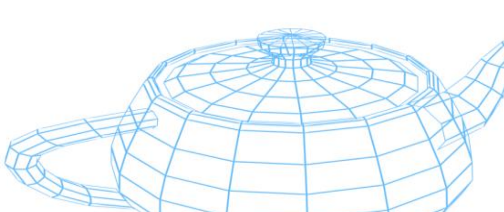
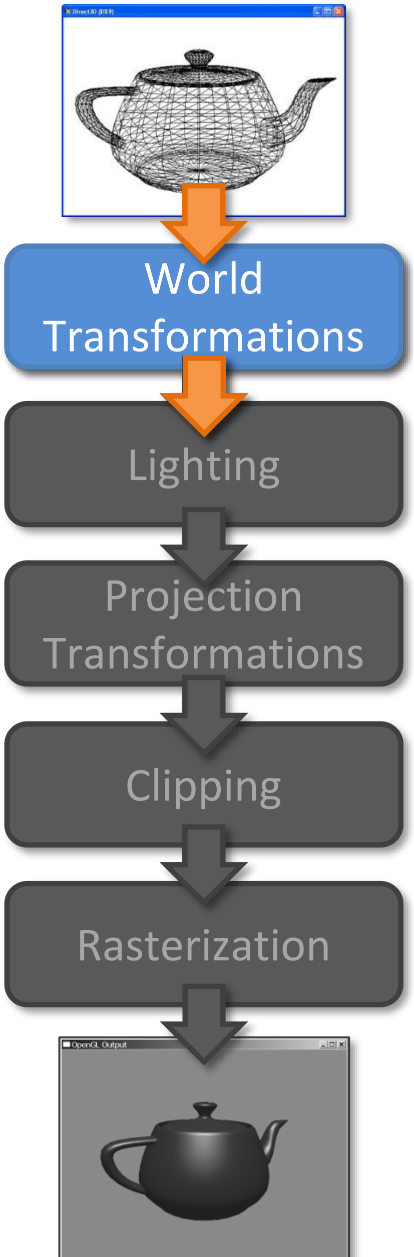
SUPSI

Computer Graphics

Mathematics for Computer Graphics (2)

Achille Peternier, adjunct professor



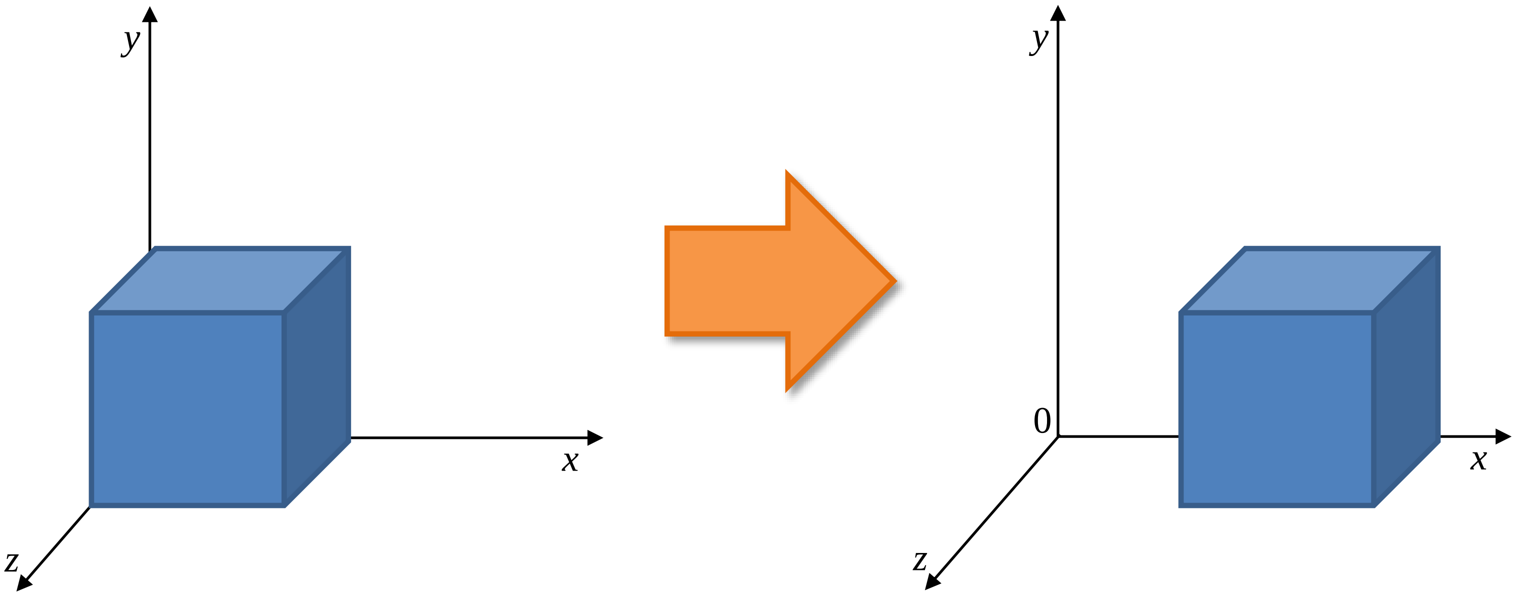

World Transformations
Lighting
Projection Transformations
Clipping
Rasterization

Translation

$$\mathbf{v}_n = \mathbf{v}_p + \mathbf{t}$$

$$\text{e.g.:}\quad \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 5 \\ 7.5 \end{bmatrix}$$

Translation

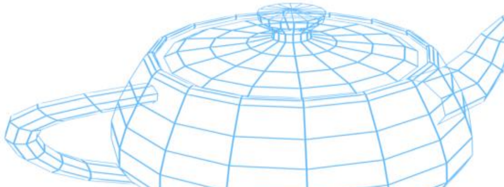


Rotation

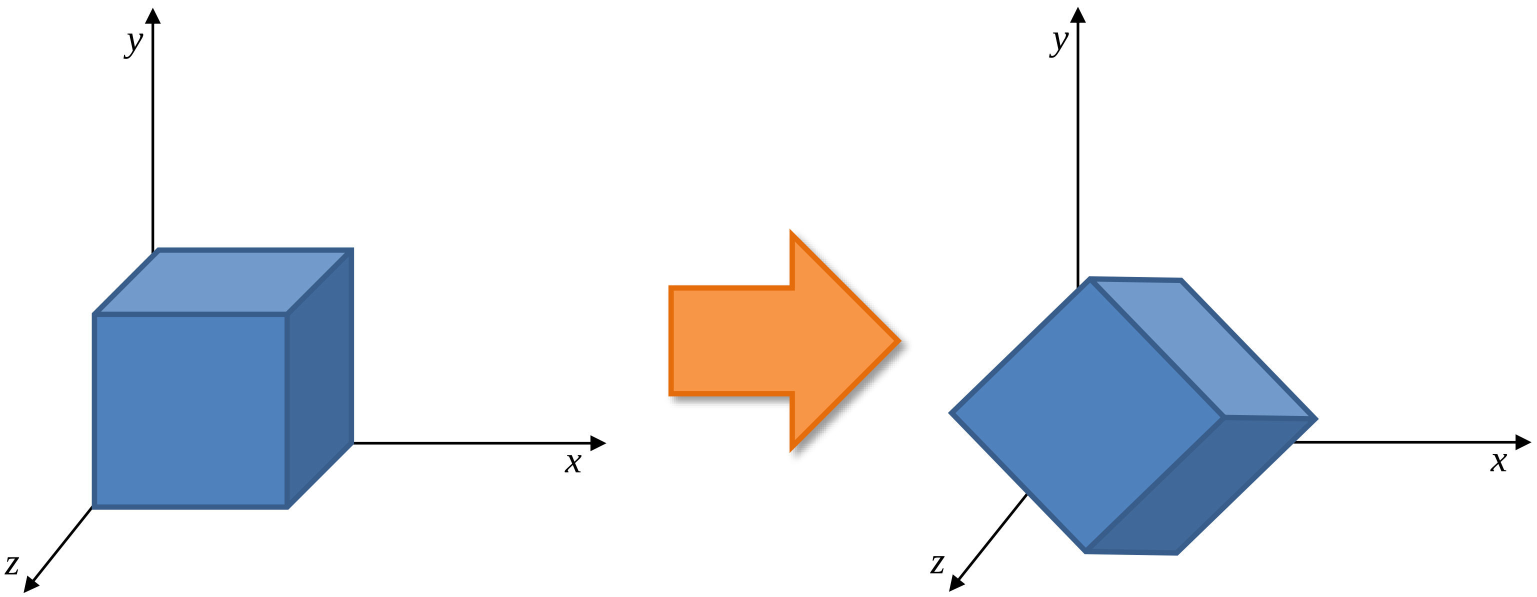
$$\mathbf{v}_n = \mathbf{R}\mathbf{v}_p$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \propto & -\sin \propto \\ 0 & \sin \propto & \cos \propto \end{bmatrix} \qquad \mathbf{R}_y = \begin{bmatrix} \cos \propto & 0 & \sin \propto \\ 0 & 1 & 0 \\ -\sin \propto & 0 & \cos \propto \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos \propto & -\sin \propto & 0 \\ \sin \propto & \cos \propto & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation



