Algebraic geometry 2
Exercise sheet 10

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Exercise 3.

1. Taking global sections we obtain a long exact sequence

$$0 \to F(X) \to F'(X) \to F''(X) \to H^1(X, F) \to H^1(X, F') \to \dots$$

We quickly see that dimensions satisfy

$$0 = \dim F(X) - \dim F'(X) + \dim F''(X) - \dim H^1(X, F) + \dim H^1(X, F') - \dots$$

which shows what we wanted to show.

We can prove this for example by induction. Clearly it holds for base cases. And the induction step: Suppose we have an exact sequence of k-vector spaces

$$0 \to C_0 \to \dots \to C_n \to 0$$

By induction hypothesis for

$$0 \to C_0 \to \dots \to C_{n-2} \to \operatorname{im}(d_{n-2}) \to 0$$

we get

$$0 = \dim C_0 - \dim C_1 + \dots \pm \dim C_{n-2} \mp \dim \operatorname{im}(d_{n-2})$$

Then just substitute

$$\dim C_{n-1} = \dim \operatorname{im}(d_{n-2}) + \dim \operatorname{coker}(d_{n-1}) = \dim \operatorname{im}(d_{n-2}) + \dim C_n$$

and we get what we want.

2. We started solving the exercise with $d \in \mathbb{N}$ in mind, so first solution is only valid for $d > 0$ (although we could probably somehow extend it).

We do induction on the sum $n + d$.

The base cases: When $n = 0$, then $\mathbb{P}^n_k = \operatorname{Spec}(k)$, so $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = k$ and all higher cohomologies vanish. Integer d here doesn't make a difference because there is no nontrivial line bundles on a point. So $\chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = 1$.

When $d = 0$ and $n > 0$ we are working with structure sheaf and in that case we know

$$H^q(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = \begin{cases} k & q = 0 \\ 0 & q > 0 \end{cases}$$

So $\chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = 1$.

The induction step: Let $i\colon V(x_n) \cong \mathbb{P}^{n-1}_k \to \mathbb{P}^n_k$ be the closed immersion. Then we have an exact sequence of sheaves on $\mathbb{P}^n_k$

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-1) \to \mathcal{O}_{\mathbb{P}^n_k} \to i_*\mathcal{O}_{\mathbb{P}^{n-1}_k} \to 0$$

Tensor this sequence with $\mathcal{O}_{\mathbb{P}^n_k}(d)$ to obtain

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(d-1) \to \mathcal{O}_{\mathbb{P}^n_k}(d) \to (i_*\mathcal{O}_{\mathbb{P}^{n-1}_k}) \otimes \mathcal{O}_{\mathbb{P}^n_k}(d) \to 0$$

The last term

$$(i_*\mathcal{O}_{\mathbb{P}^n_k})\otimes\mathcal{O}_{\mathbb{P}^n_k}(d) = i_*(\mathcal{O}_{\mathbb{P}^n_k}\otimes i^*\mathcal{O}_{\mathbb{P}^n_k}(d)) = i_*(\mathcal{O}_{\mathbb{P}^n_k}\otimes\mathcal{O}_{\mathbb{P}^{n-1}_k}(d)) = i_*\mathcal{O}_{\mathbb{P}^{n-1}_k}(d)$$

Now use the previous part of the exercise to obtain

$$\chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = \chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d-1)) + \chi(\mathbb{P}^n_k, i_*\mathcal{O}_{\mathbb{P}^{n-1}_k}(d))$$

The cohomology of the pushforward along a closed immersion is the same as the cohomology of the original sheaf, so the last term above is $\chi(\mathbb{P}^{n-1}_k, \mathcal{O}_{\mathbb{P}^{n-1}_k}(d))$ By induction hypothesis we obtain

$$\chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = \binom{n+d-1}{n} + \binom{n-1+d}{n-1} = \binom{n+d}{n}$$

which is what we needed to show.

At the end let us treat the case when $d < 0$. Recall the explicit cohomology groups that we calculated in the lecture 17:

$$H^q(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = \begin{cases} k[x_0, \dots, x_n]_d & q = 0 \\ (\frac{1}{x_0 \dots x_n} k[\frac{1}{x_0}, \dots, \frac{1}{x_n}])_d & q = n \\ 0 & \text{else} \end{cases}$$

Immediately we see that for $d < 0$ we have

$$\chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = (-1)^n H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$$

Now we just have to count the size of basis (monomials) in $(\frac{1}{x_0 \dots x_n} k[\frac{1}{x_0}, \dots, \frac{1}{x_n}])_d$.

If $d \in \{-n, \ldots, -1\}$, then there are no polynomials in $(\frac{1}{x_0 \ldots x_n} k[\frac{1}{x_0}, \ldots, \frac{1}{x_n}])_d$, so $\chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = 0$.

If $d < -n$, then there are $\binom{n+(-d-n-1)}{n}$ monomials in $(\frac{1}{x_0 \ldots x_n} k[\frac{1}{x_0}, \ldots, \frac{1}{x_n}])_d$. We see that

$$\begin{aligned}
\binom{n+(-d-n-1)}{n} &= \Pi_{i=1}^n \frac{-d-n-1+i}{i} \\
&= (-1)^n \ \Pi_{i=1}^n \frac{d+n+1-i}{i} \\
&= (-1)^n \ \Pi_{i=1}^n \frac{d+i}{i} \\
&= (-1)^n \ \binom{n+d}{n}
\end{aligned}$$

So

$$\chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = (-1)^n (-1)^n \ \binom{n+d}{n} = \binom{n+d}{n}$$

which is what we needed to show.

3. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2_k}(-d) \to \mathcal{O}_{\mathbb{P}^2_k} \to i_* \mathcal{O}_X \to 0$$

where $i \colon X \to \mathbb{P}^2_k$ is a closed immersion. Now taking cohomology we get

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \arrow[r] & H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \arrow[r] & H^0(\mathbb{P}^2_k, i_* \mathcal{O}_X) \arrow[dll] & \\
& H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \arrow[r] & H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \arrow[r] & H^1(\mathbb{P}^2_k, i_* \mathcal{O}_X) \arrow[dll] & \\
& H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \arrow[r] & H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \arrow[r] & H^2(\mathbb{P}^2_k, i_* \mathcal{O}_X) \arrow[r] & 0
\end{tikzcd}$$

We know that $H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k})$ and $H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k})$ are both 0. So $H^1(\mathbb{P}^2_k, i_* \mathcal{O}_X) = H^1(X, \mathcal{O}_X)$ is isomorphic to $H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d))$. And for the latter one we know $\dim_k H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) = \binom{2-d}{2} = \frac{(d-1)(d-2)}{2}$.

Exercise 4.

1. First thing to note: We know that colimits commute with stalks, so $\operatorname{colim} F_{i,x} \to (\operatorname{colim} F_i)_x$ is an isomorphism.

   Pick now $s \in F_i(U)$ that is in the kernel of the map Ψ_U. That means for every $x \in U$ the section s vanishes in $\operatorname{colim} F_{i,x} = (\operatorname{colim} F_i)_x$. So there exists i_x such that s vanishes in $F_{i_x,x}$. And then there also exists a neighbourhood $x \in U_x$ such that s vanishes in $F_{i_x}(U_x)$. By quasi-compactness we cover U with finitely many such U_x. Pick j to be the maximal i_x from the covering. Then s gets mapped to 0 by the transition map $F_i(U) \to F_j(U)$. So s is already equal to 0 in $\operatorname{colim}_i F_i(U)$. (I omitted transition maps, hopefully its clear what was meant. Otherwise it becomes messy.)

2. We again start with the observation that $\operatorname{colim} F_{i,x} \to (\operatorname{colim} F_i)_x$ is an isomorphism.

   Take $s \in (\operatorname{colim} F_i)(U)$. For every stalk we find i_x, U_x and $s_x \in F_{i_x}(U_x)$ such that the image of s_x by Ψ_U is equal to restriction of s to U_x. By quasi-compactness there is finite subcover. Now we need to glue these $s_x \in F_{i_x}(U_x)$. By our choice, the difference $s_x|_{U_x \cap U_y} - s_y|_{U_x \cap U_y}$ (viewing both inside $\operatorname{colim} F_i(U_x \cap U_y)$) is in the kernel of $\Psi_{U_x \cap U_y}$. Because intersection $U_x \cap U_y$ is quasi-compact, we have by previous part that $s_x|_{U_x \cap U_y} = s_y|_{U_x \cap U_y}$ (as elements of $\operatorname{colim} F_i(U_x \cap U_y)$). So there exists some $j_{x,y} \geq i_x, i_y$ such that $s_x = s_y \in F_{j_{x,y}}(U_x \cap U_y)$. Now just take the maximum $j = \max j_{x,y}$ over all pairs x, y and we get a section in $s' \in F_j(U)$ that gets mapped to s by (Ψ_U). (Again, sorry for leaving out transition maps.)

3. So we assume that the category of abelian sheaves on X has enough injectives. Let $F_i \to G_i$ be an injective embedding of directed systems and G_i injective sheaves.

   Let $H_i = \operatorname{coker}(F_i \to G_i)$ so we have an exact sequence

   $$0 \to F_i \to G_i \to H_i \to 0$$

   for every i. Taking finite limits commutes with filtered colimits, so we have an exact sequence

   $$0 \to \operatorname{colim} F_i \to \operatorname{colim} G_i \to \operatorname{colim} H_i \to 0.$$

   Lets now do induction. The base case was done in first two parts (spectral space is qcqs).

   So assume the statement holds for n.

   By injectivity $H^n(X, G_i) = 0$ for every $n > 0$ and $i \in I$. So $\operatorname{colim} H^n(X, G_i) = 0$ but only for $n > 0$. We get a diagram

   $$\begin{tikzcd}
   H^n(X, \operatorname{colim} G_i) \arrow[r] \arrow[d] & H^n(X, \operatorname{colim} H_i) \arrow[r] \arrow[d] & H^{n+1}(X, \operatorname{colim} F_i) \arrow[r] \arrow[d] & H^{n+1}(X, \operatorname{colim} G_i) \arrow[d] \\
   \operatorname{colim} H^n(X, G_i) \arrow[r] & \operatorname{colim} H^n(X, H_i) \arrow[r] & \operatorname{colim} H^{n+1}(X, F_i) \arrow[r] & \operatorname{colim} H^{n+1}(X, G_i)
   \end{tikzcd}$$

   We have $\operatorname{colim} H^{n+1}(X, G_i) = 0$ by injectivness and because $n + 1 > 0$.

Lets show that also $H^{n+1}(X, \operatorname{colim} G_i) = 0$. I dont know how to show this.

Assuming we've shown $H^{n+1}(X, \operatorname{colim} G_i) = 0$, we conclude by five lemma that $H^{n+1}(X, \operatorname{colim} F_i) \cong \operatorname{colim} H^{n+1}(X, F_i)$.