# Combinatorial optimization
# Exercise sheet 1

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18. Oktober 2023

**Exercise 1.** We can define a graph in the following way: Let $A = \{x_i\}_{i\in\mathbb{N}}$, $B = \{y_i\}_{i\in\mathbb{N}}$ and $E = \{\{x_i, y_{i+1}\} \mid i \in \mathbb{N}\} \cup \{\{x_i, y_1\} \mid i \in \mathbb{N}\}$. Clearly any subset of $A$ or $B$ satisfies Hall's condition.

Suppose it has a perfect matching $M$. Then $\{x_1, y_{i_0}\} \in M$ for some $i_0 \in \mathbb{N}$. Since vertex $x_{i_0+1}$ is of degree 1 and his only neighbour is already matched, he cannot be covered. Therefore a perfect matching $M$ cannot exist.

**Exercise 2.**

a) Suppose we have a bipartite graph $G = (A\dot\cup B, E)$, two subsets $A' \subseteq A$ and $B' \subseteq B$ and matchings $M_{A'}$ and $M_{B'}$ that cover $A'$ and $B'$ respectively.

   To create a matching that covers $A' \cup B'$ we can take the union $M_{A'} \cup M_{B'}$ and pick out a subset in the following way. First observe that the degree of every edge $v \in V(G)$ is in $\{0, 1, 2\}$, so the graph $G' = (A\dot\cup B, M_{A'} \cup M_{B'})$ is union of circles and paths.

   Because the graph is bipartite, circles are of even length and we can pick every second edge to get a matching that covers all edges in a circle.

   Regarding the paths, observe that edges alternate (w.r.t. coming from $M_{A'}$ or $M_{B'}$). Let $(e_i = \{v_i, v_{i+1}\})_{i=1,\dots,k-1}$ be one of the paths in $G'$ (we assume this path to be the whole connected component in $G'$).

   - If $k$ is even, we can pick every second edge and get a perfect matching of the path.
   - If $k$ is odd: first and last vertex must lie in the same half of (bipartite) graph, WLOG $v_1, v_k \in A$. Assume $v_1, v_k \in A'$. Since edges alternate, either $e_1$ or $e_{k-1}$ lies in $M_{B'}$. WLOG $e_1 \in M_{B'}$. Because $v_1$ is covered by the matching $M_{A'}$, there must exist an edge $\{v_1, u\} \in M_{A'}$, which is a contradiction with assumption $v_1, v_k \in A'$. Therefore at least one of $v_1, v_k$ does not lie in $A'$ in which case we do not have to cover it. We can remove it and perfectly cover the remaining (even length) path.

b) Suppose for every non-empty $E' \subseteq E(G)$ we have $\tau(G - E') < \tau(G)$. We want to show $E(G)$ is a matching in $G$.

Let $M$ be the maximum matching in bipartite graph $G$. By Königs theorem we have $\tau(G) = \nu(G)$. Take an edge $\{u, v\} \in E(G) \setminus M$. Matching $M$ is still a maximum matching in $G - \{u, v\}$ and thus $\nu(G) = \nu(G - \{u, v\})$. But according to the assumption $\tau(G - \{u, v\}) < \tau(G) = \nu(G - \{u, v\})$, which cannot hold. Therefore $E(G)$ must be equal to $M$ and thus a matching.

**Excersice 3(a):** To show that a regular bipartite graph have perfect matching first proving the Halls condition using Konig theorem

Let G be a bipartite graph with V(G)=A∪B. Let G satisfy hall condition i.e G has a matching covering A if and only if $|\Gamma(X)| \geq |X|$.
By contradiction G satisfies the Hall's condition but has no matching covering A i.e, $\nu(G) < |A|$. By Konig"s Theorem $\tau(G) < |A|$ and let U be a cover with $|U| < |A|$. Let$A' \subset A$ and $B' \subset B$ and let U=A'+ B' so

$$|A'| + |B'| = |U| < |A|$$
$$|B'| < |A| - |A'| = |A \backslash A'|$$

Since U cover edges, there is no edge between A\A' and B'. So $\Gamma(A\backslash$A') $<$ $|B'| < |A \backslash A'|$. Contradicting the halls condition.

For a regular k bipartite graph G we have $|E(G)| = k|A| = k|B|$. So $|A| = |B|$. Consider X ⊂ A, the number of edges adjacent to X is k$|X|$. These edges are adjacent to the vertices in $\Gamma(X)$ in B. But if $|\Gamma(X)| < |X|$ then some vertex in $\Gamma(X)$ must have degree strictly more than k contradicting the graph is k-regular. So $|\Gamma(X)| \geq |X|$ for every X ⊂ A. Hence by Hall's Theorem there is a matching saturating A and since $|A| = |B|$ there is matching saturating in B. Hence it is a perfect matching.

**(b):** First consider for the case for k- regular bipartite graph G. From (a) we know that G has a perfect matching. Remove a perfect matching to get a k-1 regular graph. Repeating the process to remove another perfect matching until the graph is empty. This gives the partition of edges into k disjoint matching.

For any bipartite graph add edges and vertices to obtain k regular bipartite graph and repeating the above argument to obtain partition of edges and then removing the added edges to obtain the edge set of maximum degree k partitioned into k matching

**Exercise 4(a):**
First prove the claim, Claim : If S is stable set then V\S is a vetex cover
If S is a stable set of a graph G. Let e=(u,v) be an edge. By definition of stable set at most one of u or v can be in S. Hence one of u or v is in V\S. Hence V \S is a vertex cover

Conversely suppose V \S is a vertex cover and let u,v ∈ S. Then there can't be an edge between u and v otherwise the edge wouldn't be covered in V\S.
Hence we get that $|S| + |V \backslash S| = |V(G)|$
Therefore we get $\alpha(G) + \tau(G) = |V(G)|$

**(b):** Let M be a matching of $\nu(G)$. Consider $|V(G)| - 2|M|$ vertices. For each vertex v of $|V(G)| - 2|M|$ not covered by M add to M an edge covering v.
Hence obtain an edge cover F of size
$|M| + (|V(G)| - 2|M|) = |V(G) - |M|$
Therefore, $\zeta(G) \leq |F| = |V(G) - |M| = |V(G)| - \nu(G)$

For the other direction, let F be an edge cover of size $\zeta(G)$. Choose one edge from each component of (V,F) to obtain matching M. As the graph (V,F) has atleast $|V| - |F|$ components we get
$\nu(G) \geq |M| \geq |V| - |F| = |V| - |\zeta(G)|$
Hence $\nu(G) + \zeta(G) = |V(G)|$

**(c):** By (a) $\alpha(G) + \tau(G) = |V(G)|$
$\alpha(G) = |V(G)| - \tau(G)$
Using Konig theorem we get $\alpha(G) = |V(G)| - \nu(G)$
By part (b) $\alpha(G) = \zeta(G)$