\mathscr{A}(U) = \prod_{x \in U} A_x,$$

and whose restriction maps are defined as the natural projections

$$\rho_{UV} \colon \prod_{x \in U} A_x \to \prod_{x \in V} A_x,$$

where $V \subseteq U$ is any pair of open subsets of X. The restriction map just 'throws away' the components at points in U not lying in V.

Roger Godement (1921–2016)

PROPOSITION 12.7 *$\mathscr{A}$ is a sheaf.*

PROOF: The Locality axiom holds, because if the family $\{U_i\}_{i \in I}$ of open sets covers U, any point $x_0 \in U$ lies in some U_{i_0}, so if $s = (a_x)_{x \in U} \in \mathscr{A}(U)$ is a section, the component a_{x_0} survives in the projection onto $\mathscr{A}(U_i) = \prod_{x \in U_i} A_x$. Hence if $s|_{U_i} = 0$ for all i, it follows that $s = 0$.

The Gluing condition holds: Assume we are given an open cover $\{U_i\}_{i \in I}$ of U and sections $s_i = (a^i_x)_{x \in U_i} \in \prod_{x \in U_i} A_x$ over U_i matching on the intersections $U_i \cap U_j$. The matching conditions imply that the component of s_i at a point x is the same as that of s_j provided that $x \in U_i \cap U_j$. Hence we get a well-defined section $s \in \mathscr{A}(U)$ by using this common component as the component of s at x. It is clear that $s|_{U_i} = s_i$. ❑

a^i_x is an element of A_x

DEFINITION 12.8 *The sheaf $\mathscr{A}$ is called the* Godement sheaf *of the collection* $\{A_x\}$.

So what is the stalk of $\mathscr{A}$ at a point x? It is tempting to think that it should be A_x, but in fact the group can be extremely complicated. Of course there is a map $\mathcal{A}_x \to A_x$ from the stalk to A_x, but that is in general the best you can say. For example, suppose $A_y = \mathbb{Z}/2$ for all $y \in X$, and that there is a sequence x_n of points in X converging to $x \in X$, with $x_n \neq x_m$ for all $n \neq m$. Then $\mathcal{A}_x$ maps onto the (infinite) set of *tails* of sequences of 0's and 1's.* Namely, every open neighbourhood of x contains almost all of the x_n.

The construction is not confined to abelian groups, but works for any category where general products exist (like sets, rings, *etc).*

On the other hand, if every neighbourhood of x contains a point y such that $A_y = 0$, then $\mathcal{A}_x = 0$.

**Two sequences are said to define the same tail if they merely differ at finitely many places. This an equivalence relation with uncountably many equivalence classes*

EXAMPLE 12.9 *((Skyscraper sheaves).)* We met this kind of sheaves already in Example 3.10. They are very special instances of Godement sheaves where the abelian groups A_x are zero for all x except for one, where it takes the value A. The sections are described by

$$\Gamma(U, A(x)) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

If $x \in X$ is a closed point, the stalks are easy to describe: they are zero everywhere except at x, where the stalk equals A. Indeed, if $y \neq x$, then y lies in the open set $X \backslash \{x\}$ over which all sections of $A(x)$ vanish.

However, if $\{x\}$ is non-closed point (i.e., $X \backslash \{x\}$ is not open), one still has the Godement sheaf $A(x)$, but description of the stalks is more complicated. ☆

EXAMPLE 12.10 Slightly generalizing the construction of a skyscraper sheaf, one may form the Godement sheaf $\mathscr{A}$ defined by a finite set of distinct closed points $x_1, \ldots, x_r$ and corresponding abelian groups $A_1, \ldots, A_r$. Then one sees, bearing in mind that an empty direct sum is zero, that the sections of $\mathscr{A}$ over an open set U is given as $\Gamma(U, \mathscr{A}) = \bigoplus_{x_i \in U} A_i$. The stalks of $\mathscr{A}$ are

$$\mathscr{A}_x = \begin{cases} 0 & \text{when } x \neq x_i \text{ for all } i, \\ A_i & \text{when } \; x = x_i. \end{cases}$$

☆

A barcode sheaf ?

EXERCISE 12.2 Assume that $x \in X$ is not closed and let $Z = \overline{\{x\}}$ be the closure of the singleton $\{x\}$. Show that the stalks of $A(x)$ are $(A(x))_y = 0$ if $y \notin Z$ and $(A(x))_y = A$ for points y belonging to Z ★

EXERCISE 12.3 Find examples of sheaves $\mathcal{F}$ (*e.g.* Godement sheaves), where the support is not closed. ★

The Godement sheaf associated with a presheaf

Let $\mathcal{F}$ be a presheaf on X. The stalks $\mathcal{F}_x$ of $\mathcal{F}$ give a collection of abelian groups indexed by points in X, as good as any other collection, and we may form the corresponding Godement sheaf, which we denote by $\Pi(\mathcal{F})$. The sections of $\Pi(\mathcal{F})$ are given by

$$\Pi(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x, \tag{12.1}$$

and the restriction maps are the projections, as for any Godement sheaf.

This sheaf is sometimes called the sheaf of discontinuous sections *of $\mathcal{F}$.*

There is an obvious and canonical map

$$\kappa_{\mathcal{F}} \colon \mathcal{F} \to \Pi(\mathcal{F})$$

that sends a section $s \in \mathcal{F}(U)$ to the element $(s_x)_{x \in U}$ of the product in (12.1). This map is functorial in $\mathcal{F}$, because if $\phi \colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves, one has the stalkwise maps $\phi_x \colon \mathcal{F}_x \to \mathcal{G}_x$, and by taking appropriate products of these, one obtains a map $\Pi(\phi) \colon \Pi(\mathcal{F}) \to \Pi(\mathcal{G})$. Over an open set U, it holds that

$$\Pi(\phi)\big((s_x)_{x\in U}\big) = \big(\phi_x(s_x)\big)_{x\in U},$$

and there is thus a commutative diagram of sheaves

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\kappa_{\mathcal{F}}"] \arrow[d, "\phi"'] & \Pi(\mathcal{F}) \arrow[d, "\Pi(\phi)"] \\ \mathcal{G} \arrow[r, "\kappa_{\mathcal{G}}"'] & \Pi(\mathcal{G}). \end{tikzcd} \tag{12.2}$$

It is not hard to check that $\Pi(\mathrm{id}_{\mathcal{F}}) = \mathrm{id}_{\Pi(\mathcal{F})}$ and that $\Pi(\psi \circ \phi) = \Pi(\psi) \circ \Pi(\phi)$ for two composable morphisms between presheaves on X, so that Π defines a functor from the category of presheaves on X to the category of sheaves on X.

12.2 *Sheafification*

Given a presheaf $\mathcal{F}$, there is a canonical way of defining an sheaf $\mathcal{F}^+$ that in some sense is the sheaf that best approximates $\mathcal{F}$. The main properties of $\mathcal{F}^+$ are summarized in the following proposition.

Proposition 12.11 *Given a presheaf $\mathcal{F}$ on X, there is a sheaf $\mathcal{F}^+$ and a natural presheaf map $\kappa_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{F}^+$ such that:*

i) $\kappa_{\mathcal{F}}$ is functorial in $\mathcal{F}$: A map of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[d] & \mathcal{G} \arrow[d] \\ \mathcal{F}^+ \arrow[r, "\phi^+"] & \mathcal{G}^+ \end{tikzcd}$$

ii) $\kappa_{\mathcal{F}}$ enjoys the universal property that any map of presheaves $\mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf, factors through $\mathcal{F}^+$ in a unique way. This property characterizes $\mathcal{F}^+$ up to a unique isomorphism.

iii) If $\mathcal{G}$ is a sheaf, there is a natural bijection

$$\mathrm{Hom}_{\mathsf{AbPrSh}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathsf{Sh}_X}(\mathcal{F}^+, \mathcal{G}) \tag{12.3}$$

where on the left hand side, $\mathcal{G}$ is considered as a presheaf;

iv) $\kappa_{\mathcal{F}}$ induces an isomorphism on stalks: $\mathcal{F}_x \simeq \mathcal{F}^+_x$ for every $x \in X$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\kappa"] \arrow[dr] & \mathcal{F}^+ \arrow[d, dashed] \\ & \mathcal{G} \end{tikzcd}$$

We will explain how to construct $\mathcal{F}^+$ and $\kappa_{\mathcal{F}}$ from $\mathcal{F}$ below. If you find the construction a bit daunting, don't worry – we will never need the explicit construction again. All of the arguments using $\mathcal{F}^+$ in this book use only the four properties in the proposition, which uniquely characterize $\mathcal{F}^+$ and $\kappa_{\mathcal{F}}$.

Compare this to the definition of the tensor product: we never use the explicit construction in terms of generators and relations; all we ever use is its formal properties.

Exercise 12.4 Prove that, if it exists, the sheafification is unique up to a unique isomorphism. ★

When $\mathcal{F}$ is a subpresheaf of a sheaf

What prevents the presheaf $\mathcal{F}$ from being a sheaf is of course the failure of one or both of the sheaf axioms. To remedy this, one must factor out all sections of $\mathcal{F}$ whose germs are everywhere zero, and one has to enrich $\mathcal{F}$ by adding enough new sections so that the Gluing axiom holds.

The simplest case of this is when $\mathcal{F}$ comes embedded as a subpresheaf of a sheaf $\mathcal{G}$. Then the Locality axiom already holds for $\mathcal{F}$, because it holds for $\mathcal{G}$. In this special case, we have an explicit description of the sheafification of $\mathcal{F}$; it will be equal to the image sheaf that we constructed in Definition 12.3 applied to the inclusion $\mathcal{F} \to \mathcal{G}$. Concretely, $\mathcal{F}^+(U)$ consists of the sections that locally lie in $\mathcal{F}$, that is, sections $s \in \mathcal{G}(U)$ such that there is a covering U_i of U such that $s|_{U_i} \in \mathcal{F}(U)$. Note that $\mathcal{F}^+$ is a subsheaf of $\mathcal{G}$ which contains $\mathcal{F}$ as a subpresheaf. Moreover, if $\mathcal{F}$ is already a sheaf, we don't add anything new, so that $\mathcal{F}^+ = \mathcal{F}$.

Recall that the image sheaf only required $\mathcal{G}$ to be a sheaf.

Let us check that this sheaf indeed satisfies the universal property of sheafification in Proposition 12.11. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves, where $\mathcal{G}$ a sheaf. If $U \subset X$ is open, and $s \in \mathcal{F}^+(U)$, then there is an open covering U_i of U such that $s|_{U_i} \in \mathcal{F}(U_i)$ for each i. Then $\phi_{U_i}(s|_{U_i})$ are elements of $\mathcal{G}(U_i)$ which agree on the overlaps, so they glue to a section $t \in \mathcal{G}(U)$, and we define the map $\mathcal{F}^+(U) \to \mathcal{G}(U)$ by sending s to t.

EXAMPLE 12.12 (*Bounded continuous functions.*) Consider the sheaf $C(X, \mathbb{R})$ from Example 3.4 and the subpresheaf $\mathcal{F}$, defined by setting $\mathcal{F}(U) = C_b(U, \mathbb{R})$, the group of *bounded* continuous functions. Then $C_b(X, \mathbb{R})$ is not a sheaf, because the Gluing axiom fails: the open sets $U_i = (-i, i)$ form a covering of $X = \mathbb{R}$, and the function $f_i(x) = x|_{U_i}$ defines a well defined element $C_b(U_i)$ for each i. The f_i's agree with the function $f(x) = x$ on each overlap $U_i \cap U_j$, but they do not glue to an element in $C_b(\mathbb{R})$, because the function x is not bounded.

In this example, the sheafification of the presheaf $C_b(X, \mathbb{R})$ is in fact all of $C(X, \mathbb{R})$, because any continuous function is locally bounded. ☆

EXERCISE 12.5 Prove the uniqueness part for the universal map of sheaves $\mathcal{F}^+ \to \mathcal{G}$. ★

The general construction

If $\mathcal{F}$ is a general presheaf, there is a nice and canonical way to map it into a sheaf using the Godement sheaf $\Pi(\mathcal{F})$ associated to $\mathcal{F}$. Recall the canonical map $\kappa \colon \mathcal{F} \to \Pi(\mathcal{F})$ that sends a section s of $\mathcal{F}$ over an open U to the sequence of germs $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x = \Gamma(U, \Pi(\mathcal{F}))$. This map certainly kills the 'doomed' sections, *i.e.* those whose germs all vanish. And we can get an actual sheaf by taking the image of κ in $\Pi(\mathcal{F})$:

DEFINITION 12.13 *For a presheaf $\mathcal{F}$ on X, we define its* sheafification *$\mathcal{F}^+$ as the image sheaf of the map $\kappa : \mathcal{F} \to \Pi(\mathcal{F})$.*

For simplicity, we also write $\kappa_{\mathcal{F}}$ or simply κ, for the canonical map $\mathcal{F} \to \mathcal{F}^+$.

It might help to unravel this definition. Over an open set $U \subseteq X$ the sections of $\mathcal{F}^+$ are given by

$$\begin{aligned} \mathcal{F}^+(U) &= \left\{ (s_x) \in \prod_{x \in U} \mathcal{F}_x \;\middle|\; (s_x) \text{ locally lies in } \operatorname{Im} \kappa \right\} \\ &= \left\{ (s_x) \in \prod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{l} \text{for each } x \in U \text{, there exists an open } V \subseteq U \\ \text{containing } x \text{ and } t \in \mathcal{F}(V) \text{, such that} \\ \text{for all } y \in V \text{ we have } s_y = t_y \text{ in } \mathcal{F}_y \end{array} \right\}. \end{aligned}$$

Thus elements of $\mathcal{F}^+(U)$ can be thought about as sequences $(s_x)_{x \in U}$ of germs of $\mathcal{F}$, but only those sequences that arise from local sections of $\mathcal{F}$ are allowed.

LEMMA 12.14 *The sheafification $\mathcal{F}^+$ depends functorially on $\mathcal{F}$. Moreover, if $\mathcal{F}$ is a sheaf, $\kappa \colon \mathcal{F} \to \mathcal{F}^+$ is an isomorphism, so that $\mathcal{F}$ and $\mathcal{F}^+$ are canonically isomorphic.*

PROOF: Suppose that $\phi \colon \mathcal{F} \to \mathcal{G}$ is a map between two presheaves. Let s be a section of $\Pi(\mathcal{F})$ over some open set U so that s locally lies in $\mathcal{F}$. In other words, there is a covering $\{U_i\}$ of U and sections s_i of $\mathcal{F}$ over U_i with $s|_{U_i} = \kappa_{\mathcal{F}}(s_i)$. Hence by (12.2) one has

$$\Pi(\phi)(s|_{U_i}) = \Pi(\phi)(\kappa_{\mathcal{F}}(s_i)) = \kappa_{\mathcal{G}}(\phi(s_i)).$$

This means that $\Pi(\phi)(s)$ lies locally in $\mathcal{G}$, and hence $\Pi(\phi)$ takes $\mathcal{F}^+$ into $\mathcal{G}^+$. Moreover, there is a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\kappa_{\mathcal{F}}"] \arrow[d, "\phi"'] & \mathcal{F}^+ \arrow[r] \arrow[d, "\phi^+"] & \Pi(\mathcal{F}) \arrow[d, "\Pi(\phi)"] \\ \mathcal{G} \arrow[r, "\kappa_{\mathcal{G}}"'] & \mathcal{G}^+ \arrow[r] & \Pi(\mathcal{G}). \end{tikzcd}$$

In case $\mathcal{F}$ is a sheaf, the map $\kappa_{\mathcal{F}}$ maps $\mathcal{F}$ injectively into $\Pi(\mathcal{F})$, and $\mathcal{F} = \operatorname{Im} \kappa_{\mathcal{F}}$, hence $\kappa_{\mathcal{F}}$ is an isomorphism. ❑

LEMMA 12.15 *Given a presheaf $\mathcal{F}$ on X. Then the sheaf $\mathcal{F}^+$ and the natural map $\kappa_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{F}^+$ enjoys the universal property that any map of presheaves $\mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf, factors through $\mathcal{F}^+$ in a unique way. This property characterizes the pair $\kappa_{\mathcal{F}}$ and $\mathcal{F}^+$ up to a unique isomorphism.*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\kappa"] \arrow[dr] & \mathcal{F}^+ \arrow[d, dashed] \\ & \mathcal{G} \end{tikzcd}$$

PROOF: If $\mathcal{G}$ in the commutative diagram above is a sheaf, the map $\kappa_{\mathcal{G}} \colon \mathcal{G} \to \mathcal{G}^+$ is an isomorphism and $\phi^+ \circ \kappa_G^{-1}$ provides the wanted factorization. The uniqueness statement follows formally: given two sheaves $\mathcal{F}^+$ and $\mathcal{F}'$ satisfying the above, we get by the universal properties two maps $\mathcal{F}^+ \to \mathcal{F}'$ and $\mathcal{F}' \to \mathcal{F}^+$, whose compositions are the identity by uniqueness. ❑

This also gives item *iii)* in Proposition 12.11: for a presheaf $\mathcal{F}$ and a sheaf $\mathcal{G}$ there is a natural bijection

$$\operatorname{Hom}_{\mathsf{AbPrSh}_X}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{\mathsf{AbSh}_X}(\mathcal{F}^+, \mathcal{G}), \tag{12.4}$$

where on the left $\mathcal{G}$ is considered as a presheaf.

We now turn to point *iv)* of Proposition 12.11:

A fancy way of restating this is to say that the sheafification functor $\mathcal{F} \mapsto \mathcal{F}^+$ is an adjoint *to the forgetful functor $\iota\colon \mathsf{AbSh}_X \to \mathsf{AbPrSh}$ from sheaves to presheaves.*

LEMMA 12.16 *Sheafification preserves stalks: $\mathcal{F}_x = (\mathcal{F}^+)_x$ via κ_x.*

PROOF: The map $\kappa_x\colon \mathcal{F}_x \to (\mathcal{F}^+)_x$ is injective, because $\mathcal{F}_x \to (\Pi(\mathcal{F}))_x$ is injective. To show that it is surjective, suppose that $\bar{s} \in (\mathcal{F}^+)_x$. We can find an open neighbourhood U of x such that $\bar{s}$ is the equivalence class of (s, U) with $s \in \mathcal{F}^+(U)$. By definition, this means there exists an open neighbourhood $V \subseteq U$ of x and a section $t \in \mathcal{F}(V)$ such that $s|_V$ is the image of t in $\Pi(\mathcal{F})(V)$. Clearly the class of (t, V) defines an element of $\mathcal{F}_x$ mapping to $\bar{s}$. ❑

Here are two examples where one can describe the sheafification explicitly.

EXAMPLE 12.17 (*Constant sheaves.*) Recall Example 3.9 in which we showed that the constant presheaf given by $\mathcal{F}(U) = A$ is usually not a sheaf (where A is an abelian group). In this case, the sheafification $\mathcal{F}^+$ is exactly the constant sheaf A_X defined by

$$A_X(U) = \{\, f\colon U \to A \mid f \text{ continuous} \,\}. \tag{12.5}$$

where A is given the discrete topology. To prove this, consider the map $\iota : A_X \to \mathcal{F}^+$ which over an open set $U \subset X$, sends $f\colon U \to A$ to the element $(f(x))_{x\in U} \in \prod_{x\in U} A$ (note that this element lies in $\mathcal{F}^+$, since f is locally constant). Furthermore, on stalks this is simply the identity map $\iota_x\colon A \to A$. Since ι is a map of sheaves which is an isomorphism on stalks, it is an isomorphism. ☆

See Proposition 12.6.

EXAMPLE 12.18 (*Regular functions and sheafification.*) Consider again the sheaf of regular functions on an algebraic variety, as defined in Example 3.7. There is a subtlety in the definition of a regular function; the function is only *locally* expressible as a quotient g/h; we do not require a global representative. This is related to the concept of saturation.

To see the issue, consider the case where X is an affine variety over k with coordinate ring $A(X)$, and consider the 'naive sheaf of regular functions' given by

$$o_X(U) = \left\{ f : U \to k \;\middle|\; \begin{array}{l} f = g/h \text{ for polynomials } g, h \\ \text{with } h(x) \neq 0 \text{ for every } x \in U \end{array} \right\}$$

This is not a sheaf in general. For instance, consider $X = Z(xy - zw) \subset \mathbb{A}^4$. Then x/z defines a section of o_X over $U = \{z \neq 0\}$ and w/y defines a section over $V = \{y \neq 0\}$. Using the defining relation $xy = zw$ we see that these two sections agree on the overlap $U \cap V$. However, they do not glue together to an element of $o_X(U \cup V)$. Indeed, there is no quotient g/h of polynomials in x, y, z, w representing the corresponding regular function $f\colon U \cup V \to k$.

See also [EO, Example 3.5].

The presheaf o_X sits naturally in the sheaf $C(X, k)$ of continuous maps $U \to k$, and the sheafification is exactly the usual structure sheaf $\mathcal{O}_X$ on X. Therefore, while the two expressions x/z and w/y can not be glued together to an element of $o_X(U \cup V)$, they can be glued together to an element in $\mathcal{O}_X(U \cup V)$. ☆

Quotient sheaves and cokernels

One of the main applications of the sheafification process is to be able to define quotient sheaves. So assume that $\mathcal{G} \subseteq \mathcal{F}$ is an inclusion of sheaves and define a presheaf whose sections over U is the quotient $\mathcal{F}(U)/\mathcal{G}(U)$. The restriction maps of $\mathcal{F}$ and $\mathcal{G}$ respect the inclusions $\mathcal{G}(U) \subseteq \mathcal{F}(U)$, hence passing to the quotients $\mathcal{F}(U)/\mathcal{G}(U)$, and we may use these maps as restriction maps for the quotient presheaf. The *quotient sheaf* $\mathcal{F}/\mathcal{G}$ is by definition the sheafification of this quotient presheaf. It sits in the exact sequence

Quotient sheaf

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{G} \longrightarrow 0.$$

The *cokernel* $\operatorname{Coker}\phi$ of a map $\phi\colon \mathcal{F} \to \mathcal{G}$ of sheaves is then defined as the quotient sheaf $\mathcal{G}/\operatorname{Im}\phi$; it sits in the exact sequence

$$0 \longrightarrow \operatorname{Ker}\phi \longrightarrow \mathcal{F} \xrightarrow{\ \phi\ } \mathcal{G} \longrightarrow \operatorname{Coker}\phi \longrightarrow 0.$$

EXAMPLE 12.19 To see why we have to sheafify in these constructions, consider again the exponential map from Example 12.6. The naive presheaf $U \mapsto \operatorname{Coker}\exp(U)$ is not a sheaf: the class of the function $f(z) = z$ restricts to zero in $\operatorname{Coker}\exp$ on sufficiently small open sets, but it is itself not zero (since otherwise we would be able to define a global logarithm on $\mathbb{C}\backslash 0$). ☆

Exact sequences of sheaves

A *complex of sheaves* is a sequence

$$\ldots \xrightarrow{\phi_{i-2}} \mathcal{F}_{i-1} \xrightarrow{\phi_{i-1}} \mathcal{F}_i \xrightarrow{\phi_i} \mathcal{F}_{i+1} \xrightarrow{\phi_{i+1}} \ldots$$

of maps of sheaves where the composition of any two consecutive maps equals zero, *i.e.* $\phi_{j-1} \circ \phi_j = 0$ for all j. We say that the sequence is *exact* at $\mathcal{F}_i$ if $\operatorname{Ker}\phi_i = \operatorname{Im}\phi_{i-1}$. The *short exact sequences* are the ones one most frequently encounters. They are the sequences of the form

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\ \phi\ } \mathcal{F} \xrightarrow{\ \psi\ } \mathcal{F}'' \longrightarrow 0 \tag{12.6}$$

that are exact at each stage. This is just another and very convenient way of simultaneously saying that ϕ is injective; that ψ is surjective; and that $\operatorname{Im}\phi = \operatorname{Ker}\psi$.

LEMMA 12.20 *A three term complex like the one in* (12.6) *is exact if and only if for each point $x \in X$, the induced sequence*

$$0 \longrightarrow \mathcal{F}'_x \xrightarrow{\ \phi\ } \mathcal{F}_x \xrightarrow{\ \psi\ } \mathcal{F}''_x \longrightarrow 0$$

of stalks is exact.

Proof: Bearing in mind that both forming kernels and images commute with forming stalks, this follows readily from the fact that a sheaf equals zero precisely when all stalks are zero. ❑

The following proposition will be very important:

Proposition 12.21 *For a short exact sequence* $0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ *and an open subset* U*, the following induced exact sequence is exact:*

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{\phi_U} \mathcal{F}(U) \xrightarrow{\psi_U} \mathcal{F}''(U). \tag{12.7}$$

Proof: By Lemma 12.2, the map ϕ is injective as a map of sheaves, hence injective on all open sets U, so the sequence above is exact at $\mathcal{F}'(U)$. To see it is exact in the middle as well, we need to show that $\operatorname{Ker}(\psi_U) = \operatorname{Im}(\phi_U)$. (The image presheaf is then given by the kernel of a morphism of sheaves, which is indeed a sheaf.)

That $\operatorname{Im}(\phi_U) \subseteq \operatorname{Ker}(\psi_U)$ is a consequence of taking sections being functorial: since $\psi \circ \phi = 0$, it follows that $\psi_U \circ \phi_U = (\psi \circ \phi)_U = 0$, so everything in $\operatorname{Im} \phi_U$ lies in the kernel of ψ_U.

For the opposite inclusion, $\operatorname{Ker}(\psi_U) \subseteq \operatorname{Im}(\phi_U)$, it may be helpful to look at the following diagram for $x \in U$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(U) \arrow[r, "\phi_U"] \arrow[d] & \mathcal{F}(U) \arrow[r, "\psi_U"] \arrow[d] & \mathcal{F}''(U) \arrow[d] & \\
0 \arrow[r] & \mathcal{F}'_x \arrow[r, "\phi_x"] & \mathcal{F}_x \arrow[r, "\psi_x"] & \mathcal{F}''_x \arrow[r] & 0
\end{tikzcd}$$

Note that the bottom row is exact since the sheaf sequence is exact.

Let $t \in \mathcal{F}(U)$ be so that $\psi_U(t) = 0$. Then for all $x \in U$ we have that $\psi_x(t_x) = (\psi_U(t))_x = 0$, and the germ t_x is an element in $\operatorname{Ker}(\psi_x) = \operatorname{Im}(\phi_x)$ (here we use exactness at the stalks). This means that for every $x \in U$ there is an element $s'_x \in \mathcal{F}'_x$, say represented by $(s'_{(x)}, V_{(x)})$ for some open neighbourhood $V_{(x)} \subseteq U$ of x and section $s'_{(x)} \in \mathcal{F}'(V_{(x)})$, such that $\phi_x(s'_x) = t_x$. For any two points $x, y \in U$ we then have

$$\phi_{V_{(x)} \cap V_{(y)}}(s'_{(x)}|_{V_{(x)} \cap V_{(y)}}) = t|_{V_{(x)} \cap V_{(y)}} = \phi_{V_{(x)} \cap V_{(y)}}(s'_{(y)}|_{V_{(x)} \cap V_{(y)}}),$$

so that by the injectivity of $\phi_{V_{(x)} \cap V_{(y)}}$ (which we have already proved), the required gluing condition

$$s'_{(x)}|_{V_{(x)} \cap V_{(y)}} = s'_{(y)}|_{V_{(x)} \cap V_{(y)}}$$

is satisfied, and the $s_{(x)}$'s patch together to give a section $s \in \mathcal{F}(U)$ with the property that for all $x \in U$

$$s|_{V_{(x)}} = s'_{(x)}.$$

We then conclude that for every $x \in U$

$$(\phi_U(s))_x = \phi_x(s_x) = \phi_x(s'_x) = t_x,$$

since $s_x = s'_x$. This gives $\phi_U(s) = t$ as desired. ❑

One way of phrasing Proposition 12.21 is to say that taking sections over an open set U is a *left exact functor*; that is, the functor $\Gamma(U, -)$ is left exact. This functor, however, is not right exact in general. The defect of this lacking surjectivity is a fundamental problem in every part of mathematics where sheaf theory is used, and to cope with it one has developed cohomology.

Examples

Let us give a few examples where the surjectivity on the right fails:

(12.22) (*Differential operators.*) Let $X = \mathbb{C}$ and recall the sheaf $\mathscr{A}_X$ of holomorphic functions and the map $D\colon \mathscr{A}_X \to \mathscr{A}_X$ sending $f(z)$ to the derivative $f'(z)$. There is an exact sequence

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathscr{A}_X \xrightarrow{D} \mathscr{A}_X \longrightarrow 0.$$

This hinges on the two following facts. Firstly, a function whose derivative vanishes identically is locally constant; hence the kernel $\operatorname{Ker} D$ equals the constant sheaf $\mathbb{C}_X$. Secondly, in small open disks any holomorphic function is a derivative: *e.g.* if $f(z) = \sum_{n\geqslant 0} a_n(z-a)^n$ in a small disk around a, the function $g(z) = \sum_{n\geqslant 0} a_n(n+1)^{-1}(z-a)^{n+1}$ has $f(z)$ as derivative. Thus the sequence is exact on stalks. However, taking sections over open sets U we merely obtain the sequence

$$0 \longrightarrow \Gamma(U, \mathbb{C}_X) \longrightarrow \Gamma(U, \mathscr{A}_X) \xrightarrow{D_U} \Gamma(U, \mathscr{A}_X).$$

Whether D_U is surjective or not, depends on the topology of U. If U is simply connected, one deduces from Cauchy's integral theorem that every holomorphic function in U is a derivative, so in that case D_U is surjective. On the other hand, if U is not simply connected, D_U is not surjective; *e.g.* if $U = \mathbb{C} - \{0\}$, the function z^{-1} is not a derivative in U (there is no globally defined logarithm on U).

(12.23) (*Ideal sheaves of points.*) Consider an algebraic variety X with the sheaf $\mathcal{O}_X$ of regular functions on X. For any point $p \in X$, let $k(p)$ denote the skyscraper sheaf whose only non-zero stalk is the field k located at p. There is a map of sheaves $\mathrm{ev}_p\colon \mathcal{O}_X \to k(p)$ sending a function that is regular in a neighbourhood of p to the value it takes at p. This map sits in the exact sequence of sheaves

See Example 3.10

$$0 \longrightarrow \mathfrak{m}_p \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{ev}_p} k(p) \longrightarrow 0,$$

where $\mathfrak{m}_p$ by definition is the kernel of ev_p (the sections of $\mathfrak{m}_p$ are the functions vanishing at p). Taking two distinct points p and q in X, we find a similar exact sequence

$$0 \longrightarrow \mathscr{I}_{p,q} \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{ev}_{p,q}} k(p) \oplus k(q) \longrightarrow 0,$$

where $\mathscr{I}_{p,q}$ is the sheaf of functions vanishing on p and q, and $\mathrm{ev}_{p,q}$ is the evaluation at the two points.

If for example $X = \mathbb{P}^1_{\mathbb{C}}$ (or any other projective variety), there are no global regular functions on X other than the constants, and hence $\Gamma(X, \mathcal{O}_X) = k$. But of course, $\Gamma(\mathbb{P}^1, k(p) \oplus k(q)) = k \oplus k$, so the map $\mathrm{ev}_{p,q}$ can not be surjective on global sections.

☆

Exercise 12.6 *(The exponential sequence.)* Let $X = \mathbb{C} - \{0\}$. The non-vanishing holomorphic functions in an open set $U \subseteq X$ form a *multiplicative* group, and there is a sheaf $\mathscr{A}_X^*$ with these groups as spaces of sections.

i) Show that $\mathscr{A}_X^*$ is a sheaf.

ii) For any holomorphic function f on U, show that the exponential $\exp f(z)$ is a section of $\mathscr{A}_X^*(U)$.

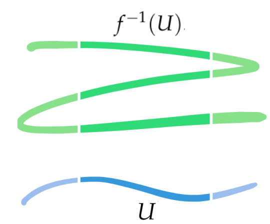
iii) Show that there is an exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathscr{A}_X \xrightarrow{\exp} \mathscr{A}_X^* \longrightarrow 0,$$

where the first map sends 1 to $2\pi i$.

★

12.3 *Direct and inverse images*



In Chapter 3 we defined the pushforward $f_*\mathcal{F}$ of a sheaf via a continuous map $f : X \to Y$. If $U \subseteq Y$ is an open set, the sections of $f_*\mathcal{F}$ over U was defined by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U),$$

and letting the restriction maps $\mathcal{F}(f^{-1}U) \to \mathcal{F}(f^{-1}V)$ be the ones from $\mathcal{F}$. This is covariant in $\mathcal{F}$: If $\mathcal{F} \to \mathcal{G}$ is a map of sheaves on X, we have an induced map $f_*\mathcal{F} \to f_*\mathcal{G}$ of sheaves on Y.

Lemma 12.24 *The functor f_* is left exact.*

Proof: This follows immediately from Proposition 12.21. ❑

Inverse images of sheaves

For a morphism $f : X \to Y$ and a sheaf $\mathcal{G}$ on Y, there is also an *inverse image*, $f^{-1}\mathcal{G}$ which is a sheaf on X. The definition is slightly more involved than the construction of the direct image.

We want to define the sheaf $f^{-1}\mathcal{G}$ over any open set $U \subset X$. However, the sheaf $\mathcal{G}$ only knows about the open sets in Y, and typically subsets of the form $f(U)$ will not be open in Y. The solution is to look at "germ-like" equivalence classes of sections $\mathcal{G}(V)$ as V runs over the collection of open sets containing $f(U)$. That is, we start out with the presheaf $f_p^{-1}\mathcal{G}$ on X defined by

$$\Gamma(U, f_p^{-1}(\mathcal{G})) = \varinjlim_{f(U) \subseteq V} \mathcal{G}(V), \tag{12.8}$$

where the indexing set consists of all open set V in Y containing $f(U)$. Every section of $f_p^{-1}\mathcal{G}$ over U is induced by a section of $\mathcal{G}$ over some V containing $f(U)$, and two such sections s and s' over V and V' induce the same section of $f_p^{-1}\mathcal{G}(U)$ if they agree over some open V'' with $f(U) \subseteq V'' \subseteq V \cap V'$. The restriction maps come from the universal property of the direct limit, since if $U' \subseteq U$, the set of opens containing $f(U)$ is contained in the set of those containing $f(U')$. And this leads us to the following:

DEFINITION 12.25 *Let $f\colon X \to Y$ be a continuous map and $\mathcal{G}$ a presheaf on Y. The* inverse image *$f^{-1}\mathcal{G}$ is the sheafification of the presheaf* (12.8)*.*

Note that while $\mathcal{G}$ may only be a presheaf, the resulting inverse image $f^{-1}\mathcal{G}$ is always a sheaf (sheafification is the special case when f is the identity map $f : X \to X$).

Forming the inverse image sheaf is again a functorial construction: Given a map $\phi\colon \mathcal{G} \to \mathcal{H}$ of sheaves on Y, we get a collection of maps $\phi_V\colon \mathcal{G}(V) \to \mathcal{H}(V)$ for open sets V containing $f(U)$ which are compatible with restrictions, and hence they pass to the direct limit and induce a map $f^{-1}\phi\colon f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$. So we get a functor $f^{-1}\colon \mathsf{AbPrSh}_Y \to \mathsf{AbSh}_X$.

While the sections of $f^{-1}\mathcal{G}$ may be hard to understand, the stalks are easy to compute: the stalk of $f^{-1}\mathcal{G}$ at a point $x \in X$ is isomorphic to $\mathcal{G}_{f(x)}$. Indeed, since sheafification preserves stalks, it suffices to verify this on the level of presheaves:

$$(f_p^{-1}\mathcal{G})_x = \varinjlim_{x \in U} f_p^{-1}\mathcal{G}(U) = \varinjlim_{x \in U} \varinjlim_{f(U) \subseteq V} \mathcal{G}(V) = \varinjlim_{f(x) \in V} \mathcal{G}(V) = \mathcal{G}_{f(x)}.$$

As we can detect exactness of a sequence of sheaves by looking at stalks, this has the following consequence:

LEMMA 12.26 *The functor f^{-1} is exact.*

EXAMPLE 12.27 If $\mathcal{F}$ is a sheaf on X and $U \subseteq X$ is an open subset, then $\mathcal{F}$ defines a sheaf on U by restriction of sections. For an arbitrary subset $Z \subseteq X$, this naive restriction does not directly give a sheaf on Z, because an open subset $V \subseteq Z$ is usually not open in X. However, letting $i : Z \to X$ denote the inclusion, we can form the inverse image $i^{-1}\mathcal{F}$ which is a sheaf on Z. The sheaf $i^{-1}\mathcal{F}$ is defined via the direct limit of $\mathcal{F}(V)$ where V runs over the open sets containing Z. In particular, when $Z = \{x\}$, we recognize the definition of the stalk, so that if $\iota\colon \{x\} \to X$ denotes the inclusion, it holds that $\iota^{-1}\mathcal{F} = \mathcal{F}_x$. ☆

EXAMPLE 12.28 The presheaf defined by (12.8) is not in general a sheaf. For instance, if $f\colon X \to Y$ is the constant map with image y, the inductive limit in (12.8) will be the stalk $\mathcal{G}_y$ whatever the open $U \subseteq X$ is. So (12.8) defines the constant presheaf with value $\mathcal{G}_y$, and as we observed in Example 3.9 on page 46, this is not always a sheaf. ☆

If you want a more concrete description of the sheaf $f^{-1}\mathcal{G}$, it is possible to define it as follows. For an open set $U \subset X$, the group $(f^{-1}\mathcal{G})(U)$ equals the set of elements $s = (s_x)_{x \in U} \in \prod_{x \in U} \mathcal{G}_{f(x)}$ with the following property: for each $x \in U$, there is a neighbourhood V of $f(x)$; a section $t \in \mathcal{G}(V)$; and a neighborhood $W \subset f^{-1}(V) \cap U$ of x such that $s_x = t_{f(x)}$ for every $x \in W$. It is not hard to check that this gives a sheaf, and that it coincides with the sheaf $f^{-1}\mathcal{G}$ given above.

The adjoint property

The definition of $f^{-1}\mathcal{G}$ is natural, but a little bit hard to work with for actual computations, as it involves both taking a direct limit over open sets and a sheafification. What's much more important is what this sheaf does: It is the adjoint of the functor f_* as a functor from AbSh_X to AbPrSh_Y (and the latter is a functor we understand well). The precise meaning behind that statement is the following:

THEOREM 12.29 *Let $f : X \to Y$ be a continuous map, let $\mathcal{F}$ be a sheaf on X and let $\mathcal{G}$ be a presheaf on Y. Then there is a natural bijection*

$$\mathrm{Hom}_{\mathsf{AbPrSh}_Y}(\mathcal{G}, f_*\mathcal{F}) \simeq \mathrm{Hom}_{\mathsf{AbSh}_X}(f^{-1}\mathcal{G}, \mathcal{F}),$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

This shows that the sheaf $f^{-1}\mathcal{G}$ constructed before satisfies a very natural universal property: sheaf morphisms $\phi\colon \mathcal{G} \to f_*\mathcal{F}$ correspond bijectively to maps $f^{-1}\mathcal{G} \to \mathcal{F}$. In particular, applying this to $\mathcal{F} = f^{-1}\mathcal{G}$, and $\mathcal{G} = f_*\mathcal{F}$ with the identity maps, we see that there is a canonical map of sheaves

$$\eta : \mathcal{G} \to f_* f^{-1}\mathcal{G},$$

which is functorial in $\mathcal{G}$, and

$$\nu\colon f^{-1} f_*\mathcal{F} \to \mathcal{F},$$

which is functorial in $\mathcal{F}$.

✲ EXERCISE 12.7 Prove Theorem 12.29. ★

Chapter 13

Sheaves of modules

$\mathcal{O}_X$*-modules*
$\bigcup$
Quasi-coherent sheaves
$\bigcup$
Coherent sheaves

When you study commutative algebra you might be primarily interested in the rings and ideals, but probably you start turning your interest towards the modules pretty quickly; they are an important part of the world of rings, and to get the results one wants, one can hardly do without them. The category Mod_A of A-modules is a fundamental invariant of a ring A; and in fact, it may be is the principal object of study in commutative algebra. There is also an analogue viewpoint for schemes for which the so called *quasi-coherent* $\mathcal{O}_X$-modules form an important attribute, if not a decisive part of the structure. They constitute a category QCoh_X with many properties paralleling those of the category Mod_A. In fact, in case the scheme X is affine, *i.e.* $X = \operatorname{Spec} A$, the two categories Mod_A and QCoh_X are equivalent as we will prove shortly. Imposing finiteness conditions on the $\mathcal{O}_X$-modules, one arrives at the category Coh_X of the so-called *coherent* $\mathcal{O}_X$-modules, which in the Noetherian case parallel the finitely generated A-modules.

We start out the chapter by describing the much broader concept of an $\mathcal{O}_X$-module. The theory here is presented for schemes, but the concept is meaningful for any locally ringed space.

13.1 Sheaves of modules

A module over a ring is an additive abelian group equipped with a multiplicative action of A. Loosely speaking, we can multiply members of the module by elements from the ring, and of course, the usual set of axioms must be satisfied. In a similar way, if X is a scheme, an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ whose sections over an open set U can be multiplied by sections of $\mathcal{O}_X(U)$.

More formally, we define an $\mathcal{O}_X$*-module* as a sheaf $\mathcal{F}$ equipped with multiplication maps $\mathcal{F}(U) \times \mathcal{O}_X(U) \to \mathcal{F}(U)$, one for each open subset U of X, making the space of sections $\mathcal{F}(U)$ into a $\mathcal{O}_X(U)$-module and this in a way compatible with all restrictions. In other words, for every pair of open subsets $V \subseteq U$, the natural diagram below is required to commute $\mathcal{O}_X$*-modules*

$$\begin{array}{ccc} \mathcal{F}(U) \times \mathcal{O}_X(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) \times \mathcal{O}_X(V) & \longrightarrow & \mathcal{F}(V). \end{array}$$

where vertical arrows represent restrictions maps and horizontal ones multiplication maps.

Homomorphisms of $\mathcal{O}_X$-modules

Maps, or *homomorphisms of $\mathcal{O}_X$-modules* are simply maps $\alpha\colon \mathcal{F} \to \mathcal{G}$ between $\mathcal{O}_X$-modules considered as sheaves such that for each open U the map $\alpha_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a map of $\mathcal{O}_X(U)$-modules. Thus we obtain a category of $\mathcal{O}_X$-modules, which we denote by Mod_X.

The category Mod_X is an *additive category*: the sum of two $\mathcal{O}_X$-homomorphisms as maps of sheaves is again an $\mathcal{O}_X$-homomorphism. So for all $\mathcal{F}$ and $\mathcal{G}$ the set $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ of $\mathcal{O}_X$-homomorphisms from $\mathcal{F}$ to $\mathcal{G}$ is an abelian group, and one verifies readily that the compositions maps are bilinear. Moreover, the direct sum of two $\mathcal{O}_X$-modules as sheaves has an obvious $\mathcal{O}_X$-structure with multiplication being defined componentwise. In fact, this definition works for arbitrary direct sums (or *coproducts* as they also are called). For any family $\{\mathcal{F}_i\}_{i\in I}$ of $\mathcal{O}_X$-modules the direct sum $\bigoplus_{i\in I}\mathcal{F}_i$ is an $\mathcal{O}_X$-module (see Exercise 13.3 below).

The notions of submodules, kernels, cokernels and images of $\mathcal{O}_X$-module homomorphisms now appear naturally. Each of these corresponding abelian constructions are compatible with the multiplication by sections of $\mathcal{O}_X$, and therefore they have $\mathcal{O}_X$-module structures. The respective defining universal properties (in the category of $\mathcal{O}_X$-modules) come for free, and one easily checks that this makes Mod_X an *abelian category*.

EXAMPLE 13.1 Ideal sheaves are important examples of $\mathcal{O}_X$-modules. Formally, a sheaf $\mathscr{I}$ is an ideal sheaf if $\mathscr{I}(U) \subset \mathcal{O}_X(U)$ for each open set $U \subset X$.

For a concrete example, consider $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, let p be the origin, and define for $U \subset \mathbb{A}^2$, the presheaf

$$\mathscr{I}(U) = \{ f \in \mathcal{O}_X(U) \mid f(p) = 0 \}.$$

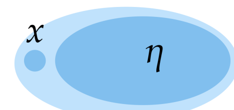
Then it is readily checked that $\mathscr{I}$ is a sheaf, and hence an ideal sheaf, because each $\mathscr{I}(U)$ is an ideal. ☆

EXAMPLE 13.2 In the same manner, the quotient sheaf $\mathcal{O}_X/\mathscr{I}$ is an $\mathcal{O}_X$-module. This is is a priori not completely obvious, because there is a sheafification involved in forming the quotient sheaf (however, see Exercise 13.2). ☆

EXAMPLE 13.3 If $\mathcal{F}$ is a sheaf obtained by gluing together sheaves $\mathcal{F}_i$ defined on a covering $\mathcal{U} = \{U_i\}$, and each $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module, then $\mathcal{F}$ is an $\mathcal{O}_X$-module. (To see this, use the explicit construction of $\mathcal{F}$ in Chapter 5). ☆

EXAMPLE 13.4 Write $\mathbb{P}^1$ for the projective line over a field k, and consider the sheaves $\mathcal{O}_X(a)$ from Section 6.9. That is, $\mathcal{O}_{\mathbb{P}^1}(a)$ is the sheaf obtained by gluing $\mathcal{O}_{U_0}$ to $\mathcal{O}_{U_1}$ using the isomorphism $\mathcal{O}_{U_1}|_{U_0\cap U_1} \to \mathcal{O}_{U_0}|_{U_0\cap U_1}$ on $U_0 \cap U_1 = \operatorname{Spec} k[x,x^{-1}]$ given by multiplication by x^a. Then $\mathcal{O}_{\mathbb{P}^1}(a)$ is an $\mathcal{O}_X$-module. As such it is a very special; it is 'locally free' in the sense that it restricts to the structure sheaf on the opens in an open covering (however, $\mathcal{O}_X(a) \simeq \mathcal{O}_X$ only for $a = 0$). ☆

EXAMPLE 13.5 (*(Modules on spectra of DVR's).)* Modules on the prime spectrum of a discrete valuation ring R are particularly easy to describe. Recall that the scheme $X = \operatorname{Spec} R$ has only two non-empty open sets, the whole space X itself and the singleton $\{\eta\}$ consisting of the generic point. The singleton $\{\eta\}$ is the underlying set of the open subscheme $\operatorname{Spec} K$, where K denotes the fraction field of R.



We claim that to give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space N and a R-module homomorphism $\rho\colon M \to N$.

Indeed, given an $\mathcal{O}_X$-module $\mathcal{F}$, we get the R-module $M = \mathcal{F}(X)$, and $N = \mathcal{F}(\{\eta\})$ which is a vector space over $K = \mathcal{O}_X(\{\eta\})$. The homomorphism ρ is just the restriction map $\mathcal{F}(X) \to \mathcal{F}(\{\eta\})$. Conversely, given the data M, N and ρ, we can define a presheaf $\mathcal{F}$ by setting $\mathcal{F}(X) = M$ and $\mathcal{F}(\{\eta\}) = N$ and as ρ is a map $\rho\colon \mathcal{F}(X) \to \mathcal{F}(\{\eta\})$, we can use it as the restriction map. If we also set $\mathcal{F}(\varnothing) = 0$, we have a presheaf $\mathcal{F}$ which satisfies the two sheaf axioms. Furthermore, since M and N are modules over $\mathcal{O}_X(X) = R$ and $O_X(\{\eta\}) = K$ respectively, this makes $\mathcal{F}$ into an $\mathcal{O}_X$-module.

Note that the restriction map can be just any R-module homomorphism $M \to N$. In particular, it can be the zero homomorphism, and in that case M and N can be completely arbitrary modules. Again, this illustrates the versatility of general $\mathcal{O}_X$-modules. ☆

EXAMPLE 13.6 (*(Godement sheaves again).*) Recall the Godement construction from Section 12.1 on page 175 in Chapter 3. Given any collection of abelian groups $\{A_x\}_{x\in X}$ indexed by the points x of X, we defined a sheaf $\mathcal{A}$ whose sections over an open subset U was $\prod_{x\in U} A_x$, and whose restriction maps to smaller open subsets were just the projections onto the corresponding smaller products. Requiring that each A_x be a module over the stalk $\mathcal{O}_{X,x}$, makes $\mathcal{A}$ into an $\mathcal{O}_X$-module; indeed, the space of sections $\Gamma(U, \mathcal{A}) = \prod_{x\in U} A_x$ is automatically an $\mathcal{O}_X(U)$-module, the multiplication being defined componentwise with the help of the stalk maps $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$. Clearly this module structures is compatible with the projections, and thus makes $\mathcal{A}$ into an $\mathcal{O}_X$-module. ☆

Exercises

(13.1) Assume that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules and that $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a map between them. Show that the kernel, cokernel and image of α as a map of sheaves indeed are $\mathcal{O}_X$-modules, and that they respectively are the kernel, cokernel and image of α in the category of $\mathcal{O}_X$-modules as well. Show that a complex of $\mathcal{O}_X$-modules is exact if and only it is exact as a complex of sheaves.

(13.2) Suppose that $\mathcal{F}$ is a *presheaf* of $\mathcal{O}_X$-modules (*i.e.* a presheaf satisfying the usual $\mathcal{O}_X$-module axioms). Show that the sheafification $\mathcal{F}^+$ naturally is an $\mathcal{O}_X$-module.

(13.3) Show that the category Mod_X has arbitrary products and coproducts, by showing that the products and coproducts in the category of sheaves AbSh_X are $\mathcal{O}_X$-modules and are the products, respectively the coproducts, in the category Mod_X.

(13.4) Assume that $p_1, \ldots, p_r$ is a set of primes, and let $\mathbb{Z}_{(p_i)}$ as usual denote the localization at the prime ideal (p_i). Let X be the scheme obtained by gluing the schemes $X_i = \operatorname{Spec} \mathbb{Z}_{(p_i)}$ together along their common open subschemes $\operatorname{Spec} \mathbb{Q}$. Describe the $\mathcal{O}_X$-modules on X.

(13.5) Let $A = \prod_{1\leqslant i\leqslant n} K_i$ be the product of finitely many fields and let $X = \operatorname{Spec} A$. Describe the category Mod_X.

★

The tilde of a module

The primary example of an $\mathcal{O}_X$-module is a sheaf of the form $\widetilde{M}$ which appeared in Chapter **??**. Let us briefly recall the construction. We work over an affine scheme $X = \operatorname{Spec} A$. For each A-module M we define a sheaf $\widetilde{M}$ on the distinguished open sets by

$$\widetilde{M}(D(f)) = M_f,$$

and letting restriction maps be the canonical localization maps: when $D(g) \subseteq D(f)$, it holds true that $g^n = af$ for some a and some n, and there is the canonical localization map $M_f \to M_g$ sending mf^{-r} to $a^r mg^{-nr}$. The same proof as for $\mathcal{O}_X$ (Proposition 4.3 on page 55) shows that this is actually a $\mathscr{B}$-sheaf, and hence gives rise to a unique *sheaf* on X, which we continue to denote by $\widetilde{M}$.

It is almost immediate that this is an $\mathcal{O}_X$-module. It is clear that $\widetilde{M}(D(f)) = M_f$ is a module over A_f, and if $U \subset X$ is any subset, we can cover it by distinguished open sets $D(f)$ and use the exact sequence of Proposition 4.11 to define the $\mathcal{O}_X(U)$-module structure on $\widetilde{M}(U)$

These sheaves enjoy a certain universal property among the $\mathcal{O}_X$-modules on affine schemes:

PROPOSITION 13.7 *Let $X = \operatorname{Spec} A$ be an affine scheme. For an A-module M and an $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural bijection*

$$\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \to \operatorname{Hom}_A(M, \mathcal{F}(X))$$

sending $\phi : \widetilde{M} \to \mathcal{F}$ to $\phi(X) : M \to \mathcal{F}(X)$.

PROOF: Given a map of A-modules $h : M \to \mathcal{F}(X)$, we need to construct the inverse $\phi : \widetilde{M} \to \mathcal{F}$ as follows. As usual, it suffices to tell what this map of sheaves does to sections over the distinguished opens $U = D(f)$. As $\mathcal{F}$ is an $\mathcal{O}_X$-module, multiplication by f^{-1} in the space of sections $\mathcal{F}(D(f))$ makes sense since $\mathcal{O}_X(D(f)) = A_f$. Therefore, we we can define a map of A_f-modules $M_f \to \mathcal{F}(D(f))$ by the composition

$$M_f \xrightarrow{h_f} \mathcal{F}(X)_f \to \mathcal{F}(D(f)).$$

This also works well with the restriction map, so we get a well defined map of sheaves $\widetilde{M} \to \mathcal{F}$. Taking $f = 1$, we see that this map on global sections coincides with h. ❑

In particular, applying this to $M = \mathcal{F}(X)$ and $\mathcal{F}(X) \to \mathcal{F}(X)$ the identity map, we find that for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a *unique* $\mathcal{O}_X$-module homomorphism

$$\beta \colon \widetilde{\mathcal{F}(X)} \to \mathcal{F} \tag{13.1}$$

that induces the identity on the spaces of global sections. Explicitly, over an open set $D(f)$ a section of $\widetilde{\mathcal{F}(X)}(D(f))$ is of the form $s \cdot f^{-n}$ for some $s \in \mathcal{F}(X)$, and β sends this to $s|_{D(f)} \cdot f^{-n} \in \mathcal{F}(D(f))$.

EXERCISE 13.6 Check that the map ϕ in the proof of Proposition 13.7 is a well defined map (there are choices involved in the definition). ★

EXERCISE 13.7 Show that, in the canonical identification of the distinguished open subset $D(f)$ with $\operatorname{Spec} A_f$, the $\mathcal{O}_X$-module $\widetilde{M}$ restricts to $\widetilde{M_f}$. ★

PROPOSITION 13.8 (PROPERTIES OF $\sim$) *i) The tilde-functor is additive; i.e. takes direct sums of modules to the direct sum and sums of maps to the corresponding sums.*

ii) For any two A-modules M and N, the association $\phi \to \widetilde{\phi}$ gives a bijection $\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ whose inverse is $\alpha \mapsto \alpha(X)$.

iii) The tilde-functor is exact.

PROOF: i) see Exercise 13.8. ii) follows from Proposition 13.7.

iii): Assume given an exact sequence of A-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

That the induced sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \widetilde{M'} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M''} \longrightarrow 0$$

is exact is a direct consequence of the three following facts. The stalk of a tilde-module $\widetilde{M}$ at the point x with corresponding prime ideal $\mathfrak{p}$ is $M_{\mathfrak{p}}$, localization is an exact functor, and finally, a sequence of sheaves is exact if and only if the sequence of stalks at every point is exact. ❑

Thus the tilde functor is fully faithful, and establishes an equivalence between Mod_A and a subcategory of Mod_X. It is a strict subcategory; most $\mathcal{O}_X$-modules are not of this form.

EXERCISE 13.8 Show that the tilde-functor is additive; *i.e.* takes direct sums of modules to the direct sum and sums of maps to the corresponding sums. ★

EXAMPLE 13.9 Assume that A is an integral domain and that K is the field of fractions of A. Show that the $\mathcal{O}_{\operatorname{Spec} A}$-module $\widetilde{K}$ is a constant sheaf in the strong sense; that is, $\Gamma(U, \widetilde{K}) = K$ for any non-empty open $U \subseteq X$, and that the restriction maps all equal the identity id_K. ☆

13.2 *Constructions involving $\mathcal{O}_X$-modules*

Tensor products

For two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ we may also define the *tensor product*, denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. As in many other cases, the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is defined by first describing a presheaf which subsequently is sheafified. The sections of the presheaf, temporarily denoted by $\mathcal{F} \otimes'_{\mathcal{O}_X} \mathcal{G}$, are defined in the natural way by

$$(\mathcal{F} \otimes'_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U). \tag{13.2}$$

The next lemma says that the tilde-functor takes the tensor product $M \otimes_A N$ of two A-modules to the tensor product $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ of the two corresponding $\mathcal{O}_X$-modules.

LEMMA 13.10 *There is a canonical isomorphism $\widetilde{M \otimes_A N} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.*

PROOF: As usual, let $\mathscr{B}$ be the basis for the Zariski topology consisting of distinguished open sets. The tensor product $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ is the sheaf associated to the presheaf T given as $U \mapsto \widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U)$. Over the distinguished open set $U = D(f)$ the sections of $\widetilde{M \otimes_A N}$ equals $(N \otimes_A M)_f$, so there is a map of $\mathscr{B}$-presheaves $T \to \widetilde{M \otimes_A N}$ coming from the assignment $m/f^a \otimes n/f^b$ to $(m \otimes n)/f^{a+b}$, which in fact induces an isomorphism $M_f \otimes_{A_f} N_f \simeq (M \otimes_A N)_f$.

After extending the $\mathscr{B}$-sheaves, we get a the desired map of sheaves

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_A N}.$$

This is an isomorphism since it is an isomorphism over every distinguished open set. ❑

EXAMPLE 13.11 Let us continue Example 13.4 on page 189. We claim that for each $a, b \in \mathbb{Z}$, there is an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_{\mathbb{P}^1}(b) \simeq \mathcal{O}_{\mathbb{P}^1}(a+b). \tag{13.3}$$

Indeed, we have $\mathcal{O}_{\mathbb{P}^1}(a)|_{U_i} = \mathcal{O}_{U_i}$ and $\mathcal{O}_{\mathbb{P}^1}(b)|_{U_i} = \mathcal{O}_{U_i}$ for each $i = 0, 1$. When we identify $\mathcal{O}_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i} = \mathcal{O}_{U_i}$, we see that the tensor product on the left-hand side is obtained by gluing the $\mathcal{O}_{U_i}$'s using the isomorphism $x^a \cdot x^b = x^{a+b}$, which is exactly the sheaf on the right-hand side.

By the way, this example also shows why the sheafification is neccessary in the definition of the tensor product. Indeed, over the open set $U = \mathbb{P}^1$, the presheaf $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes' \mathcal{O}_{\mathbb{P}^1}(1)$ has value

$$\mathcal{O}_{\mathbb{P}^1}(-1)(\mathbb{P}^1) \otimes_{\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)} \mathcal{O}_{\mathbb{P}^1}(1)(\mathbb{P}^1) = 0 \otimes_k k^2 = 0$$

On the other hand, (13.3) gives an isomorphism of sheaves $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_X$, and $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$. ☆

Hom sheaves

The sheaf of $\mathcal{O}_X$-homomorphisms

There is also a *sheaf of $\mathcal{O}_X$-homomorphisms* between $\mathcal{F}$ and $\mathcal{G}$. Recall from Example 3.5 on page 48 the sheaf $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ of homomorphisms between the sheaves $\mathcal{F}$ and $\mathcal{G}$ whose sections over an open set U is the group $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of homomorphisms between the restrictions $\mathcal{F}|_U$ and $\mathcal{G}|_U$. Inside this group one has the subgroup of the maps which are maps of $\mathcal{O}_X$-modules, and these subgroups, for different open sets U, are respected by the restriction maps. So they form the sections of a presheaf; it turns out to be a sheaf, and that is the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ of $\mathcal{O}_X$-homomorphisms from $\mathcal{F}$ to $\mathcal{G}$.

If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules of the form $\widetilde{M}$ and $\widetilde{N}$ on $X = \operatorname{Spec} A$, we would ideally like to say that $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the tilde of the module $\operatorname{Hom}_A(M, N)$. There is always a canonical map

$$\mathrm{Hom}_A(M,N)_f \to \mathrm{Hom}_{A_f}(M_f,N_f), \tag{13.4}$$

but without some finiteness condition (like being of finite presentation) on M, it is not necessarily an isomorphism, as the next example shows.

EXAMPLE 13.12 Even in the simplest case of an infinitely generated free module $M = \bigoplus_{i\in\mathbb{N}} Ae_i$ that map is not surjective. An element in $\mathrm{Hom}_A(M,N)_f$ is of the form αf^{-n} where $\alpha : M \to N$ is A-linear, as opposed to elements in $\mathrm{Hom}_{A_f}(M_f,N_f)$ which are given by their values $m_i f^{-n_i}$ on the free generators e_i, and the subtle point is that the n_i's may tend to infinity in which case we can find no n that works for all i's. ☆

EXERCISE 13.9 Recall that an A-module M is said to have *finite presentation* if there is an exact sequence of the form

$$A^m \to A^n \to M \to 0$$

Show that the map (13.4) above is an isomorphism when M is of finite presentation. Thus,

$$\mathrm{Hom}_A(M,N)^{\sim} \simeq \mathscr{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N}).$$

for any A-module N. HINT: First observe that this holds when $M = A$. Then use a presentation of M to reduce to that case. ★

Exercises

(13.10) Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules on the scheme X. Show that the tensor product $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ enjoys a universal property with respect to bilinear maps analogous to the one enjoyed by the tensor product of modules over a ring A.

✲ (13.11) Find an example of two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ so that the presheaf (13.2) is not a sheaf.

(13.12) Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules on the scheme X. Show that the stalk $(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G})_x$ at the point $x\in X$ is naturally isomorphic to the tensor product $\mathcal{F}_x\otimes_{\mathcal{O}_{X,x}}\mathcal{G}_x$ of the stalks $\mathcal{F}_x$ and $\mathcal{G}_x$. Show that the tensor product is right exact in the category of $\mathcal{O}_X$-modules.

(13.13) Show that the sheaf-hom $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ of two $\mathcal{O}_X$-modules as defined above on page 193 is a sheaf. Show that $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is right exact in the second variable and left exact in the first.

(13.14) *Adjunction between Hom and $\otimes$.* Show that for any three $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathscr{H}om_{\mathcal{O}_X}(\mathcal{G},\mathcal{H})) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G},\mathcal{H}),$$

which is functorial in all three variables. HINT: Reduce to the usual tensor product for modules over rings.

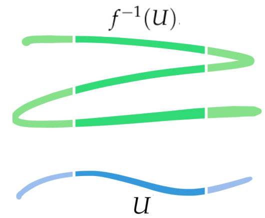
(13.15) With the notation of Example 13.11, show that

$$\mathscr{H}om_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(a),\mathcal{O}_{\mathbb{P}^1}(b)) \simeq \mathcal{O}_{\mathbb{P}^1}(b-a).$$

Conclude that there are no non-trivial maps of sheaves $\mathcal{O}_{\mathbb{P}^1}(a) \to \mathcal{O}_{\mathbb{P}^1}(b)$ when $a > b$. ★

13.3 *Pushforward and Pullback of $\mathcal{O}_X$-modules*

We have introduced two functors between the categories AbSh_X and AbSh_Y associated with a continuous map $f\colon X \to Y$ between topological spaces: the *pushforward* functor f_* and the *inverse image* functor f^{-1}. In this section we parallel these two constructions when f is a morphism of schemes to obtain functors f_* and f^* between Mod_X and Mod_Y. They form an adjoint pair of functors.



Pushforward

Let $f\colon X \to Y$ be a morphism of schemes. If $\mathcal{F}$ is a sheaf on X, recall that the pushforward $f_*\mathcal{F}$ was defined to be the sheaf on Y whose sections over an open U is $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$; in particular, we have $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}U)$. When $\mathcal{F}$ is an $\mathcal{O}_X$-module, it is then clear that each $f_*\mathcal{F}(U)$ is a module over $f_*\mathcal{O}_X(U)$ in a canonical way, and hence we may use the map of sheaves of rings $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ to equip $f_*\mathcal{F}$ with a natural $\mathcal{O}_Y$-module structure.

Pushforward

Direct images of $\mathcal{O}_X$-modules

DEFINITION 13.13 *The above $\mathcal{O}_Y$-module $f_*\mathcal{F}$ is called the* direct image *or the* pushforward *of $\mathcal{F}$ under f.*

This construction is clearly functorial in the sheaf $\mathcal{F}$, and so we obtain a functor $f_*\colon \mathsf{Mod}_X \to \mathsf{Mod}_Y$. That the pushforward f_* is left exact follows easily from Lemma **??** on page ?? in Chapter 3. It is also functorial in the morphism f in the sense that $(f \circ g)_* = f_* \circ g_*$ when f and g are composable morphism of schemes; indeed, this follows from the two equalities $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ and $(f \circ g)^\# = g^\# \circ f^\#$.

In the case when $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism between affine schemes, and $\mathcal{F} = \widetilde{M}$ for some B-module M, we can describe the pushfoward $f_*\mathcal{F}$ as follows. Let $\phi = f^\sharp(X)\colon A \to B$ be the ring map corresponding to f. The B-module M can be considered as a A-module via the map $\phi\colon A \to B$, and we denote this A-module by M_A. Then we have:

PROPOSITION 13.14 $f_*\widetilde{M} = \widetilde{M_A}$.

PROOF: Let $a \in A$. The crucial observation is that $f^{-1}(D(a)) = D(\phi(a))$ (indeed, a prime ideal $\mathfrak{p} \subset A$ satisfies $a \in \phi^{-1}(\mathfrak{p})$ if and only if $\phi(a) \in \mathfrak{p}$) from which it follows that

$$\Gamma(D(a), f_*(\widetilde{M})) = \Gamma(D(\phi(a)), \widetilde{M}) = M_{\phi(a)}.$$

Note that an element $a \in A$ acts on M_A as multiplication by $\phi(a)$: this means that the module on the right is isomorphic to $(M_A)_a = \Gamma(D(a), \widetilde{M_A})$. Thus there is an isomorphism of $\mathscr{B}$-sheaves $f_*\widetilde{M} \simeq \widetilde{M_A}$, and we are done. ❑

EXAMPLE 13.15 Consider the morphism $f : \mathbb{A}^1_k = \operatorname{Spec} k[x] \to \mathbb{A}^1_k = \operatorname{Spec} k[y]$ induced by the ring map $k[y] \to k[x]$ given by $y \mapsto x^n$ where $n > 0$. What is $f_*\mathcal{O}_{\mathbb{A}^1}$? Well, as a

$k[y]$-module, $k[x]$ is isomorphic to

$$k[y][x]/(y-x^n) \simeq k[y] \oplus k[y]x \oplus \cdots \oplus k[y]x^{n-1}.$$

Hence by Proposition13.14, we find that $f_*\mathcal{O}_{\mathbb{A}^1_k} \simeq \mathcal{O}^n_{\mathbb{A}^1_k}$. ☆

Pullback

If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, we also have a 'pullback' $f^*\mathcal{G}$, which is an $\mathcal{O}_X$-module, but this construction is a little bit more involved. Recall that we in Chapter 3 defined (Definition 12.25 on page 186) the inverse image $f^{-1}\mathcal{G}$ of a sheaf $\mathcal{G}$ by sheafifying the presheaf that assigns to an open subset $U \subseteq X$ the inverse limit $\varinjlim_{f(U)\subseteq V} \mathcal{G}(V)$ of all $\mathcal{G}(V)$ where V contains $f(U)$. When $\mathcal{G}$ is an $\mathcal{O}_Y$-module, this sheaf is naturally an $f^{-1}\mathcal{O}_Y$-module (because $f^{-1}\mathcal{O}_Y$ is expressed as an analogue inverse limit), and we can make $f^{-1}\mathcal{G}$ into an $\mathcal{O}_X$-module using the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (that makes $\mathcal{O}_X$ an $f^{-1}\mathcal{O}_Y$-algebra). Finally we take the tensor product and define:

$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}.$$

The assignment $\mathcal{G} \mapsto f^*\mathcal{G}$ is functorial, so we get a functor $f^* \colon \mathsf{Mod}_{\mathcal{O}_Y} \to \mathsf{Mod}_{\mathcal{O}_X}$. The above $\mathcal{O}_X$-module is called the *pullback* of $\mathcal{G}$ under f. *Pullbacks*

EXAMPLE 13.16 For any morphism $f : X \to Y$ we have

$$f^*\mathcal{O}_Y = \mathcal{O}_X.$$

Indeed, the tensor product identity $A \otimes_A B = B$ induces a natural isomorphism $f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = \mathcal{O}_X$ (after sheafifying the tensor product presheaf!). ☆

PROPOSITION 13.17 *For a point $x \in X$ we have the following expression for the stalk of the pullback*

$$(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}.$$

PROOF: This follows from the facts that taking stalks commutes with sheafification and tensor products, and $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$. ❑

Taking the pullback is a relatively complicated operation since it involves taking a direct limit, a tensor product, and finally a sheafification. The next result tells us that when X and Y are affine and $\mathcal{G}$ is a sheaf of the form $\widetilde{M}$ on Y, there is a much simpler description of the pullback $f^*\mathcal{G}$ which will allow us to do local computations more easily.

PROPOSITION 13.18 *Let $f \colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism induced by a ring map $\phi \colon A \to B$, and let M be an A-module. Then*

$$f^*(\widetilde{M}) = \widetilde{M \otimes_A B}. \tag{13.5}$$

PROOF: First, note that the proposition holds in the special case when M is a free module, *i.e.* $M = A^I$ (here the index set I is allowed to be infinite); this is simply because $f^*\mathcal{O}_Y = \mathcal{O}_X$

and f^* commutes with taking arbitrary direct sums. To prove it in general, we pick a presentation of M of the form

$$A^J \xrightarrow{\gamma} A^I \longrightarrow M \longrightarrow 0. \tag{13.6}$$

Applying the tilde functor followed by f^* we get a sequence

$$f^*(\widetilde{A^J}) \xrightarrow{f^*(\gamma)} f^*(\widetilde{A^I}) \longrightarrow f^*(\widetilde{M}) \longrightarrow 0$$

which is exact since both the tilde functor and f^* are right-exact. On the other hand, in a similar way, first tensorizing the sequence (13.6) by B and subsequently applying the tilde functor, we obtain the exact sequence (the tensor product is right-exact and the tilde functor exact)

$$\widetilde{A^J \otimes_A B} \xrightarrow{\gamma \otimes \mathrm{id}} \widetilde{A^I \otimes_A B} \longrightarrow \widetilde{M \otimes_A B} \longrightarrow 0.$$

Comparing the two sequences using that the proposition holds for free modules, finishes the proof. ❏

EXAMPLE 13.19 Consider again the map $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ from Example 13.15. Let us compute $f^*\widetilde{I}$ where I is the ideal $I = (y - a) \subset k[y]$. Explicitly, this is given by the tilde of the $k[x]$-module $(y-a)k[y] \otimes_{k[y]} k[x]$, a submodule of $k[y] \otimes_{k[y]} k[x]$, which is mapped to the ideal $(x^n - a)$ under the isomorphism $k[y] \otimes_{k[y]} k[x] \simeq k[x]$. Hence $f^*\widetilde{I}$ is the ideal sheaf of a subscheme of $\mathbb{A}^1_k$, the n-th roots of a when k is algebraically closed. ☆

Adjoint properties of f_* and f^*

At first sight, the definition of the pullback might seem a bit out of the blue. It is defined from $f^{-1}\mathcal{G}$, tensorizing with $\mathcal{O}_X$ over $f^{-1}\mathcal{O}_Y$ to rig it into being a $\mathcal{O}_X$-module. However, as in the case of the inverse image functor f^{-1}, the important point is what the sheaf does, rather than how it is explicitly defined. In the present case, the pullback is the adjoint of a functor which is easy to understand, namely f_*:

PROPOSITION 13.20 *The functors f_* and f^* between the categories $\mathsf{Mod}_{\mathcal{O}_X}$ and $\mathsf{Mod}_{\mathcal{O}_Y}$ are adjoint. In other words, if $\mathcal{F} \in \mathsf{Mod}_{\mathcal{O}_X}$ and $\mathcal{G} \in \mathsf{Mod}_{\mathcal{O}_Y}$, there is a functorial isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

PROOF: See Exercise 13.21. ❏

In particular, applying this to the two maps $\mathrm{id}_{f^*\mathcal{G}}$ and $\mathrm{id}_{f_*\mathcal{F}}$ provide us with the canonical maps

$$\eta \colon \mathcal{G} \to f_*f^*\mathcal{G}, \qquad \nu \colon f^*f_*\mathcal{F} \to \mathcal{F}.$$

Previously we already saw that f_* is a left-exact functor, and this implies that f^* is right-exact by the general property of adjoint functors that adjoints to left exact functors are right exact.

COROLLARY 13.21 *f^* is right exact and f_* is left-exact.*

In the case the two schemes are affine and the sheaves are of the form $\widetilde{N}$, we may understand these maps in the following way. Let the schemes be $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ and the two sheaves $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$ with M a module over B and N one over A. By what we saw above, it holds true that $f_*\widetilde{M} = \widetilde{M_A}$, and so we have

$$f^* f_* \widetilde{M} = \widetilde{M_A \otimes_A B}.$$

The point is that since the tensor product is over the ring A, we cannot move B over to the left hand side, but we do have a natural map of B-modules $M_A \otimes_A B \to M$, which is given by $m \otimes b \mapsto bm$, and the tilde of this map will be the map $\nu\colon \widetilde{M_A \otimes_A B} \to \widetilde{M}$.

To explain the other map η, note that $f^*\widetilde{N} = N \otimes_A B$. This yields

$$f_* f^* \widetilde{N} = (\widetilde{N \otimes_A B})_A,$$

and consequently η is induced by the map

$$N \to (N \otimes_A B)_A$$

given by $n \mapsto n \otimes_A 1$.

EXERCISE 13.16 Let as in the paragraph $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of affine schemes.

i) Verify the adjoint property of f_* and f^* for sheaves of the form $\widetilde{M}$: If $\mathcal{F} = \widetilde{M}$ for a B-module M and $\mathcal{G} = \widetilde{N}$ for an A-module N, show, using the adjoint properties of Hom and $\otimes$ that $\operatorname{Hom}_X(f^*\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$.

ii) By the Yoneda Lemma, an A-module P is completely determined by the functor $\operatorname{Hom}_A(P, -)$. Use this fact and the computation in *i)* to give a new proof that $f^*\widetilde{M} = \widetilde{M \otimes_A B}$.

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EXERCISE 13.17 Let $X = \operatorname{Spec} A$ and let $f \in A$ be an element. Denote by $\iota\colon D(f) \to X$ be the inclusion map. Describe the stalk of $\iota_*\mathcal{O}_{D(f)}$ at every point $x \notin D(f)$. ★

Pullback of sections

We can also pull back sections. If $\mathcal{G}$ is an $\mathcal{O}_Y$-module and $s \in \mathcal{G}(V)$, then we get a section $f^*(s) = \eta(s) \in \Gamma(f^{-1}(V), f^*\mathcal{G})$ by the map $\eta\colon \mathcal{G} \to f_* f^* \mathcal{G}$.

EXAMPLE 13.22 If $f\colon X \to Y$ is a morphism of affine varieties over k, and $\mathcal{G} = \mathcal{O}_Y$, then the above pullback coincides with the 'usual' pullback of regular functions. More precisely, if $V \subset Y$ is an open set, and $g\colon V \to k$ is a regular function, then $f^*(g)$ is an element of $\mathcal{O}_X(f^{-1}V)$, and can thus be regarded as a regular function on $f^{-1}(V)$; and $f^*(g)$ is of course nothing but the composition $g \circ f\colon f^{-1}(V) \to k$. ☆

Exercises

(13.18) Find examples of morphisms $f : X \to Y$ such that

a) $f_*\mathcal{O}_X \not\simeq \mathcal{O}_Y$
b) $f^{-1}\mathcal{O}_Y \not\simeq \mathcal{O}_X$

✲ (13.19)

a) Show that the natural map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism if and only if for each $x \in X$, the stalk map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is.
b) Let $f : X \to Y$ be a finite morphism of noetherian schemes. Show that $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism if and only if for each $x \in X$, there is an open set $U \subset X$ containing x and an open set $V \subset Y$ of $f(x)$ such that f induces an isomorphism $U \to V$.

✲ (13.20) Prove that applying f^* commutes with taking tensor products of sheaves, *i.e.* $f^*(\mathcal{G} \otimes \mathcal{H}) \simeq f^*\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{H}$ for any two $\mathcal{O}_X$-modules $\mathcal{G}$ and $\mathcal{H}$. Does the same hold for f_*?

✲ (13.21) Prove Proposition 13.20 above.

(13.22) Write out a proof that f^* is right-exact.

(13.23) Let $\iota \colon U \to X$ be the an open immersion. Show that for every $\mathcal{O}_X$-module $\mathcal{F}$ on U it holds true that $\iota^*\iota_*\mathcal{F} = \mathcal{F}$.

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Chapter 14

Quasi-coherent sheaves

The coherent and quasi-coherent sheaves are the most important types of sheaves in algebraic geometry. They form full subcategories of Mod_X denoted by Coh_X. QCoh_X. respectively. We start by defining quasi-coherent sheaves; they are the $\mathcal{O}_X$-modules which are locally of 'tilde type':

Definition 14.1 *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is a* quasi-coherent *if there is an open affine covering $\{U_i\}_{i\in I}$ of X, say $U_i = \operatorname{Spec} A_i$, and modules M_i over A_i such that $\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ for each i.*

Quasi-coherent sheaves

In particular, the modules $\widetilde{M}$ on affine schemes $\operatorname{Spec} A$ are all quasi-coherent.

Note that a priori, there could be more quasi-coherent sheaves on $\operatorname{Spec} A$. Indeed, for an $\mathcal{O}_X$-module $\mathcal{F}$ to be quasi-coherent, we require that $\mathcal{F}$ be locally of tilde-type for just *one* open affine cover. However, it turns out that this will hold for *any* open affine cover, or equivalently, that $\mathcal{F}|_U$ is of tilde-type for any open affine subset $U \subseteq X$. This is a much stronger than the requirement in the definition, and this fact will be important later on.

Lemma 14.2 *Let $X = \operatorname{Spec} A$ be an affine scheme and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for each $f \in A$, the canoical map*

$$\mathcal{F}(X)_f \to \mathcal{F}(D(f)) \tag{14.1}$$

is an isomorphism.

Proof: To prove this, we observe that since the sheaf $\mathcal{F}$ is quasi-coherent by hypothesis, and since the affine scheme $X = \operatorname{Spec} A$ is quasi-compact, there is a finite open affine covering of X by distinguished sets $D(g_i)$ such that $\mathcal{F}|_{D(g_i)} \simeq \widetilde{M}_i$ for some A_{g_i}-modules M_i.

The intuition here is that on $D(f)$ we allow sections that may have poles along $f = 0$, but they will extend to sections defined on all of X after multiplication by some power of f.

(14.1) is injective. Suppose $s \in \mathcal{F}(X)$ maps to zero in $\mathcal{F}(D(f))$. Let $s_i = s|_{D(g_i)}$ be the restriction of s to $D(g_i)$. We may view each s_i as an element of M_i. Consider now the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X) \arrow[r] \arrow[d] & \prod_i \mathcal{F}(D(g_i)) \arrow[r, equal] \arrow[d] & \prod_i M_i \arrow[d] \\
0 \arrow[r] & \mathcal{F}(D(f)) \arrow[r] & \prod_i \mathcal{F}(D(fg_i)) \arrow[r, equal] & \prod_i (M_i)_f
\end{tikzcd}$$

Further restricting $\mathcal{F}$ to the intersections $D(f) \cap D(g_i) = D(fg_i)$ yields equalities $\mathcal{F}|_{D(fg_i)} = (\widetilde{M}_i)_f$, and by hypothesis, the section s restricts to zero in $\Gamma(D(fg_i), \mathcal{F}) = (M_i)_f$. This means that the localization map sends s_i to zero in $(M_i)_f$. Hence s_i is killed by some power of f, and since there is only finitely many g_i's, there is an n with $f^n s_i = 0$ for all i; that is, $(f^n s)|_{D(g_i)} = 0$ for all i. By the locality axiom, it follows that $f^n s = 0$.

(14.1) is surjective. Let $s \in \mathcal{F}(D(f))$ be any section. Our task is to show that $f^n s$ extends to a global section t of $\mathcal{F}$ for some n; then s is the image of t/f^n. Each restriction $s|_{D(fg_i)} \in \Gamma(D(fg_i), \mathcal{F}) = (M_i)_f$ is of the form $f^{-n} s_i$ with $s_i \in M_i = \Gamma(D(g_i), \mathcal{F})$, and by the usual finiteness argument, n can be chosen uniformly for all i. This means that the different $s_i = f^n s$ and $s_j = f^n s$ match on the intersections $D(f) \cap D(g_i) \cap D(g_j)$, and by the injectivity part of the lemma applied to $\operatorname{Spec} A_{g_i g_j}$, one has $f^N(s_i - s_j) = 0$ on $D(g_i) \cap D(g_j)$ for a sufficiently large integer N. Hence the different $f^N s_i$'s patch together to give the desired global section t of $\mathcal{F}$. ❑

Theorem 14.3 *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasi-coherent if and only if for any open affine subsets $U = \operatorname{Spec} A$, the restriction $\mathcal{F}|_U$ is isomorphic to an $\mathcal{O}_X$-module of the form $\widetilde{M}$ for an A-module M.*

Proof: As quasi-coherence is conserved when restricting $\mathcal{O}_X$-modules to open sets, we may surely assume that X itself is affine; say $X = \operatorname{Spec} A$. Let $M = \mathcal{F}(X)$. We saw in (13.1), that there is a natural map of sheaves $\beta : \widetilde{M} \to \mathcal{F}$ that on distinguished open sets sends mf^{-n} to $f^{-n} m|_{D(f)}$. Over $D(f)$, this map is an isomorphism, by the previous lemma, so the two sheaves are isomorphic. ❑

Applying the theorem to affine schemes yields the important fact that any quasi-coherent sheaf $\mathcal{F}$ on an affine scheme $X = \operatorname{Spec} A$ is of the form $\widetilde{M}$ for an A-module M.

Theorem 14.4 *Assume that $X = \operatorname{Spec} A$. The tilde-functor $M \mapsto \widetilde{M}$ is an equivalence of the categories Mod_A and QCoh_X with the global section functor as inverse.*

When speaking about mutually inverse functors one should be very careful; in most cases such a statement is an abuse of language. Two functors $\mathcal{F}$ and $\mathcal{G}$ are mutually inverses when there are natural transformations, both being an isomorphism, between the compositions $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ and the appropriate identity functors. In the present case one really has an *equality* $\Gamma(X, \widetilde{M}) = M$, so that $\Gamma \circ \widetilde{(-)} = \mathrm{id}_{\mathsf{Mod}_A}$. On the other hand, the natural transformation $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$ from Lemma **??** on page ?? gives just an isomorphism of functors.

Theorem 14.3 has the important corollary that the global section is an exact functor:

Corollary 14.5 *Let $X = \operatorname{Spec} A$ be an affine scheme. Then the global section functor $\Gamma(X, -)\colon \mathsf{QCoh}_X \to \mathsf{Mod}_A$ is exact. In other words, if*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of quasi-coherent sheaves, then

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0$$

is also exact.

Proof: The inverse to any exact equivalence of categories is exact. ❏

Example 14.6 *(Quasi-coherent modules on $\mathbb{P}^1$.)* Consider the projective line $\mathbb{P}^1_k$ over k. It comes equipped with the usual affine open covering $U_0 = \operatorname{Spec} k[x]$ and $U_1 = \operatorname{Spec} k[x^{-1}]$ which are glued together along $\operatorname{Spec} k[x, x^{-1}]$. A quasi-coherent sheaf on $\mathbb{P}^1_k$ is given by a triple (M_0, M_1, τ), where

i) M_0 is a module over $\mathcal{O}_X(U_0) = k[x]$;
ii) M_1 is a module over $\mathcal{O}_X(U_1) = k[x^{-1}]$;
iii) τ is an isomorphism of modules over $k[x, x^{-1}]$:

$$\tau : M_1 \otimes_{k[x^{-1}]} k[x, x^{-1}] \to M_0 \otimes_{k[x]} k[x, x^{-1}].$$

Concrete examples are the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ from Section 6.9; where the data are $M_0 = k[x]$, $M_1 = k[x^{-1}]$ and $\tau : k[x, x^{-1}] \to k[x, x^{-1}]$ is the multiplication by x^n. ☆

Example 14.7 *(Quasi-coherent sheaves on spectra of* DVR*'s.)* The example of an discrete valuation ring is always useful to consider, and we continue exploring Example 13.5 above. Consider the $\mathcal{O}_X$- module $\mathcal{F}$ given by the data M, N, ρ. We claim that $\mathcal{F}$ is quasi-coherent if and only if $\rho \otimes K\colon M \otimes_R K \to N$ is an isomorphism (of K-vector spaces).

If $\mathcal{F}$ is quasi-coherent, then every point has a neighbourhood on which $\mathcal{F}$ is the tilde of some module. The only neighbourhood of the unique closed point is X itself, and so $\mathcal{F} = \widetilde{M}$. Therefore, $N = \mathcal{F}(U) = M_{(0)} = M \otimes_R K$ and ρ is an isomorphism. Conversely, if $\rho \otimes K : M \otimes_R K \to K$ is an isomorphism, then $\mathcal{F}$ is given by $\mathcal{F}(X) = M$ and $\mathcal{F}(\{\eta\}) = M \otimes_R N$, and so $\mathcal{F} \simeq \widetilde{M}$ is quasi-coherent. ☆

Another nice consequence of the equivalence in Theorem 14.4 is that any purely categorical construction commutes with the tilde-functor — any universal property that holds in Mod_X holds as well in QCoh_X. For instance, if $\{M_i\}_{i \in I}$ is a directed system if modules, it will be true that $(\varinjlim M_i)^{\sim}$ is the direct limit $\varinjlim \widetilde{M}_i$ in the category QCoh_X; and in fact in Mod_X as well.

Exercise 14.1 Let $X = \operatorname{Spec} A$ and let $\{M_i\}_{i \in I}$ be a directed system of modules. Then $(\varinjlim M_i)^{\sim}$ is the direct limit in Mod_X of the system $\{\widetilde{M}_i\}$. ★

Exercise 14.2 *(Direct limits of quasi-coherent sheaves.)* In general, if $\mathcal{F}$ is a directed system of $\mathcal{O}_X$-modules, one defines the direct limit $\varinjlim \mathcal{F}_i$ by sheafifying the presheaf that sends U to $\varinjlim \mathcal{F}_i(U)$.

i) Show that $\varinjlim \mathcal{F}_i$ is the direct limit in the category Mod_X; that is, it has the required universal property;

ii) Show that forming the direct limit commutes with restriction; *i.e.* for every open $U \subseteq X$ it holds that $\varinjlim(\mathcal{F}_i|_U) = (\varinjlim \mathcal{F}_i)|_U$;

iii) Show that if all the $\mathcal{F}_i$'s are quasi-coherent, then the direct limit $\varinjlim \mathcal{F}_i$ is quasi-coherent.

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EXERCISE 14.3 Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. Show that $\mathcal{F}$ is quasi-coherent if and only if for any pair $V \subseteq U$ open affine subsets the natural map

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V) \tag{14.2}$$

is an isomorphism. This is one explaination of the word 'coherence': the sections of $\mathcal{F}(V)$ can not be completely arbitrary, they fit together with the sections of $\mathcal{F}(U)$ for any larger affine $U \supset V$. ★

14.1 *Coherent sheaves*

There are many different 'finiteness conditions' one can impose on A-modules M over a ring A. The simplest is perhaps that M should be finitely generated, or equivalently, that there is a surjection $A^n \to M$ for some n. In the non-noetherian setting, there are also stronger conditions, for instance that M should be of *finite presentation*, which means that there should be an exact sequence

$$A^m \to A^n \to M \to 0.$$

(Thus M is finitely generated, and also the relations among the generators is finitely generated.) For sheaves, there are many similar conditions, some local, and some of global nature:

Definition 14.8 *Let X be a scheme and let $\mathcal{F}$ be a $\mathcal{O}_X$-module.*

- $\mathcal{F}$ *is* generated by global sections, *or,* globally generated, *if there is a surjection of $\mathcal{O}_X$-modules*
$$\mathcal{O}_X^I \to \mathcal{F}$$
This is the same thing as picking sections $s_i \in \mathcal{F}(X)$ such that the stalks $(s_i)_x$ generate $\mathcal{F}_x$ for every $x \in X$.
- $\mathcal{F}$ *is* locally generated by sections, *if, for every $x \in X$, there is an open set such that $\mathcal{F}|_U$ is generated by sections.*
- $\mathcal{F}$ *is of* finite type, *if if, for every $x \in X$, there is an open set such that $\mathcal{F}|_U$ is generated by* finitely many *sections. Equivalently, for each $x \in X$, there is a surjection $\mathcal{O}_{X,x}^n \to \mathcal{F}_x$.*
- $\mathcal{F}$ *is of* finite presentation *if for every $x \in X$, there is an open set $U \subset X$ so that $\mathcal{F}|_U$ is the cokernel of a map between two free modules of finite rank:*
$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$
- $\mathcal{F}$ *is* coherent *if it is of finite type, and, for any open $U \subset X$ the kernel of each map of $\mathcal{O}_U$-modules $\mathcal{O}_U^n \to \mathcal{F}|_U$ is of finite type.*

In contrast, Exercise 14.4 will show that $\mathcal{F}$ is quasi-coherent if for every $x \in X$, there is an open set $U \subset X$ so that $\mathcal{F}|_U$ is the cokernel of a map between two free modules, i.e., there is an exact sequence
$$\mathcal{O}_U^J \to \mathcal{O}_U^I \to \mathcal{F}|_U \to 0. \tag{14.3}$$

When X is Noetherian, which is frequently the case in algebraic geometry, the three conditions being *coherent*, being *finite type* and *being of finite presentation* coincide. The key point is that every submodule $N \subseteq M$ of a finitely generated module M over a Noetherian ring A is also finitely generated. For instance, to show that a finitely generated module M over a Noetherian ring is coherent, we just consider a map $\alpha\colon A^m \to M$. Since A is Noetherian, every submodule of A^m is finitely generated, in particular the kernel $\operatorname{Ker}\alpha$ will be.

One benefit of using coherent modules rather than finitely generated ones is that the category of coherent modules is an abelian category, even in the non-Noetherian setting. However, a problem is that coherence is very difficult to check in general, and actually, for some schemes, even affine ones, the structure sheaf $\mathcal{O}_X$ is not coherent!

Example 14.9 (*(A ring that is not coherent).*) The following is an almost tautological example of a monogenic module that is not coherent. Let $R = k[x, y, t_i, u_i | i \in \mathbb{N}]$ and $\mathfrak{a} = (t_i x - u_i y | i \in \mathbb{N})$. Then the R-module $A = R/\mathfrak{a}$ is not coherent: the ideal (x, y) is finitely generated, but the relations are not. Indeed, map the free module $Re \oplus Rf$ with basis e, f into A by sending $e \to x$ and $f \to y$. The kernel has generators $u_i e_1 + t_i e_2$ for $i \in \mathbb{N}$ and is not finitely generated; its image in R under *e.g.* the first projection equals the ideal $(u_i | i \in \mathbb{N})$ which for sure is not finitely generated. ☆

Note that each of the properties above, except being globally generated, is local. That is, $\mathcal{F}$ is (quasi)-coherent in a neighbourhood of every point, it will be quasi-coherent.

EXERCISE 14.4 Show that an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if there is an exact sequence of the form (14.3) for each $U \subset X$. ★

14.2 *Categorical and Functorial properties*

There is an important strengthening of Corollary 14.5 on page 202 to the case that merely the leftmost sheaf is quasi-coherent and without other restrictions on the two others than being $\mathcal{O}_X$-modules (and, in fact, the not yet established 'white magic of cohomology' will show that even this is not necessary). The result is a special case of a very central result that the so-called cohomology groups $H^i(X, \mathcal{F})$ vanish for $i > 0$ when $\mathcal{F}$ is quasi-coherent on an affine X; but we find in worth while to anticipate the general result in order to complete the picture of quasi-coherent modules, and it will also be required in the proof of Proposition 14.12 below.

These groups are constructed to cope with the global section functor not being exact. They will be extensively treated later on starting in Chapter 17; see also Appendix **??**

PROPOSITION 14.10 *Let X be an affine scheme and $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ three $\mathcal{O}_X$-modules. Assume they live in the short exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0.$$

If $\mathcal{F}$ is quasi-coherent, the sequence of global sections

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{H}(X) \longrightarrow 0$$

is exact.

We begin by stating a lemma:

LEMMA 14.11 *Let $\sigma \in \mathcal{H}(X)$ be a section. If $D(f)$ is a distinguished open so that $\sigma|_{D(g)}$ can be lifted to a section s of $\mathcal{G}(D(f))$, then for sufficiently large integers n, the section $f^n\sigma$ may be lifted to $\mathcal{G}(X)$*

Given the lemma, the proposition follows by a standard 'partition of unity argument': chose a finite covering $\{D(f_i)\}$ of $X = \operatorname{Spec} A$ so that each restriction $\sigma|_{D(f_i)}$ lifts to a section of $\mathcal{G}(D(f_i))$. According to the lemma there is an n so that each $f_i^n\sigma|_{D(f_i)}$ lifts to a section $\tau_i \in \mathcal{G}(X)$. Since the $D(f_i^n)$'s cover X, we may write $1 = \sum_i a_i f^n$, and then $\sum_i a_i\tau_i$ lifts σ.

PROOF OF THE LEMMA: The proof is a 'patching'-proof with two steps using a finite open affine cover $\{D(g_i)\}$ of X: firstly, we extend the sections $f^n s|D(fg_i)$ with $n >> 0$ to sections t_i of $\mathcal{G}(D(g_i))$, secondly we patch them together, which requires a new power of f as factor.

Chose the cover $\{D(g_i)\}$ of X by finitely many distinguished open affines, all so small that $\sigma|_{D(g_i)}$ extends to a section $s_i \in \mathcal{G}(D(g_i))$. Over $D(g_i f)$ the two sections $s_i|_{D(g_i f))}$ and $s|_{D(g_i f)}$ both lift $\sigma|_{D(g_i f)}$, and hence their difference belongs to the space $\mathcal{F}(D(g_i f))$. By Lemma 14.2 on page 200 and the fact that the covering $\{D(g_i)\}$ is finite, for n sufficiently large there are sections $r_i \in \mathcal{F}(D(g_i))$ such that $r_i|_{D(g_i f)} = f^n s|_{D(g_i f)} - f^n s_i|_{D(g_i f)}$. Then $t_i = r_i + f^n s_i$ are sections of $\mathcal{G}(D(g_i))$ that restrict to $f^n s|_{D(g_i f)}$.

Now, we would want to glue the t_i's together to a global section of $\mathcal{G}$, and this can be done at least after giving each t_i a factor a high power of f: let $U_{ij} = D(g_i) \cap D(g_j) = D(g_i g_j)$ and consider $t_i|_{U_{ij}} - t_j|_{U_{ij}} \in \mathcal{G}(U_{ij})$. It maps to zero in $D(g_i g_j f)$, and by Lemma 14.2 is therefore killed by a high power $f^{n_{ij}}$ of f; and as usual, we may use the same exponent, m say, for every pair i, j. It follows that the sections $f^m t_i$ of $\mathcal{G}(D(g_i))$ coincide on intersections $D(g_i) \cap D(g_j)$ and they can thus be glued together to a section t of $\mathcal{G}$, and this maps to $f^{n+m}\sigma$ since each $f^m t_i$ maps to $f^{n+m}\sigma|_{D(g_i)}$. ❑

The category of quasi-coherent sheaves has several nice properties:

PROPOSITION 14.12 *Suppose that $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent sheaves on the scheme X.*

i) The kernel, cokernel and the image of α are all quasi-coherent.

ii) The category QCoh_X is closed under extensions; that is, if

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{14.4}$$

is a short exact sequence of $\mathcal{O}_X$-modules with the two outer sheaves $\mathcal{F}$ and $\mathcal{H}$ being quasi-coherent, the middle sheaf $\mathcal{G}$ is quasi-coherent as well.

PROOF: If $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent $\mathcal{O}_X$-modules, on any open affine subsets $U = \operatorname{Spec} A$ of X it may be described as $\alpha|_U = \tilde{a}$ where $a\colon M \to N$ is a A-module homomorphism and M and N are A-modules with $\mathcal{F}|_U = \widetilde{M}$ and $\mathcal{G}|_U = \tilde{N}$. Since the tilde-functor is exact, one has $\operatorname{Ker}\alpha|_U = (\operatorname{Ker} a)\tilde{}$. Moreover, by the same reasoning, it holds true that $\operatorname{Coker}\alpha|_U = (\operatorname{Coker} a)\tilde{}$ and $\operatorname{Im}\alpha|_U = (\operatorname{Im} a)\tilde{}$.

Suppose now that an extension like (14.4) is given with $\mathcal{F}$ and $\mathcal{H}$ quasi-coherent, the two other cases are cover by the first statement. By Proposition 14.10 above the induced sequence of global sections is exact; that is, the upper horizontal sequence in the diagram below is exact. The three vertical maps in the diagram are the natural maps from Lemma **??** on page **??**. Since $\mathcal{F}$ and $\mathcal{H}$ both are quasi-coherent sheaves, the two outer vertical maps are isomorphisms, and the snake lemma implies that the middle vertical map is an isomorphism as well. Hence $\mathcal{G}$ is quasi-coherent.

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathcal{F}(X)} \arrow[r] \arrow[d] & \widetilde{\mathcal{G}(X)} \arrow[r] \arrow[d] & \widetilde{\mathcal{H}(X)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0
\end{tikzcd}$$

❏

Thus also for a general scheme X, the category QCoh_X is a category with very nice properties: it is an abelian category with tensor products and internal Hom's.

The categories of coherent and quasi-coherent sheaves

Our work in the previous sections imply the following theorem

Theorem 14.13 *The category of quasi-coherent sheaves QCoh_X is an abelian category and is closed under direct limits.*

By definition, being an abelian category entails that the hom-sets $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ are abelian groups; finite direct sums exist; kernels and cokernels of morphisms exist; every monomorphism $\mathcal{F} \to \mathcal{G}$ is the kernel of its cokernel; every epimorphism is the cokernel of its kernel; and every morphism can be factored into an epimorphism followed by a monomorphism. The hard part is thus in the last part of the statement, that any direct limit of quasi-coherent sheaves is again quasi-coherent.

One reason why we prefer the notion of 'coherence' used here (rather than the one in Hartshorne ([**?**])) is that the category of coherent sheaves Coh_X is also an abelian category, even in the non-noetherian case. Note that it does *not* contain all its direct limits, simply because an arbitrary product of coherent A-modules is typically not coherent (not even finitely generated!)

Quasi-coherence of pullbacks

Recall that for a morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes, the pullback of a quasi-coherent sheaf $\widetilde{M}$ is again quasi-coherent which follows from the formula

$$f^*(\widetilde{M}) = \widetilde{M \otimes_A B}$$

in Theorem 13.18. In fact, the same conclusion holds quite generally:

Proposition 14.14 *Let $f\colon X \to Y$ be a morphism of schemes.*

i) If $\mathcal{G}$ is a quasi-coherent sheaf on Y, then $f^*\mathcal{G}$ is quasi-coherent on X;
ii) If moreover X and Y are Noetherian, then $f^*\mathcal{G}$ is coherent if $\mathcal{G}$ is.

Proof: The first of these statements follows from the affine case and the formula above since being quasi-coherent is a local property. Moreover, since $M \otimes_A B$ is a finitely generated B-module if M is a finitely generated A-module, the second follows from the affine case as well. ❏

Quasi-coherence of pushforwards

Likewise, we showed that for a map $f : \operatorname{Spec} B \to \operatorname{Spec} A$, the pushforward $f_*\mathcal{F}$ is quasi coherent if $\mathcal{F}$ is quasi-coherent (since $f_*\widetilde{M} = \widetilde{M_A}$). The same holds true for a large class of morphisms:

Theorem 14.15 *Let $f \colon X \to Y$ be a morphism of schemes and that $\mathcal{F}$ is a quasi-coherent sheaf on X. If X is Noetherian, then the direct image $f_*\mathcal{F}$ is quasi-coherent on Y.*

Proof: We may assume that $Y = \operatorname{Spec} B$, and X is the quasi-compact and may be covered it by finitely many open affines U_i. Each intersection $U_i \cap U_j$ is again quasi-compact and we cover it with finitely many open affines U_{ijk}.

For any open $V \subseteq Y$, one has the exact sequence

$$0 \to \Gamma(f^{-1}V, \mathcal{F}) \to \prod_i \Gamma(U_i \cap f^{-1}V, \mathcal{F}) \to \prod_{i,j,k} \Gamma(U_{ijk} \cap f^{-1}V, \mathcal{F}). \tag{14.5}$$

The sequence is compatible with the restriction maps induced from inclusions $U \subseteq V$, hence gives rise to the following exact sequence of sheaves on X:

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \prod_i f_{i*}\mathcal{F}|_{U_i} \longrightarrow \prod_{i,j,k} f_{ijk*}\mathcal{F}|_{U_{ijk}} \tag{14.6}$$

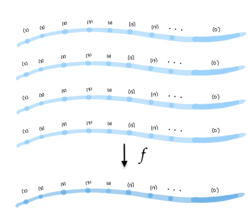
where $f_i = f|_{U_i}$ and $f_{ijk} = f|_{U_{ijk}}$. Now, each of the sheaves $f_i{*}\mathcal{F}|_{U_i}$ and $f_{ij}{*}\mathcal{F}|_{U_{ij}}$ are quasi-coherent by the affine case of the theorem (Proposition 13.14 on page 195. They are finite in number as the covering U_i is finite. Hence $\prod_i f_{i*}\mathcal{F}|_{U_i}$ and $\prod_{i,j} f_{ij*}\mathcal{F}|_{U_{ij}}$ are finite products of quasi-coherent $\mathcal{O}_X$-modules and therefore they are quasi-coherent. Now, the sheaf $f_*\mathcal{F}$ equals the kernel of a homomorphism between two quasi-coherent sheaves, and so the theorem follows from Proposition 14.12 on page 206. ❑

The following example shows the proposition fails if X is not assumed to be Noetherian:

Example 14.16 Let $X = \coprod_{i \in I} \operatorname{Spec} \mathbb{Z}$ be the disjoint union of countably infinitely many copies of $\operatorname{Spec} \mathbb{Z}$ and let $f \colon X \to \operatorname{Spec} \mathbb{Z}$ be the morphism that equals the identity on each of the copies of $\operatorname{Spec} \mathbb{Z}$ which constitute X. Then $f_*\mathcal{O}_X$ is not quasi-coherent. Indeed, the global sections of $f_*\mathcal{O}_X$ satisfy

$$\Gamma(\operatorname{Spec} \mathbb{Z}, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \prod_{i \in I} \mathbb{Z}.$$

On the other hand if p is any prime, one has



$$\Gamma(D(p), f_*\mathcal{O}_X) = \Gamma(f^{-1}D(p), \mathcal{O}_X) = \prod_{i \in I} \mathbb{Z}[p^{-1}].$$

It is not true that $\Gamma(D(p), f_*\mathcal{O}_X) = \Gamma(\operatorname{Spec} \mathbb{Z}, f_*\mathcal{O}_X) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$. Indeed, elements in $\prod_{i \in I} \mathbb{Z}[p^{-1}]$ are sequences of the form $(z_i p^{-n_i})_{i \in I}$ where $z_i \in \mathbb{Z}$ and $n_i \in \mathbb{N}$. Such an element lies in $(\prod_{i \in I} \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ only if the n_i's form a bounded sequence, which is not the case for general elements of shape $(z_i p^{-n_i})_{i \in I}$ when I is infinite. In particular, $f_*\mathcal{O}_X$ is not quasi-coherent. ☆

Coherence of pushforwards

For morphisms of schemes $f\colon X \to Y$ in general it cannot be expected that the pushforward of a coherent sheaf is again coherent, even for very 'nice' morphisms f. A simple example is the following:

EXAMPLE 14.17 Let $X = \operatorname{Spec} k[t]$ and consider the structure morphism $f\colon X \to \operatorname{Spec} k$, which is induced by the inclusion $k \subseteq k[t]$). The sheaf $\mathcal{O}_X$ is certainly coherent, but $f_*\mathcal{O}_X$ is not. Indeed, the latter sheaf equals $\widetilde{k[t]}$, and $k[t]$ is certainly not finitely generated as a k-module. ☆

However, for *finite* morphisms, we have a positive result:

LEMMA 14.18 *Let $f\colon X \to Y$ be a finite morphism of schemes. If $\mathcal{F}$ be a quasi-coherent sheaf on X, then $f_*\mathcal{F}$ is quasi-coherent on Y. If X and Y are Noetherian, $f_*\mathcal{F}$ is even coherent if $\mathcal{F}$ is.*

PROOF: Since f is finite, we can cover Y by open affines $\operatorname{Spec} A$ such that each $f^{-1}\operatorname{Spec} A = \operatorname{Spec} B$ is also affine, where B is a finite A-module. We then have $f_*\mathcal{F}(\operatorname{Spec} A) = \mathcal{F}(\operatorname{Spec} B)$. Now, since $\mathcal{F}$ is quasi-coherent, we have $\mathcal{F}|_{\operatorname{Spec} B} = \widetilde{M}$ for some B-module, which we can view as an A-module via $f^\sharp(Y)\colon A \to B$. Hence $f_*\mathcal{F}$ is quasi-coherent. If X and Y are noetherian, and $\mathcal{F}$ is coherent, the module M is finitely generated as an B-module, and hence as an A-module, since B is a finite A-module. ❑

14.3 *Closed immersions and closed subschemes*

Recall that according to Definition 4.31, a *closed subscheme* of scheme X is a closed subset $Z \subseteq X$ equipped with a sheaf of rings $\mathcal{O}_Z$ that makes $(Z, \mathcal{O}_Z)$ into a scheme in a way that $i_*\mathcal{O}_Z \simeq \mathcal{O}_X/\mathcal{I}$ for some sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. In Chapter 2 we considered the prototypical example when $X = \operatorname{Spec} A$ and $Z = V(I)$ for some ideal $I \subseteq A$; the closed subscheme Z is then isomorphic to $\operatorname{Spec}(A/I)$. However, it was not at all clear which ideal sheaves gave rise to closed subschemes, even in the fundamental case of affine schemes. In this section we will show that the right condition is that the ideal sheaf be quasi-coherent.