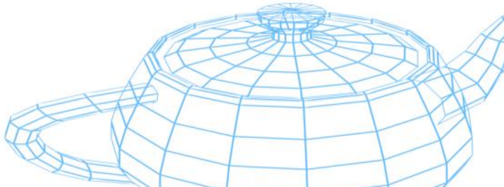
Scaling

$$\mathbf{v}_n = \mathbf{S}\mathbf{v}_p$$

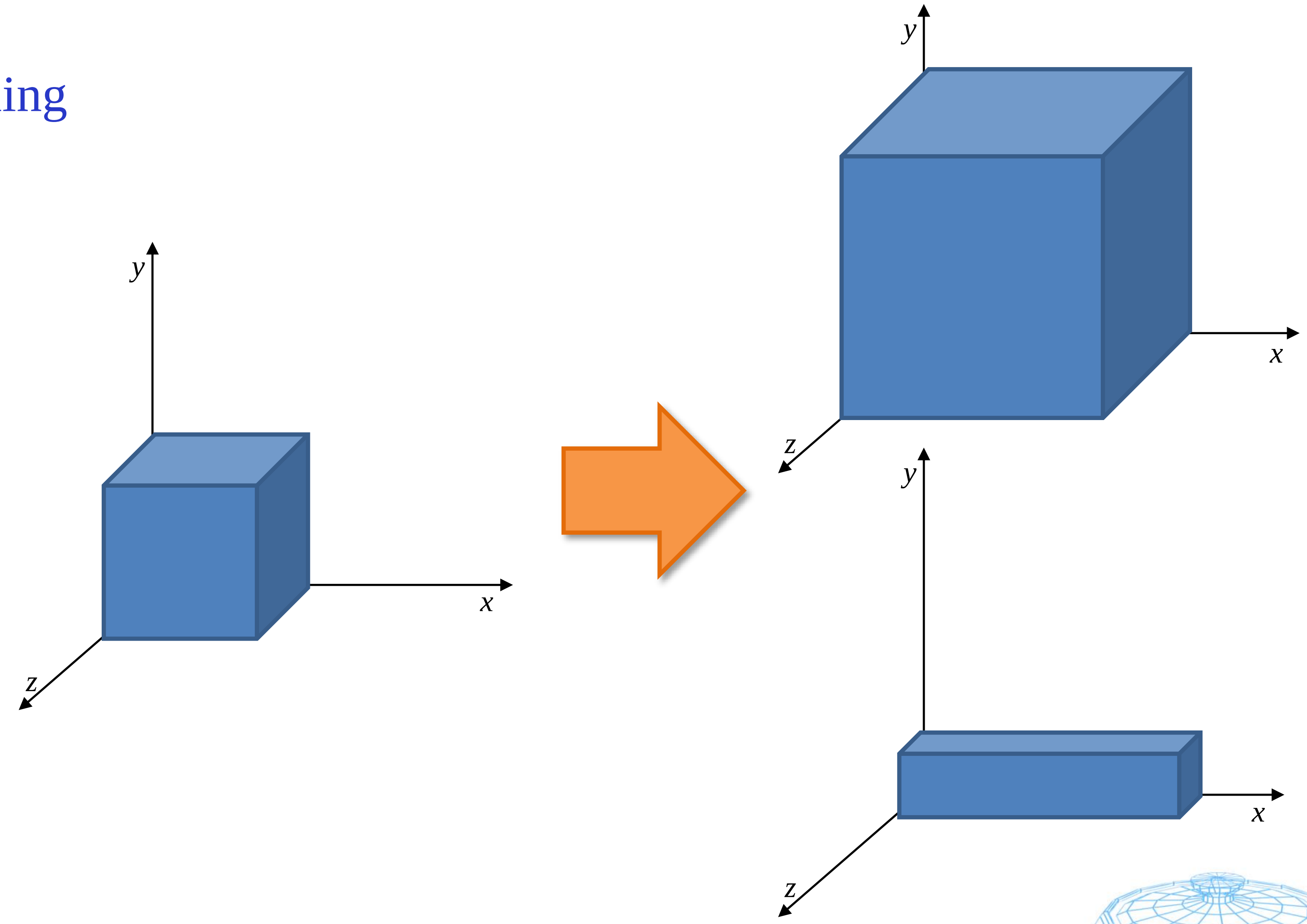
$$\mathbf{S} = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

When *x=y=z* → **uniform/isotropic** scaling

Otherwise → **non-uniform/anisotropic** scaling

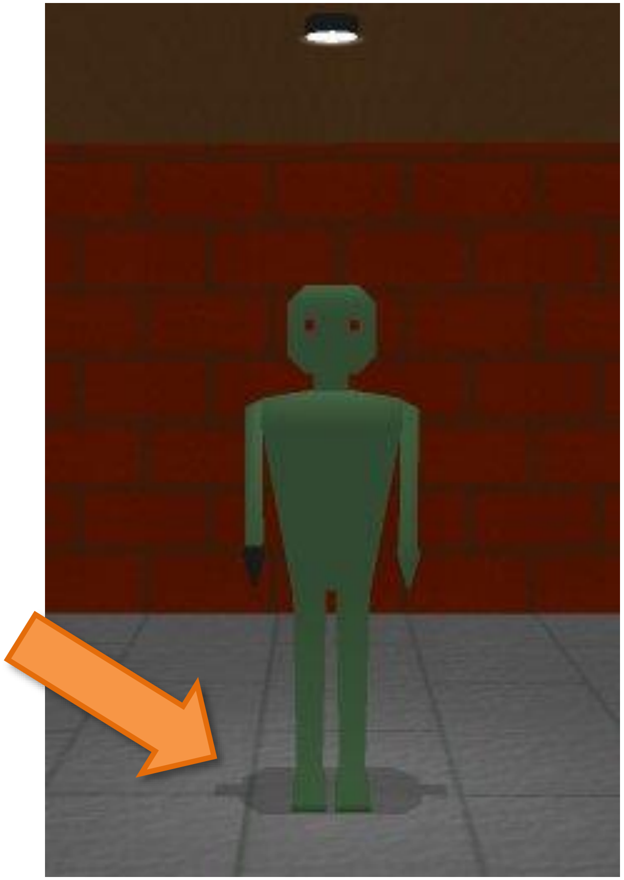


Scaling



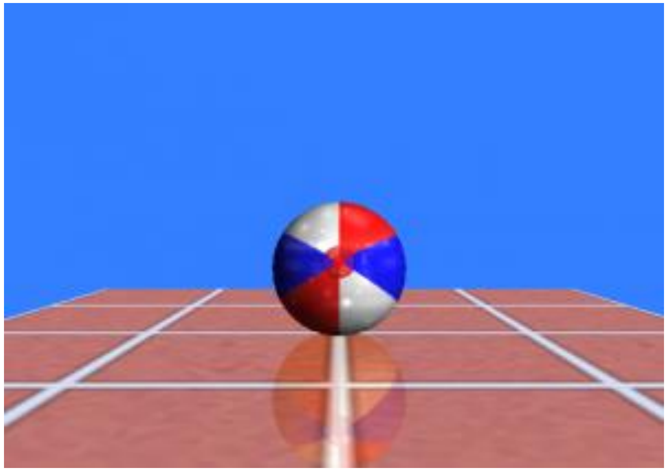
Scaling

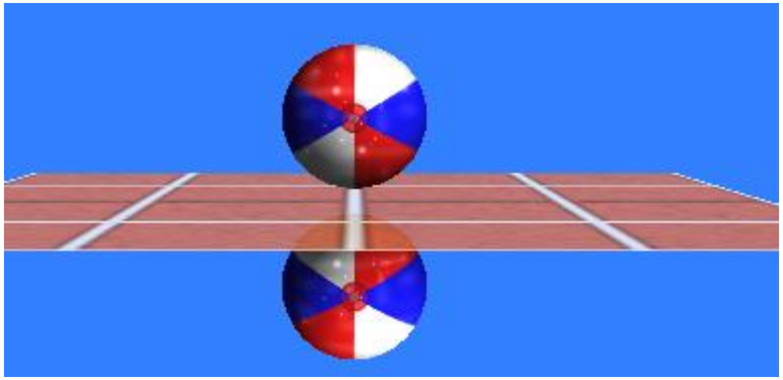
$$\boldsymbol{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Scaling

$$\boldsymbol{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





Homogeneous coordinates

- Without homogeneous coordinates:
  - Translation = vector by vector addition.
  - Rotation = matrix by vector multiplication.
  - Scaling = matrix by vector multiplication.

