Combinatorial optimization
Exercise sheet 7

Solutions by: Anjana E Jeevanand and David Čadež

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Exercise 7.1. Hint already gives us the graph, we just have to prove it satisfies the requirements. So let (K_n, c) be a graph with weights $c(\{i,j\}) = \lambda_{i,j}$ (we use $\lambda_{i,j} = \lambda_{j,i}\ \forall i,j \in K_n$ for this to be well-defined) and T be a maximum weight spanning tree in (K_n, c). Lets show local edge-connectivities in T are exactly $\lambda_{i,j}$.

Since T is a tree, local edge-connectivity for any pair of vertices is the minimum of weights of edges on the path between them.

Take $i, j \in T$.

Condition $\lambda_{i,k} \geq \min\{\lambda_{i,j}\lambda_{j,k}\}$ clearly implies $\lambda_{i,k} \geq \min_{e \in P_{i,k}} \lambda_e$, where $P_{i,k}$ is the edge set of the path between i and k. This already proves that local edge-connectivity for a pair i, j is smaller or equal to $\lambda_{i,j}$.

Now suppose the inequality would be strict. Let $\{k,l\} \in P_{i,j}$ be an edge on the path between i and j with $\lambda_{k,l} < \lambda_{i,j}$. Then we could simply replace $\{k,l\} \in T$ with $\{i,j\}$ and obtain a tree with strictly bigger weight, which contradicts our assumption that T is maximum weight.

Therefore local edge-connectivity is exactly $\lambda_{i,j}$ for every $i, j \in T$.

Exercise 7.3.

We can translate this problem (deciding whether perfect b-matching exists) to the problem of perfect matching using constructions we used in the lectures to prove theorem about b-matching polytopes. First we have to encode u into vertex bounds, by subdividing the edges and setting b on the new middle vertices to be equal to the edge capacity of the edge that we subdivided. And secondly we have to somehow get a graph with $b \equiv 1$, which we do by replacing every vertex v in this graph by $b(v)$ vertices and replace every edge $e = \{u, v\}$ by a full bipartite graph between set of vertices that replaced u and set of vertices that replaced v.

Notation: Let G be the original graph and $\tilde{G}$ the graph we get after these two transformations. We have vertices

$$V(\tilde{G}) = \{(v,i) \mid v \in V(G), i \in [b(v)]\} \cup \\ \cup \{((e,u),i),((e,v),i) \mid e = \{u,v\} \in E(G), i \in [u(e)]\}$$

and edges

$$E(\tilde{G}) = \{\{(v,i),((e,v),j)\} \mid v \in V(G), e \in \delta(v), i \in [b(v)], j \in [u(e)]\} \cup \\ \cup \{\{((e,u),i),((e,v),i)\} \mid e = \{u,v\} \in E(G), i \in [u(e)]\}$$

We show that sets $X, Y \subseteq V(G)$ disjoint and violating the property described in the exercise exist if and only if there exists a set $Z \subseteq \tilde{G}$ violating the Tutte condition (in graph $\tilde{G}$).

Suppose first that there exist $X, Y \subseteq V(G)$ disjoint which violate the condition described in the exercise. We want to define a set $Z \subseteq V(\tilde{G})$, for which number of odd components in $G - Z$ will be greater than size of Z (failing Tutte condition). The expression in the exercise already hints what to take. Namely we take:

$$Z = \{(v,i) \in \tilde{G} \mid v \in X, i \in [b(v)]\} \cup \{((e,v),i) \in \tilde{G} \mid v \in Y, e \in \delta(v), i \in [u(e)]\}.$$

This amounts to exactly $\sum_{v \in X} b(v) + \sum_{y \in Y} \sum_{e \in \delta(y)} u(e)$ vertices. It is not exactly like the expression in the exercise, but the difference will be made up by new odd components (singletons even), arising from vertices $(v,i) \in \tilde{G}$ for $v \in Y$ and from vertices $((e,u),i) \in \tilde{G}$ for $u \in X$.

