Combinatorial optimization
Exercise sheet 7

Solutions by: Anjana E Jeevanand and David Čadež

28. November 2023

Exercise 7.1. Hint already gives us the graph, we just have to prove it satisfies the requirements. So let (K_n, c) be a graph with weights $c(\{i, j\}) = \lambda_{i,j}$ (we use $\lambda_{i,j} = \lambda_{j,i}$ $\forall i, j \in K_n$ for this to be well-defined) and T be a maximum weight spanning tree in (K_n, c). Lets show local edge-connectivities in T are exactly $\lambda_{i,j}$.

Since T is a tree, local edge-connectivity for any pair of vertices is the minimum of weights of edges on the path between them.

Take $i, j \in T$.

Condition $\lambda_{i,k} \geq \min\{\lambda_{i,j}\lambda_{j,k}\}$ clearly implies $\lambda_{i,k} \geq \min_{e \in P_{i,k}} \lambda_e$, where $P_{i,k}$ is the edge set of the path between i and k. This already proves that local edge-connectivity for a pair i, j is smaller or equal to $\lambda_{i,j}$.

Now suppose the inequality would be strict. Let $\{k, l\} \in P_{i,j}$ be an edge on the path between i and j with $\lambda_{k,l} < \lambda_{i,j}$. Then we could simply replace $\{k, l\} \in T$ with $\{i, j\}$ and obtain a tree with strictly bigger weight, which contradicts our assumption that T is maximum weight.

Therefore local edge-connectivity is exactly $\lambda_{i,j}$ for every $i, j \in T$.

Exercise 7.3.

We can translate this problem (deciding whether perfect b-matching exists) to the problem of perfect matching using constructions we used in the lectures to prove theorems about b-matching polytopes. First we encode u into vertex bounds by subdividing the edges and setting b on the new middle vertices to be equal to the edge capacity of the edge that we subdivided. And secondly we have to somehow get a graph with $b \equiv 1$, which we do by replacing every vertex v in this graph by $b(v)$ new vertices and replace every edge $e = \{u, v\}$ by a full bipartite graph between set of vertices that replaced u and set of vertices that replaced v.

Notation: Let G be the original graph and $\tilde{G}$ the graph we get after these two transformations. We have vertices

$$V(\tilde{G}) = \{(v,i) \mid v \in V(G), i \in [b(v)]\} \cup \\ \cup \{((e,u),i),((e,v),i) \mid e = \{u,v\} \in E(G), i \in [u(e)]\}$$

and edges

$$E(\tilde{G}) = \{\{(v,i),((e,v),j)\} \mid v \in V(G), e \in \delta(v), i \in [b(v)], j \in [u(e)]\} \cup \\ \cup \{\{((e,u),i),((e,v),i)\} \mid e = \{u,v\} \in E(G), i \in [u(e)]\}$$

For $v \in G$, let

$$B_v = \{(v,i) \in \tilde{G} \mid i \in [b(v)]\}$$
$$D_{e,v} = \{((e,v),i) \in \tilde{G} \mid i \in [u(e)]\}$$

First we have to prove that there exists a perfect b-matching in G exactly when there exists a perfect matching in G.

Let M be a perfect matching in $\tilde{G}$. Then define $f\colon E(G) \to \mathbb{N}$ by $f(e) := |M \cap E_{\tilde{G}}(B_v, D_{e,v})|$, where $e = \{u,v\}$. Function f is well defined, since we clearly have $|M \cap E_{\tilde{G}}(B_u, D_{e,u}) = |M \cap E_{\tilde{G}}(B_v, D_{e,v})|$ by the construction and because M defines a perfect matching. Because $|D_{e,v}| = |D_{e,u}| = u(e)$, we have that $f(e) \leq u(e)$ for any $e = \{u,v\} \in E(G)$. And because every element of B_v is matched to some element in $D_{e,v}$ for some $e \in \delta(v) \subseteq G$, we have also $f(\delta(v)) = b(v)$.

Let now f be a perfect b-matching in G. We can define M as follows: for any $e = \{u,v\} \in G$, pick $f(e)$ edges in the set $E_G(B_v, D_{e,v})$ and $f(e)$ edges in the set $E_G(B_u, D_{e,u})$ and add all to M. Then there are $u(e) - f(e)$ unmatched vertices in $D_{e,v}$ and same number of unmatched vertices in $D_{e,u}$, so we can match them among each other (we can always do that, because the graph $\tilde{G}[D_{e,u} \cup D_{e,v}]$ is full bipartite). After doing so for every edge in $E(G)$, we have matched every element in B_v for every $v \in V(G)$, because $f(\delta(v)) = b(v)$ for every $v \in V(G)$.