PROPOSITION 14.19 *Let X be a scheme and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then the ringed space $Z = (\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I})$ is a scheme with a canonical closed immersion $\iota\colon Z \to X$.*

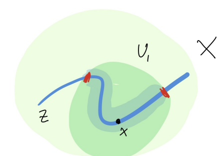
PROOF: To prove this we may assume that $X = \operatorname{Spec} A$ is affine. According to the basic Theorem 14.4 on page 201 the ideal sheaf $\mathcal{I}$ is then the tilde of some ideal $I \subseteq A$. The support of $\widetilde{A/I}$ consists exactly of the primes $\mathfrak{p}$ such that $(A/I)_\mathfrak{p} \neq 0$, or equivalently, the prime ideals so that $\mathfrak{p} \in V(I)$. Hence Z equals the closed subset $V(I)$ which is homeomorphic to $\operatorname{Spec}(A/I)$. The sheaf of rings on $\operatorname{Spec}(A/I)$ is the same as $\mathcal{O}_X/\mathcal{I}$ on Z, and consequently Z is the scheme $\operatorname{Spec}(A/I)$. The topological part of the morphism ι

is just the inclusion $Z \subseteq X$, and the algebraic part $\iota^\sharp \colon \mathcal{O}_X \to i_*(\mathcal{O}_X/\mathcal{I})$ is just the tilde of the quotient map $A \to A/I$. ❑

The converse of the previous proposition holds as well:

Proposition 14.20 *Let $Z \subset X$ be a closed subscheme of X, given by an ideal sheaf $\mathcal{I}$. Then $\mathcal{I}$ is quasi-coherent.*

Proof: On the open set $X \backslash Z$, we have $\mathcal{I}|_{X\backslash Z} \simeq \mathcal{O}_{X\backslash Z}$, and so $\mathcal{I}$ is quasi-coherent there. Let $x \in Z$. We first find an affine open neighbourhood $U = \operatorname{Spec} A$ of x such that $U \cap Z$ is an open affine in Z (recall that Z is itself assumed to be a scheme). To find U pick any affine open set $U' = \operatorname{Spec} R \subset X$ and let $V' \subseteq U' \cap Z$ be an affine open set containing x. Then pick an element $s \in \mathcal{O}_X(U') = R$ such that $s = 0$ on $U' \cap Z \backslash V'$, while $s(x) \neq 0$ (this is possible because $U' \cap Z \backslash V'$ is a closed set which does not intersect the closure $\{x\}^-$). Then let $U = D(s) \subseteq U'$. Note that $U \cap Z = D(s|_{V'}) \subseteq V'$, and it follows that $U \cap Z$ is an affine subset in Z as well. Write $U = \operatorname{Spec} A$ and $U \cap Z = \operatorname{Spec} B$, and let the inclusion $U \cap Z \to U$ correspond to the map $\phi \colon A \to B$. Let $I = \operatorname{Ker} \phi$. We claim that



$$\mathcal{I}|_U \simeq \widetilde{I},$$

which will show that $\mathcal{I}$ is quasi-coherent: indeed, for any distinguished open set $D(f)$ in U it holds true that

$$\begin{aligned} \widetilde{I}(D(f)) &= I_f = \operatorname{Ker}(A_f \to B_f) \\ &= \operatorname{Ker}\left(\mathcal{O}_U(D(f)) \to \mathcal{O}_Z(Z \cap D(f))\right) \\ &= \mathcal{I}(D(f)). \end{aligned}$$

This completes the proof. ❑

Notice that the closed subset Z can be recovered from the ideal sheaf $\mathcal{I}$ by $Z = \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$. In particular, this gives the most economic way of defining what a closed subscheme of X is: it is a subscheme of the form $(\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I})$ for some quasi-coherent sheaf of ideals $\mathcal{I}$.

Now we can finally prove Proposition 4.32 from Chapter 4.

Corollary 14.21 *Let $Z \subseteq X$ be a closed subscheme given by an ideal sheaf $\mathcal{I}$. Then for all open affines $U \subseteq X$, the intersection $U \cap Z$ is affine in Z. Moreover, if $U = \operatorname{Spec} A$, then $Z \cap U \simeq \operatorname{Spec}(A/I)$ for some ideal $I \subseteq A$.*

Proof: Since $\mathcal{I}$ is quasi-coherent, we have $\mathcal{I} = \widetilde{I}$ for some ideal $I \subseteq A$. Then for each open affine U, we have

$$\mathcal{O}_Z|_U = \operatorname{Coker}(\mathcal{I}|_U \to \mathcal{O}_X|_U) = \operatorname{Coker}(\widetilde{I} \to \widetilde{A}) = \widetilde{A/I}.$$

It follows that $(Y, \mathcal{O}_Y) = (V(I), \widetilde{A/I}) = \operatorname{Spec}(A/I)$. ❑

Corollary 14.22 *Let $X = \operatorname{Spec} A$ be an affine scheme. Associating the closed subscheme $\operatorname{Spec}(A/I)$ with the ideal I gives a one-to-one correspondence between the set of ideals of A and the set of closed subschemes of X. In particular, any closed subscheme of an affine scheme is also affine.*

Induced reduced scheme structure

We have seen that on any open subset $U \subseteq X$ of a scheme X there is a natural scheme structure induced from that of X, the structure sheaf simply being the restriction of $\mathcal{O}_X$. For $W \subseteq X$ a closed subset, there will in general be several quasi-coherent ideal sheaves $\mathcal{I}$ corresponding to W. For instance, the affine scheme $\operatorname{Spec} k[x]/(x)$ is naturally a proper subscheme of $\operatorname{Spec} k[x]/(x^2)$, but of course, they have the same underlying topological space. So, in contrast with the open 'subschemes' the underlying topological space does not determine the scheme structure. However there is one which is in some sense the 'smallest' one:

Proposition 14.23 (Induced reduced scheme structure) *Suppose that X is a scheme and that $W \subseteq X$ a closed subset. There exists a unique closed subscheme $Z \subseteq X$ such that*

i) Z is reduced;
ii) The underlying topological space of Z is W.

Proof: Let $\mathcal{I} \subseteq \mathcal{O}_X$ be the sheaf of ideals defined by

$$\mathcal{I}(U) = \{\, s \in \mathcal{O}_X(U) \mid s(x) = 0 \text{ for all } x \in U \cap W \,\}.$$

where we recall that $s(x)$ denotes the class of s in $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. We contend that $\mathcal{I}$ is quasi-coherent. On an open affine subscheme $U = \operatorname{Spec} A$ we have $W \cap U = V(I)$ for a unique radical ideal $I \subseteq A$, and it holds true that $\mathcal{I}(U) = I$; indeed, assume that $f \in A$ maps to zero in $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ for all $\mathfrak{p} \in U \cap W$; that is, for all prime ideals $\mathfrak{p} \subseteq A$ containing I. Since the preimage of $\mathfrak{p}A_\mathfrak{p}$ in $A_\mathfrak{p}$ equals $\mathfrak{p}$, it follows that $f \in \bigcap_{I \subseteq \mathfrak{p}} \mathfrak{p}$, but this intersection precisely equals $\sqrt{I} = I$ as I in the outset was radical. Hence $\mathcal{I}(U) = I$.

Moreover, for $D(g) \subseteq U$, we have $\mathcal{I}(D(g)) = I_g$ by the same argument, and so $\widetilde{I}$ and $\mathcal{I}$ are equal as sheaves on U, and hence $\mathcal{I}$ is a quasi-coherent sheaf of ideals.

Now define Z to be the closed subscheme associated with the ideal sheaf $\mathcal{I}$. Then Z is reduced and has the same underlying topological space as W (these are local statements and we just checked them on the open affines). Finally, if Z and Z' are two closed subschemes satisfying *i)* and *ii)* in the proposition, their ideal sheaves $\mathcal{I}$ and $\mathcal{I}'$ define the same radical ideal $\mathcal{I}(U) = \mathcal{I}'(U) = I$ on any open affine subscheme $U = \operatorname{Spec} A$, and so they are equal. ❑

The scheme Z comes with a canonical morphism of schemes $r\colon Z \to X$ defined as follows: We define r by the inclusion $Z \hookrightarrow X$ on the level of topological spaces. On the level of sheaves, we define $f^\sharp\colon \mathcal{O}_X \to r_*\mathcal{O}_Z$ over an open set $U \subset X$ to be the quotient map $\mathcal{O}_X(U) \to (\mathcal{O}_X/\mathcal{J})(U)$. As the induced map on stalks is a quotient map as well, it is a local homomorphism, and we obtain a morphism $r_X = (r, r^\#)\colon Z \to X$ of schemes.

In particular, we may apply this construction to $X = Z$. We denote the resulting scheme by X_{red} and refer to it as the *reduced scheme associated with* X. The scheme X_{red} and the morphism $r_X\colon X_{\text{red}} \to X$ satisfy the following universal property, which among other things, entail that X_{red} depends functorially on X (see Exercise 14.9 below).

Proposition 14.24 *Let $f\colon Y \to X$ be a morphism of schemes, with Y reduced. Then f factors uniquely through the natural map $r_X\colon X_{\text{red}} \to X$, i.e. there exists a unique morphism $g\colon Y \to X_{\text{red}}$ such that $f = r \circ g$.*

Proof: The question is easily reduced to case of affine schemes, where it follows from the fact that a map of rings $A \to B$ where B is without nilpotents, factors unambiguously through $A/\sqrt{0}$. ❑

Exercises

(14.5) Show the following:

i) The skyscraper sheaf of k on $\mathbb{A}^1_k$ at the origin 0 is quasi-coherent;

ii) The skyscraper sheaf of $k(T)$ on $\mathbb{A}^1_k$ at the origin 0 is *not* quasi-coherent.

(14.6) Let $\mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$ and consider the *twisted cubic curve* C given by the ideal

$$I = (y - x^2, z - x^3)$$

Let $\pi : C \to \mathbb{A}^1_k = \operatorname{Spec} k[z]$ be the projection from the line $L = V(x,y)$.

i) Show that π is a finite morphism;

ii) Compute $\pi_*\mathcal{O}_C$, $\pi^*\mathcal{O}_{\mathbb{A}^1_k}$ and $\pi^*\mathcal{J}$ where $\mathcal{J}$ is the ideal sheaf of the closed point $0 \in \mathbb{A}^1_k$.

(14.7) Let $f : X \to Y$ be a morphism of schemes and let $x \in X$ be a point. We say that:

- ❑ A quasi-coherent sheaf $\mathcal{F}$ on X is *flat over* Y at x if $\mathcal{F}_x$ is flat as a $\mathcal{O}_{Y,f(x)}$-module (where $\mathcal{F}_x$ is considered as a $\mathcal{O}_{Y,f(x)}$-module via the natural map $f_x^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,f(x)}$);
- ❑ $\mathcal{F}$ is *flat* if it is flat at every point in X;
- ❑ f is flat if $\mathcal{O}_X$ is flat over Y

i) Show that open embeddings are flat. What about closed immersions?

ii) Show that a morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if the map of rings $A \to B$ is flat. More generally, a quasi-coherent sheaf $\widetilde{M}$ on $\operatorname{Spec} B$ is flat over $\operatorname{Spec} A$ if and only if M is flat as an A-module;

iii) Which of the morphisms in Exercise 8.26 are flat?

iv) Prove that the blow-up morphism $\pi : Bl_0\mathbb{A}^2 \to \mathbb{A}^2$ is not flat.

(14.8) Prove that the morphism $r\colon X_{\mathrm{red}} \to X$ is a closed immersion.

(14.9) *Functoriality of $(-)_{\mathrm{red}}$.* If $f\colon X \to Y$ is a morphism, show that there is a unique morphism $f_{\mathrm{red}}\colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$ so that $f_{\mathrm{red}} \circ r_X = r_Y \circ f_{\mathrm{red}}$. Show that assignments $X \mapsto X_{\mathrm{red}}$ and $f \mapsto f_{\mathrm{red}}$ defines a functor Sch to RedSch which is adjoint to the inclusion functor RedSch $\to$ Sch, where RedSch is the full subcategory of Sch whose objects are the reduced schemes.

(14.10) Prove Proposition 14.24

✲ (14.11) *Morphisms to a closed subscheme.* Let Z be a closed subscheme of X given by sheaf of ideals $\mathcal{I}$. Suppose $f : Y \to X$ is a morphism of schemes. Show that f factors through a map $g : Y \to Z$ if and only if

i) $f(Y) \subseteq Z$;

ii) $\mathcal{I} \subseteq \operatorname{Ker}(f^\sharp\colon \mathcal{O}_X \to f_*(\mathcal{O}_Y))$.

For a morphism of schemes $f : Y \to X$, we can define the *scheme-theoretic image* of f as a subscheme $Z \subseteq X$ satisfying the universal property that if f factors through a subscheme $Z' \subseteq Z$, then $Z \subseteq Z'$. To define Z it is is tempting to use the ideal sheaf $\mathcal{I} = \operatorname{Ker}(\mathcal{O}_X \to f_*(\mathcal{O}_Y))$ — but this may fail to be quasi-coherent for a general morphism f. However, one can show that there is a largest quasi-coherent sheaf of ideals $\mathcal{J}$ contained in $\mathcal{I}$, and we then define Z to be associated to $\mathcal{J}$.

(14.12) *Noetherian induction.* Let X is a scheme. The closed subschemes form a partially ordered set when one lets $Z \subseteq Z'$ mean that the closed immersion $Z \hookrightarrow X$ factors through the immersion $Z' \hookrightarrow X$.

i) Show that $Z \subseteq Z'$ if and only it $\mathcal{I}(Z') \subseteq \mathcal{I}(Z)$;

ii) Assume X to be Noetherian. Show that any non-empty set Σ of closed subschemes contains a minimal element.

✲ (14.13) *Generic freeness of coherent sheaves.* Assume that X is a reduced and irreducible scheme and let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is 'generically free', or phrased differently, 'up to coherent sheaves with proper support it may be approximated by a free sheaf'. In precise terms, show that there is a coherent sheaf $\mathcal{H}$ on X and a map $\alpha\colon \mathcal{F} \to \mathcal{H}$ with the two properties

i) Both supports $\operatorname{Supp}\operatorname{Ker}\alpha$ and $\operatorname{Supp}\operatorname{Coker}\alpha$ are proper subschemes of X;

ii) There is an integer and an inclusion $\mathcal{O}_X^r \subseteq \mathcal{H}$ of a free sheaf such that the quotient $\mathcal{H}/\mathcal{O}_X^r$ has proper support.

(14.14) *An ideal sheaf which is not quasi-coherent.* Let $X = \operatorname{Spec} k[T] = \mathbb{A}^1_k$ and consider the origin $P \in X = \mathbb{A}^1_k$ corresponding to the maximal ideal $(T) \subset k[T]$. Define the presheaf $\mathcal{I}$

of $\mathcal{O}_X$ by for each open subset $U \subseteq X$ letting$\mathcal{I}(U) \subset \mathcal{O}_X(U)$ be given as

$$\mathcal{I}(U) = \begin{cases} \mathcal{O}_X(U) \text{ if } P \notin U; \\ 0 \text{ if } P \in U. \end{cases}$$

a) Show that $\mathcal{I}$ is an ideal sheaf, and $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ is not a closed subset of X.
b) Show directly that $\mathcal{I}$ is not quasi-coherent by showing that $\mathcal{I}(X) = 0$, but $\mathcal{I} \neq 0$.

★

Chapter 15

Sheaves on projective schemes

Projective schemes are to affine schemes what projective varieties are to affine varieties. The construction of the projective spectrum $\operatorname{Proj} R$ is similar to that of the affine spectrum $\operatorname{Spec} R$: the underlying topological space is defined with the help of prime ideals and the structure sheaf from localizations of R. However, there are some fundamental differences between the two: in the proj-construction one only considers *graded* rings R, and only homogeneous prime ideals that do not contain the irrelevant ideal R_+. As we saw, this reflects the construction of the projective spectrum $\operatorname{Proj} R$ as a quotient space

$$\pi\colon \operatorname{Spec} R - V(R_+) \to \operatorname{Proj} R.$$

Given this, we can pull back a quasi-coherent sheaf to $\operatorname{Spec} R - V(R_+)$ and extend it to a sheaf on $\operatorname{Spec} R$ via the inclusion map. Thus, it is natural to expect that quasi-coherent sheaves on $\operatorname{Proj} R$ should be in correspondence with 'equivariant' modules on $\operatorname{Spec} R$; *i.e.* the *graded*[1] R-modules. The irrelevant subscheme $V(R_+)$ complicates the picture and makes the classification a little bit more involved than the one for affines schemes. In particular, we will see that different graded R-modules may correspond to the same quasi-coherent sheaf on $\operatorname{Proj} R$.

Another important feature of $\operatorname{Proj} R$ is that it comes equipped with a canonical invertible sheaf which we will denote by $\mathcal{O}_{\operatorname{Proj} R}(1)$. This is the geometric manifestation of the fact that R is graded. Unlike the case of affine schemes, $\operatorname{Proj} R$ can typically not be recovered from the global sections of the structure sheaf. It is the sheaf $\mathcal{O}_{\operatorname{Proj} R}(1)$, or rather, the various tensor powers $\mathcal{O}_{\operatorname{Proj} R}(d) = \mathcal{O}_{\operatorname{Proj} R}(1)^{\otimes d}$, that will play the role of the affine coordinate ring in the affine case. So it is rather from the pair $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R}(1))$ one may hope to recover R.

15.1 The graded tilde-functor

Let R be a graded ring and let GrMod_R denote the category of graded R-modules. Just as in the case of the affine spectrum $\operatorname{Spec} A$, we shall set up a tilde-construction which produces sheaves on $\operatorname{Proj} R$ from graded R-modules, and in this way gives a functor GrMod_R to $\mathsf{Mod}_{\mathcal{O}_X}$. However, in contrast to the affine case, this will not be an equivalence of categories.

[1] In the model case of the projective spaces, the variety $\mathbb{P}^n$ is the quotient of $\mathbb{A}^n \backslash \{0\}$ by the group k^* acting by scalar multiplication, so in this case, the notion 'equivariant' is precise and pertinent.

Homogenization & dehomogenization

Back in in Chapter 15 on page 145, we utilized a homogenization-dehomogenization process to fabricate the structure sheaf on $\operatorname{Proj} R$, and we shall rely on a similar technique in the tilde-construction.

Recall that if for two homogeneous elements $f, g \in R_+$ there is an inclusion $D_+(g) \subseteq D_+(f)$ of the corresponding distinguished sets, then $g^r = vf$ for some homogeneous $v \in R$ and some natural number r . And as f becomes invertible in R_g, there is a canonical map $M_f \to M_g$ between the localized modules; it respects the gradings since both f and g are homogeneous, and its action on the degree zero parts yields a canonical map $\rho_{f,g}\colon (M_f)_0 \to (M_g)_0$. That map sends an element xf^{-n} with x homogeneous and $\deg x = n \deg f$ to the element $v^n x g^{-nr}$.

$D_+(g) \subseteq D_+(f)$ is equivalent to $\sqrt{(g)} \subseteq \sqrt{(f)}$

Letting $\mathscr{B}$ be the basis for the Zariski topology whose elements are the distinguished open subsets, this permits us to define a $\mathscr{B}$-presheaf $\widetilde{M}$: sections over $D_+(f)$ are to be given by

$$\widetilde{M}(D_+(f)) = (M_f)_0,$$

and the restriction maps $\widetilde{M}(D_+(f)) \to \widetilde{M}(D_+(g))$, when $D_+(g) \subseteq D_+(f)$, are to be the maps $\rho_{f,g}$ above (the two requirements to be a presheaf are easily verified).

Recall the canonical isomorphism $D_+(f) \simeq \operatorname{Spec}(R_f)_0$ from Proposition 10.8 on page 145, and it will be important to see what sheaf $\widetilde{M}$ will yield when restricted to $D_+(f)$ and transported to $\operatorname{Spec}(R_f)_0$. The answer is given in the following proposition.

PROPOSITION 15.1 *Under the isomorphism between $D_+(f)$ and $\operatorname{Spec}(R_f)_0$ one has $\widetilde{M}|_{D_+(f)} \simeq (\widetilde{(M_f)_0})$.*

Here a distinguished subset $D_+(g)$ of $D_+(f)$ is mapped isomorphically onto the distinguished open subset $D(u)$ of $\operatorname{Spec}(R_f)_0$ where $u = g^{\deg f} f^{-\deg g}$ (which is the simplest degree zero element in R_f one can fabricate out of f and g). The propostion follows directly from the following lemma, which has a slightly technical proof:

LEMMA 15.2 *With the notation above, the canonical homomorphism $\rho_{f,g}\colon (M_f)_0 \to (M_g)_0$ induces an isomorphism $((M_f)_0)_u \simeq (M_g)_0$;*

PROOF: The element u is invertible in $(R_g)_0$, so the map $\rho_{f,g}\colon (M_f)_0 \to (M_g)_0$ factors via a map

$$\rho : ((M_f)_0)_u \to (M_g)_0,$$

which explicitly is given as

$$\rho\left(xf^{-n}u^{-m}\right) = xf^{m\deg g - n} g^{-m\deg f}$$

where $x \in M$ is homogeneous of degree $n \deg f$. We contend that this is an isomorphism, and begin with showing that ρ is surjective. To that end, note that each element in $(M_g)_0$

is on the form $y \cdot g^{-l}$ with $y \in M$ homogeneous and $\deg y = l \deg g$. With an integer m so large that $m \deg f \geqslant l$, one has the equality

$$g^{-l}y = u^{-m}u^m g^{-l}y = u^{-m}(g^{m\deg f - l}y)f^{-m\deg g}.$$

The right side is an element in $((R_f)_0)_u$ since $m \deg f - l \geqslant 0$, and the equality means it maps to $g^{-l}y$.

To see that the map ρ is injective assume that an element $xf^{-n} \in (M_f)_0$ maps to zero in $(M_g)_0$; this means that there is an integer $l > 0$ so that $g^{lr}v^n x = 0$ in M. Multiplying up by appropriate powers of v and f, we get a relation in M of the form $g^{(l+n)r}x = 0$, and consequently it holds that $u^{(l+n)r}x = 0 \in (M_f)_0$. But then $xf^{-n} = 0$ in $((M_f)_0)_u$, and we are through. ❏

As an immediate consequence of Proposition 15.1 we obtain the desired

Proposition 15.3 *The $\mathscr{B}$-presheaf $\widetilde{M}$ is a $\mathscr{B}$-sheaf, and extends to a quasi-coherent sheaf on* $\operatorname{Proj} R$*; which we continue to denote $\widetilde{M}$.*

Properties of the tilde-functor

As is the case for the tilde-construction for affine spectra, the assignment $M \mapsto \widetilde{M}$ is functorial and gives a functor $\mathsf{GrMod}_R \to \mathsf{QCoh}_{\operatorname{Proj} R}$. This is close to obvious as a map $M \to N$ homogeneous of degree zero keeps being homogeneous of degree zero when localized and so induces maps $(M_f)_0 \to (N_f)_0$.

In some apects the projective tilde-functor behaves as the affine one, but in other apsects, the behaviour deviates seriously; the most striking difference being that different modules may yield isomorphic sheaves, and this is inherent, not accidental.

The following proposition summarizes the basic properties of the tilde-functor.

Proposition 15.4 *Let R be a graded ring. The functor $\mathsf{GrMod}_R \to \mathsf{QCoh}_{\operatorname{Proj} R}$ that sends M to $\widetilde{M}$ has the following properties:*

- *i) The tide-functor is additive and exact and commutes with direct limits.*
- *ii) Sections over distinguished open sets: for $f \in R$, we have $\Gamma(D_+(f), \widetilde{M}) = (M_f)_0$.*
- *iii) Stalks: for each $\mathfrak{p} \in \operatorname{Proj} R$ it holds that $\widetilde{M}_{\mathfrak{p}} = (M_{\mathfrak{p}})_0$;*
- *iv) When the ring R is Noetherian and M is finitely generated, then $\widetilde{M}$ is coherent.*

Proving these properties is straightforward, since most of them can be checked locally. Using the isomorphisms between $D_+(f)$ and $\operatorname{Spec}(R_f)_0$ we reduce immediately to the affine case.

It is important to note that, unlike the affine case, the tilde-functor is not faithful, as several modules can correspond to the same sheaf. This is not so surprising and is rooted in the fact that primes in $V(R_+)$ are thrown away in the Proj-construction, which has the effect that modules supported in $V(R_+)$ necessarily give the zero sheaf when exposed to the tilde-functor. For any integer d we let $M_{>d}$ be the R-module $M_{>d} = \bigoplus_{i>d} M_i$ (it is an R-module because of the standing hypothesis that R be positively graded).

LEMMA 15.5 *Assume that R is a graded ring and let M and N be two graded R-modules,*
i) If $\operatorname{Supp} M \subseteq V(R_+)$, then $\widetilde{M} = 0$;
ii) Assume that $M_{>d} \simeq N_{>d}$ for some d. Then $\widetilde{M} \simeq \widetilde{N}$.

PROOF: To prove *i)*, suppose that $\operatorname{Supp} M \subseteq V(R_+)$. Statement *iii)* of Proposition 15.4 above then entails that $\widetilde{M} = 0$ since $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Proj} R$.

To prove *ii)*, note that the quotient $M/M_{>d}$ is killed by the power $(R_+)^d$ and consequently has support in $V(R_+)$. By *i)* its tilded sheaf vanishes, and hence $\widetilde{M_{>d}} = \widetilde{M}$. As this holds for both M and N we are through. ❑

EXAMPLE 15.6 On $X = \operatorname{Proj} k[x_0, x_1]$, the module $M = k[x_0, x_1]/(x_0^2, x_1^2)$ has $\widetilde{M} = 0$, but it is non-zero. ☆

The next lemma is sometimes useful when working with the localization of M when R is generated in degree one. It says essentially that we are allowed to 'substitute 1 for f' when restricting a module to an affine chart $D_+(f) \subset \operatorname{Proj} R$.

LEMMA 15.7 *Suppose that M is a graded R-module and that $f \in R$ homogeneous of degree one. Then there are natural isomorphisms of $(R_f)_0$-modules*

$$(M_f)_0 \simeq M/(f-1)M \simeq M \otimes_R R/(f-1)R.$$

In particular, $(R_0)_f \simeq R/(f-1)R$.

PROOF: The element f acts as the identity on the R-module $M/(f-1)M$, so $M/(f-1)M$ is a module over R_f. Plainly sending xf^{-r} to x yields an R_f-linear homomorphism $M_f \to M/(f-1)M$, as one easily verifies, and restricting it to the degree zero piece one obtains an $(R_f)_0$-homomorphism $(M_f)_0 \to M/(f-1)M$. It is surjective: the class of a homogeneous element x is the image of $xf^{-\deg x}$, and every element in $M/(f-1)M$ is the sum of classes of homogenous elements. To check it is injective, assume that $xf^{-\deg m}$ maps to zero; *i.e.* that $x = (f-1)y$ for some $y \in M$. Expanding y in homogeneous components we may write $y = \sum_{s \leqslant i \leqslant t} y_i$ with neither y_s nor y_t equal to zero. Then

$$x = (f-1)y = -y_s + \sum_{s}^{t-1}(fy_i - y_{i+1}) + fy_t.$$

Because x is homogeneous and $y_s \neq 0$, we may infer that $y_s = -x$, but also that $fy_t = 0$ and $y_{i+1} = fy_i$. A straightforward induction then yields equalities $y_t = f^{t-s}y_s = -f^{t-s}x$; consequently x is killed by a power of f and vanishes in M_f. ❑

EXAMPLE 15.8 That f is of degree one is essential. To give an example where the above lemma fails, let $M = R = k[x]$ and $f = x^2$. We find $k[x]_{x^2} = k[x, x^{-2}] = k[x, x^{-1}]$ so that $(k[x]_{x^2})_0 = k$. But $k[x]/(x^2-1) \simeq k \oplus k$. ☆

Tensor product & Hom's

Let M and N be two graded modules over the graded ring R. There is a natural way of giving the tensor product a graded structure; a decomposable tensor $x \otimes y$ is precisely homogenous when x and y are, and, of course, it is of degree $\deg x + \deg y$. Homogenous tensors will be the sums of decomposables of the same degree; *i.e.* they form the image $\bigoplus_{i+j=n} M_i \otimes_{R_0} M_j$, and this will be the graded piece of $M \otimes_R N$ of degree n. One may check that $M \otimes_R N$ as an R_0-module is the direct sum of these graded parts (that they generate is obvious; that the pairwise intersections are zero is slightly more subtle).

The tilde-functor is in the case of affine spectra well-behaved when it comes to tensor products in that $\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \widetilde{N} = \widetilde{M \otimes_A N}$. In the projective case however, this is not always so. Unless R is generated in degree one, curious phenomena take place. The following simple example may be instructive, which also illustrates the subtlety of the proj-construction for rings not generated in degree one.

EXAMPLE 15.9 Again we consider the ring $R = k[x^2]$ where k is a field and x^2 is of degree two. Then $\operatorname{Proj} R$ is simply a point with structure sheaf $\mathcal{O}_{\operatorname{Proj} R} = k$, and, moreover, it is covered by the sole distinguished set $D_+(x^2)$.

Consider the R-module $M = \bigoplus_{i \geqslant 0} k \cdot x^{2i+1}$, which is nothing but the submodule of $k[x]$ consisting of polynomials that has non-vanishing terms only of odd degrees. We contend that $(M_{x^2})_0 = 0$. Indeed, elements of M_{x^2} are linear combinations of terms shaped like $x^{2i+1}x^{-2s}$, and none of these can be of degree zero. So $\widetilde{M} = 0$, and thus also $\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Proj} R}} \widetilde{M} = 0$. On the other hand, the module $M \otimes_R M$ possesses elements of degree zero when localized at x^2; in fact, all elements are of even degree since they are sums of terms like $p(x) \otimes q(x)$. Hence $((M \otimes_R M)_{x^2})_0 \neq 0$; or in other words, $\widetilde{M \otimes_R M} \neq 0$.

It may seem paradoxical that redefining the grading on $k[x^2]$ by giving x^2 degree one, the tilde-construction and tensor product will commute; the explanation is that the 'counter-example' M above is no more a graded module! Well, the only sensible degree one could give x and still make the example work, would be $1/2$, which is not allowed.

Note that the example also illustrates that the converse of Lemma 15.5 does not hold unconditionally (but, again as we shall see, it holds true when R is generated in degree one). ☆

Let us proceed to compare $\widetilde{M \otimes_R N}$ with $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. For each homogeneous element $f \in R$ there is map $(M_f)_0 \times (N_f)_0 \to ((M \otimes_R N)_f)_0$ sending $(x/f^n, y/f^m)$ to $(x \otimes y)/f^{m+n}$. It is obviously $(R_f)_0$-bilinear, and consquently there is an induced map

$$(M_f)_0 \otimes_{(R_f)_0} (N_f)_0 \to ((M \otimes_R N)_f)_0.$$

Since maps between $\mathscr{B}$-sheaves induce maps between sheaves, we get a natural map

$$\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Proj} R}} \widetilde{N} \to \widetilde{M \otimes_R N}, \tag{15.1}$$

which, however, as the Example 15.9 above shows, it is not always an isomorphism; but one has the folowing:

PROPOSITION 15.10 *Suppose R is generated in degree one. Then the natural map* (15.1) *is an isomorphism.*

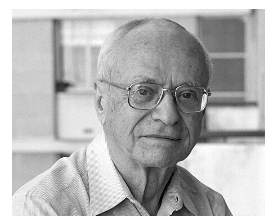
PROOF: By assumption, $\operatorname{Proj} R$ is covered by open affines of the form $D_+(f)$ where f has degree one. For such an f, the functor $M \to (M_f)_0$ coincides with the tensor-functor $(-) \otimes_R R/(f-1)R$ by the previous lemma. Furthermore, one of the standard properties of the tensor product gives that

$$(M \otimes_R (R_f)_0) \otimes_{(R_f)_0} (N \otimes_R (R_f)_0) \simeq (M \otimes_R N) \otimes_R (R_f)_0.$$

This isomorphism provides the inverse to the natural map $(M_f)_0 \otimes_{(R_f)_0} (N_f)_0 \to ((M \otimes_R N)_f)_0$ defined above. Then, since the map from (15.1) restricts to an isomorphism on each $D_+(f)$ for $f \in R$ of degree one, it is an isomorphism. ❑

15.2 Serre's twisting sheaf $\mathcal{O}(1)$

Arguably the most interesting sheaf on $X = \operatorname{Proj} R$ is the so-called *twisting sheaf*, denoted by $\mathcal{O}_X(1)$. This is a generalization of the tautological sheaf on $\mathbb{P}^n_k$, and constitutes a geometric manifestation of the fact that R is a *graded ring*. It was introduced in the groundbreaking paper **[?]** by Jean Pierre Serre. Elements in R do not define 'regular functions' on $\operatorname{Proj} R$, and we shall see that in good cases R_d will be the space of sections of the tensor power $\mathcal{O}_{\operatorname{Proj} R}(d)$ when $d \geqslant 0$, and this is a means of recovering the ring R.



Jean-Pierre Serre
(1928 –)

Let M be a graded module over the graded ring R. For each integer n, we will define an R-module $M(n)$ as follows: As an underlying R-module $M(n)$ is just M, but the grading is shifted by n:

$$M(n)_d = M_{d+n}.$$

Thus $N = M(n)$ is a graded R-module with $N_0 = M_n$, $N_1 = M_{n+1}$ and so on. The construction is functorial and is called the *shift-functor* or the *twist-functor*. Note that elements from M_d considered as element in $M(n)$ will be of degree $d - n$. In the particular case when $M = R$, this gives the naturally a graded and free R-module $R(n)$, generated by the element $1 \in R_{-n}$. Note that $M(n) = M \otimes_R R(n)$: both have M as underlying module, and $\bigoplus_{i+j=d} M_i \otimes_{R_0} R(n)_j = M_{d+n}$.

Applying the tilde-functor to $R(n)$ gives us a quasi-coherent $\mathcal{O}_{\operatorname{Proj} R}$-module on $\operatorname{Proj} R$:

DEFINITION 15.11 *For an integer n, we define*

$$\mathcal{O}_X(n) = \widetilde{R(n)}.$$

For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X, we define the twist by *n by $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.*

To each element $r \in R_d$ there is a corresponding section in $\Gamma(X, \mathcal{O}_X(d))$. This is so because, according to Proposition 15.4, we can think of an element of $\Gamma(X, \mathcal{O}_X(d))$ as a collection of pairs $(r_f, D_+(f))$ with $r_f \in ((R_f)_d)_0$ matching on the overlaps $D_+(f) \cap D_+(g)$, where f and g run through a set of homogeneous generators for R_+. Hence we can define an R_0-module homomorphism

$$R_d \to \Gamma(X, \mathcal{O}_X(d))$$

by $r \mapsto \{(r/1, D_+(f))\}$. The element $r/1$ is just the image of r under the canonical localization map $R \to R_f$, and it is of degree zero as degrees are shifted by $-d$. This also makes it clear that on the overlaps $D_+(fg)$ the two elements $(r/1, D_+(f))$ and $(r/1, D_+(g))$ become equal, and so we obtain an actual global section of $\mathcal{O}(d)$. Abusing notation, we will also denote this section by r.

In the special case that the element f is of degree one; that is, when $f \in R_1$, we have the equality $(R(n)_f)_0 = f^n \cdot (R_f)_0$; indeed, for each N the equalities $rf^{-N} = rf^n \cdot f^{-N+n}$ with $r \in R_{N+n}$ hold true. Thus, on the distinguished affine open set $D_+(f)$ it holds true that $\mathcal{O}_X(n)|_{D_+(f)} = f^n \mathcal{O}_X|_{D_+(f)}$. In particular, $\mathcal{O}_X(n)|_{D_+(f)} \simeq \mathcal{O}_{D_+(f)}$. In other words, if R is generated in degree one, the sheaf $\mathcal{O}_X(n)$ is locally free of rank one, that is, it is an invertible sheaf.

PROPOSITION 15.12 *When R is generated in degree one, the sheaf $\mathcal{O}_X(n)$ is invertible for every n. Moreover, there are canonical isomorphisms*

$$\mathcal{O}_X(m+n) \simeq \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

PROOF: Indeed, if R is generated in degree one, Proposition 15.10 shows that $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n)$ is the sheaf associated to $R(m) \otimes_R R(n) \simeq R(n+m)$; that is, associated to $\mathcal{O}_X(n+m)$. ❑

So this is a big difference between affine schemes and projective schemes: $\operatorname{Proj} R$ comes equipped with lots of invertible sheaves.

EXAMPLE 15.13 ($\mathbb{P}^1_A$ *once more.*) Let $X = \operatorname{Proj} R$ where $R = A[x_0, x_1]$ be the projective line $\mathbb{P}^1_A$ over A. Let us compute the global sections of the sheaf $\mathcal{O}_X(d)$. Our scheme X is covered by the two distinguished opens $D_+(x_0) = \operatorname{Spec} A[x_1 {x_0}^{-1}]$ and $D_+(x_1) = \operatorname{Spec} A[x_0 {x_1}^{-1}]$, and the following hold true

$$\Gamma(D_+(x_0), \mathcal{O}_X(d)) = (R_{x_0})_d = A[x_1 {x_0}^{-1}] x_0^d$$

and

$$\Gamma(D_+(x_1), \mathcal{O}_X(d)) = (R_{x_1})_d = A[x_0 {x_1}^{-1}] x_1^d.$$

On the overlap $D_+(x_0) \cap D_+(x_1) = D_+(x_0 x_1)$, we have

$$\Gamma(D_+(x_0 x_1), \mathcal{O}_X(d)) = A[x_0 x_1^{-1}, x_1 x_0^{-1}] x_0^d = A[x_0 x_1^{-1}, x_1 x_0^{-1}] x_1^d$$

and we find that two sections

$$s_0 = p_0\left(\frac{x_1}{x_0}\right) x_0^d \quad \text{and} \quad s_1 = p_1\left(\frac{x_0}{x_1}\right) x_1^d$$

agree on the overlap if and only if

$$x_1^d p_1\left(\frac{x_0}{x_1}\right) = x_0^d p_0\left(\frac{x_1}{x_0}\right). \tag{15.2}$$

Here p_0 and p_1 are polynomials with coefficients in A. Thus for such an equality to hold in $A[x_0, x_0^{-1}, x_1, x_1^{-1}]$, we immediately see that d must be non-negative; that p_0 and p_1 must have degree d and that each side of (15.2) must be a homogeneous polynomial of degree d. Conversely, any homogeneous polynomial of degree d

$$P(x_0, x_1) = a_0 x_0^d + a_1 x_0^{d-1} x_1 + \cdots + a_d x_1^d$$

gives rise to a global section through the assignments $p_0 = x_0^{-d} P$ and $p_1 = x_1^{-d} P$. Thus $\Gamma(X, \mathcal{O}_X(d))$ can be naturally identified with the A-module of homogeneous polynomials in x_0, x_1 of degree d. ☆

As alluded to above, the main point of the sheaves $\mathcal{O}_X(d)$ is that they help us recover the ring R; for instance, while x_0^d does not correspond to a regular function on $X = \operatorname{Proj} k[x_0, x_1]$, it gives a section of the sheaf $\mathcal{O}_X(d)$.

15.3 *The associated graded module*

We have associated to a graded R-module M a sheaf $\widetilde{M}$ on $X = \operatorname{Proj} R$. To classify quasi-coherent sheaves on X we would, as in the case of affine schemes, like to give some sort of inverse to this assignment. However, as opposed to the case for $X = \operatorname{Spec} A$, simply using the global sections functor will not work. Indeed, even for $\mathcal{F} = \mathcal{O}_{\mathbb{P}_k^1}$ on $\mathbb{P}_k^1$, it holds that $\Gamma(\mathbb{P}_k^1, \mathcal{F}) = k$, from which we certainly cannot recover $\mathcal{F}$. The remedy is to look at the various Serre twists $\mathcal{F}(d)$ of $\mathcal{F}$; in fact all of them at once:

Definition 15.14 *Let R be a graded ring and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on $X = \operatorname{Proj} R$. We define the* graded R-module associated *to $\mathcal{F}$, denoted $\Gamma_*(\mathcal{F})$ as*

$$\Gamma_*(\mathcal{F}) = \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{F}(d)).$$

In particular, from X alone we get the associated graded ring

$$\Gamma_*(\mathcal{O}_X) = \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)).$$

The associated graded module has the structure of an R-module: If $r \in R_d$, we have a corresponding section $r \in \Gamma(X, \mathcal{O}_X(d))$ (abusing notation, as before). So if $\sigma \in \Gamma(X, \mathcal{F}(n))$, then we may define $r \cdot \sigma \in \Gamma(X, \mathcal{F}(n+d))$ as $r \otimes \sigma$ via the isomorphism $\mathcal{F}(n) \otimes \mathcal{O}(d) \simeq \mathcal{F}(n+d)$.

If R is generated in degree one, and M is a graded R-module, we can define a homomorphism of graded R-modules

$$\alpha : M \to \Gamma_*(\widetilde{M}).$$

To define it, it is useful to think of elements in $\Gamma(X, \widetilde{M}(n))$ as a collection of elements $(m_f, D_+(f))$ for $m \in (M_f)_n$ and $f \in R_1$, matching on the various overlaps.

With this in mind, we can send an element $m \in M_n$ to the collection given by $(m/1, D_+(f))$, where f ranges over the degree one piece R_1. On the overlaps $D_+(f) \cap D_+(g) = D_+(fg)$ the two elements $(m/1, D_+(f))$ and $(m/1, D_+(g))$ become equal so this defines an actual global section of $\widetilde{M}(n)$. It is clear that this is a graded homomorphism. Moreover, it is functorial in M.

PROPOSITION 15.15 *Let R be a graded integral domain, finitely generated over R_0 in degree 1 by elements $x_0, \ldots, x_n$, and let $X = \operatorname{Proj} R$. Then*

i) $\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^n R_{x_i} \subset K(R)$;

ii) If each x_i is a prime element, then $R = \Gamma_*(\mathcal{O}_X)$.

PROOF: Cover X by the opens $U_i = D_+(x_i)$. We have, since $\Gamma(D_+(x_i), \mathcal{O}(m)) \simeq (R_{x_i})_m$, that the sheaf axiom sequence takes the following form

$$0 \to \Gamma(X, \mathcal{O}(m)) \to \bigoplus_{i=0}^{n} (R_{x_i})_m \to \bigoplus_{i,j} (R_{x_i x_j})_m.$$

Taking directs sums over all m, we get

$$0 \to \Gamma_*(\mathcal{O}_X) \to \bigoplus_{i=0}^{n} R_{x_i} \to \bigoplus_{i,j} R_{x_i x_j}.$$

So a section of $\Gamma_*(\mathcal{O}_X)$ corresponds to an $(n+1)$-tuple $(t_0, \ldots, t_n) \in \bigoplus_{i=0}^n (R_{x_i})$ such that t_i and t_j coincide in $R_{x_i x_j}$ for each $i \neq j$. Now, the x_i are not zero-divisors in R, so the localization maps $R \to R_{x_i}$ are injective. It follows that we can view all the localizations R_{x_i} as subrings of $R_{x_0 \ldots x_n}$, and then $\Gamma_*(\mathcal{O}_X)$ coincides with the intersection

$$\bigcap_{i=0}^{n} R_{x_i} \subset R_0[x_0, x_0^{-1}, \cdots, x_n, x_n^{-1}].$$

In the case each x_i is prime, this intersection is just R. ❑

COROLLARY 15.16 *Let $X = \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$ for a ring A. Then*

$$\Gamma_*(\mathcal{O}_X) \simeq A[x_0, \ldots, x_n]$$

In particular we can identify $\Gamma(\mathbb{P}^n_A, \mathcal{O}(d))$ with the A-module generated by homogeneous degree d polynomials.

When R is not a polynomial ring, it can easily happen that $\Gamma_*(\mathcal{O}_X)$ is different than R. Here is a concrete example:

EXAMPLE 15.17 (*A quartic rational curve.*) Let k be a field and let R be the k-algebra $R = k[s^4, s^3t, st^3, t^4] \subset k[s,t]$. Note that the monomial s^2t^2 is missing from the set of generators of R. Define the grading such that R_1 is the vector space generated by s^4, s^3t, st^3 and t^4.

We can also think of R as the graded ring

$$R = k[x_0, x_1, x_2, x_3]/(x_0^2x_2 - x_1^3, x_1x_3^2 - x_2^3, x_0x_3 - x_1x_2).$$

The radical of the ideal (s^4, t^4) equals R_+ so $\operatorname{Proj} R$ is covered by $U_0 = D_+(s^4) = D_+(x_0)$ and $U_1 = D_+(t^4) = D_+(x_3)$; that is, $\operatorname{Proj} R = U_0 \cup U_1$. Furthemore we have

$$U_0 = \operatorname{Spec}(R_{x_0})_0 \text{ and } U_1 = \operatorname{Spec}(R_{x_3})_0.$$

Here $(R_{x_0})_0 = k[ts^{-1}, t^3s^{-3}, t^4s^{-4}] = k[ts^{-1}]$ and similarly $(R_{x_3})_0 = k[st^{-1}]$. So $\operatorname{Proj} R$ is in fact isomorphic to $\mathbb{P}^1_k$. We have thus shown that $X = \operatorname{Proj} R$ embeds as a rational (degree 4) curve in $\mathbb{P}^3_k$.

What is $\Gamma(X, \mathcal{O}_X(1))$? On the two opens we find $\mathcal{O}_X(1)(U_0) = k[ts^{-1}] \cdot s^4$ and $\mathcal{O}_X(1)(U_1) = k[st^{-1}] \cdot t^4$. So using the sheaf sequence, we get

$$0 \to \Gamma(X, \mathcal{O}_X(1)) \to k[ts^{-1}]s^4 \oplus k[st^{-1}]t^4 \to k[st^{-1}, ts^{-1}]s^4.$$

Note that the monomial s^2t^2 belongs to both the rings $k[st^{-1}]t^4$ and $k[ts^{-1}]s^4$, and so defines an element in $\Gamma(X, \mathcal{O}_X(1))$. In fact,

$$\Gamma(X, \mathcal{O}_X(1)) = k\{s^4, s^3t, s^2t^2, st^3, t^4\}$$

even though $R_1 = k\{s^4, s^3t, st^3, t^4\}$.

In this example, the graded ring $\Gamma_*(\mathcal{O}_X) = k[s^4, s^3t, st^3, t^4]$ is the integral closure of R. Exercise 15.2 below shows that this is not a coincidence. ☆

EXAMPLE 15.18 Let $X = \mathbb{P}^1_k$. The sheaves $\mathcal{O}_X(n)$ give another example why one has to sheafify in the definition of the tensor product. If T denotes the naive presheaf

$$T(U) = \mathcal{O}_X(-1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1),$$

then clearly $\Gamma(X, T) = 0 \otimes_k k^2 = 0$. However, the sheafification T^+ is isomorphic to $\mathcal{O}_X$, so $\Gamma(X, T^+) = k$, and thus $T \neq T^+$. ☆

EXERCISE 15.1 Let k be a field and let $R = k[x_0, \ldots, x_n]$. Let $\pi : \mathbb{A}^{n+1} - 0 \to \mathbb{P}^n_k = \operatorname{Proj} R$ denote the 'quotient morphism' from Exercise 10.9. Show that for a graded R-module M, we have

$$\pi_*(\widetilde{M}|_{\mathbb{A}^{n+1}_k - 0}) = \bigoplus_{n \in \mathbb{Z}} \widetilde{M}(d)$$

★

✲ EXERCISE 15.2 Let R be a graded Noetherian integral domain generated in degree one. Show that $R' = \Gamma_*(\mathcal{O}_X)$ is an integral extension of R. (Hint: Use the Cayley–Hamilton theorem.) ★

15.4 *Quasi-coherent sheaves on* Proj R

As before, we assume that R is a graded Noetherian ring generated in degree 1. The main theorem of this section says that any quasi-coherent sheaf $\mathcal{F}$ on $X = \operatorname{Proj} R$ is the graded $\sim$ of some graded R-module M. Not surprisingly, this R-module is exactly the associated graded module $M = \Gamma_*(\mathcal{F})$. The following is the main theorem of this section:

PROPOSITION 15.19 *Let R be a graded ring, finitely generated in degree 1 over R_0. Suppose $\mathcal{F}$ is a quasi-coherent sheaf on* Proj R *and let $M = \Gamma_*(\mathcal{F})$. Then there is a canonical isomorphism*

$$\beta : \widetilde{M} \to \mathcal{F}. \tag{15.3}$$

To define β, we will need some notation. Let $x_1, \dots, x_r$ be generators of degree 1 for R and let $U_i = \operatorname{Spec}(R_{x_i})_0$. Note that $\mathcal{O}(1)$ is invertible, and that $\mathcal{O}(1)|_{U_i} \simeq \mathcal{O}_{U_i}$. Under this isomorphism, $f|_{U_i}$ may be written as $f_i x_i$ for some $f_i \in \Gamma(U_i, \mathcal{O}_X) = (R_{x_i})_0$. Also note that $D_+(f) \cap U_i = D(f_i) \subset U_i$ is a distinguished open in U_i. Multiplication by x_i^n also gives an isomorphism

$$\phi_{i,n} : \mathcal{F}|_{U_i} \to (\mathcal{F} \otimes \mathcal{O}_X(n))|_{U_i}.$$

By construction, $M = \Gamma_*(\mathcal{F})$ is a graded R-module, so the the sheaf $\widetilde{M}$ is naturally a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

Let us first define the map (15.3). As usual, it suffices to define this map over the distinguished opens $D_+(f)$, and since we assume that R is generated in degree 1, we may assume that $f \in R_1$. On $D_+(f)$, a section of $\widetilde{\Gamma_*(\mathcal{F})}$ is represented by a fraction m/f^d where $m \in M_d = \Gamma(X, \mathcal{F}(d))$. Note that $D_+(f)$ is affine, so the isomorphism $\mathcal{F}(d) \otimes \mathcal{O}(-d) \simeq \mathcal{F}$ induces an isomorphism

$$\iota_d : \Gamma(D_+(f), \mathcal{F}(d)) \otimes_{(R_f)_0} \Gamma(D_+(f), \mathcal{O}(-d)) \to \Gamma(D_+(f), \mathcal{F}) \tag{15.4}$$

Thus, if we regard f^{-d} as a section in $\mathcal{O}_X(-d)(D_+(f))$, then the tensor product $m|_{D_+(f)} \otimes f^{-d}$ can be regarded as a section of $\mathcal{F}$. This allows us to define (15.3)

$$(M_f)_0 \to \Gamma(D_+(f), \mathcal{F}) \tag{15.5}$$

by sending m/f^d to $\iota_d(m|_{D_+(f)} \otimes f^{-d})$.

PROOF (*of Theorem 15.19*): Injectivity of (15.5): Suppose that m/f^d maps to zero via (15.5), i.e., $m \in \Gamma(X, \mathcal{F}(d))$ is a section such that

$$m|_{D_+(f)} \otimes f^{-d} = 0 \tag{15.6}$$

as a section of $\mathcal{F}(d) \otimes \mathcal{O}(-d)$ over $D_+(f)$. Then also $m|_{D_+(f)} \cdot f^{-d} = 0$ in $\Gamma(D_+(f), \mathcal{F})$ (via the isomorphism (15.4)), so since f is invertible over $D_+(f)$, we must have $m|_{D_+(f)} = 0$. This also implies that $m|_{D(f_i)} = 0$. Since U_i is affine, and $D(f_i)$ is a distinguished open subset, we find by Lemma 14.2ii) that there is an $n > 0$ such that

$$f_i^n \cdot m|_{U_i} = 0$$

as a section of $\mathcal{F}(d)$ over U_i. (Here we pick an n that works for all i simultaneously.) This in turn implies that, in $\Gamma(U_i, \mathcal{F} \otimes \mathcal{O}(n))$,

$$m \otimes f^n|_{U_i} = m|_{U_i} \otimes f_i^n x_i^n = m|_{U_i} f_i^n \otimes x_i^n = \phi_{i,n}(f_i^n \cdot m|_{U_i}) = 0.$$

holds for every i. By Locality, this implies that $m \otimes f^n = 0$ and so $m/f^d = (m \otimes f^n)/f^{d+n} = 0$ in $(M_f)_0$, and the map is injective.

Surjectivity of (15.5). Let $t \in \Gamma(D_+(f), \mathcal{F})$ and consider the restrictions $t_i = t|_{D(f_i)}$. Again, since U_i is affine for each $i \in \{1, \dots, r\}$, we know from Lemma 14.2iii) that some $t_i \cdot f_i^M$ extends to a section m_i' in $\Gamma(U_i, \mathcal{F}(M))$ (and we as before choose an $M > 0$ that works for all i). Define

$$m_i = \phi_{i,M}(m_i') \in \Gamma(U_i, \mathcal{F} \otimes \mathcal{O}(M))$$

A potential problem is that the m_i might not necessarily agree on $U_i \cap U_j$, hindering them to be glued together. However, it holds that

$$m_i|_{D(f_i)} = t|_{D(f_i)} \otimes f^M|_{D(f_i)}$$

so at least $m_i = m_j$ on $U_i \cap U_j \cap D_+(f)$. Now, $U_i \cap U_j$ is also affine (because X is separated), and $U_i \cap U_j \cap D_+(f)$ is a distinguished open subset of it, so arguing as in the injectivity part shows that there is a large integer $N > 0$ such that

$$(m_i|_{U_i \cap U_j} - m_j|_{U_i \cap U_j}) \otimes f^N = 0$$

in $\Gamma(U_i \cap U_j, \mathcal{F} \otimes \mathcal{O}(M) \otimes \mathcal{O}(N))$. It then follows that $m_i \otimes f^{M+N}$ can be glued to a section $m \in \Gamma(X, \mathcal{F} \otimes \mathcal{O}(M+N))$. By construction, this section has the property that it restricts to $t \otimes f^{M+N}|_{D_+(f)}$ over $D_+(f)$. Hence m/f^{M+N} maps to t via the map 15.5. ❑

We have now defined two functors

$$\sim : \mathsf{GrMod}_R \to \mathsf{QCoh}_X$$

and

$$\Gamma_* : \mathsf{QCoh}_X \to \mathsf{GrMod}_R$$

Since $\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ is an isomorphism, it follows that $\sim$ is essentially surjective. However, unlike the affine case, the functors do not give mutual inverses. This is because, as we have seen, that $\sim$ is not faithful; the $\sim$ of any module M which is finite over R_0 is the zero sheaf.

By Lemma 15.5 however, this is the only source of ambiguity here. We can define an equivalence relation on graded modules by setting $M \sim N$ if $\bigoplus_{i \geqslant i_0} M_i \simeq \bigoplus_{i \geqslant i_0} N_i$ for some $i_0 \in \mathbb{Z}$. For two finitely generated graded R-modules M, N we have $M \sim N$ if and only if $\widetilde{M} \simeq \widetilde{N}$, so we have identified precisely the 'kernel' of the functor $\sim$.

Putting everything together, we find

Theorem 15.20 *Let R be a graded ring, finitely generated in degree 1 over R_0 and let $X = \operatorname{Proj} R$. Then the functors*

$$\sim : \mathsf{GrMod}_R \to \mathsf{QCoh}_X$$

and

$$\Gamma_* : \mathsf{QCoh}_X \to \mathsf{GrMod}_R$$

satisfy $\widetilde{\Gamma_*(\mathcal{F})} = \mathcal{F}$ for all $\mathcal{F} \in \mathsf{QCoh}_X$, and give an equivalence between the categories of quasi-cohoherent sheaves on X and graded R-modules modulo the equivalence relation $M \sim N$.

15.5 Closed subschemes of projective space

Having discussed what quasi-coherent sheaves are on projective spectra, we will now use this to study closed subschemes. We saw earlier that given a graded ideal $I \subset R$ we could associate a closed subscheme $V(I) \subset \operatorname{Proj} R$ and a closed immersion $\operatorname{Proj}(R/I) \to \operatorname{Proj} R$. On the other hand, we also saw above that many graded modules M could give rise to the same quasi-coherent sheaf $\widetilde{M}$. This is also the case for graded ideals, as we shall see, but luckily we are again able to completely identify which ideals give rise to the same closed subscheme.

In the discussion it will be convenient to introduce the *saturation* of an ideal. The upshot will be that this will serve as the 'largest' ideal corresponding to a given subscheme. We fix an ideal $B \subset R$ (the case to have in mind is the irrelevant ideal $B = R_+$). Then for a graded ideal $I \subset R$, we define the *saturation* of I with respect to an ideal B as the ideal

$$I : B^\infty := \bigcup_{i \geqslant 0} I : B^i = \{r \in R | B^n r \in I \text{ for some } n \geqslant 0\}.$$

We say that I is B-saturated if $I = I : B^\infty$ and more concisely, *saturated* if it is R_+-saturated. We will here denote $I : (R_+)^\infty$ by $\overline{I}$. It is not hard to check that $\overline{I}$ is homogeneous if I is.

Example 15.21 In $R = k[x_0, x_1]$, the (x_0, x_1)-saturation of (x_0^2, x_0x_1) is the ideal (x_0). Note that both (x_0) and (x_0^2, x_0x_1) define the same subscheme of $\mathbb{P}^1_k$, but in some sense the latter ideal is inferior, since it has a component in the irrelevant ideal (x_0, x_1). This example is typical; the saturation is a process which throws away components of I supported in the irrelevant ideal. ☆

PROPOSITION 15.22 *Let A be a ring and let $R = A[x_0, \ldots, x_n]$.*

i) If Y is a closed subscheme of $\mathbb{P}^n_A = \operatorname{Proj} R$ defined by an ideal sheaf $\mathcal{I}$, then the ideal

$$I = \Gamma_*(\mathcal{I}) \subset R$$

is a homogeneous saturated ideal. In this setting, Y corresponds to the subscheme $\operatorname{Proj}(R/I) \to \operatorname{Proj} R$.

ii) Two ideals I, J defined the same subscheme if and only if they have the same saturation.

In particular, there is a 1-1 correspondence between closed subschemes $i : Y \to \mathbb{P}^n_A$ and saturated homogeneous ideals $I \subset R$.

PROOF: (i) Let $i : Y \to \mathbb{P}^n_A$ be a closed subscheme of $\mathbb{P}^n_A = \operatorname{Proj} R$ and let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^n_A}$ denote the ideal sheaf of Y. Using the fact that global sections is left-exact, we have $\Gamma_*(\mathcal{I}) \subset \Gamma_*(\mathcal{O}_{\mathbb{P}^n_A}) = R$. $I = \Gamma_*(\mathcal{I})$ is naturally a graded R-module, so in fact I is a homogeneous ideal of R.

Any such ideal I gives rise to a closed subscheme $i' : \operatorname{Proj}(R/I) \to \mathbb{P}^n_A$ and hence an ideal sheaf $\mathcal{J}$ satisfying $\widetilde{I} = \mathcal{J}$. By Proposition 15.19, we also have $\widetilde{I} = \mathcal{I}$, so the two quasi-coherent ideal sheaves coincide and i is indeed the same as i'. By construction $I = \Gamma_*(\widetilde{I})$.

Let us show that I is saturated. Let $f \in \overline{I}$ be homogeneous of degree q. Then there is an m such that $f \cdot x_i^m \in I_{q+m}$ for all i. Since $\frac{f}{1} = \frac{f \cdot x_i^m}{x_i^m} \in (I_{x_i})_q$ for all i, the $\frac{f \cdot x_i^m}{x_i^m} \in \Gamma(U_i, \mathcal{I})$ glue to an element s in $\Gamma(X, \mathcal{I}(q)) = I_q$. Since f and s restrict to the same element in each $\Gamma(U_i, \mathcal{I})$, we see that $f \in I_q$. Since $\overline{I}$ is homogeneous, we are done.

(ii) If I, J define the same subscheme, they have the same ideal sheaf $\mathcal{I}$ and so $\overline{I} = \Gamma_*(X, \mathcal{I}) = \overline{J}$. ❑

EXAMPLE 15.23 Let k be a field and let $R = k[u, v]$. Moreover introduce the graded ring $S = R^{(n)} = k[u^n, u^{n-1}v, \ldots, v^n]$. We have a graded surjection

$$\phi : k[x_0, \ldots, x_n] \to S$$

given by $x_i \mapsto u^i v^{n-i}$ for $i = 0, \ldots, n$. The ideal $I = \operatorname{Ker} \phi$ is generated by the 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}.$$

Thus we have an embedding of $\mathbb{P}^1_k = \operatorname{Proj} S$ into $\mathbb{P}^n$ with image $V(I)$. The image is called a *rational normal curve of degree n*. Note that for $n = 2$, the image of $\mathbb{P}^1_k \to \mathbb{P}^2_k$ is the conic given by $x_1^2 = x_0 x_2$. ☆

EXERCISE 15.3 Check that the saturation $\overline{I}$ is homogeneous if I is. ★

15.6 Two important exact sequences

Hypersurfaces

Let $R = k[x_0, \dots, x_n]$ and $\mathbb{P}^n_k = \operatorname{Proj} R$. Let $F \in R$ denote an homogeneous polynomial of degree $d > 0$. F determines a projective hypersurface $X = V(F)$, which has dimension $n-1$. We then have an isomorphism

$$R(-d) \to I(X)$$

given by multiplication with F. Note the shift here: The constant '1' gets sent to F, which should have degree d on both sides! This gives the sequence of R-modules

$$0 \to R(-d) \to R \to R/(F) \to 0.$$

We have $\widetilde{R(-d)} = \mathcal{O}_{\mathbb{P}^n_k}(-d)$ and $\widetilde{(R/F)} = i_*\mathcal{O}_X$, where $i : X \to \mathbb{P}^n_k$ is the inclusion, so we get the exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \to \mathcal{O}_{\mathbb{P}^n_k} \to i_*\mathcal{O}_X \to 0.$$

Complete intersections

Let F, G be two homogeneous polynomials without common factors of degrees d, e respectively. Let $I = (F, G)$ and $X = V(I) \subset \mathbb{P}^n_k$. X is called a 'complete intersection' – it is the intersection of the two hypersurfaces $V(F)$ and $V(G)$. To study X we have exact sequences

$$0 \to R(-d-e) \xrightarrow{\alpha} R(-d) \oplus R(-e) \xrightarrow{\beta} I \to 0.$$

The maps here are defined by $\alpha(h) = (-hG, hF)$ and $\beta(h_1, h_2) = h_1F + h_2G$. These maps preserve the grading.

To prove exactness, we start by noting that α is injective (since R is an integral domain) and β is surjective (by the defintion of I). Then if $(h_1, h_2) \in \operatorname{Ker}\beta$, we have $h_1F = -h_2G$, which by the coprimality of F, G means that there is an element h so that $h_1 = -hG$, $h_2 = hF$.

Applying $\sim$, we obtain the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d-e) \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^n_k}(-e) \to \mathcal{I}_X \to 0.$$

These sequences are fundamental in computing the geometric invariants from X. We will see several examples of this later.

Sheaves on $\mathbb{P}^n_k$

We recall the following fundamental theorem in commutative algebra:

Theorem 15.24 (Hilbert's syzygy theorem) *Let k be a field and let $R = k[x_0, \dots, x_n]$. Then if M is a finitely generated graded R-module, then there is a finite free resolution (that is, an exact sequence)*

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

where $F^{b_k} = \bigoplus_{i=1}^{b_k} R(-d_i)$ is a free graded R-module.

David Hilbert
(1862 – 1943)

F_i is called the i-th *syzygy module* of the resolution. The minimal integer n that appears in such a resolution is called the *projective dimension of M*.

If we apply the $\sim$-functor here we obtain an exact sequence of sheaves on $\mathbb{P}^n_k$

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \widetilde{M} \to 0$$

where $\mathcal{E}_i = \bigoplus_{i=1}^{b_k} \mathcal{O}_{\mathbb{P}^n_k}(-d_i)$ is a direct sum of sheaves of the form $\mathcal{O}(d)$.

Thus any coherent sheaf can be resolved by locally free sheaves – in fact direct sums of invertible sheaves. This shows why the invertible sheaves $\mathcal{O}(d)$ are so important: They are the building blocks of all coherent sheaves on $\mathbb{P}^n$. We already saw some examples where having such a presentation was convenient. Let us give one more:

Example 15.25 (*The twisted cubic curve.*) Let k be a field and consider $\mathbb{P}^3 = \operatorname{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$. We will consider the *twisted cubic curve* $C = V(I)$ where $I \subset R$ is the ideal generated by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

i.e., $I = (q_0, q_1, q_2) = (x_1^2 - x_0 x_2, x_0 x_3 - x_1 x_2, -x_2^2 + x_1 x_3)$.

Consider the map of R-modules $R^3 \to I$ sending $e_i \mapsto q_i$. This is clearly surjective, since the q_i generate I. Let us consider the kernel of this map, that is, the module of relations of the form $a_0 q_0 + a_1 q_1 + a_2 q_2 = 0$ for $a_i \in R$. There are two obvious relations of this form, i.e., the ones we get from expanding the determinants of the two matrices

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \qquad \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

(So first matrix gives $x_0 q_2 - x_1 q_1 + x_2 q_2 = 0$ for instance). These give a map $R^2 \xrightarrow{M\cdot} R^3$, where M is the matrix above. This map is injective, and it turns out that there is an exact sequence of R-modules

$$0 \to R^2 \xrightarrow{M} R^3 \to I \to 0.$$

Again, if we want to be completely precise, we should consider these as *graded* modules, so we must shift the degrees according to the degrees of the maps above

$$0 \to R(-3)^2 \xrightarrow{M} R(-2)^3 \to I \to 0.$$

This gives the resolution of the ideal I of C. Then applying $\sim$, and using the fact that $\mathcal{I} = \widetilde{I}$, we get a resolution of the ideal sheaf of C:

$$0 \to \mathcal{O}_{\mathbb{P}^3_k}(-3)^2 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^3_k}(-2)^3 \to \mathcal{I} \to 0.$$

We will see later in Chapter 18 how to use sequences like this to extract geometric information about C. ☆

Chapter 16

Locally free sheaves

The most important examples of quasi-coherent sheaves are the locally free sheaves. As the name suggests, these are sheaves which are locally isomorphic to a direct sum of copies of the structure sheaf of the scheme. Because of this 'freeness' property, these sheaves are in many respects the nicest examples of sheaves on a scheme and the easiest to work with. They are also the algebraic counterpart to the vector bundles in topology.

An $\mathcal{O}_X$-module $\mathcal{E}$ is called *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is *locally free* if there exists a *trivializing cover*; that is, an open cover $\{U_i\}_{i\in I}$ such that $\mathcal{E}|_{U_i}$ is free for each i. The *rank* of $\mathcal{E}$ at a point $x \in U_i$ is the number $r_x(\mathcal{E})$ of copies of $\mathcal{O}_{U_i}$ needed (this may be finite or infinite). If X is connected, the rank of $\mathcal{E}$ will be the same everywhere, but in general we allow variation.

Locally free sheaf

We will almost always concern ourself with the case $\mathcal{E}$ has finite rank. In fact, we will mostly be interested in the case when the rank is equal to some fixed integer r. A locally free sheaf of rank one is called an *invertible sheaf*.

Invertible sheaf

In general, a locally free sheaf of rank r, $\mathcal{E}$, is obtained by gluing together copies of the trivla sheaf $\mathcal{O}_U^r$. More precisely, a sheaf $\mathcal{E}$ is locally free of rank r if and only if there is an open cover (which we well may take to be affine) U_i and isomorphisms of $\mathcal{O}_{U_i}$-modules

$$\phi_i : \mathcal{O}_{U_i}^r \to \mathcal{E}|_{U_i} \tag{16.1}$$

Given such isomorphisms ϕ_i, note that $\tau_{ji} = \phi_j^{-1} \circ \phi_i$ defines an isomorphism

$$\mathcal{O}_{U_{ij}}^r \to \mathcal{O}_{U_{ij}}^r$$

over $U_{ij} = U_i \cap U_j$. Conversely, we know from the Gluing lemma for sheaves that given isomorphisms τ_{ji} as above, satisfying the cocycle condition

$$\tau_{ki} = \tau_{kj} \circ \tau_{ji} \tag{16.2}$$

on the triple overlaps $U_i \cap U_j \cap U_k$, the sheaves $\mathcal{O}_{U_{ij}}^r$ glue to a unique locally free sheaf $\mathcal{E}$.

Note that any isomorphism of $\mathcal{O}_{U_{ij}}$-modules $\mathcal{O}_{U_{ij}}^r \to \mathcal{O}_{U_{ij}}^r$ is given by some $r \times r$-matrix with entries in $\mathcal{O}_X(U_{ij})$. Thus, we will sometimes specify $\mathcal{E}$ by giving the gluing maps τ_{ji} as matrices satisfying the cocycle condition (16.2).

If $\mathcal{E}$ is a locally free sheaf, the stalk $\mathcal{E}_x$ is a free $\mathcal{O}_{X,x}$-module for every $x \in X$. In fact, under some coherence conditions, the converse holds:

LEMMA 16.1 *Suppose that X is a locally Noetherian scheme. A coherent sheaf $\mathcal{E}$ on X having the property that $\mathcal{E}_x \simeq \mathcal{O}_{X,x}^r$ for every $x \in X$ for some fixed r, is locally free.*

However, the converse of this statement does not hold in general. A simple counterexample appears already on the spectrum of a DVR, a continuation of Example 13.5 on page 189:

EXAMPLE 16.2 Let A be DVR with fraction field K, and let x and η be respectively the closed and the open point of $X = \operatorname{Spec} A$. Let $\mathcal{E}$ be the $\mathcal{O}_X$-module defined by $\Gamma(X, \mathcal{E}) = A$ and $\Gamma(\{\eta\}, \mathcal{E}) = K$, and the restriction map being zero. Then $\mathcal{E}$ is an $\mathcal{O}_X$-module with exactly the same stalks as the structure sheaf $\mathcal{O}_X$, but it is not locally free (in fact, it is not even quasi-coherent). ☆

✲ EXERCISE 16.1 Prove Lemma 16.1. ★

16.1 Examples

EXAMPLE 16.3 The sheaf $\mathcal{O}_X^r = \bigoplus_{i=1}^r \mathcal{O}_X$ is a locally free sheaf of rank r. As this is globally a free sheaf, it is sometimes called 'trivial'. ☆

EXAMPLE 16.4 Let $X = \mathbb{P}_A^1$ and consider the sheaves $\mathcal{O}_{\mathbb{P}_A^1}(m)$ constructed on page 90. These sheaves were made by gluing together trivial sheaves of rank one, so $\mathcal{O}_{\mathbb{P}_A^1}(m)$ is locally free of rank one; that is, it is an invertible sheaf. Moreover, we showed $\mathcal{O}_{\mathbb{P}_A^1}(m) \not\simeq \mathcal{O}_{\mathbb{P}_A^1}$ for $m \neq 0$. ☆

EXAMPLE 16.5 (The tangent bundle of the n-sphere) Let $X = \operatorname{Spec} A$ where we put $A = \mathbb{R}[x_0, \ldots, x_n]/(x_0^2 + \cdots + x_n^2 - 1)$, and consider the A-module homomorphism $f : A^{n+1} \to A$ given by $f(e_i) = x_i$. Then $M = \operatorname{Ker} f$ gives rise to a quasi-coherent sheaf $\mathcal{T} = \widetilde{M}$. Any element in the kernel corresponds to a vector of elements $v = (a_0, \ldots, a_n) \in A^{n+1}$ so that

$$a_0 x_0 + \cdots + a_n x_n = 0$$

On $U = D(x_0)$ we may divide by x_0, and solve for a_0, so v is uniquely determined by the elements $(a_1, \ldots, a_n)$. Conversely, given any such an n-tuple of elements in A, we may define an element $v \in M_{x_0}$ using the above relation. In particular, $M_{x_0} \simeq A^n$. A similar argument works for the other x_i, showing that $\mathcal{T}$ is locally free of rank n. ☆

It is a hard theorem that $\mathcal{T}$ is not free, if $n \notin \{0, 1, 3, 7\}$!

EXAMPLE 16.6 *($\mathbb{P}^n$.)* Let k be a field and write $\mathbb{P}^n = \operatorname{Proj} R$ where $R = k[x_0, \ldots, x_n]$. Consider the map of graded modules $\phi : R(-1) \to R^{n+1}$ sending $1 \in R$ to the element $(x_0, \ldots, x_n) \in R^{n+1}$. This map is clearly injective, so we get an exact sequence

$$0 \to R(-1) \xrightarrow{\phi} R^{n+1} \to M \to 0$$

where $M = \operatorname{Coker} \phi$. Applying $\sim$, we get an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}_k^n}(-1) \to \mathcal{O}_{\mathbb{P}_k^n}^{n+1} \to \mathcal{E} \to 0 \tag{16.3}$$

where $\mathcal{E} = \widetilde{M}$. We claim that $\mathcal{E}$ is locally free of rank n. Indeed, on the distinguished open set $D_+(x_0) = \operatorname{Spec}(R_{x_0})_0$, we have

$$\begin{aligned}
\mathcal{E}(D_+(x_0)) &= \left(\bigoplus_{i=0}^{n} R_{x_0}/(x_0 e_0 + \cdots + x_n e_n)\right)_0 \\
&= \left(\bigoplus_{i=0}^{n} k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]e_i\right) / (e_0 + \frac{x_1}{x_0}e_0 \cdots + \frac{x_n}{x_0}e_n) \\
&\simeq \bigoplus_{i=1}^{n} k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]e_i.
\end{aligned}$$

Hence $\mathcal{E}|_{D_+(x_0)} \simeq \mathcal{O}_{U_0}^n$. By a symmetric argument, $\mathcal{E}$ is free also on the other $D_+(x_i)$, so it is locally free of rank n. We will show in Section 18.7 that $\mathcal{E}$ is not free, and in fact not even isomorphic to a direct sum of invertible sheaves. ☆

Example 16.7 (*The four-dimensional quadric hypersurface.*) Let k be a field and let $R = k[p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}]$. Consider the matrix

$$M = \begin{pmatrix} 0 & p_{01} & p_{02} & p_{03} \\ -p_{01} & 0 & p_{12} & p_{13} \\ -p_{02} & -p_{12} & 0 & p_{23} \\ -p_{03} & -p_{13} & -p_{23} & 0 \end{pmatrix}.$$

Let us consider the closed subschemes in $\mathbb{P}^5 = \operatorname{Proj} R$ defined by the conditions that this matrix has a given rank. Note that M has rank $\leqslant 3$ precisely when the determinant vanishes. In fact, this matrix M has the special property that the determinant is a square: $\det M = q^2$ where

$$q = p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$$

The locus of points where M has rank 2 is given by the ideal generated by the 2×2-minors, which by direct calculation has radical equal to the irrelevant ideal R_+. Consider the exact sequence

$$0 \to R(-1)^4 \xrightarrow{M} R^4 \to \operatorname{Coker} M \to 0.$$

Applying $\sim$ we obtain an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-1)^4 \to \mathcal{O}_{\mathbb{P}^5}^4 \to \mathcal{F} \to 0 \tag{16.4}$$

where $\mathcal{F} = \widetilde{\operatorname{Coker} M}$.

Consider the quadric hypersurface $X = V(q)$ and let $i : X \to \mathbb{P}^5$ denote the inclusion. Applying, i^* we arrive at an exact sequence of sheaves on X

$$\mathcal{O}_X(-1)^4 \to \mathcal{O}_X^4 \to \mathcal{E} \to 0$$

where $\mathcal{E} = i^*\mathcal{F}$ (recall that i^* is only right-exact). Now the discussion above shows that $\mathcal{E}$ is locally free of rank 2 (as it has rank 2 at all closed points). ☆

16.2 *Locally free sheaves and projective modules*

On an affine scheme $X = \operatorname{Spec} A$ every quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ is isomorphic to $\widetilde{M}$ for some A-module M. Thus a natural question is which A-modules give rise to locally free sheaves. The main result of this section is that $\mathcal{E}$ is locally free of finite rank if and only if M is finitely generated and projective.

We recall a few basic facts about projective modules (for a more extensive treatment see Chapter **??** in [**?**]). An A-module M is called *projective* if there is another module N so that $M \oplus N \simeq A^I$ is free. M being projective can further be characterized by saying that the functor $N \mapsto \operatorname{Hom}_A(M, N)$ is exact. It is clear that free modules have this property, but there are many examples of projective modules which are not free. However, over local rings the two notions are the same for finitely generated modules:

LEMMA 16.8 *Let A be a local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated projective A-module. Then M is free.*

PROPOSITION 16.9 *Let $X = \operatorname{Spec} A$ where A is Noetherian, and let $\mathcal{F} = \widetilde{M}$ be a coherent sheaf. The following are equivalent:*

i) *$\mathcal{F}$ is locally free;*
ii) *$\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$;*
iii) *M is* locally free, *i.e., $M_\mathfrak{p}$ is free for all $\mathfrak{p} \in \operatorname{Spec} A$;*
iv) *M is* projective, *i.e. there is a module N such that $M \oplus N \simeq A^I$ is free.*

PROOF: This is really a result in commutative algebra, so we only say a few words. We have already seen the equivalence *i)* $\iff$ *ii)*. The equivalence *ii)* $\iff$ *iii)* follows by definition of $\widetilde{M}$, and finally, *iii)* $\Rightarrow$ *iv)* follows because being 'projective' is a local property, *i.e.* M is projective if and only if $M_\mathfrak{p}$ is for every $\mathfrak{p} \in \operatorname{Spec} A$. The implication *iv)* $\Rightarrow$ *iii)* follows from the lemma above. ❏

EXAMPLE 16.10 Let $X = \operatorname{Spec} A$, where $A = \mathbb{Z}/2 \times \mathbb{Z}/2$ and consider the module $M = \mathbb{Z}/2 \times (0)$ which has the structure of an A-module. Then M is projective, since if $N = (0) \times \mathbb{Z}/2$, we have $M \otimes N \simeq A$ (as A-modules!). However, M is clearly not free, because any free A^I module must have at least four elements! The sheaf $\mathcal{E} = \widetilde{M}$ is thus locally free, but not free on X. Note that X consists of two copies of $\operatorname{Spec} \mathbb{Z}/2$. $\mathcal{E}$ restricts to the structure sheaf on one of these and to the zero sheaf on the other. ☆

EXAMPLE 16.11 A less trivial example arises in number theory. We consider $A = \mathbb{Z}[i\sqrt{5}]$ and the ideal $\mathfrak{a} = (2, 1 + i\sqrt{5})$. Then a direct computation shows that $\mathfrak{a} \otimes \mathfrak{a} \simeq A \otimes A$, so $\mathfrak{a}$ is projective (see Example **??** in CA). However, $\mathfrak{a}$ is an ideal in A, so it is free if and only if it is principal. We therefore conclude that it is not free. ☆

✲ EXERCISE 16.2 Let $X = \operatorname{Spec} A$, where $A = \prod_{i=0}^{\infty} \mathbb{Z}$. Show that $M = \mathbb{Z}$ is naturally an A-module which is projective, but not free. ★

EXERCISE 16.3 *(Torsion sheaves.)* Let X be an integral scheme, and let $\mathcal{F}$ be a quasicoherent sheaf on X. Define for each open set $U \subset X$, a subgroup $T(U) \subset \mathcal{F}(U)$ consisting of all the elements $m \in \mathcal{F}(U)$ such that the germ m_x is torsion in $\mathcal{F}_x$ for all $x \in X$, i.e., $a_x \cdot m_x = 0$ for some non-zero $a_x \in \mathcal{O}_{X,x}$.

a) Show that T is a subsheaf of $\mathcal{F}$. Also, show that T is quasi-coherent. T is called the *torsion subsheaf of* $\mathcal{F}$; another notation for it is $\mathcal{F}_{\text{tors}}$.
b) Let $\mathscr{K}$ denote the constant sheaf on $K = k(X)$. Define a map of sheaves

$$\nu : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathscr{K}.$$

Show that $T = \operatorname{Ker} \nu$.
c) A sheaf is callled *torsion free if* $\mathcal{F}_{\text{tors}} = 0$. Show that the quotient $\mathcal{F}/T$ is always torsion free, i.e., $(\mathcal{F}/T)_{\text{tors}} = 0$.
d) Show that any locally free sheaf is torsion free.

★

16.3 *Properties of locally free sheaves*

From the previous proposition, local properties of coherent locally free sheaves are obtained from corresponding properties of coherent projective modules. And by using sufficiently fine affine covers, one may even (at least, when maps are globally defined) reduce to the case of free modules.

Recall the hom-sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ whose sections over an open U is $\operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U), \mathcal{F}(U))$ and which is compatible with the tilde functor. The case that $\mathcal{F} = \mathcal{O}_X$ is of particular importance and $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is called *the dual* of $\mathcal{E}$ and is denoted $\mathcal{E}^\vee$.

The dual of a locally free sheaf

For each open affine U, there is a map of $\mathcal{O}_X(U)$-modules

$$\operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U), \mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \to \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U), \mathcal{F}(U))$$

given by $\phi \otimes s \mapsto (x \mapsto \phi(x)s)$. It is compatible with the localization maps, and therefore induces a map of $\mathcal{O}_X$-modules.

$$\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$

The next proposition summarizes some of the basic properties of locally free coherent sheaves. The proofs are immediate – just reduce to the affine and free case by restricting to a sufficiently fine covering, and for free modules the statements are well-known.

PROPOSITION 16.12 *Let X be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be two coherent locally free $\mathcal{O}_X$-modules.*

i) The direct sum $\mathcal{E} \oplus \mathcal{F}$ is locally free of rank $r_x(\mathcal{E}) + r_x(\mathcal{F})$;
ii) The tensor product $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally free of rank $r_x(\mathcal{E}) \cdot r_x(\mathcal{F})$;
iii) The dual sheaf $\mathcal{E}^\vee$ is locally free of rank $r_x(\mathcal{E})$, and the canonical evaluation map $(\mathcal{E}^\vee)^\vee \to \mathcal{E}$ is an isomorphism;
iv) The canonical map $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is an isomorphism; and rank of $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ equals $r_x(\mathcal{E}) r_x(\mathcal{F})$.

EXAMPLE 16.13 Suppose $\mathcal{E}$ is locally free of rank r. Let U_i be a trivializing cover, and let τ_{ji} denote the gluing functions for $\mathcal{E}$. As before, we interpret τ_{ji} as an $r \times r$ matrix with entries in $\mathcal{O}_X(U_i \cap U_j)$. Then $\mathcal{E}^\vee$ is obtained by the gluing functions $\nu_{ji} = (\tau_{ji})^{-1}$. ☆

EXAMPLE 16.14 Suppose $\mathcal{E}$ and $\mathcal{F}$ are locally free of ranks r and s respectively. After refining, we may assume that they admit the same trivializing cover. Suppose that the gluing functions are given by τ_{ji} and ν_{ji} respectively. Then $\mathcal{E} \oplus \mathcal{F}$ is obtained by gluing together $\mathcal{O}_{U_i}^r \oplus \mathcal{O}_{U_i}^r$ via the matrix

$$\Phi_{ji} = \begin{pmatrix} \tau_{ji} & 0 \\ 0 & \nu_{ji} \end{pmatrix}$$

If $\mathcal{F} = L$ is an invertible sheaf, then $\mathcal{E} \otimes L$ is obtained using the gluing functions $\tau_{ji} \cdot \eta_{ji}$.

Thus, for instance $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ is obtained using the gluing matrix

$$\tau_{10} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

over $U_0 \cap U_1 = \operatorname{Spec} k[t, t^{-1}]$. ☆

Pushforwards and pullbacks

A word of warning: the pushforward of a locally free sheaf is not locally free in general. For instance, if $i : \operatorname{Spec} k \to \mathbb{A}_k^1$ is the inclusion of a closed point p in $\mathbb{A}_k^1$, $\mathcal{F} = i_* \mathcal{O}_{\operatorname{Spec} k}$ has $\mathcal{F}_p = \mathcal{O}_{\mathbb{A}^1, p}$ but zero stalks outside of p, so $\mathcal{F}$ is not locally free in a neighbourhood of p. In general, if $\mathcal{F}$ is locally free of positive rank, then $\operatorname{Supp}(\mathcal{F}) = X$.

For pullbacks, however, we have the following:

PROPOSITION 16.15 *Let $f \colon X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a locally free $\mathcal{O}_X$-module (of the same rank).*

PROOF: Let U_i be a local trivialization of $\mathcal{G}$ on Y, so that $\mathcal{G}|_{U_i} \simeq \bigoplus_I \mathcal{O}_{U_i}$. Then, since $f^*\mathcal{O}_Y = \mathcal{O}_X$ for any morphism, we see that $f^*\mathcal{G}|_{f^{-1}U_i} \simeq \bigoplus_I \mathcal{O}_{f^{-1}U_i}$ is free over $f^{-1}(U_i)$. Hence $f^*\mathcal{G}$ is locally free. ❑

Let us take a closer look at the pullback of sections of $\mathcal{G}$. As explained in Chapter 13, we have the canonical map of sheaves on Y

$$f^\sharp : \mathcal{G} \to f_*(f^*\mathcal{G}).$$

Evaluating this over the open set $U = Y$, we see that for a global section $s \in \Gamma(Y, \mathcal{G})$, the pullback $f^*(s) = f^\sharp(Y)(s)$ is a global section in $f^*\mathcal{G}$.

This pullback section is especially simple in the case where X and Y are both affine; f is induced by a ring map $\phi : A \to B$; and $L = \mathcal{O}_Y^r$ (which is always the case locally). In this case, $f^*\mathcal{O}_Y^r = \mathcal{O}_X^r$ and the pullback of a section $s = (s_1, \ldots, s_r) \in A^r = \Gamma(Y, \mathcal{O}_Y^r)$ is simply $\phi(s) = (\phi(s_1), \ldots, \phi(s_r)) \in B^r = \Gamma(X, \mathcal{O}_X^r)$.

EXAMPLE 16.16 Consider the morphism $f : \mathbb{A}^1 \to \mathbb{A}^1$ induced by $k[y] \to k[x]$ $y \mapsto x^2$. Then $s = y^3 + 1$ defines a global section of $\mathcal{O}^1_{\mathbb{A}}$, and $f^*(s) = x^6 + 1$. ☆

EXAMPLE 16.17 Consider the 'squaring-morphism'

$$f \colon \mathbb{P}^1_k \to \mathbb{P}^1_k$$

from Example 10.20 on page 152. We claim that the pushforward $f_*\mathcal{O}_{\mathbb{P}^1_k}$ is locally free of rank two.

Over the local chart $U_0 = \operatorname{Spec} k[u]$ the map f is induced by $k[u] \mapsto k[t]$ with $u \mapsto t^2$, and over the chart $U_1 = \operatorname{Spec} k[u^{-1}]$ it is given by the map $k[u^{-1}] \to k[t^{-1}]$ such that $u^{-1} \mapsto t^{-2}$.

It follows that the restriction $f_*\mathcal{O}_{\mathbb{P}^1}|_{U_0}$ to U_0 equals the tilde of $k[t]$ as a $k[u]$-module which clearly is free with basis 1 and t; indeed, one has $k[t] = k[u] \oplus k[u]t$. In a symmetric way, on the chart $U_1 = \operatorname{Spec} k[u^{-1}]$ the pushforward $f_*\mathcal{O}_{\mathbb{P}^1}$ restricts to the tilde of the module $k[u^{-1}] \oplus k[u^{-1}]t^{-1}$. Hence $f_*\mathcal{O}_{\mathbb{P}^1}$ is locally free of rank 2.

In fact, one can readily check that there is an isomorphism $f_*\mathcal{O}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)$ where $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ is the invertible sheaf constructed in Example 6.3. Indeed, the factors $k[u]$ and $k[u^{-1}]$ patch up over $U_0 \cap U_1$ to give $\mathcal{O}_{\mathbb{P}^1_k}$, whereas for the other factor the gluing map is multiplication by u^{-1} since the equality $q(u^{-1})t^{-1} = q(u^{-1})u^{-1}t$ holds true. ☆

EXERCISE 16.4 *(The projection formula.)* Let $f : X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathcal{E}$ a locally free sheaf of finite rank. Show that there is a natural isomorphism of $\mathcal{O}_Y$-modules

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) \simeq f_*(\mathcal{F}) \otimes \mathcal{E}.$$

★

16.4 *Invertible sheaves and the Picard group*

Invertible sheaves

Invertible sheaves correspond to 'line bundles', as we will see later

Recall that an *invertible sheaf* on a scheme X is a coherent locally free sheaf of rank one. We usually use the letter L for such sheaves. By definition, L is invertible whenever there exists a covering $\mathcal{U} = \{U_i\}$ and isomorphisms

$$\phi_i \colon \mathcal{O}_{U_i} \to L|_{U_i}.$$

We say that $g_i = \phi_i(1) \in L(U_i)$ is a *local generator* for L. By Lemma 16.1 on page 233 a coherent $\mathcal{O}_X$-module L is invertible if and only if the stalk L_x is isomorphic to $\mathcal{O}_{X,x}$ for every $x \in X$.

Proposition 16.18 *Let X be a scheme and L and M two invertible sheaves on X. Then we have*

i) $L \otimes_{\mathcal{O}_X} M$ *is also an invertible sheaf. If g and h are local generators for L and M respectively, then $g \otimes h$ is a local generator for $L \otimes_{\mathcal{O}_X} M$;*

ii) $\mathscr{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$ *is invertible and* $\mathscr{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X) \otimes_{\mathcal{O}_X} L \simeq \mathcal{O}_X$. *If g is a local generator for L, then ψ_g defined by $\psi_g(ag) = a$ is a local generator for* $\mathscr{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$;

iii) $\mathscr{H}om_{\mathcal{O}_X}(L, M) \simeq \mathscr{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X) \otimes M$.

Proof: This follows from Proposition 16.12 on page 236. ❑

This proposition explains the term 'invertible'. Indeed, the tensor product acts as a sort of binary operation on the set of invertible sheaves; $L \otimes M$ is invertible if L and M are, and the tensor product is associative. Tensoring an invertible sheaf by $\mathcal{O}_X$ does nothing, so $\mathcal{O}_X$ serves as the identity. Moreover, for an invertible sheaf L we will define $L^{-1} = \mathscr{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$; by the proposition, L^{-1} is again invertible, and serves as a multiplicative inverse of L under $\otimes$. We can make the following definition:

Émile Picard (1856–1941)

Definition 16.19 *Let X be a scheme. The* Picard group $\operatorname{Pic}(X)$ *is the group of isomorphism classes of invertible sheaves on X under the tensor product.*

Note that it is the set of isomorphism classes of invertible sheaves that form a group, not the invertible sheaves themselves: $L \otimes_{\mathcal{O}_X} L^{-1}$ is isomorphic, but strictly speaking, not *equal* to $\mathcal{O}_X$. Note also that $\operatorname{Pic}(X)$ is an *abelian* group because $L \otimes_{\mathcal{O}_X} M$ is canonically isomorphic to $M \otimes_{\mathcal{O}_X} L$.

The Picard group depends functorially on the schemes; that is, a morphism of scheme induces a morphism between the Picard groups. This is just the rank one version of Proposition 16.15:

Lemma 16.20 *For a morphism of schemes $f\colon X \to Y$, the assignment $L \mapsto f^*L$ induces a morphism of groups*

$$f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X).$$

If f and g are two composable morphisms, it holds that $(f \circ g)^* = g^* \circ f^*$.

Example 16.21 Let $X = \operatorname{Spec} \mathbb{Z}$. If $\mathcal{E}$ is any coherent sheaf on X, then $\mathcal{E} = \widetilde{M}$ for some finitely generated $\mathbb{Z}$-module M, and by the structure theorem for finitely generated abelian groups, we may write $M = \mathbb{Z}^r \oplus T$, where T is a fine direct product of groups of the form $\mathbb{Z}/n\mathbb{Z}$. If $\mathcal{E}$ in addition is required to be locally free, it must hold that $T = 0$ (otherwise, if p is a prime factor of an n appearing in one of the summands of T, the stalk at (p) will not be free). Thus $\mathcal{E} = \widetilde{\mathbb{Z}^r} = \mathcal{O}_X^r$, and we conclude that every coherent locally free sheaf on $\operatorname{Spec} \mathbb{Z}$ is trivial. In particular, we get that

$$\operatorname{Pic}(\operatorname{Spec} \mathbb{Z}) = 0.$$

On the other hand, $\operatorname{Pic}(\mathbb{Z}[\sqrt{-5}]) \neq 0$, by Example 16.11. ☆

EXAMPLE 16.22 (*Locally free sheaves on the affine line.*) The argument of the previous example in fact applies over any PID A: every coherent sheaf on $X = \operatorname{Spec} A$ must have the form $\widetilde{M}$ for $M = A^r \oplus T$ where T is a finitely generated torsion module, and if we require $\widetilde{M}$ to be locally free, the torsion part must vanish; *i.e.* it must hold that $T = 0$. In particular, this applies to locally free sheaves on the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$:

PROPOSITION 16.23 *Any coherent locally free sheaf over $\mathbb{A}^1_k$ is trivial. Hence, in particular, it holds that* $\operatorname{Pic}(\mathbb{A}^1_k) = 0$.

In higher dimension the Quillen–Suslin theorem asserts that any locally free sheaf on $\mathbb{A}^n_k$ is trivial. This is a much deeper result than the above. In particular the Quillen–Suslin implies that $\operatorname{Pic}(\mathbb{A}^n_k) = 0$; we will see a direct proof of the latter statement in Chapter 20. ☆

16.5 Locally free sheaves on $\mathbb{P}^1_k$

Invertible sheaves

On page 90 in Chapter 6 we constructed the family $\mathcal{O}_{\mathbb{P}^1_A}(m)$ of sheaves on the projective line over a ring A. They are all invertible, as we showed in Chapter 6, and in this section we intend to show there are no others when A is a field.

Recall that $\mathbb{P}^1_k$ is obtained by gluing together the two open affine subsets $U_0 = \operatorname{Spec} k[u]$ and $U_1 = \operatorname{Spec} k[u^{-1}]$ along $V = \operatorname{Spec} k[u, u^{-1}]$. Given an invertible sheaf L on $\mathbb{P}^1$, the restriction of it to each of the two opens must be trivial since $\operatorname{Pic}(\mathbb{A}^1_k) = 0$, so there are isomorphisms $\phi_i \colon L|_{U_i} \to \mathcal{O}_{U_i}$. Over the intersection $V = U_0 \cap U_1$ we thus obtain two isomorphisms $\phi_i|_V \colon L|_V \to \mathcal{O}_V$. In particular, the composition $\phi_1|_V \circ \phi_0|_V^{-1} \colon \mathcal{O}_V \to \mathcal{O}_V$ is an isomorphism. Like any such map, it is induced by a module homomorphism $k[u, u^{-1}] \to k[u, u^{-1}]$ which is just multiplication by some unit in $k[u, u^{-1}]$. But all units in $k[u, u^{-1}]$ are of the form αu^m for an integer m and non-zero scalar α, the latter can be ignored (incorporate it in one of the ϕ_i's), and we recognize L to be the sheaf $\mathcal{O}_{\mathbb{P}^1_k}(m)$ from Chapter 6.

With the present set-up we also obtain in a natural way an isomorphism $\mathcal{O}_{\mathbb{P}^1_k}(m) \otimes \mathcal{O}_{\mathbb{P}^1_k}(m') \simeq \mathcal{O}_{\mathbb{P}^1_k}(m + m')$: the patching map over V for the tensor product equals the tensor product of the two patching maps (which are multiplication by s^m and $s^{m'}$ respectively), and when we identify $\mathcal{O}_V \otimes \mathcal{O}_V$ with $\mathcal{O}_V$, it becomes the product of the two; that is, it becomes multiplication by $s^{m+m'}$. In particular, it holds that $\mathcal{O}_{\mathbb{P}^1_k}(m) \otimes \mathcal{O}_{\mathbb{P}^1_k}(-m) \simeq \mathcal{O}_{\mathbb{P}^1_k}$.

Back in Chapter 6 we verified that the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(m)$ are not isomorphic when $m \geqslant 0$; *e.g.* since they have different spaces of global sections, and what we just did, extends this to all m. We thus have shown:

Proposition 16.24 *Every invertible sheaf on $\mathbb{P}^1_k$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(m)$ for some $m \in \mathbb{Z}$, and sending $\mathcal{O}_{\mathbb{P}^1_k}(m)$ to m yields an isomorphism $\operatorname{Pic} \mathbb{P}^1_k \simeq \mathbb{Z}$.*

We will prove a generalization of this in Proposition 20.44.

Locally free sheaves of higher rank

In 1955, Grothendieck wrote his paper "Sur la classification des fibres holomorphes sur la sphere de Riemann", showing that any locally free sheaf on the projective over a field splits as a sum of invertible sheaves:

Theorem 16.25 *Let $X = \mathbb{P}^1_k$ and let $\mathcal{E}$ be a locally free sheaf of rank r. Then there are integers $a_1, \ldots, a_r$ such that*

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_r). \tag{16.5}$$

Grothendieck's proof was sheaf-theoretic, but in fact this is a rather elementary result which has been rediscovered and reproved several times. For instance, Grothendieck was not aware of the following result, due to Dedkind–Weber from 1882.

Dedekind, Weber. Theorie der algebraischen Funktionen einer Veränderlichen', Crelle's Journal, 1882

Lemma 16.26 (Dedkind–Weber) *Let k be a field and let $A \in GL_r(k[x, x^{-1}])$. Then there exist matrices $B \in GL_r(k[x])$ and $C \in GL_r(k[x^{-1}])$ such that*

$$BAC = \begin{pmatrix} x^{a_1} & & 0 \\ & \ddots & \\ 0 & & x^{a_r} \end{pmatrix}. \tag{16.6}$$

This lemma is completely elementary, and can be proved by induction on r with only basic row-operations on matrices (see for instance Exercise **??** in [**?**]).

In any case, Theorem 16.25 follows immediately from the description of quasi-coherent sheaves on $\mathbb{P}^1_k$ from Example 14.6. In the notation of that example, we have $M_0 = k[x]^r$, $M_1 = k[x^{-1}]^r$ and $\tau : k[x^{\pm 1}]^r \to k[x^{\pm 1}]^r$. The lemma above implies that after changing bases, the map τ is given by a diagonal matrix 16.6. Hence $\mathcal{E}$ splits as (16.5).

Exercise 16.5 Prove Lemma 16.26 for $r = 2$. ★

16.6 *Zero sets of sections*

Let $\mathcal{E}$ be a locally free sheaf on a scheme X and let $x \in X$ be a point. We will call the *fiber of $\mathcal{E}$ at x* the $k(x)$-vector space $\mathcal{E}(x)$

$$\mathcal{E}(x) = \mathcal{E}_x / \mathfrak{m}_x \mathcal{E}_x \simeq \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

If $U \subseteq X$ is an open subset containing x and $s \in \Gamma(U, \mathcal{E})$ is a section of $\mathcal{E}$ over U, we shall denote by $s(x)$ the image of the germ $s_x \in \mathcal{E}_x$ in the fibre $\mathcal{E}(x)$. This is in close analogy with what we called the 'value' of a regular function in Chapter 2.

Definition 16.27 *Let $\mathcal{E}$ be an locally free sheaf on the scheme X, and suppose $s \in \Gamma(X, \mathcal{E})$ is a global section. We define the* zero set of s *by*

$$V(s) = \{x \in X \,|\, s(x) = 0\}.$$

Also, we define the open set X_s by

$$X_s = \{\, x \in X \mid s(x) \neq 0 \,\}.$$

Equivalently, X_s is the set of points x where $s \notin \mathfrak{m}_x \mathcal{E}_x$.

The set $V(s)$ is indeed a closed subset of X: the sheaf $\mathcal{E}$ is locally free, so every point has an open affine neighbourhood U such that $\mathcal{E}|_U \simeq \mathcal{O}_X|_U$, and we may safely assume that $\mathcal{E} = \mathcal{O}_X^r$ with $X = \operatorname{Spec} A$. This brings us back to the 'function case': the section s is an element in A^r, and $V(s)$ coincides with the usual closed set. It follows that $X_s = X - V(s)$ is also open in X.

Proposition 16.28 *Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{E}$ be a locally free sheaf on Y. Then*

$$f^{-1}(V(s)) = V(f^*s) \quad \textit{and } f^{-1}(X_s) = X_{f^*s}.$$

Proof: For each of these statements, we may reduce to the case $X = \operatorname{Spec} B$; $Y = \operatorname{Spec} A$ and $L = \mathcal{O}_Y^r$. In that case (i) follows from the fact that $f^{-1}(V(a)) = V(\phi(a))$ for $a \in A$, which we have seen several times before. ❑

The set $V(s)$ just defined is a priori just a closed subset of X, but we can put a canonical scheme structure on it as follows. We may view a global section $s \in \Gamma(X, \mathcal{E}) = \operatorname{Hom}(\mathcal{O}_X, \mathcal{E})$, as a map of $\mathcal{O}_X$-modules $s : \mathcal{O}_X \to \mathcal{E}$. Applying $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$, we get a map

$$s^\vee : \mathcal{E}^\vee \to \mathcal{O}_X \tag{16.7}$$

The image of $s^\vee$ is a quasi-coherent ideal sheaf of $\mathcal{O}_X$. We define the *subscheme of zeroes of s* to be the closed subscheme $Z(s)$ of X.

Example 16.29 Let $X = \operatorname{Spec} A$, and $\mathcal{E} = \mathcal{O}_X^r$. Then a section $s \in \Gamma(X, \mathcal{E})$ is given by an r-tuple $(f_1, \ldots, f_r) \in A^r$ of elements in A. The map $s^\vee$ is simply the tilde of the map $A^r \to A$, that sends the i-th basis vector e_i to f_i. Therefore, $Z(s)$ is simply the usual subscheme given by the ideal $I = (f_1, \ldots, f_r)$. Locally, any subscheme $Z(s)$ looks like this example. ☆

Exercise 16.6 Show that the subscheme $Z(s)$ satisfies the following universal property: A morphism $f : T \to X$ satisfies $f^*s = 0$ if and only if it factors through $Z(s)$. (Hint: Understand the subscheme on each open affine $\operatorname{Spec} A \subset X$ first. Reduce to the case $\mathcal{E} = \mathcal{O}_X$.) ★

16.7 Globally generated sheaves

Definition 16.30 *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say $\mathcal{F}$ is* globally generated *(or* generated by global sections*) if there is a family of sections $s_i \in \mathcal{F}(X)$, $i \in I$, such that the germs of s_i generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module for each $x \in X$.*

Equivalently, $\mathcal{F}$ is globally generated if there is a surjection

$$\mathcal{O}_X^I \to \mathcal{F} \to 0$$

for some index set I. In particular, any quotient of a globally generated sheaf is also globally generated.

Let us consider a few examples:

Example 16.31 On an affine scheme any quasi-coherent sheaf is globally generated. Indeed, if $X = \operatorname{Spec} A$, $\mathcal{F} = \widetilde{M}$, for some A-module M, then picking any presentation $A^I \to M \to 0$ for M and applying tilda shows that $\mathcal{F}$ is globally generated. ☆

Example 16.32 Let R be a graded ring generated in degree 1 and set $X = \operatorname{Proj} R$. Then $\mathcal{F} = \mathcal{O}(1)$ is globally generated. Indeed, the only way $\mathcal{F}$ could fail to be globally generated is that there is a point $x \in X$ for which all sections $s \in \Gamma(X, \mathcal{O}(1)) = R_1$ simultaneously vanish. However, by assumption R_1 generates the irrelevant ideal, so this is impossible.

