Combinatorial optimization
Exercise sheet 4

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Exercise 4.1. By greedily and heuristically looking at the graph we find the following matching $M = \{\{4, 15\}, \{1, 5\}, \{2, 6\}, \{7, 8\}, \{3, 10\}, \{9, 12\}, \{11, 16\}\}$. It has size 7, so we can try to show $\nu(G) = 7$. We can do that by finding a set X such that $q_G(X) = |X| + 2$. Again heuristically looking at the graph yields $X = \{1, 16, 8\}$, for which the graph $G \setminus X$ contains 5 odd components, namely $\{5\}, \{13\}, \{11\}, \{2, 6, 7\}, \{3, 9, 10, 12, 14\}$. So $\max_{X\subseteq V(G)}(q_G(X)-|X|) \geq 2$. But also, using Berge-Tutte formula and $\nu(G) \geq 7$, we have $\max_{X\subseteq V(G)}(q_G(X) - |X|) \leq 2$. So $\nu(G) = 7$ and M is maximum.

Exercise 4.2. I assume the exercise means we can use that fact that there exists and algorithm that finds maximal family of disjoint M-augmenting paths (in a general graph) and we do not have to make it.

So let $\epsilon > 0$ and G an indirected graph. WLOG assume $\frac{1}{\epsilon} = k \in \mathbb{N}$.

Start with $M = \emptyset$ and use the algorithm on G and M to find a family $\mathcal{P} = \{P_1, \dots, P_k\}$ of augmenting paths, such that after augmenting the matching, there are no more augmenting paths of that length in G. So in every loop, we strictly increasy the legths of path in $\mathcal{P}$ by at least 2.

Repeat this loop k times. The length of paths in the last step was at least $2k-1$. So now there is no M-augmenting paths of length $2k-1$ in G anymore (all, if any, are strictly longer).

Time complexity of this scheme is $\mathcal{O}(k(m + n))$ because we simply run a $\mathcal{O}(m + n)$ algorithm k times.

Now all that is left, is to prove this matching M is suficiently big, ie that $\frac{|M|}{\nu(G)} \geq 1 - \epsilon$. Indeed, take any maximum matching N in G and look at the graph $G' = (V(G), M \triangle N)$. Every cycle in G' contains same amount of edges from M as from N. And every path, say of length $2l + 1$, contains l edges from M and $l + 1$ from N, meaning the ratio $\frac{\#\text{ edges from } M}{\#\text{ edges from } N}$ is $\frac{l}{l+1}$. Using that paths in G' are at least of length $2k + 1$, we get that the ratio is at least $\frac{k}{k+1}$. The fraction $\frac{|M\setminus N|}{|N\setminus M|}$ is then clearly at least $\frac{k}{k+1}$ (since it is at least that much on every connected component). Then

$$\frac{k}{k+1} \leq \frac{|M \setminus N|}{|N \setminus M|} \leq \frac{|M| - |M \cap N|}{|N| - |M \cap N|} \leq \frac{|M|}{|N|}$$

So we have shown that $|M| \geq (1 - \frac{1}{k+1})\nu(G) \geq (1 - \epsilon)\nu(G)$, which we had to show.