We show that sets $X, Y \subseteq V(G)$ disjoint and violating the property described in the exercise exist if and only if there exists a set $Z \subseteq \tilde{G}$ violating the Tutte condition (in graph $\tilde{G}$).

Suppose first that there exist $X, Y \subseteq V(G)$ disjoint which violate the condition described in the exercise. We will to define a set $Z \subseteq V(\tilde{G})$, for which number of odd components in $G - Z$ will be greater than size of Z (failing Tutte condition). The expression in the exercise already hints what to take, namely

$$Z = \{(v,i) \in \tilde{G} \mid v \in X, i \in [b(v)]\} \cup \{((e,v),i) \in \tilde{G} \mid v \in Y, e \in \delta(v), i \in [u(e)]\}.$$

The cardinality of Z is $\sum_{v \in X} b(v) + \sum_{y \in Y} \sum_{e \in \delta(y)} u(e)$. It is not exactly same as the expression in the exercise, but the difference will be made up by new odd components, arising from vertices $(v,i) \in \tilde{G}$ for $v \in Y$ and from vertices $((e,u),i) \in \tilde{G}$ for $u \in X$.

By assumption there is an odd number of components C in $G - X - Y$ with $\sum_{v \in C} b(v) + \sum_{e \in E_G(V(C),Y)} u(e)$, each of which corresponds to an odd connected component in $\tilde{G} - Z$. The component $\tilde{C} \subseteq \tilde{G}$ it corresponds to, has

$$\sum_{v \in C} b(v) + \sum_{e \in E_G(V(C),Y)} u(e) + 2 \sum_{e \in E_G(V(C),X)} u(e)$$

vertices. The last summand is even, so it does not change the parity.

But as mentioned above, there are more odd components in $\tilde{G} - Z$ than there were in $G - X - Y$. Specificaly, every $(v,i) \in \tilde{G}$ for $v \in Y$ is an isolated vertex in $\tilde{G} - Z$, since $((e,v),i) \in Z$ for all $e \in \delta(v)$ and $i \in u(e)$. And, for every $e \in E_G(X,Y)$ we have isolated vertices $((e,u),i)$ for $u \in X$, since all $(u,i) \in Z$ and $((e,v),i) \in Z$ where $e = \{u,v\}$. Summing all odd components, we get that there are strictly more than

$$\sum_{v \in X} b(v) + \sum_{y \in Y} \left(\sum_{e \in \delta(y)} u(e) - b(y) \right) - \sum_{e \in E_G(X,y)} u(e) + \sum_{y \in Y} b(y) + \sum_{e \in E_G(X,y)} u(e)$$

of them. This expression simplifies to

$$\sum_{v \in X} b(v) + \sum_{y \in Y} \sum_{e \in \delta(y)} u(e).$$

But this is exactly the size of Z, so we've shown that Z violates Tutte condition. That means $\tilde{G}$ does not have a perfect matching and therefore G does not have a perfect b-matching.

Now assume G does not have a perfect b-matching. Therefore $\tilde{G}$ does not have a perfect matching. By Tutte's theorem there exists a set of vertices Z that violates Tutte condition.

We can assume the following: If $(v,i) \in Z$ for some $i \in [b(v)]$ then $B_v \subseteq Z$. We can assume this, because if $(v,j) \notin Z$, then removing all (v,i) from Z does not join any components that were previously disconnected. Though it may change an odd component into an even component, that is mitigated by removing a vertex from Z.

Then we can define

$$X = \{v \in V(G) \mid (v,i) \in Z \text{ for some } i \in [b(v)]\}$$

and

$$Y = \{v \in V(G) \mid (v,i) \text{ isolated vertex in } \tilde{G} - Z \text{ for some } i \in [b(v)]\}.$$

Note that $(v,i) \in \tilde{G}$ is an isolated vertex in $\tilde{G} - Z$ for some $i \in b(v)$ if and only if (v,j) are isolated vertices in $\tilde{G} - Z$ for all j. Clearly both are equivalent to $D_{e,v} \subseteq Z$ for all $e \in \delta(v) \subseteq G$.

Now we have to show is that these X and Y fail the condition described in the exercise.