On the other hand, if R is not generated in degree 1, then it can happen that the sheaf $\mathcal{O}(1)$ has no global sections at all. This happens for instance for the weighted projective space $\mathbb{P}(2,3,4) = \operatorname{Proj} k[x_2, x_3, x_4]$ (with $\deg x_i = i$). The sheaf $\mathcal{O}(-1)$ is likewise not typically globally generated (unless, say, X is a point). ☆

Example 16.33 For a closed subscheme $Y \subset X$, the structure sheaf $i_*\mathcal{O}_Y$ is globally generated (generated by the section '1'). On the other hand the corresponding ideal sheaf $\mathcal{I}$ is typically not globally generated. For instance, if Y a closed point in $\mathbb{P}^1_k$, then $\mathcal{I}_Y \simeq \mathcal{O}(-1)$, which has no global sections. ☆

Example 16.34 The locally free sheaves from Section **??** are both globally generated. For instance, the sheaf $\mathcal{E}$ from (16.3) admits a surjection $\mathcal{O}^{n+1} \to \mathcal{E} \to 0$. ☆

Proposition 16.35 *Let $f : X \to Y$ be a morphism of schemes and let L be an invertible sheaf on Y. Then if L is generated by global sections $s_0, \dots, s_n$, then f^*L is generated by the sections $t_0 = f^*s_0, \dots, t_n = f^*s_n$, and X is covered by the open sets $X_{t_0}, \dots, X_{t_n}$.*

Proof: For each of these statements, we may reduce to the case $X = \operatorname{Spec} B$; $Y = \operatorname{Spec} A$ and $L = \mathcal{O}_Y$. In that case (i) follows from the fact that $f^{-1}(V(a)) = V(\phi(a))$, which we have seen several times before.

For (ii), we note that hypothesis gives that the sections $s_0, \dots, s_n$ are elements in A that generate the unit ideal. But then clearly the same holds for the pullbacks $\phi(s_0), \dots, \phi(s_n)$. ❑

Example 16.36 For the pushforward, $f_*\mathcal{F}$ may fail to be globally generated even when $\mathcal{F}$ is the structure sheaf. For example, if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is the 'squaring map', i.e., the map induced by $k[u^2, v^2] \subset k[u, v]$, then $f_*\mathcal{O}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The latter sheaf is not globally generated, since it has $\mathcal{O}(-1)$ as a quotient. ☆

EXERCISE 16.7 We say that scheme X has the *resolution property* if any coherent sheaf is a quotient of a locally free sheaf.

i) Show that projective schemes of finite type over a field have the resolution property.

Let $X = \mathbb{A}^2 \cup_U \mathbb{A}^2$, where $U = \mathbb{A}^2 - 0$, be the affine 2-space with two origins. Show that X does not admit the resolution property, by the following steps:

Let $p : \mathbb{A}^2 \to X$ and $q : \mathbb{A}^2 \to X$ be the two inclusions, and let $\mathcal{E}$ be a locally free sheaf on X.

ii) Explain why $p^*\mathcal{E}$ and $q^*\mathcal{E}$ are free sheaves on $\mathbb{A}^2$.

iii) $p^*\mathcal{E}$ and $q^*\mathcal{E}$ become isomorphic on U; use Hartog's Lemma to show that $p^*\mathcal{E}$ and $q^*\mathcal{E}$ are in fact isomorphic via the identity map. Conclude that $\mathcal{E}$ is trivial.

iv) Show that the exist sheaves on X which are not quotients of locally free sheaves.

v) Show that the affine line $X = \mathbb{A}^1 \cup_U \mathbb{A}^1$, in contrast, satisfies the resolution property. ★

16.8 Maps to projective space

Given a scheme X it is natural to ask when there is a morphism to a projective space

$$f : X \to \mathbb{P}^n,$$

or when there is a closed immersion $X \hookrightarrow \mathbb{P}^n$. Given such a morphism, we get geometric information about X using this map, e.g., by studying the fibers $f^{-1}(y)$; pulling back sheaves from $\mathbb{P}^n$; or describing the equations of the image.

The corresponding question for $\mathbb{A}^n$ has already been answered. Morphisms $X \to \mathbb{A}^n$ are in one-to-one correspondence with elements of $\Gamma(X, \mathcal{O}_X)^n$, i.e., an n-tuple of regular functions on X.

Even for projective space itself, there is not so much information in the space of global sections of the structure sheaf. However, we do have something canonical associated to $\mathbb{P}^n$, namely the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$. Given a morphism $f : X \to \mathbb{P}^n$, we get an invertible sheaf $L = f^*\mathcal{O}(1)$ on X. We even get $n+1$ distinguished global sections $s_i = f^*x_i$ by pulling back the sections $x_0, \ldots, x_n$ of $\mathcal{O}(1)$.

Note that there is no point of $\mathbb{P}^n$ where the x_i simultaneously vanish. More precisely, for every $y \in \mathbb{P}^n$, the stalk $\mathcal{O}_{\mathbb{P}^n}(1)_y$ is generated by the germ of one of the x_i. So by the properties of the pullback, we see that the same statement holds for L and the sections s_i on X. We say that L is *globally generated* by the sections s_i.

The main result in this chapter is that there is a way to reverse this process. In other words, from a given invertible sheaf L and $n+1$ global sections $s_i \in \Gamma(X, L)$ with the above property, we can uniquely reconstruct a morphism $f : X \to \mathbb{P}^n$ so that $f^*\mathcal{O}_{\mathbb{P}^n}(1) = L$ and $f^*x_i = s_i$. Thus $(L, s_0, \ldots, s_n)$ is the exactly the data we are after.

THEOREM 16.37 *Let X be a scheme over a ring A, and let L be an invertible sheaf on X with global sections $s_0, \dots, s_n \in \Gamma(X, L)$ which generate L. Then there is a* unique *morphism*

$$f : X \to \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$$

so that $f^* x_i = s_i$ for $i = 0, \dots, n$.

First an easy lemma:

LEMMA 16.38 *Let X be a scheme and let L be an invertible sheaf on X. If $s \in \Gamma(X, L)$ is a global section, then there is an isomorphism*

$$\phi : \mathcal{O}_X|_{X_s} \to L|_{X_s}$$

which sends 1 to s.

PROOF: We define ϕ over an open set $U \subset X_s$, by sending $1 \in \mathcal{O}_X(U)$ to $s \in L(U)$, which is a map of $\mathcal{O}_X$-modules. This is an isomorphism if and only if it is an isomorphism locally, so we may reduce to the case where $X = \operatorname{Spec} A$ and $L = \mathcal{O}_X$. In that case $X_s = D(s) = \operatorname{Spec} A_s$, and $s \in A$ is a unit in A_s, so multiplication by s is an isomorphism $A_s \to A_s$. ❑

PROOF (*of the theorem*): We first prove uniqueness. Let $f : X \to \mathbb{P}^n_A$ be a morphism, and consider the pulled back sections $s_i = f^* x_i$ for $i = 0, \dots, n$. Write for simplicity $X_i = X_{s_i}$ for each i. From Proposition 16.35 we have $f^{-1}(D_+(x_i)) = X_i$ for each i, so X is covered by the $n+1$ subsets X_i. We can regard the morphism as glued together from the morphisms $f_i : X_i \to D_+(x_i) = \operatorname{Spec} (R_{x_i})_0$, where $R = A[x_0, \dots, x_m]$. This in turn corresponds to a morphism of A-algebras

$$f^\sharp : (R_{x_i})_0 \to \Gamma(X_i, \mathcal{O}_X).$$

Note that x_i generates $\mathcal{O}(1)$ on $D_+(x_i)$ and $x_j = \frac{x_j}{x_i} x_i$ in $R_{(x_i)}$ for $j = 0, \dots, n$. Similarly, pulling back via $f^\sharp$ gives

$$s_j = f_i^*(x_j) = f_i^\sharp\left(\frac{x_j}{x_i} x_i\right) = f_i^\sharp\left(\frac{x_j}{x_i}\right) s_i$$

(Here we interpret the fraction $\frac{x_j}{x_i}$ as a section of $\Gamma(D_+(x_i), \mathcal{O}_{\mathbb{P}^n_A})$.) It follows that from each morphism $f : X \to \mathbb{P}^n_A$, we get $n+1$ distinguished sections $s_0, \dots, s_n$, from which we can determine the morphisms f_i. Hence f is uniquely determined from the data $(L, s_0, \dots, s_n)$.

To prove existence, we suppose that we are given $n+1$ sections $s_0, \dots, s_n$ of a globally generated invertible sheaf L, we will construct a morphism to $\mathbb{P}^n_A$, such that s_i is the pullback of x_i. As in the above example, we define this morphism on an open cover. Let $X_i = X_{s_i} = \{x \in X | s_i(x) \neq 0\}$. Since the s_i globally generate L, it follows from Lemma 16.38 that the X_i provide a local trivializing cover of L: namely there is an isomorphism $\psi_i : \mathcal{O}_X|_{X_i} \to L|_{X_i}$ which sends 1 to the section s_i. In particular, if we restrict the global section s_j to X_i, we have $s_j = r_{ij}s_i$ for some $r_{ij} \in \Gamma(X_i, \mathcal{O}_X)$. We denote this section r_{ij} by $\frac{s_j}{s_i}$. These define a map of A-algebras

$$\begin{aligned} (R_{x_i})_0 &\to \Gamma(X_i, \mathcal{O}_{X_i}) \\ \frac{x_j}{x_i} &\mapsto \frac{s_j}{s_i} \end{aligned}$$

By the correspondence between ring homomorphisms and maps into affine schemes, we obtain a morphism of schemes $f_i : X_i \to D_+(x_i)$. On $X_i \cap X_k$, the map sends $\frac{x_j}{x_k} = \frac{x_j/x_i}{x_k/x_i}$ to $\frac{s_j}{s_k} = \frac{s_j/s_i}{s_k/s_i}$. In other words, the following diagram commutes:

$$\begin{tikzcd}
(R_{x_i})_0 \arrow[rr] \arrow[dr] & & \Gamma(X_i, \mathcal{O}_X) \arrow[dr] & \\
& \left(R_{x_i x_j}\right)_0 \arrow[rr] & & \Gamma(X_i \cap X_j, \mathcal{O}_X) \\
\left(R_{x_j}\right)_0 \arrow[rr] \arrow[ur] & & \Gamma(X_j, \mathcal{O}_X) \arrow[ur] &
\end{tikzcd}$$

That means that the morphisms glue to a morphism $f : X \to \mathbb{P}^n$. It is clear that $f^*\mathcal{O}(1) \simeq L$ and that the x_i pull back to the s_i, since this is true over the principal opens $D_+(x_i)$. ❏

Abusing notation, we will refer to a morphism $\phi : X \to \mathbb{P}^n_A$ as given by the data $(L, s_0, \ldots, s_n)$ and write

$$\begin{aligned} X &\to \mathbb{P}^n_A \\ x &\mapsto (s_0(x) : \cdots : s_n(x)) \end{aligned}$$

One should still keep in mind that the sections s_i are sections of L, not regular functions. In fact, from the above proof, we see that it is the ratios s_j/s_i which can be interpretated as regular functions, locally on $X_i = \{x \in X \mid s_i(x) \neq 0\}$.

We also see that two sets of data $(L, s_0, \ldots, s_n)$, $(L, t_0, \ldots, t_n)$ give rise to the same morphism $f : X \to \mathbb{P}^n_A$ if and only there is a section $\lambda \in \mathcal{O}_X^\times(X)$ so that $t_i = \lambda s_i$ for each i. Thus morphisms $f : X \to \mathbb{P}^n_A$ are in bijective correspondence with the data $(L, s_0, \ldots, s_n)$ modulo this equivalence relation.

Given a scheme X with $s_0, \ldots, s_n$ of a line bundle L, there is a maximal open subset U such that the sections generate L for all points in U, namely $U = \bigcup_{i=0}^n X_i$. Not assuming that the s_i globally generate L, we still get a morphism $\phi : U \to \mathbb{P}^n_A$. In other words, ϕ defines a *rational map* $\phi : X \dashrightarrow \mathbb{P}^n_A$, which is a morphism when restricted to U.

EXAMPLE 16.39 Let $X = \mathbb{P}^1_k = \operatorname{Proj} k[s, t]$ and $L = \mathcal{O}_{\mathbb{P}^1_k}(2)$. Then L is globally generated by s^2, st, t^2 and the corresponding morphism

$$\begin{aligned} \phi : X &\to \mathbb{P}^2_k \\ (s : t) &\mapsto (s^2 : st : t^2) \end{aligned}$$

has image $V(x_0x_2 - x_1^2)$ which is a smooth conic. ☆

Example 16.40 (*Cuspidal cubic.*) Let $X = \mathbb{A}_k^1$ and $L = \mathcal{O}_X$. Then, $\Gamma(X, L) = k[t]$ is infinite dimensional over k. Choosing the three sections $1, t^2, t^3$, we get a map of schemes

$$\begin{aligned} X &\to \mathbb{P}_k^2 \\ t &\mapsto (1 : t^2 : t^3) \end{aligned}$$

whose image in $\mathbb{P}^2$ is the cuspidal cubic minus the point at infinity. ☆

Example 16.41 ($\mathbb{P}^n$ *as a quotient space.*) Let $X = \mathbb{A}_k^{n+1}$, and $L = \mathcal{O}_X$. Then, $\Gamma(X, L) = k[x_0, \dots, x_n]$. If we take the sections $x_0, \dots, x_n$, then they generate L outside $V(x_0, \dots, x_n)$. Hence we get a morphism of schemes

$$\begin{aligned} \mathbb{A}_k^{n+1} \backslash V(x_0, \dots, x_n) &\to \mathbb{P}_k^n \\ (x_0, \dots, x_n) &\mapsto (x_0 : \dots : x_n) \end{aligned}$$

which is exactly the 'quotient space' description of $\mathbb{P}^n$ from Exercise 10.9. ☆

Example 16.42 (*Projection from a point.*) Consider the projective space $X = \mathbb{P}_A^n$ and sections $x_1, \dots, x_n$ of $\mathcal{O}(1)$, then these sections generate $\mathcal{O}(1)$ outside the point p corresponding to $I = (x_1, \dots, x_n)$ (that is, the closed point $p = (1 : 0 : \dots : 0)$). The induced morphism $\mathbb{P}_A^n - V(I) \to \mathbb{P}_A^{n-1}$ is the *projection from p*. ☆

Example 16.43 (*Cremona transformation.*) Consider the projective space $X = \mathbb{P}_A^2$ and sections x_0, x_1, x_2 of $\mathcal{O}(1)$, then the sections x_0x_1, x_0x_2, x_1x_2 generate $\mathcal{O}(2)$ outside $V(x_0x_1, x_0x_2, x_1x_2)$ corresponding to the three points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$. The induced rational map $\mathbb{P}_A^2 \dashrightarrow \mathbb{P}_A^2$ is the *Cremona transformation*. ☆

Example 16.44 (*The Veronese surface.*) Consider $X = \mathbb{P}^2$, and $L = \mathcal{O}_{\mathbb{P}^2}(2)$. If x_0, x_1, x_2 are projective coordinates on X, then the quadratic monomials

$$x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2$$

form a basis for $H^0(X, L)$, and generate L at every point. The corresponding map $\phi : X \to \mathbb{P}^5$ is in fact a closed immersion; the image is the Veronese surface. It is a classical fact that the image is defined by the 2×2 minors of the matrix

$$\begin{pmatrix} u_0 & u_1 & u_2 \\ u_1 & u_3 & u_4 \\ u_2 & u_4 & u_5 \end{pmatrix}$$

☆

Example 16.45 (*The quadric surface.*) Let us consider again the case $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Keeping the notation from Section 20.5, we have two divisors, $L_1 = (0 : 1) \times \mathbb{P}^1$, $L_2 = \mathbb{P}^1 \times (0 : 1)$. Note that each L_i is globally generated (being the pullback of a base point free divisor on $\mathbb{P}^1$). The corresponding map is of course the i-th projection map $p_i : Q \to \mathbb{P}^1$.

If x_0, x_1 is a basis for $\Gamma(X, L_1)$, and y_0, y_1 is a basis for $\Gamma(X, L_2)$, we find that $\Gamma(X, L_1 + L_2)$ is spanned by the sections

$$s_0 = x_0y_0, s_1 = x_0y_1, s_2 = x_1y_0, s_3 = x_1y_1$$

Moreover, these sections generate $\mathcal{O}_Q(L_1 + L_2)$ everywhere, and so we get a map

$$Q \to \mathbb{P}^3$$

This is of course nothing but the Segre embedding; note the quadratic relation between the four sections $s_0 s_3 - s_1 s_2 = 0$. ☆

16.9 *Application: Automorphisms of* $\mathbb{P}^n_k$

If k is a field, then any invertible $(n+1) \times (n+1)$ matrix A with entires in k acts on $k[x_0, \dots, x_n]$ and thus gives rise to a linear automorphism $\mathbb{P}^n_k \to \mathbb{P}^n_k$. Moreover, two matrices A and A' determine the same automorphism if and only if $m = \lambda m'$ for some non-zero scalar $\lambda \in k^*$. So we are led to consider the *projective linear group*

$$PGL_n(k) = GL_n(k)/k^*$$

We will now prove that all automorphisms of $\mathbb{P}^n_k$ are given by linear transformations.

THEOREM 16.46 *$Aut_k(\mathbb{P}^n) = PGL_n(k)$.*

PROOF: The above shows that there is an injective map from the righthand side to the left. To show the reverse inclusion, let $\phi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ be any automorphism. Then we get an induced map

$$\phi^* : \mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(\mathbb{P}^n)$$

which must also be an isomorphism. Since $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$, we must have either $\phi^* \mathcal{O}_{\mathbb{P}^n}(1) = O_{\mathbb{P}^n}(1)$ or $\phi^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}(-1)$. The latter case is impossible, since $\phi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ has a lot of global sections, whereas $\mathcal{O}_{\mathbb{P}^n}(-1)$ has none. So $\phi^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_{\mathbb{P}^n}(1)$. In particular, taking global sections ϕ^* gives a map

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)),$$

which is a isomorphism of k-vector spaces. However, we may choose $\{x_0, \dots, x_n\}$ as a basis for $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and so in this basis ϕ^* gives rise to an invertible $(n+1) \times (n+1)$-matrix m. By construction m induces the same linear transformation $\mathbb{P}^n_k \to \mathbb{P}^n_k$ as ϕ, and so ϕ comes from an element of $PGL_n(k)$. ❑

Chapter 17

First steps in sheaf cohomology

One of the main challenges when working with sheaves is that surjective maps of sheaves do not always induce surjections on global sections. We have seen several examples of this failure of the global sections functor Γ to be exact when applied to a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

One has a sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \tag{17.1}$$

which is exact at each stage except on the right. In many situations in algebraic geometry, knowing that $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ is surjective is of fundamental importance. For instance, if $Z \subset X$ is a subscheme, we would sometimes like to lift sections of $\mathcal{F}|_Z$ to sections of $\mathcal{F}$ on X (this is often useful in induction proofs).

The cohomology groups are defined as a partial response to this behavior of Γ; and in some good situations, these groups allow us to say something about the missing cokernel. This is done by continuing the sequence (17.1), giving rise to a *long exact sequence* of cohomology:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathcal{F}') \arrow[r] & \Gamma(X, \mathcal{F}) \arrow[r] & \Gamma(X, \mathcal{F}'') \arrow[dll, out=-30, in=150] \\
 & H^1(X, \mathcal{F}') \arrow[r] & H^1(X, \mathcal{F}) \arrow[r] & H^1(X, \mathcal{F}'') \arrow[dll, out=-30, in=150] \\
 & H^2(X, \mathcal{F}') \arrow[r] & H^2(X, \mathcal{F}) \arrow[r] & H^2(X, \mathcal{F}'') \arrow[r] & \cdots
\end{tikzcd}$$

In other words, the failure of surjectivity of the above is controlled by the group $H^1(X, \mathcal{F}')$ and the other groups in the sequence.

Cohomology groups can be defined in a completely general setting, for any topological space X and a (pre)sheaf $\mathcal{F}$ on it. With this as input we will define the *cohomology groups* $H^k(X, \mathcal{F})$, which will capture the main geometric invariants of $\mathcal{F}$. These should also be functorial in $\mathcal{F}$, meaning that we want to construct additive functors

$$\begin{aligned} H^q(X, -) \colon \mathsf{AbSh}_X &\to \mathsf{Ab} \\ \mathcal{F} &\mapsto H^q(X, \mathcal{F}) \end{aligned}$$

satisfying the following properties:

i) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$;

ii) A morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ induces for all i group homomorphisms $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$ which are functorial; in other words, each $H^i(X, -)$ is a functor;

iii) For each short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, there are maps $\delta\colon H^i(X, \mathcal{F}'') \to H^{i+1}(X, F')$ giving a *long exact sequence* as above.

The maps δ are called *connecting homomorphisms*. They depend on both i and the given short exact sequence.

There are several ways to define these groups. The modern approach, and the one summarized in Hartshorne Chapter III ([**?**]), takes the approach of using derived functors. This is in most respects the 'right way' to define the groups in general, but going through the whole machinery of derived functors and homological algebra would take us too far astray. We therefore begin with taking a more down-to-earth approach using *Čech cohomology* and later on will reintroduce the cohomology via the *Godement resolution*. There are good reasons for the double introduction. The Godement resolution has the advantage that it is completely canonical, and we can prove the main theorems we need by hand. On the other hand, the Čech resolution, which depends on the choice of a covering of X, is more intuitive and better suited for computations. Of course, the two notions of cohomology turn out to be the same, as we shall prove in Appendix A.

17.1 *Some homological algebra*

In this section we recall the most rudimentary elements from homological algebra. For a slightly more extensive presentation see Section **??** in CA (or any of the uncountably many texts on homological algebra).

Complexes of abelian groups

Recall that a *complex of abelian groups* $A^\bullet$ is a sequence of groups A^i together with maps between them

$$\cdots \xrightarrow{d^{i-2}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots$$

such that $d^{i+1} \circ d^i = 0$ for each i. A *morphism of complexes* $A^\bullet \xrightarrow{f} B^\bullet$ is a collection of maps $f_p : A^p \to B^p$ making the following diagram commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f_{i-1}"] & A^i \arrow[r, "d^i"] \arrow[d, "f_i"] & A^{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

In this way, we can talk about kernels, images, cokernels, exact sequences of complexes, etc.

We say that an element $\sigma \in A^p$ is a *cocycle* if it lies in the kernel of the map d^p i.e., $d^p\sigma = 0$. A *coboundary* is an element in the image of d^{p-1}, i.e., $\sigma = d^{p-1}\tau$ for some $\tau \in A^{p-1}$. These form subgroups of A^p, denoted by $Z^pA^\bullet$ and $B^pA^\bullet$ respectively. Since $d^p(d^{p-1}a) = 0$ for all a, all coboundaries are cocycles, so that $Z^pA^\bullet \supseteq B^pA^\bullet$. The *cohomology groups* of the complex $A^\bullet$ are set up to measure the difference between these two notions. We define the *p-th cohomology group* as the quotient group

$$H^pA^\bullet = Z^p(A^\bullet)/B^p(A^\bullet) = \operatorname{Ker} d^p/\operatorname{Im} d^{p-1}.$$

One thinks of $H^pA^\bullet$ as a group that measures the failure of the complex $A^\bullet$ of being exact at stage p: $A^\bullet$ is exact if and only if $H^pA^\bullet = 0$ for every p.

The following fact is very important:

Proposition 17.1 *Suppose that* $0 \to F^\bullet \xrightarrow{f} G^\bullet \xrightarrow{g} H^\bullet \to 0$ *is an exact sequence of complexes. Then there is a* long exact sequence *of cohomology groups*

$$\begin{tikzcd}
\cdots \arrow[r] & H^pF^\bullet \arrow[r] & H^pG^\bullet \arrow[r] & H^pH^\bullet \arrow[dll, out=-10, in=170] & \\
& H^{p+1}F^\bullet \arrow[r] & H^{p+1}G^\bullet \arrow[r] & H^{p+1}H^\bullet \arrow[r] & \cdots
\end{tikzcd}$$

Proof: For each $p \in \mathbb{Z}$, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^p \arrow[r, "f_p"] \arrow[d, "d^p"'] & G^p \arrow[r, "g_p"] \arrow[d, "d^p"] & H^p \arrow[r] \arrow[d, "d^p"] & 0 \\
0 \arrow[r] & F^{p+1} \arrow[r, "f_{p+1}"] & G^{p+1} \arrow[r, "g_{p+1}"] & H^{p+1} \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact by assumption. By the Snake lemma, we obtain a sequence

$$\begin{tikzcd}
0 \arrow[r] & Z^p(F^\bullet) \arrow[r, "f_p"] & Z^p(G^\bullet) \arrow[r, "g_p"] & Z^p(H^\bullet) \arrow[dll, out=-10, in=170, "\delta"] & \\
& F^{p+1}/B^{p+1}(F^\bullet) \arrow[r, "f_{p+1}"] & G^{p+1}/B^{p+1}(G^\bullet) \arrow[r, "g_{p+1}"] & H^{p+1}/B^{p+1}(H^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Consider now the diagram

$$\begin{tikzcd}
& F^p/B^p(F^\bullet) \arrow[r, "f_p"] \arrow[d, "d^p"'] & G^p/B^p(G^\bullet) \arrow[r, "g_p"] \arrow[d, "d^p"] & H^p/B^p(H^\bullet) \arrow[r] \arrow[d, "d^p"] & 0 \\
0 \arrow[r] & Z^{p+1}(F^\bullet) \arrow[r, "f_{p+1}"] & Z^{p+1}(G^\bullet) \arrow[r, "g_{p+1}"] & Z^{p+1}(H^\bullet) &
\end{tikzcd}$$

where the rows are exact by the above. For the maps in this diagram, $H^pF^\bullet = \operatorname{Ker} d^p$ and $H^{p+1}F^\bullet = \operatorname{Coker} d^p$. Hence applying the Snake lemma one more time, we get the desired exact sequence. ❑

Complexes of sheaves

The definitions and arguments of the previous subsection apply much more generally (to any abelian category). In particular, we make the following sheaf analogue. A *complex of sheaves* $\mathcal{F}^\bullet$ is a sequence of sheaves with maps between them

$$\cdots \xrightarrow{d^{i-2}} \mathcal{F}_{i-1} \xrightarrow{d^{i-1}} \mathcal{F}_i \xrightarrow{d^i} \mathcal{F}_{i+1} \xrightarrow{d^{i+1}} \cdots$$

such that $d^{i+1} \circ d^i = 0$ for each i. Given such a complex, we define the *cohomology sheaves* $H^p\mathcal{F}^\bullet$ as $\operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$. As above, a short exact sequence of complexes of sheaves gives rise to a long exact sequence of cohomology sheaves.

17.2 *Čech cohomology of a covering*

Let X be a topological space, and let $\mathcal{F}$ be a presheaf on it. Let $\mathcal{U} = \{U_i\}$ be an open cover of X, indexed by a linearly ordered set I. We shall assume that I is finite (and hence we may identify I with a finite segment in $\mathbb{N}_0$). As we saw previously, if $\mathcal{F}$ is a sheaf, the sequence (3.1)

$$0 \to \mathcal{F}(X) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact (whether I is finite or not). The Čech complex is essentially the continuation of this sequence; it is a complex obtained by adjoining all the groups $\mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_r})$ over all intersections $U_{i_1} \cap \cdots \cap U_{i_r}$ where the indices will be strictly increasing sequences $i_0 < \cdots < i_p$ of elements from I (in contrast to the convention in (3.1) where i, j is just a pair).

Definition 17.2 *For a presheaf $\mathcal{F}$ on X we define the* Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ *of $\mathcal{F}$ (with respect to the open covering $\mathcal{U}$) as*

Čech complex

$$C^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{U},\mathcal{F}) \xrightarrow{d^2} \cdots$$

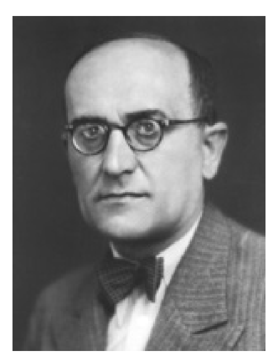
where

$$C^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<i_1<\cdots<i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}),$$

and the coboundary maps $d^p \colon C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$ *by*

$$(d^p\sigma)_{i_0,\dots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\dots\hat{i}_j,\dots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_p}}$$

where $i_0, \dots \hat{i}_j, \dots, i_{p+1}$ means $i_0, \dots, i_{p+1}$ with the index i_j omitted.



Eduard Čech
(1893–1960)

For small p we have

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{i_0} \mathcal{F}(U_{i_0}) \quad \text{and} \quad C^1(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1} \mathcal{F}(U_{i_0} \cap U_{i_1}).$$

Also note that since the covering is assumed to be finite, say having r elements, $C^p(\mathcal{U}, \mathcal{F}) = 0$ if $p \geqslant r$, simply because empty products are zero. So the Čech complex is a finite complex.

EXAMPLE 17.3 Let us look at the first few maps in the complex:

i) The coboundary map in degree zero $d^0\colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ is given as follows: if $\sigma = (\sigma_i)$, then

$$(d^0\sigma)_{ij} = \sigma_j - \sigma_i$$

where $i < j$;

ii) The one in degree one $d^1\colon C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$ satisfies: if $\sigma = (\sigma_{ij})$, then

$$(d^1\sigma)_{ijk} = \sigma_{jk} - \sigma_{ik} + \sigma_{ij}$$

where $i < j < k$.

☆

By direct substitution we see that $d^1 \circ d^0 = 0$ (all the σ_{ij} cancel). This happens also in higher degrees as a basic computation using the definition of d^i shows; that is one has:

LEMMA 17.4 $d^{p+1} \circ d^p = 0$.

In particular, the $C^\bullet(\mathcal{U}, \mathcal{F})$ forms a *complex of abelian groups*. As before, we say that an element $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ is a *cocycle* if $d^p\sigma = 0$, and a *coboundary* if $\sigma = d^{p-1}\tau$, and denote the sets of these by $Z^p(\mathcal{U}, \mathcal{F})$ and $B^p(\mathcal{U}, \mathcal{F})$ respectively. The *Čech cohomology groups* of $\mathcal{F}$ with respect to $\mathcal{U}$ are set up to measure the difference between these two notions:

DEFINITION 17.5 *The p-th* Čech cohomology *of $\mathcal{F}$ with respect to $\mathcal{U}$ is defined as* *Čech cohomology*

$$H^p(\mathcal{U}, \mathcal{F}) = Z^p(\mathcal{U}, \mathcal{F}) / B^p(\mathcal{U}, \mathcal{F}) = (\operatorname{Ker} d^p)/(\operatorname{Im} d^{p-1}).$$

It is not hard to check that a sheaf homomorphism $\mathcal{F} \to \mathcal{G}$ induces a mapping of Čech cohomology groups, so we obtain functors $\mathcal{F} \to H^p(\mathcal{U}, \mathcal{F})$ from sheaves to abelian groups. In fact, it is clear that it induces maps $C^i(\mathcal{U}, \mathcal{F}) \to C^i(\mathcal{U}, \mathcal{G})$ (it does so factor-wise), and an easy computation shows that the induced maps commutes with the coboundary maps; hence pass to the cohomology.

Examples

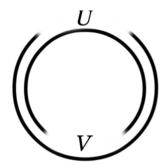
(17.6) Again it is instructive to consider the group $H^0(\mathcal{U}, \mathcal{F})$. It is governed by the map $d^0\colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$, which is simply the usual map

$$\prod_i F(U_i) \to \prod_{i<j} \mathcal{F}(U_i \cap U_j),$$

whose kernel is $\mathcal{F}(X)$ by the sheaf axioms. It follows that $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

(17.7) The most interesting cohomology group is arguably $H^1(\mathcal{U}, F)$. It is the group of cochains σ_{ij} such that $\sigma_{ik} = \sigma_{ij} + \sigma_{jk}$ modulo the cochains of the form $\sigma_{ij} = \tau_j - \tau_i$.

(17.8) (*The unit circle.*) Consider the unit circle $X = S^1$ (with the Euclidean topology), and equip it with a standard covering $\mathcal{U} = \{U, V\}$ consisting of two intervals (intersecting in two intervals S and T) and let $\mathcal{F} = \mathbb{Z}_X$ be the constant sheaf.



Here we have

$$C^0(\mathcal{U}, \mathcal{F}) = \mathbb{Z}_X(U) \times \mathbb{Z}_X(V) \simeq \mathbb{Z} \times \mathbb{Z} \qquad C^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}_X(U \cap V) \simeq \mathbb{Z} \times \mathbb{Z}.$$

The map $d^0 \colon C^0(\mathcal{U}, \mathbb{Z}_X) \to C^1(\mathcal{U}, \mathbb{Z}_X)$ is the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ given by $d^0(a, b) = (b - a, b - a)$. Hence

$$H^0(\mathcal{U}, \mathbb{Z}_X) = \operatorname{Ker} d^0 = \mathbb{Z}(1, 1) \simeq \mathbb{Z},$$

and

$$H^1(\mathcal{U}, \mathbb{Z}_X) = \operatorname{Coker} d^0 = (\mathbb{Z} \times \mathbb{Z})/\mathbb{Z}(1, 1) \simeq \mathbb{Z}.$$

Students familiar with algebraic topology, may recognize that this gives the same answer as singular cohomology.

(17.9) (*The projective line.*) Consider the projective line $\mathbb{P}^1 = \mathbb{P}^1_k$ over a field k. It is covered by the two standard affines $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[t^{-1}]$ with intersection $U_0 \cap U_1 = \operatorname{Spec} k[t, t^{-1}]$. For the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$, the Čech-complex takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(U_0) \times \mathcal{O}_{\mathbb{P}^1}(U_1) \arrow[r, "d^0"] & \mathcal{O}_{\mathbb{P}^1}(U_0 \cap U_1) \arrow[r] & 0 \\
& k[t] \times k[t^{-1}] \arrow[u, "\simeq"'] \arrow[r, "d"'] & k[t, t^{-1}], \arrow[u, "\simeq"] &
\end{tikzcd}$$

where d sends a pair $(p(t), q(t^{-1}))$ to $q(t^{-1}) - p(t)$. As we saw in Chapter 6 we have $\operatorname{Ker} d = k$, and, on the other hand, it is clear that any element of $k[t, t^{-1}]$ can be written as a sum of a polynomial in t and one in t^{-1}. Hence d is surjective, and we have

$$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Coker} d = 0.$$

(17.10) (*The sheaves $\mathcal{O}_{\mathbb{P}^1}(m)$.*) Continuing the above example, let us compute the Čech-cohomology for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(m)$. We use the same affine covering, so the Čech-complex still appears as

$$0 \longrightarrow k[t] \times k[t^{-1}] \xrightarrow{d} k[t, t^{-1}] \longrightarrow 0,$$

but the coboundary map d^0 is different; there is a multiplication by t^m in one of the restrictions, so the coboundary map is now given by

$$d(p(t), q(t^{-1})) = t^m q(t^{-1}) - p(t).$$

(see Section **??**). As we computed in Proposition 6.2, it holds true that $\operatorname{Ker} d \simeq k^{m+1}$ if $m \geqslant 0$, and $\operatorname{Ker} d = 0$ otherwise. The computation of $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1})$ is slightly more subtle.

Consider first the case when $m \geqslant 0$. As before, it is easy to see that any polynomial in $k[t, t^{-1}]$ can be written in the form $t^m q(t^{-1}) - p(t)$. In fact, this also works for $m = -1$; indeed, one has $t^{-k} = t^{-1} \cdot t^{-k+1} - 0$ and $t^k = t^{-1} \cdot 0 - t^k$. Hence $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(m)) = 0$ for $n \geqslant -1$. For $m \leqslant -2$ however, no linear combination of the monomials

$$t^{-1}, t^{-2}, \ldots, t^{m+1}$$

lies in the image, but combinations of all the others do, and it follows that $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(m))$ is a k-vector space of dimension $-m+1$.

☆

Exercises

(17.1) Let $X = S^1$ and let $\mathcal{U}$ be the covering of X with three pairwise intersecting open intervals with empty intersection. Show that the Čech-complex is of the form

$$\mathbb{Z}^3 \xrightarrow{d^0} \mathbb{Z}^3 \to 0.$$

Compute the map d^0 and use it to verify again that $H^i(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}$ for $i = 0, 1$ as above.
(17.2) Let $X = S^2$ and let $\mathcal{U}$ be the covering of X with four pairwise intersecting open sets with empty quadruple intersection; Show that the Čech-complex takes the form

$$\mathbb{Z}^4 \xrightarrow{d^0} \mathbb{Z}^6 \xrightarrow{d^1} \mathbb{Z}^4 \to 0.$$

Compute the matrices d^0, d^1 and show that $H^i(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}$ for $i = 0, 2$ and $H^i(\mathcal{U}, \mathbb{Z}_X) = 0$ for $i \neq 0, 2$.
(17.3) Prove Lemma 17.4.

★

Example: Constant sheaves

In the examples above, we considered S^1 and S^2 with the standard topology and the constant sheaves $\mathbb{Z}_X$ on them, and, in fact, the cohomology of constant sheaves will give singular cohomology for most topological spaces. However, the following proposition shows that constant sheaves are not so interesting in algebraic geometry, as we would like to study spaces which are *irreducible* as topological spaces. Then any open set $U \subset X$ is connected and the constant sheaves are effectively constant (in general a constant sheaf A_X takes the value A merely on connected sets).

PROPOSITION 17.11 *Let X be an* irreducible *topological space. Then for any finite covering $\mathcal{U}$ of X we have for a constant sheaf A_X*

$$H^p(\mathcal{U}, A_X) = 0$$

for $p > 0$.

Proof: In this case the Čech complex takes the form

$$\prod_i A \to \prod_{i<j} A \to \prod_{i<j<k} A \to \cdots$$

Note that this complex of groups does not depend on X or the covering $\mathcal{U}$, only the index set I plays a role. In particular, the complex is the same as the Čech complex of A on a one-point space (which makes it plausible that all the higher cohomology should vanish). In this case it is easy to show by hand that any p-cocycle is the coboundary of some $(p-1)$-cochain for $p > 0$.

For instance, given a 1-cocycle $g = (g_{ij}) \in \prod_{i<j} A$, let $n \in I$ be the largest element and define the element $h = (h_i) \in C^0(\mathcal{U}, A) = \prod_{i \in I} A$ by the assignment

$$h_i = \begin{cases} g_{in} & \text{when } i < n, \\ 0 & \text{when } i = n. \end{cases}$$

The cocycle condition

$$0 = (d^1 g)_{ijn} = g_{ij} - g_{jn} + g_{in},$$

where $i < j < n$, translates into $0 = g_{ij} - h_j + h_i$ or in other words $g_{ij} = h_j - h_i$, and by definition it holds that $g_{jn} = h_j = h_j - h_n$. This proves that the cocycle $g = (g_{i,j})$ is the coboundary of the element $h = (h_i)$, and thus that the class of that cocycle is zero in $H^1(\mathcal{U}, A_X)$.

The same trick works for higher $p > 0$ as well. Let again $n \in I$ be the largest element, and suppose that we are given a cocycle $g = (g_{i_0,\ldots,i_p}) \in \prod_{i_0<\cdots<i_p} A$. Setting $i_{p+1} = n$ in the differential, we see that

$$0 = (d^p g)_{i_0 \cdots i_p n} = \sum_{j=0}^{p} (-1)^j g_{i_0 \cdots \hat{i}_j \cdots i_p n} + (-1)^{p+1} g_{i_0 \cdots i_p}.$$

Now, mimicking what we did for $p = 2$, we define $h \in \prod_{i_0<\cdots<i_{p-1}} A$ by the assignment

$$h_{i_0,\ldots,i_{p-1}} = \begin{cases} (-1)^p g_{i_0,\ldots,i_{p-1} n} & \text{when } i_{p-1} < n, \\ 0 & \text{when } i_{p-1} = n. \end{cases}$$

Solving the previous equation for $g_{i_0 \cdots i_p}$ we see that when $i_p < n$

$$\begin{aligned} (d^{p-1} h)_{i_0 \cdots i_p} &= \sum_{j=0}^{p} (-1)^j h_{i_0 \ldots \hat{i}_j \ldots i_p} \\ &= (-1)^p \sum_{j=0}^{p} (-1)^j g_{i_0 \cdots \hat{i}_j \cdots i_{p-1} n} = g_{i_0 \cdots i_p}, \end{aligned}$$

and for $i_p = n$ one finds

$$(d^{p-1} h)_{i_0 \ldots i_{p-1} n} = \sum_{j=0}^{p-1} (-1)^j h_{i_0 \ldots \hat{i}_j \ldots i_{p-1} n} + (-1)^p h_{i_0 \ldots i_{p-1}} = g_{i_0 \ldots i_{p-1} n}.$$

It follows that $g = d^{p-1} h$, so that the class of g is zero in $H^p(\mathcal{U}, A_X)$. ❑

17.3 Čech cohomology of a sheaf

As seen in the examples above, the groups $H^p(\mathcal{U},\mathcal{F})$ are easily computable if one is given a nice open cover of X. Indeed, the maps in the Čech complex are completely explicit, and computing their kernels and images involves only basic row operations from linear algebra.

On the other hand, the definition of the cohomology groups is unsatisfactory for various reasons. First of all, the groups $H^p(\mathcal{U},\mathcal{F})$ depend on the open cover $\mathcal{U}$, whereas we want something canonical that only depends on $\mathcal{F}$. More importantly, it is not clear that the definition above really captures the desired information about $\mathcal{F}$. For instance, $\mathcal{U}$ could consist of the single open set X, and so $H^i(\mathcal{U},\mathcal{F}) = 0$ for all $i \geqslant 1$! Finally, it is not at all clear if these groups satisfy the functorial requirements mentioned in the introduction.

There is a fix for all of these problems which involves passing to finer and finer 'refinements' of the covering. However, this introduces a new complication, making explicit computation almost impossible: So the Čech cohomology is most useful when this limit process is not needed; that is if one particular open covering yields the limit.

Refinement of a cover

We say that an open covering $\mathcal{V} = \{V_j\}_{j\in J}$ is a *refinement* of $\mathcal{U} = \{U_i\}_{i\in I}$ if for every $V_j \in \mathcal{V}$, there is a $i \in I$ so that $V_j \subseteq U_i$. This defines a preordering on the set of coverings which we denote by $\mathcal{V} \leqslant \mathcal{U}$, and the order is directed since if $\{U_i\}$ and $\{V_j\}$ are two open covers, the cover $\{U_i \cap V_j\}$ is finer than both. If we chose a map $\sigma\colon J \to I$ so that $V_j \subset U_{\sigma(j)}$ for every j (there are several such, but we single out one), we can define a *refinement homomorphism*

$$\mathrm{ref}_{\mathcal{U}\mathcal{V}} : C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{V},\mathcal{F})$$

by setting

$$(\mathrm{ref}_{\mathcal{U}\mathcal{V}}(\sigma))_{j_0\dots j_p} = \left(\sigma_{\epsilon(j_0)\dots\epsilon(j_p)}\right)|_{V_{j_0}\cap\cdots\cap V_{j_p}}.$$

A straightforward computation yields that $d \circ \mathrm{ref}_{\mathcal{U}\mathcal{V}} = \mathrm{ref}_{\mathcal{U}\mathcal{V}} \circ d$, so that ref induces a map on cohomology groups:

$$\mathrm{ref}_{\mathcal{U}\mathcal{V}} : H^p(\mathcal{U},\mathcal{F}) \to H^p(\mathcal{V},\mathcal{F}).$$

Moreover, while the refinement maps between the complexes depend on the choice of the function $\sigma\colon J \to I$, the map induced between the cohomology groups does not.

One may then define a group $H^p(X,\mathcal{F})$ to be the direct limit of all $H^p(\mathcal{U},\mathcal{F})$ as $\mathcal{U}$ runs through all possible open covers $\mathcal{U}$ ordered by refinement. The resulting groups are indeed canonical, and they turn out to give a good answer for cohomology.

Definition 17.12 *The groups $H^p(X,\mathcal{F})$ are called the* Čech cohomology groups *of $\mathcal{F}$. In symbols,*

$$H^p(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} H^p(\mathcal{U},\mathcal{F}).$$

The main properties of Čech cohomology are summarized in the following theorem:

THEOREM 17.13 *Let X be a topological space and let $\mathcal{F}$ be a sheaf on X.*

i) The Čech cohomology groups are functors $H^i(X,-) : \mathsf{AbSh}_X \to \mathsf{Ab}$;

ii) $H^0(X,\mathcal{F}) = \mathcal{F}(X)$;

iii) Short exact sequences of sheaves induce long exact sequences of cohomology;

iv) (Leray's theorem). If $\mathcal{F}$ is a sheaf and $\mathcal{U}$ is a covering such that $H^i(U_{i_1} \cap \cdots U_{i_p}, \mathcal{F}) = 0$ for all $i > 0$ and multi-indices $i_1 < \cdots < i_p$, then

$$H^i(X,\mathcal{F}) = H^i(\mathcal{U},\mathcal{F}).$$

We have already proved the first two of these properties. The next two require more work, but the arguments are mostly formal, and we postpone the proof until Appendix **??**. We will nevertheless shortly derive the long exact sequence for sequences involving quasi-coherent sheaves $\mathcal{F}$ on separated schemes, which will be our main concern.

The last statement (Leray's theorem) is very important. It says that even though $H^i(X,\mathcal{F})$ is defined as an infinite directed limit over coverings $\mathcal{U}$, it suffices to compute it at a covering which is 'sufficiently fine' in the sense that the higher groups $H^i(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{F}) = 0$ vanish for $i > 0$. In practice, the latter condition is sometimes rather easy to check: it holds for instance if $\mathcal{F}$ is quasi-coherent and all of the intersections are affine schemes (see Corollary 18.2).

The long exact sequence for quasi-coherent sheaves

Let X be a scheme and consider a short exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

In Proposition 14.10 we proved that whenever the $U = \operatorname{Spec} A$ is an open affine in X, the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0 \tag{17.2}$$

is exact. This means that if an affine cover $\mathcal{U} = \{U_i\}_{i \in I}$ has the property that each intersection

$$U_{i_0} \cap \cdots \cap U_{i_p}$$

is affine, as taking products do not disturb exactness, there is an exact sequence

$$0 \longrightarrow C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^p(\mathcal{U},\mathcal{G}) \longrightarrow C^p(\mathcal{U},\mathcal{H}) \longrightarrow 0,$$

and consequently the sequence of *Čech complexes*

$$0 \longrightarrow C^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow C^\bullet(\mathcal{U},\mathcal{G}) \longrightarrow C^\bullet(\mathcal{U},\mathcal{H}) \longrightarrow 0$$

is also exact. Thus we are in position to apply Lemma 17.1 to obtain a long exact sequence of Čech cohomology groups

$$\cdots \longrightarrow H^i(\mathcal{U},\mathcal{F}) \longrightarrow H^i(\mathcal{U},\mathcal{G}) \longrightarrow H^i(\mathcal{U},\mathcal{H}) \longrightarrow \cdots.$$

If X is a separated scheme, such coverings are cofinal in the directed system of coverings (*i.e.* every open covering has a refinement of the kind), so in fact we have a proof of *iii)* — that is, of the existence of the long exact sequence (17.1) for quasi-coherent sheaves on separated schemes.

In general, it may certainly happen that the restriction map (17.2) is *not* surjective — one can for instance take the open covering of X with just one open set X. This explains why the Čech cohomology groups $H^i(\mathcal{U}, \mathcal{F})$ do not give long exact sequences in general. However, by passing to smaller refinements $\mathcal{V} \leqslant \mathcal{U}$, we may arrange that any section lifts, and we can use the above approach to construct the connecting homomorphisms δ.

Chapter 18

Computations with cohomology

In this chapter, we begin with more explicit computations with sheaf cohomology. We will prove two main theorems. The first result is that all higher cohomology groups of quasi-coherent sheaves on affine schemes vanish. This in turn has important foundational consequences for sheaf cohomology. For instance, together with Leray's theorem, it implies that cohomology (which is defined by a direct limit over the set of all open coverings) can be computed as Čech-cohomology with respect to any affine covering (assuming that the scheme is separated).

The second result is a complete computation of the cohomology groups of the sheaves $\mathcal{O}(d)$ on projective space $\mathbb{P}^n_A$. Using Hilbert's syzygy theorem, this in turn will allow us, at least in principle, to compute sheaf cohomology of any coherent sheaf on $\mathbb{P}^n_A$.

Towards the end of the chapter we will study many explicit examples.

18.1 Cohomology of sheaves on affine schemes

The following result is fundamental in the study of sheaf cohomology groups. It is the first example of a 'vanishing theorem' for cohomology. Recalling that cohomology groups were defined to measure the 'failure' of certain desirable statements (*e.g.* restriction maps being surjective), we are in general happy if cohomology groups are zero.

Theorem 18.1 *Let $X = \operatorname{Spec} A$ and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for the Čech cohomology one has*

$$H^p(X, \mathcal{F}) = 0 \text{ for all } p > 0.$$

Proof: Recall that we defined the groups $H^i(X, \mathcal{F})$ by taking the direct limit of $H^i(\mathcal{U}, \mathcal{F})$ over finer and finer coverings $\mathcal{U}$ of X. Since the distinguished open subsets form a basis for the topology on X, it suffices to prove that

$$H^p(\mathcal{U}, \mathcal{F}) = 0 \text{ for all } p > 0$$

for a covering $\mathcal{U} = \{D(g_i)\}$ where the g_i are finitely many elements of A generating the unit ideal. (We may consider only finitely many g_i since X is quasi-compact.)

As $\mathcal{F}$ is quasi-coherent, we may write $\mathcal{F} = \widetilde{M}$ for some A-module M, and $M = \Gamma(X, \mathcal{F})$. Note that $\mathcal{F}(D_{g_i}) = M_{g_i}$, that $\mathcal{F}(D_{g_i} \cap D_{g_j}) = M_{g_i g_j}$ and so on, so the sheaves appearing in the Čech complex are products of localizations of the module M. Explicitly, the fact that the higher Čech-cohomology groups vanish is equivalent to the statement that the following sequence is exact:

$$0 \to M \to \prod_{i \in I} M_{g_i} \to \prod_{i<j} M_{g_i g_j} \to \prod_{i<j<k} M_{g_i g_j g_k} \to \dots$$

Here the boundary maps are given as alternating sums of localization maps. For example,

$$d^1 \colon \prod_{i<j} M_{g_i g_j} \to \prod_{i<j,<k} M_{g_i g_j g_k}$$

maps the cochain $(\sigma_{ij})_{ij} \in M_{g_i g_j}$ to the cochain $(\sigma_{jk} - \sigma_{ik} + \sigma_{ij})_{i,j,k}$, viewed as an element of $\prod M_{g_i g_j g_k}$.

Notice that the beginning of the exact sequence

$$0 \to M \to \prod_{i} M_{g_i} \to \prod_{i<j} M_{g_i g_j}$$

appeared already in Proposition *iii)* on page 59 when we computed sections of the quasi-coherent module $\widetilde{M}$. The proof for the exactness of this sequence is similar to the general case.

To prove that the cohomology groups vanish, we must to each cocycle σ (that is, a cochain σ such that $d\sigma = 0$) find a cochain τ which makes $\sigma = d\tau$ a coboundary. The proof is a direct calculation; one constructs the element τ by hand.

To see how this can be done, let us for simplicity consider the case $p = 1$ first. Let $\sigma \in \prod_{i,j} M_{g_i g_j}$ be in the kernel of d. We may write

$$\sigma_{ij} = \frac{m_{ij}}{(g_i g_j)^r} \text{ where } m_{ij} \in M$$

for some r (since the index set I is finite, we may choose r independent of i and j). The relation $d\sigma = 0$ gives the relation

$$\frac{m_{jk}}{(g_j g_k)^r} - \frac{m_{ik}}{(g_i g_k)^r} + \frac{m_{ij}}{(g_i g_j)^r} = 0$$

in $M_{g_i g_j g_k}$. This implies that we have the following relation in $M_{g_j g_k}$

$$\frac{g_i^{r+l} m_{jk}}{(g_j g_k)^r} = \frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r} \tag{18.1}$$

for some $l \geqslant 0$. Now, as the open sets $D(g_i) = D(g_i^{r+l})$ cover X, we have a relation

$$1 = \sum_{i \in I} h_i g_i^{r+l}$$

where $h_i \in A$. Let us define the cohain $\tau = (\tau_j) \in \prod M_{g_j}$ by

$$\tau_j = \sum_{i \in I} h_i \frac{g_i^l m_{ij}}{g_j^r}.$$

In $\prod M_{g_j g_k}$ we may write this as

$$\tau_j = \sum_{i \in I} h_i \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r}.$$

We want to show that $d\tau = \sigma$. This is a basic computation using the relation (18.1) above. We find

$$\begin{aligned}
(d\tau)_{jk} &= \tau_k - \tau_j \\
&= \sum_{i \in I} h_i \frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \sum_{i \in I} h_i \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r} \\
&= \sum_{i \in I} h_i \left(\frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r} \right) \\
&= \sum_{i \in I} h_i \frac{g_i^{r+l} m_{jk}}{(g_j g_k)^r} = \frac{m_{jk}}{(g_j g_k)^r} \sum_{i \in I} h_i g_i^{r+l} \\
&= \frac{m_{jk}}{(g_j g_k)^r} = \sigma_{jk}
\end{aligned}$$

as desired. Hence $H^1(\mathcal{U}, \mathcal{F}) = 0$.

The proof in the general case is quite similar, but there are more indices to keep track of. For every cohain $\sigma \in \prod_{i_0,\ldots,i_p} M_{g_{i_0} \cdots g_{i_p}}$ we may then write

$$\sigma_{i_0,\ldots,i_p} = \frac{m_{i_0,\ldots,i_p}}{(g_{i_0} \cdots g_{i_p})^r} \text{ where } m_{i_0,\ldots,i_p} \in M$$

for some r (again, since the index set I is finite, we may choose r independent of i and j), and if σ is a cocycle, the relation $d\sigma = 0$ gives the following relation in $M_{g_{i_0} \cdots g_{i_p}}$:

$$\frac{g_i^{r+l} m_{i_0,\ldots,i_p}}{(g_{i_0} \cdots g_{i_p})^r} = \sum_{k=0}^{p} (-1)^k \frac{g_i^l g_{i_k}^r m_{ii_0,\ldots,\hat{i_k}\ldots i_p}}{(g_{i_0} \cdots g_{i_p})^r}. \tag{18.2}$$

As the sets $D(g_i) = D(g_i^{r+l})$ cover X, we have a relation

$$1 = \sum_{i \in I} h_i g_i^{r+l}$$

where each $h_i \in A$. Now, define the cochain $\tau \in \prod M_{g_{i_0} \cdots g_{i_{p-1}}}$ by

$$\tau_{i_0,\ldots,i_{p-1}} = \sum_{i \in I} h_i g_i^l \frac{m_{ii_0,\ldots i_p}}{(g_{i_0} \cdots g_{i_{p-1}})^r}.$$

Localizing to $\prod M_{g_{i_0},\ldots,g_{i_p}}$, we may write this as

$$\tau_{i_0,\ldots,i_p} = \sum_{i \in I} h_i g_i^l g_{i_k}^r \frac{m_{ii_0,\ldots i_p}}{(g_{i_0} \cdots g_{i_p})^r}.$$

We want to show that $d\tau = \sigma$. As before, we can check this using the relation (18.2) above:

$$\begin{aligned}(d\tau)_{i_0\cdots i_p} &= \sum_{k=0}^{p}(-1)^k\tau_{i_0\ldots\hat{i}_k\ldots i_p}\\ &= \sum_{i\in I} h_i g_i^{r+l}\sigma_{i_0\ldots i_p} = \sigma_{i_0\ldots i_p}.\end{aligned}$$

This completes the proof. ❑

Čech cohomology and affine coverings

As a corollary of the previous theorem, we see that affine coverings of schemes satisfy the conditions of Leray's theorem (see Theorem 17.13). This implies

Corollary 18.2 *Let X be a Noetherian scheme and let $\mathcal{U} = \{U_i\}$ be an affine covering such that all intersections $U_{i_0} \cap \cdots \cap U_{i_s}$ are affine. Then*

$$H^i(X, \mathcal{F}) = H^i(\mathcal{U}, \mathcal{F}).$$

In particular, the theorem applies to *any* open affine covering on a Noetherian separated scheme.

Example 18.3 (*The affine line with two origins.*) Consider the 'affine line with two origins' X from Example 6.4 on page 91. It is covered by two affine subsets $X_1 = \operatorname{Spec} k[u]$ and $X_2 = \operatorname{Spec} k[u]$ and these are glued together along their common open set $X_{12} = D(u) = \operatorname{Spec} k[u, u^{-1}]$ with the identity as gluing map. The Čech complex for this covering looks like

$$0 \longrightarrow k[u] \times k[u] \xrightarrow{d^1} k[u, u^{-1}] \xrightarrow{d^2} 0$$

where $d^1(p, q) = q - p$, and is nothing but the standard sequence that appeared in the example, and as we checked in there, it holds that $\mathcal{O}_X(X) = \operatorname{Ker} d^1 = k[u]$.

More strikingly, the cokernel $H^1(X, \mathcal{O}_X) = \operatorname{Coker} d^1$ of the map $k[u] \oplus k[u] \to k[u, u^{-1}]$ is rather big. It equals $k[u, u^{-1}]/k[u] = \bigoplus_{i>0} ku^{-i}$, so that $H^1(X, \mathcal{O}_X)$ is not finitely generated as a vector space over k. This gives another proof that X is not isomorphic to an affine scheme. ☆

18.2 Cohomology and dimension

The next result is another 'vanishing theorem'. It is a general result, due to Grothendieck, that the cohomology groups vanish above the dimension of X, at least for spaces X that are Noetherian and the dimension is interpreted as the Krull-dimension.

THEOREM 18.4 *Let X be a Noetherian topological space of dimension n, and let $\mathcal{F}$ be a sheaf on X. Then*
$$H^p(X, \mathcal{F}) = 0$$
for all $p > n$.

A proof valid in the general case may be found in Godement (Theorem 4.5.12), but we contend ourself with proving it in the special case when X is a quasi-projective variety. We begin with an easy lemma:

LEMMA 18.5 *Let X be a topological space and let $Z \subset X$ be a closed subset. Then for any sheaf $\mathcal{F}$ on Z, it holds true that $H^p(Z, \mathcal{F}) = H^p(X, i_* \mathcal{F})$.*

PROOF: Observe that each open cover $\{U_i\}$ of X induces an open cover $\{U_i \cap Z\}$ of Z, and all open covers of Z arise like this. The lemma then follows from the basic fact that for each open subset $U \subset X$ it holds that $\Gamma(U, i_*\mathcal{F}) = \Gamma(Z \cap U, \mathcal{F})$, so the two cohomolgy groups arise from the same Čech complexes. ❑

THEOREM 18.6 *Let X be a quasi-projective scheme of dimension n. Then X admits an open cover $\mathcal{U}$ consisting of at most $n+1$ affine open subsets. In particular, it holds true that*
$$H^p(X, \mathcal{F}) = 0 \text{ for } p > n$$
for any quasi-coherent sheaf $\mathcal{F}$ on X.

PROOF: Let X be a quasi-projective scheme, *i.e.* X appears as $X = Y \backslash W$ where $Y, W \subseteq \mathbb{P}^r_A$ are closed subschemes, and we may assume that no irreducible component of Y is contained in W, simply by discarding such components. Using induction on $\dim X$ we will prove that X is covered by $n+1$ open affines induced from open affines in $\mathbb{P}^r_A$.

Consider the irreducible decomposition $Y = \bigcup_i Y_i$ and observe that by prime avoidance $I_W \nsubseteq \bigcup I_{Y_i}$ where $I_T \subseteq A[x_0, \ldots, x_N]$ denotes the radical homogeneous ideal of a set $T \subseteq \mathbb{P}^N$. Pick a homogenous polynomial f such that $f \in I_W \backslash (\bigcup_i I_{Y_i})$, and let $H = V(f)$. Then we infer that the set $\mathbb{P}^r_A \backslash H = D_+(f)$ is affine and hence so is $Y \backslash H$, being a closed subscheme of an affine scheme.

By construction $Y \backslash H \subseteq Y \backslash W = X$ and $H \nsupseteq Y_i$ for any i by the choice of f. Therefore $\dim(Y_i \cap H) < \dim Y_i$ so we may use induction on the dimension to cover $Y \cap H$ by fewer that n open affines, all induced from the ambient projective space, which together with $D(f)$ gives a covering of X with $n+1$ open affine subsets. This shows the first claim.

For the second, note that in a Čech complex built on a covering consisting of at most $n+1$ affines open subsets, terms $C^p(X, \mathcal{F})$ with $p > n$ will vanish, from which follows that $0 = H^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$ for each $\mathcal{F}$ and each $p > n$. ❑

18.3 Cohomology of sheaves on projective space

In Examples 17.9 and 17.10 we computed the sheaf cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(m)$ on $X = \mathbb{P}^1_k$. For $d \geqslant 0$, we found that $H^0(X, \mathcal{O}_X(m))$ could be identified with the space of homogeneous polynomials of degree d, and $H^1(X, \mathcal{O}_X(m)) = 0$. On the other hand, for $d \leqslant -2$, $H^0(X, \mathcal{O}_X(m)) = 0$, while $H^1(X, \mathcal{O}_X(m))$ was non-zero.

We will now carry out a more general computation for the cohomology groups for $\mathcal{O}_{\mathbb{P}^n_A}(m)$ for any projective space $\mathbb{P}^n_A$ over a ring A. The strategy is however the same, we have a distinguished covering via the open sets $D_+(x_i)$ and we use Čech complex associated to this covering to compute the cohomology.

Theorem 18.7 *Let $X = \mathbb{P}^n_A = \operatorname{Proj} R$ where $R = A[x_0, \ldots, x_n]$ where A is a ring.*

i)

$$H^0(X, \mathcal{O}_X(m)) = \begin{cases} R_m & \text{for } m \geqslant 0; \\ 0 & \text{otherwise;} \end{cases}$$

ii)

$$H^n(X, \mathcal{O}_X(m)) = \begin{cases} A & \text{for } m = -n-1; \\ 0 & \text{for } m > -n-1; \end{cases}$$

iii) When $m \geqslant 0$, there is a perfect pairing[a] of A-modules

$$H^0(X, \mathcal{O}_X(m)) \times H^n(X, \mathcal{O}_X(-m-n-1)) \to H^n(X, \mathcal{O}_X(-n-1)) \simeq A;$$

iv) For $0 < i < n$ and all $m \in \mathbb{Z}$, we have

$$H^i(X, \mathcal{O}_X(m)) = 0.$$

[a] Recall that a bilinear map $M \times N \to A$ is a *perfect pairing* if the induced map $M \mapsto \operatorname{Hom}_A(N, A)$ is an isomorphism.

Proof: The simplifying trick is to consider the $\mathcal{O}_X$-module $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}(m)$. We would like to show for instance that $H^i(X, \mathcal{F}) = 0$ for $i \neq 0, n$, and since taking Čech cohomology commutes with forming direct sums, this is equivalent to $H^i(X, \mathcal{O}_X(m)) = 0$ for all m, but $\mathcal{F}$ has the advantage that it is a graded $\mathcal{O}_X$-*algebra*.

Consider the Čech-complex associated with the standard covering $\mathcal{U} = \{U_i\}$ where $U_i = D_+(x_i) = \operatorname{Spec} R_{(x_i)}$. The salient observation is that

$$\Gamma(U_i, \mathcal{F}) = \bigoplus_{m \in \mathbb{Z}} \Gamma(U_i, \mathcal{O}_{U_i}(m)) = \bigoplus_{m \in \mathbb{Z}} ((R(m)_{x_i})_0) = R_{x_i}$$

and similar equalities hold for all the intersections among the U_i's. This implies that the Čech-complex for $\mathcal{F}$ has the following form:

$$\prod_i R_{x_i} \xrightarrow{d^0} \prod_{i,j} R_{x_i x_j} \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} R_{x_0 \cdots x_n}$$

where the maps as usual are composed of localization maps. We have a graded isomorphism of R-modules:

$$\begin{aligned} H^0(X, \mathcal{F}) &= \operatorname{Ker} d^0 \\ &= \{(r_i)_{i \in I} | r_i \in R_{x_i}, r_i = r_j \in R_{x_i x_j}\} \\ &\simeq R. \end{aligned}$$

This isomorphism preserves the grading, so we get *i)*.

To prove *ii)*, note that $R_{x_0 \cdots x_n}$ is a free graded A-module spanned by monomials of the form

$$x_0^{a_0} \cdots x_n^{a_n}$$

with multidegrees $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$. The image of d^{n-1} is spanned by such monomials where at least one a_i is non-negative. Hence

$$\begin{aligned} H^n(X, \mathcal{F}) &= \operatorname{Coker} d^{n-1} \\ &= A \left\{ x_0^{a_0} \cdots x_n^{a_n} \;\middle|\; a_i < 0 \text{ for all } i \right\} \subset R_{x_0 \ldots x_n}. \end{aligned}$$

This means that

$$\begin{aligned} H^n(X, \mathcal{O}_X(m)) &= H^n(X, \mathcal{F})_m \\ &= A \left\{ x_0^{a_0} \cdots x_n^{a_n} \;\middle|\; a_i < 0 \text{ for all } i \text{ and } \sum a_i = m \right\} \subset R_{x_0 \ldots x_n}. \end{aligned}$$

In degree $-n-1$ there is only one such monomial, namely $x_0^{-1} \cdots x_n^{-1}$.

For *iii)*: If we identify $H^n(X, \mathcal{O}_X(-m-n-1))$ with

$$A \left\{ x_0^{a_0} \cdots x_n^{a_n} \;\middle|\; a_i < 0 \text{ for all } i \text{ and } \sum a_i = m \right\}$$

and $H^0(X, \mathcal{O}(m))$ with R_m, we may define the pairing by means of multiplication of Laurent polynomials:

$$\begin{aligned} H^0(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}_X(-m-n-1)) &\to R_{x_0 \cdots x_n} \\ (x_0^{m_0} \cdots x_n^{m_n}) \times (x_0^{a_0} \cdots x_n^{a_n}) &\mapsto x_0^{a_0+m_0} \cdots x_n^{a_n+m_n}. \end{aligned}$$

Here the exponents satisfy $m_i \geqslant 0$, $a_i < 0$ $\sum a_i = -m-n-1$, $\sum m_i = m$. This gives a map

$$H^0(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}_X(-m-n-1)) \to H^n(X, \mathcal{O}_X(-n-1)) = A x_0^{-1} \cdots x_n^{-1}$$

sending $(x_0^{m_0} \cdots x_n^{m_n}) \times (x_0^{a_0} \cdots x_n^{a_n})$ to zero if $m_i + a_i \geqslant 0$ for some i. This pairing is perfect; indeed. the dual of a monomial $(x_0^{m_0} \cdots x_n^{m_n})$ is represented by $(x_0^{-m_0-1} \cdots x_n^{-m_n-1})$.

For *iv)*: This last point is more involved.

Note first that $H^i(X, \mathcal{F})_{x_n} = H^i(U_n, \mathcal{F}|_{U_n})$ for all i: since $\Gamma(U_n, \mathcal{F}) = \mathcal{F}(X)_{x_n}$, the Čech-complex of $\mathcal{F}|_{U_n}$ with respect to the covering $U_i \cap U_n$ is just the localization of $C^\bullet(X, \mathcal{F})$ at x_n. Localization is exact, so it preserves cohomology, which gives the claim.

The proof of *iv)* hinges on the multiplication by $\cdot x_n$ being an isomorphism of $H^i(X, \mathcal{F})$ for $0 < i < n$, as we are going to show. Claim *iv)* follows from this, because

$$H^i(X, \mathcal{F})_{x_n} = H^i(U_n, \mathcal{F}|_{U_n}) = 0$$

for all $i > 0$ as U_n is affine; indeed, this means that each element η of $H^i(X, \mathcal{F})$ is killed by a high power of x_n. Hence $\eta = 0$ as multiplication by x_n is an isomorphism.

To show that multiplication by x_n on the relevant cohomology groups are isomorphisms, we proceed by induction on n. For $n = 1$, there is nothing to prove. For $n > 1$, let $H = V(x_n) \simeq \mathbb{P}^{n-1}$ be the hyperplane determined by x_n. There is an exact sequence

$$0 \to R(-1) \xrightarrow{\cdot x_n} R \to R/(x_n) \to 0, \tag{18.3}$$

and when applying $\sim$ to it, we obtain the exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to i_*\mathcal{O}_H \to 0,$$

where $i\colon H \to X$ denotes the inclusion. Taking the direct sum of all the twists of this sequence yields the excat sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to i_*\mathcal{F}_H \to 0,$$

where $\mathcal{F}_H = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_H(m)$. By induction on n, it holds true that $H^i(X, i_*\mathcal{O}_H(m)) = H^i(H, \mathcal{O}_H(m)) = 0$ for $0 < i < n-1$ and all $m \in \mathbb{Z}$, so the long exact sequence of cohomology gives isomorphisms

$$H^i(X, \mathcal{F}(-1)) \xrightarrow{\cdot x_n} H^i(X, \mathcal{F})$$

for $1 < i < n-1$. We claim that we have isomorphisms also for $i = 1$ and $i = n-1$. For $i = 1$, this follows because the sequence

$$0 \to H^0(X, \mathcal{F}(-1)) \to H^0(X, \mathcal{F}) \to H^0(X, i_*\mathcal{F}_H) \to 0$$

is exact; indeed, this sequence is just the same sequence as (18.3).

When $i = n-1$, we need to show that

$$0 \to H^{n-1}(X, i_*\mathcal{F}_H) \xrightarrow{\delta} H^n(X, \mathcal{F}(-1)) \xrightarrow{\cdot x_n} H^n(X, \mathcal{F})$$

is exact. The kernel of $\cdot x_n$, is generated by monomials $x_0^{a_0} \cdots x_n^{a_n}$ with $a_i < 0$ for all i. So it suffices to show that the connecting map δ is just multiplication by x_n^{-1}. Define $R' = R/x_n$. Writing the arrows in the Čech complex vertically we get the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \prod_i R(-1)_{x_0\cdots\hat{x}_i\cdots x_n} \arrow[r, "\cdot x_n"] \arrow[d] & \prod_i R_{x_0\cdots\hat{x}_i\cdots x_n} \arrow[r] \arrow[d] & R'_{x_0\cdots x_{n-1}} \arrow[r] \arrow[d] & 0\,. \\
0 \arrow[r] & R_{x_0\cdots x_n}(-1) \arrow[r, "\cdot x_n"] & R_{x_0\cdots x_n} \arrow[r] & 0 &
\end{tikzcd}$$

If $x_0^{a_0} \cdots x_{n-1}^{a_{n-1}}$ is a monomial in $H^{n-1}(H, \mathcal{F}_H)$ with $a_i < 0$ for all $1 \leqslant i \leqslant n-1$, then it comes from an $(n+1)$-tuple in $\prod_i R_{x_0\cdots\hat{x}_i\cdots x_n}$ which maps to $\pm x_0^{a_0} \cdots x_{n-1}^{a_{n-1}}$ in $R_{x_0\cdots x_n}$, which is in turn mapped onto by the monomial $x_0^{a_0} \cdots x_{n-1}^{a_{n-1}} x_n^{-1}$ in $R_{x_0 \cdots x_n}(-1)$. So $\delta(x_0^{a_0} \cdots x_{n-1}^{a_{n-1}})$ is represented by the monomial $x_0^{a_0} \cdots x_{n-1}^{a_{n-1}} x_n^{-1}$ in $H^n(X, \mathcal{F}(-1))$. ❑

COROLLARY 18.8 *Let k be a field. Then for $m \geqslant 0$*

i) $\dim_k H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(m)) = \binom{m+n}{n}$;

ii) $\dim_k H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-m)) = \binom{m-1}{n}$.

All other cohomology groups are 0.

18.4 Cohomology groups of coherent sheaves on projective schemes

The cohomology groups of $\mathcal{O}(m)$ on $\mathbb{P}_k^n$ are always finitely generated k-vector spaces. This is part of a more general result, saying that on projective schemes over a field, the cohomology groups of coherent sheaves are always finite dimensional. Note that this is definitely not the case for affine schemes: Even the H^0 of the structure sheaf on $\mathbb{A}_k^1$ is infinite-dimensional: $H^0(\mathbb{A}_k^1, \mathcal{O}_{\mathbb{A}_k^1}) = k[t]$.

THEOREM 18.9 ((SERRE)) *Let $X \subset \mathbb{P}_A^n$ be a projective scheme of finite type over a ring A and let $\mathcal{F}$ be a coherent sheaf on X.*

i) Then the cohomology groups $H^i(X, \mathcal{F})$ are finitely generated A-modules for each i.

ii) There exists an $n_0 > 0$ such that

$$H^i(X, \mathcal{F}(n)) = 0.$$

for all $n \geqslant n_0$ and $i > 0$.

PROOF: Let $i : X \to \mathbb{P}_A^n$ denote the inclusion and let $\mathcal{G} = i_*\mathcal{F}$. The sheaf $\mathcal{G}$ is again coherent and $H^i(X, \mathcal{F}) = H^i(\mathbb{P}_A^n, \mathcal{G})$, so we immediately reduce to the case $X = \mathbb{P}^n$.

Note that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_A^n$ is of the form $\widetilde{M}$ for some finitely graded module M over $R = A[x_0, \ldots, x_n]$. The theorem is clear for $i > n$, because $H^i(\mathbb{P}_A^n, \mathcal{F}) = 0$ in this range.

(i): For $i \leqslant n$, we proceed by downwards induction. Pick a graded presentation $\bigoplus_i R(-a_i) \to M$ for M and let K be the kernel, so we have an exact sequence of finitely generated graded R-modules

$$0 \to K \to \bigoplus_i R(-a_i) \to M \to 0$$

Applying tilde, we have a sequence of sheaves

$$0 \to \mathcal{K} \to \bigoplus_i \mathcal{O}(-a_i) \to \mathcal{F} \to 0$$

If we take the long exact sequence of cohomology, we get

$$\cdots \to H^i(\mathbb{P}^n_A, \mathcal{K}) \to \bigoplus_i H^i(\mathbb{P}^n_A, \mathcal{O}(-a_i)) \to H^i(\mathbb{P}^n_A, \mathcal{F}) \to H^{i+1}(\mathbb{P}^n_A, \mathcal{K}) \to \dots$$

By induction on i, the group $H^{i+1}(\mathbb{P}^n_A, \mathcal{K})$ is a finitely generated A-A, as is $\bigoplus_i H^i(\mathbb{P}^n_A, \mathcal{O}(-a_i))$. $H^i(\mathbb{P}^n_A, \mathcal{F})$ is therefore squeezed between two finitely generated A-modules, so by exactness, it is itself finitely generated.

(ii): Twist the above sequence by $\mathcal{O}_X(m)$ and take the long exact sequence in cohomology to get

$$H^i(X, \bigoplus_i (m - a_i)) \to H^i(X, \mathcal{F}(m)) \to H^{i+1}(X, \mathcal{K}(m))$$

By downward induction on i, and the fact that $H^i(X, \mathcal{E}(m))$ for any m, we find that $H^i(X, \mathcal{F}(m)) = 0$. ❑

Euler characteristic

Let k be a field and consider a coherent sheaf $\mathcal{F} = \widetilde{M}$ on a projective scheme $X \subset \mathbb{P}^n_k$. By Serre's theorem, we know that that the cohomology groups $H^i(X, \mathcal{F})$ are *finite* as k-vector spaces. In particular, we can ask about their dimensions. It turns out that the alternating sum of these dimensions has very good functorial properties, so we make the following definition:

Definition 18.10 *Let X be a projective scheme of finite type over a field k. We define the* Euler characteristic *of $\mathcal{F}$ as*

$$\chi(\mathcal{F}) = \sum_{k \geqslant 0} (-1)^k \dim_k H^k(X, \mathcal{F}).$$

This sum is well-defined, as there are only finitely many non-zero cohomology groups appearing on the right hand side.

Proposition 18.11 *The Euler characteristic χ is additive on exact sequences, i.e., if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves, then*

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Proof: This follows because if $0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$ is an exact sequence of k-vector spaces, then $\sum_i (-1)^i \dim_k V = 0$. Applying this to the long exact sequence in cohomology gives the claim. ❑

Example 18.12 Let $X = \mathbb{P}^n_k$ and $\mathcal{F} = \mathcal{O}(d)$ for $d \geqslant 0$. Then $\dim_k H^0(\mathbb{P}^n_k, \mathcal{F}) = \binom{n+d}{n}$ and all of the higher cohomology groups are zero. In the case when $d < 0$, only $H^n(X, \mathcal{F})$ can be non-zero, and the rank is given by $\binom{n+d}{n}$, where we use the formula $\binom{x}{d} = x(x -$

$1) \cdots (x-d+1)/d!$ for any $x \in \mathbb{R}$. In particular, $\chi(\mathcal{O}_{\mathbb{P}^n_k}(d)) = \binom{n+d}{d}$ is a polynomial in d of degree n. ☆

More generally on $\mathbb{P}^n_k$ we can take any coherent sheaf $\mathcal{F}$ and a free resolution of it:

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

where the $\mathcal{E}_i$ are direct sums of invertible sheaves of the form $\mathcal{O}_X(d)$. If we tensor this sequence by $\mathcal{O}(m)$, we get

$$0 \to \mathcal{E}_n(m) \to \cdots \to \mathcal{E}_1(m) \to \mathcal{E}_0(m) \to \mathcal{F}(m) \to 0$$

We claim that also $\chi(\mathcal{F}(m))$ is a polynomial in m. Note that this is true for the terms $\chi(E_i(m))$. Then since the Euler characteristic is additive on exact sequences, $\chi(\mathcal{F}(m))$ is also a polynomial in m. Moreover, again by Serre's theorem, we have $H^i(X, \mathcal{F}(m)) = 0$ for $m \gg 0$ and $i > 0$, and so $\chi(\mathcal{F}(m)) = H^0(\mathcal{F}(m))$ for m large.

If we start with a coherent sheaf $\mathcal{F}$ on a $X \subset \mathbb{P}^n_k$, applying the previous discussion to $i_*\mathcal{F}$ on $\mathbb{P}^n_k$ gives the following:

Corollary 18.13 *Let $X \subset \mathbb{P}^n$ be a projective scheme and let $\mathcal{O}(1)$ be the Serre twisting sheaf. Then the function*

$$P_{\mathcal{F}}(m) = \chi(\mathcal{F}(m))$$

is a polynomial in m, and for large m, $H^0(X, \mathcal{F}(m)) = P_{\mathcal{F}}(m)$.

This polynomial is called the *Hilbert polynomial* of $\mathcal{F}$. When $\mathcal{F} = \widetilde{M}$ for a graded module M, this coincides with the usual Hilbert polynomial of M as defined in commutative algebra.

18.5 Extended example: Plane curves

Let $X = V(f) \subset \mathbb{P}^2_k$ be a plane curve, defined by an homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d. Let us compute the groups of the structure sheaf $H^i(X, \mathcal{O}_X)$. We have the ideal sheaf sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_X \to 0$$

where the ideal sheaf $\mathcal{I}_X$ is the kernel of the restriction $\mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_X$. By Section 15.6, $\mathcal{O}_{\mathbb{P}^2}(-X) \simeq \mathcal{O}_{\mathbb{P}^2}(-d)$, and the sequence can be rewritten as

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0.$$

From the short exact sequence, we get the long exact sequence as follows:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{P}^2,\mathcal{O}(-d)) \arrow[r] & H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}) \arrow[r] & H^0(X,\mathcal{O}_X) \arrow[dll, out=-10, in=170] \\
& H^1(\mathbb{P}^2,\mathcal{O}(-d)) \arrow[r] & H^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[dll, out=-10, in=170] \\
& H^2(\mathbb{P}^2,\mathcal{O}(-d)) \arrow[r] & H^2(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}) \arrow[r] & 0.
\end{tikzcd}$$

Using the results on cohomology of line bundles on $\mathbb{P}^2$, we deduce the equality $H^0(X,\mathcal{O}_X) \simeq k$ and hence

$$H^1(X,\mathcal{O}_X) \simeq k^{\binom{d-1}{2}}.$$

The dimension of the cohomology group on the left is the *genus* of the curve X (it will be introduced properly in Chapter 23). So the above can be rephrased as saying *the genus of a plane curve of degree d is $\frac{1}{2}(d-1)(d-2)$*.

18.6 Extended example: The twisted cubic in $\mathbb{P}^3$

Let k be a field and consider $\mathbb{P}^3 = \operatorname{Proj} R$ where $R = k[x_0,x_1,x_2,x_3]$. We will continue Example 15.25 and consider the twisted cubic curve $C = V(I)$ where $I \subset R$ is the ideal generated by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Let us by hand compute the group $H^1(X,\mathcal{O}_X)$. Of course we know what the answer should be, since $X \simeq \mathbb{P}^1$, and $H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}) = 0$. Indeed, $S = R/I$ is isomorphic to the third Veronese subring $k[s,t]^{(3)} = k[s^3,s^2t,st^2,t^3]$; the Proj of this ring is $\mathbb{P}^1_k$.

Now, to compute $H^1(X,\mathcal{O}_X)$ on X, it is convenient to relate it to a cohomology group on $\mathbb{P}^3$. We have $H^1(X,\mathcal{O}_X) = H^1(\mathbb{P}^3, i_*\mathcal{O}_X)$ where $i : X \to \mathbb{P}^3$ is the inclusion. The sheaf $i_*\mathcal{O}_X$ fits into the ideal sheaf sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^3} \to i_*\mathcal{O}_X \to 0$$

where $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^3$. Applying the long exact sequence in cohomology, we get

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(\mathbb{P}^3,\mathcal{I}) \arrow[r] & H^1(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}) \arrow[r] & H^1(\mathbb{P}^3,i_*\mathcal{O}_X) \arrow[dll, out=-10, in=170] \\
& H^2(\mathbb{P}^3,\mathcal{I}) \arrow[r] & H^2(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}) \arrow[r] & \cdots
\end{tikzcd}$$

By our description of sheaf cohomology on $\mathbb{P}^3$, $H^1(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}) = 0$, which implies that $H^1(X,\mathcal{O}_X) = H^2(\mathbb{P}^3,\mathcal{I})$. We can compute the latter cohomology group using the exact sequence of Example 15.25:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^2 \to \mathcal{O}_{\mathbb{P}^3}(-2)^3 \to \mathcal{I} \to 0.$$

Now, taking the long exact sequence we get

$$\begin{tikzcd}
\cdots \arrow[r] & H^2(\mathbb{P}^3, \mathcal{O}(-3)^2) \arrow[r] & H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \arrow[r] & H^2(\mathbb{P}^3, \mathcal{I}) \arrow[dll, out=-30, in=150] \\
& H^3(\mathbb{P}^3, \mathcal{O}(-3)^2) \arrow[r] & H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \arrow[r] & H^3(\mathbb{P}^3, \mathcal{I}).
\end{tikzcd}$$

Here $H^2(\mathbb{P}^3, \mathcal{O}(-2)) = 0$ and $H^3(\mathbb{P}^3, \mathcal{O}(-3)) = 0$ by our previsous computations. Hence by exactness, we find $H^2(\mathbb{P}^3, \mathcal{I}) = 0$. It follows that $H^1(X, \mathcal{O}_X) = 0$ also, as expected.

Exercise 18.1 Prove Lemma 18.5 in more detail. ★

Exercise 18.2 Using the sequences above, show that

- $H^0(\mathbb{P}^3, \mathcal{I}(2)) = k^3$ (find a basis!)
- $H^1(\mathbb{P}^3, \mathcal{I}(m)) = 0$ for all $m \in \mathbb{Z}$.
- $H^2(\mathbb{P}^3, \mathcal{I}(-1)) = k$.

★

18.7 Extended example: Non-split locally free sheaves

A locally free sheaf is said to be *split* if it is isomorphic to a direct sum of invertible sheaves. We have seen several examples of locally free sheaves that are not free, even on affine schemes, but a priori it is not so clear whether these are direct sums of projective modules of rank 1. In this section we will study the sheaf $\mathcal{E}$ from Section **??** and show that it is indeed non-split.

The sheaf $\mathcal{E}$ is the locally free sheaf of rank n on $\mathbb{P}^n_k$ sitting in the exact sequence (16.3)

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-1) \to \mathcal{O}_{\mathbb{P}^n_k}^{n+1} \to \mathcal{E} \to 0.$$

Suppose that $\mathcal{E}$ is not split, i.e., $\mathcal{E}$ is not isomorphic to a direct sum of invertible sheaves. Since $\operatorname{Pic}(\mathbb{P}^n_k) = \mathbb{Z}$ is generated by the class of $\mathcal{O}(1)$, this would mean that $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^n_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n_k}(a_n)$ for some integers $a_1, \ldots a_n \in \mathbb{Z}$.

Recall that for $n \geqslant 2$, we have $H^{n-1}(\mathbb{P}^n_k, \mathcal{O}(m)) = 0$ for any $m \in \mathbb{Z}$. So if we could show that $H^{n-1}(\mathbb{P}^n_k, \mathcal{E}) \neq 0$, we would have a contradiction. Actually, it is the case that $H^{n-1}(\mathbb{P}^n_k, \mathcal{E}) = 0$, but we can instead consider $\mathcal{F} = \mathcal{E}(-n)$, which fits into the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-n-1) \to \mathcal{O}_{\mathbb{P}^n_k}(-n)^{n+1} \to \mathcal{F} \to 0.$$

Taking the long exact sequence in cohomology, we get

$$\cdots \to H^{n-1}(\mathcal{O}_{\mathbb{P}^n_k}^{n+1}) \to H^{n-1}(\mathcal{F}) \xrightarrow{\delta} H^n(\mathcal{O}_{\mathbb{P}^n_k}(-n-1)) \to H^n(\mathcal{O}_{\mathbb{P}^n_k}^{n+1}) \to \cdots$$

Here the two outer cohomology groups are zero, by Theorem 18.7. Hence, by exactness, we find that $H^{n-1}(\mathbb{P}^n_k, \mathcal{F}(-1)) \simeq H^0(\mathbb{P}^n_k \mathcal{O}_{\mathbb{P}^n_k}) = k$. This implies that $\mathcal{F} = \mathcal{E}(-n)$, and hence $\mathcal{E}$ cannot be a sum of invertible sheaves, and we are done.

The above gives an example of a non-split locally free sheaf of rank n. However, coming up with examples of non-split sheaves of *low rank* is a notoriously difficult problem, even for projective space. In fact, a famous conjecture of Hartshorne says that any rank 2 vector bundle on $\mathbb{P}^n$ for $n \geqslant 5$ is split. (On $\mathbb{P}^4$ this statement does not hold, as shown by the so-called *Horrocks–Mumford bundle*). This is related to the long-standing conjecture that any smooth codimension 2 subvariety of $\mathbb{P}^n$ is a complete intersection.

18.8 Extended example: Hyperelliptic curves

Let us recall the hyperelliptic curves defined in Chapter 6.

Let k be a field. For an integer $g \geqslant 1$, we consider the scheme X glued together by the affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, where

$$A = \frac{k[x,y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1 x)} \text{ and } B = \frac{k[u,v]}{(-v^2 + a_{2g+1}u + \cdots + a_1 u^{2g+1})}.$$

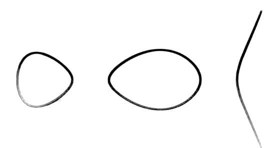


Figure 18.1: One of the affine charts of X

and before, we glue $D(x) \subset U$ to $D(u) \subset V$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

Let us compute the Čech cohomology groups of $\mathcal{O}_X$ with respect to the affine covering $\mathcal{U} = \{U, V\}$ above. Viewing the ring A as a $k[x]$-module, we can write

$$\frac{k[x,y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1 x)} = k[x] \oplus k[x]y$$

and similarly $B \simeq k[u] \oplus k[u]v$ as a $k[u]$-module.

As $\mathcal{U}$ has only two elements, the Čech complex of $\mathcal{O}_X$ has only two terms, $\mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ and $\mathcal{O}_X(U \cap V)$ and the differential between them,

$$d^0\colon\ (k[x] \oplus k[x]y) \oplus \left(k[x^{-1}] \oplus k[x^{-1}]x^{-g-1}y\right) \to k[x^{\pm 1}] \oplus k[x^{\pm 1}]y,$$

is given by by the assignment

$$\begin{aligned} &d^0(p(x) + q(x)y, r(x^{-1}) + s(x^{-1})x^{-g-1}y) \\ &\qquad\qquad = p(x) - r(x^{-1}) + (q(x) - s(x^{-1})x^{-g-1})y. \end{aligned}$$

Comparing monomials $x^m y^n$ on each side, we deduce that

$$H^0(X, \mathcal{O}_X) = \operatorname{Ker} d^0 = k$$

and

$$H^1(X, \mathcal{O}_X) = \operatorname{Coker} d^0 = k\{yx^{-1}, yx^{-2}, \ldots, yx^{-g}\} \simeq k^g.$$

In particular, $\dim_k H^1(X, \mathcal{O}_X) = g$. The latter invariant is usually referred to as the *arithmetic genus* of a curve; we have shown that the hyperelliptic curve X has arithmtic genus g.

For $g = 2$, we get a particularly interesting curve – an irreducible projective curve which cannot be embedded in $\mathbb{P}^2$. Indeed, we showed that for any irreducible curve in $\mathbb{P}^2$ of degree d and the corresponding arithmetic genus equals $\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$. However, there is no integer solution to $\frac{1}{2}(d-1)(d-2) = 2$. This implies the following:

PROPOSITION 18.14 *There exist non-singular projective curves which cannot be embedded in $\mathbb{P}^2$.*

Note that we still haven't proved that X is projective. Actually, it is not hard to see that X can be embedded into the *weighted* projective space $\mathbb{P}(1, 1, g+1) = \operatorname{Proj} k[x_0, x_1, w]$ given by the equation

$$w^2 = a_{2g+1} x_0^{2g+1} x_1 + \cdots + a_1 x_0 x_1^{2g+1}. \tag{18.4}$$

Note that this makes sense if w has degree $g+1$, but it does not define a subscheme of $\mathbb{P}^2$.

Exercises

(18.3) Let $X \subset \mathbb{P}^5$ denote a quadric hypersurface (i.e., $X = V(q)$ for a homogeneous degree 2 polynomial). Recall the exact sequence 16.4

$$0 \to \mathcal{O}_X(-1)^4 \to \mathcal{O}_X^4 \to \mathcal{E} \to 0$$

where $\mathcal{E}$ is a locally free sheaf of rank 2.

(i) Use the exact sequences (15.6) to show that

$$H^i(X, \mathcal{O}_X(-1)) = 0$$

for all $i \geqslant 0$.

(ii) Use the exact sequence (16.4) to show that $\mathcal{E}$ is not split.

✲ (18.4) Let $n > 0$ be an integer and consider the integral projective scheme $X = \operatorname{Proj}(R)$, where R is the ring

$$R = k[x, y, z, w]/(x^2, xy, y^2, u^n x - v^n y).$$

a) Show that X is irreducible, non-reduced, and of dimension 1.

b) Compute $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$.

★

18.9 *Extended example: Bezout's theorem*

Let k be an algebraically closed field. Let C and D be two curves in $\mathbb{P}_k^2$ of degrees d and e respectively. We assume here that C and D have no common component, so that Z is a 0-dimensional subscheme.

Let us compute the cohomology group $H^0(Z, \mathcal{O}_Z)$. If we assume $Z = \{x_1, \ldots, x_r\}$ is contained in $D(x_0) \simeq k[x, y]$ (which we may arrange by a linear coordinate change), then

$$\mathcal{O}_Z(Z) = \bigoplus_{i=1}^{r} \left(\frac{k[x,y]}{(f,g)} \right)_{\mathfrak{m}_{x_i}} \tag{18.5}$$

where f, g are the dehomogenized equations for C and D. In other words, $\dim_k H^0(Z, \mathcal{O}_Z)$ is the sum of the *multiplicities* at the points x_i:

$$\dim H^0(Z, \mathcal{O}_Z) = \sum_{i=1}^{r} \dim_k \left(\frac{k[x,y]}{(f,g)} \right)_{\mathfrak{m}_{x_i}}.$$

On the other hand, we can compute $H^0(Z, \mathcal{O}_Z)$ using the ideal sheaf sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^2} \to i_* \mathcal{O}_Z \to 0,$$

and we deduce that $\dim_k H^0(Z, \mathcal{O}_Z) = \dim_k H^1(\mathbb{P}^2, \mathcal{I}_Z) - 1$. We proceed to study the latter cohomology group. Recall the exact sequence from Section 15.6,

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d-e) \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^n_k}(-e) \to \mathcal{I}_Z \to 0.$$

Taking the long exact sequence of cohomology we obtain

$$0 \to H^1(\mathbb{P}^2, \mathcal{I}) \to H^2(\mathbb{P}^2, \mathcal{O}(-d-e) \to H^2(\mathbb{P}^2, \mathcal{O}(-d)) \oplus H^0(\mathbb{P}^2, \mathcal{O}(-e)) \to 0.$$

From which we get the triumphant conclusion that

$$\begin{aligned}
\dim_k H^0(Z, \mathcal{O}_Z) &= \dim_k H^1(\mathbb{P}^2_k, \mathcal{I}_Z) + 1 \\
&= \dim_k H^2(\mathcal{O}(-d-e) - \dim_k H^2(\mathcal{O}(-d)) - \dim_k H^2(\mathcal{O}(-e)) + 1 \\
&= \binom{d+e-1}{2} - \binom{d-1}{2} - \binom{e-1}{2} + 1 \\
&= de
\end{aligned}$$

In other words, we have proved Bezout's theorem for $\mathbb{P}^2_k$:

$$\sum_{i=1}^{r} \dim_k \left(\frac{k[x,y]}{(f,g)} \right)_{\mathfrak{m}_{x_i}} = de$$

18.10 Čech cohomology and the Picard group

The Picard group $\mathrm{Pic}(X)$ is an important invariant of a scheme X, but since it is defined as an abstract group of invertible sheaves on X, it may not be so obvious how to compute it. In this section, we remedy this, by relating it to a cohomology group.

Let L be an invertible sheaf on X and let $\mathcal{U} = \{U_i\}$ be a trivializing cover. Thus for each U_i there should be isomorphisms $\phi_i : L|_{U_i} \to \mathcal{O}_{U_i}$ such that $\phi_i = \phi_j$ agree on the overlaps $U_i \cap U_j$. Note that $\phi_j \circ \phi_i^{-1}$ is an isomorphism $\mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j}$. Now, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{U_i \cap U_j}, \mathcal{O}_{U_i \cap U_j}) \simeq \mathcal{O}_X(U_i \cap U_j) \tag{18.6}$$

(given by $h \mapsto h(1)$ and conversely, $s \in \mathcal{O}_X(U_i \cap U_j)$ gives the homomorphism given by multiplication by s).

Recall the sheaf of units in $\mathcal{O}_X^\times$; over an open set $U \subset X$, these where the elements s having a multiplicative inverse $s^{-1} \in \Gamma(U, \mathcal{O}_X)$. Equivalently, $\mathcal{O}_X^\times(U)$ consists of sections of $\mathcal{O}_X(U)$ such that for each $x \in U$, the germ s_x does not lie in the maximal ideal of $\mathcal{O}_{X,x}$.

From the correspondence described by the isomorphism in (18.6), we infer that the group of *isomorphisms* $\mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j}$ corresponds exactly to the group of units in $\mathcal{O}_X(U_i \cap U_j)$, *i.e.*

$$\operatorname{Isom}_{\mathcal{O}_X}(\mathcal{O}_{U_i \cap U_j}, \mathcal{O}_{U_i \cap U_j}) \simeq \mathcal{O}_X^\times(U_i \cap U_j).$$

In order to specify the invertible sheaf L, we must say how to glue together the sheaves $\mathcal{O}_{U_i}$ and $\mathcal{O}_{U_j}$ along $U_i \cap U_j$ — in other words, we have to specify a unit $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)$ for each i, j. These units s_{ij} cannot be chosen completely at random as we need to make sure they will be compatible on the triple overlaps $U_i \cap U_j \cap U_k$. It turns out to be enough that they satisfy the one constraint that $s_{ij}s_{jk}s_{ki} = 1$ on $U_i \cap U_j \cap U_k$, or in other words that

$$s_{jk}s_{ik}^{-1}s_{ij} = 1$$

in $\mathcal{O}_X^\times(U_i \cap U_j)$. Since $\mathcal{O}_X^\times$ is a sheaf of abelian groups with multiplication being the groups structure, a restatement of this is that the collection $\{s_{ij}\} \in \mathcal{O}_X(U_i \cap U_j)$ forms a 1-cocyle! This means that we get a well-defined element in $H^1(X, \mathcal{O}_X^\times)$. This is, as one verifies, independent of the choice of cover U_i, and the map $\operatorname{Pic}(X) \to H^1(X, \mathcal{O}_X^\times)$ so obtained is in fact a group homomorphism.

Conversely, any element $s \in H^1(X, \mathcal{O}_X^\times)$ is represented by a cover $\mathcal{U} = \{U_i\}$ of X and cocycles $s_{ij} \in C^1(\mathcal{U}, \mathcal{O}_X^\times)$. The cocyle condition implies that the s_{ij} define isomorphisms $\mathcal{O}_{U_i}|_{U_{ij}} \to \mathcal{O}_{U_i}|_{U_{ij}}$ which glue the $\mathcal{O}_{U_i}$ together to form an invertible sheaf. This provides an inverse to the above map. We have therefore shown:

Theorem 18.15 *Let X be a scheme. Then there is a canonical isomorphism*

$$\operatorname{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\times).$$

This result is helpful for computing Picard groups, because we can compute the H^1 using the Čech-complex. One must take a little bit care, because the sheaf $\mathcal{O}_X^\times$ is usually not quasi-coherent. Here are a few examples.

Example 18.16 Let X be the affine line with two origins. We will use the usual open covering of $\mathcal{U}$ of X by the two open subsets $U, V \subset X$ isomorphic to $\mathbb{A}_k^1$. We have

- $H^1(U, \mathcal{O}^\times) = H^1(\mathbb{A}_k^1, \mathcal{O}^\times) = 0$, because $\operatorname{Pic}(\mathbb{A}_k^1) = 0$.
- $H^1(U \cap V, \mathcal{O}^\times) = 0$, because $U \cap V \simeq \operatorname{Spec} k[x^{\pm 1}]$ which has trivial Picard group.

The higher cohomology groups $H^i(U, \mathcal{O}^\times)$, $H^i(V, \mathcal{O}^\times)$ and $H^i(U \cap V, \mathcal{O}^\times)$ also vanish, because X has dimension 1. Hence Leray's theorem says that $H^1(X, \mathcal{O}_X^\times)$ can be computed via the Čech complex

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X^\times(U) \times \mathcal{O}_X^\times(V) \arrow[r] \arrow[d, equal] & \mathcal{O}_X^\times(U \cap V) \arrow[r] \arrow[d, equal] & 0 \\ & k^\times \times k^\times & k[x^{\pm 1}]^\times & \end{tikzcd}$$

Note that $\mathcal{O}_X^\times(U \cap V)$ consists of elements of the form az^n $(a \in k^\times, n \in \mathbb{Z})$. Hence $\operatorname{Pic}(X) = \mathbb{Z}$. ☆

EXAMPLE 18.17 Here is an example with a reducible scheme. Let k be a field and set

$$X = \operatorname{Spec}\left(\frac{k[x,y]}{xy(x+y+1)}\right).$$

Consider the components $U = V(x)$, $V = V(y)$ and $W = V(x+y+1)$. As above, we have $H^1(U, \mathcal{O}_X^\times) = H^1(\mathbb{A}^1, \mathcal{O}_X^\times) = 0$, and similarly for the other components. Thus by Leray's theorem, we may compute $\operatorname{Pic}(X)$ by the Čech-complex of the coveirng $\{U, V, W\}$, which takes the form

$$0 \longrightarrow k^\times \times k^\times \times k^\times \xrightarrow{\rho} k^\times \times k^\times \times k^\times \longrightarrow 0$$

where $\rho(a,b,c) = (ba^{-1}, ca^{-1}, cb^{-1})$. From this it follows that $\operatorname{Pic}(X) \simeq \operatorname{Coker} \rho = k^\times$. ☆

Chapter 19

Properties of morphisms I

19.1 Finite morphisms

In Chapter 4 we introduced finiteness conditions for morphisms. Recall that an $f\colon X \to Y$ is *affine* if we may cover Y by open affine subsets $U_i = \operatorname{Spec} A_i$ whose inverse images $f^{-1}(U_i)$ are affine, and f is *finite* if additionally each $f^{-1}(U_i) = \operatorname{Spec} B_i$ with B_i being a finite algebra over A_i. The definitions refer a specific cover, but once they hold for one cover, they will hold for any:

Proposition 19.1 *Let $f\colon X \to Y$ be a morphism of schemes.*

i) If f is affine, then $f^{-1}(U)$ is affine for all open affine subsets $U \subseteq Y$.

ii) When f is finite, for each open affine subset $U = \operatorname{Spec} A$ of Y it holds that $f^{-1}(U) = \operatorname{Spec} B$ for a finite A-algebra B.

The proof is given as an solved exercise (Exercise 19.3 below).

In the special case that Y is a point; that is, the spectrum of a field $Y = \operatorname{Spec} k$, the scheme X is finite over Y if and only if $X = \operatorname{Spec} A$ with A being a finite dimensional k-algebra. The underlying topological space is then finite and discrete. This generalizes in the following way:

Proposition 19.2 *A finite morphism has scheme-theoretic finite fibres. In particular, the fibres are finite discrete topological spaces.*

Proof: We may certainly assume that both X and Y are affine; say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. For $\mathfrak{p} \in \operatorname{Spec} A$, the scheme-theoretical fibre over $\mathfrak{p}$ equals $\operatorname{Spec} B \otimes_A K(A/\mathfrak{p})$ where as usual $K(A/\mathfrak{p})$ denotes the fraction field. Any generator set of B as an A-module persists being a generator set of $B \otimes_A K(A/\mathfrak{p})$ as a vector space over $K(A/\mathfrak{p})$ so $B \otimes_A K(A/\mathfrak{p})$ is of finite dimension over $K(A/\mathfrak{p})$. ❑

Quasi-finite morphisms

Be aware that the converse is far from being true. One easily finds so-called *quasi-finite* morphisms; that is, morphisms all whose fibres are finite, which are not finite: every injective morphism is evidently quasi-finite, so for instance, open immersions will be, and open immersions are not finite except in trivial cases. The arch-type is the inclusion $\iota\colon D(x) \hookrightarrow \mathbb{A}^1_k$ which on the ring level corresponds to the inclusion $k[x] \hookrightarrow k[x, x^{-1}]$;

and $k[x, x^{-1}]$ is not a finite module over $k[x]$. We'll come back to the relation between quasi-finite and finite morphism when having introduced proper morphism (in Section **??**).

The fibre of a finite morphism has a lot more structure than just being finite. A finite algebra A decomposes as a finite product

$$A = \prod_{\mathfrak{m} \in \operatorname{Spec} A} A_{\mathfrak{m}}, \tag{19.1}$$

where the A_m's are Artinian local algebras; some are field extensions of k, and some have non-trivial nilpotent elements. From the decompositionin (19.1) follows that $\dim_k A = \sum_{\mathfrak{m} \in \operatorname{Spec} A} \dim_k A_{\mathfrak{m}}$, and one my heuristically think about $\dim_k A$ as the 'number of points in $\operatorname{Spec} A$ counted with multiplicity'; the multiplicity attributed to $\mathfrak{m}$ being $\dim_k A_{\mathfrak{m}}$.

If Y is an integral scheme, each finite morphism $\phi\colon X \to Y$ has a *degree*; which basically is the number of points (counted with multiplicity) in a generic fibre. It is defined as $\deg \phi = \dim B \otimes_A K(Y)$. When also X is integral, the degree equals the degree $[K(X) : K(Y)]$ of the field extension $K(Y) \subseteq K(X)$ of function fields.

The degree of a finite morphism

Proposition 19.3 *Assume that $Y = \operatorname{Spec} A$ is integral and that $\phi\colon X \to Y$ is a finite morphism. Then there is an non-empty open subset $U \subseteq Y$ so that the fibre dimension $\dim B \otimes_A K(A/\mathfrak{p}) = \deg \phi$ for all $\mathfrak{p} \in U$.*

Proof: We shall show that there is an $f \in A$ so that B_f is free over A_f of rank equal the degree $\deg \pi$, from which the proposition follows readily. Let $r = \deg \pi$. This is just the dimension of $B \otimes_A K$ as a K-vector space, and as $B \otimes_A K$ is B localized in the multiplicative set $S = A \backslash \{0\}$, we may find a basis $b_1, \ldots, b_r$ for $B \otimes_A K$ consisting of elements in some B_f (chose any basis $\{b_i\}$ and let f be a common denominator for the b_i's). Consider the map $\alpha\colon A_f^r \to B_f$ induced by the b_i's, which lives in the short exact sequence

$$0 \longrightarrow C \longrightarrow A_f^r \longrightarrow B_f.$$

As B_f is finitely generated over A_f, the cokernel of α has closed support, and after another localization, we may assume α is surjective. But as $\alpha \otimes \mathrm{id}_K$ is an isomorphism, every element in C is killed by a non-zero element in A. Now A is a domain, and the free module A^r is torsion free, and hence $C = 0$. ❑

Proposition 19.4 *A finite map is closed. In particular, if it is dominating it will be surjective.*

Proof: Let the finite morphism be $\phi\colon X \to Y$. We may assume that X and Y are affine; say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. It will be sufficient to see that $\phi(X)$ is closed since for each closed subscheme $Z \subseteq X$ the restriction $\phi|_Z$ persists being finite; indeed, if $\phi^{-1}(U)$ is affine, $\phi^{-1}(U) \cap Z$ is affine as closed subschemes of affine schemes are affine, and clearly $\mathcal{O}_Z(Z) = B/I(Z)$ is finite over A when B is.