- Goals:
  - Reducing the number of operations required.
  - Using one same method for implementing all the base transformations of the rendering pipeline (including projections).

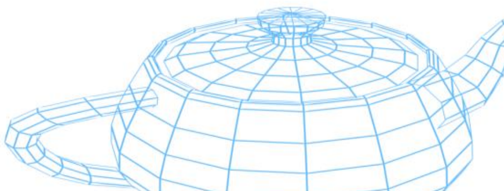
- With homogeneous coordinates:
  - Translation, rotation, scaling = matrix by vector multiplication.

Homogeneous coordinates

2D: $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

3D: $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ *w*

(`glVertex3f()` implicitly sets *w = 1*)



Homogeneous coordinates

$$\mathbf{v}_n = \mathbf{T}\mathbf{v}_p$$

where **T** can be any transformation among translation, rotation, and scaling

Homogeneous coordinates - translation

$$\mathbf{v}_n = \mathbf{T}\mathbf{v}_p$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous coordinates - translation

$$\mathbf{v}_n = \mathbf{T}\mathbf{v}_p$$

$$\begin{bmatrix} x_n \\ y_n \\ z_n \\ w_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

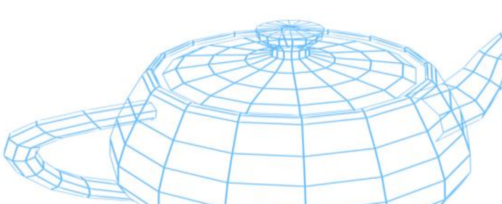
$$\begin{aligned} x_n &= x + 2 \times 1 \\ y_n &= y + (-3) \times 1 \\ z_n &= z + 4 \times 1 \\ w_n &= 1 \times 1 \end{aligned}$$

Homogeneous coordinates - rotation

$$\mathbf{v}_n = \mathbf{R}\mathbf{v}_p$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\propto & -\sin\propto & 0 \\ 0 & \sin\propto & \cos\propto & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_y = \begin{bmatrix} \cos\propto & 0 & \sin\propto & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\propto & 0 & \cos\propto & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos\propto & -\sin\propto & 0 & 0 \\ \sin\propto & \cos\propto & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Homogeneous coordinates - scaling

$$\mathbf{v}_n = \mathbf{S}\mathbf{v}_p$$

$$\mathbf{S} = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Concatenation

- Given three transformations $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$:

$$\mathbf{v}_n = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1\mathbf{v}_p$$

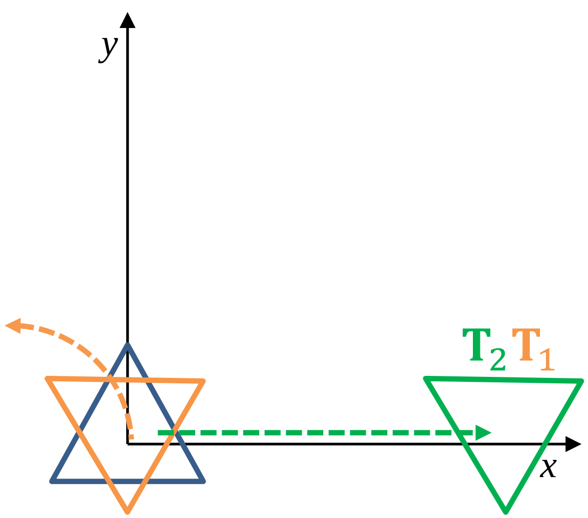
using post-multiplication and column vectors

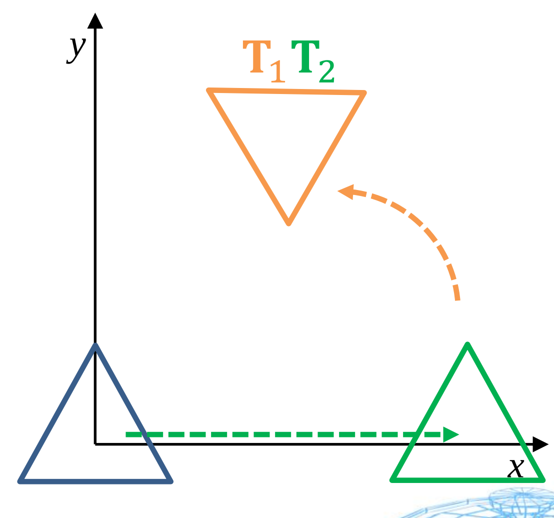
Concatenation

- Let: $\mathbf{T}_1 = \text{rotation of } 60°$

  $\mathbf{T}_2 = \text{translation}(10, 0)$

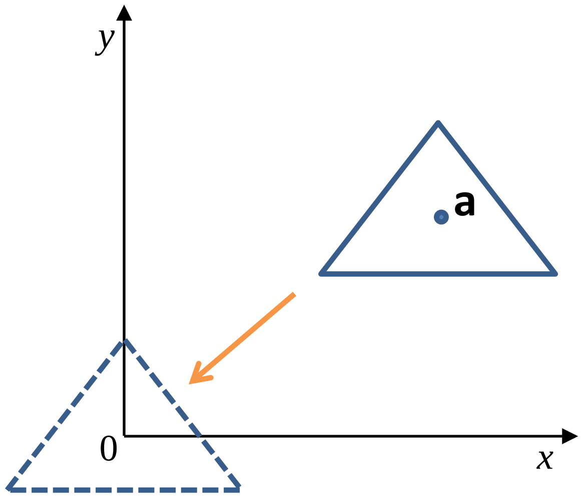
$$\mathbf{T}_2\,\mathbf{T}_1 \neq \mathbf{T}_1\,\mathbf{T}_2$$





Concatenation

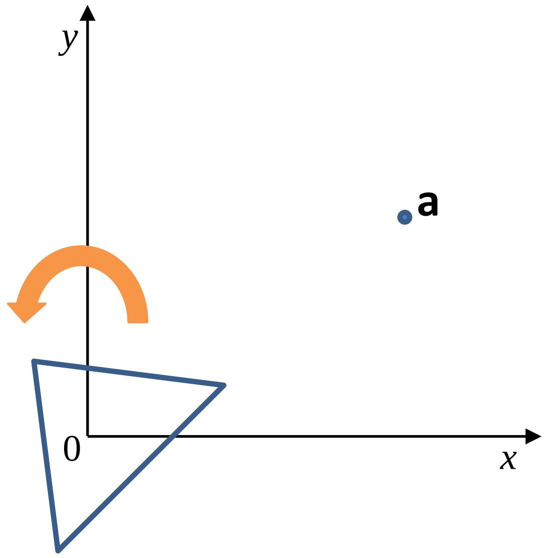
- Rotation of 45° about point **a**:



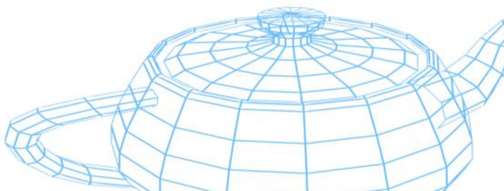
1) Subtract **a** to center the object at the origin

Concatenation

- Rotation of 45° about point **a**:

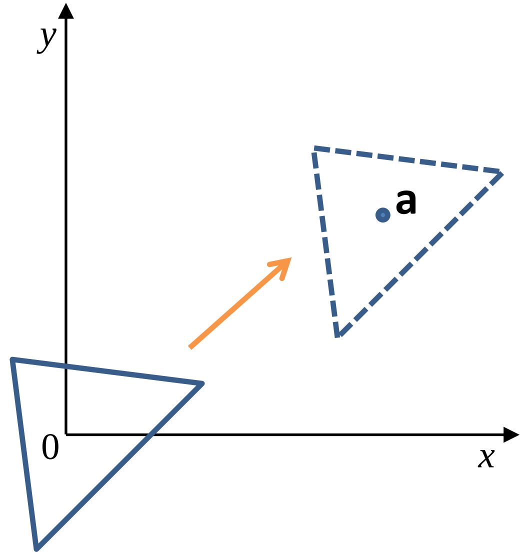


2) Rotate the object around the origin

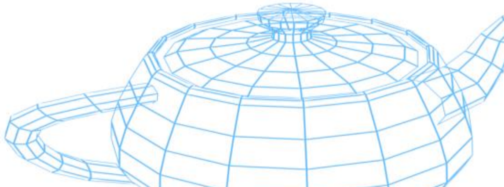


Concatenation

- Rotation of 45° about point **a**:

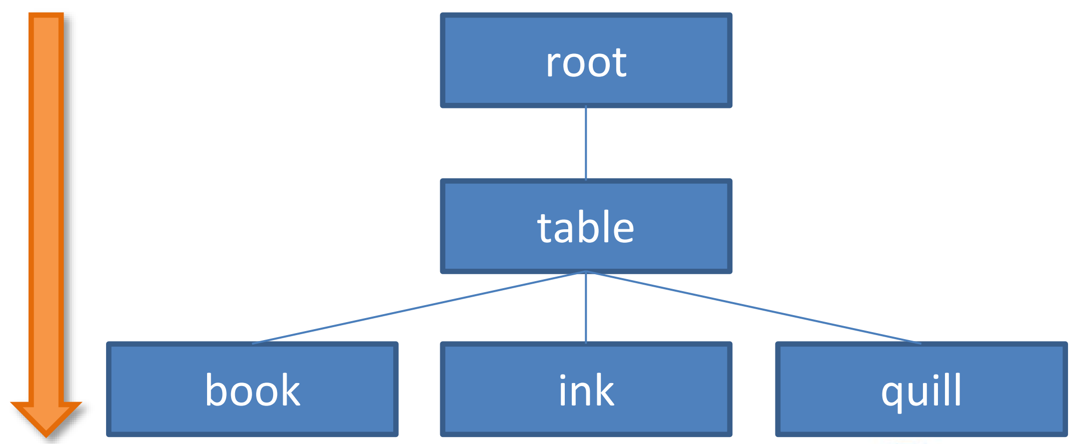


3) Add **a** to translate the object back



Scene graph

- The scene is represented as a hierarchy (tree) of dependencies:
  - Each node has its own matrix.
  - Each node multiplies the previous node’s matrix by its own matrix:
    - The cumulated resulting matrix is used by the next level.
  - Use push/pop to store/restore states as you go deeper in the tree.



Object positioning

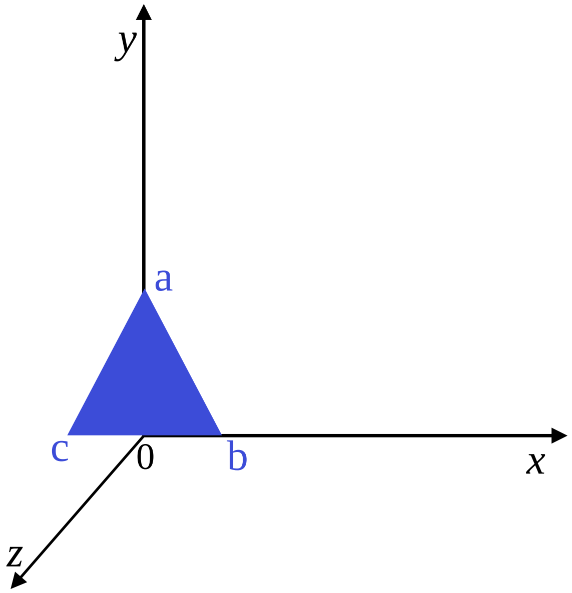
- Let put objects A, B, and C somewhere in the 3D world coordinates:
  - At each frame, we start from scratch and reprocess the scene-graph.
  - Reset the current matrix (set to identity).
  - Apply required transformations to place object A.
  - If object B depends on A’s position, apply the next transformations **without** resetting the current matrix:
    - New transformations stack on top of the previous ones.
  - If object C does not depend on previous objects’ position, reset the current matrix and start again.

Coordinate spaces

- **Object/model coordinates:**
  - The object's 3D vertices are defined as relative to the origin, i.e., the object is centered at (0, 0, 0).
- **World coordinates:**
  - 3D vertices with absolute position:
    - World center is the origin (0, 0, 0).
    - Object matrix * object coordinates = world coordinates.
- **Eye/view/camera coordinates:**
  - 3D vertices relative to the viewer's position:
    - Center is the viewer's position (0, 0, 0).
    - Camera matrix^{-1} * world coordinates = eye coordinates.

Object/model coordinates

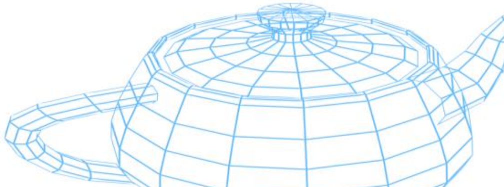
- The vertices of each 3D object are defined as relative to its origin.
- The origin usually refers to the object’s pivot point (center, barycenter or basement).
- Single 3D models are designed in object coordinates, then are moved around and their vertices become world coordinates.



a = (0, 2, 0)

b = (1, 0, 0)

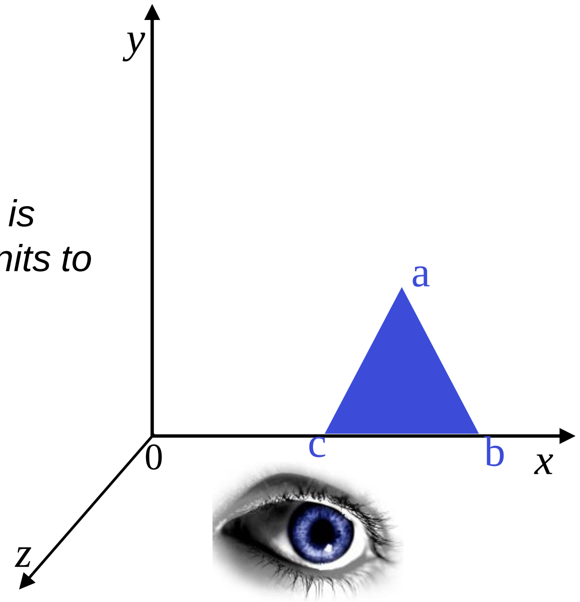
c = (-1, 0, 0)



World coordinates

- Coordinates are relative to the world's origin.
- One same object can be put at different locations in the same scene:
  - Each instance will have its own absolute coordinates.
  - Vertices are relative to the object's center in object coordinates → objects are relative to the world's center in world coordinates.

The object is translated 5 units to the right

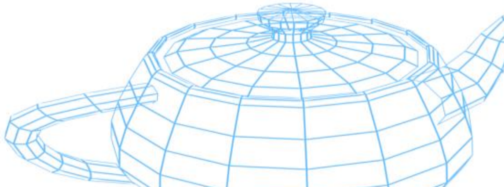


a = (5, 2, 0)

b = (6, 0, 0)

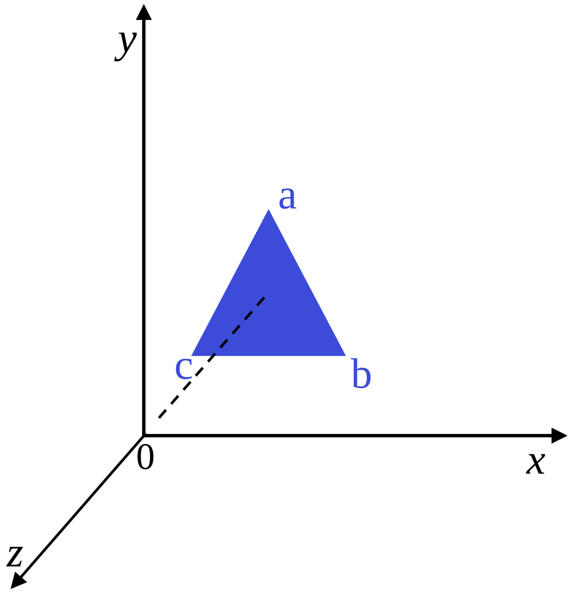
c = (4, 0, 0)

eye = (5, 1, 3)



Eye/view/camera coordinates

- The eye is now at the origin (0, 0, 0).
- Coordinates are relative to the eye's position.

