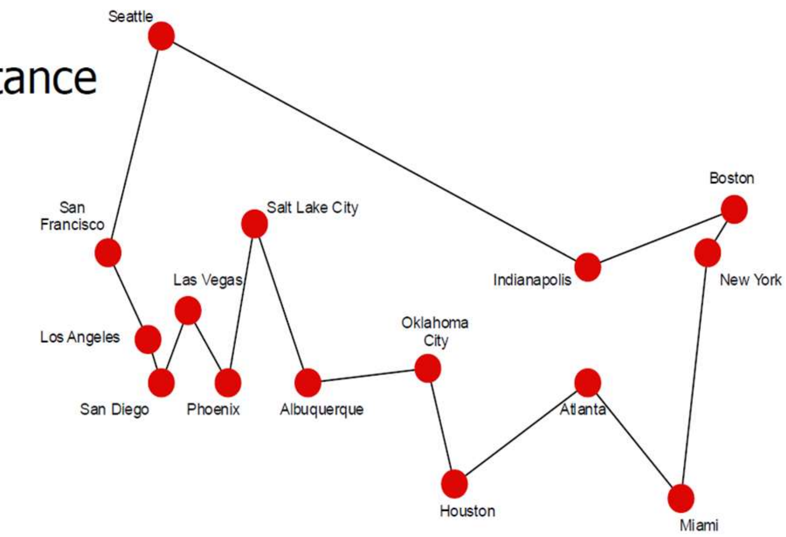
An approximation Algorithm for TSP

TSP: TRAVELING SALESPERSON PROBLEMS

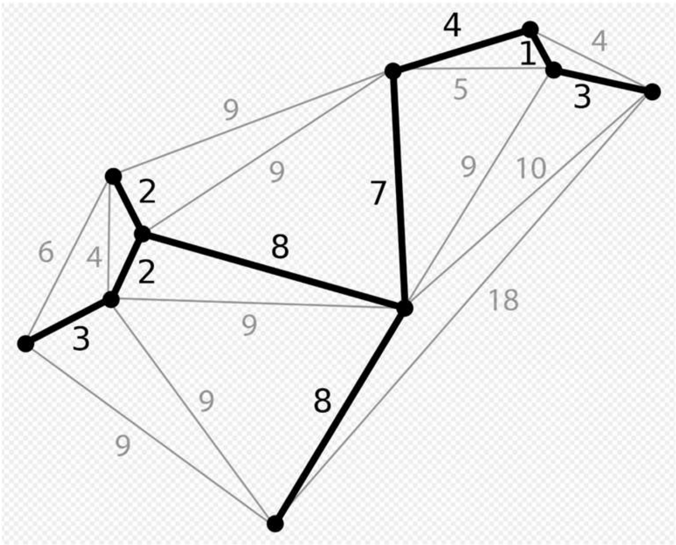
Problem: given N cities, and a distance function d between cities (usually time or kilometres), find a tour that:

- goes through every city once and only once
- minimizes the total distance



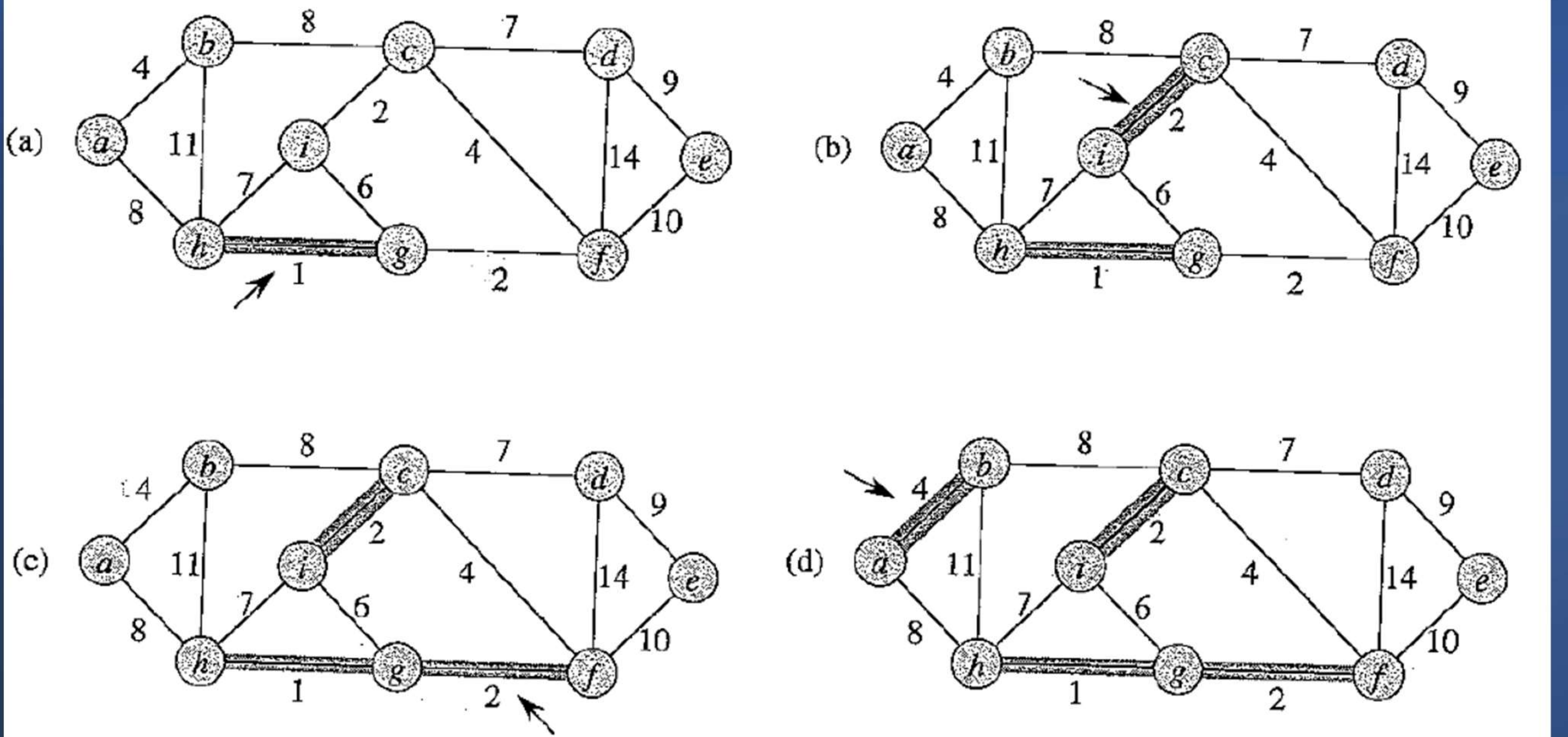
Preliminary: Min Spanning Tree

- Given an undirected graph $G=(V, E)$ with arc lengths d_{ij}'s.
- A spanning tree T is a subgraph of G that is
  - a tree (a connected acyclic graph), and
  - spans (touches) all nodes.
- Every spanning tree has $(n-1)$ arcs.
- Length of a spanning tree T is $\Sigma_{(i,j)\in T} d_{ij}$.
- The minimum spanning tree problem is to find a spanning tree of minimum length.

