Model Building

Problem

- What are the unknowns?
- What is the Objective Function?
- What are the constraints?

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Defining Objectives

- Often there are more than one possible objectives:
  - Max profit
  - Min cost
  - Max customer satisfaction
  - Max robustness of operating plan
  - Min redundancy
  - Etc…

  Possibly some of these objectives will conflict

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Defining Constraints

Some common constraints:

- Productive capacity constraints
- Raw material availabilities
- Market demands and limitations
- Material balance
- Hard and soft constraints

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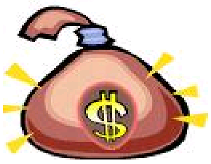
Modeling Techniques

LP: Financial Planning

- A bank makes four kinds of loans to its personal customers and these loans yield the following annual interest rates to the bank:
  - First mortgage 14%
  - Second mortgage 20%
  - Home improvement 20%
  - Personal overdraft 10%



- The bank has a maximum foreseeable lending capability of £250 million and is further constrained by the policies:

1. first mortgages must be at least 55% of all mortgages issued and at least 25% of all loans issued (in £ terms)
2. second mortgages cannot exceed 25% of all loans issued (in £ terms)
3. to avoid public displeasure and the introduction of a new windfall tax the average interest rate on all loans must not exceed 15%.

- Formulate the bank's loan problem as an LP so as to maximize interest income whilst satisfying the policy limitations.

LP: Financial Planning

- x_i = amount loaned in area i in £m (where i=1 corresponds to first mortgages, i=2 to second mortgages etc)
- limit on amount lent:

$$x_1 + x_2 + x_3 + x_4 <= 250$$

- policy condition 1:

$$x1 >= 0.55(x_1 + x_2)$$

$$x1 >= 0.25(x_1 + x_2 + x_3 + x_4)$$

- policy condition 2:

$$x_2 <= 0.25(x_1 + x_2 + x_3 + x_4)$$

- policy condition 3:

$$0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4 <= 0.15(x_1 + x_2 + x_3 + x_4)$$

- **Objective:** To maximize interest income:

Maximize $0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$

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Scheduling Postal Workers

- Each postal worker works for 5 consecutive days, followed by 2 days off, repeated weekly.

Day	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
Demand	17	13	15	19	14	16	11

- Minimize the number of postal workers (for the time being, we will permit fractional workers on each day.)

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Formulating as an LP

- Select the decision variables
  - Let x_1 be the number of workers who start working on Monday, and work till Friday
  - Let x_2 be the number of workers who start on Tuesday …
  - Let x_3, x_4, …, x_7 be defined similarly.

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The linear program

Day	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
Demand	17	13	15	19	14	16	11

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

subject to

$$
\begin{aligned}
x_1 + x_4 + x_5 + x_6 + x_7 &\geq 17 \\
x_1 + x_2 + x_5 + x_6 + x_7 &\geq 13 \\
x_1 + x_2 + x_3 + x_6 + x_7 &\geq 15 \\
x_1 + x_2 + x_3 + x_4 + x_7 &\geq 19 \\
x_1 + x_2 + x_3 + x_4 + x_5 &\geq 14 \\
x_2 + x_3 + x_4 + x_5 + x_6 &\geq 16 \\
x_3 + x_4 + x_5 + x_6 + x_7 &\geq 11
\end{aligned}
$$

$x_j \geq 0$ for $j = 1$ to 7

Solve it

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Some Enhancements of the Model

- Suppose that there was a pay differential. The cost of workers who start work on day j is c_j per worker.

Minimize $\mathbf{z = c_1\, x_1 + c_2\, x_2 + c_3\, x_3 + \ldots + c_7\, x_7}$

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Some Enhancements of the Model

- Suppose that one can hire part time workers (one day at a time), and that the cost of a part time worker on day j is PT_j.
- Let y_j = number of part time workers on day j

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What is the Revised Linear Program?

