Combinatorial optimization
Exercise sheet 6

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Exercise 6.2. We have an undirected graph G and $T \subseteq V(G)$, with $|T| = 2k$.

We have to show that minimum cardinality T-cut in G equals maximum of $\min_i \lambda_{s_i,t_i}$ over pairings $T = \{s_1, t_1, \ldots, s_k, t_k\}$ where $\lambda_{s,t}$ denotes maximum number of pairwise edge-disjoint s-t-paths.

First show that every T-cut will be bigger than $\min_i \lambda_{s_i,t_i}$ for any pairing. Take a pairing $T = \{s_1, t_1, \ldots, s_k, t_k\}$ and a T-cut $C = \delta(X)$. The cut C has separate at least one pair (say $\{s_j, t_j\}$), otherwise $|X \cap T|$ would be even. And since there are at least $\min_i \lambda_{s_i,t_i}$ edge-disjoint s_j-t_j-paths, we must have $|C| \geq \min_i \lambda_{s_i,t_i}$.

This gives us inequality that minimum cardinality of a T-cut is greater or equal to the maximum of $\min_i \lambda_{s_i,t_i}$ over pairings T.

Now we have to show that this inequality is in fact equality.

Remember (from previous courses) that for vertices $s, t \in v(G)$, the number of pariswise edge-disjoint s-t-paths is equal to the cardinality of a minimum s-t-cut.

And for computing s-t-cuts we have Gomory-Hu trees, so let $u \equiv 1$ and let H be a Gomory-Hu tree for (G, u). Then $\lambda_{s,t} = \min_{e \in P_{s,t}} u(e)$, where $P_{s,t}$ is the (unique) s-t-path in H.

Then we use a theorem from the lectures, which stated that minimum capacity T-cut can be found among fundamental ones in Gomory-Hu tree.

Define a subset of edges $F = \triangle_{s,t \in T, s \neq t} P_{s,t}$, a symmetric difference over (unique) s-t-paths in Gomory-Hu tree over all pairs $\{s, t\} \subseteq T$ (at this point T is just a set, not a pairing).

Claim. For every edge in $e \in H$, the cut at edge e is a T-cut if and only if $e \in F$.

Proof of claim. Let $e \in H$. Removing an edge e, the tree H splits into two components, say C_1 and C_2. Let $|C_1 \cap T| = p$ and $|C_2 \cap T| = r$. Since $2k = p + r$, p and r have the same parity. Observe that e lies exactly on pr paths, exactly on those, for which elements of the pair come from different components.

Therefore: $e \in F \Leftrightarrow pr$ odd $\Leftrightarrow p$ odd $\Leftrightarrow e$ defines a T-cut. $\square$ (of claim)

So minimum cardinality T-cut can be found in F. Now we just have to find a pairing, such that all paths will be contained F. This also follows from the claim above: an edge $e \in E(H) \setminus F$ always splits the tree into two components, each of which contains even number of vertices from T. We can then use the claim on components and keep on removing edges in $E(H) \setminus F$, at each step all components having even number of elements from T.

Now take a pairing so that for each pair both elements lie in the same connected component of (H, F). The paths will all lie in F, therefore

$$\min_i \lambda_{s_i,t_i} \geq \min_{e \in F} u(e) = \text{min. cardinality } T\text{-cut.}$$

Exercise 6.3. Let $R = V(G) \setminus (S_e \cup S_o)$.

First consider the existence of a solution. A solution exists exactly when there exists some T with $S_o \subseteq T \subseteq S_o \cup R$ such that there exists a T-join in G. Existence of a T-join is equivalent (as we showed in the lectures) to each connected component of G containing even number of vertices from T. Putting these two together, we get that a solution exists exactly when for every connected component C of G one of the following holds

- $|C \cap S_o|$ odd, or
- $|C \cap S_o|$ even and $|C \cap R| > 0$.

In the first case we have a $(C \cap S_o)$-join in component C and in the second case we have a $((C \cap S_o) \cup \{r\})$-join, where $r \in C \cap R$ any.

Next we contract R into a single vertex, call it r, and define edge weights in the following way:

$$c'(e) = \begin{cases} \min_{u' \in R} c(\{u', v\}) & \text{if } e = \{r, v\} \\ c(e) & \text{else} \end{cases}$$

Call this graph G'.

Then, if $|S_o|$ is odd, solve MINIMUM WEIGHT $(S_o \cup \{r\})$-JOIN PROBLEM in graph G'. Otherwise solve MINIMUM WEIGHT S_o-JOIN PROBLEM in graph G'.

Let $J' \subseteq E(G')$ be a solution to any one of the above problems. Define J in the following way:

Add every edge $e \in J'$, with $e \notin \delta(r)$, to J. For every edge $e = \{r, v\}$ add an edge $\{u, v\}$ to J, where u is one of $\operatorname{argmin}_{u' \in R} c(\{u', v\})$. Clearly we have $c(J) = c'(J')$.

Then J is clearly a partial (S_o, S_e)-join. Let us show it is indeed minimum weight.

Take any partial (S_o, S_e)-join I.

We can make the following assumptions:

- We can assume I does not contain any edges in $G[R]$. Otherwise we could remove them and not increase the weight of I.

- Also, if for some vertex $v \notin R$ we had two $e, e' \in I \cap \delta(v)$, otherwise we could remove e and e' from I and not increase the weight by doing so.
- We can assume that for each edge $e = \{u, v\} \in I$ where $u \in R$ and $v \notin R$ we have $u \in \operatorname{argmin}_w c(\{w, v\})$. Otherwise replace $\{u, v\}$ with $\{u', v\}$, where $u' \in \operatorname{argmin}_w c(\{w, v\})$. Clearly by doing so we do not increase the weight of I.

In all above modifications it is also clear that I stays partial (S_o, S_e)-join, since we only change the degrees of vertices in R.

Again contract whole R into a single vertex r and define prices same as before, again name the graph G'. By assumptions above, the price of $I' \subseteq G'$ is equal to that of I. Then $c'(J) \leq c'(I)$. And thus also $c(J) = c'(J') \leq c'(I') = c(I)$. So above constructed J was indeed minimum weight partial (S_o, S_e)-join.