SOLUZIONI

1) a) $A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$ (simmetrica reale)

$$\det(A-\lambda I) = -\lambda^3 + \frac{23}{15}\lambda^2 - \frac{127}{720}\lambda + \frac{1}{2160}$$

$$S_A = \{0{,}003 \,;\, 0{,}122 \,;\, 1{,}408\}$$

$$\|A\| \cong 1{,}408 \quad ; \quad K(A) = \frac{1{,}408}{0{,}003} \cong 469{,}333$$

b) $\vec{x} = A^{-1} \cdot \vec{b} = \begin{pmatrix} 1{,}7406 \\ -6{,}1992 \\ 12{,}864 \end{pmatrix}$

$\vec{b}_1 = \begin{pmatrix} 2{,}93 \\ 2{,}02 \\ 1{,}60 \end{pmatrix} \Rightarrow \vec{x}_1 = A^{-1} \cdot \vec{b}_1 = \begin{pmatrix} 1{,}65 \\ -5{,}64 \\ 12{,}3 \end{pmatrix}$

$$\frac{\|\vec{\Delta x}\|}{\|\vec{x}\|} \cong 5{,}56\% \quad ; \quad \frac{\|\vec{\Delta b}\|}{\|\vec{b}\|} \cong 0{,}086\%$$

$K(A)$ è molto grande, la matrice è mal condizionata. Il sistema regolato da A è perciò instabile: a fronte di un errore relativo sui dati dello 0,086%, l'errore relativo sulla soluzione potrebbe raggiungere il 40,36%!

2) $F = \begin{pmatrix} 2 & -a \\ a & -2 \end{pmatrix}$, $a > 0$

a) $F^T \cdot F = \begin{pmatrix} a^2+4 & -4a \\ -4a & a^2+4 \end{pmatrix}$

$$\det(F^T \cdot F - \mu I) = \mu^2 + (-2a^2-8)\mu + a^4 - 8a^2 + 16$$

$$S_{F^T \cdot F} = \{(a+2)^2 ; (a-2)^2\}$$

$$\|F\| = \sqrt{(a+2)^2} = a+2 \quad ; \quad K(F) = \frac{a+2}{|a-2|} \quad (a \neq 2)$$

Perciò: $K(F) = \begin{cases} \dfrac{a+2}{2-a} & 0 < a < 2 \\ \infty & a = 2 \quad (F \text{ non invertibile}) \\ \dfrac{a+2}{a-2} & a > 2 \end{cases}$

b) $\dfrac{a+2}{2-a} = 5 \iff a = \dfrac{4}{3} \in \left]0;2\right[$

$\dfrac{a+2}{a-2} = 5 \iff a = 3 \in \left]2;+\infty\right[$

c) $a = \dfrac{4}{3} \Rightarrow S_{F^TF} = \left\{ \dfrac{100}{9}; \dfrac{4}{9} \right\}$; $\|F\| = \dfrac{10}{3}$

$K(F) = 5$

$E_{\frac{100}{9}} = \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle$

$\Rightarrow \forall \vec{x} \in \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle$

$a = 3 \Rightarrow S_{F^TF} = \{25; 1\}$; $\|F\| = 5$; $K(F) = 5$

$E_{25} = \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle$

$\Rightarrow \forall \vec{x} \in \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle$

d) se $a = \dfrac{4}{3} \Rightarrow \dfrac{\|\Delta\vec{x}\|}{\|\vec{x}\|} \leq K(F) \cdot 0{,}2\% = 1\%$

se $a = 3 \Rightarrow \dfrac{\|\Delta\vec{x}\|}{\|\vec{x}\|} \leq K(F) \cdot 0{,}2\% = 1\%$

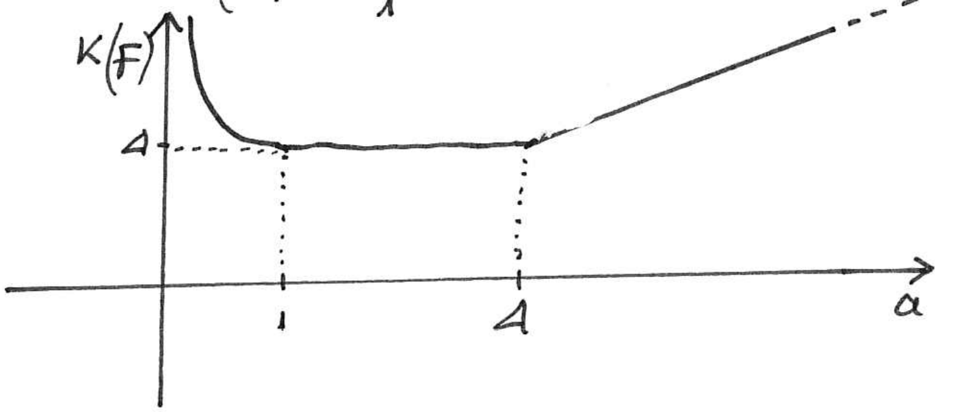
3) $F = \begin{pmatrix} 3 & 0 & 2 \\ 0 & a & 0 \\ 2 & 0 & 0 \end{pmatrix}$ simmetrica reale, $a \geq 0$