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

subject to

$$
\begin{array}{rcl}
x_1 + x_4 + x_5 + x_6 + x_7 & \geq & 17 \\
x_1 + x_2 + x_5 + x_6 + x_7 & \geq & 13 \\
x_1 + x_2 + x_3 + x_6 + x_7 & \geq & 15 \\
x_1 + x_2 + x_3 + x_4 + x_7 & \geq & 19 \\
x_1 + x_2 + x_3 + x_4 + x_5 & \geq & 14 \\
x_2 + x_3 + x_4 + x_5 + x_6 & \geq & 16 \\
x_3 + x_4 + x_5 + x_6 + x_7 & \geq & 11
\end{array}
$$

$x_j \geq 0$ for $j = 1$ to 7

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$
$+ PT_1\, y_1 + PT_2\, y_2 + \ldots + PT_7\, y_7$

subject to

$$
\begin{aligned}
x_1 + \phantom{x_2 + x_3 +{}} x_4 + x_5 + x_6 + x_7 + y_1 &\geq 17 \\
x_1 + x_2 + \phantom{x_3 + x_4 +{}} x_5 + x_6 + x_7 + y_2 &\geq 13 \\
x_1 + x_2 + x_3 + \phantom{x_4 + x_5 +{}} x_6 + x_7 + y_3 &\geq 15 \\
x_1 + x_2 + x_3 + x_4 + \phantom{x_5 + x_6 +{}} x_7 + y_4 &\geq 19 \\
x_1 + x_2 + x_3 + x_4 + x_5 + \phantom{x_6 + x_7 +{}} y_5 &\geq 14 \\
x_2 + x_3 + x_4 + x_5 + x_6 \phantom{{}+ x_7} + y_6 &\geq 16 \\
x_3 + x_4 + x_5 + x_6 + x_7 + y_7 &\geq 11
\end{aligned}
$$

$x_j \geq 0$, $y_j \geq 0$ for $j = 1$ to 7

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Another Enhancement

- Suppose that the number of workers required on day j is d_j. Let y_j be the number of workers on day j.
- The "cost" of having too many workers on day j is: $f_j(y_j – d_j)$, which is a non-linear function (e.g. if $y_i \leq d_i$ is zero otherwise positive)
- What is the minimum cost schedule?
- NOTE: this will lead to a non-linear program, not a linear program.
- We will let $s_j = y_j – d_j$ be the excess number of workers on day j.

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What is the Revised Linear Program?

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

subject to

$$
\begin{aligned}
x_1 + x_4 + x_5 + x_6 + x_7 &\geq 17\\
x_1 + x_2 + x_5 + x_6 + x_7 &\geq 13\\
x_1 + x_2 + x_3 + x_6 + x_7 &\geq 15\\
x_1 + x_2 + x_3 + x_4 + x_7 &\geq 19\\
x_1 + x_2 + x_3 + x_4 + x_5 &\geq 14\\
x_2 + x_3 + x_4 + x_5 + x_6 &\geq 16\\
x_3 + x_4 + x_5 + x_6 + x_7 &\geq 11
\end{aligned}
$$

$x_j \geq 0$ for $j = 1$ to 7

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Minimize

$$z = f_1(s_1) + f_2(s_2) + f_3(s_3) + f_4(s_4) + f_5(s_5) + f_6(s_6) + f_7(s_7)$$

subject to

$$
\begin{aligned}
x_1 + x_4 + x_5 + x_6 + x_7 - s_1 &= 17 \\
x_1 + x_2 + x_5 + x_6 + x_7 - s_2 &= 13 \\
x_1 + x_2 + x_3 + x_6 + x_7 - s_3 &= 15 \\
x_1 + x_2 + x_3 + x_4 + x_7 - s_4 &= 19 \\
x_1 + x_2 + x_3 + x_4 + x_5 - s_5 &= 14 \\
x_2 + x_3 + x_4 + x_5 + x_6 - s_6 &= 16 \\
x_3 + x_4 + x_5 + x_6 + x_7 - s_7 &= 11
\end{aligned}
$$

$$x_j \geq 0 , s_j \geq 0 \quad \text{for } j = 1 \text{ to } 7$$

A non-linear objective made linear

Suppose that one wants to minimize the maximum of the slacks, that is

$$\textbf{minimize} \quad z = \max(s_1, s_2, \ldots, s_7).$$

This is a non-linear objective.