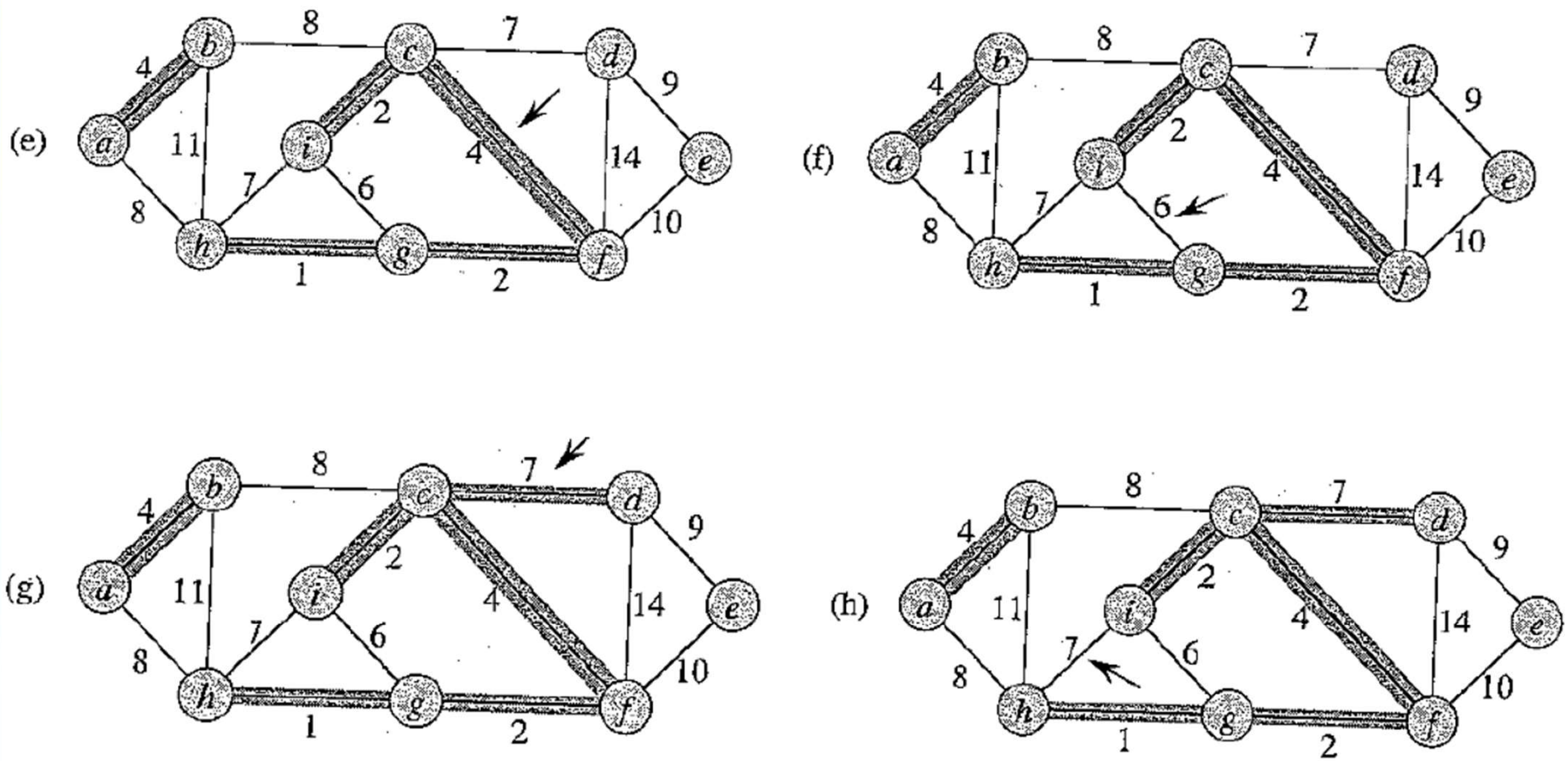


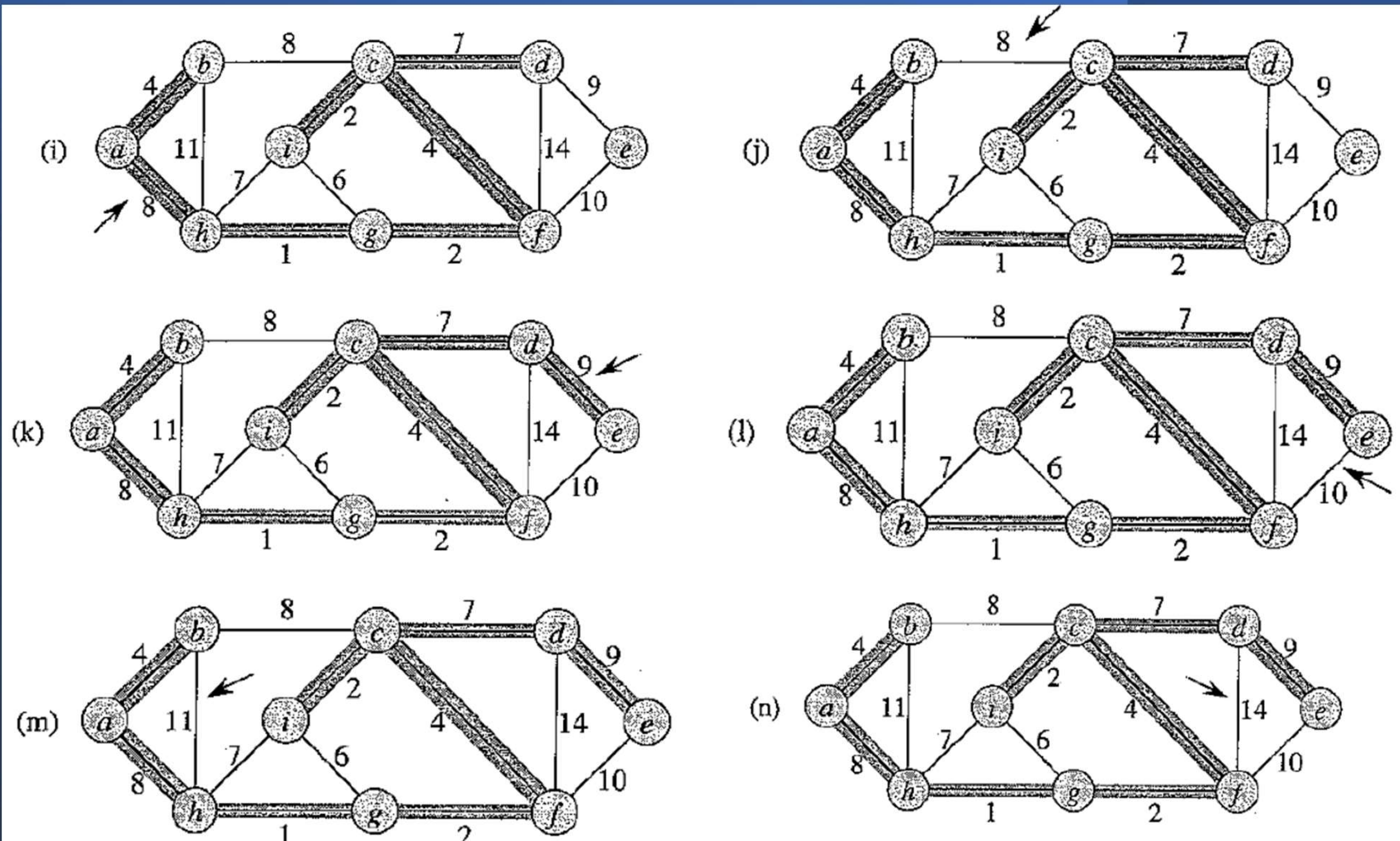
An algorithm for MST

- **Step 1:** find the shortest arc in the network. If there are more than one, pick any one randomly. Highlight this arc and the nodes connected.
- **Step 2:** Pick the next shortest arc, unless it forms a cycle with the arcs already highlighted before. Highlight the arc and the nodes connected.
- **Step 3:** If all arcs are connected, then we are done. Otherwise, repeat Step 2.



