Algebraic geometry 1
Exercise sheet 3

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Exercise 1.

1. Define

$$\pi^{-1} : U \longrightarrow \pi^{-1}(U)$$
$$(x_1, ..., x_n) \mapsto (x_1, ..., x_n)[x_1 : ... : x_n].$$

This is well-defined, because by definition of U, not all x_i can be zero at the same time, so $[x_1 : ... : x_n]$ is actually a point in projective space. We also have $(x_1, ..., x_n)[x_1 : ... : x_n] \in Z$ for $(x_1, ..., x_n) \in U$, because $x_i x_j = x_j x_i$ for all $1 \leq i, j \leq n$. To see injectivity of π^{-1}, let $(x_1, ..., x_n) \in U$ with $x_j \neq 0$. Then we have $y_j \neq 0$, because if we assume $x_j \neq 0$ and $y_j = 0$, then for some $y_i \neq 0$ (which exists since $[y_1 : ... : y_n]$ is a point in projective space) we have $0 \neq x_j y_i = x_i y_j = 0$. Therefore, we can just set $y_j = 1$. Then

$$x_i y_j = x_j y_i \implies y_1 = \frac{x_1 y_j}{x_j} = \frac{x_1}{x_j},$$

showing that all the y_i are fixed up to a scalar after fixing all the x_i.

2. Define

$$\phi : V_i \to \mathbb{A}^k_n$$
$$(x, y) \mapsto (\frac{x_1}{y_i}, \ldots, x_i, \ldots, \frac{x_n}{y_i}),$$

where the inverse map is given by

$$\phi^{-1} : \mathbb{A}^k_n \to V_i$$
$$(x_1, \ldots, x_n) \mapsto (x_1 x_i, \ldots, x_i, \ldots, x_n x_i)[x_1 : \cdots : x_{i-1} : 1 : \cdots : x_n].$$

Exercise 4.

1. Lets first prove that V_U are stable under intersections:

   Claim. Take $U, W \subseteq X$ open subsets. Then $V_{U\cap W} = V_U \cap V_W$.

   Proof of claim. Inclusion $V_{U\cap W} \subseteq V_U \cap V_W$ is clear.

   For the other inclusion take $Z \in V_U \cap V_W$. By definition $Z \cap U \neq \emptyset$ and $Z \cap V \neq \emptyset$. Suppose $Z \cap (U \cap V) = \emptyset$. Then $(Z \cap U)^c \cup (Z \cap V)^c = X$. But since Z is irreducible, and is covered by $U^c \cup V^c$, we must have (WLOG) $Z \subseteq U^c$. That is in contradiction with $Z \cap U \neq \emptyset$. $\square$(of claim)

   It also behaves well under unions:

   $$\begin{aligned} V_{U\cup W} &= \{Z \text{ cl. irred.} \mid Z \cap (U \cup W) \neq \emptyset\} \\ &= \{Z \text{ cl. irred.} \mid (Z \cap U) \neq \emptyset \text{ or } (Z \cap W) \neq \emptyset\} \\ &= \{Z \text{ cl. irred.} \mid (Z \cap U) \neq \emptyset\} \cup \{Z \text{ cl. irred.} \mid (Z \cap W) \neq \emptyset\} \\ &= V_U \cup V_W \end{aligned}$$

   and practically same argument applies to infinite unions.

   Therefore every open subset of X^{sob} can be written as V_U for some open $U \subseteq X$ (in general it could've been just a base of topology, but it is the whole topology).

   Claim. Closed irreducible subsets of X^{sob} are exactly V_U^c for open $U \subseteq X$ such that (closed) subset $U^c \subseteq X$ is irreducible.

