Combinatorial optimization
Exercise sheet 8

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Exercise 8.1. The algorithm is very similar to Christofides' Algorithm.

Algorithm 1 Metric s-t path TSP

Input (K_n, c) a metric TSP instance and two vertices $s, t \in V(K_n)$.
Output A Hamiltonian s-t path of minimum weight.

1. Compute MST T for (K_n, c).
2. Let $W := \text{odd}(T) \triangle \{s, t\}$ be the set of odd degree vertices in T, while adding s if degree of s in T is even and removing it if it is odd, and same for t.
3. Compute minimum weight W-join J.
4. Compute Eulerian s-t path in $E(T) \cup J$. We can do this because all vertices have even degree with respect to $E(T) \cup J$, except s and t.
5. Define $\overline{T}$ an s-t path in Eulerian path above. More specifically: add them in order of first appearance, except t, which should be added last.

Now we show $c(\overline{T}) \leq \frac{5}{3} c(\text{OPT})$, where OPT is minimum weight Hamiltonian s-t path in (K_n, c).

Since OPT is a tree, we have $c(T) \leq c(\text{OPT})$.

Now we want to show $c(J) \leq \frac{2}{3} c(\text{OPT})$. It suffices to show $3c(J) \leq c(T) + c(\text{OPT})$. We want to show we can find 3 disjoint W-joins in the disjoint union $E(T) \sqcup E(\text{OPT})$.

For every vertex $w \in W$ we have $\deg_{E(T) \sqcup E(\text{OPT})}(w) \geq 3$ and odd:
If $w \notin \{s, t\}$, then $\deg_{E(\text{OPT})}(w) = 2$ and $\deg_{E(T)}(w) \geq 1$ odd.
If $w \in \{s, t\}$, then $\deg_{E(\text{OPT})}(w) = 1$. By definition of W we have $w \notin \text{odd}(T)$, so also $\deg_{E(T)}(w) \geq 2$ even.

First we construct one W-join along the path OPT. This we can do by ordering $W = \{w_1, \ldots, w_{2k}\}$ in the order in which they appear on the path OPT. Then simply take segments from w_{2i-1} to w_{2i} for $i = 1, \ldots k$.

We can now remove these segments from $E(T) \sqcup E(\text{OPT})$, to obtain E'. By arguments above, every vertex in W now has even degree ≥ 2 with respect to E'. This means there exists a Eulerian tour in in E', which we can split into to W-joins by taking alternating segments.

Each of these three joins has cost greater or equal to $c(J)$, so $3c(J) \leq c(\text{OPT}) + c(T) \leq 2c(\text{OPT})$. So

$$c(\overline{T}) \leq c(T) + c(J) \leq c(\text{OPT}) + \frac{2}{3}c(\text{OPT}) \leq \frac{5}{3}c(\text{OPT})$$

Exercise 8.2. We can show this by using formulation in exercise 4.

If bounds u are finite, then P is bounded. It is also clearly closed, so it is compact.

Condition (i): if $0 \leq x \leq y \in P$, we have s-t flow f with $f(e) = y_e$ for all $e \in U$. We can reduce this flow f along s-t paths to obtain $f(e) = x_e$ on each $e \in U$. So $x \in P$.

Condition (ii): let $x \in \mathbb{R}_+^U$ with $y, z \in P$ and $y, z \leq x$, such that y and z are maximal such. We have to show that flows f and g, corresponding to y and z respectively are maximum flows. Suppose there excists an augmenting s-t path for either flow. Since its a path, it does not return to s, and thus it can only contain one edge $e \in U$. But this violates maximality of y and z. Therefore f and g are both maximum flows and thus $\mathbb{1}y = \mathbb{1}z$.