Combinatorial optimization
Exercise sheet 6

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Exercise 6.2. We have an undirected graph G and $T \subseteq V(G)$, with $|T| = 2k$.

We have to show that minimum cardinality T-cut in G equals maximum of $\min_i \lambda_{s_i,t_i}$ over pairings $T = \{s_1, t_1, \ldots, s_k, t_k\}$ where $\lambda_{s,t}$ denotes maximum number of pairwise edge-disjoint s-t-paths.

First show that every T-cut will be bigger than $\min_i \lambda_{s_i,t_i}$ for any pairing. Take a pairing $T = \{s_1, t_1, \ldots, s_k, t_k\}$ and a T-cut $C = \delta(X)$. The cut C has separate at least one pair (say $\{s_j, t_j\}$), otherwise $|X \cap T|$ would be even. And since there are at least $\min_i \lambda_{s_i,t_i}$ edge-disjoint s_j-t_j-paths, we must have $|C| \geq \min_i \lambda_{s_i,t_i}$.

This gives us inequality that minimum cardinality of a T-cut is greater or equal to the maximum of $\min_i \lambda_{s_i,t_i}$ over pairings T.

Now we have to show that this inequality is in fact equality.

Remember (from previous courses) that for vertices $s, t \in v(G)$, the number of pariswise edge-disjoint s-t-paths is equal to the cardinality of a minimum s-t-cut.

And for computing s-t-cuts we have Gomory-Hu trees, so let $u \equiv 1$ and let H be a Gomory-Hu tree for (G, u). Then $\lambda_{s,t} = \min_{e \in P_{s,t}} u(e)$, where $P_{s,t}$ is the (unique) s-t-path in H.

Then we use a theorem from the lectures, which stated that minimum capacity T-cut can be found among fundamental ones in Gomory-Hu tree.

Define a subset of edges $F = \triangle_{s,t \in T, s \neq t} P_{s,t}$, a symmetric difference over (unique) s-t-paths in Gomory-Hu tree over all pairs $\{s,t\} \subseteq T$ (at this point T is just a set, not a pairing).

Claim. For every edge in $e \in H$, the cut at edge e is a T-cut if and only if $e \in F$.

Proof of claim. Let $e \in H$. Removing an edge e, the tree H splits into two components, say C_1 and C_2. Let $|C_1 \cap T| = p$ and $|C_2 \cap T| = r$. Since $2k = p + r$, p and r have the same parity. Observe that e lies exactly on pr paths, exactly on those, for which elements of the pair come from different components.

Therefore: $e \in F \Leftrightarrow pr$ odd $\Leftrightarrow p$ odd $\Leftrightarrow e$ defines a T-cut. $\square$ (of claim)

So minimum cardinality T-cut can be found in F. Now we just have to find a pairing, such that all paths will be contained F. This also follows from the claim above: an edge $e \in E(H) \setminus F$ always splits the tree into two components, each of which contains even number of vertices from T. We can then use the claim on components and keep on removing edges in $E(H) \setminus F$, at each step all components having even number of elements from T.

Now take a pairing so that for each pair both elements lie in the same connected component of (H, F). The paths will all lie in F, therefore

$$\min_i \lambda_{s_i,t_i} \geq \min_{e \in F} u(e) = \text{min. cardinality } T\text{-cut}.$$