Combinatorial optimization
Exercise sheet 9

Solutions by: Anjana E Jeevanand and David Čadež

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Exercise 9.1.

Let U be a finite set. Define a condition for $f\colon 2^U \to \mathbb{R}$ that

$$f(X \cup \{y, z\}) - f(X \cup \{y\}) \leq f(X \cup \{z\}) - f(X) \tag{1}$$

for every $X \subseteq U$ and $y, z \in U$ with $y \neq z$.

If f is submodular, then condition 1 follows from the definition of submodularity by setting one set to be $X \cup \{y\}$ and other $X \cup \{z\}$.

Suppose now f satisfies condition 1. Take $X, Y \subseteq U$. We are trying to show

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y).$$

We do induction on $n = |X \setminus Y| + |Y \setminus X|$.

If $X \subseteq Y$ or $Y \subseteq X$, then the statement holds.

Assume now that the statement holds for $k < |X \setminus Y| + |Y \setminus X| = n$. Take $x \in X \setminus Y$. By induction hypothesis we have

$$f((X \setminus \{x\}) \cup Y) + f(X \cap Y) \leq f(X \setminus \{x\}) + f(Y). \tag{2}$$

By condition 1 we also have the following chain of inequalities

$$\begin{aligned} f(X \cup Y) - f((X \setminus \{x\}) \cup Y) &\leq f(X \cup Y_{n-1}) - f((X \cup Y_{n-1} \setminus \{x\}) \\ &\leq \dots \\ &\leq f(X \cup Y_2) - f((X \cup Y_2) \setminus \{x\}) \\ &\leq f(X \cup Y_1) - f((X \cup Y_1) \setminus \{x\}) \\ &\leq f(X) - f(X \setminus \{x\}). \end{aligned} \tag{3}$$

where $Y = \{y_1, \dots, y_n\}$ and $Y_i = \{y_1, \dots, y_i\}$.

Summing 2 and 3 yields

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y),$$

which is what we wanted to show.

Exercise 9.3. Let B_f denote the base polyhedron of f.

Take some total order $\prec$ of U. We show that $b^{\prec}$ is a vertex of U.

First we show that $b^{\prec} \in B_f$. Take some $A \subseteq U$. By definition

$$b^{\prec}(A) = \sum_{a \in A} f(U_{\preceq a}) - f(U_{\prec a}).$$

From the first exercise it follows that

$$f(U_{\preceq a}) - f(U_{\prec a}) \leq f(A_{\preceq a}) - f(A_{\prec a})$$

for every $a \in A$. So we have

$$b^{\prec}(A) = \sum_{a \in A} f(U_{\preceq a}) - f(U_{\prec a}) \leq \sum_{a \in A} f(A_{\preceq a}) - f(A_{\prec a}),$$

which is a telescoping sum that simplifies to $f(A)$ (using $f(\emptyset) = 0$).

If $A = U$, the estimation is not necessary and we have

$$b^{\prec}(U) = \sum_{a \in U} f(U_{\preceq a}) - f(U_{\prec a}) = f(U).$$

So we've shown $b^{\prec} \in B_f$.

Take any $c \in \mathbb{R}^U$. We will show that there exists a total order $\prec$ for which $b^{\prec}$ lies in the face defined by c.

Define total order $U = \{u_1, \ldots, u_n\}$ such that $c(u_1) \geq \cdots \geq c(u_n)$. Denote $c_i := c(u_i)$ and $U_i = \{u_1, \ldots, u_i\}$ for every $i \in \{1, \ldots, n\}$ (and $U_0 = \emptyset$).

Take any $x \in B_f$. We will show that $c^T b^{\prec} \geq c^T x$. Define $d_i := c_i - c_{i+1}$. By the definition of the ordering, we have $d_i \geq 0$ for all $i \in \{1, \ldots, n\}$. With some reordering (we also used this at some point during the lectures) we have

$$c^T x = \sum_{i=1}^{n} c_i x_i = \sum_{j=1}^{n} d_j \sum_{i=1}^{j} x_i.$$

Because $x \in B_f$, $x(U_i) \leq f(U_i)$ and $x(U) = f(U)$. Putting the into above equation we obtain

$$\begin{aligned}
c^T x &\leq \sum_{j=1}^{n} d_j f(U_j) \\
&= c_n f(U) + \sum_{j=1}^{n-1} (c_j - c_{j+1}) f(U_j) \\
&= \sum_{j=1}^{n} c_j (f(U_j) - f(U_{j-1})) \\
&= c^T b^{\prec}.
\end{aligned}$$

So for every face F of polyhedron, there exists a total order $\prec$, such that $b^{\prec} \in F$. If F happens to be a singleton, i.e. a vertex, then $F = \{b^{\prec}\}$.

Now we have to show that for every total order $\prec$, vector $b^{\prec}$ is a vertex. Define $c_i = n - i + 1$. Take any $x \in B_f$ with $c^T x \geq c^T b^{\prec}$. For every i we have $x(U_i) \leq f(U_i)$. We have

$$\begin{aligned} c^T x &= \sum_{j=1}^{n} \sum_{i=1}^{j} x_i \\ &\leq \sum_{j=1}^{n} f(U_j) \\ &\leq \sum_{i=1}^{n} c_i (f(U_i) - f(U_{i-1})) \end{aligned}$$

By our choice of x, we therefore have an equality at all steps. That means inequalities $x(U_i) \leq f(U_i)$ must in fact be equalities for every i. From that we can explicitly deduce that $x = b^{\prec}$. So $b^{\prec}$ is a vertex.