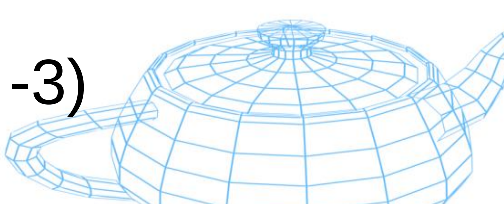


a = (0, 1, -3)

b = (1, -1, -3)

c = (-1, -1, -3)

eye^{-1} = (-5, -1, -3)



Batch transformations

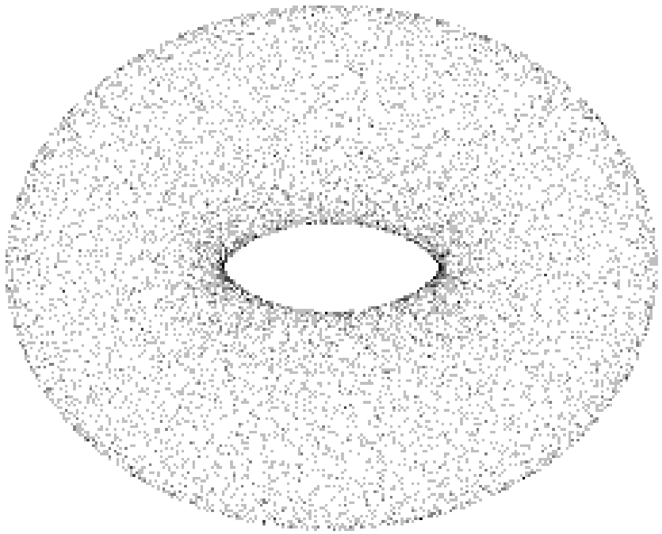
- Given the same three transformations $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ and a list of points $\mathbf{v}_{p(\mathbf{1-1000})}$ it is more efficient to:

  1) compute the final matrix $\mathbf{T}_f = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1$ just once, then

  2) multiply the 1000 points using:

$$\text{for p=1 to 1000}$$
$$\mathbf{v}_n = \mathbf{T}_f\mathbf{v}_p$$

Point cloud



Matrix interpolation

- There's no easy way to smoothly interpolate between two matrices:
  - Such interpolation is very useful in animation.

- Two options:

1) Use **matrix decomposition** on any arbitrary transformation matrix to obtain its scale, rotation, and translation parameters, which you can linearly interpolate and concatenate back into a matrix.

2) Use **quaternions** (but they only work for rotations).

Quaternions (1843)

William Rowan Hamilton
1805 - 1865

- Generalization of complex numbers.

- A quaternion is a four-tuple:

$$\mathbf{q} = (x, y, z, w) = w + xi + yj + zk$$

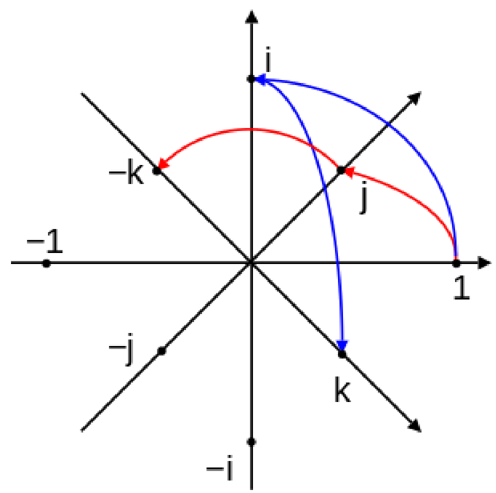
where:

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k$$

$$ji = -k$$

Not commutative (in general)

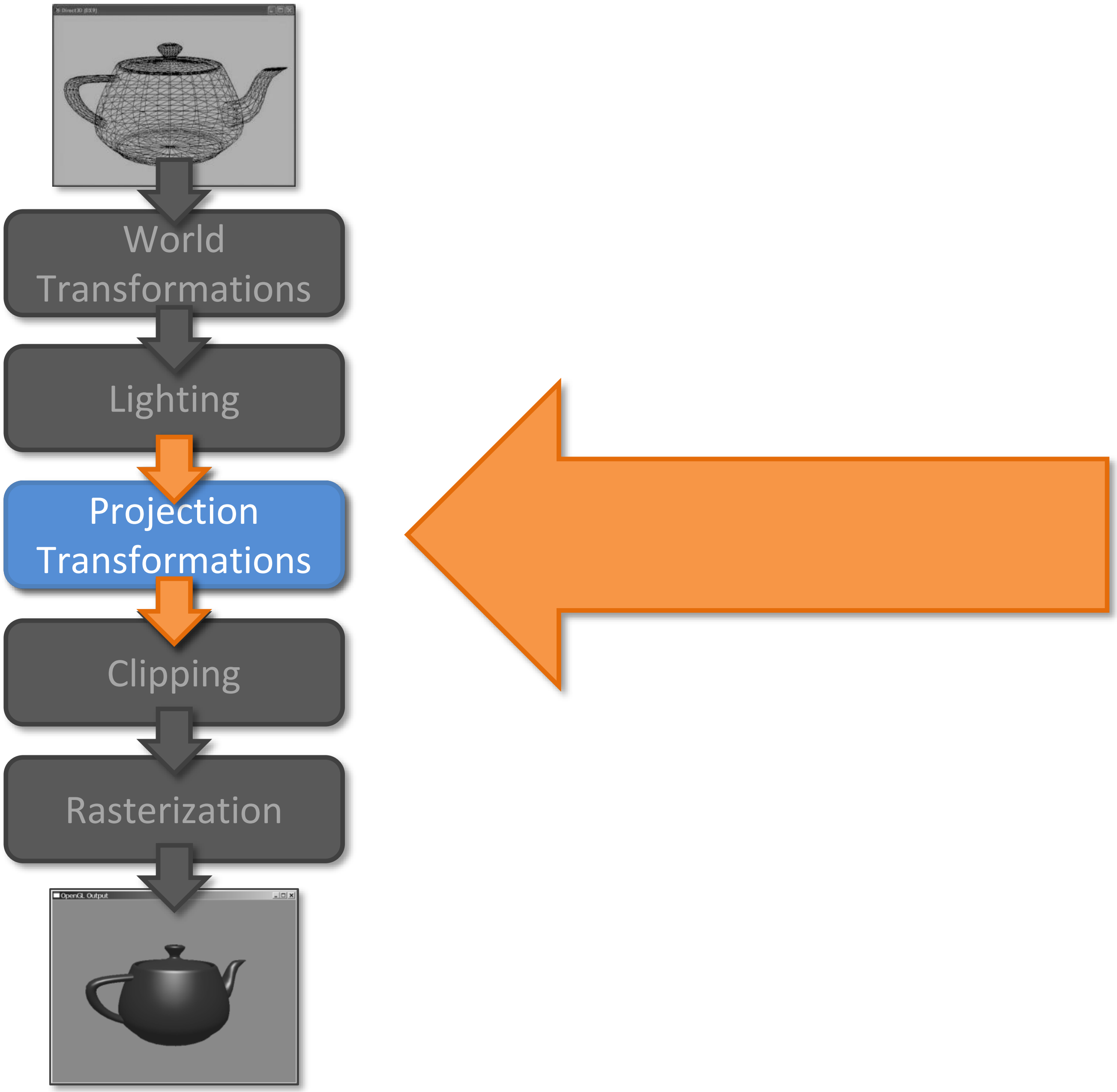


ij = k
ji = −k
ij = −ji

Quaternions

- Quaternions are useful for interpolating rotations between two matrices during animation.

- To do so:
  - Convert the two matrices into quaternions.
  - Use **s**pherical **l**inear int**erp**olation (slerp) to interpolate between the two quaternions.
  - Convert the resulting quaternion back to a matrix.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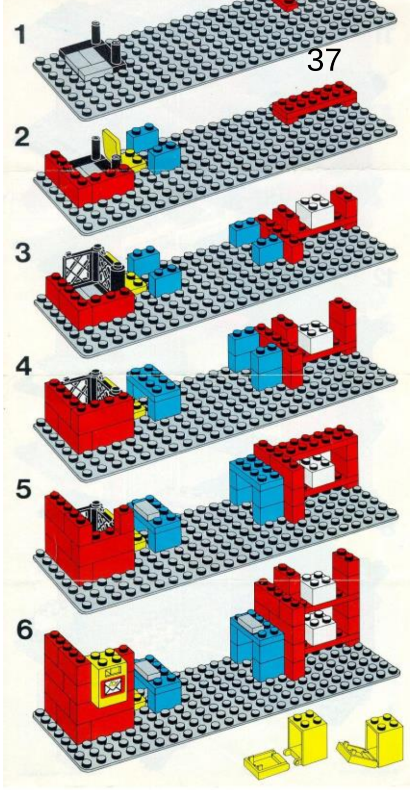
Coordinate spaces

- **Clip coordinates:**
  - Intermediate step before the divide step:
    - Projection matrix * eye coordinates = clip coordinates.
  - The goal of the projection matrix is to setup the *w* component…
    - …for the following division of *x*, *y*, *z* by *w*.
    - …for the normalization of *x*, *y*, *z*.

Projections

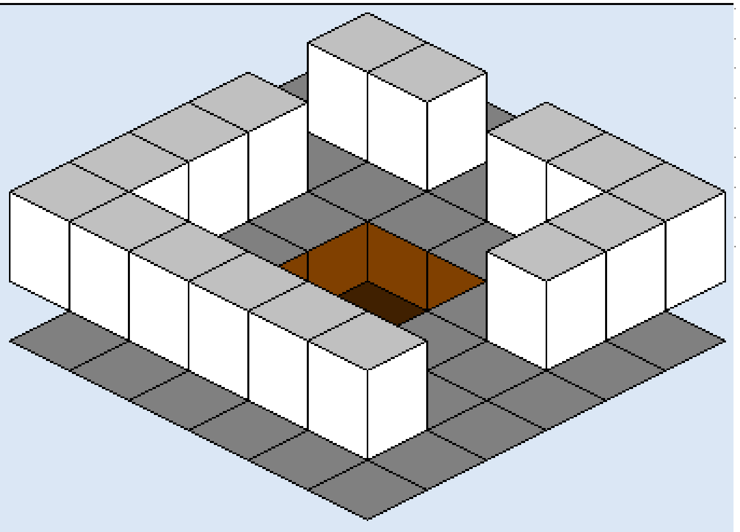
- Two main types of projection (matrices):
  - Orthographic.
  - Perspective.

