# Combinatorial optimization
# Exercise sheet 2

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**Exercise 2.1.**
**Claim.** The number of ears in an odd decomposition of a graph is uniquely determined by the following value:

$$\#\text{ of ears} = 1 + \frac{1}{2}\sum_{v\in V(G)}(\deg(v) - 2).$$

**Proof of claim.** With induction on the number of ears. Statement holds if $G$ is an odd circuit. Let $G = (\{r\}, \emptyset) + P_1 + \ldots + P_k$ be an odd ear decomposition. For subgraph $G' = (\{r\}, \emptyset) + P_1 + \ldots + P_{k-1}$ then holds

$$k - 1 = 1 + \frac{1}{2}\sum_{v\in V(G')}(\deg(v) - 2).$$

By adding $P_k$ to $G'$ we add only vertices of degree 2 (w.r.t. graph $G$), but we do increase the degree of 2 vertices (where path $P_k$ connects to $G'$) by 1. Therefore

$$\frac{1}{2}\sum_{v\in V(G)}(\deg(v) - 2) = 1 + \frac{1}{2}\sum_{v\in V(G')}(\deg(v) - 2) = k$$

which proves the claim. □ (of claim)

Claim proves that the number of ears is uniquely determined by the graph.

**Exercise 2.4.** We have a graph $G$, with $n := |V(G)|$ even, and for any set $X \subseteq V(G)$ with $|X| \leq \frac{3}{4}n$ we have

$$\left|\bigcup_{x\in X}\Gamma(x)\right| \geq \frac{4}{3}|X|. \tag{1}$$

We have to prove that $G$ has a perfect matching.

Suppose it does not have a perfect matching. Then by Tutte's theorem there exists a set $S \subseteq V(G)$ that violates Tutte condition, i.e. $q_G(S) > |S|$.

First make the following simplification: let the set $S$ be maximal, in the sense that there exists no other $S' \supseteq S$ with $q_G(S') > |S'|$. A direct consequence of that is that $G \setminus S$ does not have even connected components. If it did have an even connected component $C$, we could take any $v \in C$ and define $S' := S \cup \{v\}$. Then $C \setminus \{v\}$ would contain at least one odd connected component, simply because $C \setminus \{v\}$ has odd number of vertices. This will be useful when considering the case $|S| < \frac{n}{4}$.

- First consider the case when $|S| \geq \frac{n}{4}$. Define the set

$$I = \{v \in G \setminus S \mid v \text{ isolated in } G \setminus S\}.$$

Using condition 1 on the set $G \setminus S$ we get

$$\frac{4}{3}|G \setminus S| \leq \left| \bigcup_{x \in G \setminus S} \Gamma(x) \right|. \tag{2}$$

Observe that

$$\bigcup_{x \in G \setminus S} \Gamma(x) \subseteq G \setminus I,$$

because elements of $I$ are by definition isolated in $G \setminus S$ and therefore have no neighbors in $G \setminus S$. Therefore

$$\left| \bigcup_{x \in G \setminus S} \Gamma(x) \right| \leq |G \setminus I| = n - |I|. \tag{3}$$

We input equation 3 in equation 2 to get

$$\frac{4}{3}(n - |S|) \leq n - |I| \tag{4}$$

and thus

$$|I| \leq \frac{4}{3}|S| - \frac{1}{3}n. \tag{5}$$

Now we count all vertices in $G$ in the following way

$$|I| + 3\left(q_G(S) - |I|\right) + |S| \leq n. \tag{6}$$

Here we summed isolated vertices ($|I|$), vertices contained in odd components of $G \setminus S$ (using that their size is at least 3 and there is at least $q_G(S) - |I|$ of them) and elements in the set $S$.

We use equation 5 and inequality $q_G(S) > |S|$ to calculate

$$\begin{aligned}
|I| + 3\left(q_G(S) - |I|\right) + |S| &\leq n \\
-2|I| + 3q_G(S) + |S| &\leq n \\
-2\left(\frac{4}{3}|S| - \frac{1}{3}n\right) + 3q_G(S) + |S| &\leq n \\
-2\left(\frac{4}{3}|S| - \frac{1}{3}n\right) + 4|S| &< n \\
-\frac{8}{3}|S| + \frac{2}{3}n + 4|S| &< n \\
|S| &< \frac{n}{4}
\end{aligned}$$

which is contradiction with assumption $|S| \geq \frac{n}{4}$.