a) $\det(F - \lambda I) = -(\lambda - a)(\lambda - 4)(\lambda + 1)$

$$S_F = \{-1; 4; a\}$$

$$K(F) = \frac{\max\{1; 4; a\}}{\min\{1; 4; a\}}$$

- se $a = 0 \Rightarrow K(F) = \infty$ (F non è invertibile)
- se $0 < a < 1 \Rightarrow K(F) = \frac{4}{a}$
- se $1 \leq a < 4 \Rightarrow K(F) = \frac{4}{1} = 4$
- se $a \geq 4 \Rightarrow K(F) = \frac{a}{1} = a$



b) $K(F)$ assume valore minimo, pari a 4, se $1 \leq a \leq 4$.

4) $A = \frac{1}{16}\begin{pmatrix} 5 & 3 & -3 \\ 2 & 6 & -2 \\ -1 & 1 & 3 \end{pmatrix}$

$$\det(A-\lambda I) = -\frac{1}{64}(2\lambda-1)(4\lambda-1)(8\lambda-1)$$

$$S_A = \left\{\frac{1}{8}; \frac{1}{4}; \frac{1}{2}\right\}$$

$E_{\frac{1}{8}}$: $\operatorname{rref}\left(A-\frac{1}{8}I\right) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\qquad E_{\frac{1}{8}} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$E_{\frac{1}{4}}$: $\operatorname{rref}\left(A-\frac{1}{4}I\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\qquad E_{\frac{1}{4}} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

$E_{\frac{1}{2}}$: $\operatorname{rref}\left(A-\frac{1}{2}I\right) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\qquad E_{\frac{1}{2}} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$$\Rightarrow \vec{u}(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{\frac{1}{8}t} & 0 & 0 \\ 0 & e^{\frac{1}{4}t} & 0 \\ 0 & 0 & e^{\frac{1}{2}t} \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 2 \\ K \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{K}{2}e^{\frac{1}{8}t} + \frac{4-K}{2}e^{\frac{1}{2}t} \\ \frac{K}{2}e^{\frac{1}{4}t} + \frac{4-K}{2}e^{\frac{1}{2}t} \\ \frac{K}{2}e^{\frac{1}{8}t} + \frac{K}{2}e^{\frac{1}{4}t} \end{pmatrix}$$

$$\vec{u}(8) = \begin{pmatrix} \frac{K}{2}e + \frac{4-K}{2}e^4 \\ \frac{K}{2}e^2 + \frac{4-K}{2}e^4 \\ \frac{K}{2}e + \frac{K}{2}e^2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} e^4 + e \\ e^4 + e^2 \\ e^2 + e \end{pmatrix} \iff \boxed{K=2}$$

$$\Rightarrow \vec{u}(t) = \begin{pmatrix} e^{\frac{1}{8}t} + e^{\frac{1}{2}t} \\ e^{\frac{1}{4}t} + e^{\frac{1}{2}t} \\ e^{\frac{1}{8}t} + e^{\frac{1}{4}t} \end{pmatrix}$$

5) $A = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}$

a) $\det(A - \lambda I) = -\lambda(\lambda^2 + 1)$

$S_A = \{0; i; -i\}$

$E_0 :$ $\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad E_0 = \left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle$

$E_i :$ $\mathrm{rref}(A - iI) = \begin{pmatrix} 1 & 0 & -1-i \\ 0 & 1 & 1+i \\ 0 & 0 & 0 \end{pmatrix}$ $\quad E_i = \left\langle \begin{pmatrix} 1+i \\ -1-i \\ 1 \end{pmatrix} \right\rangle$

$E_{-i} :$ $\mathrm{rref}(A + iI) = \begin{pmatrix} 1 & 0 & -1+i \\ 0 & 1 & 1-i \\ 0 & 0 & 0 \end{pmatrix}$ $\quad E_{-i} = \left\langle \begin{pmatrix} 1-i \\ -1+i \\ 1 \end{pmatrix} \right\rangle$

$$\Rightarrow \vec{u}(t) = \begin{pmatrix} 0 & 1+i & 1-i \\ -1 & -1-i & -1+i \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-it} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1+i & 1-i \\ -1 & -1-i & -1+i \\ 1 & 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ K \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -(2K+2)\sin(t) \\ (2K+2)\sin(t) + K \\ (K+1)\cos(t) + (-K-1)\sin(t) - K \end{pmatrix}$$

$\vec{u}(t)$ contiene solo sinusoidi $\Leftrightarrow$ $K = 0$

$$\Rightarrow \vec{u}(t) = \begin{pmatrix} -2\sin(t) \\ 2\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix}$$

b) $\vec{u}\left(\frac{\pi}{2}\right) = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$