Algebraic geometry 1
Exercise sheet 5

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15. November 2023

Exercise 1.

1. We make a pushout of the diagram $U_1 \leftarrow V_1 \to U_2$, where $V_1 \to U_1$ is the inclusion and $V_1 \to U_2$ and composition of φ and inclusion.

   Let X be the pushout in terms of topological spaces and let $\alpha_1 \colon U_1 \to X$ and $\alpha_2 \colon U_2 \to X$ be the associated morphisms.

   We define a sheaf $\mathcal{O}_X$ in the following way. Take an open subset $Z \subseteq X$. Then $Z \cap \alpha_1(U_1) = Z_1$ and $Z \cap \alpha_2(U_2) = Z_2$ are an open cover of Z in X. Then let

1. Define

$$X := U_1 \coprod U_2 / \sim,$$

   where $x \sim y$ if $x = \varphi(y)$ and for $i \in \{1, 2\}$

$$\begin{aligned} \pi_i : U_i &\to X \\ x &\mapsto \bar{x}. \end{aligned}$$

   We can now give X the structure of a topological space by defining a subset $U \subset X$ to be open if $\pi^{-1}(U) \in$ are open in U_1.
   Notice, that π_i are homeomorphic onto open subsets of X. This will become important later. Next we want to define a structure sheaf on X that behaves well with restricting to U_i.
   For $U \subset X$ open, let

$$\begin{aligned} \mathcal{O}_X(U) := ker(\mathcal{O}_{U_1}(\pi^{-1}(U)) \oplus \mathcal{O}_{U_2}(\pi^{-1}(U)) &\to \mathcal{O}_{U_1}(\pi^{-1}(U) \cap U_1) \\ (x, y) &\mapsto x_{|\pi^{-1}(U) \cap V_1} - \varphi^\sharp(\pi_2^{-1}(U) \cap V_2)(y_{|\pi_2^{-1}(U) \cap V_2})), \end{aligned}$$

   where the substraction in the above term comes from the group structure of $\mathcal{O}_{U_1}(\pi^{-1}(U) \cap V_1)$. This is of course a group again, as the kernel of a ring map.

We conclude, that $(X, \mathcal{O}_X)$ is a scheme, because $X = \pi_1(U_1) \cup \pi_2(U_2)$ can be covered by affine schemes using the cover from U_1 and U_2 and since by construction of the structure sheaf $\mathcal{O}_{X|U_1} = \mathcal{O}_{x_i}$. Here we finally used, as promised, that π_i are homeomorphisms onto open subsets of X.