- Suppose now $|S| < \frac{n}{4}$. First prove the following claim.
  **Claim.** Let $G$ be a graph with $n := V(G)$ even. Assume it satisfies condition 1. Then for every set $T \subseteq V(G)$ with $|T| \leq \frac{n}{4}$ the subgraph $G \setminus T$ has no isolated vertices.
  **Proof of claim.** We suppose there exists a set $T$ with $|T| \leq \frac{n}{4}$ such that $G \setminus T$ has an isolated vertex. Fix any isolated vertex $v \in G \setminus T$.

  Consider cases when $n$ divisible by 4 and when it is not separately.

  - Let $n = 4m$ for some $m \in \mathbb{N}$. We will pick a subset $X \subseteq G \setminus T$ of exactly the size $3m$ (it exists because $T \leq \frac{n}{4}$). It fulfils the condition $|X| \leq \frac{3}{4}n = 3m$, and therefore by assumption

    $$\left|\bigcup_{x \in X} \Gamma(x)\right| \geq \frac{4}{3}|X|.$$

    But left side is at most $n-1$, because $v$ is not a neighbor of any element in $X$, and the right one is exactly $4m$. That would mean $n - 1 \geq 4m = n$, so we arrive at a contradiction.
  - Let $n = 4m + 2$ for some $m \in \mathbb{N}$. We pick a subset $X \subseteq G \setminus T$ of exactly the size $3m+1$ (it exists because $T \leq \frac{n}{4}$). Since $3m+1 \leq \frac{3}{4}n$, the set $X$ fulfils condition $|X| \leq \frac{3}{4}n$, and therefore we must have

    $$\left|\bigcup_{x \in X} \Gamma(x)\right| \geq \frac{4}{3}|X|.$$

    But left side is at most $n-1$, because $v$ is not a neighbor of any element in $X$, and the right one is exactly $\frac{4}{3}(3m+1) = 4m + \frac{4}{3}$. That would mean

    $$n - 1 \geq 4m + \frac{4}{3} = n - 1 + \frac{1}{3},$$

    so we arrive at a contradiction.

This proves the claim. $\square$ (of claim)

The claim therefore show that the subgraph $G \setminus S$ cannot have isolated vertices.

We prove another claim.

**Claim.** Assume current environment variables, mainly $|S| < \frac{n}{4}$. Then for every connected component $C$ of $G \setminus S$ we have

$$\frac{n}{4} - |S| < |C| - 1.$$

**Proof of claim.** Suppose the statement wouldn't hold. Let $C$ be a connected component in $G \setminus S$ with $\frac{n}{4} - |S| \geq |C| - 1$. Pick an element $v \in C$. Then the set $S' := S \cup (C \setminus \{v\})$ violates first claim, because $v$ by assumption does not have any neighbors in the set $G \setminus S'$ and is thus an isolated vertex in $G \setminus S'$, but $|S'| \leq \frac{n}{4}$ also holds. This proves the claim. $\square$ (of claim)

Note that $q_G(S) \geq |S| + 2$, because $q_G(S)$ and $|S|$ have same parity (i.e. $q_G(S) - |S| = 0 \mod 2$), which we argued during the lectures already when we proved Tutte's theorem. It is a direct consequence of $V(G)$ being even and all components being odd.

We consider cases when $n$ is divisible by 4 and when it is not separately.

Consider $n = 4m$ for some $m \in \mathbb{N}$. We make a simple estimate for amount of vertices in $G \setminus S$:

$$n - |S| \geq 3(|S| + 2)$$

which simplifies to $n \geq 4|S| + 6$ and further to $m \geq |S| + \frac{3}{2}$. Since all involved variables are natural numbers, we must have $m \geq |S| + 2$. We can now show that every connected component in $G \setminus S$ has to be at least of size 5. Using last claim we have that the value $\frac{n}{4} - |S| + 1 \geq |S| + 2 - |S| + 1 = 3$ must be strictly less than the size of any component. So all components must be of cardinality at least 5.

Therefore we have $n - |S| \geq 5(|S| + 2)$ (simply by giving rough lower bound for amount of vertices in $G \setminus S$).