Let C be a connected component in $\tilde{G} - Z$ with $|C| > 1$. Let $C' \subseteq V(G)$ defined as $C' = \{v \in V(G) \mid B_v \subseteq C\}$.

We can make the following assumptions about form of C (modifying Z if necessary):

- By earlier assumption we have that if $(v, i) \in C$ for some $i \in [b(v)]$, then $B_v \in C$.
- Let $e = \{u, v\} \in E_G(V(C), X)$ (with $v \in X$). Then we can assume that $D_{e,v} \cap Z = D_{e,u} \cap Z = \emptyset$. If not, we could remove all of them and not decrease the difference $q_{\tilde{G}}(Z) - |Z|$. We only have to check the case when $D_{e,u} \subseteq Z$, in which case all elements of $D_{e,v} \setminus Z$ are isolated vertices in $\tilde{G} - Z$. Removing $D_{e,v}$ from Z therefore decreases $q_{\tilde{G}}(Z)$ by at most $u(e)$ and Z by at least $u(e)$.
- Let $e = \{u, v\} \in E_G(V(C), Y)$ (with $v \in Y$). Then by definition $D_{e,v} \subseteq Z$. We can further assume $D_{e,u} \cap Z = \emptyset$, removing them if necessary. Since (u, i) are not isolated in $\tilde{G} - Z$, we do not join any (previously disconnected) components, but we may make the component C not be of odd parity anymore, which is mitigated by removing a vertex from Z.
- Let $e = \{u, v\} \in E(G[C'])$. Then we can assume $D_{e,v} \cap Z = D_{e,u} \cap Z = \emptyset$. Clearly removing these vertices does not join any (previously disconnected) components, since all neighbors of these vertices are contained in component C. But same as before, removing vertices $D_{e,v}$ and $D_{e,u}$ may change the parity of component C, but that is again mitigated by removing a vertex from Z.
- Let $e = \{u, v\} \in E(G)$ with $B_u \subseteq C$ and $B_v \subseteq E$ for some other component E with $|E| > 1$. Then we can remove $D_{e,u}$ and $D_{e,v}$. This way we connect two previously distinct components and decrease $q_{\tilde{G}}(Z)$ by at most 2 and we decrease Z by at least 2.

  There is a corner case when $u(e) = 1$ and Z contains exactly one of $D_{e,u}$ or $D_{e,v}$. In that case we component we get afterwards is odd, so the difference $q_{\tilde{G}}(Z) - |Z|$ does not decrease.

Because of the last assumption, every connected component in $G - X - Y$ corresponds to exactly one connected component C in $\tilde{G} - Z$ with $|C| > 1$. For any edge $e = \{u, v\} \in E(G - X - Y)$ the sets B_v and B_u are contained in the same connected component of $\tilde{G} - Z$.

So after these assumptions components C have a much simpler form, while Z still violates Tutte condition. Let C be a connected component with $|C| > 1$ and $C' \subseteq V(G)$ defined as $C' = \{v \in V(G) \mid B_v \subseteq C\}$.

Then calculate

$$|V(C)| = \sum_{v \in C'} b(v) + 2 \sum_{e \in E(G[C'])} u(e) + 2 \sum_{e \in E_G(C', X)} u(e) + \sum_{e \in E_G(C', Y)} u(e)$$

Therefore component C is odd exactly when $\sum_{v\in C'} b(v) + \sum_{e\in E_G(C',Y)} u(e)$ is odd.

The number of connected components in $G - X - Y$ with $\sum_{v\in C'} b(v) + \sum_{e\in E_G(C',Y)} u(e)$ odd is strictly more than

$$|Z| - \sum_{y\in Y} b(y) - \sum_{e\in E_G(X,Y)} u(e)$$

Calculating cardinality of Z (using all of the above assumptions):

$$|Z| = \sum_{v\in X} b(v) + \sum_{y\in Y} \sum_{e\in\delta(y)} u(e)$$

Joining these two equations gives us that the number of connected components C' in $G - X - Y$ with $\sum_{v\in C'} b(v) + \sum_{e\in E_G(C',Y)} u(e)$ odd is strictly more than

$$\sum_{v\in X} b(v) + \sum_{y\in Y} \sum_{e\in\delta(y)} (u(e) - b(y)) - \sum_{e\in E_G(X,Y)} u(e).$$

This finished the proof that existance of a perfect b-matching is equivalent to condition in the exercise being satisfied for every disjoint $X, Y \subseteq V(G)$.