   Proof of claim. Take V_U^c such that U^c is not irreducible. Then there exist closed subsets $U_1^c, U_2^c \subseteq X$ with $U^c = U_1^c \cup U_2^c$ meanwhile $U^c \neq U_1^c$ and $U^c \neq U_2^c$. Then $V_U = V_{U_1 \cup U_2} = V_{U_1} \cup V_{U_2}$ and we can thus cover V_U with V_{U_1} and V_{U_2}. We just have to show $V_U \neq V_{U_1}$ and $V_U \neq V_{U_2}$. Take $x \in U^c \setminus U_1^c$. Then $\overline{\{x\}} \in V_U^c \setminus V_{U_1}^c$, so $V_U \neq V_{U_1}$, which proves the claim. $\square$(of claim)

   Let us show X^{sob} is sober. Let V_U^c be closed irreducible. Then by last claim U^c is closed and irreducible. The set U^c is the generic point with $\overline{\{U^c\}} = V_U^c$. The inclusion $\overline{\{U^c\}} \subseteq V_U^c$ is obvious, because V_U^c contains the point U^c and is a closed set. For the other inclusion take a closed set that V_W^c that contains U^c. That means $U^c \cap W = \emptyset$ and thus $W \subseteq U$. Then we have $V_W \subseteq V_U$ and $V_U^c \subseteq V_W^c$. This proves that V_U^c is the closure of the point U^c.

2. Define

   $$\begin{aligned} h \colon X^{\text{sob}} &\to Z \\ W &\mapsto \text{unique generic point of } \overline{g(W)}. \end{aligned}$$

   Note that: a continuous image of an irreducible set is irreducible and the closure of an irreducible set is irreducible. So $\overline{g(W)} \subseteq Z$ is a closed irreducible subset and thus has a unique generic point in Z.

Let's now prove $g = h \circ f$. Take $x \in X$. We have to prove $g(x)$ is the unique generic point of $\overline{g\left(\overline{\{x\}}\right)}$. Clearly $g(x) \in \overline{g\left(\overline{\{x\}}\right)}$. Take any closed $W \subseteq Z$ with $g(x) \in W$. Then, by definition, $x \in g^{-1}(W)$. Because $g^{-1}(W)$ is closed, also $\overline{\{x\}} \subseteq g^{-1}(W)$. So $g\left(\overline{\{x\}}\right) \subseteq W$. But since W is closed we have $\overline{g\left(\overline{\{x\}}\right)} \subseteq W$. This proves that $g(x)$ is indeed a generic point of $\overline{g\left(\overline{\{x\}}\right)}$. So we have $g = h \circ f$.

To prove h is continuous we take an open set $U \subseteq Z$. we want to see that $h^{-1}(U)$ is open. Since $g^{-1}(U) = f^{-1}(h^{-1}(U))$ is open and f^{-1} induces a bijection of open sets, the set $h^{-1}(U)$ is open as well. So h is continuous.

We should also argue why h is unique. Take $h, h' \colon X^{\text{sob}} \to Z$ both continuous and satisfying $g = h \circ f = h' \circ f$. Pick any closed irreducible $W \subseteq X$. Suppose $h(W) \neq h'(W)$. WLOG there exists open $U \subseteq Z$ such that $h(W) \in U$ and $h'(W) \notin U$ (because requiring *unique* generic point implies T_0 property). Open sets $h^{-1}(U)$ and $h'^{-1}(U)$ therefore differ. Using one of the claims above, they are of the form $V_{U_1} = h^{-1}(U)$ and $V_{U_2} = h'^{-1}(U)$. So we have $W \in V_{U_1}$ and $W \notin V_{U_2}$. Then there exists $w \in W \cap U_1$, for which $\overline{\{w\}} \in V_{U_1}$ and $\overline{\{w\}} \notin V_{U_2}$. By definition $\overline{\{w\}} \in h^{-1}(U)$ and $\overline{\{w\}} \notin h'^{-1}(U)$ which means that $h(\overline{\{w\}}) \in U$ and $h'(\overline{\{w\}}) \notin U$. But that is a contradiction with assumption $g = h \circ f = h' \circ f$.

3.