Using all the things we calculated by now we can count vertices in $G$ once again to obtain

$$|S| + (|S| + 2)\left(\frac{n}{4} - |S| + 2\right) \leq n \tag{7}$$

where we sum $|S|$ and product of amount of connected components with lower bound for their size. Manipulating this inequality gives:

$$
\begin{aligned}
|S| + (|S|+2)\left(\frac{n}{4} - |S| + 2\right) &\leq n \\
|S| + |S|\frac{n}{4} - |S|^2 + 4 &\leq \frac{n}{2} \\
(|S|-2)\frac{n}{4} &\leq |S|^2 - |S| - 4 \\
\frac{n}{4} &\leq \frac{|S|^2 - |S| - 4}{|S|-2} \\
\frac{n}{4} &\leq |S| + 2 - \frac{|S|}{|S|-2}.
\end{aligned}
$$

Lets treat edge cases, since we divided by $|S|-2$ in the calculation, which could in general be non-positive.

- If $|S| = 0$, then $G$ would not be connected, which would clearly violate condition 1 by picking $X$ to be the smallest component.
- If $|S| = 1$, then using condition 1 on any connected component $C \subseteq G \setminus S$ yields $|C| \leq 3$. At the same time there are no isolated vertices in $G \setminus S$, so all connected components in $G \setminus S$ must be of size 3. And there are at least 3 connected components in $G \setminus S$, since $S$ violates Tutte condition. Pick $X$ to be a union of two components in $G \setminus S$, so $|X| = 6$. Then

  $$\frac{4}{3}|X| \leq |\bigcup_{x \in X} \Gamma(x)| \leq |X \cup S| = 7$$

  gives contradiction in the case $|S| = 1$.
- If $|S| = 2$, then equation 7 yields $2 \leq 0$.

Then we just plug in $n \geq 6|S| + 10$ which we calculated earlier to get

$$
\begin{aligned}
\frac{6|S|+10}{4} &\leq |S| + 2 - \frac{|S|}{|S|-2} \\
\frac{1}{2}|S| + \frac{1}{2} &\leq -\frac{|S|}{|S|-2}
\end{aligned}
$$

another contradiction, this time with existance of such $S$ in case when $n$ is divisible by 4.

Suppose now $n$ is not divisible by 4. It is even, so it must of the form $n = 4m + 2$ for some $m \in \mathbb{N}$. We again make an estimate for amount of vertices in $G \setminus S$.

$$
\begin{aligned}
n - |S| &\geq 3(|S|+2) \\
n &\geq 4|S| + 6
\end{aligned}
$$

which gives estimate $m \geq |S| + 1$. That is "worse estimate" than we got in case when $n$ was divisible by 4. But if there would exist a connected component in $G \setminus S$ with at least 5 elements, then we can get estimate $|S| + 2 \leq m$ as in the case when $n$ was divisible by 4. Concretely we get

$$\begin{aligned} |S| + 3(|S| + 1) + 5 &\leq 4m + 2 \\ |S| + \frac{3}{2} &\leq m. \\ |S| + 2 &\leq m \end{aligned}$$

If there are more than $|S| + 2$ connected components in $G \setminus S$, we also get estimate $|S| + 2 \leq m$ just like in the case when $n$ was divisible by 4, concretely

$$\begin{aligned} |S| + 3(|S| + 3) &\leq 4m + 2 \\ |S| + \frac{7}{4} &\leq m \\ |S| + 2 &\leq m. \end{aligned}$$

So in these two cases (when $G \setminus S$ has either more connected components than $|S|+2$ or it has one of size more than 5), we do exact same argument as when $n$ was divisible by 4. That argument did not use that $n$ is divisible by 4 from that point on. Therefore arrive at the contradiction with such $S$ existing and $n$ being of the form $4m + 2$ and $G \setminus S$ either having a component at least of size 5 or having strictly more than $|S|+2$ connected components.

Now we focus on the last case, if $G \setminus S$ only contains components of size 3 and exactly $|S| + 2$ of them. Then $n = |S| + 3(|S| + 2)$, so $m = |S| + 1$. Pick any connected component of the subgraph $G \setminus S$ and denote it with $C$. We use the condition 1 on the set $X := G \setminus (S \cup C)$. Note that $C$ does not have any neighbors in other connected components in $G \setminus S$, so

$$\left| \bigcup_{x \in X} \Gamma(x) \right| \leq n - 3$$

We also know $|X| = 3(|S| + 1)$. Using condition we get

$$\begin{aligned} \frac{4}{3}|X| &\leq n - 3 \\ 4(|S| + 1) &\leq n - 3 \\ 4m &\leq 4m + 2 - 3 \\ 4m &\leq 4m - 1 \end{aligned}$$

which contradicts the fact that $S$ violating Tutte condition with $|S| < \frac{n}{4}$ exists in $G$.

We covered all cases now, so there is no set $S$ that would violate Tutte condition, which means $G$ has a perfect matching by Tutte's theorem.