Let $C \subseteq B$ be the image of $\phi^\sharp \colon A \to B$, we claim that $\phi(X)$ equals the closed subscheme $\operatorname{Spec} C$ of Y; indeed, B is an integral extension of C and the Going–Up theorem yields that each $\mathfrak{p} \in \operatorname{Spec} C$ equals $\mathfrak{q} \cap C$ fo some $\mathfrak{q} \in \operatorname{Spec} B$. ❑

Proposition 19.5 *Let f and g be morphisms between schemes.*

i) If f and g are composable and f and g is finite (respectively affine), then the composition $g \circ f$ is finite (respectively affine);

ii) Being finite (respectively affine) is a property preserved under base change.

iii) If f and g are composable and $g \circ f$ is finite (respectively affine) and f is separated, then f is finite (respectively affine);

Proof: Let the morphisms be $f \colon X \to Y$ and $g \colon Y \to S$.

i) Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ the composition of two affine morphisms is affine. The finiteness part follows because if B is an algebra finite over A, and C is one finite over B, then C is finite over A: choose a finite generating sets $\{b_i\}$ of B over A and $\{c_j\}$ of C over B, then $\{b_i c_j\}$ will generate C over A.

ii): Consider the base change diagram

$$\begin{array}{ccc} X_Z & \longrightarrow & X \\ \big\downarrow{\scriptstyle f_Z} & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

We shall exhibit a covering of Z by open affines whose preimages under f_Z are affine. Note that if an open affine $V \subseteq Z$ maps into an open affine $U \subseteq Y$, then $f_Z^{-1}(V) = V \times_U f^{-1}(U)$ is affine since fibre products of affine schemes are affine. Then chose a covering $\{U_i\}$ of Y by open affines and cover each inverse image $g^{-1}(U_i)$ by open affines $\{U_{ij}\}$. Then $\{U_{ij}\}$ is a covering of Z as desired.

For the finiteness part: If B is a finite algebra over A and C is another A-algebra, clearly $B \otimes_A C$ is finite over C; indeed, if $\{b_i\}$ is a generating set for B over A, the set $\{x_i \otimes 1\}$ will generated $B \otimes_A C$ over C.

iii): The graph $\Gamma_f \colon X \to X \times_S Y$ of f is closed since it equals the pull back of the closed embedding $\Delta \colon Y \to Y \times_S Y$ (Y is separated over S). The projection $X \times_S Y \to S$ is just the pullback of $g \circ f \colon X \to S$ along $Y \to S$, and in view of *i)* therefore is affine as $g \circ f$ is. Now, composing Γ with the projection $X \times_S Y \to S$ we retrieve f, and hence f is affine by *i)*. The finiteness part is clear since if C is an algebra over the A-algebra B which is finite over A, it is all the more finite over A. ❑

A similar result applies to morphisms locally of finite type

Proposition 19.6 *Let f and g be composobale morphsms of schemes.*

i) If f and g are of finite type (respectively locally of fiite type), then $f \circ g$ is of finite type (respectively locally of fiite type);

ii) Being of finite type (respectively locally of finite type) is property preserved under base change;

iii) If $f \circ g$ is of finite type (respectively locally of fiite type), then g is of finite type (respectively locally of fiite type).

Examples

(19.7) Consider the ring map $A = k[t^2+1, t(t^2+1)] \hookrightarrow k[t] = B$ and let ϕ be the associated map between the spectra. $\operatorname{Spec} A$ is the curve in $\mathbb{A}^2_k$ whose equation is $y^2 = x^2(x+1)$ and ϕ is just the parametrisation $x = t^2+1$ and $y = t(t^2+1)$.

The maximal ideal in A corresponding to the origin, is $\mathfrak{m} = (t^2+1, t(t^2+1))$, and $\mathfrak{m}B = (t^2+1)$. So the fibre over the origin equals $k[t]/(t^2+1)$. When -1 is square root in k, this equals $k \oplus k$, otherwise it equals the quadratic field extension $k(\sqrt{-1})$.

Over the open set $D(t^2+1) \subseteq \operatorname{Spec} A$ the map is an isomorphism, indeed, $A_{t^2+1} = k[t]_{t^2+1}$ due to the stupid equality $t = yx^{-1} = t(t^2+1)(t^2+1)^{-1}$, so generically, the fibre consists of just one point.

(19.8) Assume that $Y = \operatorname{Spec} A$ is a principal ideal domain. Let X be an integral scheme and $X \to Y$ is a finite morphism. Then $X = \operatorname{Spec} B$ with B an integral domain which is a finite A-algebra. Hence B is a torsion free A-module. Since every finite torsion free module over a PID is free, it follows that the dimension $\dim_{k(\mathfrak{p})} B \otimes_A A_{\mathfrak{p}}$ is independent of the prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, so that all fibres have 'the same number of points'.

(19.9) Consider the blow-up morphism $\pi : X \to \mathbb{A}^2$ from Example 6.6. In the local charts, π is given by $\operatorname{Spec} \mathbb{Z}[x,t] \to \operatorname{Spec} \mathbb{Z}[x,y]$ induced by $y \mapsto xt$, making $\mathbb{Z}[x,t]$ into a finitely generated $\mathbb{Z}[x,t]$-algebra, so it is of finite type. However, it is not finite. In fact it is not even affine, since $\pi^{-1}(V)$ contains a copy of $\mathbb{P}^1$ for any neighbourhood V of the closed point $o \in \mathbb{A}^2$, which is not possible for affine schemes.

(19.10) Let us revisit the example of a hyperelliptic curve X from Section 6.10. In the notation from that section, the curve X has an open covering consisting of two affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ and there is a 'double cover' morphism $f : X \to \mathbb{P}^1_k$. This is a finite morphism: Over U it is induced by the inclusion

$$k[x] \subset \frac{k[x,y]}{(y^2 - a_{2g+1}x^{2g+1} - \cdots - a_1 x)},$$

and the algebra on the right is isomorphic to $k[x] \oplus k[x]y$ as a $k[x]$-module. A similar statement holds for the morphism $f|_V \colon V \to \mathbb{A}^1_k$, so f is a finite morphism.

☆

Exercises

(19.1) *Distinguished properties.* We shall say that a property $\mathcal{P}$ attributed to affine subsets of a scheme X is *distinguished*[1], if the following two requirements are fulfilled:

i) If U has $\mathcal{P}$ and $f \in \Gamma(U, \mathcal{O}_X)$, then $D(f)$ has $\mathcal{P}$;

ii) If $\{D(f_i)\}$ is a finite cover of U of distinguished open subsets each having $\mathcal{P}$, then U has $\mathcal{P}$.

Show that if $\mathcal{P}$ is distinguished and there exists one open affine cover of X with each member having $\mathcal{P}$, then all open affines in X has $\mathcal{P}$. Show that it suffices that *ii)* be satisfied for all coverings by two distinguished opens.

(19.2) *Affine morphisms.* Another big difference between morphisms of finite type and finite morphisms is that the latter are affine, in the following sense. A morphism $f\colon X \to Y$ is *affine* if $f^{-1}(U)$ is affine whenever $U \subseteq Y$ is affine and open. Show that being affine is a property local on the target; that is, if there is a cover of Y by open affines $\{U_i\}$ so that the $f^{-1}(U_i)$ are affine, then f is affine.

(19.3) *Finite morphisms.* Show that being finite is a distinguished property, so that if $f\colon X \to Y$ is finite, the inverse image of any open affine $U \subseteq Y$ is affine and $\mathcal{O}_X(f^{-1}(U))$ is a finite module over $\mathcal{O}_Y(U)$.

(19.4) *Functoriality.* Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes. Let $\mathcal{P}$ be one of the properties, locally of finite type, finite type or finite.

a) Show that if f and g are $\mathcal{P}$, then $g \circ f$ is $\mathcal{P}$;
b) Show that if $g \circ f$ is $\mathcal{P}$, then f is $\mathcal{P}$;
c) Give examples that $g \circ f$ and f are $\mathcal{P}$, but g is not.
d) If $h\colon Z \to Y$ and $f\colon X \to Y$ are a morphism and f have property $\mathcal{P}$, hen the pull back $f \times_Y Z$ has the popery $\mathcal{P}$.

19.2 *Proper morphisms*

Searching for a substitute in algebraic geometry for notion of the compact spaces in topology, it turns out that the topological notion of proper maps is suitable. These are the continuous maps with all preimages of compact sets being compact, or equivalently they are the universally closed maps, and the latter property is, together with a finiteness condition, the one adopted in scheme-theory:

DEFINITION 19.11 *A morphism $f\colon X \to Y$ is* universally closed *if it is closed and stays closed when pulled back; that is, given any morphism $T \to Y$, the pulled back map $f_T = f \times_T \mathrm{id}_T$ is closed.*
The morphism f is said to be proper *if it is separated, of finite type and universally closed. We say f is proper over Y, or just proper over A when $Y = \operatorname{Spec} A$ is the spectrum of a ring A.*

$$\begin{tikzcd} X \times_Y T \arrow[r] \arrow[d, "f_T"'] & X \arrow[d, "f"] \\ T \arrow[r] & Y \end{tikzcd}$$

[1] Vakil in his notes calls this property a "local property", but local being a overcharged name, we prefer distinguished.

EXAMPLE 19.12 A simple example of a morphism that is not proper, is the structure map $\pi\colon \mathbb{A}^1_k \to \operatorname{Spec} k$. Pulled back along itself it becomes the projection $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$ to the first factor, and within $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ there are a lot of closed sets projecting to sets not being closed; for instance, the graph of any rational function with a pole is such, the simplest being the 'hyperbola' $xy = 1$. It projects to the non-closed set $\mathbb{A}^1_k \backslash \{0\}$. ☆

EXAMPLE 19.13 Close immersions are proper; indeed they are universally closed by Proposition 8.12 on page 126 and separated by *i)* of Proposition 9.9 on page 135. They are trivially of finite type. ☆

PROPOSITION 19.14 *The following statements hold true:*

i) Pullbacks of proper maps are proper;
ii) Compositions of composable proper maps are proper;
iii) The product $f \times g$ of two proper maps is proper;
iv) Given two composable maps f and g. If $f \circ g$ is proper and f is separated then g is proper.

The proposition holds true if 'proper' is replaced by 'universally closed'.

PROOF: *i)*: Let the involved maps be $T \to Y$ and $f\colon X \to Y$ with f proper. The pullback f_T is separated and of finite type as these properties are stable under base changes, so let us check that $f_T = f \times_T \mathrm{id}_T$ is universally closed. Assuming given a morphism $U \to T$, we find using transitivity of the fibre product that $(f_T \times_U \mathrm{id}_U) = (f \times_T \mathrm{id}_T) \times_U \mathrm{id}_U = f \times_U \mathrm{id}_U$, and the latter map is closed because f is universally closed.

ii): This is undemanding the composition of two closed maps being closed and the pullback being a functor; *i.e.* $(f \circ g) \times \mathrm{id}_T = (f \times \mathrm{id}_T) \circ (g \times \mathrm{id}_T)$.

iii): It follows directly from *i)* and *ii)* that $f \times g = (f \times \mathrm{id}_{X'}) \circ (\mathrm{id}_X \times g)$ is proper, where $f\colon X \to Y$ and $g\colon X' \to Y'$ are the two given proper maps.

iv): Recall the diagram set up in Exercise 9.4 on page 138:

$$\begin{tikzcd} X \arrow[r, "\Gamma_g"] & X \times_Z Y \arrow[r, "\pi"] \arrow[d] & Y \arrow[d, "f"] \\ & X \arrow[r, "f \circ g"'] & Z. \end{tikzcd}$$

Here the square is Cartesian, and where Γ_g is the graph of g, so that the compositions of the two upper maps equals g. Since $f \circ g$ is proper and being proper is stable under base change, the projection $X \times_Z Y \to Y$ is proper. The map f is supposed to be separated so the graph Γ_g is a closed immersion. It follows that g being the composition $g = \pi \circ \Gamma_g$ of two proper maps, is proper. ❑

✲ EXERCISE 19.5 *(Locallity on the target.)* Properness is a property local on the target: Let $f\colon X \to Y$ be a morphism and assume that there is an open covering $\{U_i\}$ of Y such that each restriction $f|_{f^{-1}(U_i)}\colon f^{-1}(U_i) \to U_i$ is proper, show that then f is proper. ★

EXERCISE 19.6 Let X be a scheme, separated and of finite type over the field k, and assume that there is a closed immersion $\mathbb{A}^1_k \hookrightarrow X$. Show that X is not proper over k. ★

Projective morphisms are proper

We proceed by describing one of the most useful properties projective morphisms have: they are proper maps. And in fact, most proper morphisms one meets when practising algebraic geometry are projective. There are however others, but to construct examples is not straightforward; the first varieties being proper over $\mathbb{C}$, but not projective, were found by Hironaka in 1960.

Projective morphisms are separated and of finite type, so to prove that they are proper it remains to show that they are universally closed. This basically relies on two facts, one is about sets closed under so-called spesializations and the other about maps from spectra of valuation rings into projective spaces.

From topology we remember that compact subsets of a Hausdorff space are closed. In the world of schemes one harly meets Hausdorff spacesand quasi-compact subsets are mostly not closed (*e.g.* all open affines are quasi-compact). Fortunately there is a nice substitute (formulated in Proposition 19.17 below).

Specialization of a point

In point-set topology taking limits of points from a subset is a popular technique. This is an analogue, although vague, for schemes.

A point x in a scheme X is said to be a the *specialization* of a point y if x belongs to the closure of y; that is, if $x \in \overline{\{y\}}$. In case X is affine, this means that the inclusion $\mathfrak{p}_y \subseteq \mathfrak{p}_x$ holds and the localizations are related by a map $A_{\mathfrak{p}_x} \to A_{\mathfrak{p}_y}$ of local rings. When X is an integral scheme, all local rings $\mathcal{O}_{X,x}$ lie naturally in the function field $K(X)$, and it holds that $x \in \overline{\{y\}}$ if and only if $\mathcal{O}_{X,x} \subseteq \mathcal{O}_{X,y}$ and $\mathfrak{m}_y \cap \mathcal{O}_{X,x} \subseteq \mathfrak{m}_x$; in other words, if and only if $\mathcal{O}_{X,y}$ *dominates* $\mathcal{O}_{X,x}$.

We shall need the slightly more lax condition in the following lemma:

Lemma 19.15 *For different points x and y in an affine integral scheme $X = \operatorname{Spec} A$, it holds true that $x \in \overline{\{y\}}$ if and only if $\mathfrak{m}_y \cap \mathcal{O}_{X,x} \subseteq \mathfrak{m}_x$.*

Proof: One of the implications is generally true, if $x \in \overline{\{y\}}$ the local ring $\mathcal{O}_{X,y}$ dominates $\mathcal{O}_{X,x}$. Assume then that $\mathfrak{m}_y \cap \mathcal{O}_{X,x} \subseteq \mathfrak{m}_x$. Now $\mathcal{O}_{X,x} = A_{\mathfrak{p}_x}$ and $\mathcal{O}_{Y,y} = A_{\mathfrak{p}_y}$; Since $\mathfrak{p}A_{\mathfrak{p}} \cap A = \mathfrak{p}$ for all prime ideals in A, we have the following sequence of inclusions

$$\mathfrak{p}_y = \mathfrak{m}_y \cap A = \mathfrak{m}_y \cap \mathcal{O}_{X,x} \cap A \subseteq \mathfrak{m}_x \cap A = \mathfrak{p}_x.$$

Hence $\mathcal{O}_{X,x} \subseteq \mathcal{O}_{X,y}$ ❏

Sets closed under specialization

One says that a set $S \subseteq X$ is *closed under specialization* if $x \in \overline{\{y\}}$ for $y \in S$ implies that $x \in S$. Closed sets are naturally closed under specialization, but the converse is not generally true:

Example 19.16 Any proper infinite subset S of the affine line $\mathbb{A}^1_k$ where k is a field is an example. The only proper closed subsets of $\mathbb{A}^1_k$ being finite, S is not closed, but it is closed under specialization since all points in S are closed points. ☆

Quasi-compact morphisms

A morphism $f\colon Z \to X$ is called *quasi-compact* if X may be covered by open affines the inverse image of each under f being quasi-compact.

Proposition 19.17 *Assume that $f: Z \to X$ is a quasi-compact morphism. Then the image $f(Z)$ is closed if and only if $f(Z)$ is closed under specialization.*

Proof: One implication is trivial, so we assume that $f(Z)$ is closed under specialization. Replacing X by the closure $\overline{f(Z)}$ we may assume that f is dominating, and we are to show that f is surjective. The issue being topological, we may assume that both X and Z are reduced. Let $\{U_i\}$ be the covering such that $f^{-1}(U_i)$ is quasi-compact. The image $f(Z)$ is closed if and only if each $f(Z) \cap U_i$ is closed. Hence we may replace X by one of the U_i's and assume that X is affine and that Z is quasi-compact, say $X = \operatorname{Spec} A$. Then Z will be covered by finitely many open affines, and because taking closure commutes with forming finite unions, we may assume that Z is affine as well, say $Z = \operatorname{Spec} B$.

Our morphism f is dominant so $f^\sharp$ injective, and we may consider A as lying in B. For every prime ideal $\mathfrak{p} \subseteq A$ we have to exhibit a prime ideal $q \subseteq B$ such that $\mathfrak{p}$ contains $\mathfrak{q} \cap A$. To that end, consider the multiplicative system $S = A \backslash \mathfrak{p}$ in B. It is disjoint from $\mathfrak{p}B$, hence there is a prime ideal $\mathfrak{q}$ in B maximal among ideals disjoint from S; then by construction, $\mathfrak{q} \cap A \subseteq \mathfrak{p}$, and $\mathfrak{p}$ is a specialization of $\mathfrak{q} \cap A$, which belongs to the image. ❑

Theorem 19.18 (Fundamental theorem of elimination theory) *Every projective morphism is proper.*

Proof: We begin with a few reductions. Any projective morphism factors as the composition of a closed immersion $\iota: X \to \mathbb{P}^n_S$ and a structure map $\pi: \mathbb{P}^n_S \to S$, so it will suffice to show that each $\pi: \mathbb{P}^n_S \to S$ is proper; and it suffices to see that each π is closed. Since being closed is a property of maps local on the target, we may well assume that S is affine, say $S = \operatorname{Spec} A$.

Closed immersions are proper, and compositions of proper maps are proper.

Let $Z \subseteq \mathbb{P}^n_A$ be closed subscheme, which we may assume is reduced and irreducible. It is quasi-compact, and the image $\pi(Z)$ will be closed if and only if it is closed under specialisation (Proposition 19.17 above). So pick a point $x \in S$ that specializes from a point $y \in \pi(Z)$; in other words, we have $x \in \overline{\{y\}} = W$. There is a point z in Z that maps to y. Consider the diagram:

$$\begin{tikzcd} \mathcal{O}_{W,y} \arrow[r, hook] & \mathcal{O}_{Z,z} \arrow[r, hook] & K \\ \mathcal{O}_{W,x} \arrow[u, hook] \arrow[rr, hook] & & R \arrow[u, hook] \end{tikzcd}$$

where the vertical left map is due to x being a specialization of y. Since $\pi(z) = y$, the map $\pi^\sharp$ induces a local homomorphism $\mathcal{O}_{W,y} \to \mathcal{O}_{Z,z}$ which is injective because Z dominates W. Finally, K is the fraction field of $\mathcal{O}_{Z,z}$. In the lower right part of the diagram, R is a valuation ring with fraction field K that dominates $\mathcal{O}_{W,x}$; that is, it holds that $\mathfrak{m}_R \cap \mathcal{O}_{W,x} = \mathfrak{m}_x$. The relevant geometric version of the diagram is as follows:

$$\begin{tikzcd} \mathbb{P}^n_S \arrow[d, "\pi"'] & & \operatorname{Spec} K \arrow[ll] \arrow[d, hook] \\ S & \operatorname{Spec} \mathcal{O}_{W,x} \arrow[l] & \operatorname{Spec} R \arrow[l] \arrow[ull, dashed, "\lambda"'] \end{tikzcd} \tag{19.2}$$

where the map $\operatorname{Spec} R \to S$ sends the generic point to y and the closed point x_0 to x; the map $\operatorname{Spec} K \to S$ sends the unique point in $\operatorname{Spec} K$ to y. Now, the point is that according to the next lemma, we may fill in diagram (19.2) with the dashed map λ. Then $\pi(\lambda(x_0)) = x$, and $x \in \pi(Z)$. ❑

Lemma 19.19 (The Heavenly L'Hopitals rule) *Let R be a valuation ring and K its fraction field. Every morphism $\operatorname{Spec} K \to \mathbb{P}^n_K$ can be extended to a morphism $\operatorname{Spec} R \to \mathbb{P}^n_K$.*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & \mathbb{P}^n_K \\ \operatorname{Spec} R \arrow[ur, dashed] & \end{tikzcd}$$

Proof: As explained in Lemma 10.22 on page 153 the map $\phi\colon \operatorname{Spec} K \to \mathbb{P}^n_K$ is given by 'homogeneous coordinates' $(t_0 : \cdots : t_n)$, where the elements $t_i \in K$ are not all zero. Moreover, they may be changed by a common non-zero factor without the map ϕ changing. Let ν denote the valuation on K corresponding to the valuation ring R, and let t_r be such that $\nu(t_r)$ is the smaller of the $\nu(t_i)$'s. Then $v(t_i t_r^{-1}) = \nu(t_i) - \nu(t_r) \geqslant 0$ and each $t_i t_r^{-1}$ belongs to R. Which means that $(t_1 t_r^{-1} : \cdots : t_n t_r^{-1})$ is an R-point that coincides with ϕ on $\operatorname{Spec} K$, hence it is the desired extension. ❑

Exercises

✲ (19.7) The aim is to show that any local domain is dominated by a valuation ring. Let A be the local domain and K its fraction field.

a) *(Chevalley's lemma)* Let $x \in K$ be a non-zero element such that neither x nor x^{-1} belongs to A. Let $\mathfrak{a}$ be an ideal in A. Show that either $\mathfrak{a}A[x]$ or $\mathfrak{a}A[x^{-1}]$ is a proper ideal.
b) Use Zorn's lemma to show that there is valuation ring $R \subseteq K$ with $A \subseteq R$ and $\mathfrak{m}_R \cap A = \mathfrak{m}_A$.
c) Assume that A is Noetherian, show that A is contained in a DVR. Hint: Consider the integral closure B of A and localize B in a prime ideal of height one.

✲ (19.8) Let A be a domain.

a) Show that $A = \bigcap A_\mathfrak{p}$ where the intersection extends over all prime ideals in A that are associated to a principal ideal.
b) Show that if A is Noetherian and normal, every prime ideal $\mathfrak{p}$ associated to a principal ideal is of height one, and that $A_\mathfrak{p}$ is a DVR.
c) *(Hartog's Extension Theorem)* Conclude that a Noetherian and normal domain A is the intersection of $A_\mathfrak{p}$ with $\operatorname{ht} \mathfrak{p} = 1$, and that each $A_\mathfrak{p}$ is a DVR.

d) Show that if f is a rational function on a normal, integral and Noetherian scheme X which is defined in all points of height one, then f is defined everywhere.

✲ (19.9) This exercise is an application of Hartog's Extension theorem, and may be considered to be a modest prelude to Zariski's Main theorem. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ be two integral affine schemes and assume that Y is Noetherian and normal. Let $f\colon X \to Y$ be a birational morphism that is surjective in codimension one; that is, every point $y \in Y$ such that the prime ideal $\mathfrak{p}_y$ in B is of height one, lies in the image of f. Show that f is an isomorphism.

(19.10) A subset $S \subseteq X$ of a scheme X is *locally closed* if it is the intersection of an open and a closed set. Subsets that are finite unions of locally closed sets, are said to be *constructible*. Show that a constructible set is closed under specialization if and only if it is closed.

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Functions on proper schemes

The observation in Example 19.12 on page 283 inspires a result about a large class of schemes by way of a "hyperbola trick"—strikingly similar to the well-known "Rabinovitsch trick" from the theory of varieties. Turning the question in the example around, one concludes that functions on a scheme proper over an affine scheme $\operatorname{Spec} A$ are very restricted (we saw an example already in Proposition 6.1 on page 89). In fact they will be integral over A, and amazingly, the proof is almost completely formal, just using the hyperbola-trick inspired by Example 19.12.

The proof is quite generally valid without any finiteness condition (although universally closed morphisms are quasi-compact, see Exercise 19.13 below), except for one thing. We need to invert function somewhere on X, so nilpotence is an issue. Nilpotent functions *per se* pose no problems since they are integral over the base (they certainly fulfil an integral equation), but there are schemes having non-nilpotent function which are locally nilpotent, and we have to avoid these schemes. For instance, the disjoint union $X = \bigcup_{i \in \mathbb{N}} \operatorname{Spec} k[t]/t^i$ maps to $\operatorname{Spec} k[t]$ and t is a section in $\mathcal{O}_X(X)$ which is locally nilpotent.

In what follows we shall work in the category Sch/A of schemes over a ring A, and for simplicity we'll drop the subscript A in products and also write $\mathbb{A}^n$ for $\mathbb{A}^n_A$.

The closed subscheme $H \subseteq \mathbb{A}^2$ defined as $H = V(xy - 1)$, and which one could be tempted to call a *generalized hyperbola*, will play a central role. It is isomorphic to $\operatorname{Spec} A[t, t^{-1}]$ via the map $A[x, y]/(xy - 1) \to A[t, t^{-1}]$ defined by the assignments $x \mapsto t$ and $y \mapsto t^{-1}$; observe that both projection $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ sends H into $\mathbb{A}^1 \backslash V(t)$. Moreover, H is also the fibre over 1 of the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ which is induced by the map $A[t] \to A[x, y]$ with $t \mapsto xy$.

Theorem 19.20 *Let X be a scheme such that every locally nilpotent global section of $\mathcal{O}_X$ is nilpotent. Assume that X is universally closed over* $\operatorname{Spec} A$. *Then $\Gamma(X, \mathcal{O}_X)$ is integral over A.*

Note that on both reduced or quasi-compact schemes every locally nilpotent section of $\mathcal{O}_X$ is nilpotent, so they comply with the first reqiurement.

PROOF: Let $B = \Gamma(X, \mathcal{O}_X)$, and suppose that f is a non-zero element in B. We may certainly assume that f is not nilpotent (nilpotent elements are trivially integral), hence f is not locally nilpotent. According to Theorem 5.6 on page 81 we may view f as a map $f\colon X \to \mathbb{A}^1$. Then the distinguished open subset $D(f)$ will be an open subset of $\operatorname{Spec} B$, and $\Gamma(U, \mathcal{O}_X) = B_f$, where U denotes the inverse image of $D(f)$ in X. We may invert f over U to get a map $f^{-1}\colon U \to \mathbb{A}^1$. Moreover, U is non-empty, since f is not locally nilpotent.

This map has a graph $G \subseteq U \times \mathbb{A}^1 \subseteq X \times \mathbb{A}^1$, and the crux of the proof is that G is closed even in $X \times \mathbb{A}^1$ (as a graph it is closed in $U \times \mathbb{A}^1$, the affine line $\mathbb{A}^1$ being separated). We prove this separetely in a Lemma 19.21 below. When X is universally closed over A, the projection $\pi\colon X \times \mathbb{A}^1 \to \mathbb{A}^1$ is a closed map. Consequently (Lemma 19.21) the image $\pi(G)$ will be closed in $\mathbb{A}^1$ and contained in $\mathbb{A}^1 \backslash V(t)$. The ideal $\mathfrak{a}$ of $\pi(G)$ therefore satisfies $\mathfrak{a} + (t) = A[t]$, and we may write $1 = F(t) + tG(t)$ where F and G are polynomials in $A[t]$. Now, that $f^{-1}\colon U \to \mathbb{A}^1$ factors through $V(\mathfrak{a})$, means that $F(f^{-1}) = 0$, and It follows that $1 = f^{-1}G(f^{-1})$, which upon multiplication by a high power of f gives an integral dependence relation for f. ❏

LEMMA 19.21 *The graph G is closed in $X \times \mathbb{A}^1$ and $\pi(G)$ is contained in $\mathbb{A}^1 \backslash V(t)$.*

PROOF: This is where the hyperbola enters the scene. The point is the equality $G = g^{-1}H$ where $g\colon X \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ is given as $g = f \times \mathrm{id}$, in other words in the following diagram where π_2 denotes the second projection, the bottom square is Cartesian:

$$\begin{tikzcd}
\mathbb{A}^1 \arrow[r, "\mathrm{id}"] & \mathbb{A}^1 \\
X \times \mathbb{A}^1 \arrow[u] \arrow[r, "f\times\mathrm{id}"] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[u, "\pi_2"'] \\
G \arrow[u] \arrow[r] & H \arrow[u, "h"']
\end{tikzcd}$$

Indeed, if S is any scheme and $(a,b)\colon S \to X \times \mathbb{A}^1_k$ and $c\colon S \to H$ are morphisms such that $f \times \mathrm{id} \circ (a,b) = h \circ c$ it holds true that $(f \circ a, b) = (c, c^{-1})$. Hence a takes values in U and $(a,b) = (a, (f \circ a)^{-1})$ so (a,b) factors through G. Now, the upper square commutes and $\pi_2(H) \subseteq V(t)$ so $\pi(G)$ lies in $V(t)$ as well. ❏

There is a relation between finite and integral extensions: every finite extension is integral, but the converse does not hold. There are even examples of Noetherian domains A whose integral closure in its field of fraction is not a finite module, or for that matter, in any finite extension of the fraction field. These examples are all in characteristic p. Domains whose integral closure in any finite extensions of the fraction field is finite, are called *Japanese rings* (to honour the mathematical entourage of Nagata, the hotbed of such eye-opening examples). Examples of Japanese rings include domains of finite type over a field or over Dedekind rings of characteristic zero.

Direct from Theorem 19.20 we obtain the following corollary:

COROLLARY 19.22 *Assume that X is an integral scheme that is universally closed over the field K. Then $\Gamma(X, \mathcal{O}_X)$ is an algebraic field extension of K. If X is proper, the extension is finite.*

PROOF: The space is $\Gamma(X, \mathcal{O}_X)$ is an integral domain containing K whose elements all are integral over K according to the theorem; hence it is an algebraic field extension. If X is of finite type, the fraction field $K(X)$ of X is a finitely generated extension of K, and as $\Gamma(X, \mathcal{O}_X)$ is contained in $K(X)$, it is finitely generated over K as well. Being algebraic, it is then finite. ❏

PROPOSITION 19.23 *Let A be a Noetherian Japanese ring, and let X be an integral scheme proper over* $\operatorname{Spec} A$. *Then $\Gamma(X, \mathcal{O}_X)$ is a finite A-module.*

PROOF: We may well assume that the structure map $\pi\colon X \to \operatorname{Spec} A$ is dominating (if not, replace $\operatorname{Spec} A$ by the image of π which is closed and irreducible) with generic point y. Let K be the fraction field of A. The generic fibre $X_y = \pi^{-1}(y) = X \times_{\operatorname{Spec} A} \operatorname{Spec} K$ is proper over $\operatorname{Spec} K$ as properness is a property kept under base change, so the space of global sections $\Gamma(X_y, \mathcal{O}_{X_y})$—which lies contained in in the fraction field $K(X_y)$—is integral over K according to the theorem.

Since X is assumed to be of finite type over A, the generic fibre X_y will be of finite type over K. The fraction field $K(X_y)$ of X_y is therefore a finitely generated field extension of K, and being algebraic, it is a finite extension. By the theorem, $\Gamma(X_y, \mathcal{O}_{X_y})$ is integral over A, and thus contained in the integral closure $\bar{A}$ of A in $K(X_y)$. Therefore it is finite, since $\bar{A}$ is finite and A is Noetherian. ❏

In the same assembly line, we find the corollary that if the genertic fibre X_η of π satisfies $\Gamma(X_\eta, \mathcal{O}_{X_\eta}) = K$ and A is integrally closed, it ensues that $\Gamma(X, \mathcal{O}_X) = A$. One way to ensure that $\Gamma(X_\eta, \mathcal{O}_{X_\eta}) = K$ is to require that X_η be geometrically connected; in other words, that $\bar{X}_\eta = X_\eta \times_{\operatorname{Spec} K} \operatorname{Spec} \bar{K}$ is connected where $\bar{K}$ is an algebraic closure of K. Indeed, then $\Gamma(\bar{X}_\eta, \mathcal{O}_{\bar{X}_\eta}) = \bar{K}$ by the theorem, and since flat base change gives $\Gamma(X_\eta, \mathcal{O}_\eta) \otimes_K \bar{K} = \Gamma(\bar{X}_\eta, \mathcal{O}_{\bar{X}_\eta})$, we deduce that $\Gamma(X_\eta, \mathcal{O}_\eta) = K$. Thus we have shown

PROPOSITION 19.24 *Assume that X is proper over the integrally closed domain A and that the generic fibre is geometrically connected, then $\Gamma(X, \mathcal{O}_X) = A$.*

Proposition 19.25 *Let X and Y be two schemes and assume that Y integral and normal. Let $f\colon X \to Y$ be a proper morphism with geometrically connected generic fibre. Then $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

Proof: Since the open affines form a basis for te topology on Y, it suffices to prove that it holds that $f^\sharp_U\colon \Gamma(U, \mathcal{O}_Y) \to \Gamma(f^{-1}(U), \mathcal{O}_X)$ is an isomorphism for each open affine subset $U \subseteq Y$.

Let $U = \operatorname{Spec} A$. Because Y is assumed to be normal, A is integrally closed in K. Moreover the restriction $f|_{f^{-1}(U)}$ of f to the inverse image $f^{-1}(U)$ is proper and has the same geometric fibre as f, and we may deduce citing xxxx that $f^\sharp\colon A = \Gamma(U, \mathcal{O}_Y) \to \Gamma(f^{-1}(U), \mathcal{O}_X)$ is an isomorphism. ❑

Exercises

(19.11) If X is a scheme and $f \in \Gamma(X, \mathcal{O}_X)$, one says that f is locally nilpotent if for each $x \in X$, there is an integer n_x such that $f^{n_x} = 0$ in $\mathcal{O}_{X,x}$. It may happen that a global section f is not nilpotent but is locally nilpotent. Show that if X is quasi-compact this cannot happen. Let X be the disjoint union $X = \bigcup_n \operatorname{Spec} k[t]/t^n$. Show that t is a locally nilpotent section of $\mathcal{O}_X$ that is not nilpotent.

✲ (19.12) Let $\{K_i\}$ be an infinite collection of field extensions of a field k and let $X = \bigcup_i \operatorname{Spec} K_i$ be the disjoint union of their spectra. Show that X is not universally closed over k.

✲ (19.13) Let X and Y be two schemes. The aim of this exercise is to prove that every universally closed morphism $f\colon X \to Y$ is quasi-compact.

a) Consider, in any topological space X, the set Σ of pairs (U, Z) where $U \subseteq X$ is open and $Z \subseteq U$ is a discrete subset closed in U. Define a partial order on Σ by declaring $(U, Z) \leqslant (U', Z')$ if $U \subseteq U'$ and $Z = Z' \cap U$. Show that the set Σ has a maximal element. Hint: What else than Zorn's lemma;
b) Assume that X is not quasi-compact. Show that there exists a closed infinite and discrete subset Z of X;
c) If Y is affine, and $f\colon X \to Y$ is universally closed, then X is quasi-compact;
d) Conclude that any universally closed morphism is quasi-compact.

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Chapter 20

Divisors and linear systems

It is amazing to which extent the geometry of a variety is governed by the closed subshemes of codimension one, at least if the variety is not to singular. One heuristic reason behind this is that codimension one varieties are closely related to rational functions on X. A rational function has a zero-scheme and a pole-scheme both of codimension one (we emphasize that they are schemes; there might be multiplicities around making them non-reduced). And as we learned in complex analysis, zeros and poles are constituting invariants of a function.

Take, for instance, the quotient of two polynomials $F = f/g$ (without common factors) in $\mathbb{C}[t]$. The zero-scheme equals $V(f) \subseteq \mathbb{A}^1$ and the pole-scheme is $V(g)$. Note that if $(t-a)^n$ is a factor in f, then $V((t-a)^n) = \operatorname{Spec} \mathbb{C}[t]/(t-a)^n$ is a component of $V(f)$, so the roots of f and of g can be recovered from the zero-scheme and the pole-scheme of F. This determines even F up to a scalar factor! And amazingly, any two closed subschemes of $\mathbb{A}^1$ defines a rational function.

There is a convenient notational device: If $x \in \mathbb{P}^1$ the rational function F has an order $\operatorname{ord}_x F$ at x, and the formal sum $\sum_{x \in \mathbb{P}^1} \operatorname{ord}_x(F)x$ (which is a finite sum since F is regular, and non-zero at all but finitely many points) is called the *divisor* of F. A More generally, a *Weil divisor*, is a formal linear combination of irreducible codimension one subscheme with coefficients in $\mathbb{Z}$.

The obvious question then arises: Given a divisor $D = \sum_i n_i x_i$, is there a function having D as divisor (to which extent can we prescribe zeros and poles)? One easily checks that $\sum \operatorname{ord}_x(F) = 0$, and this is in fact when $X = \mathbb{P}^1$ the only condition. For general varieties (not to speak about schemes) the situation is vastly more complicated, but these heuristics remain the core of the theory.

We have seen that closed subschemes are in a one-to-one correspondence with quasi-coherent ideal sheaves. For (integral) subschemes of codimension one, these ideal sheaves tend to be much simpler than for higher codimension. This is essentially because of Krull's Hauptidealsatz, which says roughly that (and under certain hypotheses), such ideal sheaves are generated locally by one element. This is the prototype of a "Cartier divisor"; a subscheme which is locally cut out by one equation.

The prototype example of a divisor is a hypersurface of projective space; that is, an integral subscheme of $\mathbb{P}^n$ of codimension one. Each such subscheme is defined by a homogeneous ideal $\mathfrak{a}$ of $k[x_0, \ldots, x_n]$, and since the codimension is assumed to be one, and because the polynomial ring is factorial, the ideal $\mathfrak{a}$ is principal, *i.e.* $\mathfrak{a} = (f)$ for some homogeneous polynomial $f \in k[x_0, \ldots, x_n]$. To give a concrete example, consider the case of $\mathbb{P}^2$, and the curve

$$D_0 = V(x^3 + y^3 + z^3) \subset \mathbb{P}^2_k$$

This is clearly integral (if the characteristic of k is not three), since the defining equation is irreducible. Similarly, we can consider the subscheme

$$D_1 = V(xyz)$$

This subscheme is reduced, but not irreducible: D_1 has three irreducible components $V(x)$,$V(y)$ and $V(z)$.

The main idea of divisors is that one can talk about sums and differences of such subschemes, thereby turning them into a *group*. This is illustrated in the example above, by writing $D_1 = V(x) + V(y) + V(z)$. The sum here is completely formal — it is an element in the free group on integral subschemes of codimension one. Such a sum is by definition a *Weil divisor*.

There is an equivalence relation defined on such objects, designed to capture when two divisors belong to the same family. In the example above, $x^3 + y^3 + z^3$ and xyz define sections of the same invertible sheaf $\mathcal{O}(3)$, and span a subspace of the k-vector space $\Gamma(\mathbb{P}^n, \mathcal{O}(3))$. Geometrically, D_0 and D_1 are in fact connected by a family of subschemes in $\mathbb{P}^2$, namely

$$D_{(s:t)} = V(sxyz + t(x^3 + y^3 + z^3)),$$

where s and t are parameters. More precisely, there is a closed subscheme $D \subset \mathbb{P}^1 \times \mathbb{P}^2$ defined by the above equation, so that the fibre over a closed point $(s : t) \in \mathbb{P}^1$ is exactly the curve $D_{(s:t)}$ in $\mathbb{P}^2_k$. The intuitive picture is that we have a family of 'moving divisors', parameterized over the projective line. The key feature is that there is a morphism $f : D \to \mathbb{P}^1$, and $f^{-1}((0 : 1)) = D_0$ and $f^{-1}((1 : 0)) = D_1$. Moreover, the quotient

$$g = \frac{x^3 + y^3 + z^3}{xyz}$$

defines a rational function on $\mathbb{P}^2$. This is the pullback of the rational function s/t on $\mathbb{P}^1$: $g = f^*(s/t)$. We can use this to define an equivalence relation on divisors by declaring $D_0 \sim D_1$ if there is a rational function g so that g has zeroes along D_0 and poles along D_1 (counting with multiplicities).

The theory of divisors is closely connected to invertible sheaves. This link is motivated by the fact that integral subschemes of codimension one are typically defined by a single equation locally. Note that in the above example, the special properties of projective space imply that D_0 and D_1 are defined *globally* by a single equation. It is not hard to come up with examples of schemes with subschemes of codimension one that are not. In fact, for most schemes, the concept of a 'globally defined equation' does not make sense, since we do not have global coordinates to work with. However, *locally* this concept makes sense: we can consider subschemes $Y \subset X$ so that the ideal sheaf I_Y is locally generated by a single element $f_i \in A_i$, on some affine covering $X = \bigcup \operatorname{Spec} A_i$. In other words, the ideal sheaf I_Y is an invertible sheaf.

Conversely, given a non-zero section s of an invertible sheaf L, we can consider the zero set of s, i.e., the subscheme $Z(s)$ defined in section XXX. This subscheme has codimension 1, and gives rise to a divisor on X.

While Weil divisors are conceptual and geometric, working with invertible sheaves has some advantages. For instance, given a morphism $f\colon X \to Y$, we would like to define a 'pullback' of a divisor on Y to a divisor on X — this turns out to be much simpler when working with invertible sheaves. There are also other settings where the special properties of invertible sheaves are essential, for instance defining intersection products.

With a few exceptions, we will in this chapter assume that schemes are *integral and Noetherian*. Noetherianness is somewhat important, since we want to talk about the decomposition of a closed subschemes into its irreducible components. The assumption of integrality, especially irreducibility, is not essential for most of the statements about Cartier divisors, but it makes the definitions more transparent, and more importantly, the proofs considerably simpler (*e.g.* not having to worry about nilpotent elements or diverse components of possibly different dimensions and the zerodivisors they introduce, allows us to work with meromorphic functions as elements in a fraction field, rather some more obscure localization). Weil divisors, however, are more or less doomed to live on normal and integral schemes — there are general definitions around, but they are hardly workable unless the scheme is normal and integral.

To prove the results we want, we will need the following two facts from commutative algebra:

Theorem 20.1 (Hartog's extension theorem) *Let A be a normal integral domain. Then*

$$A = \bigcap_{ht(\mathfrak{p})=1} A_{\mathfrak{p}} \tag{20.1}$$

where the intersection is taken inside the fraction field of A.

Theorem 20.2 *A noetherian integral domain A is a unique factorization domain if and only if every prime ideal $\mathfrak{p}$ of height 1 is principal;*

Even when working with Noetherian integral domains, one should be careful with the notion of codimension. In Definition 7.14 on page 110, we defined the codimension for closed subsets of general topological spaces, and for a closed irreducible subset $V(\mathfrak{p})$ of $\operatorname{Spec} A$ this boils down to being the height of $\mathfrak{p}$. Note that this codimension does not always agree with the 'naive codimension' $\dim A - \dim A/\mathfrak{p}$ (see the discussion following Definition 7.14).

When working with domains of essential finite type over a field or over the integers, however, one does not encounter this problem: for such rings it holds true that

$$\dim A/\mathfrak{p} + \operatorname{ht}\mathfrak{p} = \dim A$$

for all prime ideals.

20.1 Subschemes of codimension one

The main theme of this section is to study closed subschemes of codimension one. We are particularly interested in the subschemes which are locally defined by a single equation which is a nonzerodivisor, *i.e.* subschemes whose ideal sheaf is an invertible sheaf. We will see that this is often the case, at least when the ambient scheme X has mild singularities. However, we shall also see simple examples of subschemes which are not possible to define locally by just one equation (see for instance Example 20.5).

For now, let us note the following characterisation of such subschemes:

Proposition 20.3 *Let X be a scheme and let $D \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I}$. Then the following are equivalent:*

i) *$\mathcal{I}$ is an invertible sheaf;*

ii) *For every $x \in X$, the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ is principal and generated by a nonzerodivisor;*

iii) *There is an open covering U_i of X and nonzerodivisors $f_i \in \mathcal{O}_X(U_i)$ such that f_i generates $\mathcal{I}(U_i)$;*

iv) *For every $x \in X$, there is an open affine neighbourhood $U = \operatorname{Spec} A$ of x such that $U \cap D = \operatorname{Spec} A/(f)$ where $f \in A$ is a nonzerodivisor.*

Proof: Clearly the first three conditions are equivalent; this is just a restatement of what it means to be locally free of rank one. Let us show the equivalence *i)* $\Leftrightarrow$ *iv)*.

We begin with the implication *i)* $\Rightarrow$ *iv)*. Let $x \in X$ and pick an open affine set $V = \operatorname{Spec} A \subset X$ so that $\mathcal{I}_D|_V \simeq \mathcal{O}_V$. This means that there is an element $f \in \mathcal{I}_D(V) \subset A = \mathcal{O}_X(V)$ which is an A-basis for $\mathcal{I}_D|_V$, and in particular, f must be a nonzerodivisor. Moreover, $D \cap V$ is the subscheme of $\operatorname{Spec} A$ defined by f, so that $D \cap V = \operatorname{Spec} A/(f)$.

For the implication *iv)* $\Rightarrow$ *i)* we need to show that $\mathcal{I}$ is invertible near every point $x \in X$. Pick an open affine $U = \operatorname{Spec} A$ neighbourhood of x so that $D \cap U = \operatorname{Spec} A/(f)$, for some nonzerodivisor f. Then $\mathcal{I}|_U \simeq \widetilde{(f)} \simeq \widetilde{A} \simeq \mathcal{O}_U$, which means that $\mathcal{I}$ is invertible. ❑

Definition 20.4 (Effective Cartier divisors) *A subscheme D satisfying the conditions of Proposition 20.3 is called an* effective Cartier divisor.

Explicitly, an effective Cartier divisor D is determined by the following data:

i) an open covering U_i;

ii) nonzerodivisors $f_i \in \mathcal{O}_X(U_i)$ that generate $\mathcal{I}_D|_{U_i}$ over U_i.

We shall call such data a set of *Cartier data* for D and denote it by $\{(U_i, f_i)\}_{i \in I}$. With this definition, we notably also accept the empty subscheme as an effective Cartier divisor; it corresponds to the unit ideal $\mathcal{O}_X$, or to the Cartier data $(X, 1)$. *Cartier data*

Informally, we say that the f_i's are *local equations* for D. Note that the collection of the f_i's cannot be chosen completely arbitrary here. If we restrict f_i and f_j to $U_i \cap U_j$, they should generate the same ideal in $\mathcal{O}_X(U_i \cap U_j)$, which means that

$$f_j = c_{ji} \cdot f_i \tag{20.2}$$

for some units $c_{ji} \in \mathcal{O}_X^\times(U_i \cap U_j)$. These units satisfy the *cocycle* condition

$$c_{jk} = c_{ji} \cdot c_{ik} \tag{20.3}$$

on $U_i \cap U_j \cap U_k$ for each triple i, j, k. Indeed, inserting the relations $f_j = c_{jk} f_k$ and $f_i = c_{ik} f_k$ in (20.2) and cancelling f_k, we get (20.3).

Note that the above data defining D is not unique: D may well be represented by a different covering V_j and elements $g_j \in \mathcal{O}_X(V_j)$. However, on intersections $U_i \cap V_j$, the elements g_j and f_i must be equal up to multiplication by a unit in $\mathcal{O}_X(U_i \cap V_j)$.

Example 20.5 Let $X = \mathbb{A}^n_k$ over a field k and let G be a polynomial. Then $D = V(G)$ is an effective Cartier divisor. It is specified by the obvious data $(\mathbb{A}^n, G)$. ☆

Example 20.6 Let $X = \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$ over a field k and let P be the point $(1:0)$. Using the standard covering $D_+(x_0)$ and $D_+(x_1)$, we see that P is the effective Cartier divisor determined by the data $(D_+(x_0), x_1 x_0^{-1})$ and $D_+(x_1, 1)$. Note that on the intersection $D_+(x_0) \cap D_+(x_1)$ the function $x_1 x_0^{-1}$ is invertible, so the data yields an effective Cartier divisor.

On the open set $D_+(x_0) = \operatorname{Spec} k[x_1 x_0^{-1}] = \mathbb{A}^1_k$ the ideal is generated by $x_1 x_0^{-1}$ which defines the point P, and on $D_+(x_1)$ the local equation is 1 which is without zeros, so the devisor defined is exactly P.

We might also consider the data $(D_+(x_0), (x_1 x_0^{-1})^n)$ and $(D_+(x_1), 1)$. In the distinguished open set $D_+(x_0) = \operatorname{Spec} k[x_1 x_0^{-1}]$ it gives the ideal $((x_1 x_0^{-1})^n)$ which defines a subscheme supported at P and of length n, and in $D_+(x_1)$ the ideal will be the unit ideal, whose zero set is empty. We denote the corresponding divisor by nP. ☆

Theorem 20.7 *Let X be a Noetherian integral scheme. Then the following two statements are equivalent:*

i) Every integral subscheme of codimension one is an effective Cartier divisor;

ii) X is locally factorial (that is, the local rings $\mathcal{O}_{X,x}$ are all UFD's).

Proof: Both conditions can be checked locally, so we may assume that $X = \operatorname{Spec} A$ is affine. Let $D = \operatorname{Spec}(A/\mathfrak{q})$ be an integral subscheme. Saying that D has codimension one is equivalent to saying that $\mathfrak{q}$ is a prime ideal of height one. If $\operatorname{Spec} A$ is factorial, then each $A_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} A$ is a UFD and $\mathfrak{q}A_\mathfrak{p}$ is principal according to Theorem 20.2. One may extend the generator to a generator for $\mathfrak{q}$ over a neighbourhood of $\mathfrak{p}$, and thence by Proposition 20.3, D will be an effective Cartier devisor.

For the converse, assume *i)*. Note first that for each $\mathfrak{p} \in \operatorname{Spec} A$ every prime ideal in $A_\mathfrak{p}$ is of the form $\mathfrak{q}A_\mathfrak{p}$ for a prime $\mathfrak{q}$ in A, and when $\mathfrak{q}A_\mathfrak{p}$ is of height one, $\mathfrak{q}$ is also of height one (there is a one-to-one correspondence between primes in A lying in $\mathfrak{p}$ and primes in

$A_\mathfrak{p}$). Consequently, if $\mathfrak{q}A_\mathfrak{p}$ is of height one, $\mathfrak{q}$ is locally principal by *i)*, which means that $\mathfrak{q}A_\mathfrak{p}$ is principal. ❑

EXERCISE 20.1 Given data $\{(U_i, f_i)\}$ as in 20.4. Assume that there are units $c_{ij} \in \mathcal{O}_X(U_i \cap U_j)$ with $f_j = c_{ji} f_i$ which satisfy the cocycle condition. Show that the data then defines a sheaf of invertible ideals. ★

EXERCISE 20.2 Check that the ideal sheaf $\mathcal{I}_{nP}$ of the divisor nP in Example 20.6 is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-n)$. ★

EXERCISE 20.3 Describe Cartier data that defines the hyperplane $V(x_i)$ in $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n]$. ★

Zero set of sections of invertible sheaves

The most important examples of effective Cartier divisors are the following. And, in fact, as it turns out, they describe every effective Cartier divisor.be a

Let X a scheme which we assume is Noetherian and reduced. Then X is the union $X = X_1 \cup \cdots \cup X_r$ of its finitely many irreducible components, and these are all integral schemes. The total ring of fractions K of X is given as $K = K_1 \times \cdots \times K_r$ where each $K_i = K(X_i)$ is the function field of X_i. An element f of K is invertible if and only if it is a nonzerodivisor, and this occurs precisely when each component of f is nonzero; in other words, when f does not vanish on any of the components of X.

Let L be an invertible sheaf on X. Each global section s of L yields a map $\mathcal{O}_X \to L$, which for open sets U sends $a \in \mathcal{O}_X(U)$ to $a \cdot s|_U$, and we say that s is a *regular section* if this injective. In the case that X is integral, this amounts to s being non-zero. *Regular section*

Recall from Section 16.6 on page 241 that each non-zero global section $s \in \Gamma(X, L)$ of L gives rise to a closed subscheme

$$V(s) \subset X$$

as the locus where s vanishes. One exhibits the coherent ideal $\mathcal{I}$ defining $V(s)$ as explained in Section **??**, it will be the image of the map $s^\vee \colon L^\vee \to \mathcal{O}_X$, dual to the map $\mathcal{O}_X \to L$ arising from s. In other words:

$$\mathcal{I} = \operatorname{Im}(s^\vee \colon L^\vee \to \mathcal{O}_X) \subseteq \mathcal{O}_X.$$

This is a coherent sheaf of ideals, and the corresponding subscheme is precisely $V(s)$. Note that the map $s^\vee$ is injective (the restriction of s to each X_i is non-zero and any non-zero map between invertible sheaves on an integral scheme is). This implies that the ideal sheaf $\mathcal{I}$ is in fact isomorphic to $L^\vee$. So in particular, $V(s)$ is an effective Cartier divisor D on X.

We can make this a little bit more explicit by writing down the Cartier data for D. Pick a covering of X by open sets U_i so that $L|_{U_i}$ is trivial, in other words, there are isomorphisms $\phi_i \colon \mathcal{O}_{U_i} \to L|_{U_i}$. On each U_i, we may write $s|_{U_i} = \phi_i(f_i)$ for some $f_i \in \mathcal{O}_X(U_i)$ which is a nonzerodivisor because s is regular. The data (U_i, f_i) then define the effective Cartier divisor D.

There is a converse to this construction. Given an effective Cartier divisor D with ideal sheaf $\mathcal{I}$. We may consider the dual sheaf

$$L = \mathcal{I}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X),$$

which again is invertible. The sheaf L admits a distinguished section $s \in \Gamma(X, L)$ that corresponds to the inclusion map $\mathcal{I} \to \mathcal{O}$. If (U_i, f_i) are local equations for D, we note that $s^\vee|_{U_i}$ corresponds to the map $\mathcal{O}_{U_i} \to \mathcal{O}_{U_i}$ which sends 1 to f_i. The image of $s^\vee$ over $\mathcal{I}(U_i)$ is therefore the ideal generated by f_i. Thus D is exactly the zero set $V(s)$. We therefore obtain:

Proposition 20.8 *Let X be a reduced Noetherian scheme. Then:*

i) Each effective Cartier divisor $D \subset X$ is the zero set $V(s)$ of some regular section $s \in \Gamma(X, L)$ of some invertible sheaf L;

ii) Two regular sections $s, t \in \Gamma(X, L)$ give rise to the same divisor if and only if $t = \lambda s$ for some unit $\lambda \in \mathcal{O}_X^\times(X)$.

Proof: We only need to prove the last statement. Suppose that s and t define the same ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$, so that we have isomorphisms

$$L^\vee \xrightarrow{s^\vee} \mathcal{I} \xrightarrow{(t^\vee)^{-1}} L^\vee.$$

Note by Proposition **??** it holds that $\mathscr{H}om_{\mathcal{O}_X}(L^\vee, L^\vee) \simeq \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{O}_X$ so that $\mathrm{Hom}_{\mathcal{O}_X}(L^\vee, L^\vee) = \Gamma(X, \mathcal{O}_X)$. Hence each isomorphism $L^\vee \to L^\vee$ is given by multiplication by some element in $\mathcal{O}_X^\times(X)$. Thus s and t differ only by a unit. ❑

20.2 *Weil divisors*

The underlying idea of divisors comes from the order of vanishing of a function, and the ideal principle would be that two functions with same orders of vanishing should be related by a unit. The correspondence is a geometric one; that is, it goes by the vanishing loci of the functions, and this requires non-reduced schemes as we aim at capturing multiplicities (the notion of 'multiplicity' will be defined below). So one problem is that an irreducible subset Y can carry several distinct non-reduced structures with the same multiplicitiy.

Example 20.9 A simple instance is the rational double point $y^2 = x^2(x+1)$. Cutting it with the x-axis we get the subscheme $\operatorname{Spec} k[x,y]/(x^2, y)$ whereas cutting it with the y-axis yields $k[x,y]/(x,y^2)$. The two differ, and both are of length two. ☆

To avoid these kind of problems, we make the assumption that X is normal. Recall that this means that all local rings $\mathcal{O}_{X,x}$ are integrally closed in the function field $K = k(X)$. One of the most important features of a Noetherian normal scheme is that it is *regular in codimension one*; that is, all the local rings $\mathcal{O}_{X,x}$ of dimension one are DVR's. These are the just local rings $\mathcal{O}_{X,\eta}$ where η is the generic point of an irreducible and reduced subscheme, say Y, of codimension one. We call such a subscheme a *prime divisor*. *prime divisor*

Let Z be a subscheme of the normal scheme X and let $\mathcal{I}$ be the coherent ideal that defines Z. The ideal $\mathcal{I}\mathcal{O}_{X,\eta}$ equals a power $\mathfrak{m}^n$ of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,\eta}$. We call n the *multiplicity* of Z at η or the *multiplicity* of Y in Z and shall write $n_Y(Z)$ for it. Certainly it might be that $n_Y(Z) = 0$, but this occurs only when Y is a not a component of Z. In this way we may associate the formal sum $\sum_Y n_Y(Z)Y$ to Z where the summation extends over all prime divisors Y in X. Since X is Noetherian, Z has only finitely many irreducible components, and so the sum is finite. This leads to

multiplicity of a prime divisor in a subscheme

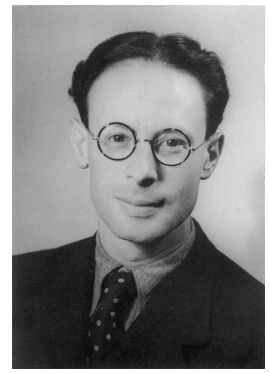
DEFINITION 20.10 (WEIL DIVISORS) *Let X be a Noetherian normal integral scheme.*

i) *A* Weil divisor *is a finite formal sum $D = \sum_i n_i Y_i$ where $n_i \in \mathbb{Z}$ and the Y_i's are prime divisors;*

ii) *We let* $\mathrm{Div}(X)$ *denote the group of Weil divisors; this is the free abelian group on prime divisors;*

iii) *We say that D is* effective *if all $n_i \geqslant 0$ and call $\bigcup_i Y_i$ the* support *of D.*

Weil divisors are named after the french mathematician André Weil; one of the founding fathers of modern algebraic geometry.



André Weil (1907– 1998)
French mathematician

The main reason why normal schemes are so desirable, is that there is one-to-one correspondence between subschemes of pure codimension one and effective Weil divisors. (Recall that a subscheme is of *pure codimension one* if all its irreducible components are of codimension one.)

PROPOSITION 20.11 *Let X be a Noetherian integral and normal scheme. Let Z and Z' be two subschemes of pure codimension one. Then $Z = Z'$ if and only if they define the same Weil divisor.*

PROOF: We may assume that X is affine*, say $X = \operatorname{Spec} A$, where A is a Noetherian normal domain. The point is that if $\mathfrak{p}$ is a height one prime in A, the only $\mathfrak{p}$-primary ideals are the symbolic powers $\mathfrak{p}^{(\nu)}$ for $\nu \in \mathcal{N}_0$ (see for instance Proposition **??** in CA). Don't get scared by these seemingly occult powers, their important feature is that $\mathfrak{p}^{(v)}$ is the only $\mathfrak{p}$-primary ideal which defines a subscheme whose multiplicity at $\mathfrak{p}$ is n.

Bing equal is a local property for closed subschemes: If $\{U_i\}$ is an open cover and $Z \cap U_i = Z' \cap U$ for all i, it holds that $Z = Z'$.

That a closed subscheme $Z = V(\mathfrak{a})$ of X is of pure codimension one, means that all the associated primes of $\mathfrak{a}$ are of height one; and in view of the discussion above, the primary decomposition of $\mathfrak{a}$ takes the form $\mathfrak{a} = \mathfrak{p}_1^{(\nu_1)} \cap \cdots \cap \mathfrak{p}_r^{(\nu_r)}$, where the ν_i's precisely are the non-zero multiplicities $n_Z(Y)$ with Y a prime divisor. So two ideals with the same multiplicities are equal. ❏

EXAMPLE 20.12 Here are a few divisors on $X = \mathbb{P}^1_k$:

$$D_1 = 3 \cdot (1:0) - 5 \cdot (0:1), \qquad D_2 = (1:1) + 5 \cdot (1:3)$$
$$D_1 + D_2 = 3 \cdot (1:0) + (1:1).$$

☆

The Weil divisors have a natural order making Div X a partial ordered set. Give two Weil divisors $D = \sum_Y n_Y Y$ and $D' = \sum n'_Y Y$, we say that $D \geqslant D'$ if $n_Y \geqslant n'_Y$ for all prime divisors Y, or equivalently, that $D - D' \geqslant 0$.

The divisor of a rational function

We continue working with a Noetherian normal integral scheme X. If $Y \subset X$ is a prime divisor with generic point $\eta_Y \in X$, the local ring $\mathcal{O}_{X,\eta_Y}$ is a discrete valuation ring, to which corresponds a normalized valuation $v_Y \colon K^\times \to \mathbb{Z}$. The concept of a valuation is a generalization of the 'order' of a zero or a pole of a meromorphic function in complex analysis. Intuitively, an element $f \in K^\times$ has positive valuation n if it vanishes to the order n along Y, and negative valuation $-n$ if it has a pole of order n there. Finally, the valuation is zero precisely when f is regular and non-vanishing near the generic point of Y; that is, in an open dense set of X meeting$Y$.

Recall that a discrete valuation ring (or a DVR in acronymic jargon) A is a local principal domain (for details see Appendix A). A generator t of the maximal ideal $\mathfrak{m}$ is called *uniformizing parameter* and has the property that every non-zero element in the fraction field K of A is of the form $\alpha t^{v(f)}$ with α a unit and $v(f) \in \mathbb{Z}$. The exponent $v(f)$ is uniquely defined by f (indeed, it equals the largest integer n so that $f \in \mathfrak{m}^n$) and yields a function $v \colon K^\times \to \mathbb{Z}$; the valuation associated with A. Note that v has the following two properties *Uniformizing parameter*

i) $v(fg) = v(f) + v(g)$;

ii) $v(f+g) \geqslant \min v(f), v(g)$.

One may recover the the ring A, the maximal ideal and the units $A^\times$ from the valuation v in that $A = \{\, f \in K \mid v(f) \geqslant 0 \,\} \cup \{0\}$, $\mathfrak{m} = \{\, f \in K \mid v(f) > 0 \,\} \cup \{0\}$ and $A^\times = \{\, f \in A \mid v(f) = 0 \,\}$.

Coming back to the Weil divisors, that the ring $\mathcal{O}_{X,\eta_Y}$ is a DVR, enables us to speak about *the order of a rational function along* Y, namely the value $v_Y(f)$, and this leads to the definition of the divisor of a non-zero rational function:

DEFINITION 20.13 (PRINCIPAL DIVISORS) *If $f \in K^\times$, we define its corresponding Weil divisor as*

$$\operatorname{div}(f) = \sum_Y v_Y(f) Y$$

where the sum extends over all integral subvarieties of codimension one. Divisors of the form $\operatorname{div}(f)$ *are called* principal divisors*, and together with* 0 *they form a subgroup of* $\operatorname{Div}(X)$.

The statement that the principal divisors form a subgroup, follows from the standard property of valuations that $v(fg) = v(f) + v(g)$ and that $v(f^{-1}) = -v(f)$; the latter follows from the former with $f = g^{-1}$.

In the definition above, the sum runs over all prime divisors of X, and this sum is well defined by the following lemma:

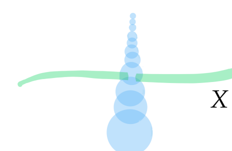
Lemma 20.14 *Suppose that X is an integral normal Noetherian scheme with fraction field K and let $f \in K^\times$. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y.*

Proof: We first reduce to the case when X is affine. Let $U = \operatorname{Spec} A$ be an open affine subset such that $f|_U \in \Gamma(U, \mathcal{O}_X)$. Since X is Noetherian and integral, the complement $Z = X - U$ is a closed subset of X of codimension at least one, which has finitely many irreducible components; in particular, only finitely many prime divisors Y are supported in Z. So we reduce to the affine case $X = \operatorname{Spec} A$ and $f \in \Gamma(X, \mathcal{O}_X) = A$, by ignoring these finitely many components. Then $v_Y(f) \geqslant 0$ automatically, and $v_Y(f) > 0$ if and only if Y is contained in $V(f)$; and since $V(f)$ has only finitely many irreducible components of codimension one, we are done. ❑

The proof of the lemma above shows where we make use of some of the finiteness assumptions on our schemes. Unfortunately, there is no getting around it, as the first of the next examples shows.

Examples

(20.15) Imitating the construction of the affine line with two origin, we can construct the *affine line X with infinitely many origins*: this scheme is integral, normal, locally Noetherian with fraction field $k(t)$, but there are infinitely many closed points $p \in X$ for which $v_p(t) = 1$.



(20.16) Let k be algebraically closed, and consider $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$. Let $K = k(x)$. Here prime divisors in X correspond to closed points $[a] \in \mathbb{A}^1_k$ associated to maximal ideals $(x-a)$ in $k[x]$. Let $f = x^2(x-1)(x+1)^{-1} \in K$. Then $v_{[a]}(f) = 0$ for all a except when $a = 0, \pm 1$, where we have

$$v_{[0]}(f) = 2, \quad v_{[1]}(f) = 1, \quad v_{[-1]}(f) = -1.$$

Hence the divisor of f is $2[0] + [1] - [-1]$.

(20.17) In fact, when k is an algebraically closed field, each divisor on $\mathbb{A}^1_k$ is principal. Indeed, if the divisor is $D = \sum_{i=1}^n n_i[p_i]$ where $n_i \in \mathbb{Z}$ and $p_i \in k$, then the rational function

$$f = \prod_{i=1}^n (t - p_i)^{n_i}$$

has $\operatorname{div}(f) = D$.

(20.18) On $X = \operatorname{Spec} \mathbb{Z}$, a Weil divisor is an expression of the form

$$D = n_1 V(p_1) + \cdots + n_1 V(p_r)$$

where the p_i are prime numbers. Note that $f = p_1^{n_1} \cdots p_r^{n_r}$ is a non-zero element of $K(\mathbb{Z}) = \mathbb{Q}$, which has $\operatorname{div} f = D$. Thus any divisor is principal also in this case.

(20.19) Consider the projective line $X = \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$ whose function field is $K = k(t)$ where $t = x_1/x_0$. Consider the function $f = t^2(t-1)^{-1} \in K$. To find the divisor of f, we treat the two affine charts $D_+(x_0)$ and $D_+(x_1)$ separately:

On $U = D_+(x_0) = \operatorname{Spec} k[t]$, the function f is an element in $\mathcal{O}_{X,p}$ for every $p \neq [1]$, and it is invertible for every $p \neq [0], [1]$. At the point $p = [0]$, the local ring equals $\mathcal{O}_{X,p} = k[t]_{(t)}$, and $t - 1$ is invertible in it, so have

$$f = t^2(t-1)^{-1} = (\text{unit}) \cdot t^2,$$

and f has valuation 2.

At the point $p = [1] \in U$, we have $\mathcal{O}_{X,p} = k[u]_{(u)}$ where $u = t - 1$. Here we may write

$$f = t^2(t-1)^{-1} = (u+1)^2 u^{-1} = (\text{unit}) \cdot u^{-1}$$

and it follows that $v_{[1]}(f) = -1$.

On the open patch $U = D_+(x_1) = \operatorname{Spec} k[u]$, where $u = x_0/x_1 = t^{-1}$, we may write $f = u^{-2}(u^{-1} - 1) = (u - u^2)^{-1}$. The only non-zero valuations are: $v_{[0]} = -1$ and $v_{[1]} = -1$. Note that the point $[1] \in D_+(x_1)$ is the point $(1:1)$ which we found also in $D_+(x_0)$ above. It follows that the divisor of f is given by

$$\operatorname{div}(f) = 2(1:0) - (1:0) - (1:1).$$

(20.20) One may consider the function from Example 20.16 as a rational function on $\mathbb{P}^1_k$. In addition to the prime divisors $(a:1)$ from the Exercise 20.16, we have the point $(1:0)$ at infinity. The function f will have pole of order two at infinity, so that

$$\operatorname{div} f = 2(0:1) + (1:1) - (-1:1) - 2(1:0)$$

Indeed, near $\infty = (1:0)$ we may use $t = x^{-1}$ as a parameter, and expressed in t, the function f becomes $f = t^{-2}(t^{-1} - 1)(t^{-1} + 1)^{-1} = t^{-2}(1-t)(1+t)^{-1}$, which vanishes to the order two at $t = 0$.

☆

Example 20.21 Let X be the curve $V(y^2 - x^3 - 1) \subset \mathbb{A}^2_k$. Then x, y and x^2/y define rational functions on X. We have $\operatorname{div} x = (0,1) + (0,-1)$ and $\operatorname{div} y = (-1,0) + (\omega,0) + (\omega^2,0)$ where ω is a primitive third root of -1, and

$$\operatorname{div}(x^2/y) = 2(0,-1) + 2(0,1) - (-1,0) - (\omega,0) - (\omega^2,0)$$

☆

The class group

One of the fundamental invariants of a scheme (or of a ring) is the class group (or its near relatives the Picard group and the ideal class group). The term 'class group' comes from algebraic number theory and its origins can be traced back to Kummer's work on Fermat's last theorem. Algebraic number theory was, and to some extent still is, largely devoted to the determination of class groups of the ring of integers in algebraic number fields. One may view the class group as the group where one finds the obstructions for a divisor being the divisor of a rational function.

Definition 20.22 (The class group) *We define the* class group of X *as*

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\{\,\mathrm{div}\, f \mid f \in K^\times\,\}.$$

Two Weil divisors D and D' are said to be linearly equivalent *(written $D \sim D'$) if they have the same image in* $\mathrm{Cl}(X)$*, or equivalently, that $D - D'$ is principal.*

Example 20.23 It follows from Example 20.17 that any Weil divisor on $\mathbb{A}^1_k$ is principal, hence $\mathrm{Cl}(\mathbb{A}^1_k) = 0$. Similarly, it holds that $\mathrm{Cl}(\operatorname{Spec}\mathbb{Z}) = 0$. ☆

If A is a Dedekind domain then $\mathrm{Cl}(\operatorname{Spec} A)$ coincides with the ideal class group $\mathrm{Cl}(A)$ of A, which measures how far A is from being a unique factorization domain. We saw earlier that $\mathrm{Cl}(\operatorname{Spec}\mathbb{Z}) = 0$, which is consistent with $\mathbb{Z}$ being a UFD. On the other hand, $\mathbb{Z}[\sqrt{-5}]$ is not an UFD, because $2 \cdot 3 = (1-\sqrt{-5})(1+\sqrt{-5}))$, and in fact $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2\mathbb{Z}$.

The group of Weil divisors $\mathrm{Div}(\operatorname{Spec} A)$ being the free abelian group on the prime divisors, has a basis consisting of the divisors $V(\mathfrak{p})$ for $\mathfrak{p}$ running through the primes of height one. The divisor $\mathrm{div}(f)$ of a rational function f is given as $\mathrm{div}(f) = \sum v_{\mathfrak{p}(f)} V(\mathfrak{p})$, so $\mathrm{div}\, f = 0$ if and only if $v_{\mathfrak{p}}(f) = 0$ for all height one primes. Each $A_{\mathfrak{p}}$ is a discrete valuation ring so $v_{\mathfrak{p}}(f) = 0$ means that $f \in A_{\mathfrak{p}}^\times$, and in view Hartog's Extension Theorem, which tells us that $\bigcap_{\mathfrak{p}} A_{\mathfrak{p}}^\times = A^\times$, we conclude that $\mathrm{div}(f) = 0$ if and only if $f \in A^\times$.

Thus the kernel of the map $\mathrm{div}\colon K^\times \to \mathrm{Div}(\operatorname{Spec} A)$ equals $A^\times$, and the cokernel is by definition the class group $\mathrm{Cl}(A)$. Hence we have the exact sequence

$$0 \to A^\times \to K^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\operatorname{Spec} A) \to \mathrm{Cl}(A) \to 0. \tag{20.4}$$

Proposition 20.24 *Let A be a Noetherian integral domain and let $X = \operatorname{Spec} A$. Then the following are equivalent:*

i) $\mathrm{Cl}(X) = 0$ *and X is normal;*

ii) Every height one prime ideal in A is principal;

iii) A is a unique factorization domain.

Proof: The equivalence of *ii*) and *iii*) is just Theorem 20.2, so the task is to show that statement *i)* is equivalent to one of the two other statements; we shall show the equivalence *i)* $\Leftrightarrow$ *ii)*.

ii) $\Rightarrow$ *i)*: If $Y \subset X$ is a prime divisor in $\operatorname{Spec} A$, then $Y = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$ of height one. Hence by assumption $Y = V(f)$ for an element $f \in A$, that is, $Y = \mathrm{div}(f)$, and so $\mathrm{Cl}(X) = 0$. Moreover, any unique factorization domain is normal.

i) $\Rightarrow$ *ii)*: Assume that $\mathrm{Cl}(X) = 0$. Let $\mathfrak{p}$ be a prime of height one, and let $Y = V(\mathfrak{p}) \subset X$. By assumption, there is an $f \in K^\times$ such that $\mathrm{div}(f) = Y$. We want to show that in fact $f \in A$ and that $\mathfrak{p} = (f)$. But the former follows from the exact sequence (20.4), since $v_{\mathfrak{q}}(f) = 0$ for $\mathfrak{q} \neq \mathfrak{p}$ and $v_{\mathfrak{p}}(f) = 1$, and so f lies in $\{a \in K^\times | v_{\mathfrak{p}}(a) \geqslant 0\} = A^\times$.

Finally, to prove that f generates $\mathfrak{p}$, consider any element $g \in \mathfrak{p}$. Then $v_{\mathfrak{p}}(g) \geqslant 1$ and $v_{\mathfrak{q}}(g) \geqslant 0$ for all $\mathfrak{q} \neq \mathfrak{p}$. It follows that $v_{\mathfrak{q}}(g/f) = v_{\mathfrak{q}}(g) - v_{\mathfrak{q}}(f) \geqslant 0$ for *all* prime ideals

$\mathfrak{q} \in \operatorname{Spec} A$. Hence $g/f \in A_{\mathfrak{q}}$ for all primes $\mathfrak{q}$ of height one, and hence $g/f \in A$, by Hartog's theorem (Theorem 20.1). It follows that $g \in fA$, and so $\mathfrak{p} = (f)A$ is principal. ❑

In particular, since $A = k[x_1, \ldots, x_n]$ is a unique factorization domain, we get

COROLLARY 20.25 $\mathrm{Cl}(\mathbb{A}_k^n) = 0$.

Projective space

Write $\mathbb{P}_k^n = \operatorname{Proj} R$, with $R = k[x_0, \ldots, x_n]$. Prime divisors on $\mathbb{P}_k^n$ correspond to homogeneous height one prime ideals in R, that is, ideals $\mathfrak{p} = (g)$ where g is a homogeneous irreducible polynomial. It is unique up to a scalar, so the degree is well defined. We can use this to define the *degree* of a divisor, by taking the sum of degrees of the corresponding polynomials. If $D = \sum_i n_i V(g_i)$, we let

$$\deg D = \deg \sum n_i V(g_i) = \sum_i n_i \deg g_i,$$

and this a group homomorphism $\deg\colon \operatorname{Div} \mathbb{P}_k^n \to \mathbb{Z}$.

Now, a rational function $f \in K(\mathbb{P}_k^n)$ is the quotient of two homogeneous polynomials of the same degree. Factoring the enumerator and the denominator we can write it as $f = \prod_i f_i^{n_i}$ where the f_i are irreducible coprime polynomials in R and the exponents $n_i \in \mathbb{Z}$, and $\sum_i n_i = 0$ as f is homogeneous of degree zero. Let us first show that

LEMMA 20.26

$$\operatorname{div}(f) = \sum n_i [V(f_i)].$$

PROOF: If$Y \subset \mathbb{P}_k^n$ is a prime divisor, let $y \in Y$ be the generic point. Since Y has codimension one, it holds that $Y = V(g)$ for some irreducible polynomial g of some degree d. For any other polynomial h of degree d, the fraction g/h is a generator of the maximal ideal $\mathfrak{m}_y \mathcal{O}_{X,y}$. We may write $f = (g/h)^r f'$ and f' a unit in $\mathcal{O}_{Y,y}$. Clearly $r = n_i$ if f_i divides g (and 0 if no f_i divides g) and f' a rational function neither containing g in the numerator nor in the denominator. This means that $v_Y(f) = r$, and hence $\operatorname{div}(f) = \sum n_i [V(f_i)]$. ❑

In view of the equality $\deg \operatorname{div}(f) = \sum n_i \deg f_i = 0$, the degree map descends to a group homomorphism

$$\deg : \mathrm{Cl}(\mathbb{P}^n) \to \mathbb{Z}$$

We claim that it is an isomorphism. It is clearly surjective since the degree of a hyperplane, for instance $V(x_0)$, equals one. Now, any $Z = \sum n_i [V(f_i)]$ in the kernel of deg, must have $\sum n_i \deg f_i = 0$. Consequently the element $f = \prod_i f_i^{n_i}$, is homogeneous of degree zero and defines an element of K. By the lemma above $Z = \operatorname{div}(f)$. Hence Z is a principal divisor, and deg is injective.

We have thus shown:

PROPOSITION 20.27 *The degree map gives an isomorphism* $\mathrm{Cl}(\mathbb{P}^n_k) \simeq \mathbb{Z}$.

EXAMPLE 20.28 $\mathrm{Cl}(\mathbb{P}^2)$ is generated by the class of line $L \subset \mathbb{P}^2$, *e.g.* $L = V(x_0)$, and any two lines L, L' are linearly equivalent. ☆

EXAMPLE 20.29 Consider the curve X as in Figure 20.1, given by

$$X = V(y^2z - x^3 - z^3) \subset \mathbb{P}^2.$$

For a line $L = V(y)$ on $\mathbb{P}^2_k$, let $L|_X$ denote the restriction of L to X (i.e., the Weil divisor $L \cap X$ on X which is of codimension 1 as X is integral). Moreover, for another line $L' = V(z)$, the two restrictions $L|_X$ and $L'|_X$ are linearly equivalent divisors on X, since $L|_X - L'|_X = \mathrm{div}(\frac{x}{z}|_X)$. This argument applies for any two lines L, L' in $\mathbb{P}^2$, so we get many relations between divisors on X. The figure below shows one example where $L|_X = P + Q + R$ and $L' = 2S + T$. ☆

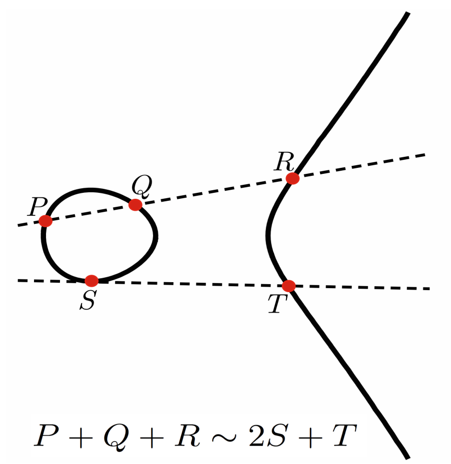


Figure 20.1: Two linearly equivalent divisors on a plane cubic

EXAMPLE 20.30 Consider the ideal $\mathfrak{p} = (2, 1 + \sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$. One easily checks that $\mathfrak{p}$ is a prime ideal, so that $Y = V(\mathfrak{p})$ is a prime divisor in $\mathrm{Spec}\,\mathbb{Z}[\sqrt{-5}]$. It holds that the square $\mathfrak{p}^2$ is principal and generated by 2. Thus $2Y = \mathrm{div}\,2$, and the class of Y in $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}])$ is two-torsion. Using the norm, one sees that Y is not principal, so its class is a genuine two-torsion element in $\mathrm{Cl}\,\mathbb{Z}[\sqrt{-5}]$. For a continuation of this example see page 319. ☆

✲ **EXERCISE 20.4** Show that all the local rings $\mathcal{O}_{X,p}$ of the curve X given by $y^2 = x^3 - 1$ in $\mathbb{A}^2_k$ are discrete valuation rings, and hence X is a normal variety. We assume that k is algebraically closed and of characteristic different from three and two. More precisely, if (a, b) is a point on X show that $x - a$ is a parameter if $b \neq 0$ and that y is one when $b = 0$. HINT: It holds true that $y^2 - b^2 = x^3 - a^3$. ★

✲ **EXERCISE 20.5** Consider the curve $y^2 = x^3 - 1$ in $\mathbb{A}^2_k$ where k is algebraically closed of characteristic different from two and three. If $(a, b) \in X$ we let $\sigma(a, p) = (a, -b)$, which also lies in X.

a) Show that for any $P \in X$, it holds that $-P \sim \sigma(P)$;
b) Show that if P, Q and R are three collinear points on X, then $P+Q+R \sim 0$;
c) Show that any Weil divisor on X is linearly equivalent to a prime divisor.

★

The sheaf associated to a Weil divisor

Let $D = \sum n_Y Y$ be a Weil divisor on the Noetherian normal and integral scheme X. We would like to form a sheaf, denoted $\mathcal{O}_X(D)$, consisting of the rational functions with 'poles at worst along D'. There are several ways of expressing this, the simplest is to require of f that $v_Y(f) \geqslant -n_Y$ for all Y so that pole order of f along Y in magnitude is bounded by n_Y. Another way, is to say that $(f)+D$ is an effective Weil divisors; *i.e.* that $(f)+D \geqslant 0$. Heuristically, one may say that the pole part of (f) is cancelled out by D. Concretely, we define the sheaf $\mathcal{O}_X(D)$ as follows:

DEFINITION 20.31 (THE SHEAF OF A WEIL DIVISOR) *Let X be a Noetherian integral and normal scheme with function field K, and let D be a Weil divisor on X. We define the sheaf $\mathcal{O}_X(D)$ by letting*

$$\mathcal{O}_X(D)(U) = \{\, f \in K \mid v_Y(f) \geqslant -n_Y \text{ for all } \ Y \text{ with } \eta_Y \in U \,\}$$

for each open subset $U \subseteq X$.
If $U' \subseteq U$ are two open subsets, the restriction map is to be the inclusion map $\mathcal{O}_X(D)(U) \subseteq \mathcal{O}_X(D)(U')$.

Note that the condition $\eta_Y \in U$ just means that U meets Y; *i.e.* that $U \cap Y$ is a dense open subset. Note also that the condition is relaxed when applied to a smaller subset $U' \subseteq U$, and hence yields a larger set of functions so that $\mathcal{O}_X(D)(U) \subseteq \mathcal{O}_X(D)(U')$. When a is a regular function U, it holds that $v_Y(af) = v_Y(a) + v_Y(f) \geqslant -n_Y$ for all Y and all $f \in \mathcal{O}_X(D)(U)$, and this makes $\mathcal{O}_X(D)$ into an $\mathcal{O}_X$-module. Even more is true, it will be quasi-coherent:

PROPOSITION 20.32 *The sheaf $\mathcal{O}_X(D)$ is quasi-coherent.*

PROOF: Exercise 20.6 ❏

There is a distinguished rational section of $\mathcal{O}_X(D)$. Namely, taking $V = X - \operatorname{Supp}(D)$, the element '1' $\in K$ defines an element of $\Gamma(V, \mathcal{O}_X(V))$. We denote this section by s_D.
EXAMPLE 20.33 Let X be the projective line $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$ over k and consider the divisor $D = V(x_1) = (1:0)$. We have the standard covering of $\mathbb{P}^1_k$ by the distinguished open sets $U_0 = \operatorname{Spec} k[x_1/x_0] = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[x_0/x_1] = \operatorname{Spec} k[s]$ (so $s = t^{-1}$ on $U_0 \cap U_1$). Let us find the global sections of $\mathcal{O}_X(D)$.

Note that the point $(1:0)$ does not lie in $U_1 = D_+(x_1)$, and this means that a rational function $f \in K$ such that $\operatorname{div}(f) + D$ is effective on U_1, must be regular on U_1; that is

$$\Gamma(U_1, \mathcal{O}_X(D)) = k[s].$$

On the patch U_0, we are looking at elements $f \in k(t)$ having valuation larger than -1 at $t = 0$. This implies that

$$\Gamma(U_0, \mathcal{O}_X(D)) = \{\, f \mid f = \alpha t^{-1} + p(t) \,\}$$

where α is a scalar and $p(t)$ a polynomial.

Now, we may think of the global sections in $\Gamma(X, \mathcal{O}_X(D))$ as pairs (f, g) with f and g elements in K that are sections of $\mathcal{O}_X(D)$ over U_0 and U_1 respectively, so that $f = g$ on $U_0 \cap U_1$. Here $g = g(s)$ is a polynomial in s, and

$$f(t) = p(t) + \alpha t^{-1} = p(s^{-1}) + \alpha s.$$

If $f = g$ in $k[t, t^{-1}]$, it is clear that p must be a constant. This implies that

$$\Gamma(X, \mathcal{O}_X(D)) = k \oplus k\, t^{-1}.$$

In fact, we will see in a bit that $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. ☆

✲ EXERCISE 20.6 Let $L = \mathcal{O}_X(D)$ for a Weil divisor D. Let $U = \operatorname{Spec} A \subset X$ be an open affine subset. Show that for $f \in A$, $\Gamma(U, \mathcal{O}_X(D))_f = \Gamma(D(f), \mathcal{O}_X(D))$. Deduce that $\mathcal{O}_X(D)$ is a quasi-coherent sheaf. ★

The Weil divisor of a section of an invertible sheaf

We continue working with a integral, normal Noetherian scheme X with function field K. Let L be an invertible sheaf on X and let s be a *rational section* of L; that is, a section of L over an open set $V \subseteq X$, or in other words an element $s \in \Gamma(V, L)$. To s one may associate a Weil divisor $\operatorname{div}(s)$ in the following way.

rational section of an invertible sheaf

Given a prime divisor Y a prime divisor, we let η_Y denote its generic point. Let $U \subset X$ be a neighbourhood of η_Y such that there is a trivialization $\phi \colon L|_U \to \mathcal{O}_X|_U$. On U, $\phi(s)$ is a rational section of $\mathcal{O}_X(U)$ (in fact a section over the dense open subset $U \cap V$), and hence it is an element f in the function field K. We define $v_Y(s) = v_Y(f)$ and summing up over all prime divisors, we get divisor of s:

$$\operatorname{div}(s) = \sum_Y v_Y(s) Y.$$

Some choices have been made on the way, but of course they don't matter, moreover the sum in 20.2 is finite:

LEMMA 20.34 *The divisor* $\operatorname{div}(s)$ *is independent of the choice of the open sets* U *and* V *and the trivialization* ϕ*. The sum in 20.2 is finite.*

PROOF: Two trivializations ϕ and ϕ' over U , differ by an invertible factor c, so that $\phi(s) = c\phi'(s)$ where c is unit in $\mathcal{O}_X(U)$; hence $v_Y(c) = 0$, and so $v_Y(\phi(s)) = v_Y(\phi'(s))$.

Let U be an affine open set over which L is trivial, there is only finitely many components of the complement U; and replacing $s|_U$ by the image under a trivialization $L|_U \simeq \mathcal{O}_U$, we are back to Lemma 20.14. ❑

EXAMPLE 20.35 Let $X = \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$ and denote its fraction field by K. Consider the invertible sheaf $L = \mathcal{O}_X(2)$. The quotient $s = \frac{x_0^3}{x_1}$ defines a section of L over $D_+(x_1)$, hence a rational section on X. Let us compute the divisor associated to s: Let $t = \frac{x_0}{x_1}$ be the coordinate on $U = D_+(x_1) = \operatorname{Spec} k[t]$.

$$\mathcal{O}_X(2)(U) = k\left[\frac{x_0}{x_1}\right] x_1^2 = k[t]x_1^2.$$

So the rational function $f = \phi(s)$ is given by $\frac{x_0^3}{x_1^3} = t^3$ which has non-zero valuation only at the point $t = 0 \in U$, where we have $v_Y(f) = 3$. To compute $\operatorname{div}(s)$, we must also consider the point outside $D_+(x_1)$. On $U = D_+(x_0)$, we use the coordinate $u = \frac{x_1}{x_0}$, and we have

$$\mathcal{O}_X(2)(U) = k[u]x_0^2.$$

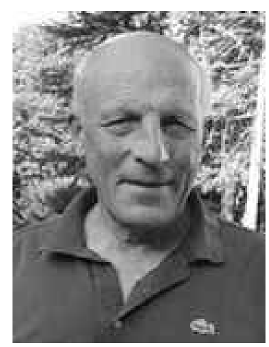
So the rational function $\phi(s)$ is given by $f = \frac{x_0}{x_1} = u^{-1}$. This has valuation $v_Y(f) = -1$ at $t = 0$ (and $v_Y(f) = 0$ at all other points). Hence we obtain

$$\operatorname{div}(s) = 3(0:1) - (1:0).$$

☆

20.3 *Cartier divisors*

Just like Weil divisors formed a group, we would like the same to apply to Cartier divisors. Note that if D and E are two effective Cartier divisors, given by ideal sheaves $\mathcal{I}_D, \mathcal{I}_E$, then $\mathcal{I}_D \cdot \mathcal{I}_E$ is again an ideal sheaf which is invertible on X. It therefore makes sense to define the sum $D + E$ to be the corresponding effective Cartier divisor. This does define a binary operation on the set of effective Cartier divisors, but it is not a group law, as there is no natural inverse $-D$ of an effective Cartier divisor.



Pierre Cartier (1932 –)
French mathematician

There is an easy fix to this, namely to let the f_i's in the specification of effective Cartier divisors on page 294 be rational functions, rather than regular functions on U_i;*i.e.* elements of $\mathcal{O}_X(U_i)$. That is, the f_i are allowed to have poles on the U_i.

DEFINITION 20.36 (CARTIER DIVISORS) *Let X be an integral scheme with function field K. We define a* Cartier divisor *by the data*

i) an open covering $\{U_i\}_{i \in I}$ of X;

ii) elements $f_i \in K$ satisfying $f_i f_j^{-1} \in \mathcal{O}_X^\times(U_i \cap U_j)$ for every i, j.

We consider two defining data $\{(U_i, f_i)\}_{i\in I}$ and $\{(V_j, g_j)\}_{i\in I}$ to define the same divisor if $f_i g_j^{-1} \in \Gamma(U_i \cap V_j, \mathcal{O}_X^\times)$ for all i, j.

The set of Cartier divisors now naturally forms a group. The identity is given by $(X, 1)$. Given D and E represented by the data $\{(U_i, f_i)\}_{i\in I}$ and $\{(V_i, g_i)\}_{i\in J}$, we can define $D + E$ as the Cartier divisor associated to the data

$$\{(U_i \cap V_j, f_i g_j)\}_{i,j}$$

Moreover, the inverse $-D$ will be defined as $\{(U_i, f_i^{-1})\}_{i\in I}$. We denote the group of Cartier divisors by $\mathrm{CaDiv}(X)$.

Definition 20.37 *Let X we an integral scheme with function field K.*

i) We say that a Cartier divisor is principal *if it is equal (as an element of* $\mathrm{CaDiv}(X)$*) to the Cartier divisor (X, f) where $f \in K^\times$.*

ii) We define $\mathrm{CaCl}(X)$ *to be the group of Cartier divisors modulo principal divisors:*

$$\mathrm{CaCl}(X) = \mathrm{CaDiv}(X)/\{(X, f)|f \in K^\times\}$$

iii) Two Cartier divisors D, D' are said to be linearly equivalent*, if $D - D' = (X, f)$ for some principal divisor (X, f), or equivalently, $[D] = [D']$ in* $\mathrm{CaCl}(X)$.

The principal Cartier divisors form a subgroup of $\mathrm{CaDiv}(X)$, which is typically smaller than $\mathrm{CaDiv}(X)$. Note on the other hand that, by definition, any Cartier divisor is 'locally principal', since it becomes principal when restricted to each U_i.

Exercise 20.7 Show that a Cartier divisor is the same thing as a global section of the sheaf $\mathscr{K}_X^\times/\mathcal{O}_X^\times$. In other words, $\mathrm{CaDiv}(X) = \Gamma(X, \mathscr{K}_X^\times/\mathcal{O}_X^\times)$. ★

Exercise 20.8 Check that the inverse of a Cartier divisors and the sum of two are well defined; that is, that all cocycle conditions are fulfilled and that the inverse, respectively the sum, is independent of choices of representatives. ★

The sheaf associated to a Cartier divisor

We just saw that each Weil divisor D gave rise to a quasi-coherent sheaf $\mathcal{O}_X(D)$. The same is true for Cartier divisors, and in this case the corresponding sheaf will turn out to be invertible.

Let D be a Cartier divisor on a scheme X given by the data $\{(U_i, f_i)\}_{i\in I}$. We will associate to it an invertible sheaf, which we denote by $\mathcal{O}_X(D)$. As in the case of Weil divisors, the intuition is that sections of $\mathcal{O}_X(D)$ should correspond to rational functions with at "worst poles along D" (at least when D is effective). That is, if $f_i \in \mathcal{O}_X(U)$ is a local equation for D, we allow rational functions like $\frac{1}{f_i}$ (but perhaps not $\frac{1}{f_i^2}$). On U_i we therefore take the subsheaf $f_i^{-1}\mathcal{O}_{U_i} \subset \mathscr{K}_{U_i}$ of the constant sheaf K on U_i. This subsheaf is isomorphic to $\mathcal{O}_{U_i}$ and has f_i^{-1} as a local generator. So over an affine subset $U = \operatorname{Spec} A \subset U_i$, the sheaf is the sheaf associated to $f_i^{-1}A \subseteq K(A)$. On the intersection, $U_i \cap U_j$, we have $f_i = c_{ij} f_j$ where c_{ij} is an invertible section of $\mathcal{O}_{U_{ij}}$. This means that $f_i^{-1}\mathcal{O}_{U_{ij}} = f_j^{-1}\mathcal{O}_{U_{ij}}$ as subsheaves of $\mathscr{K}_{U_{ij}}$. We have therefore constructed a sheaf on each U_i, and the elements coincide on the intersections U_{ij}. In order to be able to glue to a sheaf, there is a cocycle condition that has to be satisfied. But since these sheaves are all subsheaves of a fixed sheaf $\mathscr{K}$, the gluing maps are actually identity maps, and the cocycle condition is automatically satisfied. It follows that the sheaves $f_i^{-1}\mathcal{O}_{U_i}$ glue to a sheaf $\mathcal{O}_X(D)$ defined on all of X. It is by construction invertible, because it is invertible on each U_i.

Explicitly, $\mathcal{O}_X(D)$ is defined as the subsheaf of $\mathscr{K}_X$ given by

$$\Gamma(V, \mathcal{O}_X(D)) = \{f \in K \mid f_i f \in \Gamma(U_i \cap V, \mathcal{O}_X) \forall i \in I\}$$

Two different data (U_i, f_i) and (V_j, g_j) for the same divisor D give rise to the same invertible sheaf. This is because over $U_i \cap V_j$, we have $f_i = d_{ij} g_j$ for some sections $d_{ij} \in \mathcal{O}_X(U_i \cap V_j)^\times$. This means that $f_i^{-1}\mathcal{O}_{U_i \cap V_j} = g_i^{-1}\mathcal{O}_{U_i \cap V_j}$, and so the sheaf is uniquely determined as a subsheaf of $\mathscr{K}_X$.

PROPOSITION 20.38 *Let X be an integral scheme and let D and D' be two Cartier divisors.*
i) $\mathcal{O}_X(D + D') \simeq \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$
ii) $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ *if and only if D and D' are linearly equivalent.*

PROOF: We can pick a common covering U_i so that both D and D' are both represented by data (U_i, f_i), (U_i, f_i'). Then $D + D'$ is defined by $(U_i, f_i f_i')$. Locally, over U_i the sheaf $\mathcal{O}_X(D + D')$ is defined as the subsheaf of $\mathscr{K}$ given by $(f_i f_i')^{-1}\mathcal{O}_{U_i} = f_i^{-1} f_i'^{-1}\mathcal{O}_{U_i}$. The tensor product is locally(!) given as $f_i^{-1}\mathcal{O}_{U_i} \otimes f_i'^{-1}\mathcal{O}_{U_i}$, which is clearly isomorphic to $f_i^{-1} f_i'^{-1}\mathcal{O}_{U_i}$ via the mapping defined by $a f_i^{-1} \otimes b f_i'^{-1} \mapsto ab f_i^{-1} f_j'^{-1}$.

For the second claim, it suffices (by point (i)) to show that $\mathcal{O}_X(D) \simeq \mathcal{O}_X$ if and only if D is a principal Cartier divisor. So suppose that $\mathcal{O}_X(D) \subseteq \mathscr{K}_X$ is a sub $\mathcal{O}_X$-module which is isomorphic to $\mathcal{O}_X$. Then the image of 1 in $\mathcal{O}_X$ will be a section s of $\mathcal{O}_X(D)$ which generates $\mathcal{O}_X(D)$ everywhere. This means that for $f = s^{-1}$, (X, f) is the local defining data for D, and D is a principal Cartier divisor. Conversely, if $D = (X, f)$, then $\mathcal{O}_X(D) = f^{-1}\mathcal{O}_X$, and multiplication by $f \in K$ gives an isomorphism $\mathcal{O}_X(D) \simeq \mathcal{O}_X$. ❑

EXAMPLE 20.39 (*Projective space.*) Let us take a closer look at the projective space $\mathbb{P}^n_k$ over a field k. Write $\mathbb{P}^n_k = \operatorname{Proj} R$ where $R = k[x_0, \ldots, x_n]$. Consider the standard covering $U_i = D_+(x_i)$ of $\mathbb{P}^n_k$.

Let $F(x_0, \ldots, x_n) \in R_d$ denote a homogeneous polynomial of degree d. Then $F(x/x_i) = F(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$ defines a non-zero regular function on U_i, and the collection

$$(U_i, F(x/x_i))$$

forms a Cartier divisor D on X. Indeed, on the overlap $U_i \cap U_j$ we have the relation

$$F(x/x_i) = \left(x_j/x_i\right)^d F(x/x_j)$$

and x_j/x_i is a regular and invertible function on $U_i \cap U_j$. The corresponding invertible sheaf is exactly $\mathcal{O}_{\mathbb{P}^n_k}(d)$. Two homogeneous polynomials F, G of the same degree d give linearly equivalent divisors, because the quotient $F(x)/G(x)$ is a global rational function on $\mathbb{P}^n_k$. ☆

Cartier divisors vs. invertible sheaves

By the item (i) and (ii) in Proposition 20.38, we see that the natural map $\rho : \mathrm{CaDiv}(X) \to \mathrm{Pic}(X)$, which sends D to the class of $\mathcal{O}_X(D)$ in $\mathrm{Pic}(X)$ is additive and has the subgroup of principal divisors as its kernel. This means that the induced map $\rho : \mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is injective. In this section we will show that this map is also surjective, so that in fact $\mathrm{CaCl}(X) \simeq \mathrm{Pic}(X)$.

PROPOSITION 20.40 *When X is an integral scheme, the map*

$$\rho : \mathrm{CaCl}(X) \to \mathrm{Pic}(X)$$

is an isomorphism.

PROOF: We need to show that ρ is surjective. It suffices to show that any invertible $\mathcal{O}_X$-module L is isomorphic to a submodule of $\mathscr{K}_X$: If $L \subseteq \mathscr{K}$, let U_i be a trivializing cover of L and let g_i be its local generators. Then we have $L|_{U_i} = g_i \mathcal{O}_{U_i} \subset \mathscr{K}_{U_i}$ and the g_i are rational functions on U_i. On $U_{ij} = U_i \cap U_j$, we have $g_i \mathcal{O}_{U_{ij}} = L|_{U_{ij}} = g_j \mathcal{O}_{U_{ij}}$, and it follows that $g_i = c_{ij} g_j$ for units $c_{ij} \in \mathcal{O}_{U_{ij}}^\times$. Consequently, (U_i, g_i^{-1}) forms a set of local defining data for a Cartier divisor D, and of course we have $L = \mathcal{O}_X(D)$.

Let L be an invertible sheaf and consider the sheaf $L \otimes_{\mathcal{O}_X} \mathscr{K}_X$. Let $U_i \subset X$ be an open cover such that $L|_{U_i} = \mathcal{O}_X|_{U_i}$. Note that the restriction of $L \otimes_{\mathcal{O}_X} \mathscr{K}_X$ to each U_i is a constant sheaf (isomorphic to $\mathscr{K}_X$). Since X is irreducible, any sheaf whose restriction to opens in a covering is constant, is in fact a constant sheaf, and therefore $L \otimes_{\mathcal{O}_X} \mathscr{K}_X \simeq \mathscr{K}_X$ as sheaves on X. Now we can regard L as a rank 1 subsheaf of $\mathscr{K}_X$ using the composition $L \hookrightarrow L \otimes \mathscr{K}_X \simeq \mathscr{K}_X$. Hence by the above paragraph, ρ is surjective. ❑

Cartier divisors vs Weil divisors

Let X be an integral normal scheme, and let D be a Cartier divisor given by the data (U_i, f_i). If Y is a prime divisor on X, with generic point η, then since U_i is a cover, η lies in some U_i. We can then define

$$v_Y(D) = v_Y(f_i)$$

This is independent of the choice of U_i: If $\eta \in U_i \cap U_j$, then $f_i f_j^{-1}$ is an element of $\mathcal{O}_X^\times(U_i \cap U_j)$, and so $v_Y(f_i f_j^{-1}) = 0$, and hence $v_Y(f_i) = v_Y(f_j)$. Then ι is defined by

$$\iota(D) = \sum_Y v_Y(D) Y.$$

The map ι is injective, because if D is Cartier, with $v_Y(D) = 0$ for every Y, then the f_i must be units. Thus $D = 0$, as a Cartier divisor. In particular, we may view Cartier divisors as a subgroup of the group of Weil divisors.

Having defined Cartier and Weil divisors, it is natural to ask when the two coincide. We will soon see (rather simple) examples where they do not. However, when X has 'mild singularities', any Cartier divisor gives rise to a Weil divisor, and vice versa.

PROPOSITION 20.41 *Let X be an integral normal scheme. Then the following are equivalent:*

i) $\iota : \mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ *is an isomorphism;*

ii) X *is factorial (all the local rings $\mathcal{O}_{X,x}$ are UFDs).*

PROOF: We already noted that ι is always injective, so the question is about surjectivity.

Note that any Weil divisor is a linear combination of integral subschemes of codimension 1. Thus ι is surjective if and only if any such divisor is Cartier. But here the equivalence follows from Theorem 20.7. ❑

So if X is factorial, every Weil divisor comes from a Cartier divisor, and vica versa. The intuition is that this holds whenever X has 'mild' singularities. For instance, *regular* local Noetherian rings (i.e., $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$) are also UFDs ([Atiyah-MacDonald Ch. 7] or [Stacks 0AG0]). So in particular, the above applies to the main examples of interest:

COROLLARY 20.42 *On a non-singular variety X, then the map* $\iota : \mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ *is an isomorphism. Moreover, this induces natural isomorphisms between the groups of*

i) Weil divisors (up to linear equivalence)

ii) Cartier divisors (up to linear equivalence)

iii) Invertible sheaves (up to isomorphism)

From our previous computation of $\mathrm{Cl}(\mathbb{A}^n_k)$, we get the following theorem:

THEOREM 20.43 *Let k be a field. Then* $\mathrm{Pic}(\mathbb{A}^n_k) = \mathrm{Cl}(\mathbb{A}^n_k) = \mathrm{CaCl}(\mathbb{A}^n_k) = 0$.

We previously computed that $\mathrm{Cl}(\mathbb{P}^n_k) = \mathbb{Z}$, so Corollary 20.42 gives the following:

COROLLARY 20.44 *On $\mathbb{P}^n_k$ any invertible sheaf is isomorphic to some $\mathcal{O}_{\mathbb{P}^n_k}(m)$.*

20.4 *Effective divisors and linear systems*

Recall that a Cartier divisor D is effective if D can be represented by local data (U_i, f_i) where the rational functions actually lie in $\mathcal{O}_X(U_i)$. We will write $D \geqslant 0$ when this is satisfied. This makes $\mathrm{CaDiv}(X)$ into an ordered group, namely, we define $D \geqslant D'$ if $D - D' \geqslant 0$, i.e., the difference $D - D'$ is effective. Note also that if D and D' are both effective, then so is $D + D'$.

