Combinatorial optimization
Exercise sheet 1

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Exercise 1.1. We can define a graph in the following way: Let $A = \{x_i\}_{i\in\mathbb{N}}$, $B = \{y_i\}_{i\in\mathbb{N}}$ and $E = \{\{x_i, y_{i+1}\} \mid i \in \mathbb{N}\} \cup \{\{x_i, y_1\} \mid i \in \mathbb{N}\}$. Clearly any subset of A or B satisfies Hall's condition.

Suppose it has a perfect matching M. Then $\{x_1, y_{i_0}\} \in M$ for some $i_0 \in \mathbb{N}$. Since vertex x_{i_0+1} is of degree 1 and his only neighbour is already matched, he cannot be covered. Therefore a perfect matching M cannot exist.

Exercise 1.2.

a) Suppose we have a bipartite graph $G = (A\dot{\cup}B, E)$, two subsets $A' \subseteq A$ and $B' \subseteq B$ and matchings $M_{A'}$ and $M_{B'}$ that cover A' and B' respectively.

   To create a matching that covers $A' \cup B'$ we can take the union $M_{A'} \cup M_{B'}$ and pick out a subset in the following way. First observe that the degree of every edge $v \in V(G)$ is in $\{0, 1, 2\}$, so the graph $G' = (A\dot{\cup}B, M_{A'} \cup M_{B'})$ is union of circles and paths.

   Because the graph is bipartite, circles are of even length and we can pick every second edge to get a matching that covers all edges in a circle.

   Regarding the paths, observe that edges alternate (w.r.t. coming from $M_{A'}$ or $M_{B'}$). Let $(e_i = \{v_i, v_{i+1}\})_{i=1,\dots,k-1}$ be one of the paths in G' (we assume this path to be the whole connected component in G').

   - If k is even, we can pick every second edge and get a perfect matching of the path.
   - If k is odd: first and last vertex must lie in the same half of (bipartite) graph, WLOG $v_1, v_k \in A$. Assume $v_1, v_k \in A'$. Since edges alternate, either e_1 or e_{k-1} lies in $M_{B'}$. WLOG $e_1 \in M_{B'}$. Because v_1 is covered by the matching $M_{A'}$, there must exist an edge $\{v_1, u\} \in M_{A'}$, which is a contradiction with assumption $v_1, v_k \in A'$. Therefore at least one of v_1, v_k does not lie in A' in which case we do not have to cover it. We can remove it and perfectly cover the remaining (even length) path.

b) Suppose for every non-empty $E' \subseteq E(G)$ we have $\tau(G - E') < \tau(G)$. We want to show $E(G)$ is a matching in G.

Let M be the maximum matching in bipartite graph G. By Königs theorem we have $\tau(G) = \nu(G)$. Take an edge $\{u, v\} \in E(G) \backslash M$. Matching M is still a maximum matching in $G - \{u, v\}$ and thus $\nu(G) = \nu(G - \{u, v\})$. But according to the assumption $\tau(G - \{u, v\}) < \tau(G) = \nu(G - \{u, v\})$, which cannot hold. Therefore $E(G)$ must be equal to M and thus a matching.