By assumption there is an odd number of components C in $G - X - Y$ with $\sum_{v \in C} b(v) + \sum_{e \in E_G(V(C),Y)} u(e)$, each of which corresponds to an odd connected component in $\tilde{G} - Z$. The component $\tilde{C} \subseteq \tilde{G}$ it corresponds to, has

$$\sum_{v \in C} \sum_{i \in [b(v)]} 1 + \sum_{e \in E_G(V(C),Y)} \sum_{i \in [u(e)]} 1 + 2 \sum_{e \in E_G(V(C),X)} \sum_{i \in [u(e)]} 1$$

vertices. The last summand is even, so it does not change the parity.

But as mentioned above, we actually get more odd components in $\tilde{G}$ than we had in G. Namely, every $(v,i) \in \tilde{G}$ for $v \in Y$ is an isolated vertex in $\tilde{G} - Z$, since all $((e,v),i) \in Z$ for $e \in \delta(v)$. And, for every $e \in E_G(X,Y)$ we have isolated vertices $((e,u),i)$ for $u \in X$, since all $(u,i) \in Z$ and $((e,v),i) \in Z$ where $e = \{u,v\}$. Summing all odd components, we get that there is strictly more than

$$\sum_{v \in X} b(v) + \sum_{y \in Y} \left(\sum_{e \in \delta(y)} u(e) - b(y) \right) - \sum_{e \in E_G(X,y)} u(e) + \sum_{y \in Y} b(y) + \sum_{e \in E_G(X,y)} u(e)$$

which simplifies to

$$\sum_{v \in X} b(v) + \sum_{y \in Y} \sum_{e \in \delta(y)} u(e).$$

But this is exactly the size of Z. So $\tilde{G}$ does not have a perfect matching and therefore G does not have a perfect b-matching.

Now assume G does not have a perfect b-matching. Therefore $\tilde{G}$ does not a perfect matching. So by Tutte's theorem we have a set of vertices Z that is violating Tutte condition. We will first make a series of assumptions about Z, effectively modifying it into a specific form, and construct X and Y from it.

If $(v,i) \in Z$, then $\forall j \in [b(v)]$: $(v,j) \in Z$. We can assume this, because if $(v,j) \notin Z$, then removing all (v,i) from Z does not join any components that were previously disconnected. Though it may change an odd component into an even component, that is mitigated by removing a vertex from Z.

Then we can define

$$X = \{v \in V(G) \mid (v,i) \in Z \text{ for some } i \in [b(v)]\}$$

and

$$Y = \{v \in V(G) \mid (v,i) \text{ isolated vertex in } \tilde{G} - Z \text{ for some } i \in [b(v)]\}.$$

Note that $(v,i) \in \tilde{G}$ (for some i) is isolated if and only if (v,j) is isolated for all j. Clearly both are equivalent to Z containing all vertices of the form $((e,v),\_) \in \tilde{G}$, where $e \in \delta(v) \subseteq G$. And that $(v,i) \in Z$ (for some i) if and only if $(v,j) \in Z$ for all j.

Now all we have to show is that these X and Y fail the condition described in the exercise.

Let C be a connected component in $\tilde{G} - Z$ with $|C| > 1$.

We can make the following assumptions about form of C (modifying Z a little if necessary):

- By earlier assumption we have that if $(v,i) \in C$ for some $i \in [b(v)]$, then $(v,j) \in C$ for all $j \in [b(v)]$.

- Let $e = \{u,v\} \in E_G(V(C), X)$ (with $v \in X$). Then we can assume that $((e,u),i), ((e,v),i) \in Z$ for all $i \in [u(e)]$. If not, we could simply remove all of them and not decrease the difference $q_{\tilde{G}}(Z) - |Z|$. In this case only interesting situation would be when Z would contain $((e,u),i)$ for all $i \in [u(e)]$, and we would thus have $((e,v),i)$ be an isolated vertex in $\tilde{G} - Z$ for all i. But losing $u(e)$ isolated vertices is mitigated by removing $u(e)$ elements from Z.

- Let $e = \{u,v\} \in E_G(V(C), Y)$ (with $v \in Y$). Then by definition $((e,v),i) \in Z$ for all $i \in [u(e)]$. And we can further assume $((e,u),i) \notin Z$ for all $i \in [u(e)]$, removing them if necessary. Since (u,i) are not isolated in $\tilde{G} - Z$, we do not join any (previously disconnected) components, but we may make the component C not be of odd parity anymore, which is mitigated by removing a vertex from Z.