LEMMA 20.45 *If D is an effective Cartier divisor, we may identify $\mathcal{O}_X(-D)$ with the ideal sheaf of D, i.e.,*

$$\mathcal{I}_D = \mathcal{O}_X(-D)$$

Proof: If (U_i, f_i) are local defining data for D, then $\mathcal{O}_X(-D)$ is generated by f_i (as a subsheaf of $\mathcal{K}$). Since D is effective, the f_i are elements of $\mathcal{O}_X(U_i)$, so $\mathcal{O}_X(-D)$ is contained in $\mathcal{O}_X$ as an ideal sheaf. ❑

If D is an effective Cartier divisor, there is a canonical section $s \in \Gamma(X, \mathcal{O}_X(D))$. Indeed, over the open set U_i, we have

$$\mathcal{O}_X(D)|_{U_i} = \frac{1}{f_i}\mathcal{O}_X \subset \mathcal{K}$$

Each of these clearly contains the element '1', which gives us the section s_D.

What is the zero scheme of the section s_D? To answer this, we note that the inverse to the local parameter $\mathcal{O}_{U_i} \to \mathcal{O}_X(D)|_{U_i}$ maps $1 \in \Gamma(X, \mathcal{O}_X(D))$ to $f_i \in \mathcal{O}_X(U_i)$. We therefore see that

$$\operatorname{div} s = D.$$

Conversely, given a section $s \in \Gamma(X, \mathcal{O}_X(D))$, we have may write over each U_i, $s|_{U_i} = a/f_i$ for some $a \in \mathcal{O}_X(U_i)$.

$$\operatorname{div} s + D \geqslant 0$$

The inclusion $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ induces an exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

where the right hand is to be interpreted both as the cokernel of the left-most map and as the structure sheaf of the subscheme associated to D. Moreover, the support of the $i_*\mathcal{O}_D$ (i.e., the set of points $x \in X$, such that $\mathcal{O}_{D,x} \neq 0$) coincides with underlying topological space of D.

Theorem 20.46 *Let X be an noetherian, integral scheme and let D be a Cartier divisor on X. Then D is linearly equivalent to an effective divisor if and only if $\mathcal{O}_X(D)$ has a non-zero global section. For each section σ, we get a divisor denoted by* div σ. *Two such sections σ, σ' give rise to the same divisor if and only if $\sigma = c\sigma'$ where $c \in \Gamma(X, \mathcal{O}_X)^\times$.*

Proof:

❑

Definition 20.47 *The set of effective divisors D' linearly equivalent to D is denoted by $|D|$. This is called the* complete linear system *of D.*

The name 'linear system' comes from the special case when X is a projective variety X over a field k (thus X is integral, separated of finite type over k). In this case, we have

$\Gamma(X, \mathcal{O}_X)^\times = k^\times$, and the previous discussion shows that the linear system $|D|$ is given by

$$\begin{aligned} |D| &= \{D' | D' \geqslant 0 \text{ and } D' \sim D\} \\ &= (\Gamma(X, \mathcal{O}_X(D)) - 0) / k^\times \\ &= \mathbb{P}\Gamma(X, \mathcal{O}_X(D)) \end{aligned}$$

When X is projective over k, the groups $\Gamma(X, \mathcal{O}_X(D))$ are finite dimensional as k-vector spaces (we will prove this fact in Chapter **??**), so the set of effective divisors D' linearly equivalent to D is (as a set) a projective space $\mathbb{P}^n_k$.

Definition 20.48 *A* linear system of divisors *is a linear subspace of a complete linear system $|D|$.*

Example 20.49 Consider the case $X = \mathbb{P}^n_k$ and $D = dH$, where H is the hyperplane divisor (so H is a Cartier divisor with $\mathcal{O}_X(H) \simeq \mathcal{O}_X(1)$). In this case the linear system of D associated to $\mathcal{O}_X(dH)$ is given by the set of homogeneous polynomials of degree d modulo scalars, i.e.,

$$|D| = \left\{ \sum_{i_0 + \cdots + i_n = d} a_{i_0, \ldots, i_n} x_0^{i_0} \cdots x_n^{i_n} \right\} / k^\times \simeq \mathbb{P}^N_k$$

where $N = \binom{n+d}{d} - 1$. The points of this projective space correspond to degree d hypersurfaces, and the coefficients $a_{i_0, \ldots, i_n}$ give homogeneous coordinates on it. ☆

20.5 Examples

A useful exact sequence

Given a Noetherian, normal and integral X and an open subset U, the restriction of a prime divisor on X is a prime divisor on U, so it is natural to ask how the two class groups are related. The answer is given by the theorem below.

Before stating the result, we shall be precise about the restriction map, so consider a prime divisor Y in X. If $Y \cap U \neq \emptyset$, it is dense in Y, and so the generic point of Y lies in U, and the crucial point is that $\mathcal{O}_{U, \eta_Y} = \mathcal{O}_{X, \eta_Y}$ is of height one. Since $\operatorname{Div} X$ is free abelian on the prime divisors, this allows us to define a restriction map $\operatorname{Div} X \to \operatorname{Div} U$ by

$$Y \mapsto \begin{cases} Y \cap U & \text{if } Y \cap U \neq \emptyset, \\ 0 & \text{else} . \end{cases}$$

Moreover, if f is a rational function on X, the restriction $f|_Y$ is a rational function on Y, it holds that $v_{Y \cap U}(f|_Y) = v_Y(f)$ (the two valuation rings are equal), and consequently the divisor (f) restricts to the divisor $(f|_Y)$. The restriction map passes to the quotient and yields a map $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$.

Theorem 20.50 *Let X be a Noetherian, normal and integral scheme. Let $Z \subset X$ be a closed subscheme and let $U = X - Z$. If $Z_1, \ldots, Z_r$ are the prime divisors corresponding to the codimension one components of Z, there is an exact sequence*

$$\bigoplus_{i=1}^{r} \mathbb{Z} Z_i \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0, \tag{20.5}$$

where the map $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is defined by $[Y] \mapsto [Y \cap U]$.

Proof: If Y is a prime divisor on U, the closure in X is a prime divisor in X, so the map is surjective, and we just need to check exactness in the middle.

Suppose Y is a prime divisor which is principal on U. Then $Y|_U = \mathrm{div}(f)$ for some $f \in K(U) = K = k(X)$. Now $D = \mathrm{div}(f)$ is a divisor on X such that $D|_U = \mathrm{div}(f)|_U$. Hence $D - Y$ is a Weil divisor supported in $X - U$, and hence it must be a linear combination of the Z_i's. Thus $D - Y$ is in the image of the left-most map, and we are done. ❑

As a special case, we see that removing a codimension two subset does not change the group of Weil divisors. So for instance $\mathrm{Cl}(\mathbb{A}^2 - 0) = \mathrm{Cl}(\mathbb{A}^2)$.

Example 20.51 Consider the projective line $\mathbb{P}^1_k$ over a field k, and let P be a point. We have the exact sequence

$$\mathbb{Z}[P] \to \mathrm{Cl}(\mathbb{P}^1) \to \mathrm{Cl}(\mathbb{A}^1) \to 0.$$

We saw that $\mathrm{Cl}(\mathbb{A}^1) = 0$, so the map $\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^1)$ is surjective. It is also injective: If $[nP] = 0$ in $\mathrm{Cl}(\mathbb{P}^1)$ for some n, then $nP = \mathrm{div}(f)$ for some $f \in k(\mathbb{P}^1)$. Consider the open set $U = \mathbb{P}^1 - P \simeq \mathbb{A}^1$. Then $nP|_U = 0$, so we must have $\mathrm{div}(f)|_{\mathbb{A}^1} = 0$. Thus f has neither zeros, nor poles, and so $f \in \Gamma(\mathbb{A}^1, \mathcal{O}^\times_{\mathbb{A}^1}) = k^\times$. Hence f is constant, and so $n = 0$. This gives another proof of $\mathrm{Cl}(\mathbb{P}^1) = \mathbb{Z}$. ☆

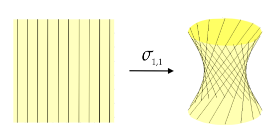
Exercise 20.9 Let $\mathbb{P}^2 = \operatorname{Proj} k[x_0, x_1, x_2]$. An irreducible homogeneous polynomial f of degree $d \geqslant 1$ determines a prime divisor $D = V(f)$. Consider the open set $U = \mathbb{P}^2_k - D$. Show that the above exact sequence above takes the form

$$0 \to \mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z} \to \mathrm{Cl}(U) \to 0.$$

Deduce that $\mathrm{Cl}(U) = \mathbb{Z}/d\mathbb{Z}$. ★

The smooth quadric surface

Let k be an field, and let $Q = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Recall that Q embeds as a quadric surface in $\mathbb{P}^3_k$ via the Segre embedding. So we can view Q both as a fiber product $\mathbb{P}^1 \times \mathbb{P}^1$ and the quadric $V(xy - zw) \subset \mathbb{P}^3$.



Since Q is a product of two $\mathbb{P}^1$s there are natural ways of constructing Weil divisors on Q from those on $\mathbb{P}^1$. For instance, we can let

$$L_1 = (0:1) \times \mathbb{P}^1 \subset Q,$$

which is a prime divisor on Q corresponding to the 'vertical fiber' of Q. Similarly, $L_2 = \mathbb{P}^1 \times (0:1)$ is a Weil divisor on Q. From these we obtain an exact sequence

$$\mathbb{Z}L_1 \oplus \mathbb{Z}L_2 \to \mathrm{Cl}(Q) \to \mathrm{Cl}(Q - L_1 - L_2) \to 0$$

Here $Q - L_1 - L_2 = U_{11} = \operatorname{Spec} k[x^{-1}, y^{-1}]$. The latter is isomorphic to $\mathbb{A}^2_k$, so $\mathrm{Cl}(Q - L_1 - L_2) = 0$. This shows that $\mathrm{Cl}(Q)$ is generated by the classes of L_1 and L_2. We claim that the first map is also injective, so that in fact that

$$\mathrm{Cl}(Q) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2.$$

If the map is not injective, there must be a relation $aL_1 - bL_2 \sim 0$, or equivalently,

$$\mathcal{O}_Q(aL_1) \simeq \mathcal{O}_Q(bL_2) \tag{20.6}$$

for some integers $a, b \in \mathbb{Z}$. We will show that this is not the case, by showing

i) $\mathcal{O}_Q(L_1)|_{L_1} \simeq \mathcal{O}_{\mathbb{P}^1}$;

ii) $\mathcal{O}_Q(L_2)|_{L_1} \simeq \mathcal{O}(1)_{\mathbb{P}^1}$

Then restricting (20.6) to L_1, we get $b = 0$, and hence also $a = 0$, by switching the roles of L_1 and L_2.

To prove i): Note that $L_1 \simeq L_1'$ where $L_1 = (1:0) \times \mathbb{P}^1$. This follows because we can consider the divisor of the rational function $x \in k(Q) = k(x, y)$:

$$\operatorname{div} x = (0:1) - (1:0) = L_1 - L_1'$$

Then note that $\mathcal{O}_Q(L_1')|_U \simeq \mathcal{O}_U$ over the open set $U = Q - L_1'$. However L_1 is contained in U, so the isomorphism i) follows.

Q is covered by four affine subsets

$$\begin{aligned} U_{00} &= \operatorname{Spec} k[x, y] & U_{10} &= \operatorname{Spec} k[x^{-1}, y] \\ U_{01} &= \operatorname{Spec} k[x, y^{-1}] & U_{11} &= \operatorname{Spec} k[x^{-1}, y^{-1}] \end{aligned}$$

Consider $\mathbb{P}^1_k = W_0 \cup W_1$, where $W_0 = \operatorname{Spec} k[t], W_1 = \operatorname{Spec} k[t^{-1}]$. The first projection $p_1 : Q \to \mathbb{P}^1_k$ is induced by the ring maps

$$\begin{aligned} k[x] &\to k[x, y] & k[x^{-1}] &\to k[x^{-1}, y] \\ k[x] &\to k[x, y^{-1}]; & k[x^{-1}] &\to k[x^{-1}, y^{-1}]; \end{aligned}$$

Let $p = (0:1)$ be the Weil divisor on $\mathbb{P}^1$. The Cartier data of p is given by $(W_0, t), (W_1, 1)$, so that $\mathcal{O}_{\mathbb{P}^1}(p) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. The pullback $D = p_1^*(p)$ is a Cartier divisor on Q, corresponding to the Weil divisor $(0:1) \times \mathbb{P}^1$. The corresponding Cartier data is given by

$$\begin{aligned} &(U_{00}, x), \quad (U_{10}, 1) \\ &(U_{01}, x), \quad (U_{11}, 1) \end{aligned}$$

Let $L_1 = (0:1) \times \mathbb{P}^1_k$ and $L_2 = \mathbb{P}^1_k \times (0:1)$. Consider the restriction of D to L_2. L_2 is covered by the two open subsets $V_0 = U_{00} \cap L_2 = \operatorname{Spec} k[x, y]/y = \operatorname{Spec} k[x], V_1 =$

$U_{10} \cap L_2 = \operatorname{Spec} k[x^{-1}, y]/(y) = \operatorname{Spec} k[x^{-1}]$. In terms of these opens, the restriction $D|_{L_2}$ has Cartier data

$$(V_0, x), (V_1, 1)$$

obtained by restricting the data above. In particular, identifying $L_2 \simeq \mathbb{P}^1$, we see that $\mathcal{O}_Q(D)|_{L_1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. In particular, since $\mathrm{Cl}(\mathbb{P}^1) = \mathbb{Z}$, no multiple nD is equivalent to 0 in $\mathrm{Cl}(Q)$: if that were the case, we would have $\mathcal{O}_Q(nD) \simeq \mathcal{O}_Q$, and hence $\mathcal{O}_Q(nD)|_{L_2} \simeq \mathcal{O}_Q|_{L_2} \simeq \mathcal{O}_{\mathbb{P}^1}$, a contradiction.

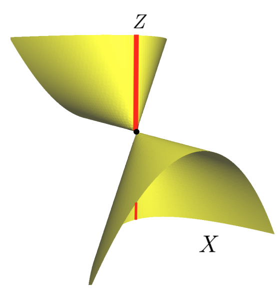
This completes the proof that

$$\mathrm{Cl}(Q) \simeq \mathbb{Z}L_1 \oplus \mathbb{Z}L_2.$$

If D is a divisor on Q, $D \sim aL_1 + bL_2$ and we call (a, b) the 'type' of D. A divisor of type $(1, 0)$ or $(0, 1)$ is a line on the quadric surface $Q \subset \mathbb{P}^3$. We have $i^*\mathcal{O}_{\mathbb{P}^3}(1) \simeq \mathcal{O}_Q(L_1 + L_2)$, so a $(1, 1)$-divisor is represented by a hyperplane section of Q (a conic). A prime divisor of type $(1, 2)$ or $(2, 1)$ is a *twisted cubic curve*.

The quadric cone

Let $X = \operatorname{Spec} R$ where $R = k[x, y, z]/(xy - z^2)$, and k has characteristic $\neq 2$. Let $Z = V(y, z)$ be the closed subscheme corresponding to the line $\{y = z = 0\}$. Note that $Z \simeq \operatorname{Spec} k[x, y, z]/(xy - z^2, y, z) = \operatorname{Spec} k[x]$, so it is integral of codimension 1.



A singular quadric surface

Note that $X - Z = X - V(y) = D(y)$, and the latter equals

$$\operatorname{Spec} k[x, y, y^{-1}, z]/(xy - z^2) = \operatorname{Spec} k[y, y^{-1}][t, u]/(t - u^2) = \operatorname{Spec} k[y, y^{-1}, u]$$

which is the spectrum of a UFD. It follows that $\mathrm{Cl}(X - Z) = 0$. Recall now the sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X - Z) \to 0$$

where the first map sends 1 to $[Z]$. Hence $\mathrm{Cl}(X)$ is generated by $[Z]$.

We first show that $2Z = 0$ in $\mathrm{Cl}(X)$. This is because we can consider the divisor of y. The rational function y is invertible in every stalk $\mathcal{O}_{X,p}$ except when $p \in V(y)$. Moreover, by the defining equation $xy = z^2$, we see that the divisor of y can only be non-zero along Z. The valuation at the generic point η of Z is 2: The local ring equals

$$\mathcal{O}_{X,\eta} = (k[x, y, z]/(xy - z^2))_{(y,z)}$$

and since x is invertible here, we see that $y \in (z^2)$ and that z is the uniformizer.

Now we show that Z is not a principal divisor. It suffices to prove that this is not principal in $\operatorname{Spec} \mathcal{O}_{X,p}$ where $p \in X$ is the singular point of X. The local ring here equals

$$\mathcal{O}_{X,p} = (k[x,y,z]/(xy - z^2))_{(x,y,z)}$$

In this ring $\mathfrak{p} = (x,z)$ is a height 1 prime ideal, but it is not principal: Let $\mathfrak{m} \subset \mathcal{O}_{X,x}$ be the maximal ideal. Note that $x, y \in \mathfrak{m}$, since x, y are not units. Moreover, it is clear that the vector space $\mathfrak{m}/\mathfrak{m}^2$ (which is the Zariski cotangent space at x) is 3-dimensional, spanned by $\{\overline{x}, \overline{y}, \overline{z}\}$. Then $\overline{x}, \overline{y}$ gives a 2-dimensional subspace of $\mathfrak{m}/\mathfrak{m}^2$. Hence, since $\overline{x}$ and $\overline{y}$ are linearly independent in this quotient, there couldn't be an non-constant element $f \in \mathcal{O}_{X,x}$ for which $x = af, y = bf$. This means that $[Z] \neq 0$ in $\mathrm{Cl}(X)$ and hence

$$\mathrm{Cl}(X) = \mathbb{Z}/2.$$

Note that the open subscheme $X - (0,0,0)$ *is* factorial. Hence removing a codimension 2 subset has an effect on $\mathrm{CaCl}(X)$. Recall however, that the class group of Weil divisors $\mathrm{Cl}(X)$ stays unchanged under removing a codimension 2 subset.

Projective quadric cone

Let $X = \operatorname{Proj} R$ where $R = k[x,y,z,w]/(xy - z^2)$. Let $H = V(w)$ be the hyperplane determined by w. We have

$$0 \to \mathbb{Z}H \to \mathrm{Cl}(X) \to \mathrm{Cl}(X - H) \to 0$$

(Here H is a divisor corresponding to the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$, hence it is non-torsion in $\mathrm{Cl}(X)$, so the first map is injective). $X - H$ is isomorphic to the affine quadric cone from before, hence $\mathrm{Cl}(X - H) = \mathbb{Z}/2$. Using this sequence, we see that $\mathrm{Cl}(X) = \mathbb{Z}$, generated by a Weil divisor D such that $H = 2D$. More precisely, D is the divisor $V(x,z)$ which is supported on a line on X.

The Weil divisor D is not Cartier; being Cartier is a local condition, so this follows from the example of the affine quadric cone above. Here is an alternative way to see it: If $D = V(x,z)$ is Cartier, the sheaf $L = \mathcal{O}_X(D)$ is invertible, and hence so is its restrtction to the line $\ell = V(x,z) \simeq \mathbb{P}^1_k$. The Picard group of $\mathbb{P}^1_k$ is $\mathbb{Z}$, generated by $\mathcal{O}_{\mathbb{P}^1}(1)$, so we have $L|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \in \mathbb{Z}$. On the other hand, we know that the divisor $H = 2D$ is Cartier and in fact $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^3}(1)|_X$ (the local generator is given by w). Restricting further to ℓ, we obtain $\mathcal{O}_{\mathbb{P}^3}(1)|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ (as the divisor of w is just one point on ℓ). But these two observations imply that $2a = 1$, a contradiction. Hence D is not Cartier.

Quadric hypersurfaces in higher dimension

Here is an application of the 'useful exact sequence' (20.5).

LEMMA 20.52 (NAGATA'S LEMMA) *Let A be a noetherian integral domain, and let $x \in A - 0$. Suppose that (x) is prime, and that A_x is a UFD. Then A is a UFD.*

PROOF: We first show that A is normal. Of course A_x is normal, being a UFD. So if $t \in K(A)$ is integral in A, it lies in A_x. We need to check that if $a/x^n \in A_x$ is integral over A and x does not divide a, then $n = 0$. If we have an integral relation

$$(a/x^n)^N + b_1(a/x^n)^{N-1} + \cdots + b_N = 0$$

Multiplying by x^{nN} we get $a^N \in xA$, so $x|a$, because A is an integral domain. Hence A is normal.

Now the Weil divisor $D = \operatorname{div} x$ is an effective divisor and so there is an exact sequence

$$\mathbb{Z}D \to \operatorname{Cl}(\operatorname{Spec} A) \to \operatorname{Cl}(\operatorname{Spec} A_x) = 0 \to 0$$

The image of the left-most map is 0, so $\operatorname{Cl}(A) = 0$, and so A is a UFD. ❑

Let $A = k[x_1, \dots, x_n, y, z]/(x_1^2 + \cdots + x_m^2 - yz)$. We will prove that A is a UFD for $m \geqslant 3$. A is a domain, since the defining ideal is prime. Apply Nagata's lemma with the element y:

$$A_y = k[x_1, \dots, x_n, y, y^{-1}, z]/(y^{-1}(x_1^2 + \cdots + x_m^2) - z) \simeq k[x_1, \dots, x_n, y, y^{-1}, z]$$

which is a UFD. We show that y is prime: Taking the quotient we get

$$A/y = k[x_1, \dots, x_n, x]/(x_1^2 + \cdots + x_m^2)$$

which is an integral domain, because $x_1^2 + \cdots + x_m^2$ is irreducible (for $m \geqslant 3$).

Note that for $m = 2$, we get the quadric cone, which we have seen is not a UFD.

Applying a change of variables, we find the following description of the class groups of quadrics in any dimension:

PROPOSITION 20.53 *Let k be a field containing $\sqrt{-1}$ and let $X = V(x_0^2 + \cdots + x_m^2) \subset \mathbb{A}_k^{n+1} = \operatorname{Spec} k[x_0, \dots, x_n]$.*

i) $m = 2$, $\operatorname{Cl}(X) = \mathbb{Z}/2$

ii) $m = 3$, $\operatorname{Cl}(X) = \mathbb{Z}$

iii) $m \geqslant 4$, $\operatorname{Cl}(X) = 0$

There is also the following statement for projective quadrics:

PROPOSITION 20.54 *Let $X = V(x_0^2 + \cdots + x_m^2) \subset \mathbb{P}^n = \operatorname{Proj} k[x_0, \dots, x_n]$.*

i) $m = 2$, $\operatorname{Cl}(X) = \mathbb{Z}$;

ii) $m = 3$, $\operatorname{Cl}(X) = \mathbb{Z}^2$;

iii) $m \geqslant 4$, $\operatorname{Cl}(X) = \mathbb{Z}$.

Exercises

✲ (20.10) Show that for the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, d)$ we have $\operatorname{Cl}(\mathbb{P}) = \mathbb{Z}D$ and $\operatorname{CaCl}(\mathbb{P}) = \mathbb{Z}H$ where $H = dD$.

✲ (20.11) The same reasoning as for $\mathbb{P}^1_k$ can be applied to the affine line X with two origins. Compute $\mathrm{Pic}(X)$ for this example.

★

An example from number theory

We turn to an example from number theory and pick up the thread from Example 20.30. There we claimed that the class group of the quadratic extension $A = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[x]/(x^2+5)$ is equal to $\mathbb{Z}/2\mathbb{Z}$. We also verified that the class of $Y = V(2, 1+\sqrt{-5}))$ was a non-trivial two-torsion element. Here we complete the claim and show that the class of Y generates $\mathrm{Cl}(A)$.

Since A is a Dedekind ring, the class group is generated by prime divisors, so we will be through when we show that $V(\mathfrak{p})$ is equivalent to Y for each prime ideal in A that is not principal. The only non-principal prime ideals in A are those of the form $(p, a \pm \sqrt{-5})$ where $p \in \mathbb{Z}$ is a prime and a is an integer that satisfy a congruence $a^2 + 5 \equiv 0 \mod p$, and altering a by a multiple of p, we may assume that $0 \leqslant a < p$.

The proof goes by induction on p, and to lubricate the induction, we shall prove a somehow more general statement. Note that the lemma with $n = p$ yields what we want; that the class of every non-principal prime divisor equals Y.

Lemma 20.55 *For each ideal $\mathfrak{a} = (n, a+\sqrt{-5})$ for any integers n and a satisfying $n \geqslant 2$ and $a^2+5 \equiv 0 \mod n$, there are non-zero elements f and g in A so that*

$$(f)(n, a \pm \sqrt{-5}) = (2, 1+\sqrt{-5})^{\epsilon}(g)$$

where either $\epsilon = 0$ or $\epsilon = 1$.

The two signs in the statement reflects the two possible choices of the square root, and it will suffice to do the case $a + \sqrt{-5}$; it is however crucial that $(2, 1+\sqrt{-5}) = (2, 1-\sqrt{-5})$.
Proof: The most significant portion of the proof is the induction part which reduces the proof to the case that $n \leqslant 5$ (which subsequently is done case by case):

So we assume $n > 5$ and proceed by induction on n; we write $a^2 + 5 = bn$, and compute

$$(b)(n, a+\sqrt{-5}) = (bn, b(a+\sqrt{-5})) = \tag{20.7}$$
$$= (a^2+5, b(a+\sqrt{-5})) = (a-\sqrt{-5}, b)(a+\sqrt{-5}). \tag{20.8}$$

Now, $bn = a^2+5 < n^2+5$, so as $n > 5$, it follows that $b < n$ and clearly $a^2+5 \equiv 0 \mod b$. By induction, it follows that for approriate elements f' and g from A, we have the equality

$$(f') \cdot (b, a+\sqrt{-5}) = (2, 1+\sqrt{-5})^{\epsilon} \cdot (g),$$

and so muliplying xxx through by f' we arrive at

$$(bf')(n, a+\sqrt{-5}) = (2, 1+\sqrt{-5})^{\epsilon} \cdot (g)(a+\sqrt{-5}).$$

It remains to treat the special cases with $n \leqslant 5$. Again, write $a^2+5 = bn$ with $0 \leqslant a < n$. When $n = 5$, it holds that $a^2 = 5(b-1)$, and this implies that $b = 1$ and $a = 0$. One easily verifies that all ideals $(n, \sqrt{-5})$ are principal (either generated by 1 or by $\sqrt{-5}$). That $n = 4$ is impossible since no square is congruent $-1 \mod 4$. Finally, if $n = 3$ and $a = 1$, we have

$$(2)(3, 1 + \sqrt{-5}) = (6, 2(1 + \sqrt{-5})) = (1 - \sqrt{-5}, 2)(1 + \sqrt{-5}),$$

and if $a = 2$, we note that $(3, 2 + \sqrt{-5}) = (3, 1 - \sqrt{-5})$. We are left with the case $n = 2$ and $a = 1$, which is exactly what we want. ❑

EXERCISE 20.12 Let d be a square free integer and assume that $d \not\equiv 1 \mod 4$ so that $\mathbb{Z}[\sqrt{d}]$ is a Dedekind ring. Show that the class group of $\mathbb{Z}[\sqrt{d}]$ is finite. HINT: Concider the indction portion of the proof above. ★

20.6 Extended example: Hirzebruch surfaces

Let $r \geqslant 0$ be an integer and consider the scheme X which is glued together by the four affine scheme charts

$$\begin{aligned} U_{00} &= \operatorname{Spec} k[x, y] \\ U_{01} &= \operatorname{Spec} k[x, y^{-1}] \\ U_{10} &= \operatorname{Spec} k[x^{-1}, x^r y] \\ U_{11} &= \operatorname{Spec} k[x^{-1}, x^{-r} y^{-1}] \end{aligned}$$

This is a non-singular, integral 2-dimensional scheme over k. When $k = \mathbb{C}$, these are the so-called r-th *Hirzebruch surfaces*. In many ways, these surface behave as the 'Möbius strips' in algebraic geometry. Note in particular, when $r = 0$, we get $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

Divisors

Let us define two divisors D_1, D_2 on X by the following Cartier divisors. Writing $K = k(x, y)$, the Cartier data is given by

$$D_1 = \begin{bmatrix} (U_{00}, x), & (U_{01}, x) \\ (U_{10}, 1), & (U_{11}, 1) \end{bmatrix}, \qquad D_2 = \begin{bmatrix} (U_{00}, y), & (U_{01}, 1) \\ (U_{10}, y), & (U_{11}, 1) \end{bmatrix}$$

We will show that D_1, D_2 generate $\operatorname{CaCl}(X) \simeq \operatorname{Pic}(X)$. Note that both of these divisors are effective, since they are defined by rational functions which are regular on the U_{ij}. In fact $D_1 \simeq \mathbb{P}^1$, since it is glued together by $V_0 = U_{00} \cap V(x) = \operatorname{Spec} k[y]$ and $V_1 = U_{01} \cap V(x) = \operatorname{Spec} k[y^{-1}]$. Moreover, D_2 restricted to D_1 is given by the Cartier data $(V_0, y), (V_1, 1)$, which corresponds to a closed point on D_1. Hence $\mathcal{O}_X(D_2)_{D_1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. This shows that D_2 is non-torsion in $\operatorname{Cl}(X)$, since no multiple of it is trivial when restricted to D_1. A similar argument shows that D_1 is non-torsion in $\operatorname{Cl}(X)$.

Let D_1' be the Cartier divisor

$$D_1' = \begin{bmatrix} (U_{00}, 1), & (U_{01}, 1) \\ (U_{10}, x^{-1}), & (U_{11}, x^{-1}) \end{bmatrix},$$

We can compute that $D_1'|_{D_1}$ equals the divisor $(V_0, 1), (V_1, 1)$ which is principal. In fact,

$$\operatorname{div} x = D_1 - D_1'$$

So, $D_1 = D_1'$ in $\mathrm{Cl}(X)$. This shows that D_1 and D_2 are independent in $\mathrm{Cl}(X)$, because $\mathcal{O}_X(D_1')$ restricts to $\mathcal{O}_X$ on D_1, but as $\mathcal{O}_{\mathbb{P}^1}(1)$ on D_2.

Now let $U = X - D_1 - D_2$. This is isomorphic to U_{11} which is the spectrum of $k[x^{-1}, x^{-r}y^{-1}]$ which is a UFD. The exact sequence

$$\mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \to \mathrm{Cl}(X) \to 0$$

and the previous analysis shows that $\mathrm{Cl}(X) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2$.

Sheaf cohomology

We now want to compute $H^i(X, \mathcal{O}_X(D))$ for a divisor $D = aD_1 + bD_2$. As usual, we utilize a Čech complex. We want to mimick the computation for $\mathbb{P}^n$. In that proof, the $\mathbb{Z}$-graded polynomial ring $k[x_0, \ldots, x_n]$ played an important role, and the groups in the Čech complex corresponded to degree 0 localizations of it. For X there is no such $\mathbb{Z}$-graded ring lying around, but we can get by by introducing the *bigraded* ring

$$R = k[x_0, x_1, y_0, y_1]$$

where the degrees of the variables are defined by

$$\deg x_0 = (1, 0), \deg x_1 = (1, 0), \deg x_2 = (0, 1), \deg x_3 = (-r, 1).$$

By identifying $x = \frac{x_1}{x_0}$, $y = \frac{x_0^r y_1}{y_0}$, we find that the Čech complex of U_{ij} can be written

$$\begin{matrix} R_{[x_0x_1]} & & R_{[x_0x_1y_0]} & & \\ \oplus & & \oplus & & \\ R_{[x_0x_1]} & & R_{[x_0x_1y_0]} & & \\ \oplus & \to & \oplus & \to & R_{[x_0x_1y_0y_1]} \\ R_{[x_0x_1]} & & R_{[x_0x_1y_0]} & & \\ \oplus & & \oplus & & \\ R_{[x_0x_1]} & & R_{[x_0x_1y_0]} & & \end{matrix}$$

where the bracket means that we take the $(0, 0)$-part of the localization. So for instance,

$$R_{[x_0y_0]} = R\left[\frac{x_1}{x_0}, \frac{x_0^r y_1}{y_0}\right]$$

As in the $\mathbb{P}^n$ case, we now have a bigraded isomorphism

$$\bigoplus_{a,b \in \mathbb{Z}} H^0(X, \mathcal{O}_X(aD_1 + bD_2)) \simeq \bigcap_{i,j} R_{[x_iy_j]} = R$$

In particular, $H^0(X, \mathcal{O}_X(aD_1 + bD_2))$ can be identified with polynomials of bidegree (a, b) in R. So for example $H^0(X, \mathcal{O}(D_1))$ corresponds to degree $(1, 0)$-polynomials, e.g., linear combinations of x_0, x_1. These two sections correspond to the sections $1, x$ above. Similarly, $H^0(X, \mathcal{O}(D_2))$ is 1-dimensional.

Perhaps the most interesting divisor is $E = D_2 - rD_1$,which is effective. When $r > 0$, this has $H^0(X, \mathcal{O}_X(nE)) = k$ for every $n \geqslant 0$, but $\mathcal{O}_X(nE) \not\simeq \mathcal{O}_X$ for $n \neq 0$. Therefore E couldn't possibly be globally generated. In fact, $E \simeq \mathbb{P}^1$ and

$$\mathcal{O}_X(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1}(-r)$$

In particular, this gives an example of an effective divisor which does not pull back (restrict) to an effective divisor.

Chapter 21

Representable functors*

A common theme in mathematics is to study a mathematical object via the set of morphisms from other objects into it. For instance, if X is a topological space, we can recover its underlying set by considering all maps $\{*\} \to X$ from a one-point-set to X. If X is a manifold, we define the fundamental group by looking at maps $S^1 \to X$. In fact, we have seen several examples of this philosophy already in the context of schemes:

EXAMPLE 21.1 (*R-valued points.*) For a ring R, we studied the set of morphisms

$$\operatorname{Spec} R \to X.$$

That is, $X(R) = \operatorname{Hom}(\operatorname{Spec} R, X)$. In particular, we saw that we could recover each point $x \in X$ by taking the residue field $R = k(x)$. ☆

EXAMPLE 21.2 (*Tangent vectors.*) Morphisms

$$\operatorname{Spec}(k[\epsilon]/\epsilon^2) \to X$$

are in bijection with pairs (x, v) where $x \in X$ is a k-point, and $v \in T_{X,x}$ is a tangent vector (i.e., a linear functional $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$) ☆

For each scheme T, the set of morphisms $T \to X$ gives us some information about X, but is unlikely that we can recover *all* the information about X from this set alone. A fundamental insight of Grothendieck was to consider all such morphisms $T \to X$ at once, i.e., to consider the whole functor $\operatorname{Hom}_{\mathsf{Sch}}(-, X)$ as a functor from schemes to sets. This is the so-called 'functor of points'.

DEFINITION 21.3 *For a scheme X, we define its* functor of points *to be the functor*

$$\begin{aligned} h_X : \mathsf{Sch}^{op} &\to \mathsf{Sets} \\ T &\mapsto \operatorname{Hom}_{\mathsf{Sch}}(T, X) \end{aligned}$$

That is $h_X(T)$ gives the set of all maps from X to T.

Note that if $S \to T$ is a morphism of schemes, there is an induced map $\operatorname{Hom}_{\mathsf{Sch}}(T, X) \to \operatorname{Hom}_{\mathsf{Sch}}(S, X)$. This is the reason for writing Sch^{op} rather than Sch, to emphasis that h_X is a *contravariant* functor.

The assignment $X \mapsto h_X$ defines a functor (the "Yoneda embedding") from schemes to the category of functors

$$h_\bullet : \mathsf{Sch} \to Fun((\mathsf{Sch})^{op}, \mathsf{Sets})$$

by associating to any morphism $f : X \to Y$ the natural transformation $h_f : h_X \to h_Y$ given by

$$\begin{aligned} h_f(Z) : h_X(Z) &\to h_Y(Z) \\ g \in h_X(Z) &\mapsto f \circ g \in h_Y(Z). \end{aligned}$$

Here $Fun((\mathsf{Sch})^{\mathrm{op}}, \mathsf{Sets})$ denotes the category of functors $F : (\mathsf{Sch})^{\mathrm{op}} \to \mathsf{Sets}$; morphisms $\eta : F \to G$ are given by natural transformations, i.e., for each morphism of schemes $h : S \to T$ there should be a commutative diagram of maps of sets

$$\begin{tikzcd} F(T) \arrow[r, "F(h)"] \arrow[d, "\eta(T)"] & F(S) \arrow[d, "\eta(S)"] \\ G(T) \arrow[r, "G(h)"] & G(S) \end{tikzcd}$$

The Yoneda lemma

A simple, but amazingly useful result is the so called *Yoneda lemma*. It says that there is a bijection between the set of arrows $X \to Y$ and the set of natural transformations of functors $h_X \to h_Y$.

"Tell me who your friends are, and I'll tell you who you are"

Lemma 21.4 (Yoneda lemma) *The functor $h_\bullet$ is fully faithful; or in other words, the map of sets*

$$\mathrm{Hom}_{\mathsf{Sch}}(X, Y) \to \mathrm{Hom}_{\mathsf{Fun}}(h_X, h_Y) \tag{21.1}$$

is bijective. Thus every natural transformation $h_X \to h_Y$ is of the form h_f for a unique morphism $f : X \to Y$.

Proof: Let $F : h_X \to h_Y$ be a natural transformation. Applying F to the scheme X, we get a map

$$F(X) : h_X(X) = \mathrm{Hom}_{\mathsf{Sch}}(X, X) \to \mathrm{Hom}_{\mathsf{Sch}}(X, Y) = h_Y(X)$$

If there is an $f : X \to Y$ such that $h_f = F$, then we must have

$$F(X)(\mathrm{id}_X) = h_f(X)(\mathrm{id}_X) = f \circ \mathrm{id}_X = f,$$

so f is determined by F. Thus (21.1) is injective.

For surjectivity, put $f = F(X)(\mathrm{id}_X)$; we will check that $h_f = F$. This means that for any scheme Z, the map of sets

$$F(Z) : h_X(Z) \to h_Y(Z)$$

is equal to the map of sets

$$h_f(Z) = - \circ f : h_X(Z) \to h_Y(Z)$$

Since F is a natural transformation, we have for any $g : Z \to X$, a commutative diagram

$$\begin{tikzcd} h_X(X) \arrow[r, "F(X)"] \arrow[d, "h_X(g)"] & h_Y(X) \arrow[d, "h_Y(g)"] \\ h_X(Z) \arrow[r, "F(Z)"] & h_Y(Z) \end{tikzcd}$$

Going through the diagram clockwise, we see that $\mathrm{id}_X Z$ gets sent to $g \circ f$, while going counterclockwise, id_X gets sent to $F(Z)(g)$. Hence

$$F(Z)(g) = g \circ f = h_f(Z)(g)$$

and so $F = h_f$.

❑

In particular, we have the following consequences:

Corollary 21.5 *i)* *h_X and h_Y are isomorphic (as functors from Sch^{op} to Sets), if and only if $X \simeq Y$.*

ii) *If a functor F is the same as h_X for some scheme X, then X is determined up to isomorphism.*

For our purposes, the important consequence of this is that instead of specifying the scheme explictly, say by giving a projective embedding and a homogeneous ideal, we can simply specify a functor equivalent to h_X, and this will precisely pin down what scheme we are talking about.

Replacing the scheme X with its associated functor of points h_X may at this point seem like just yet another jump in abstraction, but the nice thing is that you can work with functors whose values are good old sets. For instance, by the Yoneda lemma, we see that to give a morphism $f : X \to Y$ of schemes is the same thing as for each scheme Y giving a map of sets $f(T) : X(T) \to Y(T)$ which is functorial in T (i.e., any map $u : T' \to T$ induces a commutative square with $X(T), X(T'), Y(T), Y(T')$.) In fact, using that schemes are locally affine, and that morphisms of schemes glue, we see that it is even sufficient to test this condition on affine schemes $T = \operatorname{Spec} B$.

Here is another example:

Example 21.6 In the chapter on fiber products, we defined the universal property of fiber products for any two arrows $\psi_1 : X_1 \to S$, $\psi_2 : X_2 \to S$ in a category $\mathscr{C}$. This fiber product is an object $X_1 \times_S X_2$ (provided it exists!) in $\mathscr{C}$ equipped with two projections $\pi_i : X_1 \times_S X_2 \to X_i$ such that $\psi_1 \circ \pi_1 = \psi_2 \circ \pi_2$, and $X_1 \times_S X_2$ is universal as such, meaning that for any object X in $\mathscr{C}$, with two arrows $\phi_i \colon X \to X_i$ in C such that $\phi \circ \psi_1 = \phi \circ \psi_2$, there is a unique arrow $\phi \colon X \to X_1 \times_S X_2$ satisfying $\pi_i \circ \phi = \phi_i$ for $i = 1, 2$.

In the category of schemes, we saw that fiber products always exists, but the product had some surprising features. In particular, the underlying set of the fiber product was not the fiber product of the underlying sets.

There is a nice way to explain this in terms of the functors of points h_{X_i}. In this picture, the given arrows ψ_i induce natural transformations $h_{\psi_i} \colon h_{X_i} \to h_S$ sending an arrow $f \in h_{X_i}(T)$ to the composition $\psi_i \circ f$. The universal property of the fiber product translates into the following: For any scheme T, one has a bijection of sets (!)

$$h_{X_1 \times_S X_2}(T) \xrightarrow{\simeq} h_{X_1}(T) \times_{h_S(T)} h_{X_2}(T) \tag{21.2}$$

In other words, there is an isomorphism of functors in $Fun(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Sets})$.

$$h_{X_1 \times_S X_2} \xrightarrow{\simeq} h_{X_1} \times_S h_{X_2} \tag{21.3}$$

where the arrow sends an arrow $\psi \in h_{X_1 \times_S X_2}(T)$ to the pair of arrows $(\pi_1 \circ \psi, \pi_2 \circ \psi)$. The salient point is that for any scheme T, the set $\mathrm{Hom}_{\mathsf{Sch}}(T, X_1 \times_S X_2)$ is the fiber product of the two sets $\mathrm{Hom}_{\mathsf{Sch}}(T, X_1)$ and $\mathrm{Hom}_{\mathsf{Sch}}(T, X_2)$ over $\mathrm{Hom}_{\mathsf{Sch}}(T, S)$. Thus the fiber product of schemes is not so mysterious after all; it is forced upon us by the universal property of fiber products of sets! ☆

The second reason for introducing h_X is that the schemes we encounter in algebraic geometry are often defined in terms of functors in the first place: We want to construct schemes as solutions to problems in algebraic geometry, e.g., through their universal properties. The most basic example is probably projective space, which we construct as the 'universal parameter space' for lines through the origin in a vector space. More sophisticated examples are *Hilbert schemes* and *moduli spaces*, where one seeks a scheme that parameterizes geometric objects up to isomorphism. The strength here is that we are usually more interested in, and indeed better equipped to studying, the actual objects appearing in the problem than the actual underlying structure of the scheme. Thus we are often able to study a scheme X, by "probing it" via the set of morphisms $T \to X$ into it.

Definition 21.7 *We say that a functor F is* representable by a scheme *if it is isomorphic (as a functor) to h_X for some scheme X. By the Yoneda lemma, X is unique, if it exists.*

Here are a few examples:

Example 21.8 Let A be a ring and consider the functor $F : \mathsf{Sch}^{op} \to \mathsf{Sets}$ given by

$$F(S) = \mathrm{Hom}_{\mathsf{Rings}}(A, \Gamma(S, \mathcal{O}_S))$$

Using Corollary 5.6, we see that F is representable by the scheme $X = \operatorname{Spec} A$. ☆

Example 21.9 In particular, the functor $F : \mathsf{Sch}^{op} \to \mathsf{Sets}$ given by

$$F(S) = \Gamma(S, \mathcal{O}_S)$$

is representable by the scheme $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[X]$. More generally, $\mathbb{A}^n$ represents the functor $F(S) = \Gamma(S, \mathcal{O}_S)^n$; this is just a fancy way of saying that a morphism $X \to \mathbb{A}^n$ is the same thing as an n-tuple of regular functions.

☆

Example 21.10 The functor $F : \mathsf{Sch}^{op} \to \mathsf{Sets}$ given by

$$F(S) = \Gamma(S, \mathcal{O}_S)^{\times}$$

is representable by the scheme $\mathbb{A}^1 - 0 = \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$. ☆

A functor of points behaves very much like a sheaf. In fact, $F = h_Y$ induces a sheaf (of sets) on every scheme. For an open cover U_i of a scheme X, we have induced 'restriction maps' $F(X) \to F(U_i)$ (of sets). Moreover, two elements $s, t \in F(X)$ which map to the same element in each $F(U_i)$, must be equal, and, a collection of elements $s_i \in F(U_i)$ that agree on the overlaps $F(U_i \cap U_j)$ glue to a unique element $s \in F(U_i)$. This all follows from the Gluing Lemma for morphisms of sheaves.

In fact, this observation gives us examples of functors which are not representable:

EXAMPLE 21.11 The functor $F : \mathsf{Sch}^{op} \to \mathsf{Sets}$ given by

$$F(S) = \{\text{locally free sheaves of rank } r\}/\mathit{isomorphism}$$

is not representable. To see why, assume that there is a scheme Y such that $F \simeq h_Y$. For a scheme S, let $o_S \in F(S)$ denote the element corresponding to the trivial sheaf $\mathcal{O}_S^r$.

Now, let X be a scheme, and let $\mathcal{E}$ be a locally free sheaf $\mathcal{E}$ or rank r on it. We may choose a covering U_i of X such that $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^r$. This means that the corresponding element $e \in F(X)$ must map to $o_{U_i} \in F(U_i)$ via the all the maps $F(X) \to F(U_i)$. The same is of course true for the element $o_X \in F(X)$, so by the above remark, we must have that $e = o_X \in F(X)$. We have just showed that any locally free sheaf on T is trivial. However, this is clearly not the case, e.g., for $X = \mathbb{P}^n$ and $\mathcal{E} \simeq \mathcal{O}(1)^r$. ☆

EXERCISE 21.1

a) Let $n > 0$ be an integer. Find a scheme which represents the functor

$$F(X) = \{s \in \mathcal{O}_X(X) | s^n = 0\}.$$

b) Show that the functor defined by

$$F(X) = \{s \in \mathcal{O}_X(X) | s \text{ is nilpotent}\}$$

is not representable.

★

EXERCISE 21.2 Let $\phi\colon Y \to X$ be an arrow in $\mathscr{C}$. One says that ϕ is *injective* or a *monomorphism* if $\phi \circ \alpha_1 = \phi \circ \alpha_2$ implies $\alpha_1 = \alpha_2$ for any pair of arrows $\alpha_i\colon T \to Y$. Show that ϕ is a monomorphism if and only if $h_X(\phi)$ is. ★

EXERCISE 21.3 Assume that $\phi_i\colon Y_i \to X$ for $i = 1, 2$ are two monomorphisms. Assume that $h_{Y_1}(T) \subseteq h_{Y_2}(T)$ for all T. Show that one may factor $\phi_2 = \phi \circ \phi_1$ for a unique monomorphism $\phi\colon X_1 \to X_2$. ★

21.1 *Projective space as a functor*

So which functor does projective space represent? Intuitively, we search for a functor F, so that we can associate to each morphism $f : X \to \mathbb{P}^n$ to a *set* $F(X)$ of data on X. The corresponding question was already answered for $\mathbb{A}^n$: Morphisms $X \to \mathbb{A}^n$ are in bijective and functorial correspondence with the set $\Gamma(X, \mathcal{O}_X)^n$, i.e., an n-tuple of regular functions $f_1, \ldots, f_n$ on X. By Theorem 16.37 on page 245, the corresponding statement for $\mathbb{P}^n$ was: an $(n+1)$-tuple of sections of an invertible sheaf L that locally generate L anywhere.

This motivates the following functor: For a scheme X we set

$$F(X) = \big\{(L, s_0, \ldots, s_n) \big| s_0, \ldots, s_n \in \Gamma(X, L) \text{ generate } L \text{ everywhere}\big\} \,/ \sim$$

where the $\sim$ says that $(L, s_0, \ldots, s_n) \sim (M, t_0, \ldots, t_n)$ if and only if there is an isomorphism $f : L \to M$ so that $f^*(s_i) = \lambda \cdot t_i$ for some $\lambda \in \mathcal{O}_X^\times(X)$.

This is indeed a contravariant functor, because given a morphism $f : X \to Y$, and data $(L, s_0, \ldots, s_n) \in F(Y)$, we can pull back sectrions to get $(f^*L, f^*s_0, \ldots, f^*s_n)$, whose equivalence class defines an element of $F(X)$.

Note also that on projective space $\mathbb{P}^n = \operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_n]$, we have the data required in the defintion, namely

$$(L, s_0, \ldots, s_n) = (\mathcal{O}(1), x_0, \ldots, x_n).$$

THEOREM 21.12 *The functor F is represented by the scheme $\mathbb{P}^n$.*

PROOF: We need to define natural transformations of the two functors $h_{\mathbb{P}^n}$ to F. For a scheme X, we define

$$\Phi(X) : \operatorname{Hom}(X, \mathbb{P}^n) \to F(X)$$

by sending a morphism $f : X \to \mathbb{P}^n$ to the equivalence class of the data

$$(L, s_0, \ldots, s_n) = (f^*\mathcal{O}(1), f^*x_0, \ldots, f^*x_n)$$

It is clear that this assignment is functorial, i.e., it gives a natural transformation. Moreover, by definition Φ sends the identity map $\mathrm{id}_{\mathbb{P}^n}$ to the equivalence class of the sequence above.

To prove the theorem, we need to construct an inverse Ψ to Φ. To each $(L, s_0, \ldots, s_n)$, we let $\Phi(L, s_0, \ldots, s_n)$ be the corresponding morphism $f : X \to \mathbb{P}^n$. By the construction in Theorem 16.37, we do have an isomorphism $L = f^*\mathcal{O}(1)$ and $s_i = f^*x_i$ for $i = 0, \ldots, n$. Thus Ψ is the inverse to Φ. ❑

EXAMPLE 21.13 Note that elements of $F(\operatorname{Spec} k)$ are in correspondence with $(n+1)$-tuples $(a_0, \ldots, a_n) \in k^{n+1}$, so that not all a_i are zero. This follows because all invertible sheaves on $\operatorname{Spec} k$ are trivial. Here we consder two $(n+1)$-tuples $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ equivalent if there is a scalar $\lambda \in k^\times$ so that $b_i = \lambda a_i$ for all i. In other words, by the theorem, we recover the usual description of the k-points of projective space as '1-dimensional subspaces of k^{n+1}'. Everything here also works verbatim for a local ring. ☆

EXAMPLE 21.14 What about $F(\operatorname{Spec} R)$ for a ring R? The line bundle L corresponds to a projective R-module M of rank 1. The condition on the sections $s_0, \ldots, s_n$ means that there is a surjection $R^{n+1} \to M \to 0$, which, by projectivity must split. Thus $F(\operatorname{Spec} R)$ is in bijection with the set of rank 1 summands of R^{n+1}, i.e., modules of rank 1 such that $M \oplus E \simeq R^{n+1}$ for some module E. This is the right generalization of a "line in k^n" for general rings. ☆

21.2 Grassmannians

Grassmannian varieties are central objects in classical algebraic geomerty. Just like projective space parameterize lines through the origin of a vector space, Grassmannians are designed to parameterize linear subspaces of a fixed vector space. More precisely, for natural numbers r and n, there is a variety $Gr(r,n)$ whose k-points $[W]$ are in bijection with the set of linear subspaces $W \subset k^n$. Thus projective space $\mathbb{P}_k^{n-1}$ is the special case when $r = 1$. More generally, $Gr(r,n)$ is a non-singular projective variety over k of dimension $r(n-k)$.

The basic idea is that an r-dimensional linear subspace $L \subset k^n$ is determined by a basis of r vectors $w_1, \ldots, w_r \in W$. We encode these as the rows of a $r \times n$ matrix M. Of course this matrix is not unique in determining L; each choice of basis gives a matrix with the same property. In fact, two matrices M and M' give rise to the same subspace L if and only if they are related by an element of $GL_r(k)$, that is, $M' = AM$ for some $A \in GL_r(k)$.

Let $Mat(r,n) \simeq k^{rn}$ denote the space of $r \times n$-matrices and let $W \subset Mat(r,n)$ denote the open set consisting of the matrices of rank r. . Based on the above paragraph, we want to form the quotient space

W is the complement of the closed set defined by all $r \times r$-minors of the generic matrix, so it is open

$$\pi : W \to W/GL_r(k)$$

This is of course possible in the category of topological spaces; we simply give $G := W/SL_r(k)$ the quotient topology, by declaring an open set $U \subset G$ to be open if and only if $\pi^{-1}U$ is open in W). We can even define a sheaf on G by letting $\mathcal{O}_G(U)$ be the group of functions $f : U \to k$ such that $f \circ \pi \in \mathcal{O}_V(\pi^{-1}(U))$. This makes $(G, \mathcal{O}_G)$ into a locally ringed space.

What is not yet clear, is that the quotient space can be given the structure of a scheme. However, we know that this should be possible in the case $r = 1$, because taking the quotient of $W = k^{n+1} - 0$ by $GL_1(k) = k^\times$ is exactly what we did for projective space $\mathbb{P}^n$. For $\mathbb{P}^n$ we could form the affine cover by looking at the charts where some x_i was non-zero. We can generalize this to find an affine cover of G as follows.

If a subspace $L \subset k^n$ is represented by M, we can by Gaussian elimination find a more canonical representative, by putting M in reduced echelon form. In other words, we may represent each L by a matrix M with some $r \times r$ identity matrix as a submatrix. Conversely, any such matrix M determines a subspace L of k^n and now different matrices M, M' of this form give rise to different subspaces L, L'. Note that matrices M with a fixed $r \times r$ identity submatrix are parameterized by an affine space of dimension $nr - r^2 = r(n-r)$. It therefore makes sense to try to construct the variety $Gr(r,n)$ by gluing together these $\binom{n}{r}$ affine spaces.

We translate this into the language of schemes as follows. Let k be a ring and let

$$\mathbb{A}_k^{rn} = \operatorname{Spec} k \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \ddots & & \vdots \\ x_{r1} & \cdots & & x_{rn} \end{bmatrix}$$

denote the affine space of dimension rn. We write the variables in a $r \times n$ array, because we would like to think of the k-points of $\mathbb{A}_k^{rn}$ as $r \times n$-matrices with entries in k.

Write $M = (x_{ij})$ for the matrix of indeterminates. For each ordered tuple $I = (i_1, \ldots, i_r)$ with $1 \leqslant i_1 < \cdots < i_r \leqslant n$, let M_I denote the submatrix of M given by the columns in I. Also, let M^I be M with the columns indexed by I replaced by the identity matrix.

For each I, consider the closed subset $U_I \subset Mat(r,n)$ of matrices m where the submatrix m_I is the identity matrix. As these are defined by r^2 linear equations, we have

$$U_I \simeq \mathbb{A}_k^{r(n-r)}$$

for each I. We will for simplicity write $U_I = \operatorname{Spec} R_I$, where $R_I = k[M^I]$ denotes the polynomial ring with variables in the columns indexed by I^c.

EXAMPLE 21.15 For $n = 4, r = 2$, there are 6 such affine spaces each isomorphic to $\mathbb{A}^4$:

$$U_{12} = \operatorname{Spec} k \begin{bmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{bmatrix}, \quad U_{13} = \operatorname{Spec} k \begin{bmatrix} 1 & x_{12} & 0 & x_{14} \\ 0 & x_{22} & 1 & x_{24} \end{bmatrix},$$
$$U_{14} = \operatorname{Spec} k \begin{bmatrix} 1 & x_{12} & x_{13} & 0 \\ 0 & x_{22} & x_{23} & 1 \end{bmatrix}, \quad U_{23} = \operatorname{Spec} k \begin{bmatrix} x_{11} & 1 & 0 & x_{14} \\ x_{21} & 0 & 1 & x_{24} \end{bmatrix},$$
$$U_{24} = \operatorname{Spec} k \begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ x_{21} & 0 & x_{23} & 1 \end{bmatrix}, \quad U_{34} = \operatorname{Spec} k \begin{bmatrix} x_{11} & x_{12} & 1 & 0 \\ x_{21} & x_{22} & 0 & 1 \end{bmatrix}.$$

☆

Next, for two ordered pairs I, J, let us define the open set of U_I

$$U_{I,J} = D(\det M_J^I) \subset U_I,$$

corresponding to matrices $m \in U_I$ so that also the submatrix m_J is invertible. Note that if m is a matrix with $\det m_I \neq 0$, then $m_I^{-1} m$ has the identity matrix in columns I, so it belongs to U_I. This gives a map

$$\tau_{JI} : U_{I,J} \to U_{J,I}$$

sending M^J to $(M_J^I)^{-1} M^I$. More formally, τ_{JI} is induced from the ring map

$$\phi_{IJ} : (R_J)_{\det M_I^J} \to (R_I)_{\det M_J^I}$$

sending x_{ij} to the (i,j)-th entry of $(M_J^I)^{-1} M^I$. Here $\phi_{JI}(\det M_I^J) = 1/\det M_J^I$, so the map is well defined in the localization. For instance, in the example above, with $I = (1,3)$ and $J = (1,2)$, $(M_J^I)^{-1} M^I$ equals

$$\begin{pmatrix} 1 & x_{12} \\ 0 & x_{22} \end{pmatrix}^{-1} \begin{pmatrix} 1 & x_{12} & 0 & x_{14} \\ 0 & x_{22} & 1 & x_{24} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{x_{12}}{x_{22}} & \frac{-x_{12}x_{24}+x_{14}x_{22}}{x_{22}} \\ 0 & 1 & \frac{1}{x_{22}} & \frac{x_{24}}{x_{22}} \end{pmatrix}$$

which means that the ring map $\phi_{12,13} : (R_{13})_{x_{23}} \to (R_{12})_{x_{22}}$ sends

$$\begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} \mapsto \begin{pmatrix} -\frac{x_{12}}{x_{22}} & \frac{-x_{12}x_{24}+x_{14}x_{22}}{x_{22}} \\ \frac{1}{x_{22}} & \frac{x_{24}}{x_{22}} \end{pmatrix}$$

By the theory of Gaussian elimination (which we are allowed to carry out in localizations of $k[M^I]$), the maps ϕ_{JI} are isomorphisms, and satisfy the cocycle condition for gluing

$$\tau_{KI} = \tau_{KJ} \circ \tau_{JI}. \tag{21.4}$$

Indeed, if A is an $r \times n$ matrix (with identity matrix in the I-columns), (21.4) follows from the matrix identity

$$(A_K)^{-1}A = (A_J^{-1}A)_K(A_J^{-1}A)$$

We therefore obtain a gluing diagram

$$\begin{tikzcd} U_I \simeq \mathbb{A}_k^{r(n-r)} & & & U_J \simeq \mathbb{A}_k^{r(n-r)} \\ & U_{I,J} \arrow[ul, hook] \arrow[r, "\tau_{JI}"] & U_{J,I} \arrow[ur, hook] & \end{tikzcd}$$

We call the resulting scheme the *Grassmannian* $Gr_k(r,n)$ over k. The case when $k = \mathbb{Z}$ is of special interest; in this case we refer to $Gr_{\mathbb{Z}}(r,n)$ simply as the *Grassmannian*.

We will also be interested in the case when k is a field: in this case, Proposition xxx below says that the k-points of $Gr(r,n)$ is in bijection with the set of r-dimensional subspaces of k^n. In fact, the construction works over any scheme: For any scheme S, there is a relative Grassmannian scheme $Gr_S(r,n) \to S$ which is obtained by gluing copies of $\mathbb{A}_S^{r(n-r)}$.

PROPOSITION 21.16 *The Grassmannian $Gr(r,n)$ is a non-singular scheme.*

The Plucker relations

Consider the morphism

$$\Phi : Mat(r,n) \to \mathbb{A}^{\binom{n}{r}} = \operatorname{Spec} k[y_I]$$

which is induced by the $r \times r$ determinants, $\Phi^\sharp(y_I) = \det M_I$. This induces a morphism

$$\phi : U \to \mathbb{P}^{\binom{n}{r}-1}$$

where $U = Mat(r,n) - V(\det M_I)$ is the open subset of matrices of full rank.

If we consider $U_I \subset U$, we have an induced morphism

$$\phi_I : U_I \to D_+(y_I) \subset \mathbb{P}^{\binom{n}{r}-1}.$$

LEMMA 21.17 *The morphisms ϕ_I are closed immersions.*

PROOF: After re-indexing, it suffices to consider the case $I = \{1, 2, \ldots, r\}$. Then, if we expand the minors of the $r \times n$ matrix (corresponding to a matrix U_I)

$$\begin{bmatrix} 1 & 0 & 0 & x_{1,r+1} & \ldots & x_{1,n} \\ 0 & \ddots & 0 & \vdots & \ldots & \vdots \\ 0 & 0 & 1 & x_{r,r+1} & \ldots & x_{r,n} \end{bmatrix}$$

it is clear that the set of minors contains (up to sign) all the variables x_{ij} with $j \notin I$. Thus $\phi_I : \mathbb{A}^{r(n-r)} \to \mathbb{A}^{\binom{n}{r}-1}$ is given by the graph of a morphism $\mathbb{A}^{r(n-r)} \to \mathbb{A}^{\binom{n}{r}-1-r(n-r)}$, and hence is a closed immersion. ❑

This means that $Gr(r,n)$ embeds as a closed subscheme of $\mathbb{P}^{\binom{n}{r}-1}$ by the morphism ϕ. The image of ϕ is defined by all the polynomial relations between the $r \times r$-minors of the matrix M. These relations are known to be generated by the so called *Plucker quadrics*. These quadrics take the form

$$\sum_{l=1}^{r+1} (-1)^l y_{i_1,\dots,i_{r-1},j_l} y_{j_1,\dots,\hat{j_l},\dots j_{r+1}} = 0, \tag{21.5}$$

for any two ordered sequences I, J. Here $j_1,\dots,\hat{j_l}\dots j_{r+1}$ denotes the sequence $j_1,\dots,\dots j_{r+1}$ with the term j_l omitted.

The proof that these quadrics generate the entire ideal of the Grassmannian is not particularly difficult, but it requires algebraic manipulations that make it a little bit lengthy. Besides we do not need this fact in what follows. In fact, $Gr(k,n)$ is probably the best example of a projective variety that is not best studied by its equations in some projective space.

EXAMPLE 21.18 Let us continue the example of $n = 4$, $r = 2$. In this case, the image of ϕ is the quadric

$$Q = V(y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23}) \subset \mathbb{P}^5$$

This follows by the relation between the minors of M:

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix} - \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} \begin{vmatrix} x_{12} & x_{14} \\ x_{22} & x_{24} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} = 0$$

and the fact that both $Gr(2,4)$ and Q are non-singular of the same dimension (over a field, they are 4-dimensional).

Let us consider the the chart $D_+(y_{34}) \subset Q$. Here it is clear that $D(y_{34}) \simeq \operatorname{Spec} k[y_{13}, y_{14}, y_{23}, y_{24}]$, because inverting y_{34} allows us to eliminate y_{12} by the above equation. The corresponding map $U_{34} \to D(y_{34})$ is induced by the map $k[y_{13}, y_{14}, y_{23}, y_{24}] \to k[x_{11}, x_{12}, x_{21}, x_{22}]$ taking $(y_{13}, y_{14}, y_{23}, y_{24})$ to $(-x_{21}, x_{11}, -x_{22}, x_{12})$. This is clearly an isomorphism. Note that a k-point $(a : b : c : d : e : 1) \in Q$ corresponds to the matrix

$$\begin{pmatrix} c & e & 1 & 0 \\ -b & -d & 0 & 1 \end{pmatrix}$$

and a is uniquely determined by $a = be - cd$. ☆

The universal sheaves on Gr(r, n)

Just like the projective space carries a *tautological sheaf* $\mathcal{O}(-1)$, the Grassmannian carries a *universal subbundle*. This sheaf is what makes $Gr(r,n)$ represent the *Grassmannian functor*, see Section xxx.

In fact, on $G = Gr(r,n)$ there is an exact sequence

$$0 \to \mathcal{S} \to \mathcal{O}_G^n \to \mathcal{Q} \to 0$$

where $\mathcal{S}$ and $\mathcal{Q}$ are locally free sheaves of rank r and $n-r$ respectively. $\mathcal{S}$ is called the *universal subbundle* and $\mathcal{Q}$ is called the *universal quotient bundle*.

To define $\mathcal{S}$ and the map $\mathcal{S} \to \mathcal{O}_G^n$, we work over the affine open sets $U_I = \operatorname{Spec} R_I$. The idea is to take the submodule of R_I^n generated by the row vectors of M. That is, we take the rows $m_1, \ldots, m_r$ of the $r \times n$ matrix M^I and define a sub module of R_I^n:

$$S_I = R_I \cdot m_1 + \cdots + R_I \cdot m_r \subset R_I^n$$

In other words, S_I is the image of the map $R_I^r \to R_I^n$ given by the *transpose* of M^I

Note that $S_I \simeq R_I^r$ and that the quotient R_I^n / S_I is also free of rank $n-r$. This follows from the fact that M^I contains the identity matrix in columns I.

EXAMPLE 21.19 In the example above, the submodule for $I = (13)$ is given by

$$R_I \begin{pmatrix} 1 \\ x_{12} \\ 0 \\ x_{14} \end{pmatrix} + R_I \begin{pmatrix} 0 \\ x_{22} \\ 1 \\ x_{24} \end{pmatrix} \subset R_I^4$$

☆

It follows almost immediately by the 'linear algebraic' nature of the construction that these inclusions glue together to a subsheaf of $\mathcal{O}_G^n$. More precisely, we can construct gluing maps $\nu_{JI} : \mathcal{O}_{U_{IJ}}^r \to \mathcal{O}_{U_{JI}}^r$ using left multiplication by the $r \times r$-matrix $(M_I^J)^{-t}$ (which is invertible on the overlaps U_{IJ}). This makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{O}_{U_{I,J}}^r \arrow[r, "(M^I)^t"] \arrow[d, "(M_J^I)^{-t}"] & \mathcal{O}_{U_{I,J}}^n \arrow[d, equal] \\ \tau_{JI}{}^*\mathcal{O}_{U_{J,I}}^r \arrow[r, "\tau_{JI}^*(M^I)^t"] & \tau_{JI}{}^*\mathcal{O}_{U_{J,I}}^n \end{tikzcd}$$

Moreover, as a consequence of (21.4), we can check that the cocycle conditions are satisfied. So the sheaves $\widetilde{S}_I$ on U_I glue to a sheaf § together with an injective map $S \to \mathcal{O}_{Gr(r,n)}^n$. The cokernel, Q, is also locally free, since it is free over each U_I, as we remarked above.

REMARK 21.20 *We can define the map $Gr(k,n) \to \mathbb{P}_{\mathbb{Z}}^{\binom{n+r}{r}-1}$ using the Yoneda lemma, by giving a natural transformation between the functors. If S is a scheme, this transformation takes an S-valued point of $Gr(k,n)$, that is, a quotient $\mathcal{O}^n \to Q \to 0$, to $\wedge^r \mathcal{O}^n \to \wedge^r Q \to 0$, which since $\wedge^r Q$ has rank 1, defines a point in $\mathbb{P}_{\mathbb{Z}}^{\binom{n+r}{r}-1}(S)$.*

The Grassmannian functor

Theorem 21.21 *$Gr(r,n)$ represents the functor*

$$T \mapsto \{\textit{short exact sequences } 0 \to \mathcal{S} \to \mathcal{O}_T^n \to \mathcal{Q} \to 0\}/\textit{isomorphism}$$

Other examples

If we modify of the functor of points of $\mathbb{P}^n$, we obtain other interesting examples of schemes parameterizing geometric objects. The following examples are only meant as basic illustrations of this point - they will not appear later in the notes.

Example 21.22 (*Projective bundles.*) Let $\mathcal{E}$ be a locally free sheaf on a scheme X. Consider the functor on Sch/X given by

$$F(h : Y \to X) = \{\text{invertible sheaf quotients } h^*\mathcal{E} \to L \to 0\}$$

Then F is represented by a scheme $\mathbb{P}(\mathcal{E}) \to X$ over X called the projectivization of $\mathcal{E}$. One can think of closed points of this scheme as hyperplanes in the fibers of $\mathcal{E}$. ☆

Example 21.23 (Proj *of an $\mathcal{O}_X$-module.*) Let X be a scheme and let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Consider the functor on Sch/X given by

$$F(h : Y \to X) = \{\text{invertible sheaf quotients } h^*\mathcal{F} \to L \to 0\}$$

Then F is represented by a scheme $\mathrm{Proj}(\mathcal{F})$. Many geometric constructions can be formulated as such schemes. For instance, if $\mathcal{I}$ is a sheaf of ideals on X, then $\mathrm{Proj}(\mathcal{F})$ can be identified with the blow-up of X along $\mathcal{I}$ (see Hartshorne ([**?**]) II.7). ☆

Chapter 22

Differentials

So far we have defined schemes and surveyed a few of their basic properties (*e.g.* how to study sheaves on them). In this chapter, we introduce tangent spaces and *Kähler differentials*, which allow us in some sense to do calculus on schemes. This in turn will allow us to define the most important sheaves in algebraic geometry, namely, the cotangent sheaf, the tangent sheaf, and the sheaves of n-forms.

Differentials appear prominently throughout many areas of mathematics, multivariable analysis, manifolds and differential geometry to mention a few. In algebraic geometry they are introduced algebraically using their formal properties and are usually referred to as Kähler differentials after the German mathematician Erich Kähler (1906–2000).

22.1 Derivations and Kähler differentials

Derivations

We will work over a base ring A, and B will be an A-algebra. We will also need a B-module M. The geometric picture to have in mind is that $A = k$, where k is a field, and $X = \operatorname{Spec} B \to \operatorname{Spec} k$ is the structure morphism.

Definition 22.1 *An A-*derivation *(from B with values in M) is an A-linear map $D\colon B \to M$ satisfying the product rule, also called the Leibniz rule:*

$$D(bb') = bD(b') + b'D(b).$$

Given that the product rule holds, it is easy to see that D is A-linear if and only if it vanishes on all elements of the form $a \cdot 1$ with $a \in A$; indeed, if D is A-linear, we have $D(a \cdot 1) = aD(1) = 0$ since $D(1) = 0$, which follows from the product rue applied to $1^2 = 1$. If D vanishes on A, the product rule gives $D(ab) = aD(b) + bD(a) = aD(b)$. We may therefore think of the elements in B of the form $a \cdot 1$ as 'constants'; note however, that a derivation also can vanish on other elements in B (a stupid example is the zero map, which is a derivation. For a more constructive example see Example 22.9 below).

Example 22.2 The map of the polynomial ring $B = k[x]$ to itself which is given by $P(t) \mapsto P'(t)$, is a k-derivation. More generally, the partial differential operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$, as well as their k-linear combinations, are k-derivations on the polynomial ring $k[x_1, \ldots, x_n]$.

A straightforward induction shows that the good old rules from calculus have analogues in the abstract situation: it holds true that $D(b^n) = nb^{n-1}D(b)$ and, in case b is invertible in B, that $D(1/b) = -D(b)/b^2$. Moreover, if $P(t)$ is a polynomial in $A[t]$, one has the chain rule $D(P(b)) = P'(t)D(b)$, where $P'(t)$ is the formal derivative defined as $P'(t) = \sum_i ia_it^{i-1}$ when $P(t) = \sum_i a_it^i$.

The set of A-derivations $D\colon B \to M$ is usually denoted by $\operatorname{Der}_A(B, M)$. This set inherits a B-module structure from M, and it is as such naturally a submodule of $\operatorname{Hom}_A(B, M)$. This gives rise to a covariant functor $\operatorname{Der}_A(B, -)$ from Mod_B to itself. More precisely, if $\phi\colon M \to M'$ is a B-linear map, we can map a derivation $D \in \operatorname{Der}_A(B, M)$ to $\phi \circ D\colon B \to M'$, which is in turn an A-derivation of B with values in M'.

The set of derivations $\operatorname{Der}_A(B, M)$ is also functorial in the base ring A and the A-algebra B; in both cases it is contravariant. If $A \to A'$ is a ring homomorphism, any A'-derivation $B \to M$ is in turn an A-derivation. We therefore obtain an inclusion $\operatorname{Der}_{A'}(B, M) \subseteq \operatorname{Der}_A(B, M)$.

The module of Kähler differentials

The covariant functor $\operatorname{Der}_A(B, -)$ on the category of B-modules is representable. This simply means that there exists a distinguished B-module $\Omega_{B/A}$ and an isomorphism of functors

$$\operatorname{Der}_A(B, -) \simeq \operatorname{Hom}_B(\Omega_{B/A}, -). \tag{22.1}$$

In more down-to-earth terms, this condition is equivalent to there being a *universal derivation** $d_B\colon B \to \Omega_{B/A}$ that has the following property: For any A-derivation $D\colon B \to M$ there exists a unique B-module homomorphism $\alpha\colon \Omega_{B/A} \to M$ such that $D = \alpha \circ d_B$. In terms of diagrams, we have

$$\begin{tikzcd} B \arrow[r, "d_B"] \arrow[dr, "D"'] & \Omega_{B/A} \arrow[d, dashed, "\alpha"] \\ & M. \end{tikzcd}$$

**The ring A is an essential part of the structure,but for the sake of a practical notation is not shown; when it is necessary to emphasize the base ring, the notation will be $d_{B/A}$*

The slogan is: each derivation is the pushout of the universal derivation d_B by some B-linear map.

To see directly why such a module exists, we can construct it via generators and relations. For each element $b \in B$ introduce a symbol db and consider the free B-module $G = \bigoplus_{b \in B} Bdb$ they generate. Inside G we have the submodule H generated by all expressions of one of the types

$$d(b + b') - db - db', \text{ or } \quad d(bb') - bdb' - b'db, \text{ or } \quad da$$

for $b, b' \in B$ and $a \in A$. We then define $\Omega_{B/A} = G/H$, and the map $d_B\colon B \to \Omega_{B/A}$ is given as $d_B(b) = db$. It is well-defined as a group homomorphism since any $\mathbb{Z}$-linear relation among the db's maps to zero in G/H by the imposed additive constraint, and it is a derivation because all relations $d(bb') = bdb' + b'db$ are forced to hold in G/H. Finally, it will be A-linear since $da = 0$ in G/H.

It is not hard to see that this module indeed satisfies the universal property above: given an A-derivation $D\colon B \to M$, we define the B-homomorphism $\alpha\colon \Omega_{B/A} \to M$ by $\alpha(db) = D(b)$ (which is well-defined precisely because D is a derivation!).

DEFINITION 22.3 *The elements of the module $\Omega_{B/A}$ are called the* Kähler differentials, *or simply* differentials *of B over A.*

EXAMPLE 22.4 (*Change of constants.*) To any homomorphism of rings $\rho\colon A \to A'$ corresponds the natural inclusion $\mathrm{Der}_{A'}(B, M) \subseteq \mathrm{Der}_A(B, M)$, which via the isomorphism (22.1) induces a surjective B-linear map

$$\beta\colon \Omega_{B/A} \to \Omega_{B/A'}.$$

It is just the B-linear map that arises from $d_{B/A'} \circ \rho$ by the universal property of $d_{B/A}$. moreover, it renders the diagram below commutative In terms of the generating sets in the construction above, the map β simply sends db to db; note that $da' \mapsto 0$ for all $a' \in B$ coming from A'. ☆

EXAMPLE 22.5 Let $B = k[t]$ be the polynomial ring over the field k. Then $\Omega_{B/k}$ is a free module over B generated by dt, *i.e.* $\Omega_{B/k} = B \cdot dt$. This is best seen by verifying the universal property: the map $d_B\colon B \to B \cdot dt$ with $d_B(f(t))) = f'(t)dt$ is a k-derivation, any other derivation $D\colon B \to M$ comply to the same rule $D(f(t)) = f'(t)D(t)$; hence the corresponding B-linear map α may be given as $\alpha(bdt) = bD(t)$. ☆

More generally we have:

PROPOSITION 22.6 *Let A be any ring and let $B = A[x_1, \ldots, x_n]$. Then $\Omega_{B/A}$ is the free B-module generated by $dx_1, \ldots, dx_n$ and the universal derivation is given by*

$$d_B f = \sum (\partial f/\partial x_i) dx_i.$$

PROOF: The universal property follows from the general chain rule: for any A-derivation $D\colon B \to M$ into a B-module M, the formula

$$D(f) = \sum_i (\partial f/\partial x_i) D(x_i). \tag{22.2}$$

holds true. Indeed, an easy induction, using the product rule, shows it to be true when f is a monomial, and then A-linearity finishes he story. The B-linear map $\alpha\colon \bigoplus_i Bdx_i \to M$ which sends each basis element dx_i to $D(x_i)$, will be the wanted factoring map; by the general chain rule (22.2), it satisfies the equality $D = \alpha \circ d_B$. ❑

Examples

Here are some more explicit calculations of $\Omega_{B/A}$:

(22.7) (*Localization.*) If $B = S^{-1}A$ is a localization of A, then $\Omega_{B/A} = 0$. Indeed, take $b \in B$, and choose $s \in S$ so that $sb \in A$. Then $sd_B b = d_B(sb) = 0$, which implies that $d_B b = 0$ since s is invertible in B.

(22.8) (*Surjections.*) Generally, if $\phi \colon A \to B$ is surjective, then $\Omega_{B/A} = 0$, because if $b = \phi(a)$, then $d_B b = a \cdot d_B(1) = 0$ in $\Omega_{B/A}$.

(22.9) (*Separable field extensions.*) Let $K = k(a)$ be a separable field extension and let $P(t)$ be the minimal polynomial of a. For any k-derivation $D \colon K \to K$ it holds that $0 = D(0) = D(P(a)) = P'(a)D(a)$. Hence $D(a) = 0$ since $P'(a) \neq 0$ the element a being separable over k. The product rule implies that $D(a^n) = na^{n-1}D(a) = 0$ for each natural number n, and since the powers a^n generate K as a vector space over k, it follows that $D = 0$.

(22.10) (*Inseparable field extensions.*) Contrary to the separable ones, inseparable extensions have non-trivial derivations. Let us consider the simplest case when K is obtained by adjoining a p-th root to a field k of characteristic p; that is, $K = k(b)$ with $b^p = a$, where $a \in k$ is not a p-th power. The minimal polynomial of b is $P(t) = t^p - a$, and $K = k[t]/(t^p - a)$. The point is that $P'(t) = pt^{p-1} = 0$, so for each $c \in K$ the k-linear map $k[t] \to K$ given by $Q(t) \mapsto Q'(t)c$ vanishes on $P(t)$ and descends to a k-linear map $D_c \colon K \to K$. Leibniz' rules immediately yields that D_c is a derivation, and as $D_c(b) = c$, the derivation D_c does not vanish. We conclude that $\mathrm{Der}_k(K, K) \simeq K$ and that $\Omega_{K/k} \simeq K$ as well; in fact, D_b serves as a universal derivation.

(22.11) (*The differentials of a tensor product.*) Let B and C be two A-algebras. Then the map

$$d \colon B \otimes_A C \to (\Omega_{B/A} \otimes_A C) \oplus (B \otimes_A \Omega_{C/A})$$

given as $b \otimes c \mapsto b \otimes d_C c + d_B b \otimes c$ on decomposable tensors and extended by bilinearity is a universal A derivation. We compute

$$\begin{aligned} d(b'b \otimes c'c) &= bb' \otimes (c'd_C c + cd_C c') + (b'd_B b + bd_B b') \otimes cc' = \\ &= b' \otimes c' \cdot (b \otimes d_C c + d_B b \otimes c) + b \otimes c \cdot (b' \otimes d_C c' + d_B b' \otimes c') \end{aligned}$$

and δ is a derivation, and which is universal in view of the formula

$$\gamma(db \otimes c + b' \otimes dc') = 1 \otimes c' \cdot \alpha(b' \otimes 1) + b' \otimes 1 \cdot \beta(1 \otimes c'),$$

which defines the required map $\gamma \colon (\Omega_{B/A} \otimes_A C) \oplus (B \otimes_A \Omega_{C/A}) \to M$. Here $\alpha \colon \Omega_{B/A} \to M$ and $\beta \colon \Omega_{C/A} \to M$ are the linear maps corresponding to the derivations $D|_{B \otimes 1} \colon B \to M$ and $D|_{1 \otimes C} \colon C \to M$ and $D \colon B \otimes_A C \to M$ is a given A-derivation.

☆

22.2 *Properties of Kähler differentials*

There are a few useful ways for computing modules of differentials when changing rings.

Base change

The Kähler differentials behave well with respect to base change:

PROPOSITION 22.12 *Let A be a ring and B be an A-algebra, and let A' be another A-algebra. Define $B' = B \otimes_A A'$. Then there is a canonical isomorphism*

$$\Omega_{B'/A'} \simeq \Omega_{B/A} \otimes_B B'$$

PROOF: The universal derivation $d_B \colon B \to \Omega_{B/A}$ induces an A'-linear map

$$d' = d_B \otimes \mathrm{id}_{A'} \colon B' \to \Omega_{B/A} \otimes_A A' = \Omega_{B/A} \otimes_B B'$$

which clearly is a derivation. This will be the required universal derivation of $\Omega_{B'/A'}$, and so the claim follows: let $\iota \colon B \to B' = B \otimes_A A'$ be the canonical map $b \mapsto b \otimes 1$. Given an A'-derivation $D \colon B' \to M$ into a B'-module, the map $D \circ \iota \colon B \to M$ will be an A-derivation, and consequently it will factor as $\alpha \circ d_B$ for a B-linear map $\alpha \colon \Omega_{B/A} \to M$. The map $\alpha \otimes \mathrm{id}_{A'} \colon \Omega_{B/A} \otimes_A A' \to M \otimes_A A' = M$ then yields the desired factorization of D. ❑

Two exact sequences

Let A be a ring and let $\rho \colon B \to C$ be a homomorphism of A-algebras. There is natural homomorphism of C-modules

$$\rho_* \colon \Omega_{B/A} \otimes_B C \to \Omega_{C/A}$$

defined by $\rho_*(d_B b \otimes c) = c d_C \rho(b)$. The dual of ρ_* corresponds, under the identification (22.1), to the map $\mathrm{Der}_A(C, N) \to \mathrm{Der}_A(B, N)$ that sends a derivation $D \colon C \to N$ to $D \circ \rho$. (Note that $\mathrm{Hom}_B(\Omega_{A/B}, N) = \mathrm{Hom}_C(\Omega_{A/B} \otimes_B C, N)$ since N is a C-module.)

Moreover, there is a canonical 'change-of-constants-map'

$$\beta \colon \Omega_{C/A} \longrightarrow \Omega_{C/B}$$

as explained in Example 22.4 above.

The next propositions describes the kernel of this 'change-of-constants-map', and as one would suspect, it is generated by the elements shaped like db where $b \in C$ comes from B:

PROPOSITION 22.13 *The following sequence of C-modules is exact*

$$\Omega_{B/A} \otimes_B C \xrightarrow{\rho_*} \Omega_{C/A} \xrightarrow{\beta} \Omega_{C/B} \longrightarrow 0$$

PROOF: That $\beta \circ \rho_* = 0$ is clear. Checking exactness amounts to showing that for any C-module N, the dual sequence

$$0 \to \mathrm{Hom}_C(\Omega_{C/B}, N) \to \mathrm{Hom}_C(\Omega_{C/A}, N) \to \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, N)$$

is exact, and, as the map β is surjective (Example 22.4), only exactness in the middle is an issue. Note that $\operatorname{Hom}_C(\Omega_{B/A} \otimes_B C, N) = \operatorname{Hom}_B(\Omega_{B/A}, N)$, so the in view of the constituting isomorphisms (22.1), the sequence can be written as

$$0 \longrightarrow \operatorname{Der}_B(C, N) \longrightarrow \operatorname{Der}_A(C, N) \longrightarrow \operatorname{Der}_A(B, N).$$

The map on the left merely considers a B-derivation to be an A-derivation, whereas the one on the right sends $D\colon C \to N$ to the composition $D \circ \rho$. Saying that D is mapped to zero in $\operatorname{Der}_A(B, N)$, is saying that it vanishes on all elements b in C coming from B, which is equivalent to saying it is a B-derivation; indeed, it will B-linear by Leibniz rule:

$$D(bx) = bD(x) + xD(b) = bD(x),$$

for $x \in C$ and $b \in C$ coming from B. ❑

In the next proposition we establish an exact sequence that relates the differentials of an A-algebra B and those of a quotient $C = B/I$. It involves a map $\delta\colon I/I^2 \to \Omega_{B/A} \otimes_B C$ which sends the class of $f \in I$ mod I^2 to $d_B f \otimes 1$, or more formally, which results from applying the tensor functor $- \otimes_B C$ to the restriction $d_B|_I\colon I \to \Omega_{B/A}$. (Note that $I \otimes_B C = I/I^2$ as $C = B/I$).

PROPOSITION 22.14 (CONORMAL SEQUENCE) *Suppose that B is an A-algebra. Let $C = B/I$ for some ideal $I \subset B$ and let $\alpha\colon B \to C = B/I$ be the canonical map. Then there is an exact sequence of C-modules*

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \xrightarrow{\alpha_*} \Omega_{C/A} \longrightarrow 0.$$

PROOF: As in the previous proposition it suffices to check that for each C-module N, the dual sequence

$$0 \longrightarrow \operatorname{Der}_A(C, N) \longrightarrow \operatorname{Der}_A(B, N) \longrightarrow \operatorname{Hom}_C(I/I^2, N) = \operatorname{Hom}(I, N)$$

is exact. In view of Proposition 22.13 and Exampe 22.8 the map α_* is surjective, and hence the leftmost map is injective. The rightmost map associates to a derivation $D\colon B \to N$ its restriction to I. (Note that this is indeed a homomorphism of C-modules since $IN = 0$). If $D|_I = 0$, clearly D passes to the quotient and yields a $D'\colon C = B/I \to N$, which is a C-derivation since D is a B-derivation. In other words, D lies in the image of $\operatorname{Der}_A(C, N)$, and the sequence is exact in the middle as well. ❑

COROLLARY 22.15 *Let A be a ring and let B be a finitely generated A-algebra (or a localization of such). Then $\Omega_{B/A}$ is finitely generated over B.*

PROOF: Write $B = A[x_1, \ldots, x_n]/I$ for some variables $x_1, \ldots, x_n$ and apply Proposition 22.6 on page 337 and the above proposition. ❑

✲ **EXERCISE 22.1** *(The diagonal and $\Omega_{B/A}$.)* Suppose that B is an A-algebra. There is an exact sequence of A-modules

$$0 \longrightarrow I \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$$

where μ is the multiplication map, which acts as $b_1 \otimes b_2 \mapsto b_1 b_2$ on decomposable tensors, and where I is the kernel of μ. Since $B \otimes_A B / I \simeq B$, the module I/I^2 has the structure of a B-module.

a) Show that I is generated by elements of the form $a \otimes 1 - 1 \otimes a$;
b) Show that the two B-module structure on I/I^2 induced from each factor of the tensor product agree; that is, $b \otimes 1 \cdot x = 1 \otimes b \cdot x$ for all $x \in I/I^2$;
c) Show that $d \colon B \to I/I^2$ defined by $db = b \otimes 1 - 1 \otimes b$ is an A-derivation;
d) Show that d is a universal derivation so that $I/I^2 \simeq \Omega_{B/A}$ and $d = d_{B/A}$.

★

EXERCISE 22.2 Let $A \to B$ be a map of Noetherian rings, $\pi \colon X \to Y$. Assume that $\Omega_{B/A} = 0$. Show that the diagonal Δ is a connected component of $X \times_Y X = \operatorname{Spec} B \otimes_A B$.

Assume that $I \subseteq A$ is finitely generated ideal such that $I^2 = I$. Show that I is a principal ideal generated by an idempotent. HINT: Let $\{x_i\}$ generate I and write $x_i = \sum_j a_{ij} x_j$ with $a_{ij} \in I$. Consider the matrix $\Phi = (\delta_{ij} - a_{ij})$. Show that $\det \Phi$ annihilates I, and hence there is an $e \in I$ so that $(1-e)I = 0$. Show that $e^2 = e$ and that $I = (e)$. ★

Kähler differentials and localization

When we later shall globalize the construction of the Kähler differenials, the following two results about their behavior with respect to localizations are important. They both rely on the sequence in Proposition 22.13.

PROPOSITION 22.16 *Let $S \subset A$ be a multiplicative subset mapping into the group of units in B. Then 'change-of-constants-map' is an isomorphism*

$$\Omega_{B/A} \simeq \Omega_{B/S^{-1}A}.$$

PROOF: The 'change-of-constants-map' is the map β in the sequence

$$\Omega_{S^{-1}A/A} \otimes_{S^{-1}A} B \longrightarrow \Omega_{B/A} \xrightarrow{\beta} \Omega_{B/S^{-1}A} \longrightarrow 0,$$

and by Example 22.7 we have $\Omega_{S^{-1}A/A} = 0$. ❑

PROPOSITION 22.17 *Suppose S is a multiplicative system in B and let $\iota \colon B \to S^{-1}B$ be the localization map. Then the natural map ι_* yields an isomorphism*

$$\iota \colon S^{-1}\Omega_{B/A} \simeq \Omega_{S^{-1}B/A}.$$

PROOF: Note that $S^{-1}\Omega_{B/A} = \Omega_{B/A} \otimes_B S^{-1}B$, so we are in the context of Proposition 22.13 and may use the exact sequence there. We previously checked that $\Omega_{S^{-1}B/B} = 0$ (Example 22.7) and hence ι_* is surjective. Thus in view of the identity $\operatorname{Hom}_{S^{-1}B}(S^{-1}\Omega_{B/A}, M) =$

$\mathrm{Hom}_B(\Omega_{B/A}, M,)$ which is valid for any $S^{-1}B$-module M, it suffices to see that the map

$$\mathrm{Der}_A(S^{-1}B, M) \longrightarrow \mathrm{Der}_A(B, M)$$

corresponding to ι_* is surjective. This is the case since every $D\colon B \to M$ extends to a derivation $D'\colon S^{-1}B \to M$ by the formula

$$D'(bs^{-1}) = (sDb - bDs)s^{-2}, \tag{22.3}$$

some checking must be done, which is left to the reader. ❑

✲ EXERCISE 22.3 Check that the expression $D'(bs^{-1})$ in (22.3) does not depend on the choice of representative for bs^{-1} and that the resulting D' is a derivation. ★

Some explicit computations

We can use the previous exact sequences to do explicit computations with $\Omega_{B/A}$. If B is a finitely generated A-algebra, say $B = A[x_1, \ldots, x_n]/I$ where $I = (f_1, \ldots, f_r)$. Then we have

$$\Omega_{A[x_1,\ldots,x_r]/A} \otimes_A B \simeq \textstyle\bigoplus_{i=1}^n Bdx_i.$$

The conormal sequence (Proposition 22.14), takes the form

$$I/I^2 \xrightarrow{\delta} \textstyle\bigoplus_{i=1}^n Bdx_i \longrightarrow \Omega_{B/A} \longrightarrow 0$$

and as I/I^2 is generated as a B-module by the classes of the $f_1, \ldots, f_r$ modulo I^2 there is a g surjection $B^r \to I/I^2$ wich gives the exact sequence

$$B^r \xrightarrow{\delta'} \textstyle\bigoplus_{i=1}^n Bdx_i \longrightarrow \Omega_{B/A} \longrightarrow 0.$$

Explicitly, in an appropriate basis, the map d' is given by the $n \times r$ *Jacobian matrix* $J = (\partial f_j / \partial x_i)$.

THEOREM 22.18 *Let A be a ring and let $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Then*

$$\Omega_{B/A} = \frac{\bigoplus_i Bdx_i}{\sum_j B\big(\sum_i (\partial f_j/\partial x_i)\ dx_i\big)}$$

and the universal A-derivation is given as $d_{B/A}f = \sum_i (\partial f/\partial x_i)\ dx_i$.

Examples

(22.19) *(Non-singular plane curves.)* Let k be a field and let $X = \operatorname{Spec} R$ where $R = k[x,y]/(f)$. Let us compute the module of differentials $\Omega_{R/k}$. Let f_x, f_y denote the (images of the) partial derivatives of f in R. Then by the above

$$\Omega_{R/k} = Rdx \oplus Rdy/(f_x dx + f_y dy).$$

If the curve X is non-singular; *i.e.* that $V(f, f_x, f_y) = \emptyset$, the module of differentials $\Omega_{R/k}$ will be a locally free R-module of rank one: local bases over the open sets $D(f_x)$ and $D(f_y)$ respectively, are given by

$$\frac{dy}{f_x} \text{ and } -\frac{dx}{f_y}.$$

Note, that on the overlap $D(f_x) \cap D(f_y)$ the two agree since we have $f_x dx + f_y dy = 0$ in $\Omega_{R/k}$.

(22.20) (*The nodal cubic.*) The curve $X \subseteq \mathbb{A}^2$ with equation $y^2 = x^2(x+1)$ is the so-called *nodal cubic*. It has a singular point at the origin $(0,0)$ and is regular elsewhere. Let $B = k[x,y]/(y^2 - x^2(x+1)$. Then

$$\Omega_{B/k} = Bdx \oplus Bdy/(2ydy - (3x^2 + 2x)dx)$$

In this case $\Omega_{B/k}$ has rank one for every point $(x,y) \neq (0,0)$, indeed, if $y \neq 0$ dx will be a basis, and dy will be one when $x \neq 0$, except where $3x + 2 = 0$, but these points are covered by the first case as $y \neq 0$ there.

At the origin, the relation $2ydy - (3x^2 + 2x)dx$ is identically zero, so $\Omega_{B/k}$ has rank two there.

We can also view B as an algebra over $A = k[x]$. In that case, we get

$$\Omega_{B/A} = B/(2y)dy \simeq k[x]/(x^2(x+1))dy.$$

(22.21) (*The cuspidal cubic curve.*) As indicated in he previous example the module of Käher differential will be locally free when A is the coordinate ring of a regular curve, but near singular points it can be a rather complicated module. Even non-trivial torsion elements can appear, as is the case for the coordinate ring of plane curves with isolated singularities (in fact, it is a conjecture it happens for all singular curves). Here we illustrate this by the coordinate ring $A = k[x,y]/(y^2 - x^3)$ of the cuspidal cubic, and you will find a discussion of the general case in Exercise 22.6.

The differentials are given as $\Omega_{A/k} = Adx \oplus Ady/(2ydy - 3x^2dx)$, and $\eta = 3ydx - 2xdy$ is a non-zero torsion element, it is in fact killed by y and x^2:

$$x^2\eta = 3x^2ydx - 2x^3dy = y(3x^2x - 2ydy) = 0;$$
$$y\eta = 3y^2dx - 2xydy = x(3x^2dx - 2ydy) = 0.$$

And being non-zero (see Exercise 22.4 below), η generates a submodule isomorphic to $k[x,y]/(x^2,y)$, which is supported at the singular point (the origin).

✲ EXERCISE 22.4 Check that η is non-zero, and that the torsion part of $\Omega_{A/k}$ is generated by η. ★

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Exercises

(22.5) Let $B = k[x,y]/(x^2 + y^2)$. Show that if k has characteristic $\neq 2$, then

$$\Omega_{B/k} = (Bdx + Bdy)/(xdx + ydy)$$

If k has characteristic 2, then $\Omega_{B/k}$ is the free B module $Bdx + Bdy$.

✲ (22.6) *Torsion in the Kähler differentials.* (This exercise requires some knowledge of Koszul complexes and homological algebra). Let $f \in k[x,y]$ be a polynomial without multiple factors and let $A = k[x,y]/(f)$. Show that the submodule of torsion elements of $\Omega_{A/k}$ is isomorphic the quotient $\big((f_x, f_y) : f\big)/(f_x, f_y)$ of the transporter ideal $\big((f_x, f_y) : f\big)$ in the polynomial ring $R = k[x,y]$. Show that $X = V(f)$ is regular if and only if $\Omega_{A/k}$ is torsion free. Show that, more precisely, the torsion is of length $\dim_k A/(f_x, f_y)A$. (This number is the sum of a contribution from each singular point, often called *the Tjurina number* of the singular point. The formula for the length is due to Zariski ([**?**])).

(22.7) Let $f \in k[x,y]$ be the equation of a non-singular curve. Let $A = k[x,y]/(f)$ and $B = k[x,y]/(f^2)$. Show that $\Omega_{B/k} \simeq Bdx \oplus Bdy$ if k is of characteristic two and that $\Omega_{B/k} \simeq \Omega_{A/k}$ if not.

(22.8) *Transcendental extensions.* Let k be a field and $K = k(x_1, \ldots, x_n)$ a purely transcendental field extension. Show that $\Omega_{K/k} \simeq K^n$ with $dx_1, \ldots, dx_n$ as a basis. HINT: Consider $k[x_1, \ldots, x_n]$ and use (**??**), then localize and use 22.17.

✲ (22.9) Assume that $k \subseteq K$ is a finitely generated field extension.

a) Show that $\dim_K \Omega_{K/k} \geqslant \operatorname{trdeg} K/k$;
b) Show that equality holds if and only if K is separably generated* over k.
c) Show that if k is perfect, it holds that $\dim_K \Omega_{K/k} = \operatorname{trdeg} K/k$, hence K is separably generated over k. HINT: Let $P(t) = \sum_i a_i t^i$ be a minimal polynomial in for x, show that $dP = P'(t)dt + \sum_i da_i \cdot t^i \in \Omega_{K[t]/k}$ is non zero.

★

A field extension $k \subseteq K$ is separably generated if there is a transcendence basis $x_1, \ldots, x_n$ for K over k so that K is separable over $k(x_1, \ldots, x_n)$. If in addition K is finitely generated over k, the K will be finite over $k(x_1, \ldots, x_n)$.

22.3 *The sheaf of differentials*

For us, the primary motivation for studying $\Omega_{B/A}$ is that it gives us an intrisic module $\Omega_{B/A}$ associated to a ring homomorphism $A \to B$. By applying $\sim$, we thus get an intrinsic sheaf on $X = \operatorname{Spec} B$ associated to the map of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$. We would like to globalize this construction to an arbitrary morphism of schemes $f \colon X \to S$. This will lead us to form the *sheaf of relative differentials* $\Omega_{X/S}$ which will be a quasi-coherent $\mathcal{O}_X$-module.

This sheaf is locally built out of the various $\Omega_{B/A}$ on local affine charts. These are not just arbitrary modules that just happen to glue together to a sheaf; each of them come with the universal property of classifying derivations $D \colon B \to M$. For this reason, we would like to say that the $\Omega_{X/Y}$ should satisfy a similar universal property. We make the following definition:

DEFINITION 22.22 *Let $\mathcal{F}$ be a quasi-coherent (?) $\mathcal{O}_X$ module. A morphism $D \colon \mathcal{O}_X \to \mathcal{F}$ of $\mathcal{O}_X$-modules is an* S-derivation *if for all open affine subsets $V \subset S$ and $U \subset X$ with $f(U) \subset V$, the map $D|_U$ is an $\mathcal{O}_S(V)$-derivation of $\mathcal{O}_X(U)$ with values in $\mathcal{F}$. The set of all such S-derivations is denoted by* $\operatorname{Der}_S(\mathcal{O}_X, \mathcal{F})$.

Definition 22.23 *The* sheaf of relative differentials *is a pair* $(\Omega_{X/S}, d_{X/S})$ *of a quasi coherent (?)* $\mathcal{O}_X$*-module* $\Omega_{X/S}$ *and a* S*-derivation* $d_{X/S}\colon \mathcal{O}_X \to \Omega_{X/S}$ *that satisfies the following universal property: For each quasi-coherent (?)* $\mathcal{O}_X$*-module* $\mathcal{F}$*, and each* S*-derivation* $D\colon \mathcal{O}_X \to \mathcal{F}$ *there exists a unique* $\mathcal{O}_X$*-linear map* $\alpha\colon \Omega_{X/S} \to \mathcal{F}$ *such that* $D = \alpha \circ d_{X/S}$*.*
When $S = \operatorname{Spec} A$*, we sometimes write* $\Omega_{X/A}$ *for* $\Omega_{X/S}$*.*

In other words, $\Omega_{X/S}$ is a sheaf that represents the functor of S-derivations, in the sense that there is a functorial isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, -) \simeq \operatorname{Der}_S(\mathcal{O}_X, -).$$

Exercise 22.10 Prove, using the universal property of differentials, that gives that this sheaf is unique up isomorphism, if it exists. ★

In the affine situation with a morphism $X = \operatorname{Spec} B \to S = \operatorname{Spec} A$ we have the module of K differentials $\Omega_{A/B}$ and the corresponding sheaf $\widetilde{\Omega_{A/B}}$ will serve as the sheaf of relative differential on X; this is just a consequence of $\sim$ being an equivalence of categories Mod_B and QCoh_X. In the general case, gluing the local differential on affine covers works well, and the main theorem of this section says that sheaves of relative Kähler differentials exist unconditionally.

Theorem 22.24 *Let* $f\colon X \to S$ *be a morphism of schemes. Then there is a sheaf of relative differentials* $\Omega_{X/S}$*, which is a quasi-coherent sheaf on* X*.*
Moreover, $\Omega_{X/S}$ *has the property that for each open affine open* $V = \operatorname{Spec} A$ *and each open affine* $U = \operatorname{Spec} B \subset f^{-1}(V)$ *it holds that*

$$\Omega_{X/S}|_U \simeq \widetilde{\Omega_{B/A}}.$$

Also for each $x \in X$*, we have*

$$(\Omega_{X/S})_x \simeq \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}.$$

Proof: Fix an open subset $V = \operatorname{Spec} A$ of S, and let $U = \operatorname{Spec} B$ be an affine open subset in X so that $f(U) \subset V$. For these two, we define

$$\Omega_{U/V} = \widetilde{\Omega_{B/A}}$$

which is a sheaf on U. We first show that the different $\Omega_{U/V}$ glue together to an $\mathcal{O}_{f^{-1}V}$-module $\Omega_{f^{-1}(V)/V}$ when U runs through an open affine cover of $f^{-1}(V)$. This comes down to showing that if $U' = \operatorname{Spec} B'$ is a distinguished open affine subset of U, then

$$\Omega_{U/V}|_{U'} \simeq \Omega_{U'/V}.$$

But as B' is a localization of B, Proposition 22.17, tells us that ι_* is such an isomorphism with $\iota\colon B \to B' \to$ the localization map. These maps depend functorially on the inclusions, so the gluing conditions are trivially fulfilled.

Then we show that the sheaves $\Omega_{f^{-1}V/V}$ for all affine opens $V \subseteq S$ glue to a $\mathcal{O}_X$-module $\Omega_{X/S}$. This amounts to showing that for each distinguished open $V' = \operatorname{Spec} A' \subseteq V$, and all open $U = \operatorname{Spec} B$ of $f^{-1}(V')$, we have

$$\Omega_{U/V} = \Omega_{U/V'}$$

But this follows from Proposition 22.16, as A' is a localization of A in a single element (which maps to an invertible element in B).

This means that we get an $\mathcal{O}_X$-module $\Omega_{X/S}$. Let us check that it satisfies the above universal property. So we need to define the universal derivation $d_{X/S} \colon \mathcal{O}_X \to \Omega_{X/S}$.

Let $V = \operatorname{Spec} A \subseteq S$ and $U = \operatorname{Spec} B \subseteq X$ be an affine open subset such that $f(U) \subseteq V$. Define $d_{X/S}(U) = d_{B/A}$. By the gluing construction above, this map does not depend on the chosen affine open V, and it can be checked that the assignment is compatible with restriction maps. Hence this gives a morphism of sheaves $d_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$, which by construction is an S-derivation.

To check that this is universal, we again work locally. Let $d \colon \mathcal{O}_X \to \mathcal{F}$ be an S-derivation, where $\mathcal{F}$ is an $\mathcal{O}_X$-module. Let $U = \operatorname{Spec} A \subseteq S$ and $V = \operatorname{Spec} B \subseteq X$ so that $f(U) \subseteq V$. By the universal property of $\Omega_{B/A}$, we get an A-derivation $D(V) \colon B \to \mathcal{F}(V)$, and hence a unique B-linear map $\alpha(V) \colon \Omega_{X/S}(V) = \Omega_{B/A} \to \mathcal{F}(V)$ such that $D(V) = \alpha(V) \circ d_{X/V}(V)$. One has to check that these maps are compatible with restriction maps (use the universal property of $\Omega_{B/A}$), but after that, we obtain a unique $\mathcal{O}_X$-linear map $\alpha : \Omega_{X/S} \to \mathcal{F}$ so that $D = \alpha \circ d_{X/S}$. ❑

Note that the sheaf $\Omega_{X|S}$ is always quasi-coherent (it is by definition locally of the form $\widetilde{M}$ for some module). Moreover, when X is of finite type over a field, $\Omega_{B/A}$ is finitely generated, and so $\Omega_{X|k}$ is even coherent.

