Combinatorial optimization
Exercise sheet 5

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Exercise 5.3. Let G a graph and $T \subseteq V(G)$ with $|T|$ even.

i) Suppose the set $F \subseteq E(G)$ intersects every T-join. Take some $t \in T$. Now look at all $v \in V(G)$ for which there exists a t-v-path that does not intersect F:

$$X = \{v \in V(G) \mid \text{exists a } t\text{-}v\text{-path that does not intersect } F\} \cup \{t\}$$

Use induction.

Suppose $T = \{t, s\}$. Then T-joins are exactly t-s-paths. We are assuming every such path intersects F. In this case the set X then only contains one element, and it clearly defines a T-cut $C = \delta(X)$, because every edge $\{u, v\} \in C$ with $u \in X$ and $v \notin X$ it must hold $\{u, v\} \in F$, otherwise we would get $v \in X$.

And the induction step:

- If $|X \cap T|$ odd, then we found a T-cut that is contained in F, namely the cut $C = \delta(X)$. Lets quickly argument why this is true: every edge $\{u, v\} \in C$ (with $u \in X$ and $v \notin X$) has to lie in F, otherwise we could extend the path from t to u with the edge $\{u, v\}$ and get $v \in X$.
- If $|X \cap T|$ even, then for every $t_i \in X \cap T$, $t_i \neq t$, define the path p_i to be the path from t to t_i that does not intersect F. Take a symmetric difference of all these paths $\otimes p_i$, which is, using a proposition from the lectures, a $T \cap X$-join. Let $T' = T \setminus X$. If there existed a T'-join J would not intersect F, then $\otimes p_i \otimes J$ would be a T-join that does not intersect F. So every T'-join intersects F. Using inductions hypothesis we get that there exists a T'-cut $C = \delta(X')$, which is contained in F. Clearly $t \in X'$ if and only if $T \cap X \subseteq X'$, so $|X' \cap T|$ is odd, which proves it is a T-cut.

In both cases we found a T-cut that is contained in F.

The other direction is obvious using the proposition from the lectures, which says that every T-cut and T-join intersect.

ii) First remember the proposition from the lectures, saying that a subgraph $G' \subseteq G$ contains a T-join if and only if every connected component of G' contains even number of elements in T.

Define $V' = \cup_{e \in F} e$ and a graph $G' = (V', F)$. Suppose there exists a connected component $C' \subseteq G'$ for which $|V(C') \cap T|$ is odd. Then simply taking $C := \delta(V(C'))$ is a T-cut that does not intersect F (because C' is by definition a connected component in G').

The other direction is again obvious using the proposition from the lectures, which says that every T-cut and T-join intersect.