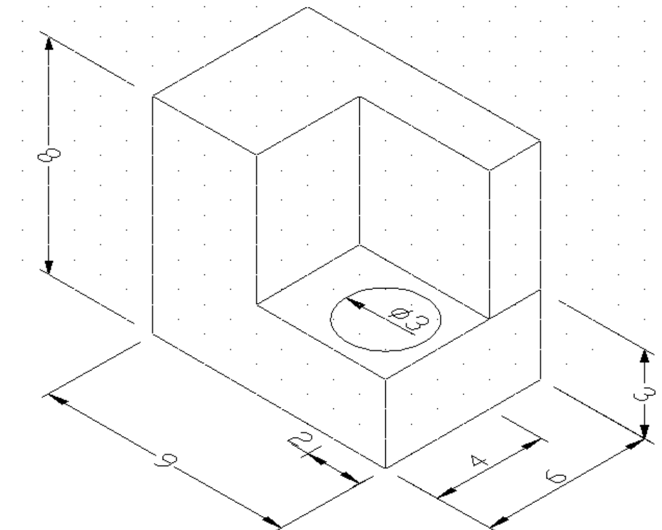
- Other kinds of projection are difficult to implement (e.g., fish-eye).

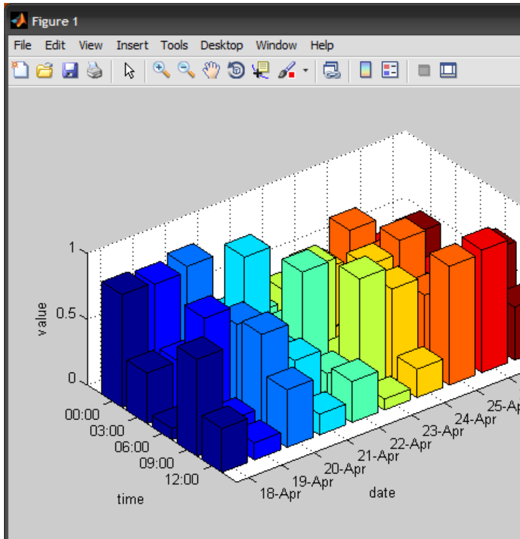


Orthographic projection

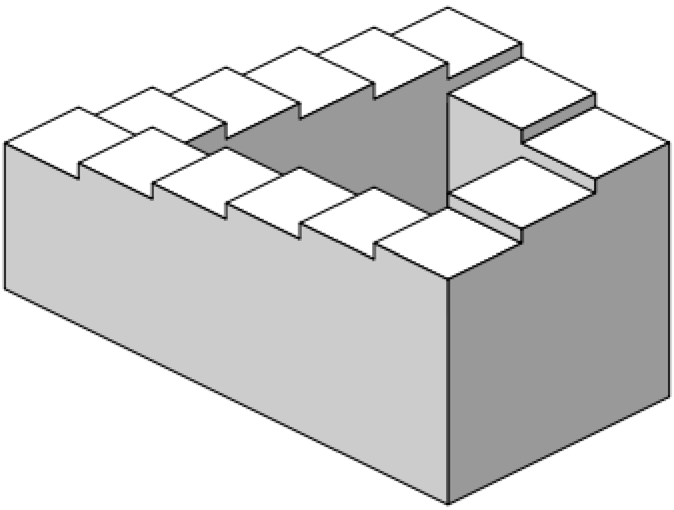
- Distant objects appear with the same size, no perspective:
  - The clipping space is a cube (and not a truncated pyramid).
  - Useful for drawing 2D graphics, diagrams, blueprints, CAD tools, etc.

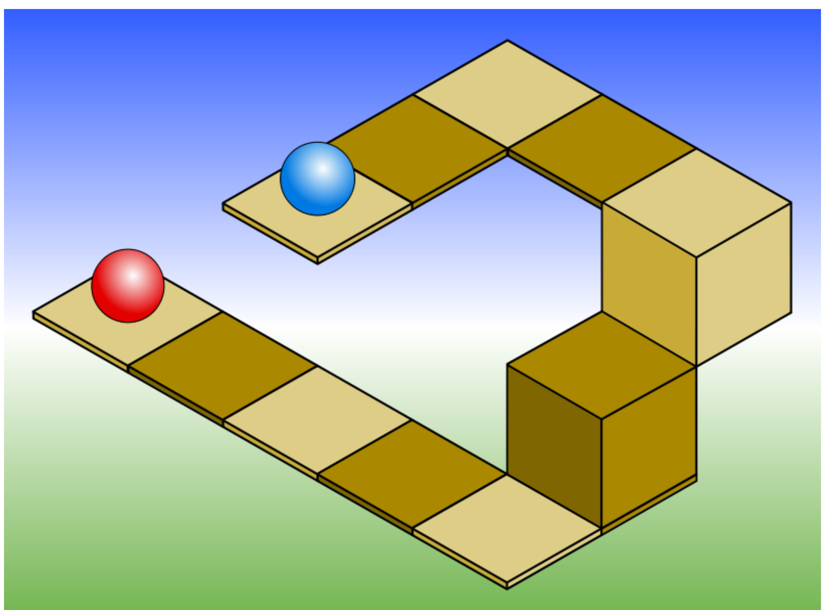






Orthographic projection limitations






SIMCITY SOCIAL
FOR SALE
FOR SALE
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FOR SALE
FOR SALE
HOTEL
HOTEL
SUPERMARKET
SIM CORP
ATM


Rene Guerin
61
66
42
64


USING
HOLDING
POWER
021940
00:03:33

https://youtu.be/Me4ymG_vnOE

Orthographic projection

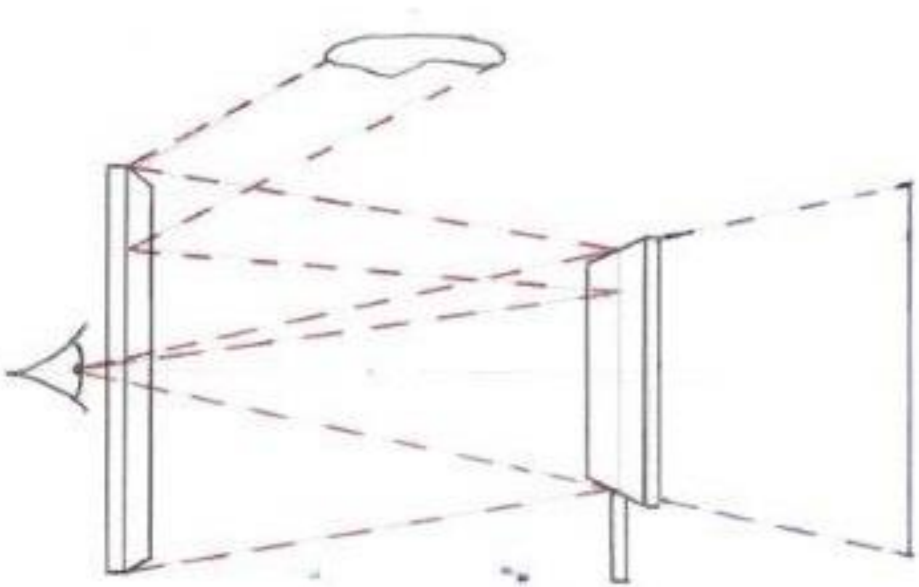
z is inverted!

$$\begin{bmatrix} \frac{2}{right-left} & 0 & 0 & -\frac{right+left}{right-left} \\ 0 & \frac{2}{top-bottom} & 0 & -\frac{top+bottom}{top-bottom} \\ 0 & 0 & \frac{-2}{far-near} & -\frac{far+near}{far-near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(as defined in *glOrtho* and *gluOrtho2D*)

- The orthographic projection is basically a scaling of the scene into the clipping space.