- Let $e = \{u,v\} \in E(\tilde{G}[C])$. Then we can assume $((e,v),i), ((e,u),i) \notin Z$ for all $i \in [u(e)]$. Clearly removing these vertices does not join any (previously disconnected) components, since all neighbors of these vertices are contained in component C, more precisely: $N(((e,v),i)) \setminus Z \subseteq C$ and $N(((e,u),i)) \setminus Z \subseteq C$ for all i. But again, removing vertices $((e,v),i)$ and $((e,u),i)$ may change the parity of component, but that is again mitigated by removing a vertex from Z.

- Let $e = \{u,v\} \in E(G)$ with $(u,i) \in C$ and $(v,i) \in C'$ for some other component C' with $|C'| > 1$. Then we can remove $((e,v),i),((e,u),i) \in Z$ for all $i \in [u(e)]$. This way we connect two previously distinct components and thus lose at most 2 odd components. If we remove at least 2 vertices, thats no problem. But if there existed was only one $((e,w),i) \in Z$ ($w \in \{u,v\}$), then after removing $((e,w),i)$ still have 1 odd component. So in no cases do we decrease the difference $q_{\tilde{G}}(Z) - |Z|$. That means two vertices $u,v \in V(G-X-Y)$ are in the same component exactly when (u,i) and $(v,j$ are in the same component in $\tilde{G} - Z$ for any pair i,j.

Because of the last assumption, we know that every connected component in $G-X-Y$ corresponds to exactly one connected component C in $\tilde{G}-Z$ with $|C| > 1$.

So after these assumptions components C have a much simpler form, while still violating Tutte condition in $\tilde{G}$. Let C be a connected component with $|C| > 1$ (avoid those components that come from vertices of Y and vertices from edges between X and Y). Then we calculate

$$|V(C)| = \sum_{\substack{v \in V(G) \\ (v,i) \in C}} b(v) + 2 \sum_{\substack{e=\{u,v\} \in E(G) \\ (u,i),(v,i) \in C}} u(e) + 2 \sum_{\substack{e=\{u,v\} \in E(G) \\ (u,i) \in C,(v,i) \in Z}} u(e) + \\ + \sum_{\substack{e=\{u,v\} \in E(G) \\ (u,i) \in C, v \in Y}} u(e)$$

Therefore component C is odd exactly when $\sum_{\substack{v \in V(G) \\ (v,i) \in C}} b(v) + \sum_{\substack{e=\{u,v\} \in E(G) \\ (u,i) \in C, v \in Y}} u(e)$ is odd.

Since connected components C' in $G-X-Y$ correspond bijectively to connected components C in $\tilde{G}-Z$ with $|C| > 1$.

So number of connected components in $G-X-Y$ with $\sum_{\substack{v \in V(G) \\ (v,i) \in C}} b(v) + \sum_{\substack{e=\{u,v\} \in E(G) \\ (u,i) \in C, v \in Y}} u(e)$ odd is strictly more than

$$|Z| - \sum_{y \in Y} b(y) - \sum_{e \in E_G(X,Y)} u(e)$$

Calculating cardinality of Z:

$$|Z| = \sum_{v \in X} b(v) + \sum_{y \in Y} \sum_{e \in \delta(y)} u(e)$$

Here we use all above assumptions concerning components.

Join these two equations and get that number of connected components in $G - X - Y$ with $\sum_{\substack{v \in V(G) \\ (v,i) \in C}} b(v) + \sum_{\substack{e=\{u,v\} \in E(G) \\ (u,i) \in C, v \in Y}} u(e)$ odd is strictly more than

$$\sum_{v \in X} b(v) + \sum_{y \in Y} \sum_{e \in \delta(y)} (u(e) - b(y)) - \sum_{e \in E_G(X,Y)} u(e).$$

This finished the proof that existance of a perfect b-matching is equivalent to condition in the exercise being satisfied for every disjoint $X, Y \subseteq V(G)$.