Example 22.25 Let A be a ring and let $X = \mathbb{A}^n_S = \operatorname{Spec} A[x_1, \ldots, x_n]$ be affine n-space over $S = \operatorname{Spec} A$. Then $\Omega_{X/S} \simeq \mathcal{O}_X^n$ is the free $\mathcal{O}_X$-module generated by $dx_1, \ldots, dx_n$. ☆

If X is a separated scheme over S then one could also define $\Omega_{X/S}$ as follows. Let $\Delta : X \to X \times_S X$ be the diagonal morphism and let $\mathcal{I}_\Delta$ be the ideal sheaf of the image of Δ. Then $\Omega_{X/S} = \Delta^*(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)$. This does in fact give the same sheaf as above, since these two definitions coincide when X and S are both affine (Exercise 22.1). This definition gives a quick way of obtaining the sheaf $\Omega_{X/S}$, but it is not very enlightening nor suited for computations.

The properties of the Kähler differentials $\Omega_{B/A}$ translate into the following results for $\Omega_{X/Y}$:/Users/ellingsr/Desktop/Minken

Proposition 22.26 (Base change) *Let $f \colon X \to S$ be a morphism of schemes and let S' be a S-scheme. Let $X' = X \times_S S'$ and let $p \colon X' \to X$ be the projection. Then*

$$\Omega_{X'/S'} \simeq p^*\Omega_{X/S}$$

Proposition 22.27 *Let X, Y, and Z be schemes along with maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then there is an exact sequence of $\mathcal{O}_X$-modules*

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0. \tag{22.4}$$

Proposition 22.28 (Conormal sequence) *Let Y be a closed subscheme of a scheme X over S. Let $\mathcal{I}_Y$ be the ideal sheaf of Y on X. Then there is an exact sequence of $\mathcal{O}_X$-modules*

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_{X/S} \otimes \mathcal{O}_Y \to \Omega_{Y/S} \to 0. \tag{22.5}$$

22.4 *The Euler sequence and differentials of $\mathbb{P}^n_A$*

We have seen that the cotangent sheaf of the affine spaces $\mathbb{A}^n$ is trivial, *i.e.* they are isomorphic to $\mathcal{O}^n_{\mathbb{A}^n}$. In this section we will give a concrete description of the cotangent bundle of projective space, suitable for explicit computations. It comes as a short exact sequence, sometimes called the *Euler-sequence*, and involves a twist of the tautological map $\mathcal{O}^{n+1}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)$ that represents a point in $\mathbb{P}^n$ as 'the corresponding quotient of A^n'.

Euler's theorem states that if f is a rational function of degree d, it holds that $\sum x_i f_{x_i} = df$, or, in particular, when f is of degree zero, one has $\sum_i x_i f_{x_i} = 0$. Now, the functions on open sets in projective space are all rational functions of degree zero, and so Euler tells us that their differentials all live in the kernel of the map

$$\bigoplus_i \mathcal{O}_{\mathbb{P}^n}(-1)dx_i \to \mathcal{O}_{\mathbb{P}^n}$$

that sends $\sum_i f_i dx_i$ to $\sum_i x_i f_i$. This gives a strong heuristic argument for the next theorem:

Theorem 22.29 *Let ring A and $X = \mathbb{P}^n_A$ the projective n-space over A. Then there is an exact sequence*

$$0 \longrightarrow \Omega_{X/A} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Proof: Choosing coordinates on $\mathbb{P}^n_A$ we have $\mathbb{P}^n_A = \operatorname{Proj} R$ where R is the graded A-algebra $R = A[x_0, \dots, x_n]$. We introduce a graded R-module M by the exact sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_i R(-1)dx_i \xrightarrow{\eta} R$$

where η is the 'Euler map' $\sum_i f_i dx_i \mapsto \sum_i f_i x_i$. It is homogenous of degree zero when we give each dx_i degree one. Note that $\operatorname{Coker} \eta = R/(x_0, \dots, x_n)$, so that when 'tilded' the sequence becomes

$$0 \longrightarrow \widetilde{M} \longrightarrow \bigoplus_i \mathcal{O}_{\mathbb{P}^n}(-1)dx_i \xrightarrow{\tilde{\eta}} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

We begin with a swift recap of the construction of the projective space $\operatorname{Proj} R$. It is covered by standard open affines $D_+(x_i)$ each equal to $\operatorname{Spec}(R_{x_i})_0$, where $(R_x)_0$ is the degree zero piece of the localization R_{x_i} (equipped with natural grading). The overlaps of the standard opens are the distinguished open sets $D_+(x_i x_j) = \operatorname{Spec}(R_{x_i x_j})_0$.

The universal derivation $d_R \colon R \to \Omega_{R/A} = \bigoplus_j R dx_j$ extends to a derivation

$$d_{R_{x_i}} \colon R_{x_i} \to \Omega_{R_{x_i}/A} = \bigoplus\nolimits_j R_{x_i} dx_j$$

by the usual rule for the derivative of a fraction, and it preserves degrees when each dx_j is given degree one; that is $(R_{x_i} dx_i)_\nu = (R_{x_i})_{\nu-1} dx_i$. Taking the degree zero part, yields a derivation

$$(R_{x_i})_0 \to \bigoplus\nolimits_j (R_{x_i}(-1))_0 dx_j;$$

that is, when exposed to tilde, a derivation

$$\mathcal{O}_{D_+(x_i)} \to \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(-1)|_{D_+(x_i)} dx_j.$$

Since these derivations for different i originate from the *same* global derivation d_R, they are forced to agree on the overlaps, and patch together to give a derivation

$$\mathcal{O}_{\mathbb{P}^n} \to \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(-1) dx_i.$$

It takes values in $\widetilde{M}$, and by universality there is a map $\Omega_{\mathbb{P}^n/A} \to \widetilde{M}$. The rest of the proof consists of checking that this is an isomorphism, which is a local issue. Both $\Omega_{\mathbb{P}^n/A}$ and $\widetilde{M}$ are locally free of rank n, so it suffices to see that α is surjective.

On the open set $D_+(x_i)$ the sheaf $\Omega_{\mathbb{P}^n/A} = \Omega_{D_+(x_i)/A}$ originates from the module $\Omega_{(R_{x_i})_0/A}$, which has a basis formed by the $d(x_j/x_i)$ for $j \neq i$, and one checks without much resistance that the map α sends $d(x_i/x_j)$ to $(x_j dx_i - x_i dx_j)/x_i^2$ (what else could it be?). But the kernel of the Euler map η is generated by the elements $x_i dx_j - x_j d_{x_i}$, and so we are through. ❑

Since $\Omega_{\mathbb{P}^n_A}$ injects into $\mathcal{O}_{\mathbb{P}^n_A}(-1)^{n+1}$ (which has no global sections), we get:

Corollary 22.30 $\Gamma(\mathbb{P}^n_A, \Omega_{\mathbb{P}^n_A}) = 0$

Exercise 22.11 Show that the kernel of η is generated by $n(n-1)/2$ expressions $x_i dx_j - x_j dx_i$. ★

22.5 *Relation with the Zariski tangent space*

The tangent space to a differentiable manifold at a point is defined at the space of 'point derivations' as the point, *i.e.* derivations from the ring of C^∞-germs near the point to $\mathbb{R}$. The analogue to this for a scheme X over a field k would be the space of derivations $\operatorname{Der}_k(\mathcal{O}_{X,x}, k(x))$, where $k(x)$ is the residue class field at x, and in view of the fundamental relation (22.3) and the equality

$$\operatorname{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, k(x)) = \operatorname{Hom}_{k(x)}(\Omega_{X/k} \otimes k(x), k(x)),$$

the cotangent space; *i.e.* the dual of the tangent space, will be $\Omega_{X/k} \otimes_{\mathcal{O}_X} k(x)$.

Another candidate is, however, the Zariski tangent space $\operatorname{Hom}_{k(x)}(\mathfrak{m}/\mathfrak{m}^2, k(x))$, In contrast to the 'point derivations', the Zariski tangent space is not a relative notion, it does not depend on the subfield k, and can be defined for any local ring. The Zariski cotangent space will simply be the dual space $\mathfrak{m}/\mathfrak{m}^2$.

These two possible tangent spaces give rise to two different notions, *regularity* and *smoothness*, which both in some sense mimic the property of being a manifold. Fortunately, in several cases the two are equivalent; one such situation is in described in the following proposition:

Proposition 22.31 *Suppose $(B, \mathfrak{m})$ is a local ring with residue field $K = B/\mathfrak{m}$ and assume that B contains a field k. If the extension $k \subseteq K$ is finite and separable, then the map from the conormal sequence*

$$\delta\colon \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B K$$

is an isomorphism.

Proof: The conormal sequence with $A = k$ and $C = K$ takes the following shape:

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes_B K \longrightarrow \Omega_{K/k} \longrightarrow 0,$$

and according to Example 22.9 on page 338 it holds that $\Omega_{K/k} = 0$, so δ is surjective.

The map δ sends $x \in \mathfrak{m}$ to dx. We shall exhibit an inverse $\psi\colon \Omega_{B|k} \otimes_B K \to \mathfrak{m}/\mathfrak{m}^2$ to δ. Constructing such a map is equivalent to constructing a map of B-modules $\Omega_{B|k} \to \mathfrak{m}/\mathfrak{m}^2$, or equivalently, a derivation $D\colon B \to \mathfrak{m}/\mathfrak{m}^2$.

The derivation $D\colon B \to \mathfrak{m}/\mathfrak{m}^2$ will be the composition $D \circ \pi$ of the canonical 'reduction-mod-$\mathfrak{m}^2$-map' $\pi\colon B \to B/\mathfrak{m}^2$ and a derivation $D_0\colon B/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$. To construct the latter, we cite the lemma below that the k-algebra $B/\mathfrak{m}^2$ splits as a direct sum $B/\mathfrak{m}^2 = K \oplus \mathfrak{m}/\mathfrak{m}^2$, and simply let D_0 be the projection onto $\mathfrak{m}/\mathfrak{m}^2$; that is

$$D_0(a + x) = x,$$

where $\alpha \in K$ and $x \in \mathfrak{m}/\mathfrak{m}^2$. The reduction map π being an algebra homomorphism, it suffices to see that D_0 is a k-derivation. To this end, we compute:

$$\begin{aligned} D_0((a+x)(a'+x')) &= D_0(aa + (ax' + a'x) + x'x) \\ &= D_0(aa) + D_0(ax' + a'x) + D_0(x'x) = ax' + a'x, \end{aligned}$$

and we get the same answer when we expand

$$(a' + x')D_0(a + x) + (a + x)D_0(a' + x')$$

since $xx' = 0$. Hence D_0 is a derivation, and we get the desired inverse. It is indeed an inverse to the map δ, since via the identification $\operatorname{Der}_A(B, M) = \operatorname{Hom}_B(\Omega_{B/A}, A)$, it sends dx to x. ❑

LEMMA 22.32 *Let B be a local ring with maximal ideal I that satisfies $I^2 = 0$. Assume that B contains a field k and that the extension $k \subseteq K = B/I$ is finite and separable. Then B contains a subring isomorphic to K; so that $B = K \oplus I$.*

PROOF: Since K is finite and separable over k, it is primitive. So let $K = k(x)$ and let P being the minimal polynomial of x over k. It is separable, so $P'(x) \neq 0$. We shall lift x to an element $y \in B/\mathfrak{m}^2$ such that that $P(y) = 0$ (meaningful as $k \subseteq B/\mathfrak{m}^2$ and P has coefficients in k). Then the the subring $k(y)$ maps isomorphically onto K.

Chose any lifting z of x. Then $P(z) = \epsilon \in I$. For any $\alpha \in I$ Taylor's formula yields

$$P(z + \alpha) = P(z) + P'(z)\alpha$$

as $\alpha^2 = 0$. Now $P'(x)$ is a unit in B/I, and as units reduce to units (Lemma 22.33 below)and hence $y = z + \alpha$ is such that $P(y) = 0$. ❑

Recall that a Noetherian local ring B is called *regular* if the Krull dimension equals the embedding dimension; or with $\mathfrak{m}$ the maximal ideal and $K = B/\mathfrak{m}$, it holds that $\dim_K \mathfrak{m}/\mathfrak{m}^2 = \dim B$.

LEMMA 22.33 *Let $\pi\colon B \to A$ be a surjective ring homomorphism with kernel I. Assume that $I^2 = 0$. Then every element in B that maps to a unit in A is invertible, and there is an exact sequence of groups*

$$1 \longrightarrow 1 + I \longrightarrow B^* \xrightarrow{\pi} A^* \longrightarrow 1\ .$$

PROOF: All elements in $1 + I$ are units, since if $x^2 = 0$, it holds that $(1 + x)(1 - x) = 1$. Assume that $\pi(x)y = 1$ and let $z \in B$ be so that $\pi(z) = y^{-1}$ Then $xz \in 1 + I$ and is therefore invertible, so *a fortiori* x is invertible. ❑

EXERCISE 22.12 Show that if K is a finitely generated extension of k with a separating basis, there is a field $K' \subseteq B$ mapping isomorphically to K. HINT: First treat the case that $K = k(x)$ with x a variable; then use induction on the cardinality of a separating basis. ★

COROLLARY 22.34 *With notation as in Proposition 22.31 but additionally with B being Noetherian, the ring B is a regular local ring if and only if*

$$\dim B = \dim_k \Omega_{B/K} \otimes_B K.$$

The separability condition in Proposition 22.31 is certainly necessary, this is already the case for fields: fields are regular local rings of dimension zero, and for a inseparable field extension $k \subseteq K$ the module of differentials $\Omega_{K/k}$ is never zero; for instance, if $K = k(x)$ with $x^p = a$, it holds that $\Omega_{K/k} = K$.

Smooth varieties

We give a definition for smoothness of varieties. In general schemes can have components of different dimension, so we if $x \in X$ is a point, we let $\dim_x X$ be the Krull dimension of a sufficiently small affine neighbourhood of x; if x is a closed point it coincides with $\dim \mathcal{O}_{X,x}$.

DEFINITION 22.35 (SMOOTHNESS OVER FIELDS) *Let X be a (separated (?)) scheme of (essential (?)) finite type over a field k and let $x \in X$ be a point. We say that X is* smooth *at x if $\Omega_{X/k}$ is locally free of rank $\dim_x X$ near x. The scheme X is called* smooth *if it is smooth at every closed point.*

THEOREM 22.36 *Let X be a variety (integral separated scheme of finite type) over a perfect field k (e.g. k algebraically closed, finite or of characteristic zero) and let $x \in X$ be a closed point. Then the following are equivalent:*

i) *X is smooth at x;*

ii) *$(\Omega_{X/k})_x$ is free of rank $\dim X$;*

iii) *X is non-singular at x.*

PROOF: *i)* $\iff$ *ii)* is just the definition of X being smooth together with the fact that a coherent module $\mathcal{F}$ over $\mathcal{O}_X$ is locally free in near x if and only $\mathcal{F}_x$ is free.

ii) $\implies$ *iii)*. Assuming that $\Omega_{\mathcal{O}_{X,x}/k}$ is free of rank $n = \dim \mathcal{O}_{X,x}$, we infer, by the above proposition, that $\dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = n$, and so $\mathcal{O}_{X,x}$ is a regular local ring.

iii) $\implies$ *ii)*. There are two salient points: The first is that if x is a regular point, the integer $d(y) = \Omega_{X/k} \otimes_{\mathcal{O}_X} k(y)$ takes on its minimal value at x, and the second is that $d(y)$ can only increase upon specialization. The details are as follows: Let K be the function field of X. If the local ring $\mathcal{O}_{X,x}$ is regular, it follows from Proposition 22.31 that $\dim (\Omega_{X/k}) \otimes k(x) = \dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim X$. From Exercise 22.9 on page 344 follows that $\dim_K \Omega_{K/k} = \dim_K \Omega_{X/k} \otimes K \geqslant \operatorname{trdeg} K/k$. The transcendence degree of the function field of a variety equals $\dim X$, and hence $(\Omega_{X/k})_x$ is a free $\mathcal{O}_{X,x}$-module by the general fact that a finite module over an integral local ring having generic fibre of the larger dimension than the special one, is free (Exercise 22.13 below). ❑

✲ **EXERCISE 22.13** *(Jumping of fibre dimension upon specialization.)* Let A be a local integral domain with maximal ideal $\mathfrak{m}$, residue field $k = A/\mathfrak{m}$ and fraction field K. Let M be a finite A-module and assume that $\dim_K M \otimes_A K \geqslant \dim_k M \otimes_A k$. Then M is a free A-module. (See also Proposition **??** in CA) ★

✲ **EXERCISE 22.14** Let X be a variety over a perfect field. Show that the function field K of X is separably generated over k and that the smooth (hence regular) closed points of X form an open dense subset. Give a counterexample if k is not perfect. ★

DEFINITION 22.37 *When X is smooth over k and $\Omega_{X/k}$ is a locally free sheaf on X, we refer to it as* the cotangent bundle *or* cotangent sheaf *of X.*
The sheaf of p-forms is defined as

$$\Omega^p_{X/k} = \bigwedge^p \Omega_{X/k}$$

In particular, if X has dimension n, $\omega_X = \Omega^n_{X/k}$ is called the canonical bundle *of X. As $\Omega_{X/k}$ has rank n, ω_X is locally free of rank one, i.e. an invertible sheaf.*
The tangent sheaf, *or the* tangent bundle *is the sheaf*

$$T_X = \mathscr{H}om_{\mathcal{O}_X}(\Omega_{X|k}, \mathcal{O}_X)$$

PROPOSITION 22.38 *Let $X = \mathbb{P}^n$. Then $\Omega^n_X = \mathcal{O}(-n-1)$.*

PROOF: Consider the Euler sequence for the cotangent bundle of $\mathbb{P}^n$

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0$$

In general, if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, we have $\bigwedge^e \mathcal{E} = \bigwedge^{e'} \mathcal{E}' \otimes \bigwedge^{e''} \mathcal{E}''$. Hence we get

$$\bigwedge^n \Omega_{\mathbb{P}^n} = \bigwedge^{n+1} \mathcal{O}(-1)^{\oplus(n+1)} = \mathcal{O}(-n-1).$$

Note by the way that the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ fits into the following sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to \mathcal{T}_X \to 0$$

where the left-most map sends 1 to the vector $(x_0, \ldots, x_n)$. ❑

EXAMPLE 22.39 For $X = \mathbb{A}^1$, we get $\Omega_{\mathbb{A}^1} = \mathcal{T}_{\mathbb{A}^1} = \mathcal{O}_{\mathbb{A}^1}$. ☆
EXAMPLE 22.40 Let A be a ring and let $X = \mathbb{P}^1_A$. Then it holds that $\Omega_{X/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ and $T_X = \mathcal{O}_{\mathbb{P}^1_A}(2)$. For this, we can use the standard covering of $\mathbb{P}^1 = \operatorname{Proj} A[x_0, x_1]$, given by $U_i = D_+(x_i)$. On U_0, we have

$$\Omega_{U_0|A} \simeq A[\frac{x_1}{x_0}]d\left(\frac{x_1}{x_0}\right)$$

, and similarly on U_1. On the intersection $D_+(x_0) \cap D_+(x_1)$, we have

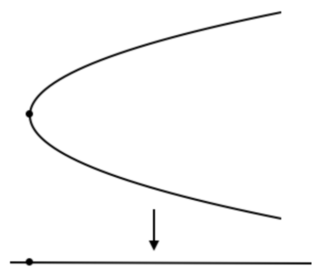
$$x_0^2 d\left(\frac{x_1}{x_0}\right) = -x_1^2 d\left(\frac{x_0}{x_1}\right)$$

This gives a non-vanishing section of $\Omega_X(2) \otimes \Omega_{X|A}$ and furthermore an isomorphism $\Omega_{X|A} \simeq \mathcal{O}_X(-2)$. ☆

Smooth morphisms

We can also use the sheaves of differentials to define a notion of smoothness for *morphisms*. In short, we say that a morphism $f : X \to S$ is *smooth* at a point $x \in X$ is if $\Omega_{X|S}$ is locally free of rank $\dim_x X - \dim_{f(x)} S$ there. If this is not the case, we say that f is *ramified* at x, and that x is a *ramification point*. f is smooth if it is smooth at every point.

EXAMPLE 22.41 Let $A = k[x]$ and let $B = k[x,y]/(x-y^2) \simeq k[y]$ where k is a field of characteristic not equal to 2. Let $X = \operatorname{Spec} B$ and let $Y = \operatorname{Spec} A$. Let $f : X \to Y$ be the morphism induced by the inclusion $A \hookrightarrow B$ (thus $x \mapsto y^2$).



Since $B \simeq k[y]$ it follows that $\Omega_X = Bdy$, the free B-module generated by dy. Similarly $\Omega_Y = Adx$. The sequence (22.4) gives us

$$\begin{array}{ccccccc} \Omega_Y \otimes_A B & \to & \Omega_X & \to & \Omega_{X/Y} & \to & 0 \\ \| & & \| & & \wr\| & & \\ Bdx & \to & Bdy & \to & Bdy/B(2ydy) & & \end{array}$$

The point is that $\Omega_{X/Y} = (k[y]/(2y))dy$ is a torsion sheaf supported on the ramification locus of the map $f : X \to Y$. (The only ramification point is above 0.) Note that $\Omega_{X/Y}$ is the quotient of Ω_X by the submodule generated by the image of dx in $\Omega_X = Bdy$. The image of dx is $2ydy$. ☆

Exercises

✲ (22.15) Let $p > 2$ be a prime number and let $k = \mathbb{F}_p(t)$. Let

$$X = \operatorname{Spec} k[x,y]/(y^2 - x^p - t)$$

Show that all the local rings of X are regular, but X is not smooth over k.

(22.16) Let $C \simeq \mathbb{P}^1$ denote the twisted cubic curve in $\mathbb{P}^3$. Show that the conormal sheaf of C is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-5) \oplus \mathcal{O}_{\mathbb{P}^1}(-5)$.

★

Chapter 23

Curves

jWe have through the course seen several examples of curves. Plane curves with conics and hyper elliptic curves have been favourites, the normal rational curves as examples of curves that are not plane. In this chapter we shall study curves more systematically and from a intrinsic point of view, that is we curves per se and not as subschemes of larger scheme.

So far we have not given a formal definition of a curve; here it comes: a *curve* is a one dimensional variety over a field k. Recall that this means that X apart from being of dimension one, is an integral scheme separated and of finite type over k. *Curve*

We shall restrict our attention to curves over perfect fields; in addition to all fields of characteristic zero this covers the cases that k is algebraically closed or a finite field.

23.1 *The local ring at regular points of a curve*

A variety X is smooth at a point x if the $\Omega_{X,x}$ is locally of rank $\dim X$ near x, and over a perfect field this is equivalent to $\mathcal{O}_{X,x}$ being a regular local ring. In other words, it is equivalent to the Zariski cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ being of dimension $\dim X$ as a vector space over $k(x)$.

For curves, the important points is that the Noetherian regular local rings of dimension one are precisely the discrete valuation rings; that is the Noetherian local PID's. The ideal structure of these rings is particularly simple, the powers of the maximal ideal are the only non-zero ideals.

Lemma 23.1 *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. The following statements are equivalent:*

i) *A is a DVR;*
ii) *the maximal ideal $\mathfrak{m}$ is principal;*
iii) *all ideals are principal and powers of the maximal ideal;*
iv) $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$.

At a regular point $x \in X$ the maximal ideal $\mathfrak{m}_x$ is principal, and any generator is called a *local parameter* or a *uniformizing parameter* at x. Each rational function on X can expressed in a unique fashion as $f = \alpha t^\nu$ where ν is an integer and α is unit in $\mathcal{O}_{X,x}$; that is, it is a regular function which does not vanish at x. To every DVR is associated a normalized valuation on the fraction field, which we in the present case denote by ν_x. Note that $\nu_x(f)$

is precisely the integer ν above. One may think about the valuation $v_x(f)$ as the order of f at x, either the order of vanishing, if f is regular at x, or the order of the pole if not.

Our ground field is assumed to be perfect, and the differential criterion for regularity in Theorem 22.36 on page 351 applies:

Proposition 23.2 *A curve X over a perfect field is regular at a closed point x if and only if the stalk $(\Omega_{X/k})_x$ is free of rank one.*

Example 23.3 (*Plane curves.*) Consider $X = \operatorname{Spec} A$ where $A = k[u,v]/(f)$. In Example 22.19 we found the following expression for the Kähler differentials of A:

$$\Omega_{A/k} = Adu \oplus Adv/(f_u du + f_v dv),$$

and this is not of rank one (*i.e.* of rank two) exactly at the points of X where the two partials f_u and f_v vanish. Hence a point $x \in X$ is a smooth point if and only if at least one of the partials does not vanish at x, and X is a regular curve when $V(f, f_u, f_v) = \emptyset$. In terms of ideals this reads $(f, f_u, f_v) = k[u,v]$. ☆

Example 23.4 (*A regular but not smooth curve.*) Note that over fields that are not perfect, being regular and being smooth are not the same. For instance, assume that k is of characteristic two and that $\alpha \in k$ is not a square root. Then $\mathfrak{m} = (v, u^2 + \alpha)$ is a maximal ideal in $k[u,v]$ and $x = V(u^2 + \alpha, v)$ a closed point in $\mathbb{A}^2_k$.

The plane affine curve $f = v^2 - u(u^2 + \alpha)$ is regular at all points: Since $df = (u^2 + \alpha)du$, it is smooth except at $x = V(u^2 + \alpha, v)$, where it, however, is regular. Indeed, f does not belong to $\mathfrak{m}^2 = (v^2, v(u^2 + \alpha), u^4 + \alpha^2)$.

The curve $X' = X \otimes_k k'$ acquires a singular point if k' is an extension of k containing a square root of α: If say $\beta^2 = \alpha$, then f takes the form $f = v^2 - u(u^2 + \beta^2) = v^2 - u(u+\beta)^2$, and X' has a node at $(-\beta, 0)$.

The moral is that regularity is not always invariant under base change. A scheme over k is called *geometrically regular* if $X \otimes_k k'$ is regular for all finite extensions k' of k. ☆

Exercise 23.1 Find the singularities of the curve in $\mathbb{P}^2_k$ whose equation is $x^2y^2 + x^2z^2 + y^2z^2 = 0$. ★

Another all important feature of one dimensional Noetherian domains is that they are regular precisely when they are normal:

Proposition 23.5 *Let A be a one-dimensional Noetherian domain A. Then A is normal if and only if it is regular.*

Proof: Being normal is a local property, and by definition a Noetherian ring is regular precisely when then all the local rings $A_{\mathfrak{p}}$ are regular, so the proposition boils down to the local case, which is standard algebra: a one-dimensional local domain is normal if and only if it is a DVR. ❑

Back in Chapter 7 we constructed the normalization $\overline{X}$ of an integral scheme X (Theorem 11.19) together with a morphism $\pi\colon \overline{X} \to X$. In view of the above proposition, $\overline{X}$ is in fact a *desingularization* of X. *Desingularization of a curve*

THEOREM 23.6 *The normalization $\overline{X}$ of a curve X over k, is a non-singular curve. The normalization map $\pi\colon \overline{X} \to X$ is finite and birational. If X is proper over k the same holds for $\overline{X}$.*

PROOF: This is just Theorem 11.20 on page 170. ❏

23.2 Morphisms between curves

Morphisms between curves are bla bla

We recall the following three fundamental facts about morphisms of curves (proved in Algebraic geometry I):

PROPOSITION 23.7 *Let X be a variety and Y a curve over k, and let $f\colon X \to Y$ be a morphism. Then either*

i) $f(X)$ is a point in Y; or

ii) $f(X)$ is open and dense in Y.

In the case ii), when X is a curve, the extension $k(Y) \subseteq k(X)$ of function fields will be a finite extension. Moreover, when X is proper over k, so is Y, and f is a finite morphism.

PROOF: The first statement follows from lemma below; indeed, let $\operatorname{Spec} A \subseteq Y$ and $\operatorname{Spec} B \subseteq X$ be open affines such that $\operatorname{Spec} B$ maps into $\operatorname{Spec} A$. The image of $\operatorname{Spec} B$ is either a point, in which case the image of X will be that point, or $\operatorname{Spec} B$ dominates $\operatorname{Spec} A$, and its image contains an open subset. The image of f will then be open because subsets of an irreducible curve containing a non-empty open set are open.

LEMMA 23.8 *Let A and B be a domains and $\phi\colon A \to B$ a ring homomorphism. Assume that A is of Krull dimension one. Then either ϕ is injective or factors by a field. In particular, the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is either dominant or has a closed point as image.*

PROOF: Since B is domain, so is also $\phi(A)$, and $\operatorname{Ker}\phi$ is a prime ideal. Since A is a domain of Krull dimension one, the kernel $\operatorname{Ker}\phi$ is either maximal or zero. ❏

Assume then that X is a curve and that f is dominant. The two function fields $k(X)$ and $k(Y)$ are both of transcendence degree one, and so $k(X)$ is algebraic over Y, but X is of finite type over Y, since it is of finite type over k, and thus $k(X)$ is a finite extension of $k(Y)$.

When f is proper, it will be surjective and by general properties of proper maps (xxxx) Y will be proper over k. Is X also a curve, every fibre of f over a closed point will be closed and so will be finite. Hence f is quasi-finite, and also being proper, it is finite (xxxx). ❏

This leads to the notion of the degree of a morphism between curves:

DEFINITION 23.9 (THE DEGREE OF A FINITE MORPHISM) *Let $f\colon X \to Y$ be a dominant morphism between curves. The degree $[k(X) : k(Y)]$ is called the* degree *of f and is denoted* $\deg f$.

Since the degree of field extensions is multiplicative in towers, one has:

LEMMA 23.10 *If f and g are dominant composable morphisms between curves, the composition $f \circ g$ is dominant and $\deg f \circ g = \deg f \deg g$.*

The fibre of a morphism

We shall examine the scheme theoretic fibre $f^{-1}(y)$ over a closed point $y \in Y$ of a morphism $f\colon X \to Y$ between two curves in more detail. The most interesting case is when both X and Y are regular curves and the morphism is finite and dominant, and we will confine the analysis to that case. The analysis is local on Y, so we may additionally assume that X and Y are affine; say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, where A and B are reular one-dimensional rings and B is a finite A-algebra. If $x \in X$ the ring $\mathcal{O}_{X,x}$ is a valuation ring and we denote by v_x the corresponding valuation on $k(X)$.

For the basic details about scheme theoretic fibres see Section 8.5

PROPOSITION 23.11 *Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a finite morphism where A is a regular one-dimensional ring. If each component of $\operatorname{Spec} B$ dominates $\operatorname{Spec} A$, then B is a locally free A module. In the case that B is integral, the rank of B equals $\deg f$.*

PROOF: The zero divisors of B is the union of the minimal prime ideals $\{\mathfrak{p}_i\}$ in B, and since each component of X dominates Y, it holds that $\mathfrak{p}_i \cap A = 0$. This means that each non-zero element t of A is a non-zero divisor on B. Hence B is a torsion free finite A-module, and as A is a Dedekind ring, it follows (see xxxx)) that B locally free. ❑

EXAMPLE 23.12 (*Illustartive example.*) Let $f(t)$ be a polynomial in $k[t]$. The assignment $t \to f(t)$ defines a map $k[t] \to k[t]$ and hence a map $\mathbb{A}^1_k \to \mathbb{A}^1_k$. The scheme theoretic fibre over the closed point $(t-a) \in \mathbb{A}^1_k$ (heuristically speaking over $\alpha \in \mathbb{A}^1(k)$) is the closed subscheme $V(f(t)-\alpha)$. The polynomial $f-\alpha$ factors as

$$f - \alpha = f_1^{\nu_1} \cdot \ldots \cdot f_r^{\nu_r}$$

where the f_i's are irreducible and pairwise distinct. One would like to think about $V(f(t)-\alpha)$ as the solutions of $f(t)-\alpha = 0$, but the roots β_i of the f_i's do not necessarily lie in k, but in extensions $k(\beta_i)$; and of course, each appears with multiplicity ν_i. The Chinese Remainder Theorem gives

$$k[f]/(f(t)-\alpha) = \prod_{1 \leqslant i \leqslant r} k[t]/(f_i^{\nu_i}).$$

And so we get the expression

$$\dim_k k[f]/(f(t)-\alpha) = \deg f = \sum [k(\beta_i : k)]\nu_i.$$

for 'the number of points in the fibre'; indeed, if all $\beta_i \in k$ and all $\nu_i = 1$, it is equal to the cardinality of the fibre. ☆

Coming back to general situation, with X and Y be regular curves over k and $f\colon X \to Y$ be a finite non-constant morphism, we shall describe the scheme fibre $f^{-1}(y)$ quit similarly as done in the illustrative example above. Let t be a uniformizer at y.

Consider a point $x \in X$ mapping to y. The induced map $f_y^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ gives rise to a field extension $k(y) \subseteq k(x)$, which is finite since the Nullstellensatz tells us that both $k(x)$ and $k(y)$ are finite extensions of k. The degree $d_x = [k(x) : k(y)]$ is called the *local degree* of f at x_i. In the case that k is algebraically closed, the two fields coincide with k, and the local degree equals one.

The local degree of a morphism

The number $e_x = v_x(f^\sharp(t))$ will be called the *ramification index* of f at x. It does not depend on the choice of parameter t, and it holds that $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x^e\mathcal{O}_{X,x}$ and we have the equality

The ramification index of a morphism

$$e_x = \dim_{k(x)} \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}.$$

We say that f *ramifies* in x when $e_x > 1$.

The scheme theoretic fibre $f^{-1}(y)$ equals $\operatorname{Spec} B/\mathfrak{m}_yB$, and as the domain B is of Krull dimension one and $\mathfrak{m}_yB \neq 0$, the ring $B/\mathfrak{m}_yB$ will be of dimension zero. It is of finite length and decomposes as the product of its localizations:

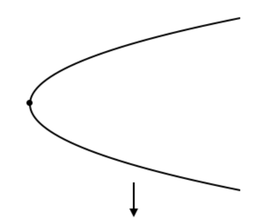
$$B/\mathfrak{m}_yB = \prod_{f(x)=y} \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}.$$

Combining this formula with Proposition 23.11 above one gets:

PROPOSITION 23.13 *Let $f\colon X \to Y$ a finite morphism between regular curves over k. For each closed point $y \in Y$, it holds that*

$$\deg f = \sum_{f(x)=y} d_x e_x.$$

EXAMPLE 23.14 Let $A = k[u]$ and let $B = k[u,v]/(u - v^2) \simeq k[v]$ where k is a field whose characteristic is not two. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let $f\colon X \to Y$ be the morphism induced by the inclusion $A \hookrightarrow B$ (thus $u \mapsto v^2$). The morphism f is ramified at the origin $x = (0,0)$, and here the ramification index is two. Indeed, u is a uniformizing parameter of $\mathcal{O}_{Y,y} = k[u]_{(u)}$ at $y = 0$, while v is the uniformizer of $\mathcal{O}_{X,x} = B_{(u,v)} = k[v]_{(v)}$. Then we have $v_y(u) = v_x(v^2) = 2$.



☆

The reader might notice a resemblance between the previous example and Example 22.41 , where ramification was defined in terms of the relative sheaf of differentials $\Omega_{X/Y}$. In that example, $\Omega_{X/Y}$ was a torsion sheaf supported on the single point $x = (0,0)$. This correspondence between the two notions of ramification is a general fact (at least in characteristic 0), and we have the useful formula for the ramification indexes of curves:

PROPOSITION 23.15 *Let : $X \to Y$ be a morphism between non-singular curves over k, and let $x \in X$ be a closed point. Assume that the ramification index e_x is invertible in k. Then one has*

$$e_x = \operatorname{length}(\Omega_{X/Y})_x + 1.$$

PROOF: From a general perspective, one has the exact sequence

$$f^*\Omega_{Y/k} \xrightarrow{df} \Omega_{Y/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0 \tag{23.1}$$

from Proposition 22.27 on page 347. In our setting Y is a regular curve, so at a point $y \in Y$ the stalk $(\Omega_{Y/k})_y$ is a free $\mathcal{O}_{Y,y}$-module with basis du for u a uniformizer at y. Similarly, at a point $x \in X$ the stalk $(\Omega_{X/k})_x$ is free $\mathcal{O}_{X,x}$-module with basis dv for v a uniformizer at x.

Now $f_y^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ acts as $u \mapsto \alpha v^e$ with α a unit and $e = e_x$, and so the stalk at x of df in (23.1) is determined by the assignment $du \mapsto d\alpha v^e$, and we compute

$$d\alpha v^e = v^e\alpha' dv + e\alpha \cdot v^{e-1}dv = v^{e-1}(v\alpha' + e\alpha)dv.$$

Now, by hypothesis, e is invertible in k so that $\alpha' v + e\alpha$ is a unit in $\mathcal{O}_{X,x}$. Consequently the image of df is the submodule generated by $v^{e-1}dv$; and so the cokernel of df (which equals $(\Omega_{X/Y})_x$) is isomorphic to $\mathcal{O}_{X,x}dv/v^{e-1}\mathcal{O}_{X,x}dv \simeq \mathcal{O}_{X,x}/\mathfrak{m}_x^{e-1}$. ❑

EXAMPLE 23.16 Continuing Example 23.14 above, we se that the origin is the only place where f ramifies since $df = 2udu$, and the characteristic of k is supposed to be different from 2. If k is not algebraically closed, the local degree may be two; this happens, for instance, at a point $a \in k$ if a does not have a square root in k. ☆

Extension of maps and The Fundamental Theorem

This section presents two basic result about non-singular curves. The first basically says that any rational map on a non-singular curve into a a projective variety is globally defined. Combining this with the Main Theorem of Birational Geometry (Theorem 11.12 on page 164) and the fact that every curve is birationally equivalent to a non-singular one, we obtain the second, which states that the category of projective non-singular curves over k with dominant maps is equivalent to the category of finitely generated field extensions of k of transcendence degree one.

Later it will turn out that all curve are projective, so in fact the claim applies to the category of non-singular curves proper over k.

PROPOSITION 23.17 *Let X be an irreducible curve over k and let $x \in X$ be a closed point where X is regular. Then any morphism $f\colon X\backslash\{x\} \to Y$ to a projective variety Y has a unique extension $\overline{f}\colon X \to Y$.*

PROOF: The salient point of the proof is precisely the same as in the proof of Lemma 19.19 on page 286. Fixing the notation, we let t be a uniformizer at x, and denote by K the function field of X.

The morphism f yields a K-point $\operatorname{Spec} K \to \mathbb{P}^n_K$ which is described by homogenous coordinates $(a_0 t^{\nu_0} : \cdots : a_n t^{\nu_n})$ where the a_i are units in $\mathcal{O}_{X,x}$ and the ν_i's are integers. After scaling through by $t^{-\min \nu_i}$ we may assume that for each i it holds that $\nu_i \geqslant 0$ and at $\nu_{i_0} = 0$ for at least one i_0.

Now the a_i are non-vanishing sections of $\mathcal{O}_X$ over some some open neighbourhood U of x and after shrinking U if need be, t will also be a section of $\mathcal{O}_X$ over U with x as the sole zero. Hence the $a_i t^{v_i}$ define a map $U \to \mathbb{P}^n_k$.

❏

COROLLARY 23.18 *Any rational map between two non-singular projective curves extends to a morphism. In particular, any birational map extends to an isomorphism.*

PROOF: The first statement is just a reformulation of Proposition 23.17.

That two curves X and Y are birationally equivalent, means that there are open subsets $U \subseteq X$ and $V \subseteq Y$ and an isomorphism $f\colon U \to V$. Now, both f and f^{-1} extends respectively to morphisms $g\colon X \to Y$ and $h\colon Y \to X$, and since $h \circ g|_U = \mathrm{id}_U$ and $g \circ h|_V = \mathrm{id}_V$, it follows that $h \circ g = \mathrm{id}_X$ and $g \circ h = \mathrm{id}_Y$; indeed, morphisms that agree on an open dense set are equal. ❏

THEOREM 23.19 (MAIN THEOREM OF NON-SINGULAR PROJECTIVE CURVES) *There is an equivalence of categories between the following categories:*

i) The category of non-singular projective curves over k and dominant morphisms;

ii) The category of finitely generated field extensions of k of transcendence degree one and k-algebra homomorphisms.

PROOF: First, if X and Y are two nonsingular projective curves, any rational map extends to a morphism. This shows, combined with Theorem 11.10 on page 163, that the functor $X \mapsto k(X)$ is fully faithful.

Next we show it is essentially surjective: each finitely generated field K of transcendence degree one over k is of the form $k(X)$ for some nonsingular projective curve X. If K is generated by $a_1, \ldots, a_r$ the k-subalgebra $A = k[a_1, \ldots, a_r]$ will be of dimension one according to Theorem 7.16 on page 111. The curve $X = \operatorname{Spec} A$ is contained in the affine space $\mathbb{A}^r_k$ in a natural way, and closing it up in $\mathbb{P}^r_k$, yields a projective curve, whose normalization is projective (after xxx) and has K as function field. ❏

EXAMPLE 23.20 (*Morphisms into* $\mathbb{P}^1_k$*.)* For a non-singular curve X over k, there is a natural one-to-one correspondence between non-constant rational functions on X and dominant maps from X to $\mathbb{P}^1_k$. A rational function on X is just a morphism from some open subset to $\mathbb{A}^1_k$ and, this extends to a morphism from the entire X to $\mathbb{P}^1_k$.

To be somehow more explicit, we let $\mathbb{P}^1_k = \operatorname{Proj} k[t_0, t_1]$ and the affine line $\mathbb{A}^1_k$ in the construction above be $D_+(t_0)$. Let $g \in k(X)^\times$ be given. To lessen the confusion, denote by G the extended map $G\colon X \to \mathbb{P}^1_k$.

Let U_g be the maximal open where g is defined; *i.e.* $U_g = G^{-1}D_+(t_1)$, then $G^\sharp(t_0/t_1) = g$ in $\mathcal{O}_X(U_g)$. Let $U_{g^{-1}}$ be the maximal open where g^{-1} is defined, then $U_{g^{-1}} = G^{-1}D_+(t_0)$, and it holds that $G^\sharp(t_1/t_0) = 1/g$ in $\mathcal{O}_X(U_{g^{-1}})$. ☆

Coherent sheaves on curves

Recall that an element of an A-module is called a *torsion element* if it is killed by a nonzerodivisor of A, and a module is a *torsion module* if all elements are torsion. On the other hand, a module is *torsion free* if no non-zero element is torsion. The sum of two torsion elements is clearly torsion, so the subset of a module M formed by the torsion elements, is a submodule T. It has the property that M/T is torsion free.

We shall need the following result from algebra:

PROPOSITION 23.21 *Let A be a* PID. *Then any finitely generated torsion free module M is free. In particular, if A is regular of dimension one, every finitely generated torsion free module is locally free.*

PROOF: Observe first that there are non-zero maps $M \to A$. Indeed, the natural map $M \to M \otimes_A K$ that sends m to $m \otimes 1$ is injective since M is torsion free. Then choose a K-linear map $M \otimes_A K \to K$ that does not vanish on M. If $\{m_i\}$ is a finite generating set for M, the images $\phi(m_i)$ may be brought on the form a_i/b with a common denominator. Then $b\phi$ is our map.

We proceed by induction on the rank of M. If the rank is one, M is an ideal in A, and hence is free since A is a PID. If the rank is superior to one, chose a non-zero map $M \to A$. The image is an ideal, hence free of rank one, and M splits as $M = \operatorname{Ker}\phi \oplus \operatorname{Im}\phi$. By induction, $\operatorname{Ker}\phi$ is free, and we are done.

Finally, that A is regular of dimension one, means that all the local rings $A_\mathfrak{p}$ with $\mathfrak{p} \in \operatorname{Spec} A$ are DVR's, and in particular, they are PID's. Hence each localization $M_\mathfrak{p}$ is free; in other words, M is locally free. ❑

Returning to the global situation, any coherent sheaf $\mathcal{F}$ on a scheme X contains a torsion subsheaf $\mathcal{T}$, whose sections over an open set $U \subset X$ equals the subgroup of $\mathcal{F}(U)$ of elements annihilated by some nonzerodivisor of $\mathcal{O}_X(U)$ (see Exercise 16.3 on page 235). The quotient $\mathcal{F}/\mathcal{T}$ is torsion free in the sense that on open affine subsets U its section space (which equals $\mathcal{F}(U)/\mathcal{T}(U)$) is a torsion free module over $\mathcal{O}_X(U)$.

When X is a curve, the support of $\mathcal{T}$ is finite, say it consists of the points $p_1, \ldots, p_r$, and $\mathcal{T}$ is the direct sum of its stalks at these points: $\mathcal{T} = \bigoplus_{i=1}^r \mathcal{T}_{p_i}$.

THEOREM 23.22 *Let X be a non-singular curve and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a decomposition*

$$\mathcal{F} = \mathcal{E} \oplus \mathcal{T}$$

where $\mathcal{T} \subset \mathcal{F}$ is the torsion subsheaf and $\mathcal{E}$ is locally free.

PROOF: The quotient $\mathcal{E} = \mathcal{F}/\mathcal{T}$ is locally free by Proposition 23.21, and are to see that the exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0 \tag{23.2}$$

is split exact. Let U be an affine neighbourhood about p_i. Since $\mathcal{F}/\mathcal{T}|_U$ is the tilde of a projective module, the sequence (23.2) splits when restricted to U. Hence there is a map $\phi_i \colon \mathcal{F}|_U \to \mathcal{T}_{p_i}$ splitting the inclusion $\mathcal{T}_{p_i} \to \mathcal{F}$. This map extended by zero is a map $\phi_i \colon \mathcal{F} \to \mathcal{T}_{p_i}$ that splits off $\mathcal{T}_{p_i}$. The sum $\sum \phi_i$ then splits of the entire torsion subsheaf $\mathcal{T}$. ❑

The torsion sheaves on X are easily classified, but only for rather few class curves are the locally free sheaves satisfactory understood, but there is a vast literature about them. For instance, back on page 241 we proved Theorem 16.25 which states that every coherent locally free sheaf on $\mathbb{P}^1_k$ decomposes as a direct sum $\bigoplus \mathcal{O}_{\mathbb{P}^1_k}(a_i)$ of line bundles. In fact this property characterises $\mathbb{P}^1_k$ among non-singular curves (even among normal projective varieties).

23.3 Divisors on regular curves

We shall mostly work with regular curves in this section, in which case there is no substantial distinction between Weil and Cartier divisors, every Weil divisor has a a set of Cartier data, and every set of Cartier data yields a Weil divisor. The distinction only shows up in the way a divisor is presented.

The codimension one-subsets of a curve are precisely the closed points, so that a Weil divisor is a finite formal combination

$$D = \sum_{x \in X} n_x x$$

of closed points in X, where the coefficients are integers. Each residue field $k(x)$ is a finite extension of the ground field k whose degree is denoted by $[k(x) : k]$. Note that in case the ground field is algebraically closed, all these degrees equal one. We define the degree of the prime divisor x as $\deg x = [k(x) : k]$, and extending this by linearity, every Weil divisor is given a *degree*, namely the sum: *The degree of a divisor*

$$\deg D = \sum [k(x) : k] n_x.$$

As noted above, every Weil divisor on a regular curve has a Cartier representation. To a given Weil divisor $D = \sum_x n_x x$ we may associate Cartier data $\{(U_x, g_x)\}$, indexed by Supp D, by letting U_x be any open affine neighbourhood of x disjoint from the rest of Supp D, and letting $g_x = t_x^{n_x}$, where t_x is a uniformizing parameter at x. In terms of the Cartier data, the degree is given as

$$\deg D = \sum_{x \in X} [k(x) : k] v_x(g_x),$$

where, as usual, $v_x$is the valuation associated to $\mathcal{O}_{X,x}$.

Each non-zero coherent sheaf of ideals on a regular curve X is invertible (all the local rings are PID's), so with each finite subscheme Z of X is associated an effective Weil divisor (as in 20.4 on page 294):

$$D_Z = \sum_{x \in X} \operatorname{length}(\mathcal{O}_{Z,x}) x,$$

where the sum is finite because $\operatorname{length}(\mathcal{O}_{Z,x}) = 0$ for x outside the finite set Z. The $\operatorname{length}(\mathcal{O}_{Z,x})$ is the number of terms in composition series, and each subquotent equals $k(x)$, so $[k(x) : k] \operatorname{length}(\mathcal{O}_{Z,x}) = \dim_k \mathcal{O}_{Z,x}$. Summing up over closed points $x \in X$ yields

$$\deg D_Z = \dim_k \mathcal{O}_Z.$$

Recall also that each Weil divisor determines an invertible sheaf $\mathcal{O}_X(D)$, which over an open set U takes the value

$$\mathcal{O}_X(D)(U) = \{ f \in K | (\operatorname{div} f + D)|_U \geqslant 0 \}$$

Then D is effective if and only if $\Gamma(\mathcal{O}_X(D)) \neq 0$. In particular, if $\Gamma(X, \mathcal{O}_X(D))$ has dimension at least 2, there is a second effective divisor $D' = \sum m_i q_i$ such that D and D' are linearly equivalent.

EXAMPLE 23.23 Consider the prime divisor x on $X = \mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[t]$ corresponding to the maximal ideal (t^2+1) in $\mathbb{R}[t]$. Then $k(x) = \mathcal{O}_{X,x}/(t^2+1)\mathcal{O}_{X,x} = \mathbb{R}[t]/(t^2+1) = \mathbb{C}$ and hence $\deg x = 2$. ☆

EXAMPLE 23.24 Consider the closed subset $D = V(t^5 - 1)$ of $X = \operatorname{Spec} \mathbb{Q}[t]$. Since $s(t) = t^4 + t^3 + t^2 + t + 1$ is an irreducible polynomial over $\mathbb{Q}$ the ideal $(s(t))$ in $\mathbb{Q}[t]$ is maximal. Thus the set D consists of the two points $p = V(t-1)$ and $q = V(s(t))$, and we may consider D as the divisor $D = p + q$. The residue fields are $k(p) = \mathbb{Q}$ and $k(q) = \mathbb{Q}(\eta)$, where η is a primitive fifth-root of unity. Consequently, the degree of $D = p + q$ is

$$\deg D = [\mathbb{Q} : \mathbb{Q}] + [\mathbb{Q}(\eta) : \mathbb{Q}] = 1 + 4 = 5,$$

which of course fits well with $t^5 - 1$ being of degree 5.

Over the field $k = \mathbb{Q}(\eta)$, the divisor $V(t^5 - 1)$ in $\operatorname{Spec} \mathbb{Q}(\eta)[t]$ splits as the sum of five different points, each with local degree one. Indeed, $t^5 - 1 = \prod_i t - \eta^i$, and letting $p_i = V(t_{\eta^i})$, we find $D = p_0 + \cdots + p_5$. ☆

EXAMPLE 23.25 *(The circle.)* Consider the circle $X = \operatorname{Spec} \mathbb{R}[u,v]/(u^2 + v^2 - 1)$. The example is about the divisors on X obtained by intersecting X with lines $u + v - a$ where a real; or in the present terminology, the principal divisor $D = \operatorname{div}(u + v - a)$. The support of D equals $V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal $\mathfrak{a} = (u+v-a, u^2+v^2-1)$. It holds that $\mathfrak{a} = (u+v-a, P(v))$ where $P(v) = 2v^2 - 2av + a^2 - 1$, so that

$$k[u,v]/\mathfrak{a} \simeq k[v]/(P(v)).$$

Now there are three cases. Firstly, if $a > \sqrt{2}$, the polynomial $P(v)$ does not have real roots, and $\mathbb{R}[u,v]/\mathfrak{a} = \mathbb{C}$. The divisor D is the prime divisor $x = V(\mathfrak{a})$ and $k(x) = \mathbb{C}$.

Secondly, if $a < \sqrt{2}$, the polynomial $P(v)$ splits as the product of two disinct linear factors l_1 and l_2, and $\mathfrak{a} = \mathfrak{m}_1 \cap \mathfrak{m}_2$ with $\mathfrak{m}_i = (u+v-a, l_i)$. Each ideal $\mathfrak{m}_i$ is maximal, and $\mathbb{R}[u,v]/\mathfrak{m}_i = \mathbb{R}$. The divisor D equals $D = x_1 + x_2$ with $x_i = V(\mathfrak{m}_i)$, and $k(x_i) = \mathbb{R}$.

Finally, when $a = \sqrt{2}$, we find $\mathfrak{a} = (u+v-a, l^2)$ where $l = v - \sqrt{2}/2$. The divisor D becomes $D = 2x$ with $x = (u - \sqrt{2}/2, v - \sqrt{2}/2)$ and $k(x) = \mathbb{R}$. ☆

Pullbacks of divisors

If $f\colon X \to Y$ is a morphism, we can pull back invertible sheaves from Y to X, as well as sections of these. By the correspondence between divisors and invertible sheaves, this gives us a way of pulling back divisors from Y to X. In the context of curves, we can make this a little bit more explicit. We assume that the morphism $f\colon X \to Y$ is finite, it is then surjective, and $f^\sharp$ induces an inclusion of function fields $k(Y) \hookrightarrow k(X)$.

We aim at defining the pull back of Weil divisors and start by just pulling back a point $y \in Y$. This pullback is just the divisor associated to the scheme theoretic fibre $f^{-1}(y)$. To give a detailed description, choswe a local parameter $t \in \mathcal{O}_{Y,y}$ at y and define

$$f^*(y) = \sum_{f(x)=y} v_x(f^\sharp t)x,$$

where as usual v_x is the valuation at x. Changing t by a unit in $\mathcal{O}_{Y,y}$ does not alter the valuation $v_x(f^\sharp t)$ because a unit in $\mathcal{O}_{Y,y}$ stays a unit in $\mathcal{O}_{X,x}$. Extending this by linearity, we obtain a well defined group homomorphism

$$f^*\colon \operatorname{Div} X \to \operatorname{Div} Y.$$

We can also understand this map on the level of Cartier divisors: if D is a Cartier divisor on Y given by the data $\{(U_i, g_i)\}$, where $g_i \in k(Y)^\times$, we can consider the data $\{(f^{-1}U_i, f^\sharp g_i)\}$, which defines a Cartier divisor on X.

EXAMPLE 23.26 (*Principal divisors.*) The principal divisor $\operatorname{div} g$ of a rational function $g \in k(X)^\times$ equals the pullback $G^*((0:1)-(1:0))$, where $G\colon X \to \mathbb{P}^1_k$ is the extension of g as in Example 23.20 on page 360. With the notation there it holds that

$$\operatorname{div} g = \sum_{x \in U_g} v_x(g) + \sum_{x \in U_{g^{-1}}} v_x(g) = \sum_{x \in U_g} v_x(g) - \sum_{x \in U_{1/g}} v_x(1/g)$$

since $v_x(g) = 0$ for $x \in U_g \cap U_{g^{-1}}$. But this is precisely the pullback $G^*((0:1)-(1:0))$ since $G^\sharp(t_0/t_1) = g$ and $g^\sharp(t_1/t_0) = 1/g$ and and t_0/t_1 and t_1/t_0 are uniformizers at $(0:1)$ and $(1:0)$ respectively. ☆

LEMMA 23.27 *If $f\colon X \to Y$ is finite and D is a divisor on Y, we have* $\deg f^*D = \deg f \cdot \deg D$.

PROOF: It suffices to treat the case of prime divisors, so let $D = y$. Now, let $\operatorname{Spec} A$ be an open neighbourhood of y and $\operatorname{Spec} B$ the inverse image of $\operatorname{Spec} A$. Then B is a torsion free A-algebra a so is locally free of rank equal to $[k(X) : k(Y)] = \deg f$. For t a uniformizer at y the value $v_x(f^\sharp t)$ is the ramification index of f at x and is written e_x. Moreover,

$$\deg x = [k(x) : k] = [k(x) : k(y)][k(y) : k] = d_x \deg y$$

where d_x is the local degree of f at x. From (23.3) and Proposition 23.13 follows that

$$\deg f^*y = \sum_{f(x)=y} v_x(f^\sharp t) \deg x = (\sum_{f(x)=y} d_x e_x) \deg y = \deg f \cdot \deg y.$$

❑

LEMMA 23.28 *For a non-zero $g \in k(X)$ and a morphism $f\colon X \to Y$, we have*

$$f^* \operatorname{div} g = \operatorname{div} g \circ f.$$

PROOF: Let $G\colon X \to \mathbb{P}^1$ be the extension of g then $G \circ f$ is the extension of $g \circ f$ and so according to Example 23.26 above, it holds that

$$f^* \operatorname{div} g = f^*(G^*((0:1) - (1:0)) = (G \circ f)^*((0:1) - (1:0)) = \operatorname{div} g \circ f.$$

❑

COROLLARY 23.29 *For a non-zero rational function $g \in k(X)$, we have* $\deg \operatorname{div} g = 0$. *Hence the degree map descends to a well-defined map*

$$\deg\colon \operatorname{Cl}(X) \to \mathbb{Z}.$$

In other words, linearly equivalent divisors have the same degree.

PROOF: This is clear if g is a constant. If not, g defines a morphism $G\colon X \to \mathbb{P}^1_k$ so that

$$\operatorname{div} g = G^*((1:0) - (0:1)).$$

Thus we are done by the above lemma. ❑

EXAMPLE 23.30 Assume that k is a field whose characteristic is different from 2 and 31. and consider the curve $X \subset \mathbb{A}^2_k = \operatorname{Spec} k[u, v]$ given by the equation

$$v^2 = u^3 + u^2 + 1 \tag{23.3}$$

which is a regular curve. Consider the the rational function $g = v + 1$ on X. What is $\operatorname{div} g$? Note that g is regular, so there are no points x for which $v_x(g) < 0$. Rewriting (23.3) as

$$(v-1)(v+1) = u^2(u+1), \tag{23.4}$$

we see that the zeros of $v+1$ are the points $x = (0,-1)$ and $y = (-1,-1)$. Near $x = (0,1)$ both $(v+1)$ and $u+1$ are invertible, and the equality

$$v - 1 = u^2(u+1)(v+1)^{-1} \tag{23.5}$$

shows that u is uniformizer there (the maximal ideal $\mathfrak{m}_x$ is generated by $v-1$ and u). In the same vein, near $y = (-1,-1)$ both $v+1$ and u are invertibel, and we infer from (23.5) that $u+1$ is a uniformizer. It follows that

$$\operatorname{div} g = v_x(u^2) + v_y(u+1) = 2x + y. \tag{23.6}$$

☆

EXAMPLE 23.31 Consider the curve $Y \subset \mathbb{P}^2_k = \operatorname{Proj} t_0, t_1, t_2$ given by the equation

$$t_2^2 t_0 = t_1^3 + t_1^2 t_0 + t_0^3$$

Note that the curve in the previous example equals $X \cap D(t_0)$, where we use coordinates $u = t_1/t_0, v = t_2/t_0$. Let us compute $\operatorname{div} g$ for the same rational function $g = t_2/t_0 + 1$ as before, but this time on Y. For this, we only need to consider the points where $t_0 = 0$. From the equation, we see that there is a single point in $Y \cap V(t_0)$, namely the point $z = (0:0:1)$. To compute $v_x(g)$ here, we use the chart $D(t_1)$. Then $Y \cap D(x_2)$ is isomorphic to the plane curve given by the equation

$$u = v^3 + v^2 u + u^3 \tag{23.7}$$

where now $u = t_0/t_2$ and $v = t_1/t_2$. The point z is then the origin $(u,v) = (0,0)$ in $D_+(t_2)$. Note that $g = u^{-1} + 1$. Rewriting (23.7) as

$$v^3 = u(1 - v^2 u - u^3),$$

we see that v is a uniformzer at z and that $v_x(u) = 3$. Hence we find we also see that $v_z(u) = 3$, and so

$$v_x(g) = v_x(u^{-1}+1) = v_x(u+1)/u) = v_x(u+1) - v_x u = -3$$

Finally, we concude thats

$$\operatorname{div} g = 2(1:0:-1) + (1:-1:-1) - 3(0:0:1). \tag{23.8}$$

Note that, since Y is projective we may use Corollary 23.29 and conclude that $\deg \operatorname{div} g = 0$, which immediately yields 23.8. ☆

Pushforward of divisors

For a morphism of curves $f\colon X \to Y$, one may also define a pushforward map $f_*\colon \operatorname{Div}(X) \to \operatorname{Div}(Y)$ as follows. If $D = \sum_{x \in X} n_x x$, we define

$$f_*(D) = \sum_{x \in X} d_x n_x x,$$

where d_x is the local degree of f at x (as defined on page 358). This defines an element of $\operatorname{Div}(Y)$ and f_* will be a linear map $\operatorname{Div} X \to \operatorname{Div} Y$. In this case it is not so obvious that the map descends to a map between the class groups $\operatorname{Cl}(X)$ and $\operatorname{Cl}(Y)$. However, it turns out that this is indeed the case: for $f \in k(X)^\times$, we have $f_* \operatorname{div}(f) = \operatorname{div} N(f)$ where $N\colon k(X) \to k(Y)$ is the *norm map* between the function fields.

Proposition 23.32 *If $f\colon X \to Y$ is a finite morphism between regular curves and $g \in k(X)$ a non-zero rational function. then $f_* \operatorname{div} f = \operatorname{div} N(f)$. In particular, f_* passes to the quotient and gives a homomorphism $f_*\colon \operatorname{Cl}(X) \to \operatorname{Cl}(Y)$.*

Quit generally there is norm for any finite field exetension $K \subseteq L$. Then norm $N(g)$ of an element $g \in L^\times$ is defined as the determinant of the multiplication map $L \to L$ given as $t \mapsto gt$. It is a multiplicative map $N\colon L^\times \to K^\times$. If $B \subseteq L$ is a subsring and $A = B \cap K$, it holds that $N(g) \in A$ for each $g \in B$.

Lemma 23.33 *If A is a DVR and $\phi\colon A^n \to A^n$ is an injective map, then $v(\det\phi) = \operatorname{length}(\operatorname{Coker}\phi)$.*

Proof: Represent ϕ by a matrix (a_{ij}). After a permutation of rows and columns we may assume that $v(a_{11}) \leqslant v(a_{ij})$ for all other entries a_{ij}. It is then straightforward to perform elementary row and column operations to make the matrix have $a_{1i} = a_{i1} = 0$ for $i \neq 1$. Repeated application of this procedure yields bases for the source and target of ϕ in which ϕ has a diagonal matrix. If the i-th diagonal element is $\alpha_i t^{e_i}$ with α_i a unit, then $\operatorname{Coker}\phi \simeq \bigoplus_i A/(t^{e_i})A$ and $\operatorname{length}(\operatorname{Coker}\phi) = \sum e_i$, which obviously equals $v(\det\phi)$. ❑

Corollary 23.34 *If A is a DVR and B a finite free A-algebra, then $\operatorname{length}_B(B/(b)B) = v_A(N(b))$ for any element $b \in B$.*

Proof: As to the proof of Proposition 23.32, it will be sufficient to establish that

$$v_y(N(g)) = \sum_{f(x)=y} d_x v_x(g)$$

for all $y \in Y$. We shall apply the Corollary with $A = \mathcal{O}_{Y,y}$ and $B = (f_*\mathcal{O}_X)_y$; the latter is finite and free over A (Proposition 23.21 on page 361).

The ring B is a one-dimensional semi-local ring whose maximal ideals correspond to the points in the fibre $f^{-1}(y)$. Hence $B/(g)B$ is Artinian, and it decomposes as $B/(g)B = \bigoplus \mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x}$. We claim that

$$\operatorname{length}_A \mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x} = d_x v_x(g),$$

from which 23.32 follows in view of the Corollary. Indeed, it holds true that $\operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x}) = v_x(g)$, which means that $\mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x}$ has a composition series of length $v_x(g)$, and the subquotients are all isomorphic to $k(x)$; hence $\operatorname{length}_A(\mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x}) = [k(x) : k(y)] \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x})$, and the claim follows. ❑

EXERCISE 23.2 *(The projection formula.)* Let $f\colon X \to Y$ be a finite morphism between regular curves and let D be a divisor on Y. Show that $f_* f^* D = \deg f \cdot D$. ★

23.4 *The canonical divisor*

In this section we exclusively work over a perfect field k. A curve over k will then be smooth if and only if it is regular, and the sheaf $\Omega_{X/k}$ of regular differential is then locally free of rank one.

The elements of $\Omega_{k(X)/k}$ are called *rational differential forms* We are going to associate a Weil divisor with each non-zero rational differential form on X. These divisors will all be rationally equivalent, and so we find a well defined divisor class in $\operatorname{Div} X$, which only depends on the curve. It is called the *canonical class* and the divisors belonging to it will be called *canonical divisors*; often denoted by K_X. The canonical class is the most important invariant of the curve.

Rational differentials forms

canonical class

Since the Kähler differentials localize well (Theorem 22.36), the module $\Omega_{k(X)/k}$ is a one-dimensional $k(X)$-vector space being the stalk of the invertible sheaf $\Omega_{X/k}$. Any local generator η of $\Omega_{X/k}$ at a point $x \in X$, is a generator for $\Omega_{k(X)/k}$ as well, so that each rational differential ω is of the form $\omega = g\eta$ for some $g \in k(X)$; indeed, $\Omega_{k(X)/k} = \Omega_{\mathcal{O}_{X,x}/k} \otimes_{\mathcal{O}_{X,x}} k(X)$.

To every rational differential one may associate a Weil divisor $\operatorname{div} \omega$ by the following procedure. For each point $x \in X$ chose a generator η_x for $\Omega_{\mathcal{O}_{X,x}/k}$ and write $\omega = g_x \eta_x$ with $g_x \in k(X)$. Then let

$$\operatorname{div} \omega = \sum_{x \in X} v_x(g_x) x.$$

Two verifications are needed. Firstly, the expression is independent of the choices of local generators; which is clear since another generator η'_x will be related to η_x through an equality $\eta_x = \alpha \eta'_x$ with α a unit in $\mathcal{O}_{X,x}$. Hence $\omega = g_x \eta_x = \alpha g_x \eta'_x$, and $v_x(\alpha g_x) = v_x(g_x)$. The second is that the sum in fact is finite. But this ensues from any local generator η_x being a generator for $\Omega_{X/k}$ over some neighbourhood U of x.

That the divisors associated with two rational differentials are linearly equivalent, is clear from the definition in (23.4); indeed, two rational differentials are proportional with a factor from $k(X)$, and for each $x \in X$ it holds that $v_x(hg_x) = v_x(h) + v_x(g_x)$, so that

$$\operatorname{div}(h\omega) = \operatorname{div} h + \operatorname{div} \omega.$$

This leads to:

DEFINITION 23.35 (THE CANONICAL CLASS) *Let X be a smooth curve over k. The* canonical class *of X in $\operatorname{Cl} X$ is the divisor class of* $\operatorname{div} \omega$ *for any non-zero $\omega \in \Omega_{k(X)/k}$. Any divisor in the canonical class is called* a canonical divisor *and often will be denoted by K_X.*

What we have done so far is valid over any field as long as the curve is smooth. When the ground field is perfect, there is as a good local description of the rational differentials in terms of uniformizers making calculations easier.

LEMMA 23.36 *Assume that X is smooth at the closed point $x \in X$ and that t is uniformizer at x. Then each element of $\Omega_{k(X)/k}$ is of the form gdt with $g \in k(X)$; in other words, $\Omega_{k(X)/k}$ is of rank one over $k(X)$ with dt as a basis.*

PROOF: The Zariski cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ at x is always generated by the class of a uniformizer, and in virtue of Proposition 22.31, if follows that dt generates $\Omega_{\mathcal{O}_{X,x}/k}$ when X is smooth and $k(x)$ is a separable extension of k. ❑

EXAMPLE 23.37 When the residue field $k(x)$ is not separable over the ground field k, it happens that $dt = 0$ for a uniformizer t at x. For instance, if k is of characteristic p and $\alpha \in k$ does not have a p-th root in k, the ideal $\mathfrak{m} = (t^p - \alpha)$ in $k[t]$ is maximal. The element $t^p - \alpha$ is a uniformizer in the local ring $k[t]_{\mathfrak{m}}$, but its derivative equals 0. ☆

Be reminded that two Weil divisors D and D' are linearly equivalent precisely when the two associated invertible sheaves are isomorphic.

PROPOSITION 23.38 *The invertible sheaf associated to* $\operatorname{div}\omega$ *equals* Ω_X.

EXAMPLE 23.39 Let $X = \mathbb{P}^1_k$ with the usual covering $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[t^{-1}]$. The differential form dt is an element of $\Omega_{k(X)/k}$, which generate $\Omega_{X/k}|_{U_0}$. This means that $v_x(dt) = 0$ for every $x \in U_0$. For the remaining point $(1:0)$ at infinity, note that t^{-1} is the uniformizer there, $(1:0)$ corresponding to the origin in U_1. We have $d(t^{-1}) = -t^{-2}dt$; hence $dt = -(t^{-1})^{-2}d(t^{-1})$. This means that $v_{(1:0)}dt = -2$, so that $\operatorname{div} dt = -2(1:0)$.

As a Cartier divisor, the corresponding divisor is given by $(U_0, 1)$, $\quad (U_1, t^2)$ This shows that $\Omega_X \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$. ☆

EXAMPLE 23.40 Assume that k is of characteristic different from 2. Let $X \subseteq \mathbb{A}_k$ be the elliptic curve given by the equation

$$v^2 = u^3 - u,$$

and consider the differential $\omega = du$. At a point $p = (a, b)$ where $b \neq 0$, the coordinate u is a uniformizer, and so $du = d(u - a)$ has zero valuation at p. When $b = 0$, the curve has three points: $p_1 = (0, 0)$, $p_2 = (-1, 0)$, and $p_3 = (1, 0)$.

At these points, v will be a uniformizer, and since $2vdv = (3u^2 - 1)du$, it holds that

$$du = 2v/(3u^2 - 1)dv.$$

Hence $v_{p_i}(du) = 1$ for all three. Summing up, we conclude that

$$\operatorname{div}\omega = p_1 + p_2 + p_3.$$

☆

EXAMPLE 23.41 We consider the projectivization $X \subset \mathbb{P}^2_k$ of the previous example, *i.e.* the curve whose homogeneous equation is

$$x_1^2 x_2 = x_0^3 - x_0 x_2^2.$$

Consider again the rational differential ω from before; that is, $\omega = d(x_0/x_2)$. We know the behaviour of ω on the distinguished open set $D_+(x_2)$, so what remains to compute the divisor of ω, is the valuation $v_p(\omega)$ for each point in $X \cap V(x_2)$, but this intersection has just one the single point $x = (0:1:0)$.

We dehomogenize in the chart $D_+(x_1)$ by setting $u = x_0/x_1$ and $v = x_2/x_1$. The equation of X in $D_+(x_1)$ becomes

$$v = u^3 - uv^2.$$

Since $1 + uv$ is invertible near x, this shows that u is a uniformizer at x and that $v_x(v) = 3$. Our differential ω takes the form $\omega = d(x_0/x_1 \cdot x_1/x_2) = d(u/v) = (udv - vdu)/v^2$.

We find

$$dv = 3u^2 - v^2 - 2uvv')dv$$

So

$$udv - vdu = (3u^3 - uv^2 - 2u^2vv' - v)du$$

The terms ot

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23.5 *The genus of a curve*

DEFINITION 23.42 *The* arithmetic genus *of X is defined as the number*

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X).$$

The geometric genus *of X is defined as*

$$p_g(X) = \dim_k H^0(X, \Omega_X).$$

Note that $\chi(\mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X)$, so we get

$$p_a(X) = 1 - \chi(\mathcal{O}_X).$$

These numbers are defined using different sheaves, and there is no a priori reason to expect that they should have anything to do with each other. However, we shall see later in the chapter that there is a strong relation between them: $p_a = p_g$ whenever X is non-singular. For the time being we will still refer to the arithmetic genus p_a as the *genus* of X.

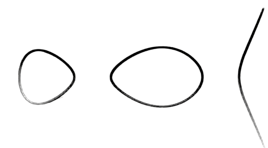
Example 23.43 When $X = \mathbb{P}^1$, we have $H^1(\mathbb{P}^1, \mathcal{O}) = 0$ so the arithmetic genus is 0. Likewise, we have that $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, so also $p_g = 0$. ☆

Example 23.44 Let $X \subset \mathbb{P}^2$ be a plane curve, defined by a homogeneous polynomial $f(x_{0,1}, x_2)$ of degree d. In Chapter 17, we computed that $H^1(X, \mathcal{O}_X) \simeq k^{\binom{d-1}{2}}$. Hence the genus of X is $\frac{(d-1)(d-2)}{2}$. ☆

23.6 *Hyperelliptic curves*

Let us recall the hyperelliptic curves defined in Chapter 3. For an integer $g \geqslant 1$ we consider the scheme X glued together by the affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, where

$$A = \frac{k[x,y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1 x)} \text{ and } B = \frac{k[u,v]}{(-v^2 + a_{2g+1}u + \cdots + a_1 u^{2g+1})}$$



As before, we glue $D(x)$ to $D(u)$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

In Chapter **??** we showed that the genus of X was g and claimed that X was actually projective.

Let us examine the last point in more detail, and give a new projective embedding of X. To do this, we will need to work out the groups $\Gamma(X, \mathcal{O}_X(nP))$ for a point $p \in X$.

Let us for simplicity assume that $a_{2g+1} = 1$. Let p be the unique closed point given by $V(u,v)$ in X. In the local ring at p, we have

$$u = v^2(1 + a_{2g}u + \cdots + a_1 u^{2g})^{-1} = v^2(\text{unit}),$$

and hence v generates $\mathfrak{m}_p$. Hence v is the local parameter. The valuations of v, u, x, y are given by

$$\nu_p(v) = 1, \quad \nu_p(u) = 2, \quad \nu_p(x) = -2, \quad \nu_p(y) = 1 + (g+1)(-2) = -(2g+1)$$

We computed in XXX that $\Gamma(X, \mathcal{O}_X) = k$, which agrees with our expectation that there are no non-constant regular function on a projective curve. Let us consider the case where the rational functions are allowed to have poles at p (and only at p). In other words, we are interested in elements $s \in \Gamma(X, \mathcal{O}_X(p))$. Note that the point p does not lie in U; this means that s is regular there, and hence can be viewed as a *polynomial* in x, y. Now, as $A = k[x] \oplus k[x]y$ as a $k[x]$-module, we can express any element s can be expressed as $f(x) + h(x)y$. We can then calculate

$$\begin{aligned} \nu_p(f(x) + h(x)y) &= \min\{\nu_p(f(x)), \nu_p(h(x))\nu_p(y)\} \\ &= \min\{-2\deg f, -(2\deg h + 2g + 1)\} \end{aligned}$$

Thus, since we assume $g \geqslant 1$, any non-constant rational function with a pole at p must have valuation $\leqslant -2$ there, and hence we have only the constants in $\Gamma(X, \mathcal{O}_X(p)) = k$.

On the other hand for the divisor $2p$ we obtain an extra section, corresponding to the rational function x:

$$\Gamma(X, \mathcal{O}_X(2p)) = k \oplus kx$$

Note that $\mathcal{O}_X(2p)_p = \mathcal{O}_{X,p} \cdot x$. The section $x \in \Gamma(X, \mathcal{O}_X(2p))$ is non-vanishing at p, while the section $1 \in \Gamma(X, \mathcal{O}_X(2p))$ is vanishing at p, since $1 = u \cdot x$ and $u \in \mathfrak{m} \subset \mathcal{O}_p$. Note that the linear series generated by $1, x$ generates $\mathcal{O}_X(2p)$ everywhere, so we get the morphism

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{P}^1 \\ (x,y) & \mapsto & (1:x) \end{array}$$

This morphism is exactly the double cover above. It gets even more interesting if we allow even higher order poles at p. The computation above shows that

$$\Gamma(X, \mathcal{O}(3p)) = \begin{cases} k \oplus kx \oplus ky & \text{if } g = 1 \\ k \oplus kx & \text{if } g > 1 \end{cases}$$

Case $g = 1$. We can show, using the embedding criterion of Chapter **??**, that the sections $x_0 = 1, x_1 = x, x_2 = y$ give an embedding

$$X \to\hookrightarrow \mathbb{P}^2_k$$

The image is even seen to be a cubic curve: One computes that $\Gamma(X, \mathcal{O}(6p))$ is 6-dimensional, but we have 7 global sections: $1, x, y, x^2, xy, x^3, y^2$. That means that there must be some relation between them of the form - of course it is just the relation in A:

$$y^2 = a_3x^3 + a_2x^2 + a_1x.$$

This gives the following defining equation of X in $\mathbb{P}^2$:

$$x_2^2x_0 = a_3x_1^3 + a_2x_0x_1^2 + a_1x_0^2x_1$$

Case $g = 2$. In this case, the divisor $3p$ does not give a projective embedding. However, the map given by $5p$ gives something interesting: We obtain

$$\Gamma(X, \mathcal{O}_X(5p)) = k \oplus kx \oplus kx^2 \oplus ky$$

These sections generate $\mathcal{O}_X(5p)$, so we obtain a morphism

$$\phi : X \to \mathbb{P}^3$$

given by the sections $u_0 = 1, u_1 = x, u_2 = x^2, u_3 = y$ of $L = \mathcal{O}_X(5p)$. Notice that $u_0u_2 - u_1^2 = 0$, so the image of X lies on a quadric surface. In fact, the image of ϕ is precisely the relations between the sections:

[]

The map ϕ is in this case a closed immersion, showing that X is projective.

Chapter 24

The Riemann–Roch theorem

When X is a projective curve over a field k, the cohomology groups $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces and we define

$$h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$$

Note that in this case, $h^i(X, \mathcal{F}) = 0$ for all $i \geqslant 2$, so we have two cohomology groups $h^0(X, \mathcal{F})$ and $h^1(X, \mathcal{F})$ to work with. We will mostly be interested in the case when $\mathcal{F} = \mathcal{O}_X(D)$ for some divisor D; any invertible sheaf on X is of this form.

Our most basic tool for studying the cohomology groups $H^0(X, \mathcal{O}_X(D))$ is the ideal sheaf sequence of a point $p \in X$, which takes the form

$$0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to k(p) \to 0 \tag{24.1}$$

where the first map is the inclusion and the second is evaluation at p. Here we have identified the ideal sheaf $\mathfrak{m}_p \subset \mathcal{O}_X$ by the invertible sheaf $\mathcal{O}_X(-p)$, and the sheaf $i_*\mathcal{O}_p$ with the skyscraper sheaf with value $k(p)$ at p. If L is an invertible sheaf, we can tensor (24.1) by L and get

$$0 \to L(-p) \to L \to k(p) \to 0 \tag{24.2}$$

where $L(-p)$ is the invertible sheaf of sections of L vanishing at p. (Here we also identify $L \otimes k(p) \simeq k(p)$, because every invertible sheaf over a point is trivial). In particular, taking $L = \mathcal{O}_X(D+p)$ in (24.2) we get the following basic bound:

Lemma 24.1 *Let X be a non-singular projective curve, and let D be a divisor on X. Then*

- $h^0(X, \mathcal{O}(D+p)) \leqslant h^0(X, \mathcal{O}(D)) + 1$ *for each $p \in X$.*
- $h^0(X, \mathcal{O}_X(D)) \leqslant \deg D + 1$.

Proof: We only need to prove the last part. Also, it suffices to constder the case when $D = \sum n_p p$ is effective (otherwise the left-hand side is 0). In that case, the inequality follows by applying the first inequality $\deg D$ times. ❑

Recall, that we defined for a sheaf $\mathcal{F}$, the Euler characteristic $\chi(\mathcal{F})$ as the alternating sum of the $h^i(X, \mathcal{F})$. One useful property of $\chi(X, -)$ is that it is additive on short exact sequences: $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. Thus applying χ to (24.2), we get

$$\chi(L(-p)) = \chi(L) - \chi(k(-p)) = \chi(L) - 1.$$

Theorem 24.2 (Easy Riemann–Roch) *Let X be a smooth projective curve of genus g and let D be a Cartier divisor on X. Then*

$$\chi(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = \deg D + 1 - g$$

Proof: Let $p \in X$ be a point and consider the sequence (24.2) with $L = \mathcal{O}_X(D+p)$. Then, as we just saw, $\chi(\mathcal{O}_X(D+p)) = \chi(\mathcal{O}_X(D)) + 1$. Also the right-hand side of the equation above increases by 1 by adding p to D (since $\deg(D+p) = \deg D + 1$). This means that the theorem holds for a divisor D if and only if it holds for $D+p$ for any closed point p. So by adding and subtracting points, we can reduce to the case when $D = 0$. But in that case, the left hand side of the formula is by definition $\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = 1 - g$, which equals the right hand side. ❏

The formula above is useful because the right hand side is so easy to compute. The number we are really after is the number $h^0(X, \mathcal{O}_X(D))$, since this is the dimension of global sections of $\mathcal{O}_X(D)$. This in turn would help us to study X geometrically, since we could use sections of $\mathcal{O}_X(D)$ to define rational maps $X \dashrightarrow \mathbb{P}^n$. So if we, for some reason, could argue that say, $H^1(X, \mathcal{O}_X(D)) = 0$, we would have a formula for the dimension of the space of global sections of $\mathcal{O}_X(D)$.

In any case, we can certainly say that $h^1(X, \mathcal{O}_X(D)) \geqslant 0$, so we get the following bound on $h^0(X, \mathcal{O}_X(D))$. It is a *lower bound* on $h^0(X, \mathcal{O}_X(D))$, which is often enough in applications.

Corollary 24.3 $h^0(X, \mathcal{O}_X(D)) \geqslant \deg D + 1 - g$.

Example 24.4 A typical feature is that $H^1(X, \mathcal{O}_X(D)) = 0$ provided that the degree $\deg D$ is large enough. This is basically a consequence of Serre's theorem. To give an example, consider again the case where X is a hyperelliptic curve of genus 2, as in XXXX. We have the following table of the various cohomology groups $H^i(X, \mathcal{O}_X(np))$ for the point $p = (u, v)$:

D	0	$1p$	$2p$	$3p$	$4p$	$5p$	$6p$	$7p$
$H^0(X, \mathcal{O}_X(D))$	1	1	2	2	3	4	5	6
$H^1(X, \mathcal{O}_X(D))$	2	1	1	0	0	0	0	0
$\chi(\mathcal{O}_X(D))$	-1	0	1	2	3	4	5	6

and it is not so hard to prove directly using the Čech complex that $H^1(X, \mathcal{O}_X(np)) = 0$ for all $n \geqslant 3$. ☆

Fortunately, there are more general results which tell us when $H^1(X, \mathcal{O}_X(D)) = 0$. This is due to the following fundamental theorem:

Theorem 24.5 (Serre duality) *Let X be a smooth projective variety of dimension n and let D be a Cartier divisor on X. Then for each $0 \leqslant p \leqslant n$,*

$$\dim_k H^p(X, \mathcal{O}_X(D)) = \dim_k H^{n-p}(X, \mathcal{O}_X(K_X - D))$$

So if X is a curve, we get that $h^1(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(K_X - D))$ and the Riemann–Roch theorem takes the following form:

Theorem 24.6 (Riemann–Roch) *Let X be a non-singular projective curve of genus g and let $D \in \operatorname{Div}(X)$ be a divisor. Then*

$$h^0(X, \mathcal{O}_X(D)) - h^0(X, \mathcal{O}_X(K_X - D)) = \deg D + 1 - g$$

This is a much stronger statement than the Riemann–Roch formula we had before. It is more applicable because the group $H^0(X, \mathcal{O}_X(K_X - D))$ is easier to interpret: it is the space of global sections of the sheaf associated to the divisor $K_X - D$, or equivalently $\Omega_X(-D)$. It is also often easier to argue that there can be no such global sections of this sheaf. For instance, in the case $\deg D > \dim K_X$ then $K_X - D$ cannot be effective: effective divisors $\sum n_i p_i$ have non-negative degree.

So what is this degree of the canonical divisor K_X? From Serre duality, we get that $H^0(X, \mathcal{O}_X(K_X))$ and $H^1(X, \mathcal{O}_X)$ have the same dimension, so the geometric genus and arithmetic genus agree:

$$p_g = p_a = g.$$

Then applying the Riemann–Roch formula to $D = K_X$, we get

$$g - 1 = \dim_k H^0(X, \mathcal{O}_X(K_X)) - \dim_k H^0(X, \mathcal{O}_X(K_X - K_X)) = \deg K + 1 - g$$

and so $\deg K_X = 2g - 2$. We summarize this in the following corollary.

Corollary 24.7 *Suppose that D is a Cartier divisor of degree $\leqslant 2g - 1$. Then $H^1(X, \mathcal{O}_X(D)) = 0$, and*

$$h^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g$$

Moreover, if $\deg D = 2g - 2$, then $H^1(X, \mathcal{O}_X(D)) \neq 0$ only if $D \sim K_X$.

Example 24.8 Let us verify the Riemann–Roch formula for $X = \mathbb{P}^1$. It suffices to check it for all divisors of the form $D = dP$ where $P \in \mathbb{P}^1$ is a point. In this case, the right-hand-side of the fomula equals $\deg D + 1 - 0 = d + 1$.

If $d \geqslant 0$, we may identify $H^0(X, D)$ with the space of homogenous degree d polynomials in x_0, x_1. Hence $h^0(X, D) = d + 1$. Moreover, $h^1(X, D) = 0$, as we saw in Chapter XXX. If $d < 0$, we have $h^0(X, D) = 0$ and $h^1(X, D) = -d - 1$. ☆

24.1 Serre duality

The aim of the next few sections is to prove the following:

Theorem 24.9 (Serre duality) *Let X be a projective curve over an algebraically closed field k. Then there is a coherent sheaf ω_X on X, together with an isomorphism $t : H^1(X, \omega_X) \to k$, such that for any locally free sheaf $\mathcal{E}$ on X, there is a perfect pairing*

$$H^0(X, \mathcal{F}) \times H^1(X, \omega_X \otimes \mathcal{E}^\vee) \to H^1(X, \omega_X) \simeq k \tag{24.3}$$

In particular, $H^1(X, \omega_X \otimes \mathcal{E}^\vee) \simeq H^0(X, \mathcal{E})^\vee$.

The sheaf ω_X is called a *dualizing sheaf*. The existence of ω_X is usually not enough for applications or explicit computations. The important point is that, in the smooth case, the dualizing sheaf equals with the cotangent sheaf, which is easier to study (e.g., because there are formulas for the canonical divisor).

Theorem 24.10 *If X is a non-singular, projective curve, the dualizing sheaf ω_X is isomorphic to the cotangent sheaf Ω_X.*

There are of course several known proofs of this result [?], [?], [?], [?]. Our proof is quite elementary, in the sense that it requires no derived functors, $\mathcal{E}xt$-sheaves, residues, adeles, etc. The ad hoc approach here is however much less conceptual than the standard proofs, and give essentially no information about the isomorphism $H^1(X, \Omega_X) \simeq k$.

We will prove the two theorems in three steps:

i) First we prove both theorems for $X = \mathbb{P}^1$. In which $\omega_X = \mathcal{O}_{\mathbb{P}^1}(-2)$ serves as a dualizing sheaf (and we know this coincides with $\Omega_{\mathbb{P}^1}$).

ii) Then we prove existence of ω_X for a general curve, using a Noether normalization $f : X \to \mathbb{P}^1$. Here the sheaf ω_X is constructed just to satisfy the formal properties of Serre duality.

iii) We finally prove that $\omega_X \simeq \Omega_X$ by a computation on the self product $X \times_k X$.

The fact that $\mathbb{P}^1$, and hence X, can be a covered by two affine open sets simplifies things a lot. In particular, we have a concrete interpretation of the first cohomology group H^1 of a sheaf, in terms of the Cech complex.

24.2 Proof of Serre duality for $X = \mathbb{P}^1$

Lemma 24.11 *Serre duality holds for $\mathbb{P}^1$ with $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ and $\mathcal{F}$ is an invertible sheaf.*

Proof: Recall that we may identify $H^0(X, \mathcal{O}_{\mathbb{P}^1}(d))$ with the k-vector space of homogeneous polynomials of degree d and $H^1(X, \mathcal{O}_{\mathbb{P}^1}(-2-d))$ with the k-vector space spanned by Laurent monomials $x_0^{-u} x_1^{-v}$ with $u + v = d + 2$ $u, v \geqslant 1$. The multiplication map

$$x_0^a x_1^b \times x_0^{-u} x_1^{-v} := \begin{cases} x_0^{-1} x_1^{-1} & \text{if } (u,v) = (a-1, b-1) \\ 0 & \text{otherwise} \end{cases}$$

defines a perfect pairing, which induces (24.3) when $\mathcal{F} = \mathcal{O}(d)$ with $d \geqslant 0$. For $d < 0$ all groups are zero. ❑

24.3 A simple cohomological lemma

In our proof, we will deduce Serre duality on a general curve X by considering a finite (hence affine) morphism $\pi : X \to \mathbb{P}^1$. The following lemma will allow us to transport computations of cohomology of sheaves on X to computations on $\mathbb{P}^1$, at the cost of replacing $\mathcal{F}$ with $\pi_*\mathcal{F}$.

Lemma 24.12 *Let $\pi : X \to Y$ be an affine morphism of varieties. Then for each coherent sheaf $\mathcal{F}$ on X, and $i \geqslant 0$, we have a canonical isomorphism*

$$H^i(X, \mathcal{F}) = H^i(Y, \pi_*\mathcal{F}).$$

Proof: Let $\mathcal{U} = \{U_i\}$ be a finite affine covering of Y such that $H^i(X, \pi_*\mathcal{F})$ is computed by the Cech complex $C^\bullet(U_i, \pi_*\mathcal{F})$. The hypotheses give that X is covered by the affine subsets $\pi^{-1}(U_i)$. The lemma follows simply because the Cech complexes of $\mathcal{F}$ and $\pi_*\mathcal{F}$ with respect to the respective coverings are the same. ❑

24.4 Curves obtained by gluing two affines

If X is a non-singular projective curve over k, we can pick a Noether normalization $\pi : X \to \mathbb{P}^1$, which is affine, finite and flat.

Recall the standard gluing construction of $\mathbb{P}^1$ as $U \cup U'$ where $U = \operatorname{Spec} A$, and $U' = \operatorname{Spec} A'$, and $A = k[a]$ and $A' = k[a']$. The gluing is defined by the isomorphism $D(a) = \operatorname{Spec} A_a \simeq \operatorname{Spec} A'_{a'} = D(a')$, using the isomorphism $\tau : A_a \to A'_{a'}$ given by $\tau(a) = a'^{-1}$.

Because the morphism π is affine, we find that also X can be covered by two affine subsets $\pi^{-1}(U)$, $\pi^{-1}(U')$. We write $V = \operatorname{Spec} B$ and $V' = \operatorname{Spec} B'$ for these subsets. Note that $\pi|_V$ (resp. $\pi|_{V'}$) is induced by a ring map $u : A \to B$ (resp. $u' : A' \to B'$), so that $b = u(a)$ (resp. $b' = u'(a')$). Thus X is obtained by gluing V and V' along $\operatorname{Spec} B_b$ and $\operatorname{Spec} B'_{b'}$ using an isomorphism $\sigma : B_b \to B_{b'}$, which is compatible with π, in the sense that the diagram below commutes:

$$\begin{tikzcd} B_b \arrow[r, "\sigma"] & B'_{b'} \\ A_a \arrow[u, "u_a"] \arrow[r, "\tau"] & A'_{a'} \arrow[u, "u'_{a'}"'] \end{tikzcd}.$$

Gluing sheaves

Given a quasi-coherent sheaf $\mathcal{G}$ on $\mathbb{P}^1$, we get an A-module $N = \Gamma(U, \mathcal{G})$, and an A'-module $N' = \Gamma(U', \mathcal{G})$. On $D(a) = \operatorname{Spec} A_a \simeq \operatorname{Spec} A'_{a'} = D(a')$, these are related by an isomorphism of $A_{a'}$-modules

$$\mu : N'_{a'} \to N_a$$

(where we view N_a as an $A_{a'}$-module using the isomorphism τ). Conversely, by the tilde-construction and the Gluing lemma for sheaves, given modules N, N' and an isomorphism μ as above, we can construct a quasi-coherent sheaf $\mathcal{G}$ on $\mathbb{P}^1$.

Similarly, giving a quasi-coherent sheaf $\mathcal{F}$ on X is equivalent to giving: A B-module M; A B'-module; and an isomorphism of $B_{b'}$-modules

$$\nu : M_{b'} \to M_b$$

$\mathcal{F}$ is coherent if and only if M and M' are finitely generated.

24.5 *The dualizing sheaf*

We will use the gluing construction of the previous section to define a sheaf ω_X on X, starting from $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1$. To define it, we need to define two modules on each affine open and check that they glue over the intersection.

The general construction goes as follows. Start with an A-module N and consider the A-module

$$M = Hom_A(B, N).$$

The crucial point is that M can be viewed as a B-module, via the rule

$$b \cdot \phi(y) := \phi(b \cdot y), \quad y \in B$$

for each A-linear map $\phi : B \to N$. Likewise, for an A'-module N', the A'-module $M' = \mathrm{Hom}_{A'}(B', N')$ can be viewed as a B'-module.

If N and N' arise from a sheaf $\mathcal{G}$ on $\mathbb{P}^1$ in the construction above, there is a natural isomorphism

$$\mathrm{Hom}_{A'_{a'}}(B'_{b'}, N'_{a'}) \to \mathrm{Hom}_{A_a}(B_b, N_a)$$

sending $\phi : B'_{b'} \to N'_{a'}$ to $\mu^{-1} \circ \phi \circ \sigma : B_b \to N_a$. One checks that this is an isomorphism of B_b-modules. Thus from any sheaf $\mathcal{G}$ on $\mathbb{P}^1$, we obtain a sheaf, denoted by $\pi^!\mathcal{G}$, on X. In fact, the map $\mathcal{G} \mapsto \pi^!\mathcal{G}$ defines a *functor* from the category of coherent $\mathcal{O}_{\mathbb{P}^1}$-modules to $\mathcal{O}_X$-modules, but we will not need this fact here.

The crucial ingredient we need is that there is a canonical isomorphism

$$\pi_* \mathscr{H}om_X(\mathcal{F}, \pi^!\mathcal{G}) \simeq \mathscr{H}om_{\mathbb{P}^1}(\pi_*\mathcal{F}, \mathcal{G}). \tag{24.4}$$

We first prove this locally:

LEMMA 24.13 *For a finitely generated B-module L, there is a natural isomorphism (of A-modules)*

$$\mathrm{Hom}_B(L, \mathrm{Hom}_A(B, N)) \to \mathrm{Hom}_A(L, N) \tag{24.5}$$

PROOF: The map is defined by sending $\phi : L \to \mathrm{Hom}_A(B, N)$ to $\ell \mapsto \phi(\ell)(1)$.

The map (24.5) is clearly an isomorphism for $L = B^{\oplus n}$. To prove it in general, pick a presentation

$$B^{\oplus m} \to B^{\oplus n} \to L \to 0.$$

Applying $\mathrm{Hom}_B(-, Hom_A(B, N))$, we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_B(L, \mathrm{Hom}_A(B,N)) \arrow[r] \arrow[d] & \mathrm{Hom}_B(B, \mathrm{Hom}_A(B,N))^{\oplus n} \arrow[r] \arrow[d, equal] & \mathrm{Hom}_B(B, \mathrm{Hom}_A(B,N))^{\oplus m} \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{Hom}_B(L, N) \arrow[r] & \mathrm{Hom}_B(B, N)^{\oplus n} \arrow[r] & \mathrm{Hom}_B(B, N)^{\oplus m} \end{tikzcd}$$

Then (24.5) is the left-most vertical map, and this is an isomorphism by the 5-Lemma. ❑

Inspecting the proof of Lemma 24.5, we note that the isomorphism in (24.5) is compatible with localizations. Thus the isomorphisms sheafify, and we get the sheaf isomorphism (24.4).

DEFINITION 24.14 *We define the* dualizing sheaf *of X as the sheaf $\omega_X = \pi^! \omega_{\mathbb{P}^1}$.*

So far we haven't used the fact that X is non-singular; any projective curve admits a dualizing sheaf ω_X. which is a coherent $\mathcal{O}_X$-module. Note that ω is a coherent sheaf on X (because locally it is constructed by $\mathrm{Hom}_A(B, N)$, which is finitely generated). In the non-singular case, we will prove in Section 24.7 that $\omega_X \simeq \Omega_X$. A first step towards this, is to show that ω_X is invertible.

PROPOSITION 24.15 *Let X be a non-singular projective curve. Then ω_X is an invertible sheaf.*

PROOF: Since X is a non-singular curve, ω_X is locally free if and only if it is torsion free. Let $\mathcal{T} = (\omega_X)_{\mathrm{tors}}$ denote the torsion subsheaf and $\mathcal{E}$ is the torsion free part, so that there is an exact sequence

$$0 \to \mathcal{T} \to \omega_X \to \mathcal{E} \to 0$$

Applying π_* to this, we get

$$0 \to \pi_* \mathcal{T} \to \pi_* \omega_X \to \pi_* \mathcal{E}$$

Then applying formula (24.4), shows that $\pi_* \omega_X = \mathscr{H}om(\pi_* \mathcal{O}_X, \omega_{\mathbb{P}^1})$. As π is finite and surjective, $\pi_* \mathcal{O}_X$ is locally free. Thus since $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ is invertible, we find that $\pi_* \omega_X$ is also locally free. Note that $\pi_* \mathcal{T}$ is again a torsion sheaf on $\mathbb{P}^1$. As $\pi_* \mathcal{T}$ maps injectively into a locally free sheaf, we must have $\pi_* \mathcal{T} = 0$. This implies that $\Gamma(X, \mathcal{T}) = \Gamma(\mathbb{P}^1, \pi_* \mathcal{T}) = 0$. On a curve, the only torsion sheaf with no global sections is the zero sheaf, so $\mathcal{T} = 0$ as well.

Finally, to compute the rank of ω_X, we use the fact that the formation of $\pi^! \mathcal{G}$ behaves well with localization. This implies that $\omega_{X,\eta}$ at the generic point $\eta = \operatorname{Spec} k(X)$ coincides with $\operatorname{Hom}_{k(\mathbb{P}^1)}(k(X), k(\mathbb{P}^1))$. The latter is a $k(\mathbb{P}^1)$-vector space of dimension equal to the degree of $k(X) : k(\mathbb{P}^1)$. Hence, as a $k(X)$-vector space it has dimension 1. ❑

24.6 *Proof of Theorem 24.9*

From here on, we can finish the proof of Serre duality on X:

$$\begin{aligned} H^1(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X) &= H^1(\mathbb{P}^1, \pi_*(\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X)) && \text{(Lemma 24.12)} \\ &= H^1(\mathbb{P}^1, \pi_* \mathscr{H}om(\mathcal{F}, \omega_X)) \\ &= H^1(\mathbb{P}^1, \mathscr{H}om(\pi_*\mathcal{F}, \omega_{\mathbb{P}^1})) && \text{(by (24.4))} \\ &= H^1(\mathbb{P}^1, (\pi_*\mathcal{F})^\vee \otimes_{\mathcal{O}_{\mathbb{P}^1}} \omega_{\mathbb{P}^1}) \\ &= H^0(\mathbb{P}^1, \pi_*\mathcal{F})^\vee && \text{(Serre duality on } \mathbb{P}^1\text{)} \\ &= H^0(X, \mathcal{F})^\vee. && \text{(Lemma 24.12)} \end{aligned}$$

24.7 *The dualizing sheaf equals the canonical sheaf*

The goal of this section is to show that the dualizing sheaf ω_X is isomorphic to the cotangent sheaf Ω_X. Note that both of these sheaves are locally free: the first by Corollary 24.15, and Ω_X because X is smooth.

We will work with the self-product $X \times X$ with the two projections $p, q : X \times X \to X$.

Lemma 24.16 ("Kunneth formula") *Let V and X be varieties over k with V affine. Let $\mathcal{F}$ denote a coherent $\mathcal{O}_V$-module and let $\mathcal{G}$ denote a coherent $\mathcal{O}_X$-module and write $p, q : X \times V \to X$ for the two projections on $V \times X$. Then there is a natural isomorphism*

$$H^i(V \times X, p^*\mathcal{F} \otimes q^*\mathcal{G}) = \Gamma(V, \mathcal{F}) \otimes_k H^i(X, \mathcal{G}) \tag{24.6}$$

Proof: Let $\mathcal{U} = \{U_i\}$ denote an open affine covering of X so that $C^\bullet(\mathcal{U}, \mathcal{G})$ computes the cohomology group $H^i(X, \mathcal{G})$. Tensoring $C^\bullet(\mathcal{U}, \mathcal{G})$ with the module $M = \Gamma(V, \mathcal{F})$ gives a complex $C^\bullet(\mathcal{U}, \mathcal{G}) \otimes_k M$ which is easily seen to compute the cohomology of both sides of (24.6). ❑

Consider the diagonal embedding $i : \Delta \to X \times X$. Recall that the normal bundle of Δ in $X \times X$ is isomorphic to the tangent bundle T_X. We thus have an exact sequence

$$0 \to \mathcal{O}_{X\times X} \to \mathcal{O}_{X\times X}(\Delta) \to i_* T_X \to 0. \tag{24.7}$$

We now tensor this sequence by $q^*\omega_X$, to we get a sequence

$$0 \to q^*\omega_X \to q^*\omega_X(\Delta) \to i_*(\omega_X \otimes T_X) \to 0 \tag{24.8}$$

Here we have used the projection formula:

$$i_* T_X \otimes q^*\omega_X = i_*(T_X \otimes i^* q^* \omega_X) = i_*(T_X \otimes \omega_X).$$

If we restrict the sequence (24.8) to the open set $V \times X$, where $V = \operatorname{Spec} R$ is affine, and take the long exact sequence in cohomology, we get

$$\Gamma(V \times X, i_*(\omega_X \otimes T_X)) \to H^1(V \times X, q^*\omega_X) \to H^1(V \times X, q^*\omega_X(\Delta)) \tag{24.9}$$

By Lemma 24.12), we may identify the first group with $\Gamma(V, \omega_X \otimes T_X)$. By Lemma 24.16, we may identity the second with $\Gamma(V, \mathcal{O}_X) \otimes_k H^1(X, \omega_X) = \Gamma(V, \mathcal{O}_X)$ (we also use the isomorphism $H^1(X, \omega_X) = k$). These identifications are compatible with restriction maps, so we get a map of sheaves $\omega_X \otimes T_X \to \mathcal{O}_X$, or equivalently,

$$\rho : \omega_X \to \Omega_X$$

We claim that ρ is surjective. Since both sheaves are locally free, it must then be an isomorphism . Thus we find that $\omega_X \simeq \Omega_X$.

because the kernel is locally free of rank 0

To conclude, it suffices to prove that the group $H^1(V \times X, q^*\omega_X(\Delta))$ in (24.9) vanishes for each affine $V \subset X$. Note that by Lemma 24.12, we have

$$H^1(V \times X, q^*\omega_X(\Delta)) = H^1(X, \omega_X \otimes q_*\mathcal{O}_{V\times X}(\Delta))$$

Note that $q_*\mathcal{O}_{V\times X}(\Delta)$ is locally free. By the duality property of ω_X, we may identify this with

$$H^0(X, \mathscr{H}om_X(q_*\mathcal{O}_{V\times X}(\Delta), \mathcal{O}_X))^\vee$$

By the change-of-rings property of Hom, the latter equals

$$H^0(X, q_* \mathscr{H}om_X(\mathcal{O}_{V\times X}(\Delta), \mathcal{O}_{V\times X}))^\vee = H^0(X, q_*\mathcal{O}_{V\times X}(-\Delta))^\vee$$

Using 24.12 agian, the latter cohomology group equals

$$H^0(V \times X, \mathcal{O}_{V\times X}(-\Delta))^\vee$$

But this last group is indeed zero: sections of $\mathcal{O}_{V\times X}(-\Delta) \simeq \mathcal{I}_\Delta$ correspond to sections of $\mathcal{O}_{V\times X}$ that vanish along the diagonal. However, we have

$$\Gamma(V \times X, \mathcal{O}_{V\times X}) = \Gamma(V, \mathcal{O}_V) \otimes_k \Gamma(X, \mathcal{O}_X) \simeq \Gamma(V, \mathcal{O}_V) \otimes_k k,$$

which implies that any such section can only vanish along a 'vertical' divisor $D \times X$ for $D \subset V$.

Exercises

(24.1)

a) Show that the pushforward of a torsion sheaf is torsion
b) Show that a sheaf $\mathcal{F}$ is torsion iff it is supported on a proper closed subset
c) Show that if $\mathcal{F}$ is a torsion sheaf on a curve X then $H^0(X, \mathcal{F}) = 0$ if and only if $\mathcal{F} = 0$.

★

Chapter 25

Applications of the Riemann–Roch theorem

In this chapter we give a few of the (many) consequences of the Riemann–Roch formula. We start by translating the results of Chapter **??** into concrete numerical criteria for a divisor D to be base point free or very ample. Then we use these results to classify all curves of all genus $\leqslant 4$.