Example from Introduction to Algorithms, Cormen et all, MIT press, 1991

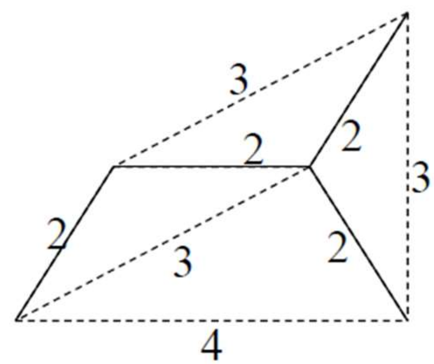

(e)
a b c d e f g h i
4 8 7 9 11 2 14 4 7 6 8 10 1 2
(f)
a b c d e f g h i
4 8 7 9 11 2 14 4 7 6 8 10 1 2
(g)
a b c d e f g h i
4 8 7 9 11 2 14 4 7 6 8 10 1 2
(h)
a b c d e f g h i
4 8 7 9 11 2 14 4 7 6 8 10 1 2


(i)
(j)
(k)
(l)
(m)
(n)

A MST based heuristic

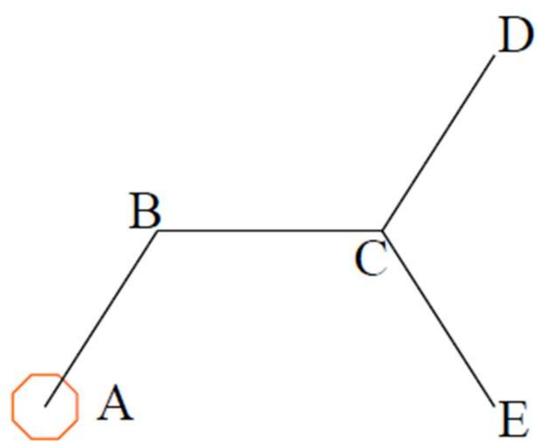
- *Step 1*: Construct a minimum spanning tree
- *Step 2*: Let the *root* be an arbitrary vertex
- *Step 3*: Traverse all the vertices by depth-first search, record the sequence of vertices (both visited and unvisited)
- *Step 4*: Use shortcut strategy to generate a feasible tour

Step 1: Construct a minimum spanning tree

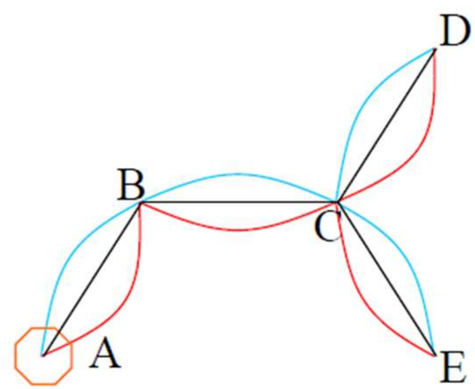


MST could be solved in $O(n^2)$, .which is also a Lower bound for TSP, $W^* \leq L^*$.

Step 2: Let the *root* be an arbitrary vertex



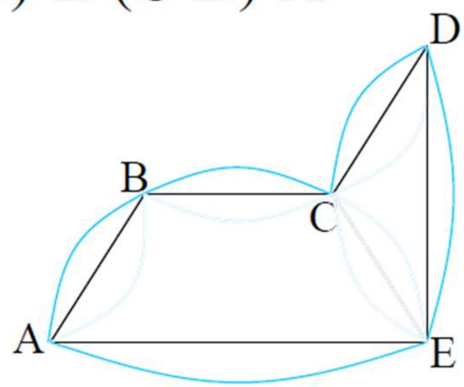
Step 3: Traverse all the vertices



The sequence is A-B-C-D-C-E-C-B-A, the length of this tour is $2W^*$.

Step 4: Use shortcut to generate the MST tour

A-B-C-D-(C)-E-(C-B)-A



The MST tour is A-B-C-D-E-A, the length of this TSP tour is less or equal to $2W^*$.

Worst-case analysis

Let L^{MST} denotes the length of the tour generated by above strategy, then we have

$$L^{MST} \leq 2W^* \leq 2L^*$$

Where W^* denotes the length of the minimum spanning tree.

And this bound is tight.

Exercise

Apply the MST algorithm to the example