But we can transform it, so the problem becomes an LP.

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The objective ensures that $z = s_j$ for some j.

Minimize z

The new constraint ensures that $z \geq \max(s_1, \ldots, s_7)$

subject to $z \geq s_j$ for $j = 1$ to 7.

$$
\begin{aligned}
x_1 + \quad\quad\quad\quad x_4 + x_5 + x_6 + x_7 - s_1 &= 17 \\
x_1 + x_2 + \quad\quad\quad\quad x_5 + x_6 + x_7 - s_2 &= 13 \\
x_1 + x_2 + x_3 + \quad\quad\quad\quad x_6 + x_7 - s_3 &= 15 \\
x_1 + x_2 + x_3 + x_4 + \quad\quad\quad\quad x_7 - s_4 &= 19 \\
x_1 + x_2 + x_3 + x_4 + x_5 - \quad\quad\quad\quad s_5 &= 14 \\
x_2 + x_3 + x_4 + x_5 + x_6 \quad\quad - s_6 &= 16 \\
x_3 + x_4 + x_5 + x_6 + x_7 - s_7 &= 11
\end{aligned}
$$

$x_j \geq 0$, $s_j \geq 0$ for $j = 1$ to 7

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Another non-linear objective made linear

Suppose that the "goal" is to have d_j workers on day j. Let y_j be the number of workers on day j.

Suppose that the objective is

$$\text{minimize} \quad \Sigma_i \ | y_j - d_j |$$

This is a non-linear objective.

But we can transform it, so the problem becomes an LP.

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The objective ensures that $z_j = | y_j – d_j |$ for each j.

Minimize $\Sigma_j z_j$

$z_j \geq d_j - y_j$ for j = 1 to 7.

$z_j \geq y_j - d_j$ for j = 1 to 7.

The new constraints ensure that $z_j \geq | y_j – d_j |$ for each j.

subject to

$$
\begin{aligned}
x_1 + \; x_4 + x_5 + x_6 + x_7 &= y_1 \\
x_1 + x_2 + \; x_5 + x_6 + x_7 &= y_2 \\
x_1 + x_2 + x_3 + \; x_6 + x_7 &= y_3 \\
x_1 + x_2 + x_3 + x_4 + \; x_7 &= y_4 \\
x_1 + x_2 + x_3 + x_4 + x_5 - &= y_5 \\
x_2 + x_3 + x_4 + x_5 + x_6 &= y_6 \\
x_3 + x_4 + x_5 + x_6 + x_7 &= y_7
\end{aligned}
$$

$x_j \geq 0$, $y_j \geq 0$ for j = 1 to 7

Solve it

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A ratio constraint:

Suppose that we need to ensure that at least 30% of the workers have Sunday off.

How do we model this?

$$(x_1 + x_2)/(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7) \geq .3$$

$$(x_1 + x_2) \geq .3\,x_1 + .3\,x_2 + .3\,x_3 + .3\,x_4 + .3\,x_5 + .3\,x_6 + .3\,x_7$$

$$-.7\,x_1 - .7\,x_2 + .3\,x_3 + .3\,x_4 + .3\,x_5 + .3\,x_6 + .3\,x_7 <= 0$$

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Overview

- scheduling problem
  - The model
  - Practical enhancements or modifications
  - Two non-linear objectives that can be made linear
  - A non-linear constraint that can be made linear

Other enhancements

- Require that each shift has an integral number of workers
  - integer program
- Consider longer term scheduling
  - model 6 weeks at a time
- Consider shorter term scheduling
  - model lunch breaks
- Model individual workers
  - permit worker preferences

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