25.1 Very ampleness criteria

Recall the criterion of Theorem **??**, that an invertible sheaf L is very ample if and only if its linear system separates points and tangent vectors. Using Riemann–Roch we can translate that result into a very simple, numerical criterion for very ampleness on a curve:

Theorem 25.1 *Let X be a non-singular projective curve and let D be a divisor on X. Then*

i) $|D|$ is base point free if and only if

$$h^0(D-P) = h^0(D) - 1 \qquad \text{for every point } P \in X.$$

ii) D is very ample if and only if

$$h^0(D-P-Q) = h^0(D) - 2 \qquad \text{for every two points } P, Q \in X$$

(including the case $P = Q$)

iii) A divisor D is ample iff $\deg D > 0$

Proof: (i) We take the cohomology of the following exact sequence

$$0 \to \mathcal{O}_X(D-P) \to \mathcal{O}_X(D) \to k(P) \to 0$$

and get

$$0 \to H^0(X, \mathcal{O}_X(D-P)) \to H^0(X, \mathcal{O}_X(D)) \to k$$

From this sequence, we get $h^0(D) - 1 \leqslant h^0(D-P) \leqslant h^0(D)$.

The right-most map takes a global section of $\mathcal{O}_X(D)$ and evaluates it at P. To prove that $|D|$ is base point free, we must prove that there is a section $s \in \mathcal{O}_X(D)$ which does not vanish at P, or equivalently, that the map $H^0(X, \mathcal{O}_X(D)) \to k$ is surjective. But this happens if and only if $h^0(D-P) = h^0(D) - 1$.

(ii) If the above inequality is satisfied, we see in particular that $|D|$ is base point free. So it determines a morphism $\phi : X \to \mathbb{P}^n$. We can use Theorem **??** that ensure that ϕ is an embedding. That is, we need to check that ϕ separates (a) points and (b) tangent vectors.

For (a), we are assuming that $h^0(D - P - Q) = h^0(D) - 2$, so the divisor $D - P$ is effective and does not have Q as a base point (by (i)). But this means that there is a section of $H^0(X, \mathcal{O}_X(D - P))$ which doesn't vanish at Q. We have $H^0(X, D - P) \subseteq H^0(X, D)$, so we get a section of $\mathcal{O}_X(D)$ which vanishes at P, but not at Q. Hence $|D|$ separates points.

For (b), we need to show that $|D|$ separates tangent vectors, i.e., the elements of $H^0(X, \mathcal{O}_X(D))$ should generate the k-vector space $\mathfrak{m}_P\mathcal{O}_X(D)/\mathfrak{m}_P^2\mathcal{O}_X(D)$ at every point $P \in X$. This condition is equivalent to saying that there is a divisor $D' \in |D|$ with multiplicity 1 at P: Note that $\dim T_P(X) = 1$, $\dim T_P D' = 0$ if P has multiplicity 1 in D' and $\dim T_p(D') = 1$ if P has higher multiplicity. But this is equivalent to P not being a base point of $D - P$. Applying (i) again, we see that $h^0(D - 2P) = h^0(D) - 2$, so we are done.

(iii) By definition, D is ample if mD is very ample for $m \gg 0$. So the result follows by the next result, since any divisor of degree $\geqslant 2g + 1$ is very ample. ❑

PROPOSITION 25.2 *Let X be a non-singular projective curve and let D be a divisor on X. Then*

i) If $\deg D \geqslant 2g$, then D is base point free.
ii) If $\deg D \geqslant 2g + 1$, then D is very ample.

PROOF: By Serre duality, $h^1(D) = h^0(K - D) = 0$ because $\deg D > \deg K = 2g - 2$. Similarly, $h^1(D - P) = 0$.

(i) Applying Riemann–Roch, we find that $h^0(D - p) = h^0(D) - 1$ for any $P \in X$, so we are done by the above theorem.

(ii) In this case we also get $h^1(D - P - Q) = 0$, so Riemann–Roch shows that $h^0(D - P - Q) = h^0(D) - 2$, which is the conclusion we want. ❑

EXAMPLE 25.3 On $X = \mathbb{P}^1$ a divisor D is base point free if and only if $\deg \geqslant 0$. Moreover, D is very ample if and only if $\deg D > 0$ ☆

EXAMPLE 25.4 If X is a curve of genus 1, a divisor D is base point free if $\deg D \geqslant 2$. We will see later that, if $D = p$ for some point p, we have $h^0(X, \mathcal{O}_X(D)) = 1$, so D can not be base point free (because the generator of $H^0(X, \mathcal{O}_X(D))$ vanishes at p).

A divisor D of degree $\geqslant 3$ is very ample if $\deg D \geqslant 3$. ☆

25.2 *Curves on* $\mathbb{P}^1 \times \mathbb{P}^1$

Let us consider one central example, namely curves on the quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Recall that $\mathrm{Cl}(Q) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$ where $L_1 = [0:1] \times \mathbb{P}^1$ and $L_2 = \mathbb{P}^1 \times [0:1]$.

We can use this to prove that Q contains non-singular curves of any genus $g \geqslant 0$. (This is in contrast with the case of $\mathbb{P}^2$, where only genera of the form $\binom{d-1}{2}$ were allowed).

To prove this, consider the divisor $D = aL_1 + bL_2$ where $a, b \geqslant 1$. D is effective, so let $C \in |D|$ be a generic element.

LEMMA 25.5 *C is non-singular.*

PROOF: D is defined by a bihomogeneous equation

$$\sum_{i+j=a, l+k=b} c_{ij,kl} x_0^i x_1^j y_0^l y_1^k = 0$$

On the open set $D_+(x_0) \cap D_+(y_0) \simeq \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ this is given by

$$\sum_{i+j=a, l+k=b} c_{ij,kl} x^j y^k = 0$$

and it is clear that if the coefficients $c_{ij,kl}$ are general, this is non-singular. By symmetry this also happens in the other charts, so C is non-singular. ❑

To compute the genus of C, we use the formula $2g - 2 = \deg \Omega_C$. So we need to find Ω_C and compute its degree. This is best computed using the Adjunction formula of Proposition **??**:

$$\begin{aligned} \Omega_C &= \omega_Q \otimes \mathcal{O}_Q(C)|_X \\ &= \mathcal{O}_Q(-2L_1 - 2L_2) \otimes \mathcal{O}(aL_1 + bL_2)|_C \\ &= \mathcal{O}_C((a-2)L_1 + (b-2)L_2) \end{aligned}$$

To compute the degree of this, we consider the degrees of $L_1|_C$ and $L_2|_C$ separately. Note that the degree $\deg L_1|_C$ is invariant under linear equivalence, so we can compute the degree of any $[s:t] \times \mathbb{P}^1$ for a general point $[s:t] \times \mathbb{P}^1$. The point is that as a Weil divisor, $L_1|_X$ is obtained by intersecting $[s:t] \times \mathbb{P}^1$ with X. When $[s:t] \in \mathbb{P}^1$ is a general point, the intersection $X \cap [s:t] \times \mathbb{P}^1$ is a reduced subscheme of X, consisting of b points (as $C \subset Q = \mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of type $aL_1 + bL_2$). Hence $\deg L_1|_C = b$ and $\deg L_2|_C = a$. It follows that

$$2g - 2 = \deg \Omega_C = (a-2)b + (b-2)a = 2ab - 2a - 2b$$

Solving for g gives us the following theorem:

THEOREM 25.6 *Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Then a generic divisor C in $|aL_1 + bL_2|$ is a smooth projective curve of genus*

$$g = (a-1)(b-1).$$

In particular, Q contains non-singular curves of any genus $g \geqslant 0$.

25.3 *Curves of genus 0*

The results of the previous results are particularly strong when the genus is small. For instance, when $g = 0$, any divisor of positive degree is very ample! We can use this to classify all curves of genus 0. First a simple lemma:

Lemma 25.7 *Let X be a non-singular curve. Then $X \simeq \mathbb{P}^1$ if and only if there exists a Cartier divisor D such that $\deg D = 1$ and $h^0(X, \mathcal{O}_X(D)) \geqslant 2$. In this case, the divisor D is even very ample.*

Proof: Let $g \in H^0(X, \mathcal{O}_X(D))$. Then $D' \sim \operatorname{div} g + D \geqslant 0$, so replacing D by D' we may assume that D is effective. Since $\deg D = 1$, we must have $D = p$ for some point $p \in X$. Now take $f \in H^0(X, \mathcal{O}_X(D)) - k$. As above, f induces a morphism $\phi : X \to \mathbb{P}^1$. This morphism has degree equal to 1, so it is birational, and hence X is isomorphic to $\mathbb{P}^1$. ❏

Proposition 25.8 *A non-singular curve X is isomorphic to $\mathbb{P}^1$ if and only if $\mathrm{Cl}(X) \simeq \mathbb{Z}$.*

Proof: We have seen that the Picard group of any $\mathbb{P}^n_k$ is isomorphic to $\mathbb{Z}$ via the degree map $\deg : \mathrm{Pic}(\mathbb{P}^n_k) \to \mathbb{Z}$.

Conversely, suppose X is a curve with $\mathrm{Cl}(X) \simeq \mathbb{Z}$. Let p, q be two distinct points on X. By assumption, p and q are linearly equivalent, so the linear system $|p| = \mathbb{P}H^0(X, \mathcal{O}_X(D))$ is at least 1-dimensional. Then $X \simeq \mathbb{P}^1_k$ by the previous lemma. ❏

Theorem 25.9 *Any curve of genus 0 over an algebraically closed field is isomorphic to $\mathbb{P}^1$.*

Proof: Let $p \in X$ be a point and consider the divisor $D = p$. If X has genus 0, then $1 = \deg D > 2g - 2 = -2$, so $H^1(X, \mathcal{O}_X(D)) = 0$. Then Riemann-Roch tells us that

$$\dim H^0(X, \mathcal{O}_X(D)) = 1 + 1 - 0 = 2$$

Hence $X \simeq \mathbb{P}^1_k$ by Lemma 25.7. ❏

We conclude by yet another characterisation of $\mathbb{P}^1$:

Lemma 25.10 *Let C be a non-singular projective curve and D any divisor of degree $d > 0$. Then*

$$\dim |D| \leqslant \deg D$$

with equality if and only if $C \simeq \mathbb{P}^1$.

Proof: Although one might guess that this lemma follows directly from Riemann-Roch, this does not seem to be the case: Riemann-Roch gives a different sort of relationship between the dimension and degree of a divisor.

We may assume that D is effective, i.e., $D = P_1 + \cdots + P_d$ for some points $P_1, \ldots, P_k \in C$ (possibly equal) (otherwise replace D by some different effective divisor $D' \in |D|$). We induct on d.

First suppose $d = 1$. There is an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(P) \to k(P) \to 0.$$

Now $h^0(\mathcal{O}_C) = 1$ and $h^0(k(P)) = 1$ therefore $h^0(\mathcal{O}_C(P)) \leqslant 2$ so $\dim |P| \leqslant 1$. If $\dim |P| = 1$ then $|P|$ has no base points so we obtain a morphism $C \to \mathbb{P}^1$ of degree $\deg P = 1$ which must be an isomorphism, and so $C \simeq \mathbb{P}^1$ is rational.

Next suppose $D = P_1 + \cdots + P_d$. Let $D' = P_1 + \cdots + P_{d-1}$. There is an exact sequence

$$0 \to \mathcal{O}_C(D') \to \mathcal{O}_C(D) \to k(P_d) \to 0.$$

Now $h^0(\mathcal{O}_C(D')) \leqslant d$ by induction and $h^0(k(P_d)) = 1$ so $h^0(\mathcal{O}_C(D)) \leqslant d+1$, therefore $\dim |D| \leqslant d$ with equality iff $h^0(\mathcal{O}_C(D')) = d$. By induction $h^0(\mathcal{O}_C(D')) = d$ iff C is rational. ❑

Non-algebraically closed fields

It is of course possible to develop the theory of curves over any field k, not just algebraically closed ones. In this case, there tend to be more divisors around than just the combinations of closed points. For instance, for $X = \mathbb{P}^1_{\mathbb{R}}$, the subscheme $D = V(x^2+1)$ is of codimension 1, so it gives a Weil divisor on X. The results of this chapter, including the Riemann–Roch theorem, still holds true, provided the degree of a divisor D is defined in terms of the degree of the field extension over which D is defined. In the above example, we would for instance have $\deg D = [\mathbb{R}(D) : \mathbb{R}] = [\mathbb{C} : \mathbb{R}] = 2$.

In this setting, a curve of genus 0, need not be isomorphic to $\mathbb{P}^1_k$ (although certainly this is true over the algebraic closure: $X \times_k \overline{k} \simeq \mathbb{P}^1_{\overline{k}}$). For instance, the curve $X = V(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^2_{\mathbb{R}}$ has genus 0, but is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$: This is because $X(\mathbb{R}) = \varnothing$, wheras $\mathbb{P}^1(\mathbb{R}))$ is infinite. A nice and surprising fact, however, is that a curve of genus 0 over a field k is at least always isomorphic to a projective conic in $\mathbb{P}^2_k$. This is because of the anticanonical divsor: $-K_X$ has degree 2 and defines an embedding $X \mapsto \mathbb{P}^2_k$.

EXAMPLE 25.11 Let k be any field, and consider the conic $X = V(x_0^2 + x_1^2 - x_2^2) \subset \mathbb{P}^2_k$. This X has a k-rational point $p_0 = (1 : 0 : 1)$. Projecting from p_0, we obtain a rational map $X \dashrightarrow \mathbb{P}^1_k$, which is birational. Hence X is isomorphic to $\mathbb{P}^1_k$. ☆

EXAMPLE 25.12 The conic $X = V(x_0^2 + x_1^2 - 3x_2^2)$ has many $\mathbb{R}$-points, but no $\mathbb{Q}$-points! ☆

25.4 *Curves of genus 1*

A plane curve $X \subset \mathbb{P}^2_k$ of degree 3 has genus 1. This follows from our earlier work on the canonical divisor, which showed $\omega_X \simeq \mathcal{O}_{\mathbb{P}^2_k}(d-3)|_X \simeq \mathcal{O}_X$, and so $g = h^0(\omega_X) = h^0(\mathcal{O}_X) = 1$. In this section, we show that in fact *every* curve of genus 1 arises this way:

> THEOREM 25.13 *Any projective curve X of genus 1 can be embedded as a plane cubic curve in $\mathbb{P}^2_k$.*

PROOF: Pick a point $P \in X$ and consider the divisor $D = 3P$. D has degree $3 \geqslant 2g+1$, so it is very ample. Furthermore, by Riemann–Roch, $h^0(3P) = 3$, so there is a projective embedding $\phi : X \to \mathbb{P}^2_k$. The image $\phi(X)$ is a smooth curve of degree equal to $\deg \phi^* \mathcal{O}_{\mathbb{P}^2}(1) = \deg D = 3$. ❑

In contrast to the $g = 0$ case however, there are many non-isomorphic genus 1 curves. For instance, in the *Legendre family* of curves in $X_\lambda \subset \mathbb{P}^2$ given by

$$y^2 z = x(x-z)(x-\lambda z)$$

where $\lambda \in k$, each X_λ is isomorphic to at most a finite number of other $X_{\lambda'}$'s.

Actually, these are essentially all the curves of genus 1.

THEOREM 25.14 *Let k be a field of char $k \neq 2,3$. Then any genus 1 curve X admits a projective model given by an homogeneous equation*

$$x_2^2 x_0 = x_1^3 + a x_1 x_0^2 + b x_0^3$$

for some $a, b \in k$ with $4a^3 + 27b^2 \neq 0$.

Divisors on X

Let X be a curve of genus 1. We will study the divisors on X. To make the discussion a bit more concrete, let $X \subset \mathbb{P}^2$ be the curve given by $y^2 z = x^3 - xz^2$. We claim that there is an exact sequence

$$0 \to X(k) \to \mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

This means that the class group $\mathrm{Cl}(X)$ is very big – its elements are in bijection with the k-points of X, of which there might be uncountably many. (In particular, this is another reason why X cannot be isomorphic to $\mathbb{P}^1$.)

If $L \subset \mathbb{P}^2$ is a line, we get a divisor $L|_X$: That is, we take a section $s \in \mathcal{O}_{\mathbb{P}^2}(1)$ defining L and restrict it to X. The divisor of $s \in \mathcal{O}_X(1)$ consists of three points P, Q, R (counted with multiplicity). In particular, since any two lines are linearly equivalent on $\mathbb{P}^2$, we get for every pair of lines L, L' and corresponding triples P, Q, R, a relation

$$P + Q + R \sim P' + Q' + R'$$

(where $\sim$ denotes linear equivalence).

Let us consider the point $O = [0,1,0]$ on X. This is a special point on X: it is an *inflection point*, in the sense that there is a line $L = V(z) \subset \mathbb{P}^2$ which has multiplicity three at O, so that L restricts to $3O$ on X. This has the property that any three collinear points P, Q, R in X satisfy

$$P + Q + R \sim 3O$$

We will use these observations to define a group structure on the set of closed points $X(k)$, using the point O as the identity. The group structure will be induced from that in $\mathrm{Cl}(X)$.

Consider the subgroup $\mathrm{Cl}^0(X) \subset \mathrm{Cl}(X)$ consisting of degree 0. This fits into the exact sequence

$$0 \to \mathrm{Cl}^0(X) \to \mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

We now define a map

$$\begin{array}{rcl} \xi : X(k) & \to & \mathrm{Cl}^0(X) \\ P & \mapsto & [P - O] \end{array}$$

LEMMA 25.15 *ξ is a bijection.*

PROOF: ξ is injective: $\xi(P) = \xi(Q)$ implies that $P \sim Q$. Then $P = Q$ (otherwise X would be rational, by Proposition 25.8). (Alternatively, it follows because $h^0(X, \mathcal{O}_X(P)) = 1$).

ξ is surjective: Take a divisor $D = \sum n_i P_i \in \mathrm{Div}(X)$ of degree 0. Then $D' = D + O$ has degree 1, so by Riemann–Roch, $H^0(X, \mathcal{O}_C(D'))$ is 1-dimensional. Hence there exists an effective divisor of degree 1 in $|D'|$, which must then be of the form $D' = Q$. But that means that $D + O \sim Q$, or, $D \sim Q - O$, as desired. ❑

Using this bijection, we can put a group structure on the set $X(k)$:

THEOREM 25.16 *The set of k-points $X(k)$ on a genus 1 form a group.*

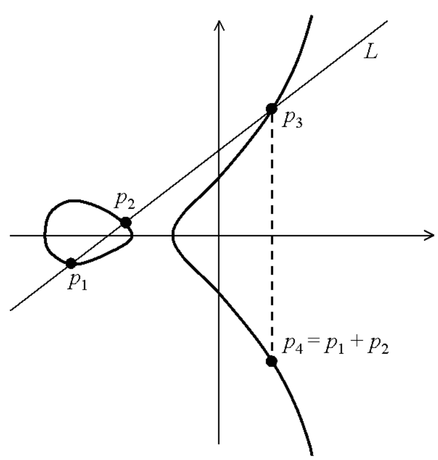
The group law has the following famous geometric interpretation. Given two points $p_1, p_2 \in X$, we draw the line L they span (see the figure below). This intersects X in one more point, say p_3. In the group $\mathrm{Cl}^0(X)$ we have

$$p_1 + p_2 + p_3 = 3O$$

To define the 'sum' $p_1 + p_2$ (which should be a new k-point of X), we then look for a point p_4 so that

$$p_4 - O = (p_1 - O) + (p_2 - O)$$

or in other words, $p_4 + O = p_1 + p_2$. By the above, this becomes $p_4 + O = 3O - p_3$ or, $p_3 + p_4 + O = 3O$. This tells us that we should define p_4 as follows: We draw the line L' from O to p_3 (shown as the dotted line in the figure), and define p_4 to be the third intersection point of L' with X. By construction, we get $(p_1 - O) + (p_2 - O) = (p_4 - O)$ in $\mathrm{Cl}^0(X)$.



Given the equation of X in $\mathbb{P}^2$, and coordinates for the points p_1, p_2, we can of course write down explicit formulas for the coordinates of p_4, and they are rational functions in the coordinates of p_1, p_2. This is almost enough to justify that X is a *group variety*, i.e., it is an algebraic variety equipped with a multiplication map $m : X \times X \to X$ satisfying the usual group axioms, and m is a morphism of algebraic varieties.

25.5 Curves of genus 2

Let X be a non-singular projective curve of genus 2. We saw one example of such a curve earlier in this chapter, namely the curve obtained by gluing two copies of the affine curve $y^2 = p(x)$ where $p(x)$ is a polynomial of degree five. The condition that X is non-singular implies that p has distinct roots.

We already saw in Chapter XX that a genus 2 curve cannot be embedded in the projective plane $\mathbb{P}^2_k$ (since 2 is not a triagonal number). However, we show the following:

Theorem 25.17 *Any curve of genus 2 is isomorphic to a hyperelliptic curve*

Here, a curve C is said to be *hyperelliptic* if there is a degree 2 map $X \to \mathbb{P}^1$. Equivalently, there is a base point free linear system of degree 2 and dimension 1. Equivalently again, there exists points $P, Q \in X$ so that the invertible sheaf $L = \mathcal{O}_X(P+Q)$ is globally generated and by two global sections.

It is classical notation that a base point free linear system of degree d and dimension r is called a g^r_d. So to say that a curve is hyperelliptic is to say that it has a g^1_2.

Example 25.18 If $g = 0$, then $X \simeq \mathbb{P}^1$. Let $D = 2P$, then $H^0(D) = kx_0^2 + kx_0x_1 + x_1^2$, so $|D| \simeq \mathbb{P}^2$ is identified with the space of quadratic polynomials up to scaling. If we take two quadratic polynomials q_0, q_1 with no common zeroes, we get a base point free linear system $g^1_2 \subset |D|$. ☆

Example 25.19 If $g = 1$ any divisor of degree 2 gives a g^1_2 by Riemann-Roch. Indeed, if D has degree 2 then

$$h^0(D) - h^0(K - D) = 2 + 1 - g = 2$$

and $\deg(K - D) = -2$ so $h^0(D) = 2$ and hence $\dim |D| = 1$. This D is base point free, since $D - p$ has degree 1, and hence since X is not rational, $h^0(D - p) = 1 = h^0(D) - 1$. ☆

Example 25.20 Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth divisor of bidegree $(2, g+1)$. Then $K_X \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, g-1)$ and X has genus g. Moreover, the projection $p_2 : X \to \mathbb{P}^1$ is finite of degree 2, which shows that X is hyperelliptic.

The projections $p_1, p_2 : Q \to \mathbb{P}^1$ give rise to a degree 2 and a degree $g+1$ morphism of X to $\mathbb{P}^1$. Thus there exists a 2:1 morphism $f : X \to \mathbb{P}^1$. f corresponds to a base point free linear system on X of degree 2 and dimension 1. Thus X is hyperelliptic.

In this example, $\Omega_X = \mathcal{O}_Q(X) \otimes \omega_Q|_X = \mathcal{O}_Q(2, g+1) \otimes \mathcal{O}_Q(-2, -2) = \mathcal{O}_X(0, g-1)$. The latter invertible sheaf has g independent global sections so X has genus g. Moreover

K_X is base point free, but not very ample, since the corresponding morphism $X \to \mathbb{P}^{g-1}$ is not an embedding (it maps X onto a conic). ☆

To prove the theorem, we must produce a degree two map $\phi : X \to \mathbb{P}^1$. We have a natural candidate: the canonical divisor K_X, which has degree $2g - 2 = 2$. We claim that K_X is base point free.

Note that we cannot apply Proposition 25.2 directly to prove this, since the degree is too small. However, we can use Riemann–Roch to check directly that the conditions in Theorem 25.1 apply. That is, we need to show that for every point $P \in X$, we have

$$h^0(X, K_X - P) = h^0(X, K_X) - 1 = 2 - 1 = 1$$

Applying Riemann–Roch to the divisor $D = P$, we also get $h^0(P) - h^0(K_X - P) = 1 + 1 - 2 = 0$. As P is effective, and X is not rational, we have $h^0(P) = 1$, and so also $h^0(X, K_X - P) = 1$, as we want.

25.6 Curves of genus 3

The case of curves of genus 3 is especially interesting. We have seen two examples of curves of genus 3 so far:

Example 25.21 A plane curve $X \subset \mathbb{P}^2$ of degree $d = 4$ has genus $\frac{1}{2}(d-1)(d-2) = 3$.

Notice that

$$\Omega_X = \mathcal{O}_{\mathbb{P}^2}(d-3)|_X = \mathcal{O}_X(1)$$

so Ω_X is very ample, since it is the restriction of the very ample invertible sheaf $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$. Hence K_X is very ample, and the corresponding morphism is exactly the given embedding $X \hookrightarrow \mathbb{P}^2$. ☆

Example 25.22 The curves in Section 23.6 on page 371 can be chosen to have genus $g = 3$. In this case, X admits a 2:1 map to $\mathbb{P}^1$, and thus X is hyperelliptic. ☆

Example 25.23 A curve X on the quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ of type (2,4) is hyperelliptic. It is a curve of degree 6 and genus 3. ☆

Thus these examples are a bit different. The curves in the first example have very ample canonical divisor K_X (they are 'canonical') whereas the two others to do not ('hyperelliptic'). We show that this distinction is a general phenomenon for curves of genus three:

Proposition 25.24 *Let X be a curve of genus 3. Then there are two possibilities:*

i) *K_X is very ample. Then X embeds as a plane curve of degree 4.*

ii) *K_X is not very ample. Then X is a hyperelliptic curve, and it embeds as a $(2,4)$ divisor in $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $K_X \sim 2F$, where $F = L_1|_X$.*

We willl deduce this from a more general result:

Theorem 25.25 *Let X be a curve of genus $\geqslant 2$. Then K is very ample if and only if X is not hyperelliptic.*

PROOF: K is very ample if and only if $h^0(K-P-Q) = h^0(K) - 2 = g - 2$ for every $P, Q \in X$. By Riemann–Roch, we compute

$$h^0(P+Q) - h^0(K-P-Q) = 2 + 1 - g = 3 - g$$

Hence K is very ample if and only if $h^0(P+Q) = 1$ for every P, Q.

If X is hyperelliptic, then there is a map $\phi : X \to \mathbb{P}^1$, so that $\phi^*([1:0]) = P + Q$ for some points $P, Q \in X$ (possibly equal). Here the linear system $|P+Q|$ is 1-dimensional, so $h^0(X, P+Q) = 2$, and hence K_X is not very ample.

If X is not hyperelliptic, we have $h^0(X, P+Q) = 1$ for any P, Q (otherwise it is $\geqslant 2$, and $P+Q$ induces a map $X \to \mathbb{P}^1$ of degree two), and hence K_X is very ample.

We still need to check the last part of the above theorem, namely that every hyperelliptic curve arises as a curve of type (2,4) on $Q \subset \mathbb{P}^3$.

We proceed as follows. Let $D = P_1 + \cdots + P_4$ denote a generic degree 4 divisor on X (so $P_1, \ldots, P_4$ are general points of X). By Riemann–Roch, we get

$$h^0(D) - h^0(K-D) = 4 + 1 - 3 = 2$$

We claim that $h^0(K-D) = 0$, so that $h^0(D) = 2$. Note that $K - D$ has degree $2g - 2 - 4 = 0$, so $K - D$ is a divisor of degree 0. This is effective if and only if $K \sim D$. However, there is a 4-dimensional family of divisors of the form $P_1 + \cdots + P_4$, wheras the space of effective canonical divisors has dimension $\dim |K| = 2$. Hence if the points P_i are chosen generically, $K - D$ will not be effective, and hence the claim holds.

From this, we obtain a linear system $|D|$ of dimension 1. We claim that D is base point free. We need to show that

$$h^0(D-P) = h^0(D) - 1 = \deg D + 1 - 3) - 1 = 1$$

for every point P. Suppose not, and let P be a base point of D. Since $D = P_1 + P_2 + P_3 + P_4$ we may suppose that $P = P_4$.

By Riemann–Roch, we are done if we can show $h^0(K - D + P) = 0$. However, $K - D + P = K - P_1 - P_2 - P_3$. There is a 3-dimensional space of effective divisors of the form $P_1 + P_2 + P_3$ for points $P_i \in X$, but only a 2-dimensional linear system of effective canonical divisors $|K|$. Hence $K - D + P$ is not effective.

We therefore have two morphisms from our hyperelliptic curve X; $f : X \to \mathbb{P}^1$ (induced by the g^1_2) and $g : X \to \mathbb{P}^1$ (induced by D). By the universal property of the fiber product, this gives a morphism

$$\phi = (f \times g) : X \to \mathbb{P}^1 \times_k \mathbb{P}^1$$

We claim that this is a closed immersion. Let $F = P + Q \in |g^1_2|$. The map $D + F$ induces the map $F : X \to \mathbb{P}^3$, which coincides with $j \circ \phi$ where $j : \mathbb{P}^1 \times \mathbb{P}^1$ is the Segre embedding. To prove the claim, it suffices to show that F is an embedding, or equivalently that $D + F$ is very ample.

First claim that $K \sim 2F$. Since both of these divisors have degree 4 it suffices to show that $K - 2F$ is effective. Note that in any case $h^0(X, 2F) \geqslant 3$, since if $H^0(X, F) = \langle x, y \rangle$, then x^2, xy, y^2 are linearly independent in $H^0(X, 2F)$ (understand why!). Now applying Riemann–Roch to $D = 2F$, we get

$$h^0(2F) - h^0(K - 2F) = 4 + 1 - 3 = 2$$

so $h^0(K - 2F) \geqslant 1$, and $K \sim 2F$ as we want.

Now, to show that $D + F$ is very ample, we need to show that

$$h^0(X, D + F - p - q) = h^0(D + F) - 2$$

for any pair of points $p, q \in X$. By Riemann–Roch again, we can conclude if we know that $h^0(K - D - F + p + q) = 0$. But since $K \sim 2F$, we have

$$K - D - F + p + q \sim F - D + p + q$$

These are divisors of degree 0, so if this is effective, we must have $D \sim F + p + q$. However, the space of effective divisors of the form $D' + p + q$ with $D' \sim F$ is 3-dimensional (since $|F|$ has dimension 1, and p and q can be chosen freely on X). On the other hand, as we have seen, the space of divisors of the form $D = P_1 + \cdots + P_4$ is of dimension 4, so choosing D generically means that this$F - D + p + q$ is not effective. It follows that $h^1(D - p - q) = h^0(D + F) - 2$ and hence D is very ample. ❑

25.7 Curves of Genus 4

Recall that curves of genus $g \geqslant 2$ split up into two disjoint classes.

i) Hyperelliptic curves: X admits a 2:1 to $\mathbb{P}^1$

ii) Canonical curves: K_X is very ample

Here's an example of a genus 4 curve in $\mathbb{P}^1 \times \mathbb{P}^1$:

EXAMPLE 25.26 Consider a type $(2,5)$ curve C on $Q \subset \mathbb{P}^3$. Then C has degree $7 = 2 + 5$ and C is hyperelliptic (because of the degree 2 map coming from projection onto the first fact $p_1 : Q \to \mathbb{P}^1$). A type $(3,3)$ curve on Q is also of genus 4. It is a degree 6 complete intersection of Q and a cubic surface. Curves of type $(3,3)$ have at least two g^1_3's. ☆

In fact, using the same strategy as for $g = 3$, one can show that any hyperelliptic curve of genus 4 arises this way.

Classifying curves of genus 4

We start with an abstract curve X of genus 4. We may assume that X is not hyperelliptic (since in that case it embeds as a $(2,5)$-divisor on $\mathbb{P}^1 \times \mathbb{P}^1$). So we assume that K_X is very ample. Therefore we have the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1} = \mathbb{P}^3$. The degree of the embedded curve is $\deg \omega_X = 2g - 2 = 6$. Thus we can view X as a degree 6 genus 4 curve in $\mathbb{P}^3$.

What are the equations of X in $\mathbb{P}^3$? To answer this question we use a very powerful technique in curve theory, namely we combine Riemann–Roch with the sheaf cohomology on $\mathbb{P}^n$. Twisting the ideal sheaf sequence of X by $\mathcal{O}_{\mathbb{P}^3}(2)$ and taking cohomology gives the exact sequence

$$0 \to H^0(\mathbb{P}^3, I_X(2)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(X, \mathcal{O}_X(2)) \to \cdots$$

Note that $\mathcal{O}_{\mathbb{P}^3}(1)|_X = K_X$. Applying Riemann-Roch states to the divisor $D = 2K_X$, we get

$$h^0(\mathcal{O}_X(2)) = \deg 2K_X + 1 - g + h^1(\mathcal{O}_X(D)) = 12 + 1 - 4 + 0 = 9.$$

(Note that $h^1(\mathcal{O}_X(2)) = h^0(K_X - 2K_X) = h^0(-K_X) = 0$ since K_X is effective). Since $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 10$ it follows that the map $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2))$ must have a nontrivial kernel so $h^0(\mathbb{P}^3, I_X(2)) > 0$.

The upshot of this is that we now know that X lies in some surface of degree 2. Since X is integral, this surface cannot be a union of hyperplanes. So X lies on either a singular quadric cone $Q_0 = V(xy - z^2)$ or the nonsingular quadric surface $Q = V(xy - zw)$.

If C lies on Q then it must have a type (a, b) which must satisfy $a + b = 6$ and $(a-1)(b-1) = 4$. The only solution is $a = b = 3$. Since $\mathcal{O}_Q(3,3) \simeq \mathcal{O}_{\mathbb{P}^3}(3)|_Q$, this implies that C is the restriction of a divisor on $\mathbb{P}^3$, that is, $C = Q \cap S$ for a degree 3 surface S.

The other possibility is that C lies on Q_0. Computing as before, we obtain

$$0 \to H^0(\mathcal{O}_X(3)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_X(3)) \to \cdots$$

As before one sees that $h^0(\mathcal{O}_X(3)) = 15$ and $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20$. Thus $h^0(\mathcal{O}_C(3)) \geqslant 5$. Let $q \in H^0(\mathcal{O}_C(2))$ be the defining equation of Q_0. Then $xq, yq, zq, wq \in H^0(\mathcal{O}_C(3))$. But $h^0(\mathcal{O}_C(3)) \geqslant 5$ so there exists an $f \in H^0(\mathcal{O}_C(3))$ so that the global sections xq, yq, zq, wq, f are independent. Thus there is an f not in (q). Since $f \notin (q)$ we see that $S = Z(f) \not\supset Q$ so $C' = S \cap Q$ is a degree 6 not necessarily nonsingular or irreducible curve. Since $C \subset S$ and $C \subset Q$ it follows that $C \subset C'$. Since these are both integral curves of the same degree $\deg C = 6 = \deg C'$, we must have $C = C'$. Thus in the case that C lies on Q_0 we see that C is also a complete intersection $C = Q_0 \cap S$ for some cubic surface S.

This proves the following theorem:

Theorem 25.27 *Let X be a non-singular curve of genus 4. Then either*

i) X is hyperelliptic (in which case X embeds as a $(2,5)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^1$); or

ii) $X = Q \cap S$ is the intersection of a quadric surface and a cubic surface in $\mathbb{P}^3$.

25.8 *Automorphisms of plane curves are linear*

Let C be a smooth curve of degree d and genus g embedded in projective space $\mathbb{P}^2$. The divisor $D = \mathcal{O}_C(1)$ is defined as the line bundle corresponding to the hyperplane class on C.

The exact sequence

$$0 \mapsto \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \mapsto 0$$

can be twisted by $\mathcal{O}_{\mathbb{P}^2}(1)$ to get

$$0 \mapsto \mathcal{O}_{\mathbb{P}^2}(-d+1) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_C(1) \mapsto 0.$$

Taking cohomology of this sequence gives $H^0(C, \mathcal{O}_C(1)) \equiv H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Therefore, the linear system cut out on C by the lines in $\mathbb{P}^2$ is complete, and $h^0(C, \mathcal{O}_C(1)) = 3$.

Let A be another effective divisor on C such that $deg(A) = d$ and $h^0(C, A) = 3$. Then the degree of $D - A$ is o and $h^0(C, D - A) \neq 0$. Therefore, $D - A$ is linearly equivalent to $\mathcal{O}_C$, and A is linearly equivalent to D. This means that $D = \mathcal{O}_C(1)$ is the unique effective divisor of degree d and with $h^0(C, D) = 3$.

Finally, let $\phi : C \to C$ be an automorphism. Then $\phi^* \mathcal{O}C(1) \equiv \mathcal{O}_{\mathcal{C}}(1)$. Therefore, ϕ acts on the sections of $\mathcal{O}\mathbb{P}^2(1)$. Since $\mathbb{P}^2 = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}\mathbb{P}^2(1))) = \mathbb{P}(H^0(C, \mathcal{O}C(1)))$, it follows that ϕ is the restriction to C of a linear automorphism of $\mathbb{P}^2$.

Chapter 26

Further constructions and examples

26.1 Some explicit blow-ups

Let X be a noetherian integral scheme and let $\mathcal{I}$ be a quasi-coherent ideal sheaf on X corresponding to a closed subscheme $Y \subset X$. We will associate to this data a new scheme $\widetilde{X}$, called the *blow-up of X along Y* and a morphism $\pi : \widetilde{X} \to X$, called the *blow-up morphism*. This will have the main property that the scheme theoretic preimage $E = \pi^{-1}(Y)$ is a Cartier divisor, and that π is an isomorphism outside E. Moreover, $\widetilde{X}$ will be universal with respect to these properties in the following sense:

Theorem 26.1 *Let X and $\mathcal{I}$ be as above. There is a scheme $\widetilde{X}$ admitting a morphism $\pi : \widetilde{X} \to X$ so that*

i) The inverse image $\pi^{-1}(Y)$ of Y is an effective Cartier divisor on $\widetilde{X}$.

ii) $\pi : \widetilde{X} \to X$ is an isomorphism outside the support of E.

Moreover, for any morphism $g : Z \to X$ from an integral scheme with the property that $g^{-1}(Y)$ is an effective Cartier divisor, there is a unique $\tilde{g}$ making the following diagram commute

$$\begin{tikzcd} Z \arrow[r, "\tilde{g}"] \arrow[dr, "g"'] & \widetilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

The scheme $\widetilde{X}$ is constructed via a gluing operation that we now explain. Given X and the ideal sheaf $\mathcal{I}$, we form the *Rees algebra* of $\mathcal{I}$ given by

$$R(\mathcal{I}) = \bigoplus_{d \geqslant 0} \mathcal{I}^d t^d = \mathcal{O}_X \oplus \mathcal{I}t \oplus \mathcal{I}^2 t^2 \oplus \cdots .$$

This is again a quasi-coherent $\mathcal{O}_X$-module, and as such it is a *graded $\mathcal{O}_X$-algebra*. For an affine open set $U = \operatorname{Spec} A \subset X$, $I = \mathcal{I}(U)$ is an ideal of A, and $R(\mathcal{I})(U) = \bigoplus_{d \geqslant 0} I^d$ is a graded A-algebra. It follows that we obtain a projective scheme $\operatorname{Proj}(R(I))$ which is a scheme over $\operatorname{Spec} A$. Moreover, it is not so hard to check that the natural morphisms $\operatorname{Proj} R(I) \to \operatorname{Spec} A$ glue to a morphism

$$\pi : \widetilde{X} \to X.$$

Let us now consider the scheme theoretic image E of Y. Over the open set $U = \operatorname{Spec} A$, this is defined by the fibre product $\operatorname{Proj}(R(I)) \times_U (Y \cap U)$, or in other words Proj of the *associated graded ring*

$$R(I) \otimes_A A/I = \bigoplus_{d \geqslant 0} I^d/I^{d+1}t^d = A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

Let us compute these rings in a few examples.

EXAMPLE 26.2 Consider $\mathbb{A}^2_{\mathbb{Z}} = \operatorname{Spec} A$ where $A = \mathbb{Z}[x, y]$. The ideal $\mathfrak{m} = (x, y)$ corresponds to the origin in $\mathbb{A}^2_k$. The Rees algebra is given by

$$R(\mathfrak{m}) = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2 t^2 \oplus \cdots$$

Consider the homomorphism of graded A-modules

$$\begin{array}{rcl} \psi : A[U, V] & \to & R(\mathfrak{m}) \\ U & \mapsto & tx \\ V & \mapsto & ty \end{array}$$

This is clearly surjective, since $R(\mathfrak{m})$ is generated in degree 1. The kernel of ψ is generated by $xV - yU$, giving a graded isomorphism

$$R(I) \simeq A[U, V]/(xV - yU)$$

We consider the blow-up $X = \operatorname{Proj}(R(I))$ which is a projective scheme over A. Moreover, the graded surjection $A[U, V] \to R(I)$ induces a closed immersion

$$X \subset \mathbb{P}^1_A 4 = \mathbb{P}^1 \times \mathbb{A}^2$$

Here the section projection gives the blow-up morphism $\pi : X \to \mathbb{A}^2$. Note that the first projection $p : X \to \mathbb{P}^1$ has fibers isomorphic to a line in $\mathbb{A}^2$. In fact, p is exactly the tautological bundle $\mathcal{O}(-1)$. ☆

EXERCISE 26.1 Consider $\mathbb{A}^2 = \operatorname{Spec} A$ as above. Show that the blow-up of the ideal $(x, y)^d$ is isomorphic to the blow-up X in Example 26.2 (i.e., with $d = 1$). ★

EXAMPLE 26.3 We now projectivize the previous example. Consider $\mathbb{P}^2 = \operatorname{Proj} A$ where $A = \mathbb{Z}[x_0, x_1, x_2]$. The ideal $\mathfrak{m} = (x_0, x_1)$ corresponds to the point $(0 : 0 : 1) \in \mathbb{P}^2$. As above, there is a surjection of graded A-modules

$$\begin{array}{rcl} \psi : A[U, V] & \to & R(\mathfrak{m}) \\ U & \mapsto & tx_0 \\ V & \mapsto & tx_1 \end{array}$$

giving us an isomorphism of graded rings

$$R(\mathfrak{m}) \simeq A[U, V]/(x_0 V - x_1 U)$$

This in turn gives a closed embedding of the blow-up X

$$X \subset \mathbb{P}^1_{\mathbb{P}^2} = \mathbb{P}^1 \times \mathbb{P}^2$$

Here the section projection gives the blow-up up morphism $\pi : X \to \mathbb{P}^2$. The first projection $p : X \to \mathbb{P}^1$ has fibers isomorphic to $\mathbb{P}^1$. ☆

EXAMPLE 26.4 If we in Example 26.2 consider instead the ideal $I = (x^2, y^2)$, we get a singularity on the blow-up X. Note that I corresponds to a non-reduced subscheme of $\mathbb{A}^2$ supported at the origin. Since I is generated by two elements, we can still carry out the same trick and obtain an isomorphism of graded A-modules

$$R(\mathfrak{m}) \simeq A[U,V]/(x^2V - y^2U)$$

Note that in the chart $D_+(U)$, this is isomorphic to the affine scheme

$$\operatorname{Spec} \mathbb{Z}[x,y,v]/(x^2v - y^2)$$

This scheme is not regular: It is singular along the v-axis $V(x,y)$ (in other words, the preimage of the origin). The hypersurface $y^2 = x^2v$ is known as *Whitney's umbrella*. ☆

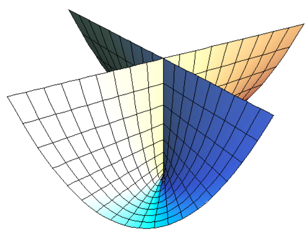


Figure 26.1: The Whitney Umbrella

EXERCISE 26.2 Show that also the blow-up of $\mathbb{A}^2$ along the ideal (x^2, y) is singular and describe the singular locus. ✯

EXAMPLE 26.5 (*Blow-ups of Complete intersections.*) Let us generalize the previous examples, and consider $\mathbb{A}^n = \operatorname{Spec} A$, $A = \mathbb{Z}[x_1, \ldots, x_n]$ and let $f_1, \ldots, f_r \in R$ be a regular sequence for elements (i.e., such that the image of each f_i is a non-zero divisor in $A/(f_1, \ldots, f_{i-1})$. Then the Rees Algebra of $I = (f_1, \ldots, f_r)$ is given by

$$R(I) = A[w_1, \ldots, w_r]/J$$

where J is the ideal generated by the 2×2-minors of the matrix

$$\begin{pmatrix} w_1 & w_2 & \ldots & w_r \\ f_1 & f_2 & \ldots & f_r \end{pmatrix}$$

In particular, the blow-up $\operatorname{Proj} R(I)$ embeds into $\mathbb{A}^n \times \mathbb{P}^{r-1}$. ☆

EXAMPLE 26.6 (*A line in $\mathbb{P}^3$.*) Consider $\mathbb{P}^3_k = \operatorname{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$ and let ℓ denote the line $V(x_0, x_1) \subset \mathbb{P}^3$. The blow-up of $\mathbb{P}^3_k$ along ℓ is the closed subscheme of $\mathbb{P}^3 \times \mathbb{P}^1$ defined by the bigraded polynomial $x_0V - x_1U = 0$. Note that the second projection $q : X \to \mathbb{P}^1$ has fibers of dimension 2: They correspond to planes $H \subset \mathbb{P}^3_k$ containing ℓ. ☆

EXAMPLE 26.7 (*The quadric cone again.*) Consider the quadratic cone $Q = \operatorname{Spec}(S)$ where $S = k[x,y,z]/(xz - y^2))$. We saw in Section 20.5 that the line $\ell = V(x,y)$ defined a Weil divisor which was not Cartier. What happens if we blow up Q along this line? Let $\pi : X \to Q$ denote this blow-up. By the universal property of blowing up, the inverse image $\pi^{-1}(\ell)$ must be a Cartier divisor on X – so π transforms a non-Cartier divisor into a Cartier divisor. Let us verify this claim directly.

Consider the graded ideal $I \subset S$. The Rees algebra of ℓ is given by

$$R = S[w_0, w_1]/(yw_0 - zw_1, xw_0 - yw_1)$$

Let us check that X is in fact regular. Note that X is covered by the two affine open subsets $D_+(w_0)$ and $D_+(x_1)$).

On the open set $D_+(w_0)$, we have

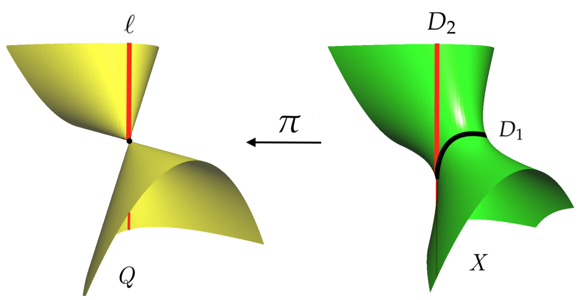
$$R_{(w_0)} = S\left[\frac{w_1}{w_0}\right]/(y - \frac{w_1}{w_0}z, x - y\frac{w_1}{w_0}) \simeq k[z]\left[\frac{w_1}{w_0}\right]$$

So $\operatorname{Spec} R_{(w_0)} \simeq \mathbb{A}^2_k$. Thus X is regular in this chart. The same happens in the other chart $D_+(w_1)$. Thus $D = \pi^{-1}(\ell)$, being a subscheme of codimension 1 in a non-singular variety, should be a Cartier divisor.

Note that $D = \pi^*(\ell)$ corresponds to the ideal generated by x, y in R. In the ring R, this ideal decomposes as

$$(x, y) = (x, y, z) \cap (x, y, w_1)$$

So that D has two components: $D_1 = V(x, y, z)$ and $D_2 = V(x, y, w_1)$. Note that $\pi(D_1)$



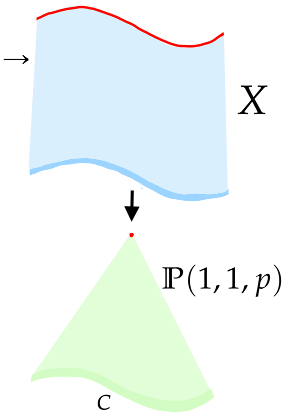
is the point $(0,0,0) \in Q$, whereas $\pi_1(D_2) = \ell$. To show that D is Cartier, it suffices to show that the ideals of D_1 and D_2 are both locally generated by one element. Using the description of $D_+(w_0)$ above, we see that D_1 is described by $z = 0$, and D_2 by $\frac{w_1}{w_0} = 0$, so they are both principal. On $D_+(w_1) = \operatorname{Spec} k[x][\frac{w_0}{w_1}]$, $D_1 = V(x)$ and $D_2 = 0$ (since $w_1 = 0$ defines the empty scheme on $D_+(w_0)$). Hence D is a Cartier divisor.

In this example, we could also blow up the ideal $\mathfrak{m} = (x, y, z)$ corresponding to the origin $o = (0,0,0)$. The resulting blow-up X' would again be a regular scheme, with the property that $\pi^{-1}(o)$ is Cartier. In fact, using the universal property of blowing up, one can show that X and X' are isomorphic (however the two Rees algebras $R(I)$ and $R(\mathfrak{m})$ are very different). ☆

EXAMPLE 26.8 (*The weighted projective space $\mathbb{P}(1,1,p)$.)* Let $S = k[x_0, x_1, x_2]$ where the variables x_0, x_1, x_2 have degrees $1, 1, p$ respectively. In Chapter 10, we defined the weighted projective plane $\mathbb{P} = \mathbb{P}(1,1,p)$ as $\operatorname{Proj} S$. To study $\mathbb{P}$ more explicitly, it will be conventient to study the Veronese subring $S^{(p)}$, which is generated in degree 1 by the elements

$$x_0^p, x_0^{p-1}x_1, \ldots, x_1^p, x_2 \tag{26.1}$$

Using these as a basis for $\Gamma(\mathbb{P}, \mathcal{O}(p))$, we get a rational map $\phi : \mathbb{P} \dashrightarrow \mathbb{P}^{p+1}$. These sections are clearly base point free, so the map ϕ is a morphism. Furthermore, working locally, one can check that ϕ is in fact an embedding.



The image of ϕ is defined by the kernel of the corresponding surjection $k[u_0, \ldots, u_p, u_{p+1}] \to S$, or in other words, the relations between the monomials (26.1). Note that the first $p+1$ monomials define the rational normal curve C in $\mathbb{P}^d$. Realizing this, we see that ϕ embeds $\mathbb{P}$ as the *projective cone* over the curve C. In particular, $\mathbb{P}$ is singular at the vertex of the cone, corresponding to the the closed point $P = V(x_0, x_1)$.

We will now consider the blow-up X of $\mathbb{P}$ at the point P. We claim that X is in fact regular. ☆

EXAMPLE 26.9 (*The Twisted Cubic.*) Let k be a field and consider the twisted cubic curve $C \subset \mathbb{P}^3_k$. Recall that C is defined by three quadrics

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

From the resolution of the ideal in Section xxx, it is not so hard to show that the Rees algebra is isomorphic to

$$R(I) = \frac{A[w_0, w_1, w_2]}{(x_1 w_0 - x_2 w_1 + x_3 w_2, x_0 w_0 - x_1 w_1 + x_2 w_2)}$$

This presentation shows that the blow-up X of $\mathbb{P}^3_k$ along C embeds into $\mathbb{P}^3_k \times \mathbb{P}^2_k$. Here the first projection is the blow-up morphism $\pi : X \to \mathbb{P}^3$.

The blow-up has the following geometric description. Consider $a_0, \ldots, a_3 \in k$ and the corresponding closed point $a = (a_0 : a_1 : a_2 : a_3)$ in $\mathbb{P}^3$. If $a \notin C$, the matrix $M = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{pmatrix}$ has rank 2 (since some 2×2-minor does not vanish at a). This means that the equation $M \cdot w = 0$ has only one non-trivial solution up to scaling. The fiber of $\pi : X \to \mathbb{P}^3$ is exactly the corresponding point $(w_0 : w_1 : w_2)$ in $\mathbb{P}^2$. On the other hand, if $a \in C$, then the two rows of M are proportional and there is a 2-dimensional null space $V \subset k^3$ of solutions. The fiber of π over a is exactly the projectivization of V, i.e., a line in $\mathbb{P}^2$. In fact, using the equations above, it is possible to see that the restriction of π to $E \subset X$ gives a '$\mathbb{P}^1$-bundle' over C (i.e., π is Zariski-locally isomorphic to $C \times \mathbb{P}^1 \to C$).

The other projection $q : X \to \mathbb{P}^2$ is also interesting. Since the relations in the Rees algebra are linear in the x_i, it means that also q is a '$\mathbb{P}^1$-bundle' over $\mathbb{P}^2$, i.e., every closed fiber is isomorphic to $\mathbb{P}^1$, and it is locally trivial. If we start with $\mathcal{O}(1)$ on $\mathbb{P}^3$, we get an invertible sheaf $L = q^*\mathcal{O}(1)$ on X, and the pushforward $q_* L$ is a locally free sheaf on $\mathbb{P}^2$, called the *Bordiga bundle*. ☆

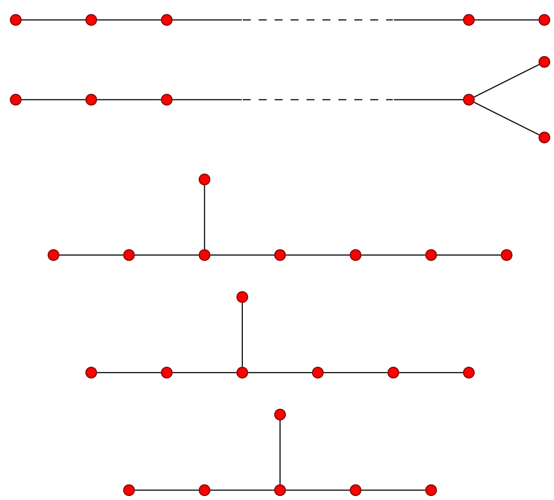
26.2 *Resolution of some surface singularities*

The quadratic cone we studied on page 6.9, is the simplest entry in a series of singular surfaces in $\mathbb{A}^3$, which also are prototypical examples of a class of surface singularities called A_n-singularities. These again form one series among the so-called Du-Val singularities (also called the rational double points). There is another infinite series D_n and three exceptional ones, E_6, E_7 and E_8, but we will only study the A_n's, which are the simplest.

Generally speaking a resolution of the singularities of a variety X is a smooth variety $\widetilde{X}$ and a birational morphism $\pi\colon \widetilde{X} \to X$. One may specify different additional desired properties of π, the strongest being that π is the composition of blow ups with smooth centers contained in singular part of X. It is natural to include at least one example of this important technique in a section about blow-ups.

Du Val singularities

The du Val singularities are isolated surface singularities characterized by the the configurations of the components of the exceptional divisor. These configurations are best described in terms of th dual graphs, where each node represents an irreducible component of E and two nodes are connected by an edge precisely when the correponding components intersect. All components are isomorphic to $\mathbb{P}^1$, and when meeting they meet transversally. Finally, their self-intersection is -2; meaning that the restriction $\mathcal{O}_X(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$.



The A_n singularity

For each natural number n, we consider the surface $Y_n = V(xz - y^{n+1}) \subseteq \mathbb{A}^3$. By the Jacobian criterion one easily checks it is singular at the origin $P = (0,0,0)$, but smooth elsewhere. The surface Y_n is a normal as is any surface in $\mathbb{A}^3$ whose singularities are isolated, but it is not factorial. The elements x, y and z are irreducible members of the coordinate ring $A_n = k[x,y,z]/(xz - y^{n+1})$ so by the very definition of A_n unique factorization does not hold.

Unlike what is true for the the quadratic cone, when $n \geqslant 2$ lines through the origin, apart from the two lines $l_x = V(x,y)$ and $l_z = V(z,y)$, are not contained in Y_n; they meet Y_n in at most $n-1$ other points. The two lines are Weil divisors on Y_n that are not Cartier; however their union is given by the single equation $y = 0$, and is Caartier. We'll need to refer to the two lines l_y and l_z later, so we'll call them the *special lines*.

For notational reasons, we extend the class A_n and include the two surfaces Y_0 and Y_{-1}, which both are non-singular surfaces, the equations being $xz - 1$ and $xz - y$.

We aim at describing such a resolution of the singularities of the surface Y_n, and will prove the following proposition.

PROPOSITION 26.10 *There is a nonsingular surface W_n and a birational morphism $\pi\colon W_n \to Y_n$ which is the succession of blow-ups of single points.*

i) The number of blow-ups is $n/2$ blow-ups when n is even and $(n+1)/2$ if n is odd.

ii) The exceptional divisor E is

$$E = F_1 + \cdots + F_n$$

with each $F_i \simeq \mathbb{P}^1$ and two components F_i and F_j are disjoint unless $|i-j| = 1$ and in that case F_i and F_j meet transversally in one point. Further, it holds that $\mathcal{O}(F_i)|_{F_i} = \mathcal{O}_{\mathbb{P}^1}(-2)$.

The exceptional divisor is the scheme theoretical inverse image of the singular point in Y_n, and π induces an isomorphism between $W_n \backslash E$ and $Y_n \backslash P$.

Blow-up of a point in $\mathbb{P}^3$

To begin with we describe the blow-up $\pi\colon \widetilde{Y}_n \to Y_n$ of origin P_n in detail. A practical way of doing that is to use that $\widetilde{Y}_n$ equals the proper transform of Y_n in the blow-up $\widetilde{\mathbb{A}^3}$ of the origin. This blow-up of $\mathbb{A}^3$ is given as the closed subscheme

$$\widetilde{\mathbb{A}}^3 \subseteq \mathbb{A}^3 \times \mathbb{P}^2,$$

where $\mathbb{P}^2 = \operatorname{Proj} k[u,v,w]$, and which is given by the vanishing of the 2×2-minors of the matrix

$$\begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}.$$

In other words the equations of $\widetilde{\mathbb{A}}^3$ in $\mathbb{A}^3 \times \mathbb{P}^2$ are the equations

$$uy - vx = uz - wx = vz - wy = 0. \tag{26.2}$$

We let $p\colon \widetilde{\mathbb{A}}^3 \to \mathbb{A}^3$ denote the blow-up map, which is just the restriction of the first projection, and further we let $q\colon \widetilde{\mathbb{A}^3} \to \mathbb{P}^2$ denote the restriction of the second projection.

The blow-up $\widetilde{\mathbb{A}}^3$ is covered by the three distinguished open affine subsets $D_+(u)$, $D_+(v)$ and $D_+(w)$. Each is isomorphic to $\mathbb{A}^3$, the isomorphism is obtained from (26.2) after inverting the actual variable; for instance, if $v \neq 0$, solving (26.2) for x and z yields

$x = uv^{-1}y$ and $z = wv^{-1}y$. In order to simplify the notation (and in accordance with standard dehomogenization principles), we set the inverted variable, in this case v, equal to one. Then $x = uy$ and $z = wy$ and $D_+(v) = \operatorname{Spec} k[u, y, w]$. It may be described as the subset of $\mathbb{A}^3 \times \mathbb{P}^2$ consisting of the points

$$(uy, y, wy) \times (u : 1 : w)$$

with y, v and w varying freely.

The exceptional fibre $E \cap D_+(v)$ is given as $(x, y, z)k[u, y, w]$ which in view of the relations $x = uy$ and $z = wy$ becomes

$$(x, y, z)k[u, y, w] = (uy, y, wz) = (y),$$

and the restriction of q to $E \cap D_+(v)$ yields an isomorphism with the standard distinguished set $D_+(v) \subseteq \mathbb{P}^2$.

The two other open affines $D_+(u)$ and $D_+(w)$ have similar properties. It holds that $D_+(u) = \operatorname{Spec} k[x, v, w]$ with $y = vx$ and $z = wx$ whilst $D_+(w) = \operatorname{Spec} k[u, v, z]$ with $x = uz$ and $y = vz$. The exceptional divisor satisfies $E \cap D_+(u) = V(x)$ and $E \cap D_+(w) = V(z)$, and they are mapped isomorphically by q onto the standard distinguished sets $D_+(u)$ and $D_+(w)$ in $\mathbb{P}^2$.

The blowing up the singular point of $\widetilde{Y}_n$

Let us trace what happens to the blow-up of Y_n in each of the open sets in the previous paragtraph, and we begin with most interesting one, namely $D_+(v)$. Here the affine coordinates are y, u and w and $x = uy$ and $z = wy$. With these substitutions the equation $xz - y^{n+1}$ of $p^{-1}(Y_n) \cap D_+(v)$ takes the form

$$xz - y^{n+1} = yu \cdot yw - y^{n+1} = y^2(u \cdot w - y^{n-1}).$$

Discarding y^2 we find that the equation becomes $uw - y^{n-1}$, which describes that part of the proper transform $\widetilde{Y}_n$ lying in $D_+(v)$. We note that $\widetilde{Y}_n \cap D_+(v)$ is just the surface Y_{n-2} (and this is a crucial observation for the later iteration).

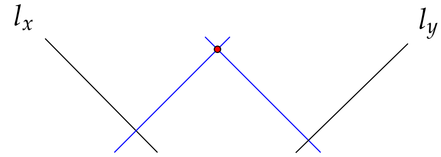
It is straightforward to determine the equation for Y_n in the two remaining distinguished opens and verify that Y_n has no singularities in either: In $D_+(u)$, which is an $\mathbb{A}^3$ with coordinates x, v and w and transition formulas $y = vx$ and $z = wx$, the equation of $\widetilde{Y}_n \cap D_+(u)$ takes the form

$$xz - y^{n+1} = wx^2 - v^{n+1}x^{n+1} = x^2(w - v^{n+1}x^{n-1}).$$

Discarding x^2 leaves us with the equation $w - v^{n+1}x^{n+1} = 0$, which describes a smooth surface. Now, x and z appears in a symmetric way, so by symmetry $\widetilde{Y_n} \cap D_+(w)$ is smooth as well. We have proven:

Lemma 26.11 *When $n \geqslant 3$, the blow-up of Y_n in the singular point has just one singular point, which has an affine open neighbourhood isomorphic to Y_{n-2}. The blow-up of Y_1 and Y_2 are smooth.*

The exceptional divisor is an important part of the description, so we note that $E \cap D_+(v)$ is the union of the special lines l_x and l_y in Y_{n-2}, and closing them up, we see that E is the union of two $\mathbb{P}^1$'s meeting in the point P_{n-2}. The two original special lines l_x and l_y in Y_n are split apart, and their inverse images each meet one of the $\mathbb{P}^1$'s in one point.



The final iteration

To arrive at the nonsingular surface W_n, we shall iterate the blow-up procedure and for each natural number $r \leqslant (n+1)/2$ recursively construct a tower

$$Z_{n-2r} \longrightarrow Z_{n-2r+2} \longrightarrow \ldots\ldots \longrightarrow Z_{n-2} = \widetilde{Y}_n \longrightarrow Z_n = Y_n$$

of birational maps each being the blow-up of a singular point of type A_{n-2i}.

It begins with the blow-up $Z_n = \widetilde{Y_n}$ of Y_n in the singular point. As we saw, it is covered by three open affines and unless $n=1$ or $n=2$, has one singular point P_{n-2} lying on an affine piece isomorphic to Y_{n-2} (In case $n=1$ or $n=2$ the surface Y_{n-2} is non-singular). The same holds for the blow-up Z_{n-4} of $\widetilde{Y}_{n-2}$ in the singular point, it is cover by five affines of which four are smooth and the fifth is isomorphic to Y_{n-4}.

Each Z_{n-2r} is covered by $2r+1$ affine opens, and has just one singular point lying in an affine piece isomorphic to Y_{n-2r}, unless of course $n-2r=0$ or $n-2r=-1$, and those cases apart we apply Lemma 26.11 and let Z_{n-2r-2} be the blow-up of Z_{n-2r} in the singular point.

Finally, when $r = n/2$ or $r = (n+1)/2$ according to n being even or odd, the top surface Z_{n-2r} will be smooth; indeed, the critical open affine, *i.e.* the one which is not *a priori* smooth, will either be isomorphic to Y_0 or to Y_{-1}, and either one is smooth. Letting $W_n = Z_{n-2[(n+1)/2]}$ and π the composition of the maps in the tower (26.2), we arrive at the first part of Proposition 26.10.

The exceptional divuisor

Pulling back $L_0 = V(y)$ in Y_n to $\widetilde{Y}_n$ results in the divisor L_1 whose ideal is y. The affine coordinates in $D_+(v) = \mathbb{A}^3$ are u, y, w and y 'persist being y', that is $L_1 \cap D_+(y)$ are exactly the two special lines through the singular point. So iterating this, at stage r, we we find the expression for the exceptional divisor:

$$E_n = F_1 + \cdots + (F_{2r+1} + F_{2r}) + \cdots + F_2$$

and

$$E_n + l_x + l_y = 0$$

When reaching the final stage, this gives *ii)*. Using 26.2 we find

$$\mathcal{O} = \mathcal{O}(E_n + l_x + l_y)|_{F_i} = \begin{cases} \mathcal{O}(F_i + F_i + F_{i-1})|_{F_i} & 1 < i < n \\ \mathcal{O}(l_x + F_n + F_{n-1})|_{F_n} \\ \mathcal{O}(F_2 + F_1 + l_z)|_{F_1} \end{cases}$$

and the *ii)* follows in view of F_i intersecting F_{i+1} transversally in one point and likewise l_x (resp l_y) meets F_n (resp l_z) transversally in one point.

EXAMPLE 26.12 We underline that the surface $xz + y^{n-1}$ is a *prototypical* A_n-singularity. In general an A_n-singularity is one with a resolution as in the proposition, and in general they are not even locally isomorphic. For instance a surface Y in $\mathbb{A}^3$ with equation $xzf + y^{n+1}$, where f is function that does not vanish at the origin, has an A_n singularity at the origin and will be resolved in the same manner as the one given by $xz + y^{n+1}$, but it requiers a little additional work. Of course, the resolution is local and only affects the singularity at the origin, any other singularities Y might have remains unaltered. ☆

EXERCISE 26.3 Resolve the singulary in ther example by mimicking what we did for $xy + z^{n+1}$. ★

26.3 *Unexpected behaviour*

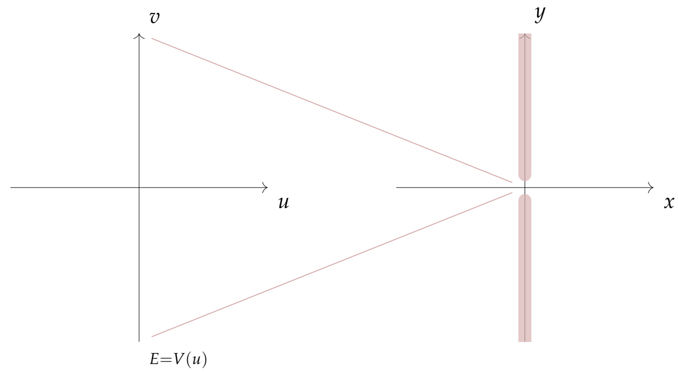
Many a young geometer has struggled with the subtleties in the examples of Nagata of counterintuitive behaviour of Noetherian rings, and finally written them off as belonging to the kingdom of non-geometry, but indeed, they have a pronounced geometric flavour. We shall present one in a simple geometric way as the limit of a series of blow ups of the affine plane $\mathbb{A}^2_k$. It serves as the basis for examples of some of the classical pathologies among Noetherian rings: of a non-catenary Noetherian ring and a Noetherian ring whose integral closure is not finite.

They are given in an appendix called Examples of bad rings' *of his book* [?]

Even though the ring in the example is as close to being of finite type over the field k as it can be without being, it suffers these shortcomings, and is a reminder of our limited intuition about Noetherian rings which are not of finite type over a field,

An infinite sequence of blow ups

We have, several times in this book, encountered the map $\pi\colon \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ that sends the point (x, y) to (x, xy). The map collapses the line $E = V(x)$ to the origin (ande therefore E is called the exeptional divisor). Outside E, the map is an isomorphism, and the image is just the subset $D(x) \cup \{(0, 0)\}$. We shall refer to π is as the *affine blow up* of the origin, since it is the restriction to one of the affine charts of the blow up map of the projective plane from Section 6.6. On the level of rings, the map corresponds to the inclusion $k[x, xy] \subseteq k[x, y]$.



In what follows, we shall work over an arbitrary algebraically closed ground field k.

The behaviour of an irreducible curve C in $\mathbb{A}^2_k$ when pulled back along π is simple. There are two cases according to whether C passing by the centre or not.

If C does not pass by the centre of blow up, *i.e.* the origin, its inverse image in $\mathbb{A}^2$ remains irreducible and $V(x) \cap \pi^{-1}(C) = \emptyset$; heuristically, any intersection C has with the line $V(x)$ is pushed out to infinity. For instance, the line $y = c$ becomes the hyperbola $x \cdot y = c$ when y is replaced by xy. Indeed, if f is the equation for C, the inverse image is given as $f(x, xy)$, and setting $x = 0$, yields $f(0,0)$, which is non-zero when f does nor pass by the origin. So $\pi^{-1}(C)$ is entirely contained in $D(x)$, where π is an isomorphism, and hence $\pi^{-1}(C)$ is isomorphic to C.

In the second case, when C passes through the centre, so that $f \in (x, y)$, the polynomial $f(x, y)$ is without constant term, and $f(x, xy) = x^{\mu} f_1(x, y)$ for some $\mu \geqslant 1$ and with $f_1 \notin (x, y_1)$. Then $\pi^{-1}(C)$ has E as a component of multilicity μ.

The examples we have in mind, will all be the limit of a sequence of iterated birational maps

$$\mathbb{A}_\infty \longrightarrow \ldots\ldots \longrightarrow \mathbb{A}_i \longrightarrow \mathbb{A}_{i-1} \longrightarrow \ldots \longrightarrow \mathbb{A}_1 \longrightarrow \mathbb{A}_0, \qquad (26.3)$$

where all the $\mathbb{A}_i$'s are isomorphic to $\mathbb{A}^2_k$. We let $\mathbb{A}_i = \operatorname{Spec} A_i$ with $A_i = k[x, y_i]$. The maps between the $\mathbb{A}_i$'s are basically affine blow ups, but at each stage we translate along the exceptional line $E_{i+1} = V(x)$ by an amount of a_{i+1} before composing with the blow up map. In other words, the map $\pi_i \colon \mathbb{A}_{i+1} \to \mathbb{A}_i$ will be given by the assignment $(x, y_{i+1}) \mapsto (x, x(y_{i+1} + a_{i+1}))$. The scheme $\mathbb{A}$ is the inverse limit of the sequence, and will be the spectrum of the direct limit (in fact the union as it will turn out) of the rings A_i.

The map $\pi_i \colon \mathbb{A}_{i+1} \to \mathbb{A}_i$ is induced by the inclusion $k[x, y_i] \hookrightarrow k[x, y_{i+1}]$ where

$$y_i = x(y_{i+1} + a_{i+1}). \qquad (26.4)$$

The A_i's share the rational function field $k(x, y)$ as their common quotient field where they form an ascending chain of rings. Their union is the ring $A = k[x, y, y_1, y_2, \ldots]$, and the projective limit appearing in (26.3) is just $\mathbb{A}_\infty = \operatorname{Spec} A$.

$$k[x, y] \hookrightarrow \ldots \hookrightarrow k[x, y_i] \hookrightarrow k[x, y_{i+1}] \hookrightarrow \ldots \hookrightarrow A \hookrightarrow k(x, y)$$

Note that $k[x,x^{-1},y_i] = k[x,x^{-1},y_{i+1}]$ so the induced map $(A_i)_x \to (A_{i+1})_x$ between localizations are not only isomorphisms, but equalities. In the same vein, $A_x = k[x,x^{-1},y]$. The open subset $U_\infty = D(x) \subseteq \mathbb{A}_\infty$ maps isomorphically to each $U_i = D(x) \subseteq \mathbb{A}_i$ (and in fact, the map is the identity).

One easily verifies that the principal ideal $(x)A$ is a maximal—indeed, killing x entails killing all y_i—and if $p_\infty \in \mathbb{A}_\infty$ denotes the corresponding closed point, it holds true that $\mathbb{A}_\infty = \{p_\infty\} \cup D(x)$. The point p_∞ maps to the centre of blow up in each $\mathbb{A}_i$ and is the inverse image of each exceptional line E_i.

In general the ring A is not Noetherian, for instance if all the a_i's are zero, it holds true that $\bigcap_i (x^i) = (x,y,y_1,\dots)$. The astonishing point, however, is that for a sufficiently generic choice of the points $\{a_i\}$, the ring A will be Noetherian. We shall prove

PROPOSITION 26.13 *If the power series $\tau = \sum_i a_i x^i$ is transcendental, the ring A is Noetherian. It is an an integral domain of dimension two, and all its local rings are regular, and A is even a* UFD. *Furthermore, it has a closed point of height one through which no curve passes.*

Technically speaking, the closed point p_∞ is a Cartier divisor, as it is given by one equation!

The hypothesis in the proposition enters the proof in the following way. For any power series τ there is a corresponding ring homomorphism $\iota\colon A \to k[[x]]$, which is injective when τ is transcendental. To define the map, define for each $r \in \mathbb{N}$ the "tail" of $\tau \in k[[x]]$ by:

$$\tau_r = \sum_{i \geqslant 1} a_{i+r} x^i = a_r x + a_{r+1} x^2 + \dots.$$

Then $\tau = \sum_{i<r} a_i x^i + x^{r-1}\tau_r$, and it holds true that $\tau_r = x(\tau_{r+1} + a_{r+1})$, which permits us to define the map $\iota\colon A \to k[[x]]$ by the assignments $x \mapsto x$ and $y_r \mapsto \tau_r$.

LEMMA 26.14 *If the power series $\tau = \sum_i a_i x^i$ is transcendental, the map ι is injective.*

PROOF: If ι is not injective, for some r there is a polynomial $F(x,y_r)$ with $F(x,\tau_r) = 0$. Since $\tau = \sum_{i<r} a_i x^i + x^{r-1}\tau_r$, setting $G = x^N F$ for a sufficiently large natural number N, we wil have $G(x, \tau - \sum_{i<r} a_i x^i) = 0$, which shows that τ is algebraic. ❑

PROOF OF OF THE PROPOSITION: We first recall Cohen's criterion which says that a ring is Noetherian if all prime ideas are finitely generated. So consider a prime ideal $\mathfrak{p} \subseteq A$. There are two cases, either $\mathfrak{p} \subseteq (x)$ or $\mathfrak{p} \nsubseteq (x)$. For each $i \in \mathbb{N}$, define $\mathfrak{p}_i = \mathfrak{p} \cap A_i$.

The case that $\mathfrak{p}$ is not contained in (x) is the easy one. Pick an ielement $f \in \mathfrak{p}$, but with $f \notin (x)$. For some large i, we have $f \in A_i$, but $f \notin (x,y_i) = (x) \cap A_i$, and we may certainly assume that f is irreducible. Since $V(f)$ does not pass by the centre of blow up, it persists being irreducibel (and prime) in A_j for all $j > i$, hence f is irreducible and prime in A. So if $\mathfrak{p}$ is a height one ideal, we readily get $\mathfrak{p} = (f)$. If the height is two, $\mathfrak{p} \cap A_i = (x-a, y-b)$ and one easily obtains $\mathfrak{p} = (x-a, y-b)$.

For future reference, we treat the salient case when $\mathfrak{p} \subseteq (x)$, in a separate lemma. ❑

LEMMA 26.15 *If for a prime ideal $\mathfrak{p}$ in A it holds $\mathfrak{p} \subset (x)$, then $\mathfrak{p} = 0$.*

PROOF: Assume $\mathfrak{p} \subseteq (x)$. Then the ideal $\mathfrak{p}_0 = \mathfrak{p} \cap k[x,y]$ is strictly contained in (x,y) and is therefore principal generated by an irreducible polynomial $f(x,y)$. To keep the geometric intuition, we let C denote the corresponding curve in $\mathbb{A}_0 = \mathbb{A}^2$. For each index i the polynomial f belongs to A_i and may be expressed as $f(x,y) = f_i(x,y_i)$. Geometrically speaking, the polynomial f_i defines the inverse image C_i of C in $\mathbb{A}_i$. We contend that for each i there is a factorisation $f_i = x^i g_i$ in A_i; of course, this entails that $g_i \in \mathfrak{p}$, and consequently g_i must be without a constant term. Geometrically, these factorisations correspond to chopping a copy of the exceptional divisor off the inverse images of C at each stage in the tower (26.3).

Proceeding by induction we find

$$f_{i+1}(x,y_{i+1}) = f_i(x, x(y_{i+1}+a_{i+1})) = x^i g_i(x, x(y_{i+1}+a_{i+1})).$$

Since g_i does not have a constant term, $g_i(x, x(y_{i+1}+a_i))$ has x as a factor, and we may write $g_i(x, x(y_{i+1}+a_i)) = x g_{i+1}(x, y_{i+1})$, and the claim is established.

So we have $f \in \bigcap_i (x^i)$. Now, the crux is the inclusion $\iota \colon A \hookrightarrow k[[x]]$. Of course, in the power series ring $k[[x]]$ it holds true that $\bigcap (x)^i = 0$. It follows that $\iota(f) = 0$; consequently $f = 0$ and $\mathfrak{p} = 0$. ❑

A Noetherian domain with intergral closure not finite

In this section k will be a perfect field of positive characteristic p.

We aim at examining the image of the Frobenoius map $A \to A$ that sends and element f to its p-th power f^p. It will be a ring $B = (A)^p$ with $k(x^p, y^p)$ as fraction field. We will show that A is the integral closure of B in he ield $k(x,y)$ and that it is not finite over B.

Consider for each i the subring $B_i = k[x^p, y_i^p]$ of $A_i = k[x, y_i]$. The gist of the example is that the inclusion map $A_i \hookrightarrow A_{i+1}$ takes B_i into B_{i+1}; indeed, since we are in characteristic p it holds true that

$$y_i^p = (x(y_{i+1}+a_i))^p = x^p(y_{i+1}^p + a_i^p).$$

An even more holds true, the resulting chain of the B_i's is of the same shape as the chain (**??**) save being built with the sequence $\{a_i^p\}$ in stead of $\{a_i\}$ and the variables x^p and y_i^p i stead of x and y_i. The situation is summarized by the diagram:

$$\begin{tikzcd}
\dots \arrow[r] & k[x^p, y_i^p] \arrow[r] \arrow[d, hook] & k[x^p, y_{i+1}^p] \arrow[r] \arrow[d, hook] & \dots \\
\dots \arrow[r] & k[x, y_i] \arrow[r] & k[x, y_{i+1}] \arrow[r] & \dots
\end{tikzcd}$$

Note that $\tau' = \sum_i a_i^p x^{ip}$ will be transcendental when $\tau = \sum_i a_i x^i$ is (because k is prfect), so the game played in the upper chain is the same as the one played in the bottom one, just with slightly altered players; in particular, $B = k[x^p, y_0^p, \dots]$ is normal and Noetherian.

PROPOSITION 26.16 *The ring B is Noetherian with quotient field $k(x^p, z^p)$. The integral closure of B in $k(x,z)$ equals A, and it is not a finite module over B.*

PROOF: Aiming for a contradiction, let us assume that A is finitely generated over B. It will then be a Noetherian B-module since B is Noetherian, and the ascending chain

$$A_n = B[x, y_0, \ldots, y_n] = k[x, y_n, y_{n+1}^p, \ldots]$$

of sub B-modules of A will be stationary. Hence for some $n >> 0$ it holds that $y_{n+1} \in A_n$.

Now, A_n is generated as a ring over B by the monomials $x^i y_n^j$, with $0 \leqslant i, j < p$. The coefficients in a relation expressing y_{n+1} as a combination of these generators, involve only finitely many of the y_i's, and they will all lie in $k[x^2, y_r^2]$ for $r >> 0$; hence we may write

$$y_{n+1} = \sum_{i,j<p} x^i y_n^j h_{ij}(x^p, y_r^p), \tag{26.5}$$

where the coefficients belong to $k[x^p, y_r^p]$. Now, $y_{n+1} = x^{r-n-1} y_r + s(x)$ and $y_n = x^{r-n} y_r + t(x)$ where $s(x)$ and $t(x)$ are polynomials. Inserting this in (26.5), we arrive, after some reorganization, at an expression

$$x^{r-n-1} y_r = g_0(x, y_r^p) + x^{r-n} y_r g(x, y_r^p) + x^{2(r-n)} y_r^2 g_2(x, y_r^p) + \ldots$$

between polynomials in $k[x, y_r]$. The only the term on the left side that can have terms linear in y_r, is $x^{r-n} y_r g(x, y_r^p)$, and so all the other vanish, so that

$$x^{r-n-1} y_r = x^{r-n} y_r g_1(x, y_r^2).$$

Cancelling $x^{r-n-1} y_r$, we obtain $1 = x y_r g(x, y_r^2)$, which is absurd. ❑

If one insists on having an example where the rings involved are DVR's just observe that $(x,z) \cap B = (x^p, z_p)$ so denoting this maximal ideal by $\mathfrak{m}$, we see that the discrete valuation ring $B_\mathfrak{m}$ does not have a finite integral closure in the field $k(x,z)$.

As a spin off, the example easily yields examples of domains whose normalization is not finite: any ring lying between B and A which has fraction field $k(x,y)$ and is finite as a B-module will do. For instance, the ring $C = B[x,y] = k[x, y, y_1^p, \ldots]$ will be one. It is a finite B-module generated by $x^i y_0^j$ with $0 \leqslant i, j < p$, so it is Noetherian, and A can not be finite over C (if it where, A would be finite over B). Thus

PROPOSITION 26.17 *The ring $C = k[x, y, y_1^p, y_2^p]$ is Noetherian but its normalization is not finitely generated as a C-module.*

Krull showed that a local Noetherian one dimensional domain A, then the normalization $\widetilde{A}$ is finite over A if and only if the completion $\widehat{A}$ is without nilpotent elements (there are generalisation to other dimension of the same flavour although not as clean cut). As an illustration of Krull's result, let us point to a nilpotent element in $\widehat{C}$, the completion of C

with respect to the maximal ideal $\mathfrak{m} = (x, y_0)$. Not to create unnecessary confusion, and to underline that y_i^p is not a p^{th}-power in C when $i > 0$, we introduce the notation $w_i = y_i^p$. In C the relation

$$y_0^p = x^p(w_1 + a_0^p)$$

holds. Note that if w_1 were a p^{th}-power—which it is not—say $w_1 = \omega^p$, we would have a nilpotent element since then $(y_0 - x(\omega + a_1))^p = 0$. However, in the completion $\widehat{C}$ the element w_1 has a p^{th}-root. In view of the relations (26.3) a straightforward induction yields the following equality which is valid for all $r \geqslant 1$:

$$w_1 = x^{pr} w_{r+1} + \sum_{j=1}^{r} x^{jp} a_j^p.$$

Now, $x^{pr} w_{i+r} \in \mathfrak{m}^{pr}$, so the right hand side converges to $\sum_{j=1}^{\infty} x^{jp} a_j^p$, and allows us to exhibit w_1 as a p^{th}-power:

$$w_1 = (\sum x^j a_j)^p.$$

A Noetherian domain that is not catenary

The making of such an example needs two ingredients. The first is a construction very similar to the previous one but with a parameter z. That is, instead of blowing up a sequence of points in $\mathbb{A}^2$, we blow up a sequence of lines in $\mathbb{A}^3$. The second ingredient is a pinching manoeuvre as in Section **??**.

In algebraic terms we start out with an ascending chain

$$\mathbb{A}_0 \subseteq \ldots \subseteq A_i \subseteq A_{i+1} \subseteq \ldots \subseteq A$$

of rings where $A_i = k[x, y_i, z]$ and $y_{i+1} = x(y_i + a_i)$, and we put $A = \bigcup_i A_i = k[x, z, y_0, \ldots]$. Geometrically we have a sequence

$$\mathbb{A}_\infty \longrightarrow \ldots\ldots \longrightarrow \mathbb{A}_i \longrightarrow \mathbb{A}_{i-1} \longrightarrow \ldots \longrightarrow \mathbb{A}_1 \longrightarrow \mathbb{A}_0,$$

where this time $\mathbb{A}_i \simeq \mathbb{A}^3$ and each map is the translation $y_i \mapsto y_i + a_i$ followed by the affine blow up of the line $L_i = V(x, y_i)$. The exceptional divisors $\mathcal{E}_i$ are in this case the planes $V(x) = \operatorname{Spec} k[z, y_i]$ and the restriction of the map π_i is just the projecion onto $\operatorname{Spec} k[z] = L_{i-1}$

As before one checks that $(x, z - c) \subseteq A$ is a maximal (killing x kills all y_i and setting $z = c$ transform A into k) and that the ideal (x) is prime with $A/(x) = k[z]$. The closed subset $L_\infty = V(x)$ maps to each of the lines $L_i = V(x, y_i) \subseteq \mathbb{A}_i$, which are blown up. And the $(x, z - c) \subseteq A$ constitute the closed points of $L_{(\infty)}$.

PROPOSITION 26.18 *If the power series $\tau = \sum_i a_i x^i$ is transcendental, the ring A is Noetherian.*

Before proceeding to the proof, let us finish with the application and from A derive a ring which is not catenary. Observe that the maximal ideal $\mathfrak{m} = (x,z)$ defines a closed point p although being of height two. Take any closed point q not lying on L_∞. Its maximal ideal $\mathfrak{n}$ is of height three since q is lying in open part $D(x)$ of $\mathbb{A}$ which is isomorphic to $\mathbb{A}^3_k$. Now, the idea is to coalesce p and q to one point r, just like we did in Section **??**. To this end, consider the pushout diagram of rings

$$\begin{array}{ccc} A & \xrightarrow{\phi} & k \oplus k \\ \uparrow & & \uparrow \iota \\ B & \longrightarrow & k \end{array}$$

where ϕ is the evaluation at p and q and ι the canonical diagonal map. The ring B is the pushout ring defined by the diagram. Clearly A is generated over B by two elements, and citing the Eakin–Nagat theorem we deduce that B is Noetherian.

So why is B not catenary? From (0) to $\mathfrak{m}_p$ we have the staurated chain $0 \subset (x) \subset (x,z) = \mathfrak{m}$ and from (0) to $\mathfrak{n}$ we have saturated chains of length three, more over $\operatorname{Spec} A$ and $\operatorname{Spec} B$ differ merely in that the two pints p and q are identified to say r, and the two chains survives intactly as chains from (0) to $\mathfrak{m}_\mathfrak{r}$; one is of length two and the other has length three.

PROPOSITION 26.19 *The ring B is a Noetherian domain that is not catenary. It has a maximal ideal joined to zero by saturated chains of length two and three.*

PROOF OF PROPOSITION 26.18: We shall reduce to the previous case through a projection onto the plane $z = 0$; algebriacally this correponds to the ideal $\mathfrak{q} = \mathfrak{p} \cap k[x,y,y_1,\dots]$ and $\mathfrak{q}_i = \mathfrak{p} \cap k[x,y_i]$. There are two cases to consider:

i) $\mathfrak{p} \cap k[x,y,y_1,\dots] \neq 0$;

ii) $\mathfrak{p} \cap k[x,y,y_1,\dots] = 0$;

In the first case, when $\mathfrak{q} \neq 0$, it follows from the previous case that $\mathfrak{q}_i \nsubseteq (x,y_i)$ for $i >> 0$ and hence that $V(\mathfrak{p}_i)$ eventually will be disjoint from L_i. By the little lemma 26.20 below it ensues that $\mathfrak{p}_i A_{i+1} = \mathfrak{p}_i$ (two primes one contained in the other which are equal in $D(x)$, must be equal). A set $f_1,\dots,f_r$ of generators for $\mathfrak{p}_i$, will generate $\mathfrak{p}_j$ for $j \geqslant i$ as well and hence also $\mathfrak{p}$; indeed, any element in $\mathfrak{p}$ lies in $\mathfrak{p}_i$ for some $i >> 0$. .

In the second case, it must so that $\mathfrak{p}_i$ is of height one; indeed $k[x,y_i] \subseteq k[x,y_i,z]/\mathfrak{p}_i$ show that $\dim k[x,y_i,z]/\mathfrak{p}_i = 2$—and it is therefore generated by an irreducible $f \in k[x,y_i]$. Now by the same argument as above, f stays irreducible (and hence prime) in A_j for $j \geqslant i$ and we deduce that $\mathfrak{p} = (f)$. ❏

An easy little lemma

Our situation may be summarized by the diagram where $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are two Noetherian affine schemes, and π is a dominant map corresponding to an inclusion

$A \subseteq B$:

$$\begin{tikzcd} X \arrow[r, "\pi"] & Y \\ E \arrow[u, hook] \arrow[r, "\pi"] & F \arrow[u, hook] \end{tikzcd}$$

Moreover, $F = V(x)$ is a divisor and $E = V(x)$ is the inverse image of F; finally we make the a crucial assumption that π induces an isomorphism $X\backslash E \simeq Y\backslash F$. The following little lemma is heuristically concincing, but needs a proof, which is a nice recapitulation of primary decomposition:

Lemma 26.20 *In the staging just described, if $Z = V(\mathfrak{p})$ is disjoint from F, then $\mathfrak{p}B$ is a prime ideal.*

Proof: Since X is Noetherian, there is an irredundant primary decomposition $\mathfrak{p}B = \mathfrak{q} \cap \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, where $\mathfrak{q}$ is such that $V(\mathfrak{q}) \cap D(x) = \pi^{-1}Z$, and $\mathfrak{q}$ is a prime ideal because π is an isomorphisn between $X\backslash E$ and $Y\backslash F$. Let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$.

In $X\backslash E = D(x)$, the q_i's disappear from the decomposition. Hence for each i it holds that $V(\mathfrak{q}_i) \subseteq E$, which is impossible since $V(\mathfrak{p}B)$ does not meet E. Indeed, cover $X\backslash V(\mathfrak{p}B)$ by distinguished open subsets $V(f_i)$ with $1 \leqslant i \leqslant s$, which means that $(f_1, \ldots, f_s) = \mathfrak{p}B$. Then $\mathfrak{p}B \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r$ and prime avoidance gives $\mathfrak{p}B \subseteq \mathfrak{p}_i$ for at least one i, which contradicts the decomposition being irredundant. ❑

Appendix A

Some results from Commutative Algebra

A.1 Direct and inverse limits

Direct limits

Recall that a *preordered* set is a set endowed with a relation $i \leqslant j$ which is symmetric; that is, $i \leqslant i$ for all i, and transitive; that is, if $i \leqslant j$ and $j \leqslant k$, then $i \leqslant k$. A preordered set resembles a partially ordered set, but lacks the anti-symmetry property: it might be that $i \leqslant j$ and $j \leqslant i$ without i and j being equal. We say that a preordered set I is *directed*if the following condition holds: for any two indices i and j there is a $k \in I$ such that $k \geqslant i$ and $k \geqslant j$.

Directed orders

A *directed system of modules* (M_i, ϕ_{ij}) is a collection $\{M_i\}_{i \in I}$ of A-modules, indexed by an directed index set I, and a collection of A-linear maps $\phi_{ij} \colon M_j \to M_i$, one for each pair (i, j) so that $j \leqslant i$, satisfying the two conditions

Directed systems of modules

$$\begin{tikzcd} M_k \arrow[r, "\phi_{jk}"'] \arrow[rr, bend left, "\phi_{ik}"] & M_j \arrow[r, "\phi_{ij}"'] & M_i \end{tikzcd}$$

- $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ whenever $k \leqslant j \leqslant i$;
- $\phi_{ii} = \mathrm{id}_{M_i}$.

The *direct limit* of the system (M_i, ϕ_{ij}) is an A-module $\varinjlim M_i$ together with a collection of A-linear maps

Direct limits

$$\begin{tikzcd} M_j \arrow[dd, "\phi_{ij}"'] \arrow[dr, "\phi_j"] & \\ & \varinjlim M_i \\ M_i \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

$$\phi_i \colon M_i \to \varinjlim M_i$$

which satisfy $\phi_i \circ \phi_{ij} = \phi_j$, and which are universal with respect to this property. That is, for any A-module N and any given system of A-linear maps

$$\psi_i \colon M_i \to N$$

such that $\psi_i \circ \phi_{ij} = \psi_j$, there is a unique map $\eta \colon \varinjlim M_i \to N$ satisfying $\psi_i = \eta \circ \phi_i$.

$$\begin{tikzcd} M_i \arrow[r, "\phi_i"] \arrow[dr, "\psi_i"'] & \varinjlim M_i \arrow[d, dashed, "\eta"] \\ & N \end{tikzcd}$$

(A.1) The definition of the direct limit may be formulated in any category: just replace the words 'A-module' with 'object' and A-linear by 'arrow'. In general categories it may easily happen that direct limits do not exist. However, the category of modules over a ring is a well behaved category, and here all limits exist:

> **Proposition A.2** *Let A be any ring. Every directed system (M_i, ϕ_{ij}) of modules over A has a direct limit, which is unique up to a unique isomorphism.*

PROOF: We begin with introducing an equivalence relation on the disjoint union $\coprod_i M_i$. Essentially, two elements are to be equivalent if they become equal somewhere far out in the hierarchy of the M_i's. In precise terms, $x \in M_i$ and $y \in M_j$ are to be equivalent when there is an index k larger than both i and j such that x and y map to the same element in M_k; that is, $\phi_{ki}(x) = \phi_{kj}(y)$. We write $x \sim y$ to indicate that x and y are equivalent; the first point to verify is that this is an equivalence realtion. Obviously the relation is symmetric, since $\phi_{ii} = \mathrm{id}_{M_i}$ it is reflexive, and it being transitive follows from the system being directed: Assume that $x \sim y$ and $y \sim z$, with x, y and z sitting in respectively M_i, M_j and M_k. This means that there are indices l dominating i and j, and m dominating j and k so that the two equalities $\phi_{li}(x) = \phi_{lj}(y)$ and $\phi_{mj}(y) = \phi_{mk}(z)$ hold true. Because the system is directed, there is an index n larger than both l and m, and by the first requirement above, we find

$$\begin{tikzcd} & & M_n & & \\ & M_l \arrow[ur] & & M_m \arrow[ul] & \\ M_i \arrow[ur] & & M_j \arrow[ul] \arrow[ur] & & M_k \arrow[ul] \end{tikzcd}$$

$$\phi_{ni}(x) = \phi_{nl}(\phi_{li}(x)) = \phi_{nl}(\phi_{lj}(y)) = \phi_{nm}(\phi_{mj}(y)) = \phi_{nm}(\phi_{mk}(z)) = \phi_{nk}(z),$$

and so $x \sim z$. The underlying set of the A-module $\varinjlim M_i$ is the quotient $\coprod_i M_i / \sim$, and the maps ϕ_i are the ones induced by the inclusions of the M_i's in the disjoint union.

The rest of the proof consists of putting an A-module structure on $\varinjlim M_i$ and checking the universal property. To this end, the salient observation is that any two elements $[x]$ and $[y]$ in the limit may be represented by elements x and y from the same M_k: Indeed, if $x \in M_i$ and $y \in M_j$, choose a k that dominates both i and j and replace x and y by their images in M_k. Forming linear combinations is then possible by the formula $a[x] + b[y] = [ax + by]$ where the last combination is formed in any M_k where both x and y live; this is independent of the particular k used (the system is directed, and the ϕ_{ij}'s are A-linear). The module axioms follow since any equality involving a finite number of elements from the limit may be checked in an M_k where all involved elements have representatives.

Finally, checking the universal property is straightforward: The obvious map from the disjoint union $\coprod_i M_i$ into N induced by the ψ_i's is compatible with the equivalence relation and hence passes to the quotient; that is, it gives the desired map $\eta\colon \varinjlim M_i \to N$. ❑

Apart from the universal property, there are two 'working principles', reflecting the working principles for stalks, one should bear in mind when computing with direct limits:

- Every element in $\varinjlim M_i$ is of the form $\phi_j(x)$ for some j and some $x \in M_j$.
- An element $x \in M_j$ maps to zero in $\varinjlim M_i$ if and only if $\phi_{ij}(x) = 0$ for some $i \geqslant j$.

EXAMPLE A.3 (*Union as a direct limit.*) If each M_i are submodules of some A-module M, and the maps $M_j \to M_i$ are given by inclusions $M_j \subset M_i$, then the direct limit is simply the union:

$$\varinjlim_i M_i = \bigcup_i M_i.$$

☆

Example A.4 (*Stalks as a direct limit.*) Let X be a topological space, and consider the directed set I of open neighbourhoods U of a point $x \in X$ ordered by inclusion. If $\mathcal{F}$ is a presheaf on X, then setting $M_U = \mathcal{F}(U)$, the above construction of the direct limit $\varinjlim_U M_U$ is exactly the same as the previous definition of the stalk $\mathcal{F}_x$. ☆

Example A.5 (*Localization as a direct limit.*) Let A be a ring and S a multiplicative subset. We put a preorder on S by declaring $s \leqslant t$ when $t = us$ for some $u \in S$, and this makes S a directed set. Next, for $s \leqslant t$ with $t = us$, there exists a ring homomorphism $f_{ts}\colon A_s \to A_t$, which is defined by $f_{ts}(as^{-n}) = au^n t^{-n}$. In this way the family of rings $\{A_s\}_{s\in S}$ forms a directed system of rings — one easily checks that the properties required of a directed family hold.

For each $s \in S$, there is a localization map $A_s \to S^{-1}A$, so from the universal property of the direct limit, we obtain a canonical A-linear map

$$\phi : \varinjlim_{s\in S} A_f \to S^{-1}A.$$

We contend this is an isomorphism. The map ϕ is surjective: any element in $S^{-1}A$ is of the form as^{-1} with $s \in S$; this element lies in A_s and hence in the image of ϕ. The map ϕ is injective: if $as^{-n} \in A_f$ is mapped to 0 in $S^{-1}A$, then for some $t \in S$ it holds that $ta = 0$, hence $as^{-n} = 0 \in A_{st}$, and ϕ is injective. ☆

Inverse limits

The dual concept of a direct limit are the *inverse limit* (also called the *projective limit* or just the *limit*) of an *inverse system* $\{M_i\}_{i\in I}$. These systems and their limits are defined similarly to the direct systems, just with the arrows reversed. In fact, an inverse system indexed by I is nothing but a direct system indexed by the opposit ordered set I^{op}, though the limits will have rather different properties.

Inverse limits

Inverse systems

To be precise, the staging is as follows: we are given a collection of A-modules M_i, indexed by a directed preordered set I and for every pair i, j from I with $i \leqslant j$, we are given an A-linear map $\phi_{ji}\colon M_j \to M_i$ (note that they go 'backwards') which satisfy the compatibility conditions:

- $\phi_{ki} \circ \phi_{jk} = \phi_{ji}$ when $i \leqslant k \leqslant j$;
- $\phi_{ii} = \mathrm{id}_{M_i}$.

$$\begin{tikzcd} & M_j \arrow[d, "\phi_{ji}"] \\ \varprojlim M_i \arrow[ur, "\phi_j"] \arrow[dr, "\phi_i"'] & \\ & M_i \end{tikzcd}$$

$$\begin{tikzcd} N \arrow[r, dashed, "\eta"] \arrow[dr, "\psi_i"'] & \varinjlim M_i \arrow[d, "\phi_i"] \\ & M_i \end{tikzcd}$$

The inverse limit of the system is a module $\varprojlim_{i\in I} M_i$ together with a collection of universal maps $\phi_i\colon \varprojlim_{i\in I} M_i \to M_i$ satisfying $\phi_i = \phi_{ji} \circ \phi_j$. That is, for any other module N together with maps $\psi_i\colon N \to M_i$ such that $\psi_i = \phi_{ji} \circ \psi_j$ there is a unique A-linear map $\eta\colon N \to \varprojlim_{i\in I} M_i$ satisfying $\psi_i = \phi_i \circ \eta$.

Proposition A.6 *Every directed inverse system of modules has a limit.*

PROOF: Consider the product $\prod_i M_i$ and define a submodule by

$$L = \{ (x_i) \mid x_i = \phi_{ji}(x_j) \text{ for all pairs } i,j \text{ with } i \leqslant j \}. \tag{A.1}$$

The projections induce maps $\phi_i \colon L \to M_i$, and we claim that L together with these maps constitute the inverse limit of the system. A family of maps $\psi_i \colon N \to M_i$ gives rise to a map $\eta \colon N \to \prod_i M_i$ by the assignment $x \mapsto (\psi_i(x))$, and it takes values in L when the ψ_i's satisfy the compatibility constraints $\psi_i = \phi_{ji} \circ \psi_j$. This map is clearly unique, and hence we get the desired universal property. ❑

EXAMPLE A.7 (*Inverse limits and intersections.*) If all the M_i are submodules of some fixed module M, and the maps $M_j \to M_i$ are the inclusion, then the inverse limit is simply the intersection:

$$\varprojlim_{i \in I} M_i = \bigcap_{i \in I} M_i \subset M.$$

☆

EXAMPLE A.8 (*p-adic integers.*) An important application of inverse limits is to form so-called 'completions of rings'. The primary example is the p-adic numbers. Let p be a prime number and consider the modules $M_i = \mathbb{Z}/p^i\mathbb{Z}$. They form an inverse system indexed by $\mathbb{N}_0$ with ϕ_{ij} being just the canonical reduction map $\mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ that for $j \geqslant i$ sends a class $[n]_{p^j}$ mod p^j to the class $[n]_{p^i}$ mod p^i. The system may be visualized by the sequence

$$\ldots \longrightarrow \mathbb{Z}/p^{i+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^i\mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}.$$

The inverse limit is denoted by $\mathbb{Z}_p$ and is called the ring of p-adic integers. ☆

EXAMPLE A.9 (*Inverse limits and sections.*) Whereas direct limits gives us stalks, inverse limits give a way to compute sections. In the context of sheaves, the slogan is: 'Direct limits have a localizing effect, while inverse limits effectuate globalizations.'

Consider an open set U of the topological space X and a sheaf $\mathcal{F}$ on X. Assume given an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of U which is directed under inclusion; *i.e.* the intersection of two members from $\mathcal{U}$ contains a third, then the restriction maps induce an isomorphism $\mathcal{F}(U) \simeq \varprojlim_{i \in I} \mathcal{F}(U_i)$. Indeed, the maps $\rho_{UU_i} \colon \mathcal{F}(U) \to \mathcal{F}(U_i)$ comply with the compatibility request $\rho_{UU_i} = \rho_{U_jU_i} \circ \rho_{UU_j}$ for $U_i \subseteq U_j$, and they thus give a canonical map $\mathcal{F}(U) \to \varprojlim_{i \in I} \mathcal{F}(U_i)$.

In view of the description (A.1) this is an isomorphism: that s maps to zero, means that $\rho_{UU_i}(s) = s|_{U_i} = 0$ for each i, which by the Locality axiom entails that $s = 0$. Furthermore, sections $s_i \in \mathcal{F}(U_i)$ so that $s_j|_{U_i} = s_i$ for each inclusion $U_i \subseteq U_j$ may, by the Gluing axiom, be glued together to give a section of $\mathcal{F}$ over U, and the map is surjective.

In fact, with slightly more care one can establish that if $\mathcal{F}$ is a presheaf, the sections of the sheafification $\mathcal{F}^+$ is given as the inverse limit

$$\mathcal{F}^+(U) \simeq \varprojlim_{i \in I} \mathcal{F}(U_i). \tag{A.2}$$

☆

EXERCISE A.1 Convince yourself that (A.2) holds true. ★

Exercises

(A.2) Let A be a ring and $a \in A$ an element. Let a direct system indexed by $\mathbb{N}$ be given by $G_i = A$ for all i and $f_{ij}(x) = a^{j-i}x$ for $i \leqslant j$. Determine the direct limit $\varinjlim_{i\in I} G_i$.

(A.3) Let A be a ring. Show that the inverse limit of the inverse system

$$\ldots \to A[x]/\mathfrak{m}^{i+1} \to A[x]/\mathfrak{m}^i \to \ldots \to A[x]/\mathfrak{m}^2 \to A[x]/\mathfrak{m}$$

where $\mathfrak{m} = (x)$, and the maps are the canonical reduction maps, is isomorphic to the ring of formal power series $A[[x]]$.

(A.4) Show that the map $\mathbb{Z} \to \mathbb{Z}_p$ sending n to $([n]_{p^i})_i$ is an injective ring homomorphism. Show that the assignment $x \mapsto p$ defines an isomorphism $\mathbb{Z}[[x]]/(x-p) \to \mathbb{Z}_p$.

(A.5) Assume that $\{G_i\}_{i\in I}$ is a directed (resp. inverse) system with transitions morphisms f_{ij} in a category C. Let $J \subseteq I$ be a subset which is directed when endowed with the ordering induced from I. Then $\{G_j\}_{j\in J}$ is a directed (resp. inverse) system.

a) Show that there is a canonical morphisms $\varinjlim_{j\in J} G_j \to \varinjlim_{i\in I} G_i$, respectively a canoncal morphism $\varprojlim_{i\in I} G_i \to \varprojlim_{j\in J} G_j$, whenever the involved limits exist.

b) The subset J is said to be *cofinal* if there for every $i \in I$ is a $j \in J$ with $j \geqslant i$. Show that the morphisms defined in a) are isomorphisms whenever J is cofinal in I. *Cofinal subsets*

(A.6) Let $\{G_i\}_{i\in I}$ be a directed (respectively inverse) system of abelian groups. Assume that I is *discrete*, that is no two elements are comparable; that is, $i \leqslant j$ only when $i = j$. Show that $\varinjlim_{i\in I} G_i = \bigoplus_i G_i$, respectively $\varprojlim_{i\in I} G_i = \prod_i G_i$, provided the sum respectively the product exists in C.