Perspective (~1413)



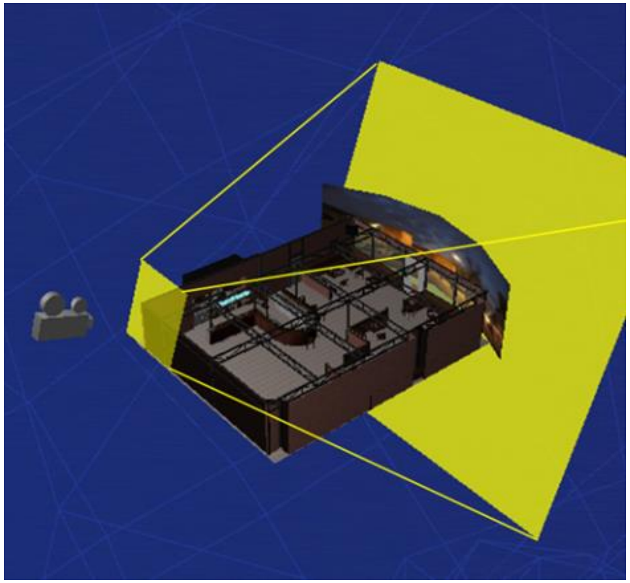
Filippo Brunelleschi
1377 - 1446

Città ideale, anon., ~1480-90

Andrea Pozzo, church of S. Ignazio, Rome (1691-1694)

Perspective projection

- The *w* component of each vertex increases with its distance from the near plane:
  - Division of *x*, *y*, *z* by *w* is done just after.
- Points converge to the center according to their distance.
- The clipping space is a truncated pyramid.



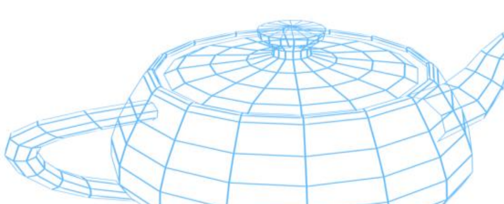
Perspective projection

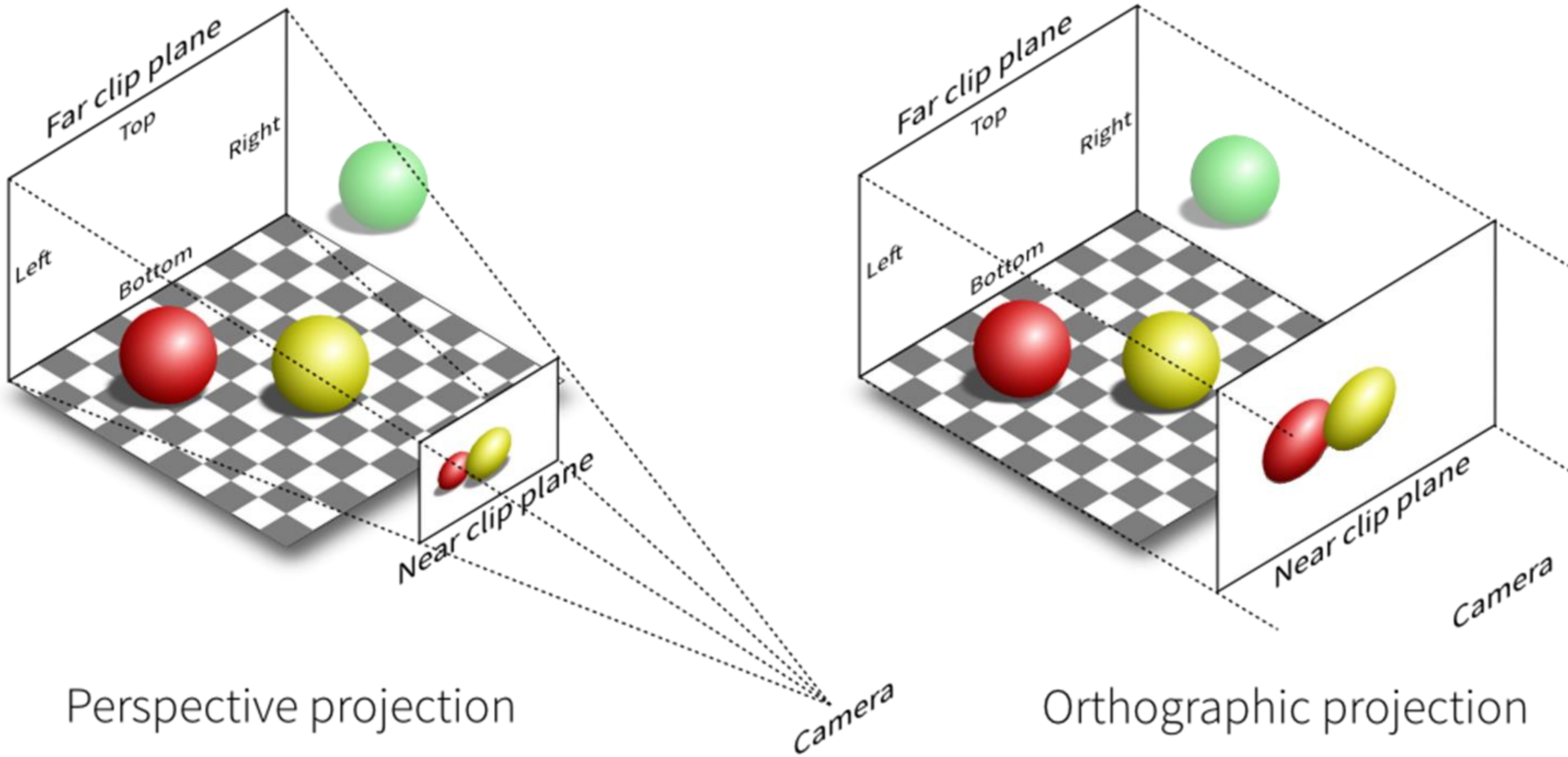
z is inverted and copied to w!

$$\begin{bmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{far + near}{near - far} & \frac{2 \times far \times near}{near - far} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

fieldOfView	vertical (y) view angle
f	cotangent(*fieldOfView*/2)
aspect	aspect ratio (4:3, 16:9, etc.)

(as defined in *gluPerspective*)



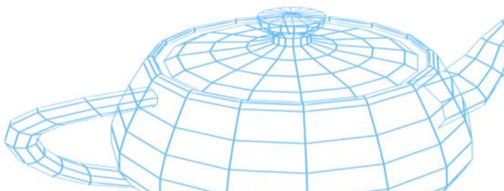


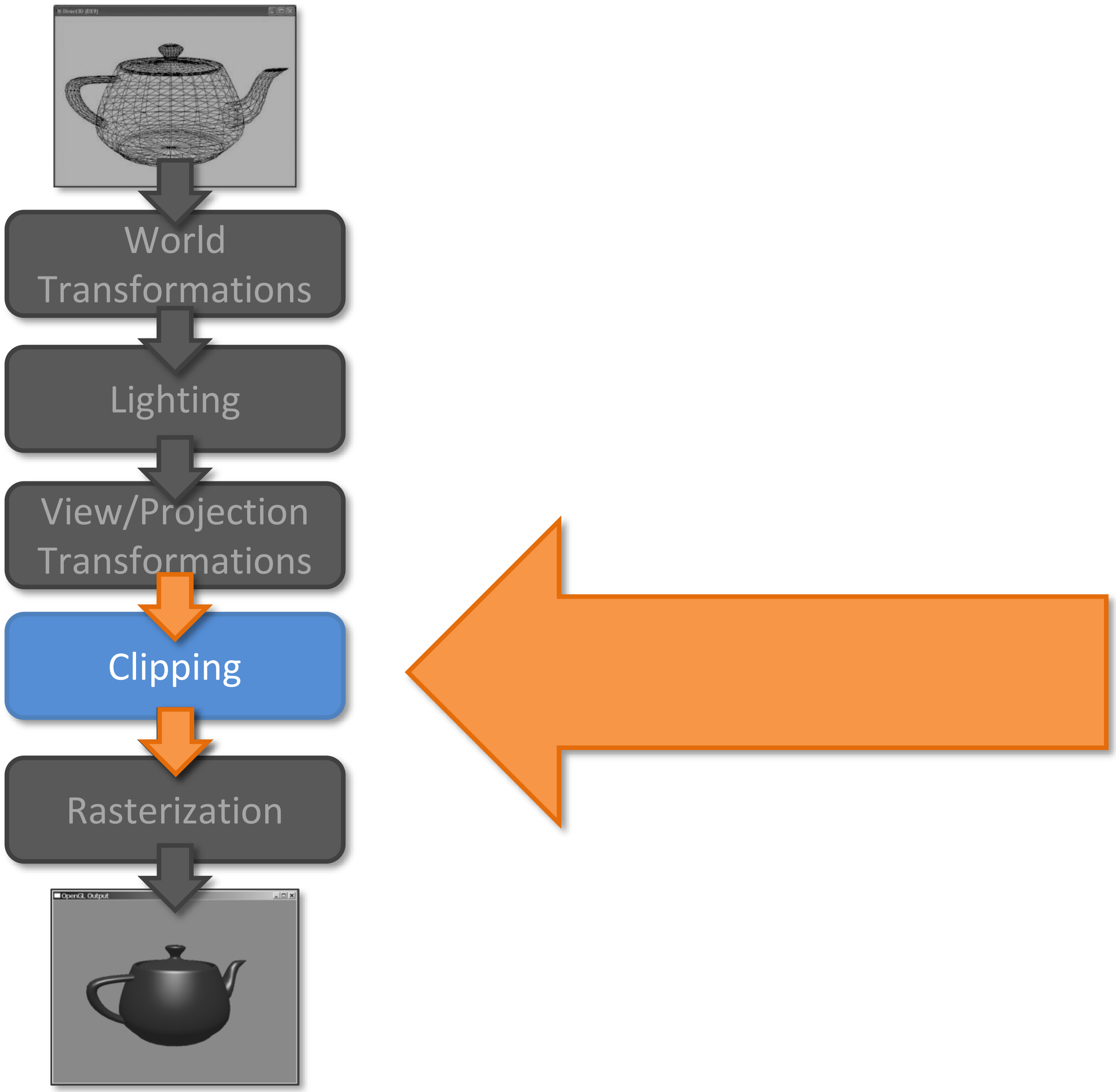
(source: Nicolas Rougier [modified])

So far…

clip coords — eye coords — world coords — object coords

$$\begin{bmatrix} clip_x \\ clip_y \\ clip_z \\ clip_w \end{bmatrix} = \text{projMat} * \text{cameraMat}^{-1} * \overbrace{\text{transMat} * \text{rotMat} * \text{scaleMat}}^{\textit{at will and in any order…}} * \begin{bmatrix} obj_x \\ obj_y \\ obj_z \\ 1 \end{bmatrix}$$

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Coordinate spaces

- **Normalized device coordinates:**
  - 4D → 3D:
    - Clip coordinates *x*, *y*, *z* divided by *w*.
  - In the range (-1, -1, -1) to (1, 1, 1): vertices not within this range are clipped.
  - *z* coordinate is still present.

Coordinate spaces

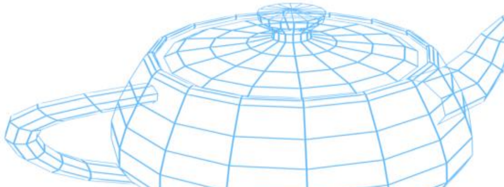
- **Screen/window coordinates:**
  - Final XY(Z) pixel coordinates:
    - Viewport transformation * normalized device coordinates = screen pixels
    - Z used for z-buffer and perspective-correct texture mapping.

$$x_{sc} = (x_{ndc}+1) \times \frac{screenWidth}{2}$$

$$y_{sc} = (y_{ndc}+1) \times \frac{screenHeight}{2}$$

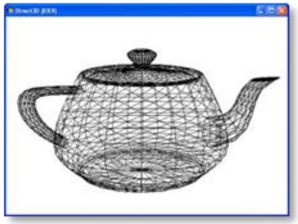
$$z_{sc} = \frac{z_{ndc} + 1}{2}$$

ndc = normalized device coordinates
sc = screen coordinates



Full pipeline

$$\begin{bmatrix} clip_x \\ clip_y \\ clip_z \\ clip_w \end{bmatrix} = \text{projMat} * \text{cameraMat}^{-1} * \text{transMat} * \text{rotMat} * \text{scaleMat} * \begin{bmatrix} obj_x \\ obj_y \\ obj_z \\ 1 \end{bmatrix}$$



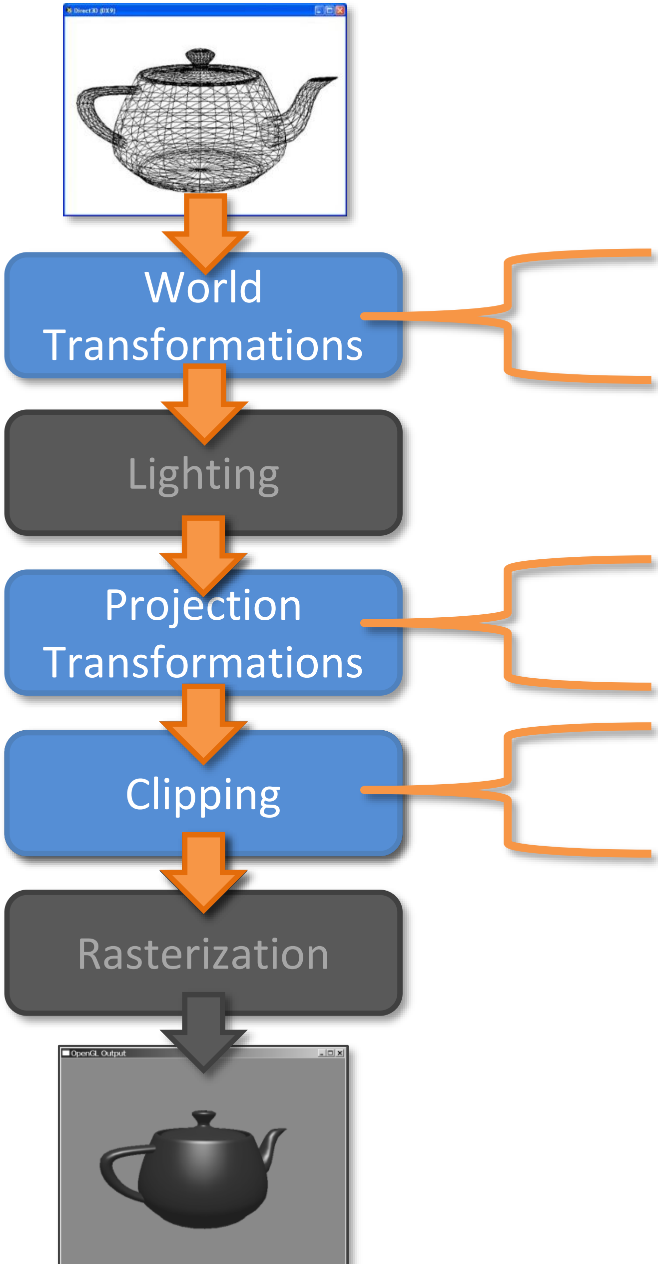
$$division\ by\ W$$

$$x_{sc} = (x_{ndc}+1) \times \frac{screenWidth}{2}$$

$$y_{sc} = (y_{ndc}+1) \times \frac{screenHeight}{2}$$

$$z_{sc} = \frac{z_{ndc}+1}{2}$$

Coordinate spaces (summary)



Object/model coordinates
World coordinates
Eye/view/camera coordinates

Clip coordinates
(Division by w)

Normalized device coordinates
Screen/window coordinates

An in-depth summary on OpenGL coordinate spaces:
http://unspecified.wordpress.com/2012/06/21/calculating-the-gluperspective-matrix-and-other-opengl-matrix-maths/

Camera demo

$$\textbf{Perspective} = \begin{bmatrix} \dfrac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \dfrac{zFar + zNear}{zNear - zFar} & \dfrac{2 \times zFar \times zNear}{zNear - zFar} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

GLM

- Transformations:

```
#include <glm/glm.hpp>
#include <glm/gtc/matrix_transform.hpp>

int foo()
{
  glm::vec4 v_obj = glm::vec4(glm::vec3(0.0f), 1.0f);
  glm::mat4 M_rot = glm::rotate(glm::mat4(1.0f),
                                glm::radians(90.0f),
                                glm::vec3(0.0f, 1.0f, 0.0f));
  glm::mat4 M_trans = glm::translate(glm::mat4(1.0f),
                                     glm::vec3(10.0f, 0.0f, 0.0f));
  glm::vec4 v_world = M_trans * M_rot * v_obj;

  return 0;
}
```

GLM

- Constants:

```
#include <glm/glm.hpp>
#include <glm/gtc/constants.hpp>

double squarePi()
{
   return glm::pi<double>() * glm::pi<double>();
}
```

GLM

- Quaternions:

```
#include <glm/glm.hpp>

int foo2()
{
   glm::mat4 M1 = …
   glm::mat4 M2 = …
   glm::quat qM1 = glm::quat(M1);
   glm::quat qM2 = glm::quat(M2);
   glm::quat qR = glm::slerp(qM1, qM2, 0.5f);
   glm::mat4 R = glm::mat4(quat_R);

   return 0;
}
```

GLM

- OpenGL (thus GLM) accesses matrices in column-major order, e.g.:

$$\begin{bmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{bmatrix}$$

← *in the documentation*

- But C arrays are stored in row-major order:

```
glm::mat4 mat( a, b, c, d,
               e, f, g, h,
               i, j, k, l,
               m, n, o, p );
```

← *in the code*