(A.7) Assume that I is a directed set in which every element is dominated by a maximal element. Let $\{G_i\}_{i\in I}$ be a direct (respectively inverse) system of abelian groups indexed by I. Show that $\varinjlim_{i\in I} G_i$ is isomorphic to the direct sum $\bigoplus G_j$, respectively $\varprojlim_{i\in I} G_i$ is isomorphic to the product $\prod G_j$, where the sum, respectively the product, extends over all maximal elements in I.

(A.8) Show that arbitrary direct and inverse limits exist in the category Sets and Rings of sets, respectively of rings. HINT: Adapt the proofs above.

✲ (A.9) Exhibit a directed system in the category sets of finite sets that does not have a direct limit in sets.

★

A.2 Regular local rings

A Noetherian local ring A with $\dim A = n$ and with maximal ideal $\mathfrak{m}$ is said to be *regular local ring* if the maximal ideal can be generated by n elements; that is, by as many elements as the dimension indicates. Nakayama's lemma tells us that the minimal number of generators of $\mathfrak{m}$ equals the so called embedding dimension $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ of A, so A is regular precisely when the Krull dimension and the embedding dimension coincide. A general ring A is *regular* if all the local rings $A_\mathfrak{p}$ are regular. *regulære lokale ringer*

Discrete valuation rings

When it comes to one-dimensional rings, A is regular if and only if $\mathfrak{m}$ is principal. This has many equivalent formulations, and we list the few we shall need.

> **Proposition A.10** *Let A be a Noetherian local domain with maximal ideal $\mathfrak{m}$ of dimension one. Then the following are equivalent*
> *i) The maximal ideal $\mathfrak{m}$ is principal;*
> *ii) A is a* PID *and all ideals are powers of $\mathfrak{m}$;*
> *iii) A is integrally closed.*
> *iv) A is regular, i.e., $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.*

Proof: $i) \Rightarrow ii)$. Let x a generator for the maximal ideal $\mathfrak{m}$ and let $\mathfrak{a} \subseteq A$ be a non-zero ideal. Let n be the largest integer such that $\mathfrak{a} \subseteq \mathfrak{m}^n$. Krull's intersection theorem asserts that $\bigcap_i \mathfrak{m}^i = 0$, and the ideal $\mathfrak{a}$ is therefore not contained in all powers of $\mathfrak{m}$ and such an n exists. Since $\mathfrak{a} \nsubseteq \mathfrak{m}^{n+1}$, there is an $a \in \mathfrak{a}\mathfrak{b}$ such that $a = bx^n$ with $b \notin \mathfrak{m}$; that is, b is a unit since the ring is local. It follows that $(x^n) \subseteq \mathfrak{a}$, and we are done.

$ii) \Rightarrow iii)$. Every PID is a UFD and all UFD's are integrally closed.

$iii) \Rightarrow i)$. Finally, assume that A is integrally closed in its fraction field K and let $x \in \mathfrak{m}$ be any element. Since A is Noetherian and of dimension one, there is an element $y \in A$ not in (x) such that $(x : y) = \mathfrak{m}$. This means that $yx^{-1}\mathfrak{m} \subseteq A$. We contend that $yx^{-1}\mathfrak{m} = A$. If not, one would have $yx^{-1}\mathfrak{m} \subseteq \mathfrak{m}$ and since $\mathfrak{m}$ is a finitely generated and faithful A-module it would follow that yx^{-1} is integral over A. Hence it holds that $yx^{-1} \in A$ since A is integrally closed, and therefore also $y \in (x)$, which is not the case.

$(i) \Leftrightarrow (iv)$. If $\mathfrak{m} = (x)$, then $\mathfrak{m}/\mathfrak{m}^2$ is generated by the class of x modulo $\mathfrak{m}^2$. We also have $\mathfrak{m} \neq \mathfrak{m}^2$ (since A has dimension 1), so $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$. The converse implication follows by Nakayama's lemma. ❑

A ring as in the proposition is also a *discrete valuation ring*. If t is a generator for the maximal ideal $\mathfrak{m}$, one calls t a *uniformizing parameter* of A. In fact, the above proof shows that any element of $\mathfrak{m} - \mathfrak{m}^2$ is a uniformizing parameter.

In a discrete valuation ring A, all non-zero ideals are of the form (t^ν) with $\nu \in \mathcal{N}_0$, and therefore any non-zero element in the fraction field $K = K(A)$ may be written as αt^ν with α a unit in A and ν an integer. Indeed, if $f \in A$ and $f \neq 0$, we let $v(f)$ be the unique non-negative integer such that $(f) = \mathfrak{m}^{\nu(f)}$, then $f = \alpha t^{v(f)}$ with α being a unit, and for a general non-zero element fg^{-1} of the fraction field, one finds $fg^{-1} = \alpha t^{\nu(f)-\nu(g)}$ with α a unit.

The function $v\colon A\backslash\{0\} \to \mathbb{Z}$ sending f to the unique integer such that $f = \alpha t^{v(f)}$ with α a unit, is called the *valuation* associated to A. It resembles the well-known order function from complex analysis (recall that every meromorphic function has an order at a point, positive if its a zero and negative in case of a pole), and it share several of its properties. For instance, the two following identities hold:

- $\nu(fg) = \nu(f) + \nu(g)$;

- $\nu(f+g) \geqslant \min\{\nu(f), \nu(g)\}$,

with equality in the latter when $\nu(f) \neq \nu(g)$. Any function $A\backslash\{0\} \to \mathbb{Z}$ satisfying these two properties is called a *discrete valuation* on A. We sometimes extend this definition to include 0, by assigning $v(0) = \infty$; in that case v is a map from $v : A \to \mathbb{Z} \cup \infty$. We will also sometimes extend the valuation to the whole fraction field $K = K(A)$ by defining $v(a/b) = v(a) - v(b)$.

Given the valuation $v : K \to \mathbb{Z} \cup \infty$, we can recover the valuation ring as the subring of K given by

$$A = \{x \in K^{\times} | v(x) \geqslant 0\} \cup \{0\}$$

and the maximal ideal is given by

$$\mathfrak{m} = \{x \in K^{\times} | v(x) \geqslant 1\} \cup \{0\}$$

The group of units in A is given by the subgroup

$$A^{\times} = \{x \in K | v(x) = 0\}.$$

Note also that for any $x \in K$, either $x \in A$ or $x^{-1} \in A$.

EXAMPLE A.11 Let $K = k(x)$ be the field of rational functions in one variable. Let $f \in k[x]$ be an irreducible polynomial. Then any element $y \in K$ can be written as $y = f^d g/h$ where $d \in \mathbb{Z}$; and g, h are coprime to f. We can define a valuation $v_f : K^{\times} \to \mathbb{Z}$ by setting $v(y) = d$. In this case, the valuation ring is the localization of $k[x]$ at f:

$$A = k[x]_{(f)}$$

☆

EXAMPLE A.12 Let $K = k(x)$ be the field of rational functions in one variable. Define the valuation $v_\infty : K^{\times}\mathbb{Z}$ by setting

$$v_\infty\left(\frac{f}{g}\right) = \deg g - \deg f$$

One can check that this defines a valuation on $k(x)$. The valuation v_∞ is supposed to measure the order of a pole 'at infinity'. The corresponding valuation ring is

$$R = \{f/g \in k(x) | \deg f \leqslant \deg g\}.$$

with maximal ideal $\mathfrak{m} = \{f/g \in k(x) | \deg f < \deg g\}$. ☆

EXAMPLE A.13 Let $K = \mathbb{Q}$ be the field of rational numbers, and let p be a prime number. Any $y \in \mathbb{Q}$ can be expressed as $y = p^d a/b$ where $d \in \mathbb{Z}$ and a, b are coprime to p. We can define the *p-adic* valuation $v_p : \mathbb{Q}^{\times} \to \mathbb{Z}$ by setting $v(y) = d$. In this case, the valuation ring is the localization of $\mathbb{Z}$ at (p):

$$A = \mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \in \mathbb{Q} \,|\, \gcd(p, n) = 1 \right\}$$

☆

EXERCISE A.10 Assume that ν is a discrete valuation on a field K. Show that the set $A = \{ x \in K \mid \nu(x) \geqslant 0 \}$ is discrete valuation ring by showing that $\{ x \in K \mid \nu(x) > 0 \}$ is a maximal ideal generated by one element. ★

A.3 Unique factorization domains

Lemma A.14 *Let A be a noetherian domain. Then A is a UFD if and only if every height 1 prime ideal is principal*

Proof: Suppose that A is a UFD. Let $\mathfrak{p}$ be a height 1 prime ideal. Take $x \in \mathfrak{p}$ non-zero and let $x = x_1 \cdots x_n$ be a factorization into irreducible elements. Since $\mathfrak{p}$ is prime, we must have, say, $x_1 \in \mathfrak{p}$. However, also (x_1) is prime (since A is UFD), so since $\mathfrak{p}$ has height 1, we must have $\mathfrak{p} = (x_1)$.

Conversely, suppose that every height 1 prime is principal. Since A is noetherian, every non-zero non-unit x has a factorization into irreducible elements. It suffices to prove that an irreducible element is prime. Let $(x) \subset \mathfrak{p}$ be a minimal prime over (x). Then $\mathfrak{p}$ has height 1 (localize at $\mathfrak{p}$ and use minimality to see why).

❑

A.4 Hartog's extension theorem

Proposition A.15 *Let A be a noetherian normal integral domain of dimension $\geqslant 1$ with fraction field K. Then*

$$\bigcap_{\mathfrak{p} \in \operatorname{Spec} A,\, ht(\mathfrak{p})=1} A_{\mathfrak{p}} = A$$

A.5 Projective modules

Lemma A.16 *Let A be a local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated projective A-module. Then M is free.*

Proof: This is a standard application of Nakayama's lemma. Let $k = A/\mathfrak{m}$ denote the residue field, and consider the module $M \otimes_A k = M/\mathfrak{m}M$. Since M is finitely generated, this is a finite dimensional vector space over k. Let $m_1, \ldots, m_r \in M$ denote a collection of elements in M that map to a basis for $M \otimes_A k$. We obtain a map $\phi : A^r \to M$ sending the standard basis vector e_i to m_i for each $i = 1, \ldots, r$. Note that $\phi \otimes id_k$ is an isomorphism, so by Nakayama's lemma ϕ is surjective. We thus get a short exact sequence

$$0 \to K \to A^r \xrightarrow{\phi} M \to 0,$$

where $K = \operatorname{Ker} \phi$. When M is a projective module, this sequence splits [?]. Hence it stays exact when tensorized by k. Again, since $\phi \otimes id_k$ is an isomorphism, we get that $K \otimes_A k = 0$, and hence $K = 0$, once more by Nakayama's lemma (note that K is finitely generated, being a direct summand of a finitely generated module). It follows that $M \simeq A^r$ is free. ❑

A.6 Dimension theory

Appendix B

Solutions

Solutions for exercises in Chapter 2

EXERCISE 2.1 $f \in \sqrt{\mathfrak{a}} \Rightarrow f^n \in \mathfrak{a}$ for some $n > 0 \Rightarrow f^n \in \mathfrak{p}$ for all primes $\mathfrak{p} \supset \mathfrak{a} \Rightarrow f \in \mathfrak{p}$ for all $\mathfrak{p} \supset \mathfrak{a} \Rightarrow f \in \bigcap_{\mathfrak{p}\supset\mathfrak{a}} \mathfrak{p}$.

We follow the hint. Let $\iota\colon A \to A_f$ denote the localization map. If $f \notin \sqrt{\mathfrak{a}}$ then $\mathfrak{a}A_f$ is a proper ideal ($1 = \sum a_i f^{-n}$ implies that $f^n \in \mathfrak{a}$ for some n). Hence there is a maximal ideal $\mathfrak{m} \supset \mathfrak{a}A_f$. The preimage $\mathfrak{p} = \iota^{-1}(\mathfrak{m})$ is then a maximal ideal containing $\mathfrak{a}$, but not f. Hence $f \notin \bigcap_{\mathfrak{p}\supseteq\mathfrak{a}} \mathfrak{p}$.

EXERCISE 2.2 f nilpotent $\Rightarrow f^n = 0$ for some $n > 0 \Rightarrow D(f) = D(f^n) = D(0) = \varnothing$. Conversely, $D(f) = \varnothing \Rightarrow V(f) = X \Rightarrow f \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A \Rightarrow f \in \bigcap_{\mathfrak{p}} \mathfrak{p} = \sqrt{0} \Rightarrow f$ is nilpotent.

EXERCISE 2.4 Assume first that $\bigcup_i D(f_i) = \operatorname{Spec} A \backslash V(\mathfrak{a})$. Then

$$V(\mathfrak{a}) = (\bigcup_i D(f_i))^c = \bigcap_i V(f_i) = V\big((f_i | i \in I)\big),$$

and consequently $\mathfrak{a}$ and $(f_i | i \in I)$ have the same radical. On the other hand, if they share radicals, the same equalities hold true (but read in a different order):

$$\bigcup_i D(f_i) = V((f_i|i \in I))^c = V(\mathfrak{a})^c = \operatorname{Spec} A \backslash V(\mathfrak{a}).$$

EXERCISE 2.5 Let U and V be two non-empty open subsets of $\overline{Z}$. Then both $U \cap Z$ and $V \cap Z$ are nonempty, and being open in Z, they have a nonempty intersection since Z is assumed to be irreducible. For the second statement assume that Z is irreducible and that $f(Z) = Z' \cup Z''$ with Z' and Z'' closed sets. Then $Z = f^{-1}(Z') \cup f^{-1}(Z'')$, and it follows that either $Z = f^{-1}(Z')$ or $Z = f^{-1}(Z'')$. In the former case $f(Z) = Z'$, and in the latter $f(Z) = Z''$.

EXERCISE 2.6 Let $\{Z_i\}$ be an ascending chain of irreducible subsets containing Z. We contend that the union $W = \bigcup_i Z_i$ is irreducible. Indeed, if U and V are open subsets of W, there must be an index ν so that both $U \cap Z_\nu$ and $V \cap Z_\nu$ are non-empty. Both are open in Z_ν and Z_ν being irreducible, their intersection is non-empty. Hence by Zorn's lemma, there is a maximal irreducible set containing Z. To second task, any $x \in X$ is contained in an irreducible set; indeed, the closure $\overline{x}$ is irreducible.

Exercise 2.8 Assume first that $\operatorname{Spec} A$ is disconnected; say it is the disjoint union $\operatorname{Spec} A = U_1 \cup U_2$ with U_i a proper open set. Then each U_i is closed as well, and hence it is shaped like $U_i = V(\mathfrak{a}_i)$ for some radical ideal $\mathfrak{a}_i$ in A. Since $U_1 \cap U_2 = V(\mathfrak{a}_1) \cap V(\mathfrak{a}_2) = \emptyset$, it holds that $\mathfrak{a}_1 + \mathfrak{a}_2 = A$, and since $U_1 \cup U_2 = \operatorname{Spec} A$ it holds that $\sqrt{\mathfrak{a}_1 \cap \mathfrak{a}_2} = \sqrt{\mathfrak{a}_1} \cap \sqrt{\mathfrak{a}_2} = \mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$. Then the Chinese Remainder theorem yields that $A \simeq A/\mathfrak{a}_1 \times A/\mathfrak{a}_2$.

If e is an idempotent, $1 - e$ is also idempotent, so when e is distinct from 0 and 1, the pair $1 - e$ and e form a pair of non-trivial orthogonal idempotent with sum equal to unity, and they determine a non-trivial representation of A as a direct product. Hence $\operatorname{Spec} A$ is disconnected according to Example 2.24.

Exercise 2.9 Assume to begin with that X is Noetherian and let Σ be a family of closed sets without a minimal element. One then easily constructs a strictly descending chain that is not stationary by recursion. Assume a chain

$$X_r \subset X_{r-1} \subset \cdots \subset X_1$$

of length r has been found; to extend it just append any subset in Σ strictly contained in X_r, which does exist since Σ by assumption has no minimal member.

Next, assume that every non-empty family of closed subsets has a minimal member and let an open covering $\{U_i\}$ of an open subset U of X be given. Introduce the family Σ consisting of the closed sets that are finite intersections of complements of members of the covering; *i.e.* the sets of the shape $U_{i_1}^c \cap \cdots \cap U_{i_r}^c$. It has a minimal element Z. If U_j is any member of the covering, it follows that $Z \cap U_j^c = Z$, hence $U_j \subseteq Z^c$, and by consequence $U = Z^c$.

Finally, suppose that every open U in X is quasi-compact and let $\{X_i\}$ be a descending chain of closed subsets. The open set $U = X \setminus \bigcap_i X_i$ is quasi-compact by assumption, and it is covered by the ascending collection $\{X_i^c\}$, hence it is covered by finitely many of them. The collection $\{X_i^c\}$ being ascending, we can infer that $X_r^c = U$ for some r; that is, $\bigcap_i X_i = X_r$ and consequently it holds that $X_i = X_r$ for $i \geqslant r$.

Exercise 2.11

a)

$$I = (x^2, y^2) \cap (x^2, z) \cap (y^2, z)$$

b)

$$\begin{aligned} I &= (x^2, y^2x) \cap (y, y^2x) \\ &= (x) \cap (x^2, y^2) \cap (y) \cap (x, y^2) \\ &= (x) \cap (y) \cap (x, y^2) \end{aligned}$$

Another primary decomposition is given by

$$I = (x) \cap (y) \cap (x - y, y^2)$$

c)

$$\begin{aligned} I &= (x^3, xy^4) \cap (y, xy^4) \\ &= (x) \cap (x^3, y^4) \cap (y) \\ &= (x) \cap (y) \cap (x^3, y^4) \end{aligned}$$

d)

$$\begin{aligned} I &= (x^2 + (x^2-1)^2 - 1, y - x^2) \\ &= (x^2, y - x^2) \cap (x-1, y-x^2) \cap (x+1, y-x^2) \\ &= (x^2, y) \cap (x-1, y-1) \cap (x+1, y-1) \end{aligned}$$

Exercise 2.12 $X = \coprod_{i \in I} \operatorname{Spec} A_i$ is not quasi-compact, hence not homeomorphic to the spectrum of any ring (the cover $U_i = \operatorname{Spec} A_i$ has no finite subcover).

Exercise 2.13 The implication $\Rightarrow$: Given a $\mathbb{Q}$-algebra A there is a canonical structure map $\phi : \mathbb{Q} \to A$. This induces maps $\mathbb{Z} \hookrightarrow \mathbb{Q} \to A$ and hence maps $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$, which is the desired factorization.

Assume given a factorization

$$\begin{tikzcd} \operatorname{Spec} A \arrow[rr] \arrow[dr] & & \operatorname{Spec} \mathbb{Z} \\ & \operatorname{Spec} \mathbb{Q} \arrow[ur, hook, "\text{canonical}"'] & \end{tikzcd}$$

We apply the global section functor Γ and get the diagram

$$\begin{tikzcd} A & & \mathbb{Z} \arrow[ll] \arrow[dl, hook, "\text{canonical}"] \\ & \mathbb{Q} \arrow[ul, "\phi"] & \end{tikzcd}$$

which gives us a morphism $\phi : \mathbb{Q} \to A$ showing that A is a $\mathbb{Q}$-algebra.

Exercise 2.14

a) By Lemma 2.8 on page 27 prime ideal $\mathfrak{p}$ is maximal if and only if $V(\mathfrak{p})$ is a closed point.

b) Some possibilities include: *i)* $A = \mathbb{Q}$, *ii)* $A = \mathbb{C}[x]_{(x)}$ (or any DVR) and *iii)* $A = \mathbb{C}[x,y]_{(x)}$.

c) The point is that because $\mathfrak{m}_B$ is a proper ideal and elements in A not in $\mathfrak{m}_A$ are units, it always holds that $\phi^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$. Hence $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ if and only if $\phi(\mathfrak{m}_A) = \mathfrak{m}_B$; ot in other words if and only if $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

d) The inclusion $\mathbb{C}[x]_{(x)} \to \mathbb{C}(x)$ is not local: $\mathbb{C}(x)$ is a field and has (0) as its only (maximal) ideal, but this is not maximal in $\mathbb{C}[x]_{(x)}$. Geometrically this corresponds to mapping the point $\operatorname{Spec} \mathbb{C}(x)$ to the open point $\eta \in \operatorname{Spec} \mathbb{C}[x]_{(x)}$.

Exercise 2.15 Assume first that Spec A is a singleton and let $N = \sqrt{0}$ denote the nilradical of A. Clearly A is a local ring (all rings have at least one maximal ideal) with maximal say $\mathfrak{m}$. We have $N = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \mathfrak{m}$. Hence $A \backslash N = A \backslash \mathfrak{m} = A^*$, and every element in $\mathfrak{m}$ is nilpotent.

Next, assume that A is local and that all non-units are nilpotent. Consider the quotient map $\pi : A \to A/N$ where $N = \sqrt{0}$. It is surjective, and maps units in A to units in A/N. By assumption $A \backslash N = A^*$, so each element in A/N is either zero or a unit. Hence A/N is a field, and consequently N must be the maximal ideal. Every prime ideal $\mathfrak{p}$ is contained in a maximal ideal; that is, contained in N, and on the other hand the nilradical N is contained in every prime ideal, so we conclude that $\mathfrak{p} = N$. Hence there is only one prime ideal.

Solutions for exercises in Chapter 3

Exercise 3.1 One can for instance take the constant presheaf with $\mathcal{F}(U) = \mathbb{Z}$ for every non-empty U: this does not satisfy the gluing axiom. For an example which fails Locality: write $X = \{p, q\}$ and define $\mathcal{F}$ by $\mathcal{F}(X) = \mathbb{Z}^3$, $\mathcal{F}(\{p\}) = \mathbb{Z}$, $\mathcal{F}(\{q\} = \mathbb{Z}$ and $\mathcal{F}(\varnothing) = 0$. Also define the restriction maps $\rho_p : \mathcal{F}(X) \to \mathcal{F}(\{p\})$ and $\rho_q \colon \mathcal{F}(X) \to \mathcal{F}(\{q\})$ by the first and second projection map $\mathbb{Z}^3 \to \mathbb{Z}$ respectively. This is easily seen to be a presheaf, but it is not a sheaf since the two elements $(0, 0, 0), (0, 0, 1) \in \mathcal{F}(X)$ both restrict to the same element 0 in $\{p\}$ and $\{q\}$.

Exercise 3.2 Let $U \subset X$ be a connected open set. If the derivative Df is zero, the function f is locally constant, hence constant (since U is connected). Therefore $\mathscr{A}(U) = \mathbb{C}$.

If U has connected components $\{U_i\}_{i \in I}$, we can define a map $\mathscr{A}(U) \to \prod_i \mathbb{C}$ by sending $(f \colon U \to \mathbb{C})$ to the tuple $(f(x_i))_{i \in I}$ where $x_i \in U_i$ is any point. This is clearly an injective and surjective map which commutes with the restriction mappings.

Exercise 3.3 Locality holds because $\mathcal{G}$ is a subpresheaf of the sheaf $\mathcal{F}$ of holomorphic maps $U \to Y$.

Gluing holds, because maps $f_i : U_i \to Y$ agreeing on the overlaps can be glued in $\mathbb{A}$ and if $\pi \circ f = \mathrm{id}_U$ holds on a covering, it holds globally as well.

$\mathcal{G}(X)$ is the empty set: Any $f : X \to Y$ in $\mathcal{G}(X)$ is of the form $f(z) = (z, h(z))$ where h is holomorphic, satisfying $h(z)^2 = z$ for each $z \in X$. But there is no globally defined square root function $\sqrt{z}$ on $\mathbb{C} \backslash 0$.

Exercise 3.5 Let us verify the two sheaf axioms, and we begin with the Locality axiom. Assume that $\phi \colon \mathcal{F}|_U \to \mathcal{G}|_U$ is a section and that $\phi|_{U_i} = 0$ for all members from an open cover $\{U_i\}_{i \in I}$ of U. For every open $V \subseteq U$ and every section $s \in \mathcal{F}(V)$ it then holds that $\phi(s)|_{V \cap U_i} = 0$ for all i, and hence $\phi(s) = 0$ by the Locality axiom for $\mathcal{G}$.

Then to the Gluing axiom: we are given a bunch of maps $\phi_i \colon \mathcal{F}_{U_i} \to \mathcal{G}_{U_i}$ which coincides on the overlaps $U_{ij} = U_i \cap U_j$, and we are to define a map $\phi \colon \mathcal{F}|_U \to \mathcal{G}|_U$ restricting to each ϕ_i. This amounts to giving appropriate maps from $\mathcal{F}_V$ to $\mathcal{G}_V$ for each open $V \subseteq U$, but replacing each U_i by $V \cap U_i$ we may well assume that $V = U$. The Gluing axiom for $\mathcal{G}$ permits us to construct ϕ_U: pick a section $s \in \mathcal{F}(U)$, and form $s_i = \phi(s|_{U_i})$. On the overlap

U_{ij} one has

$$s_i|_{U_{ij}} = \phi_i(s|_{U_i})|_{U_{ij}} = \phi_i|_{U_{ij}}(s|_{U_{ij}}) = \phi_j|_{U_{ij}}(s|_{U_{ij}}) = \phi_j(s|_{U_s})|_{U_{ij}} = s_j|_{U_{ij}},$$

and the s_i's may be glued together to give the section $\phi_U(s)$ in $\mathcal{G}(U)$. The required properties, that the ϕ_U's are compatible with restrictions and each is additive, follow readily, but industrious students are recommended to check it.

Exercise 3.6 The 'only if'-direction is obvious.

For the converse, let $U \subset X$ be an open set, and let $s \in \mathcal{F}(U)$ be a section. Let $x \in U$ be any point. By assumption $\phi_x(s) = \psi_x(s)$, which means that there is an open neighborhood $V \subset U$ of x such that

$$\phi_U(s)|_V = \psi_U(s)|_V.$$

As this is true for every x, we must have $\phi_U(s) = \psi_U(s)$, because both sides restrict to the same sections locally, and $\mathcal{G}$ is a sheaf. This means that $\phi = \psi$.

Exercise 3.7 The forward direction is obvious. For the converse, if ϕ_U is an isomorphism for every open set U, then $\mathcal{F}_x \to \mathcal{G}_x$ for all x, and hence ϕ is an isomorphism, by Exercise 3.6.

Solutions for exercises in Chapter 4

Exercise 4.2 Starting with $(f, f^\sharp) : X \to Y$ and $(g, g^\sharp) : Y \to Z$, we define $h = (g \circ f) : X \to Z$ by $g \circ f : X \to Z$ on the level of topological spaces, and $h^\sharp : \mathcal{O}_Z \to h_* \mathcal{O}_X$ by

$$\mathcal{O}_Z(W) \to \mathcal{O}_Y(g^{-1}(W)) \to \mathcal{O}_X(f^{-1}(g^{-1}(W)),$$

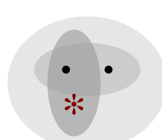
that is, $h^\sharp = f^\sharp_{g^{-1}W} \circ g^\sharp_W$. The induced map $\mathcal{O}_{Z,h(x)} \to \mathcal{O}_{X,x}$ coincides with $f^\sharp_x \circ g^\sharp_{f(x)}$, so being a composition of morphisms of local rings, it is a morphism of local rings.

Exercise 4.3 Since f is local, the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ maps $\mathfrak{m}_y$ into $\mathfrak{m}_x$, and hence $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k(x)$.

When X and Y are affine varieties over an algebraically field k, this map is simply the identity map $k \to k$.

Exercise 4.8 (*Three-point-schemes*)

Let X be a space with three points and organize the possible topologies according to the number of closed sets. The discrete topology, which has all eight subsets closed, are realized as the spectrum of products of three fields $k_1 \times k_2 \times k_3$. It there are seven, the topology is also discrete; indeed, either all points are closed or all doubletons are, and in both cases the topology will be discrete.



There is merely one topology on a three space X having six closed sets: X must have two closed points and two closed doubletons; and the union of the two closed points is one of the closed doubletons. The point not lying in a doubleton is both open and closed and is a connected component of X. One may *e.g.* realize X as $\operatorname{Spec}(V \times k)$ where V is a DVR and k a field.

There are two topologies on X having five closed sets: one with two and one with just one closed point, but with two closed doubletons. The latter is irreducible, but has no generic point, so ot is excluded from being a scheme. The former can be realized as the spectrum of any one dimensional semilocal integral domain with two maximal idea, l*e.g.* $\mathbb{Z}_6$.

Having four closed subsets forces X to have exactly one closed point and one closed doubleton. They can be organised in two ways, either the closed doubleton contains the closed point or not. In the latter case X is irreducible but has no generic point (and the closed set has two!), so it is not underlying a scheme. The former is realized as the spectrum of some non-noetherian valuation rings (for an explicit example, see CA Example **??** or Section **??** on page **??** below.)

No three-point-space with three closed subsets underlies a scheme. If it has a one closed point, there are two generic points, and if it has a closed doubleton, that doubleton has two generic points. The trivial topology, has three generic points, and is of course not a scheme.

Exercise 4.9

a) If X is affine, say $X = \operatorname{Spec} A$, an irreducible Z is of the form $V(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ which is the unique generic point of Z. In general, if $U \subseteq X$ is an open and affine subset meeting Z, the set $U \cap Z$ has a generic point z in U; that is, the closure of $\{z\}$ in U equals $Z \cap U$. Now $U \cap Z$ is dense in Z since Z is irreducible, so that the closure of $\{z\}$ in X must be equal to Z. If z_1 and z_2 are two generic points of Z, both must lie in $U \cap Z$ since its complement is a proper closed set, hence they coincide by the affine case.
b) Any scheme having a closed irreducible subset with more than one point is not Hausdorff. Indeed, if $Z = \overline{\{z\}}$ and $y \in Z$ is different from z, any open neighbourhood of y contains z. In a Hausdorff and sober topology the irreducible components are the singletons, and if additionally the space is quasi-compact, they must be finite in number. It follows that the space discrete (and finite); hence the spectrum of an Artinian ring.

c) Given different points x and y in X we are to exhibit an open set containing one of them but not the other. If $x \notin \{\bar{y}\}$, the open set $\{\bar{y}\}^c$ contains x but not y. If $x \in \{\bar{y}\}$, it holds that $y \notin \{\bar{x}\}$ since otherwise we would have $\{\bar{x}\} = \{\bar{y}\}$ and $x = y$ by uniqueness of generic points; hence $\{\bar{x}\}^c$ is an open subset as desired.

d) In a Quasi-compact set every descending chain $\{Z_i\}$ of closed sets has a non-empty intersection, and by Zorn's lemmas we deduce that there are minimal non-empty closed sets. Such a minimal closed set Z has a unique generic point z, and being minimal Z reduces to $\{z\}$.

Solutions for exercises in Chapter 5

Exercise 5.5 Starting with a morphism $(f, f^\sharp)\colon \operatorname{Spec} K \to X$, we let x be the image of f (Spec K consists of a single point x_0, so this is a well-defined point of X.) The map between stalks is just $f^\sharp_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_{\operatorname{Spec} K, x_0} = K$, which gives a map between residue fields $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \to \mathcal{O}_{\operatorname{Spec} K, 0} = K$. As always with non-zero maps of fields, this has to be an injection.

Conversely, suppose we are given $x \in X$ and $k(x) \to K$. We can define the corresponding map of topological spaces $f\colon \operatorname{Spec} K \to X$, which takes Spec K to $x \in X$. We also construct a map of structure sheaves $f^\#\colon \mathcal{O}_X \to f_*\mathcal{O}_{\operatorname{Spec} K}$ in the following way: for opens $U \subseteq X$ not containing x, the map $f^\#_U$ is the zero map (which it has to be, as $f^{-1}(U) = \varnothing$), while for opens U with $x \in U$, we need maps $\mathcal{O}_X(U) \to K$. These maps are constructed using the given map $k(x) \to K$ via the compositions

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x} \longrightarrow k(x) \longrightarrow K,$$

and thus we obtain the desired morphism of schemes $(f, f^\#)\colon \operatorname{Spec} K \to X$. It is clear that these two constructions are mutually inverse.

Solutions for exercises in Chapter 6

Exercise 6.1 $\mathbb{A}^1_k$ minus the origin is naturally identified with the open set $D(x) \subset \mathbb{A}^1_k$, and $D(x) = \operatorname{Spec} k[x, x^{-1}]$ is affine.

Exercise 6.2 Let $X = \operatorname{Spec} k[x, y, z, w]/(xw - yz)$ and consider the open set $U = X - V(x, y) = D(x) \cup D(y)$. We have

$$\mathcal{O}_X(D(x)) = (k[x, y, z, w]/(xw - yz))_x = k[x, x^{-1}, y, z, w]/(xw - yz) \simeq k[x, x^{-1}, y, z]$$

and similarly, $\mathcal{O}_X(D(y)) \simeq k[x, y, y^{-1}, w]$. We also have $D(x) \cap D(y) = D(xy)$, and

$$\mathcal{O}_X(D(xy)) = k[x, x^{-1}, y, y^{-1}, z]$$

The sheaf sequence gives

$$0 \to \mathcal{O}_X(U) \xrightarrow{\alpha} k[x, x^{-1}, y, z] \times k[x, y, y^{-1}, w] \xrightarrow{\beta} k[x, x^{-1}, y, y^{-1}, z]$$

where $\beta(p(x^{\pm 1}, y, z), q(x, y^{\pm 1}, w)) = p(x^{\pm 1}, y, z) - q(x, y^{\pm 1}, x^{-1}yz))$. We see that $\mathcal{O}_X(U)$, the kernel of β, is identified with elements p in $k[x^{\pm 1}, y^{\pm 1}, z]$ which are simoultaneously polynomials in $x^{\pm 1}, y, z$ as well as $x, y^{\pm 1}, x^{-1}yz$. We leave it to the reader to check that p must be a polynomial in $x, y, z, x^{-1}yz$, i.e.,

Hint: view monomials as elements in $\mathbb{Z}^3$.

$$\mathcal{O}_X(U) \simeq k[x, y, z, x^{-1}yz] \simeq k[x, y, z, w]/(xw - yz)$$

Thus shows that $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is an isomorphism, so U is not affine, by the argument of Section 6.2.

Exercise 6.3 Morphisms $f : \mathbb{P}^1_k \to \operatorname{Spec} A$ are in one-to-one correspondence with ring maps $A \to \Gamma(\mathbb{P}^1, \mathcal{O}) = k$. However, each ring map $A \to k$ must correspond to the 'constant map' $\operatorname{Spec} k \to \operatorname{Spec} A$.

Exercise 6.7 Answer: $\Gamma(X, \mathcal{O}_X) = \mathbb{Z}[x, y]$ and $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ is exactly the blow-up morphism p.

Exercise 6.11 With reference to the diagram on page 93 that gives the gluing recipe for the blow-up, we put $u = ts^{-1}$; the relation $xt = ys$ defining the blow-up then becomes $y = ux$. The blow-up will be glued together of the two affine planes $\operatorname{Spec} k[x, u]$ and $\operatorname{Spec} k[y, u^{-1}]$ along the common open subset $\operatorname{Spec} k[x, y, u, u^{-1}] = \operatorname{Spec} k[x, u, u^{-1}] = \operatorname{Spec} k[y, u, u^{-1}]$. So in terms of the L_n-terminology x plays the role of s and y that of t; and the glue derives from the relation $y = ux$ and becomes $s \mapsto u^{-1}t$. In other words, the result of the gluing will be L_{-1}. The exceptional divisor E is given by respectively $x = 0$ and $y = 0$ in the two $\mathbb{A}^2$'s; that is respectively by $s = 0$ and $t = 0$, which are the equations for the zero section.

Exercise 6.14 Using homogeneous coordinates $(x : u)$ on $\mathbb{P}^1_k$, points (x, y) in X_1 map to $(x : 1)$ and points in X_2 to $(1 : u)$. Therefore the image of $D(x)$ is $D_+(x)$, and of $D(u)$ it is $D_+(u)$.

Assume first that the characteristic of k is not two. The fibre over a point $(x : 1)$ are the points (x, y) where y is a solution of the quadratic equation $y^2 = (x^{2g} - 1)x$. There are precisely two, save when the right side vanishes, and this occurs at $(0 : 1)$ and the $2g$ points $(\xi : 1)$ with ξ a $2g$-th root of unity. The fibre over $(1 : u)$ in X_2 are points (u,v) with $v^2 = u(1 - u^{2g})$. We have already accounted for all of them except over $(1 : u)$ which reduces to the one point $(0, 0)$. Hence we find all together $2g + 2$ "ramification points", as they are called.

If the characteristic of k equals two, numbers have at most one square root, so in that case all fibres have just one point (but there is of course a multiplicity around).

Exercise 6.15 Let $g(n) = 4n^4 - 4n^3 + 12n^2 + 20$ and note that $g(n)$ is a perfect square if and only if $n^4 - n^3 + 3n^2 + 5$ is (*i.e.* there is a solution to $n^4 - n^3 + 3n^2 + 5 = y^2$). It is easy to show that

$$(2n^2 - n + 2)^2 < g(n) < (2n^2 - n + 5)^2$$

so the only way that $g(n)$ can be a perfect square is if it equals $(2n^2 - n + 3)^2$ or $(2n^2 - n + 4)^2$. Trying both of these possibilities gives that $n = 2$ is the only solution. In that case $2^4 - 2^3 + 3 \cdot 2^2 + 5 = 25$. Hence $X_1(\mathbb{Z}) = \{(2, -5), (2, 5)\}$. To determine $X(\mathbb{Z})$, we need only find $X_2(\mathbb{Z}) - X_1(\mathbb{Z})$. But this corresponds to $\mathbb{Z}$-points where $u = 0$, i.e., $(u, v) = (0, \pm 1)$. We conclude that $X(\mathbb{Z})$ consists of the four points $(2, -5), (2, 5)$ and (0 ± 1).

Solutions for exercises in Chapter 7

Solutions for exercises in Chapter 8

Exercise 8.6 (*Basic properties*)

The verifications are entirely functorial only relying on the universal property and so are valid in any category (where involved products exist).

a) It is totally tautological that the bottom square in diagram

$$\begin{tikzcd} & Z \arrow[dl, dashed, "\psi"'] \arrow[d, "\psi"] \\ X \arrow[r, "="'] \arrow[d, "\phi_X"'] & X \arrow[d, "\phi_X"] \\ S \arrow[r, "="'] & S \end{tikzcd} \tag{B.1}$$

is Cartesian; indeed, the upper part shows that any map $\psi\colon Z \to X$ is its proper lifting to X.

b) The order of X and Y is just apparent and a typographical phenomenon; X and Y enter the formulation of the universal property in a complete symmetric way; hence $X \times_S Y$ and $X \times_S Y$ are identical, just denoted in two different ways.

c) There are three natural maps from $X \times_S (Y \times_S Z)$ to respectively X, Y and Z; the first, p_X, is just the projection onto X, the two next, p_Y and p_Z, are the projection onto $Y \times_S Z$ followed respectively by the projections onto Y and Z. For $(X_S \times_S Y) \times_S Z$ there are corresponding maps p'_X, p'_Y and p'_Z.
We contend that giving an S-map $\psi\colon Z \to X \times_S (Y \times_S Z)$ is the same as giving three S-maps ψ_X,ψ_Y and ψ_Z from Z to X, Y and Z respectively. Indeed, given ψ, one just composes with the maps p_X, p_Y and p_Z; and i the triple ψ_X, ψ_Y and ψ_Z is given, the map $\big(\psi_X, (\psi_y, \psi_Z)\big)$ is as desired. The analoguos statement clearly holds for the product $(X_S \times_S Y) \times_S Z$. Hence we obtain maps $\big(p_X, (p_Y, p_Z)\big)$ and $((p'_X, p'_Y), p'_Z)$ between the products and one easily verifies they are mutually inverse maps.

d) On the spot one obtains a map $X \times_S Y \to X \times_S T$, which enters in the commutative diagram

$$\begin{tikzcd} & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[ur, dashed] \arrow[urr] \arrow[r] & X \times_S T \arrow[r] \arrow[d] & T \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

and using that the lower and the large square both are Cartesian, one infers easily that the upper square is Cartesian.

Exercise 8.13 By Noether normalization, there are finite morphisms $X \to \mathbb{A}^m$, and $Y \to \mathbb{A}^n$. Then $X \times Y \to \mathbb{A}^{m+n}$ is also finite, which gives the claim.

Exercise 8.15 (*Flat base change*)

Let $\{U_i\}$ be an affine covering of X. Letting $U_{i,B} = U_i \times_{\operatorname{Spec} A} \operatorname{Spec} B$ and $U_{ij,B} = U_{ij} \times_{\operatorname{Spec} A} \operatorname{Spec} B$, then $\{U_{i,B}\}$ is an open affine covering of X_B and one verifies that $U_{ij,B} = U_{i,B} \cap U_{j,B}$ Moreover, the sequences (5.2) on page 80 for X and X_B give rise to the commutative diagram below, where β_1 and β_2 are isomorphisms (check all the details!):

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X,\mathcal{O}_X)\otimes_A B \arrow[r] \arrow[d, "\beta"] & \bigoplus_i \Gamma(U_i,\mathcal{O}_{U_i})\otimes_A B \arrow[r] \arrow[d, "\beta_1"] & \bigoplus_{i,j} \Gamma(U_{ij},\mathcal{O}_{U_{ij}})\otimes_A B \arrow[d, "\beta_2"] \\ 0 \arrow[r] & \Gamma(X_B,\mathcal{O}_{X_B}) \arrow[r] & \bigoplus_i \Gamma(U_{i,B},\mathcal{O}_{U_{i,B}}) \arrow[r] & \bigoplus_{i,j} \Gamma(U_{ij,B},\mathcal{O}_{U_{ij,B}}) \end{tikzcd}$$

It follows that we have a map β as desired, and if B is A-flat, the upper sequence is exact and the Five-lemma shows that β is an isomorphism.

Exercise 8.17 (*Answer:*)

The fibre product equals the empty scheme $\operatorname{Spec} 0 = \varnothing$. Geometrically, this means that $\operatorname{Spec}(\mathbb{Z}/2) \to \operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}(\mathbb{Z}/3) \to \operatorname{Spec}\mathbb{Z}$ have distinct images.

Exercise 8.18 Let ϕ be the morphism $\operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$ be induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}[i]$, and note that $\mathbb{Z}[i] = \operatorname{Spec}\mathbb{Z}[x]/(x^2+1)$.

a) The preimage of ϕ over $(p) \in \operatorname{Spec}\mathbb{Z}$ consists of prime ideals $\mathfrak{q} \subset \mathbb{Z}[i]$ such that $\mathfrak{q} \cap \mathbb{Z} = (p)$; that is, prime ideals in $\mathbb{Z}[i]/(p) = \mathbb{F}_p[i]$, or in other words, prime ideals in $\mathbb{F}_p[x]/(x^2+1)$. We also have
$$\dim_{\mathbb{F}_p}\left(\mathbb{F}_p[x]/(x^2+1)\right) = \dim_{\mathbb{F}_p}(\mathbb{F}_p + \mathbb{F}_p x) = 2.$$

b) The ring $A = \mathbb{F}_p[x]/(x^2+1)$ is a field if and only if x^2+1 does not have a root in $\mathbb{F}_p$: Assume that $a \in \mathbb{F}_p$ is a root of x^2+1 in $\mathbb{F}_p$; that is, $x^2+1 = (x-a)(x-b)$ for some $b \in \mathbb{F}_p$. Hence $(x-a)$ is not a unit in A, and A is not a field.
Assume that A is not a field, and let $a \in A\backslash 0$ be so that $(a) \subset A$ is a proper ideal. Then $A/(a)A$ is a $\mathbb{F}_p$-vector space of dimension one, and there is an isomorphism $A/(a) \simeq \mathbb{F}_p$. Thus we get a surjective ring homomorphism $\phi : \mathbb{F}_p[x]/(x^2+1) \to \mathbb{F}_p$. Note that $0 = \phi(x^2+1) = \phi(x)^2+1$, so $a = \phi(x)$ is a root of x^2+1 in $\mathbb{F}_p$.

c) The ideal $(p)\mathbb{Z}[i]$ is prime if and only if $A = \mathbb{Z}[i]/(p) \simeq \mathbb{F}_p[x]/(x^2+1)$ is an integral domain. If A is not a field, then there is an $a \in A$ which generates a proper ideal, so we conclude as in b).

Exercise 8.20

a) The prime ideal of A are all maximal (CA Proposition 10.55) since A is of finite type over k. Then the Chinese Remainder Theorem gives a surjection $\phi\colon A \to A/\mathfrak{m}_1 \times \cdots \times A/\mathfrak{m}_r$ where the $\mathfrak{m}_i$'s are the maximal ideals in A. Hence $r \leqslant \dim A$. In case $r = \dim A$, the map ϕ is an isomorphism and each $A/\mathfrak{m}_i$ is of dimension one over k.

b) The fibre over a point $\mathfrak{p}$ in $\operatorname{Spec} A$ equals $\operatorname{Spec} B \otimes_A K(A/\mathfrak{p})$ and the ring $B \otimes_A K(A/\mathfrak{p})$ is a vector pace of dimension n over $K(A/\mathfrak{p})$ since A is free of rank over B. It follows from a) that the fibre has at most n points.

Solutions for exercises in Chapter 9

Exercise 9.5 That the diagram is Cartesian is just the statement that two T-morphisms f and g are equal independently of wether one regards them as T-morphisms or S-morphisms. All diagonals are separable and pullbacks of separable maps are separable according to *iii)* of Proposition 9.9. If X separable, the diagonal $\Delta_{X/S}$ is a closed immersion, hence ι is a closed immersion.

Exercise 9.8 Consider the morphism $h : X \to Y \times_S Y$ induced by the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "f"] \arrow[drr, bend left] \arrow[ddr, bend right, "g"'] & & \\
& Y \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

We claim that $Z = h^{-1}(\Delta(Y))$. $\subseteq$. Let $x \in Z$, so that $f(x) = g(x) = y$. This gives $\pi_1(\Delta(Y)) = y = f(x) = \pi_1(h(x))$ and $\pi_2(\Delta(Y)) = \pi_2(h(x))$. Since π_1 and π_2 are monic, we conclude that $\Delta(y) = h(x)$, so that $x \in h^{-1} * (\Delta(Y))$.

$\supseteq$. Let $x \in h^{-1}(\Delta(Y))$. $h(x) = \Delta(y)$ for some $y \in Y$. Hence $f(x) = \pi_1(h(x)) = \pi_1(\Delta(y)) = y$ and similarly $g(x) = y$, so $x \in Z$.

Now, since Y is separated, $\Delta(Y)$ is closed in $Y \times_S Y$, so h is continuous, $Z = h^{-1}(\Delta(Y))$ is closed in X.

Exercise 9.9 cacac

Exercise 9.12

a) The salient point in the solution of this exercise, is the diagram

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] & X \times_Z Y \arrow[r, "\eta"] \arrow[d] & Y \arrow[d, "g"] \\
& X \arrow[r, "f \circ g"'] & Z
\end{tikzcd} \tag{B.2}$$

where the square is Cartesian and Γ_f is the graph of f. It follows that η is a separated, being the pullback of the separated map $f \circ g$, and all graphs being separated $f = \eta \circ \Gamma_f$ is separated.

b) For the example just take Y to be the disjoint union $Y = Y_1 \cup Y_2$ and $f_1 \colon Y_1 \to Z$ being separable while $f_2 \colon Y_2 \to X$ is not.

Exercise 9.12 Hint: Reduce to the affine case, showing that $\mathcal{O}_S \to f_* \mathcal{O}_X$ is surjective.

Exercise 9.13 There is a general fact that if $f \colon X \to Y$ and $g \colon Y \to Z$ are morphisms of schemes, and P is a property of morphisms preserved under composition and base change, then under the hypothesis that $g \circ f$ has property P and the diagonal of g has property P, we can conclude f has property P. In our case, we can take P to be the property of separatedness, and then for $Z = \operatorname{Spec} \mathbb{Z}$, $g \circ f$ must be the unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ (which is separated by assumption), and the diagonal of g is separated because it is a locally closed immersion (this is true of the diagonal of any morphism), so we conclude f is separated.

Solutions for exercises in Chapter 10

Exercise 10.1 One easily checks that $\mathfrak{q}$ is an ideal. Every $x \in (R_\mathfrak{p})_0$ is shaped like $x = fg^{-1}$ with $\deg f = \deg g$. When $x \notin \mathfrak{q}$ it holds that $f \notin \mathfrak{p}$, and f is invertible in $R_\mathfrak{p}$; it follows that $f^{-1}g \in (R_\mathfrak{p})_0$, hence $\mathfrak{q}$ is maximal.

Exercise 10.2 One way is trivial, so let us prove that $xy \in \mathfrak{p}$ implies that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ when x and y are homogeneous. Assume that neither x nor y belongs to $\mathfrak{p}$. By subtracting all homogeneous component belonging to $\mathfrak{p}$, we may assume that no component of x or of y lies in $\mathfrak{p}$. But if x_n and y_m are the components of highest degree of x and y, the product $x_n y_m$ is a component of xy and consequently lies in $\mathfrak{p}$ because $\mathfrak{p}$ is homogeneous. Contradiction.

Exercise 10.6 This follows from the 'homogenization' procedure: Since X is integral, each open affine $U = D_+(f)$ is dense and have the same function field. Since U is the spectrum of an integral domain, we have $k(U) = K((R_f)_0)$.

Exercise 10.8 The following ring works in all three cases. Let t and x_i for $i \in \mathbb{N}_0$ be variables and let $\mathfrak{a}$ be the homogeneous ideal in $k[t, x_0, x_1, \ldots]$ generated by all products $x_i x_j$ and tx_i. Then $\operatorname{Proj} R/\mathfrak{a}$ is just one point; indeed, $D_+(t) = \operatorname{Spec} k$ since inverting t kills all the x_i's, and $D_+(x_i) = \varnothing$ since inverting a nilpotent kills everything.

Exercise 10.9 Let $R = A[x_0, \ldots, x_n]$. There is a commutative diagram of localizations

$$\begin{tikzcd}
(R_{x_i})_0 \arrow[rr, hook] \arrow[dr, hook] & & R_{x_i} \arrow[dr, hook] & \\
& (R_{x_i x_j})_0 \arrow[rr, hook] & & R_{x_i x_j} \\
(R_{x_j})_0 \arrow[rr, hook] \arrow[ur, hook] & & R_{x_j} \arrow[ur, hook] &
\end{tikzcd}$$

which shows that the morphisms $\operatorname{Spec}(R_{x_i}) \to \operatorname{Spec}(R_{x_i})_0$ glue to a morphism from $\bigcup_{i=0}^n D(x_i) = \mathbb{A}_A^{n+1} - V(x_0, \ldots, x_n)$ to $\mathbb{P}_A^n = \bigcup_{i=0}^n D_+(x_i)$.

Exercise 10.12 Let $X = \operatorname{Proj} R$, $Y = \operatorname{Proj} R'$ and $Z = \operatorname{Proj} S$. Let $f \in R$ be a homogeneous element. Define

$$Z_f = \bigcup_{g \in R'} \operatorname{Spec} S_{f^{\deg g} \otimes g^{\deg f}}$$

We claim that there is a natural isomorphism of graded rings

$$\begin{aligned} S_{f'\otimes g'} &\to (R_f)_0 \otimes_A (R'_g)_0 \\ \frac{r\otimes r'}{(f'\otimes g')^s} &\mapsto \frac{r}{f'^s}\otimes\frac{r'}{g'^s} \end{aligned}$$

where $f' = f^{\deg g}$ and $g' = g^{\deg f}$. Indeed, the inverse is given by the map $(R_f)_0 \otimes_A (R'_g)_0 \to S_{f'\otimes g'}$ defined by

$$\frac{r}{f^r}\otimes\frac{r'}{g^t} \mapsto \frac{r^{t\deg g}\otimes r'^{r\deg f}}{(f'\otimes g')^{rt}}$$

Hence we see that

$$Z_f = \bigcup_{g\in R'} \operatorname{Spec}((R_f)_0 \otimes_A (R'_g)_0)$$

On the overlaps, we have

$$\begin{aligned} \operatorname{Spec}((R_f)_0 \otimes_A (R'_g)_0) \cap \operatorname{Spec}((R_f)_0 \otimes_A) R'_h)_0) &= \operatorname{Spec}\left(S_{f^{\deg g+\deg h}\otimes (gh)^{\deg f}}\right) \\ &= \operatorname{Spec}\left((R_f)_0 \otimes_A R'_{gh}\right) \end{aligned}$$

From this is is clear that

$$Z_f = D_+(f)\times_R Y$$

Moreover, for any other $f'\in R$ we have $Z_{f'} = Z_{ff'} = X_{ff'}\times_R Y$. Hence

$$Z = \bigcup_{f\in R} Z_f = X\times_R Y.$$

Exercise 10.13 Consider the map of graded A-algebras $\phi : A[u,v]\to R$ given by $u\mapsto xt$ and $v\mapsto yt$. It is clearly surjective, and it will suffice to show that $\operatorname{Ker}\phi = \mathfrak{a}$ where $\mathfrak{a} = (xv - yu)$. The inclusion $\mathfrak{a} \subseteq \operatorname{Ker}\phi$ is clear. Conversely, we can write, modulo $\mathfrak{a}$, any element p of $k[x,u,y,v]$ as

$$p = \sum a_{i,j,k} x^i u^j v^k + \sum b_{i',j',k'} x^{i'} y^{j'} v^{k'}.$$

If now $p\in\operatorname{Ker}\phi$, we have

$$0 = \phi(p) = \sum a_{i,j,k} x^{i+j} y^k t^{j+k} + \sum b_{i',j',k'} x^{j'} y^{i'+j'} t^{k'}.$$

The monomials in x, y and t being linearly independent over k, the coefficients $a_{i,j,k}, b_{i',j',k'}$ must vanish except possibly when the same monomials appear in each sum; *i.e.* when $i+j = i', k = j'+k', i = i'+j'$ and $j+k = k'$, and in which case we must have $a_{i,j,k} = -b_{i',j',k}$. These conditions imply that $j = -j'$ which must then both be 0, and so $i = i', k = k'$. It follows that $p = 0 \mod \mathfrak{a}$ and so $\operatorname{Ker}\phi \subset \mathfrak{a}$.

Exercise 10.14 The map α sends the irrelevant ideal $R_+ = (x,y,z)$ into the ideal (x,y,w^p) whose radical equals $(x,y,w) = A_+$. Hence $G(\alpha) = \operatorname{Proj} A = \mathbb{P}^2_k$, and we get a morphism $\pi\colon \mathbb{P}^2_k \to \mathbb{P}(1,1,p)$.

If k is of characteristic different from p, the best way of thinking about the map π is by introducing an action of the cyclic group μ_p of roots of unity. An element $\eta \in \mu_p$ acts on $\mathbb{P}^2_k$ by $(x : y : w) \mapsto (x : y : \eta w)$ which on on the ring level is expressed as $w \mapsto \eta w$. Then clearly $k[x, y, w^p]$ is the ring of invariants.

To identify the fibres, we examine π over each of the distinguished open sets. We start with $D_+(x)$ which equals Spec $(R_x)_0$ and maps into Spec $(A_x)_0$. On the level of rings this is the map $k[yx^{-1}, zx^{-p}] \to k[yx^{-1}, wx^{-1}]$ that sends $z \mapsto w^p$; *i.e.* $zx^{-p} \mapsto (wx^{-1})^p$. Simplifying the notation, it presents itself as $k[u, v] \to k[u, w]$ with v mapping to w^p. On the geometric level it sends (u, v) to (u, v^p). The fibre over $\mathfrak{m} = (u - a, v - b)$ will be $\operatorname{Spec} k[u, w]/(u - a, w^p - b)$. If the characteristic is different from p, and $b \neq 0$, the fibre has p distinct points corresponding to the p distinct p^{th}-roots of b. If $b = 0$ or k has characteristic p, the fibre is isomorphic to the non-reduced scheme $\operatorname{Spec} k[t]/t^p$.

Next, let us see what happens in $D_+(w)$, which is more interesting. After the coordinates being simplified, the map will be

$$k[t_0, \ldots, t_n] \to k[u, v] \quad t_i \mapsto u^{p-i}v^i$$

If fibre $\mathfrak{m} = (u - a, v - b)$ belongs equals $\bigcap_{\eta \in \mu_p} (u - \eta a, v - \eta b)$. If $(a, b) \neq (0, 0)$, the points $(\eta a, \eta b)$ are different and fibre has exactly p-point. In case $(0, 0)$ the fibre is reduced to one point, but the scheme-theoretical is highly non-reduced fibres being equal to $\operatorname{Spec} k[u, v]/(u^{p-i}v^i | i = 1, \ldots, p)$.

Exercise 11.4 I) A is not integrally closed because $y \in Frac(A) \backslash A$ is integral over A (it satisfies $T^2 - y^2 = 0$) II) The morphism

$$f : \mathbb{A}^2_k \to \mathbb{A}^4_k : (x, y) \mapsto (u = x, v = xy, w = y^2, z = y^3)$$

has as image X, the surface $X \subset \mathbb{A}^4_k$ defined by the three equations

$$u^2 w = v^2,\ u^3 z = v^3,\ w^3 = z^2$$

III) If we put $O = (0, 0) \in \mathbb{A}^2_k$ and $O' = (0, 0, 0, 0) \in X \subset \mathbb{A}^4_k$, the morphism f restricts to an isomorphism $f_0 : \mathbb{A}^2_k \backslash \{O\} \overset{\equiv}{\to} X \backslash \{O'\}$. Its inverse

$$f_0^{-1} : X \backslash \{O'\} \overset{\equiv}{\to} \mathbb{A}^2_k \backslash \{O\} : (u, v, w, z) \mapsto (x, y)$$

is given by:

$$x = uy = v/u \text{ if } u \neq 0 \quad \text{or} \quad y = z/w \text{ if } w \neq 0$$

Exercise 11.5 (*Noether normalization*)

(i) This follows immediately from the statement of the Noether Normalization lemma in Commutative Algebra.

(ii) Write $A = B[z]/(z^2 - xy)$, where

$$B = \mathbb{C}[x, y]/(x^2 y - xy^3 + x^2 y^2 - 1)$$

If we perform a change of variables $u = -x$, $v = x + y$, then the relation here becomes

$$\begin{aligned} x^2 y - xy^3 + x^2 y^2 - 1 &= u^2(u + v) + u(u + v)^3 + u^2(u + v)^2 - 1 \\ &= 2u^4 + (\text{lower order terms in } u) \end{aligned}$$

If we divide by 2 we see that u is integral over $\mathbb{C}[v]$. Hence a Noether normalization is given by

$$\operatorname{Spec} A \to \operatorname{Spec} \mathbb{C}[x+y]$$

Solutions for exercises in Chapter 12

EXERCISE 12.1 Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves. If $s \in \operatorname{Ker} \phi(V)$, then clearly $s_x \in \operatorname{Ker} \phi_x$. We can therefore define $\Phi : (\operatorname{Ker} \phi)_x \to \operatorname{Ker} \phi_x$ by sending (s, V) to s_x. Φ is clearly a group homomorphism. Φ is injective: If (s, V) maps to zero, then $(s|_W, W) = (0, W)$ for some $W \subset V$ and hence the element is zero in the stalk $(\operatorname{Ker} \phi)_x$. Φ is surjective: any element $s_x \in \operatorname{Ker} \phi_x \subset \mathcal{F}_x$ is induced by some section (s, V) for some $V \subset X$. $\phi(s)$ is an element so that $\phi(s)_x = 0$, so by shrinking V we may assume that $\phi(s) = 0$ on V, and hence $s \in \operatorname{Ker} \phi_V$. In particular, s_x is induced by (s, V).

EXERCISE 12.3 Take any non-closed subset Z of your favourite ringed space, and define a Godement sheaf $\mathscr{A}$ with the property that $\mathscr{A}_x \neq 0$ if and only if $x \in Z$.

EXERCISE 12.6 The rightmost map exp is surjective as a map of sheaves, because non-vanishing functions *locally* have logarithms. However, over the open set $U = X$, the map is not surjective: the non-vanishing function $f(z) = z$ is not the exponential of a global holomorphic function.

EXERCISE 12.7 To prove Theorem 12.29, it will be convenient to introduce some notation. Let us define an *f-map* $\Lambda\colon \mathcal{G} \to \mathcal{F}$ to be a collection of homomorphisms $\Lambda_V\colon \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ indexed by open subsets $V \subseteq Y$ such that

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\Lambda_V"] \arrow[d, "\rho_{\mathcal{G}}"'] & \mathcal{F}(f^{-1}V) \arrow[d, "\rho_{\mathcal{F}}"] \\ \mathcal{G}(V') \arrow[r, "\Lambda_{V'}"'] & \mathcal{F}(f^{-1}V') \end{tikzcd}$$

commutes for all inclusions $V' \subseteq V$ of open sets in Y. Bearing this definition in mind, we can now prove the following lemma, which implies Theorem 12.29:

LEMMA B.1 *Let $f\colon X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf on X and let $\mathcal{G}$ be a sheaf on Y. There are canonical bijections between the following four sets:*

i) The set of f-maps $\Lambda\colon \mathcal{G} \to \mathcal{F}$;
ii) The set of sheaf maps $\mathcal{G} \to f_*\mathcal{F}$;
iii) The set of sheaf maps $f^{-1}\mathcal{G} \to \mathcal{F}$;
iv) The set of presheaf maps $f_p^{-1}\mathcal{G} \to \mathcal{F}$.

PROOF: The bijection between *iii)* and *iv)* follows by the adjoint property of sheafification (as in 12.4) because $\mathcal{F}$ is a sheaf, so it suffices to consider the sets in *i)*, *ii)* and *iv)*.

Let us first consider *i)* and *ii)*. If we are given a map of sheaves $\phi : \mathcal{G} \to f_*\mathcal{F}$, we have a map $\phi_V : \mathcal{G}(V) \to f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$ for each open set $V \subseteq Y$. By the definition of a sheaf map, this commutes with the various restriction mappings to smaller opens $V' \subseteq V$, so we get a well-defined f-map $\Lambda : \mathcal{G} \to \mathcal{F}$. Conversely, it is clear that any f-map Λ appears from a map of sheaves in this manner, so we have established the desired bijection.

For the bijection between the sets *i)* and *iv)*, suppose we are given a map of presheaves $f_p^{-1}\mathcal{G} \to \mathcal{F}$, so that we have a map

$$\varinjlim_{W \supseteq f(U)} \mathcal{G}(W) \to \mathcal{F}(U).$$

Applying this to $U = f^{-1}(V)$, and compositing with the map $\mathcal{G}(V)$ into the direct limit, we get a map $\mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$. Again, this is compatible with the restriction maps to smaller open sets $V' \subseteq V$, so we get a well-defined f-map $\Lambda : \mathcal{G} \to \mathcal{F}$. Conversely, it is clear that any f-map Λ arises in this manner, i.e., each Λ_V factors as

$$\begin{tikzcd} \mathcal{G}(V) \arrow[rr, "\Lambda_V"] \arrow[dr] & & \mathcal{F}(f^{-1}V) \\ & \varinjlim_{W \supseteq V} \mathcal{G}(W) \arrow[ur, "\phi_{f^{-1}V}"'] & \end{tikzcd}$$

for some map of presheaves $\phi : f_p^{-1}\mathcal{G} \to \mathcal{F}$: Just define ϕ_U for $U \subseteq X$ by composing Λ_W with the restriction map to get a map $\mathcal{G}(W) \to \mathcal{F}(f^{-1}W) \to \mathcal{F}(U)$ for $W \supseteq f(U)$ – the fact that Λ is an f-map means that we get an induced map in the direct limit. Over $U = f^{-1}V$, this ϕ makes the above diagram commute. This completes the proof of the lemma. ❑

Solutions for exercises in Chapter 13

Exercise 13.7 As $\Gamma(D(f), \widetilde{M}) = M_f$, there is a map $\beta_f : \widetilde{M_f} \to \widetilde{M}|_{D(f)}$ that on distinguished open subsets $D(g) \subseteq D(f)$ induces an isomorphism between the two spaces of sections, both being equal to the localization M_g.

Exercise 13.11 One can for instance take $\mathcal{F} = \mathcal{G}$ equal to the constant sheaf on $\mathbb{Z}_X$ on a space X with two connected components.

Exercise 13.16

$$\begin{aligned} \mathrm{Hom}_X(f^*\mathcal{G}, \mathcal{F}) &= \mathrm{Hom}_X(f^*\widetilde{N}, \widetilde{M}) \\ &= \mathrm{Hom}_X((N \otimes_A B)^{\sim}, \widetilde{M}) \\ &= \mathrm{Hom}_B(N \otimes_A B, M) \\ &= \mathrm{Hom}_A(N, \mathrm{Hom}_A(B, M)) \\ &= \mathrm{Hom}_A(N, M_A) \\ &= \mathrm{Hom}_Y(\widetilde{N}, \widetilde{M}_B) \\ &= \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \end{aligned}$$

Exercise 13.18

a) Let $Y = \mathbb{A}^1 = \operatorname{Spec} k[x]$, $X = \operatorname{Spec} k$ and $f : X \to Y$ the inclusion of a k-point y. Then $f_*\mathcal{O}_X$ is a skyscraper sheaf at y, which is certainly not isomorphic to $\mathcal{O}_Y$. f to be the inclusion of a point $x \in X$; then $i_*\mathcal{O}_X$

b) $k \subset K$ induces $f : \operatorname{Spec} K \to \operatorname{Spec} k$; $\mathcal{F} = f^{-1}\mathcal{O}_Y$ is a sheaf satisfying $\mathcal{F}(\operatorname{Spec} K) = k$, where as $\mathcal{O}_X(\operatorname{Spec} K) = K$.

Exercise 13.19

a) A morphism of sheaves is an isomorphism if and only if it is an isomorphism on stalks. The stalk of $(f^{-1}\mathcal{O}_Y)_{Y,f(x)}$ and the induced map is the stalk map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.

b) See https://mathoverflow.net/questions/286828/when-is-the-inverse-image-of-the-structure-sheaf-the-structure-sheaf

Exercise 13.20 From its definition, it is straightforward to check that applying f^* commutes with taking tensor products of sheaves. On the level of presheaves, we have

$$\begin{aligned} f^*(\mathcal{G} \otimes \mathcal{H}) &= f^{-1}(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{H}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\ &= (f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{H}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\ &= (f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \otimes (f^{-1}\mathcal{H}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &= f^*\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{H}. \end{aligned}$$

and sheafifying, we get an isomorphism of the corresponding sheaves.

However, the pushforward f_* rarely commutes with taking tensor products of sheaves (we will see several examples of this later). There is however, at least, a map $f_*(\mathcal{F}) \otimes f_*(\mathcal{G}) \to f_*(\mathcal{F} \otimes \mathcal{G})$ for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$: If $U \subseteq Y$ is an open, and $s \in f_*(\mathcal{F}), t \in f_*(\mathcal{G})$, then $s \otimes t$ is an element of $\mathcal{F} \otimes \mathcal{G}$ over $f^{-1}(U)$, and hence $s \otimes t$ defines a section of $f_*(\mathcal{F} \otimes \mathcal{G})$ over U.

Exercise 13.21 For every open sets $U \subseteq X, V \subseteq Y$ such that $f(U) \subseteq V$, we have a map $\mathcal{F}(V) \to \mathcal{G}(U)$. Note that both terms here are $\mathcal{O}_Y(V)$-modules (in view of the map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$, and the map is a homomorphism of $\mathcal{O}_Y(V)$-modules.

Suppose $\phi\colon \mathcal{G} \to f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module homomorphism. Then by the adjoint property of f_* and f^{-1} (in the categories Sh_X and Sh_Y), we get a map $f^{-1}\mathcal{G} \to \mathcal{F}$, which is $f^{-1}\mathcal{O}_Y$-linear. Now $\mathcal{F}$ is an $\mathcal{O}_X$-module, so we get an $\mathcal{O}_X$-linear map $f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{G}$ by the universal property of the tensor product. Hence we obtain a map of $\mathcal{O}_X$-modules $f^*\mathcal{G} \to \mathcal{F}$.

Conversely, let $\phi\colon f^*\mathcal{G} \to \mathcal{F}$ be $\mathcal{O}_X$-linear. Again by properties of the tensor product, there is a map $f^{-1}\mathcal{G} \to f^*\mathcal{G}$ which is $f^{-1}\mathcal{O}_Y$-linear. Consequently there is a $f^{-1}\mathcal{O}_Y$-linear map $f^{-1}\mathcal{G} \to \mathcal{F}$. This induces a $\mathcal{O}_Y$-linear map $\mathcal{G} \to f_*\mathcal{F}$ by the earlier adjointness property of f_* and f^{-1}.

Solutions for exercises in Chapter 14

Exercise 14.1 That limit in QCoh_X is also the limit in the larger category Mod_X follows leisurely by Lemma **??** on page ??; details are left to the students.

Exercise 14.3 By Theorem 14.3 it suffices to treat the case that X is affine, say $X = \operatorname{Spec} A$. Assume first that the maps (14.2) are isomorphisms. We may take $V = D(f)$ and $U = X$ and $M = \mathcal{F}(X)$. Then from (14.2) it follows that $\mathcal{F}(D(f)) = M_f$ which shows that the canonical map $\beta : \widetilde{M} \to \mathcal{F}$ is an isomorphism over all distinguished open subsets, and therefore an isomorphism. Hence $\mathcal{F}$ is quasi-coherent.

To argue for the reverse implication, we may again assume $X = \operatorname{Spec} A$, $U = X$ and $V = \operatorname{Spec} B$. So suppose that $\mathcal{F}$ is quasi-coherent; that is, $\mathcal{F} = \widetilde{M}$ for some A-module M. Let $\iota \colon V \to X$ denote the inclusion map. We have $\iota^* \widetilde{M} = \widetilde{M}|_U \simeq \widetilde{M \otimes_A B}$. Taking global sections, this isomorphism turns into exactly the map (14.2), so we get our desired isomorphism.

Exercise 14.4 Suppose $\mathcal{F}$ is quasi-coherent as in Definition 14.8. Then locally, $\mathcal{F}$ sits in an exact sequence

$$\mathcal{O}_U^J \to \mathcal{O}_U^I \to \mathcal{F}|_U \to 0.$$

This means that for any $x \in X$, there is *some* affine open subscheme $V = \operatorname{Spec} A \subset U$ such that the sequence evaluated at V stays exact. In other words, there is an exact sequence

$$A^J \xrightarrow{\phi} A^I \to \mathcal{F}|_V \to 0.$$

Let $M = \operatorname{Coker} \phi$. We claim that $\mathcal{F}|_V \simeq \widetilde{M}$. But this is clear because the tilde-functor is exact, so both $\mathcal{F}_V$ and $\widetilde{M}$ is the cokernel of the map $\widetilde{\phi} : \mathcal{O}_X^J \to \mathcal{O}_X^I$.

Conversely, suppose that the condition above holds. That means that, for $x \in X$, there is some affine $U = \operatorname{Spec} A$ containing x such that $\mathcal{F}|_U = \widetilde{M}$ for some A-module A. We may choose a (possibly infinite) presentation of the module M

$$A^J \to A^I \to M \to 0.$$

Then again since $\sim$ is exact, we see that

$$\mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathcal{F}_U \to 0$$

which means that $\mathcal{F}$ is quasi-coherent.

Exercise 14.10

Proof: The only difference between X and X_{red} is the structure sheaf, so define g on the level of topological spaces by f. On the level of sheaves we find that (over any open $U \subseteq X$) the map $f^\#(U) : \mathcal{O}_X(U) \to f_* \mathcal{O}_Y(U)$ takes all nilpotents to zero, as Y is reduced. By the universal property of quotients there must exist a unique morphism of rings $g^\#(U) : \mathcal{O}_{X_{\text{red}}}(U) \to f_* \mathcal{O}_X(U)$ such that $f^\#(U) = g^\#(U) \circ r^\#(U)$. This gives the required morphism $(g, g^\#)$ of schemes. ❑

Exercise 14.11 The condition (i) is clearly necessary. If there is a sequence $Y \to Z \to X$, then there is a sequence of sheaves $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to f_*(\mathcal{O}_Y)$, which means that the map $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ factors through $\mathcal{O}_X/\mathcal{I}$, and so also (ii) holds.

Conversely, we define the map g on topological spaces by the inclusion (i). To define it on sheaves, we use the map $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$. This annihilates $\mathcal{I}$, so we thus get a map $\mathcal{O}_X/\mathcal{I} \to f_*(\mathcal{O}_Y) = g_*(\mathcal{O}_Y)$. This gives us the map $g : Y \to Z$ factoring f.

Exercise 14.13 Pick a non-empty open affine $U' = \operatorname{Spec} A$ (where A is an integral domain) and represent $\mathcal{F}|_{U'}$ as $\mathcal{F}|_{U'} = \widetilde{M}$ where M is finitely generated A- module. By a general principle in commutative algebra there is an $f \in A$ so that $M_f \simeq A_f^r$ for some non-negative integer r (it might be zero!!). Let $U = \operatorname{Spec} A_f$ and $\iota \colon U \to X$ the inclusion. Now, it holds that $\mathcal{H}' = \iota_* \mathcal{F}|_{\mathcal{U}} = \iota_* \mathcal{O}_U^r$ is quasi-coherent by Theorem 14.15 on page 208, and by the adjoint property, as in (13.3) on page 13.3, there are canonical maps $\alpha \colon \mathcal{F} \to \iota_*^* \mathcal{F} = \mathcal{H}'$ and $\beta \colon \mathcal{O}_X^r \to \iota_*^* \mathcal{O}_X^r = \mathcal{H}'$. Let $\mathcal{H}$ be $\operatorname{Im} \alpha + \operatorname{Im} \beta$. It is quasi-coherent being a subscheme of the quasi-coherent sheaf $\mathcal{H}'$ and being the quotient of a coherent sheaf it is coherent. Moreover, both α and β are isomorphisms when restricted to U, so their respective kernels and cokernels are supported in the proper closed subset $X \subset U$.

Solutions for exercises in Chapter 15

Exercise 15.2 Let $x_1, \ldots, x_r$ be degree one generators of R. Let $\alpha : R \to R' = \Gamma_*(\mathcal{O}_X)$, be the map above. It is clear that the map is injective: If $r \in R$ is an element so that $r/1 = 0$ over every $(R_f)_0$, then $r = 0$.

To show integrality, let $s \in R'$ be a homogeneous element of non-negative degree. By quasi-compactness, we can find an $n > 0$, so that $\alpha(x_i^n)s \in \alpha(R)$ for every i. R_m is generated by monomials in x_i of degree m, so $\alpha(R_m)s \subset \alpha(R)$ for m large (e.g., $m \geqslant kn$). Let $R^{\geqslant kn}$ be the ideal of R generated by elements of degree $\geqslant kn$. We have that $\alpha(R^{\geqslant kn})s \subset \alpha(R^{\geqslant kn})$. Moreover, since R is noetherian, $R^{\geqslant rn}$ is finitely generated, so applying the Cayley–Hamilton theorem, we get that s satisfies an integral equation over R. Hence R' is integral over R.

Solutions for exercises in Chapter 16

Exercise 16.1 We may assume that $X = \operatorname{Spec}(A)$, where A is Noetherian, and $F = \widetilde{M}$, where M is a finitely generated A-module. Let $x_1, \ldots, x_n$ be generators for M as an A-module. We have $\mathcal{E}_x = M_\mathfrak{p}$, for the prime ideal $\mathfrak{p} \subset A$ corresponding to $x \in X$. By assumption, $M_\mathfrak{p} \simeq A_\mathfrak{p}^r$ is free, so let $m_1, \ldots, m_r$ be a basis of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. We can write, in $M_\mathfrak{p}$:

$$x_i = \sum c_{ij} m_j$$

Clearing denominators, we see that some multiple $d_i x_i$ (with $d_i \in A - \mathfrak{p}$) is a linear combination of the elements m_i with coefficients in A. Let $s = d_1 \cdots d_r$, and consider the open subset $D(s) \subseteq X$. Now, s is invertible in A_s, so there is a surjective map $A_s^r \to M_s$. This is also injective, since any relation between the m_i in M_s must survive in $M_\mathfrak{p}$ (since $s \notin \mathfrak{p}$). Hence $M_s \simeq A_s^r$. It then follows that $F|_{D(s)} \simeq \widetilde{M_s} \simeq \mathcal{O}_X^r|_{D(s)}$, is free on the open neighbourhood $D(s)$ of x.

Exercise 16.2 Let $A = \prod_{i=0}^\infty \mathbb{Z}$. We may regard $M = \mathbb{Z}$ as an A-module, by embedding it as the 0-th component in $\prod_{i=0}^\infty \mathbb{Z}$. Thus M is projective, since $\mathbb{Z} \oplus \prod_{i=1}^\infty \mathbb{Z} = A$. However, M is not free, since A (and hence any free module) is uncountable. In this example $\operatorname{Spec} A$ is an infinite disjoint union of $\operatorname{Spec} \mathbb{Z}$'s; $\mathcal{F} = \tilde{M}$ restricts to $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}$ on one of these components and 0 on the others.

Exercise 16.7 i) This is a consequence of the Hilbert syzygy theorem

ii) Any locally free sheaf on $\mathbb{A}^n$ is trivial (the Quillen–Suslin theorem).

iii) The glueing condition on $X \backslash \{o_1, o_2\}$ (o_1, o_2 are the two origins) is an automorphism of the trivial line bundle on $A^2 \backslash \{o\}$, hence extends to an automorphism on A^2 by Hartog's Lemma. This implies that the initial vector bundle is trivial.

iv) The ideal sheaf $\mathcal{I}_x$ of a closed point $x \in X$ is not a quotient of a locally free sheaf (e.g., since it is not globally generated).

Solutions for exercises in Chapter 17

Solutions for exercises in Chapter 18

Exercise 18.4 (*Hints:*)

a) Consider $D(x)$ and compute the reduction of $R_{(x)}$.
b) X is covered by $U = \operatorname{Spec} k[\frac{y}{x}, \frac{w}{x}]/((\frac{w}{x})^2)$ and $V = \operatorname{Spec} k[\frac{x}{y}, \frac{z}{y}]/((\frac{z}{y})^2)$. Using the Čech complex, compute that

$$\dim_k H^0(X, \mathcal{O}_X) = n + 1.$$

Solutions for exercises in Chapter 19

Exercise 19.1 By property *i)* there is basis $\mathcal{B}$ for the topology of X consisting of open affine subsets all having $\mathcal{P}$. Let $V \subseteq X$ be an open affine subset. It can be covered by members of the basis $\mathcal{B}$ each of which in its turn can be covered by finitely many distinguished open subsets of V; hence there is a finite covering $\{D(f_i)\}$ of V with each $D(f_i)$ being contained in some U from $\mathcal{B}$. Now, $D(f_i) \cap U = D(f_i|_U)$, so if $D(f_i) \subseteq U$, it holds that $D(f_i) = D(f_i|_U)$, and by *i)* the distinguished open $D(f_i)$ has $\mathcal{P}$, and finally, since the $D(f_i)$'s cover V, property *ii)* gives that U has $\mathcal{P}$.

As to the latter, use induction on the number r of f_i's. Write $a_1 f_1 + \cdots + a_r f_r = 1$, and let $g = a_2 f_2 + \cdots + a_r f_r$. Then each $D(f_i g)$ with $i \geqslant 2$ is distinguished in $D(f_i)$ and hence has $\mathcal{P}$ by *i)*; on the other hand, they are also distinguished in $D(g)$ and cover $D(g)$. Hence $D(g)$ has $\mathcal{P}$ by induction, and U being the union of $D(f_1)$ and $D(g)$ has $\mathcal{P}$ by $r = 2$ case.

Exercise 19.2 We appeal to the previous exercise and shall verify that $f^{-1}(U)$ being affine, is a distinguished property. So we have to see that the two requirements are fulfilled. Number one is the easiest: if $f^{-1}(U) = \operatorname{Spec} B$ and $s \in \Gamma(U, \mathcal{O}_X)$, it holds true that $f^{-1}(D(s)) = B(f^\sharp(s))$.

The second requirements is more demanding. Let $U = \operatorname{Spec} A$ and let $\{D(s_1), D(s2)\}$ be a finite cover of U by two distinguished opens so that $f^{-1}(D(s_i)) = \operatorname{Spec} B_i$. A crucial observation is that $f^{-1}(D(s_1)) \cap f^{-1}(D(s_2)) = f^{-1}(D(s_1 s_2)) = \operatorname{Spec}(B_2)_{s_1}$. Now consider the "gluing sequence" associated to the covering $\{f^{-1}(D(s_1)), f^{-1}(D(s_2))\}$ and which computes the space $B = \Gamma(f^{-1}(U), \mathcal{O}_X)$:

$$0 \longrightarrow B \longrightarrow B_1 \times B_2 \longrightarrow B_{12}.$$

It is a sequence of A-modules, and the right hand map sends $(b_1, b_2) \to \iota_1 b_1 - \iota_2 b_2$ where $\iota_i \colon B_i \to B_{12}$ are the localization maps. When being localized in s_1, the sequence takes the form

$$0 \longrightarrow B_{s_1} \longrightarrow (B_1)_{s_1} \times (B_2)_{s_1} \longrightarrow (B_2)_{s_1}$$

and becomes split exact, hence it follows that $B_{s_1} = (B_1)_{s_1} = B_1$. It also holds true that $B = \Gamma(f^{-1}(U), \mathcal{O}_X)$, hence there is a map $X \to \operatorname{Spec} B$, inducing open embedding on $\operatorname{Spec} B_i$. Hence it is an isomorphism.

Exercise 19.5 If $g\colon T \to Y$ is a morphism, the inverse images $g^{-1}(U_i)$ form an open cover of T. Each restriction $f_T|_{f_T^{-1}g^{-1}U_i}$ being the pullback of f along the $g^{-1}(U_i) \to U_i \to S$, is a closed map since $f|_{f^{-1}(U_i)}$ is supposed to be proper. Hence f_T is closed since being closed is local on the target.

Exercise 19.7 If both $A[x] = \mathfrak{a}A[x]$ and $A[x^{-1}] = \mathfrak{a}A[x^{-1}]$ hold true, there are relations

$$1 = a_0 + a_1 x + \cdots + a_r x^r$$
$$1 = b_0 + b_1 x^{-1} + \cdots + b_s x^{-s}$$

where $a_r \neq 0$ and $a_s \neq 0$. We may assume $s \leqslant r$ and that r is minimal. Multiplying the second relation by $a_r b_s^{-1} x^r$ and subtracting from the first, we obtain a relation of degree less than r; contradiction. As to the second claim, consider the set Σ of local subrings of K dominating A. It is non-empty since A itself belongs to it and the union of an ascending chain of rings dominating A clearly dominates A. So there is a maximal one, say R in Σ. If R is not a valuation ring there is an $x \in K$ with neither x nor x^{-1} lying in R, by Chevalley's lemma, either $\mathfrak{m}_R R[x]$ or $\mathfrak{m}_R R[x^{-1}]$ is a proper ideal, say $\mathfrak{m}_r R[x]$. Then $\mathfrak{m}_R R[x] \cap R = \mathfrak{m}_R$ and by localizing we obtain a local domain strictly larger than R that dominates A.

Exercise 19.8

a) Let $u = ab^{-1}$ be an element in the intersection $\bigcap A_\mathfrak{p}$ not lying in A. Let $\mathfrak{a} = \{\, x \mid xa \in (b)\,\}$. Then clearly $(b) \subseteq \mathfrak{a}$. Moreover $a \notin (b)$ since $u \notin A$, so that a is not a zero-divisor of $A/(b)$, and hence there is a prime ideal $\mathfrak{p}$ associated to (b) not containing a. It is straightforward that $\mathfrak{a} \subseteq \mathfrak{p}$. Now, that $u \in A_\mathfrak{p}$ means that $u = cd^{-1}$ with $d \notin \mathfrak{p}$; that is, $ab^{-1} = cd^{-1}$. This gives $ad = bc$, and so $d \in \mathfrak{a}$ contradicting that $d \notin \mathfrak{p}$.
b) If a principal ideal $(a) \subseteq A$ is prime, the maximal ideal of $A_{(a)}$ will be generated by a, and hence $A_{(a)}$ being Noetherian is a dvr. So assume that (a) is not a prime ideal and let $\mathfrak{p}$ be associated to (a); because A Noetherian $\mathfrak{p} = (a : b)$ for some $b \notin (a)$. We contend that $\mathfrak{p}A_\mathfrak{p}$ is generated by ab^{-1}. It holds true that $ba^{-1}\mathfrak{p} \subseteq A$ so that also $ba^{-1}\mathfrak{m} \subseteq A_\mathfrak{p}$. If $ba^{-1}\mathfrak{m}$ were contained in $\mathfrak{m}$, $ba^{-1}\mathfrak{p} \subseteq A$ would be contained in $\mathfrak{p}$ and the element ba^{-1} would be integral over A; hence it would belong to A which it does not. We deduce that $ba^{-1}\mathfrak{m} = A_\mathfrak{p}$; or in other words, $\mathfrak{m} = (ab^{-1})$. Thus the maximal ideal of $A_\mathfrak{p}$ is principal, and since $A_\mathfrak{p}$ also is Noetherian, it is a dvr.
c) Combine *a*) and *b*).
d) Translate *c*) into geometry.

Exercise 19.9 It holds that $A \subseteq B$, and they have a common fraction field K . If $\mathfrak{p}$ is a height one prime ideal in A, the local ring $A_\mathfrak{p}$ is dvr because A is normal. Let $\mathfrak{q} \subseteq B$ be a prime ideal such that $\mathfrak{q} \cap A = \mathfrak{p}$, and let V be valuation ring in K with $B \subseteq V$ and $\mathfrak{m}_V \cap B = \mathfrak{q}$. Then, since $A \subseteq B \subseteq V$ and $\mathfrak{m}_V \cap A = \mathfrak{p}$, it holds that $A_\mathfrak{p} \subseteq V$; but dvr's are maximal rings in their fraction fields so that $A_{\mathfrak{p}} = V$. It follows by Hartog's therorem that $A \subseteq B \subseteq \bigcap A_{\mathfrak{p}} = A$; where the intersection extends over all primes of height one in A.

Exercise 19.12 Let K be an algebraically closed field of sufficiently heigh transcendence degree that it contains all the K_i's, and choose an embedding $K_i \subseteq K$ for each i. These embeddings give rise to sections σ_i of the projections from $\mathbb{A}^1_K \times_k \operatorname{Spec} K_i$ onto $\mathbb{A}^1_K$. Perform now the base change

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{A}^1_{\operatorname{Spec} K} & \longrightarrow & \operatorname{Spec} k, \end{array}$$

where the resulting A is the disjoint union $A = \bigcup_i \mathbb{A}^1 \times_k \operatorname{Spec} K_i$. Chose infinitely many (for instance countably many, not to run into cardinality issues) closed points x_i in $\mathbb{A}^1_K$ and consider $Z = \{\sigma_i(x_i)\}$ in A. It is closed, but maps to the non-closed set $\{x_i\}$ in $\mathbb{A}^1_K$, so f is not universally closed.

Exercise 19.13

a) If $\{(U_i, Z_i)\}_{i \in I}$ be an ascending chain in Σ and let $U = \bigcup_i U_i$ and $Z = \bigcup_i Z_i$. We contend that Z is discrete and closed in U. Indeed, let $x \in Z$; then for some ν, it holds that $x \in Z_\nu$, and as Z_ν is discrete, there is an open V in U_ν with $V \cap Z_\nu = \{x\}$. But then $V \cap Z = V \cap U_\nu \cap Z = V \cap Z_\nu = \{x\}$. This shows that Z is discrete. Similarly, if $x \in U \backslash Z$ for it will, for some index ν, hold that $x \in U_\nu$, and Z_ν being closed in U_ν, there is an open neighbourhood V of x in $U_\nu$not meeting Z_ν. But now $Z \cap U_\nu = Z_\nu$, so V does not meet Z either. Hence Z is closed in U, and (U, Z) belongs to Σ. By Zorn's lemma there is a maximal pair in Σ.

b) For a scheme X, let (U, Z) be a maximal pair as in previous point. If U is a proper subset, pick a point $x \in X$ not in U and let V be an open affine containing x. Then $U \cap V$ is a non-empty proper open subset of the affine V, and its complement has a closed point y. The set $Z \cup \{y\}$ is then discrete and closed in $V \cup U$, contradicting the maximality of (U, Z), and we may conclude that $U = X$. Finally, if every such Z is finite, X would be covered by finitely many U_i's and hence quasi-compact.

c) Let Z be as in b). The image $f(Z)$ is thus a union of closed points which is closed because f is universally closed, and as every closed subset of an affine scheme is quasi-compact, $f(Z)$ must be finite. Hence infinitely many of the members of Z map to a closed point x in Y, and we let Z' be the union of those. From the following commutative diagram and Proposition 19.14 above

$$\begin{array}{ccc} Z' & \xrightarrow{\iota} & X \\ {\scriptstyle f|_{Z'}}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} k(x) & \xrightarrow{i_x} & Y \end{array}$$

we deduce that the restriction $f|_{Z'}\colon Z' \to \operatorname{Spec} k(x)$ is universally closed, which is absurd in view of the previous exercise (Exercise 19.13).

d) Follows directly from the definition of a quasi-compact map.

Solutions for exercises in Chapter 20

Exercise 20.4 We'll do this by checking that the maximal ideals $\mathfrak{m}_p$ are principal for all $p \in X$ (which suffices by Proposition A.10 in Appendix A) So, take a point $(a,b) \in X$, it satisfies the relation

$$(y-b)(y+b) = x^3 - a^3 = (x-a)(x^2+ax+a^2).$$

Each maximal ideal $\mathfrak{m}_p$ is generated by $x-a$ and $y-b$. If $b \neq 0$, it holds that $y+b$ is invertible in $\mathcal{O}_{X,p}$ (the characteristic is not two) and (B) yields that $y-b = (x-a)(x^2 + ax + x^2)(y+b)^{-1}$. So $x-a$ generate $\mathfrak{m}_p$. If $b=0$ it holds that $3a^2 \neq 0 \neq$ (the characteristic is not three), and x^2+ax+a^2 is invertible in $\mathcal{O}_{X,p}$. So $x-a = y^2(x^2+ax+a^2)^{-1}$ there, and $\mathfrak{m}_p$ is generated by y.

Exercise 20.5

a) Let $P=(a,b)$. The function $f = x-a$ has divisor $\operatorname{div} f = P + \sigma(P)$. Indeed, if $b \neq 0$, then $x-a$ is a uniformazing parameter in both P and $\sigma(P)$. If $b=0$, it holds that $P = \sigma(P)$, and y is a uniformazing parameter. Moreover $x - a = (\text{ unit }) \cdot y^2$.

b) Let $g = y - \alpha x + \beta = 0$ be the equation of the line through the three points, then $\operatorname{div} g = P+Q+R$.

c) By replacing 'negative' contributions $-P$ with $\sigma(P)$, we may assume that $D = \sum n_i P_i$ with each $n_i > 0$. Induction on the degree $\sum n_i$. If $d=1$ we are done. If $d \geqslant 2$, any pair of the points P_i has a sum linearly equivalent to a prime divisor, and replacing the sum by the prime divisor, we reduce d by one.

Exercise 20.6 There is always an injective map $\Gamma(\operatorname{Spec} A, \mathcal{O}_X(D))_f \to \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X(D))$. Conversely, take $s \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X(D))$. Then there is an open set $U \subset \operatorname{Spec} A_f$ such that s is regular on U, so that $\operatorname{div} s + D \geqslant 0$ on $\operatorname{Spec} A_f$. This implies that $\operatorname{div} s + D \geqslant 0$ can fail only over $V(f) \subset \operatorname{Spec} A$. This means that

$$(\operatorname{div} f^n s + D) \geqslant 0$$

over $\operatorname{Spec} A$. Hence the map above is surjective.

Exercise 20.10 Using the exact sequence ?? we find that $\operatorname{Cl}(\mathbb{P}) = \mathbb{Z}D$. As explained in xxx, $\mathbb{P}$ is isomorphic to the projective cone over the rational normal curve $C \subset \mathbb{P}^d$ of degree d. Let $D \subset \mathbb{P}$ correspond to a line $\ell \simeq \mathbb{P}^1$ passing through the vertex. Clearly dD is Cartier, since $\mathcal{O}_{\mathbb{P}}(dD) \simeq \mathcal{O}_{\mathbb{P}^d}(1)|_{\mathbb{P}}$ is invertible. On the other hand we have $\mathcal{O}_{\mathbb{P}^n}(1)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on the line ℓ. If D is Cartier, then $\mathcal{O}_X(D)|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \in \mathbb{Z}$. However, this a would have to satisfy $da = 1$, which is a contradiction.

Exercise 20.11 (*The Picard group of the affine line with two origins*) Recall that X is obtained by gluing $U_0 = \operatorname{Spec} k[s]$ and $U_1 = \operatorname{Spec} k[t]$ along $U_{01} = \operatorname{Spec} k[s, s^{-1}] = \operatorname{Spec} k[t, t^{-1}]$

using the identification $t = s$. So much of the same argument applies: Given an invertible sheaf L on X, the restriction of it to each U_0, U_1, U_{01} must be trivial and over U_{01} we obtain and automorphism $\psi : \phi_1 \circ \phi_0^{-1} : \mathcal{O}_{U_{01}} \to \mathcal{O}_{U_{01}}$. Again this is induced by a map $k[s,s^{-1}] \to k[s,s^{-1}]$ which must be of the form $p(s,s^{-1}) = s^n$ for some $n \in \mathbb{Z}$. As for $\mathbb{P}^1_k$, the sheaves we obtain from s^n are non-isomorphic (e.g., since they have non-isomorphic $\Gamma(X,L)$). So we have $\mathrm{Pic}(X) = \mathbb{Z}$.

Solutions for exercises in Chapter 21

Solutions for exercises in Chapter 22

Exercise 22.1

a) Pick an element $\sum_i a_i \otimes b_i$ that belongs to I, which means that $\sum_i a_i b_i = 0$. Then

$$\textstyle\sum_i a_i \otimes 1(1 \otimes b_i - b_i \otimes 1) = \sum_i a_i \otimes b_i - (\sum_i a_i b_i \otimes 1) = \sum_i a_i \otimes b_i.$$

b) This is trivial: $b \otimes 1 - 1 \otimes b$ is a member of I.

c) Since the two B-module-structures coincide, one has

$$adb + bda = a \otimes 1(b \otimes 1 - 1 \otimes b) + 1 \otimes b(a \otimes 1 - 1 \otimes a) = ab \otimes 1 - 1 \otimes ab = d(ab).$$

d) Let $D\colon B \to M$ be an A-derivation and define an A-linear map $\alpha'\colon B \otimes_A B \to M$ by $a \otimes b \mapsto bD(a)$.
If we give $B \otimes_A B$ the B-module structure from the second factor, α' will be B-linear, and it vanishes on I^2; indeed, we have

$$(a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b) = ab \otimes 1 - a \otimes b - b \otimes a + 1 \otimes ab$$

and applying α' to this we obtain

$$D(ab) - bD(b) - aD(b) + abD(1) = 0.$$

By *a*) we infer that $\alpha'(I^2) = 0$. The map α' thus passes to the quotient and gives a map $\alpha\colon I/I^2 \to M$, which satisfies $\alpha(d(b)) = \alpha(b \otimes 1 - 1 \otimes b) = 1 \cdot D(b) - bD(1) = D(b)$, and we are through.

Exercise 22.3 We begin by checking that the choice of representative for bs^{-1} does not matter. So assume that $bs^{-1} = at^{-1}$; that is, $uas = ubt$ for some $u \in S$. Leibniz' rule gives

$$asD(u) + uaD(s) + usD(a) = btD(u) + ubD(t) + utD(b).$$

After having multiplied through by u, we may cancel the terms $uasD(u)$ and $ubtD(u)$, and we find the equality

$$u^2(aD(s) + sD(a)) = u^2(bD(t) + bD(t)).$$

Multiplied through by st it becomes

$$u^2(staD(s) + ts^2D(a)) = u^2(stbD(t) + st^2D(b)),$$

and after a slight reorganizing this gives

$$u^2s^2(tD(a) - aD(t)) = u^2t^2(sD(b) - bD(s)).$$

To see that the map defined by (22.3) abide by Leibniz' rule, we compute and find

$$\begin{aligned} s^2t^2d(at^{-1} \cdot bs^{-1}) &= D(ab)st - abD(st) = \\ &= staD(b) + stbD(a) - absD(t) - abtD(s) = \\ &= sa(tD(b) - bD(t)) + tb(sD(a) - aD(s)), \end{aligned}$$

from which the desired equality follows upon division by s^2t^2.

Exercise 22.4 The exercise is trivial if k is of characteristic two or three, so assume this is not the case. To see η is non-zero, consider $\Omega_{A/k[x]}$. On one hand, it equals $A/(2y)Ady = k[x,y]/(y, y^2 - x^3)dy = k[x]/(x^3)dy$, and on the other hand, it lives in the sequence

$$\begin{tikzcd} 0 \arrow[r] & k[x]dx \arrow[r] & \Omega_{A/k} \arrow[r] & \Omega_{A/k[x]} \arrow[r] \arrow[d, equal] & 0 \\ & & & k[x]/(x^3)dy & \end{tikzcd}$$

Our torsion element $\eta = 3ydx - 2xdy$ maps to $2xdy$ in $\Omega_{A/k[x]} = k[x]/x^3dy$, which is non-zero, and so must be non-trivial.

In fact, η generated the torsion part of $\Omega_{A/k}$: The torsion part maps injectively into $\Omega_{A/k[x]}$ since $k[x]dx$ is torsion free, so the torsion is bounded by $k[x]/x^3dy$. If say ξ is maps to dy, we have $dy = \xi - p(x)dx$, with $y\xi = 0$. Then $ydy = p(x)dx$ hence $(x^3 - ypx)dx = 0$ impossible since $x^3 - yp(x)$ can not be zero in A.

Exercise 22.6 The Kähler differentials live in the diagram with exact rows and columns

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d, "\simeq"] & K \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r, "\alpha"] & Adx \oplus Ady \arrow[r] \arrow[d, "\beta"] & \Omega_{A/k} \arrow[r] \arrow[d] & 0 \\ & & A \arrow[r, "\simeq"] & A & \end{tikzcd}$$

where $\alpha(a) = (f_x dx + f_y dy)a$ and $\beta(adx + bdy) = af_y - bf_x$. The lower left 'hook' in the diagram is the Koszul complex on the partial derivatives f_x and f_y tensorized by A. It follows that $H = \operatorname{Tor}_1^R(T, A)$ where $T = R/(f_x, f_y)$: indeed, since f does not have multiple components, the singularities are isolated. Hence f_x and f_y form a regular sequence, and T is resolved by the Koszul complex built on them. On the other hand, this Tor-module can be computed as the kernel of the 'multiplication-by-f-map' in the sequence

$$R/(f_x, f_y) \xrightarrow{f} R/(f_x, f_y) \longrightarrow R/(f, f_x, f_y) \longrightarrow 0,$$

and this kernel is the image in $R/(f_x, R_y)$ of the transporter $\{\, x \mid fx \in (f_x, f_y)\,\}$.

Finally, since $T = R/(f_x, f_y)$ is Artinian, multiplication by f in T is injective if and only if it is an isomorphism; that is, if and only if f is a unit mod (f_x, f_y), which is equivalent to $(f, f_x, f_y) = 1$.

Exercise 22.9

a) By induction on the number of generators, it suffices to consider the case $L = K(x)$. If x is transcendent the dimension $\Omega_{L/k}$ increases by one as does the transcendence degree, so we may assume that x is algebraic. Then $L = K[t]/(P(t))$, let $I = (P(t))$. There is a diagram

$$\begin{tikzcd} I/I^2 \arrow[r, "\delta"] & \Omega_{K[t]/k} \otimes_{K[t]} L \arrow[r] & \Omega_{L/k} \arrow[r] & 0 \\ & \Omega_{K/k} \otimes_K L \arrow[u, "\phi"] \arrow[ur, "\psi"'] & & \end{tikzcd}$$

where the upper row is the conormal sequence and ϕ is induced by the inclusion $K \subseteq K[t]$; *i.e.* it sends $d_K f \otimes 1$ to $d_{K[t]} f \otimes 1$. It holds that $\Omega_{K[t]/k} = \Omega_{K/k} dt$, hence $\dim_L \Omega_{K[t]/k} \otimes_{K[t]} L = \dim_L \Omega_{K/k} \otimes_K L + 1$. Now, I/I^2 is a one-dimensional vector space over L since I is principal. It follows that

$$\dim_L \Omega_{L/k} \geqslant \dim_L \Omega_{K[t]/k} \otimes_{K[t]} L - 1 = \dim_L \Omega_{K/k} \otimes_K L.$$

b) Assume then that $\dim_K \Omega_{K/k} = \operatorname{trdeg} K/k = r$, and let $x_1, \ldots, x_r$ be elements in K so that $dx_1, \ldots, dx_r$ is a basis for $\Omega_{K/k}$ over K. It follows that $\Omega_{K/k(x_1,\ldots,x_r)} = 0$, and hence K is separable and algebraic (*e.g.* by *a*)) over $k(x_1, \ldots, x_r)$. But since $\operatorname{trdeg} k(x_1, \ldots, x_r) = r$, the elements $x_1, \ldots, x_r$ are transcendent over k.

c) Finally, assume that k is perfect. Now $\delta[P]$ is the class of $dP \otimes 1$. One computes $dP(x) = P'(x)dx + \sum_i da_i x^i$. If $P'(x) \neq 0$, it follows that $dP \neq 0$. If $P'(x) = 0$, the element x is inseparable over K, but is not inseparabel over k, which is perfect, hence not all the coefficients of $P(t)$ lie in k: it follows that at least one $da_i \neq 0$, and hence $dP \neq 0$.

Exercise 22.13 Since M is finitely generated one may lift a basis for $M \otimes_k k$ and obtain a a sequence

$$0 \longrightarrow E \longrightarrow A^r \xrightarrow{\ \phi\ } M \longrightarrow F \longrightarrow 0$$

where $\phi \otimes k$ is an isomorphism and $r \dim_k M \otimes_A k$. Nakayama's lemma yields that ϕ is sutjecyive and hence $F = 0$. Tenzored with K sequence B becomes

$$0 \longrightarrow E \otimes_A K \longrightarrow K^r \longrightarrow M \otimes_A K \longrightarrow 0.$$

Now $\dim_K M \otimes_A K \geqslant r$, so $E \otimes_A K = 0$. Since A being an integral domain, is torsion free, it follows that $E = 0$.

Exercise 22.14 We may assume that $X = \operatorname{Spec} A$. Let K be the fraction field of A. By Exercise *c*) it holds that K is separably generated over k (which is perfect by hypo).

Hence $\dim \Omega_{K/k} = \operatorname{trdeg} K/k = \dim X$ and in view of $\Omega_{K/k} = \Omega_{A/k} \otimes_A K$, it holds that $\Omega_{X/k} \otimes_{\mathcal{O}_X} K$ is free of rank $\dim X$. A basis extends to a basis for $\Omega_{X/k}|_U$ for some open dense subset U. For $x \in U$. we have $\dim X = \dim_{k(x)} \Omega_{X/k} \otimes k(x) = \operatorname{trdeg}_k K \leqslant \dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim_{K(x)} \Omega_{K/k}$. By definition X is smooth at each $x \in U$, and it follows by Theorem 22.36 that x is regular.

Exercise 22.15 (*Hint:*)

Computing the local ring explicitly, shows that it is a domain, and every local ring with dimension 1 is a DVR hence regular.

Solutions for exercises in Chapter 23

Solutions for exercises in Chapter 24

Solutions for exercises in Chapter 24

Solutions for exercises in Chapter 25

Solutions for exercises in Chapter 26

Solutions for exercises in Chapter A

Exercise A.4 Consider the ring maps $v_n : \mathbb{Z}[[x]] \to \mathbb{Z}/p^n$ given by sending x to p. These maps are consistent with the maps $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$, so by the universal property of the inverse limit we get a map

$$\mathbb{Z}[[x]] \to \varprojlim \mathbb{Z}/n^k\mathbb{Z} = \mathbb{Z}_p.$$

The map is clearly surjective, and the kernel is indeed

$$\cap_{k=1}^{\infty}(x-p, x^k) = (x-p).$$

Hence we get the desired isomorphism $\mathbb{Z}[[x]]/(x-p) \simeq \mathbb{Z}_p$.

Exercise A.5 a) Let $G = \varinjlim_{i \in I} G_i$. Each G_j admits a map to G_i for each $i \geqslant j$, and these are consistent with the directed maps of the directed system. Hence we get for each j, an induced map $v_j : G_i \to G$. The maps v_j are compatible with the directed maps $G_j \to G_{j'}$ for $j' \geqslant j$. So by the universal property of $\varinjlim$, we get an induced map $\varinjlim_{j \in J} G_j \to G$.

b) We only need to construct an inverse to the map in a). Since J is cofinal, for each i, we have a map $G_i \to \varinjlim_{j \in J} G_j$ which is compatible with the directed maps $G_i \to G_{i'}$. Hence by the universal property, we get an induced map $\varinjlim_{i \in I} G_i \to \varinjlim_{j \in J} G_j$.

Exercise A.9 Let $G_i = \{0, 1, \ldots, i\}$ and define for each