Università della Svizzera italiana

Scuola universitaria professionale della Svizzera italiana

IDSIA
Istituto Dalle Molle di studi sull'intelligenza artificiale

Linear Programming

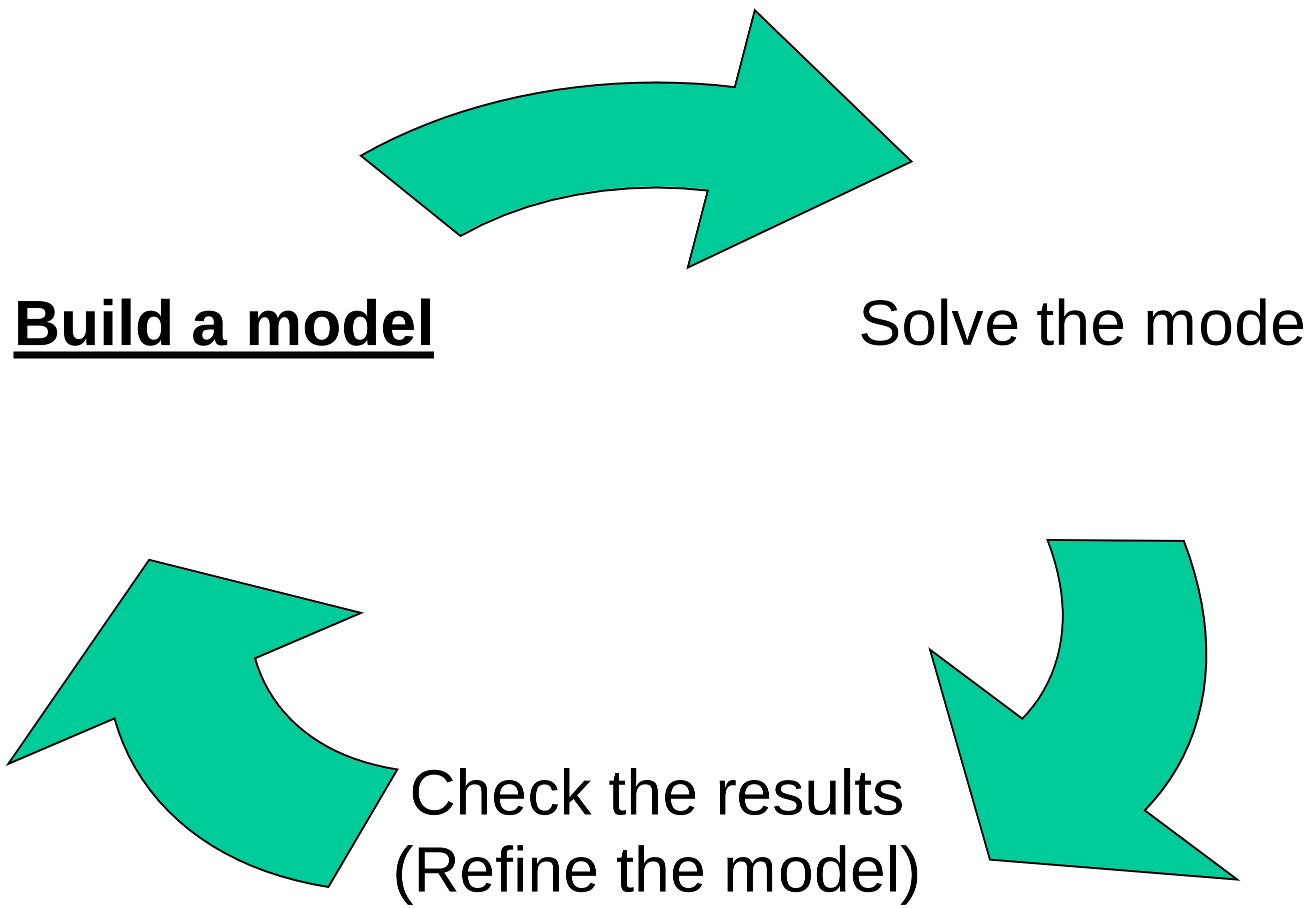
First Example

Monaldo Mastrolilli

SUPSI - IDSIA

monaldo@idsia.ch

How to solve it



Monaldo Mastrolilli
IDSIA

The Concept of a Model

- Many applications of science make use of ***models***
- Sometimes such models are concrete (e.g. model aircraft for wind tunnel exp.)
- In operational research we will be concerned with abstract models (algebraic symbolism will be used to mirror the internal relationships in the object)
- Our attention will be on mathematical models that involve mathematical relationships (such as equations, inequalities, logical dependencies, etc…)

Monaldo Mastrolilli
IDSIA

Motivations for Building Models

- The exercise of building a model often reveals relationships which were not apparent. Better understanding.
- Having built a model it is possible to analyze it
- Experimentation is possible (different scenarios)
- If we can model a problem as a Linear Program then we can solve it by using very effective solver (like CPLEX, GLPK, LPsolve, etc…)

Monaldo Mastrolilli
IDSIA

Mathematical Programming

- Definition
  - "A **mathematical programming model** is a mathematical decision model for planning (programming) decisions that optimize an objective function and satisfy limitations imposed by mathematical constraints."[1]
- General symbolic model:

Maximize (or minimize): $f(x_1, x_2 \ldots x_n)$ — Objective

Subject to:

$$\left.\begin{aligned} &g_1(x_1, x_2 \ldots x_n) \quad \{\le, \ge, =\} \quad b_1 \\ &g_2(x_1, x_2 \ldots x_n) \quad \{\le, \ge, =\} \quad b_2 \\ &\vdots \\ &g_m(x_1, x_2 \ldots x_n) \quad \{\le, \ge, =\} \quad b_m \end{aligned}\right\} \text{Constraints}$$

… where $x_1, x_2 \ldots x_n$ are the **decision variables**.

[1] T.W. Knowles, Management Science: Building and Using Models, Irwin, 1989.

Mathematical Programming (2)

- The common feature of mathematical programming models is that they all involve <u>OPTIMIZATION</u>
- We wish to *maximize* or *minimize* something (<u>objective function</u>)
- Types of mathematical programs:
  - **Linear Programs (LP):** the objective and constraint functions are linear and the decision variables are continuous.
  - **Integer Linear Programs (ILP):** one or more of the decision variables are restricted to integer values only and the functions are linear.
    - Pure IP: all decision variables are integer.
    - Mixed IP (MIP): some decision variables are integer, others are continuous.
    - 1/0 MIP: some or all decision variables are further restricted to be valued either “1” or “0”.
  - **Nonlinear Programs:** one or more of the functions is not linear.
- In this course we confine our attention to LP and ILP

Monaldo Mastrolilli
IDSIA

Linear Programs (LP)

Monaldo Mastrolilli
IDSIA

Linear Programming (LP)

- General symbolic form

$$\text{Maximize:} \quad c_1 x_1 + c_2 x_2 + \ldots c_n x_n \qquad \Big\} \text{ Objective}$$

$$\left.\begin{aligned} \text{Subject to:} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \quad \{\le, \ge, =\} \quad b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \quad \{\le, \ge, =\} \quad b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \quad \{\le, \ge, =\} \quad b_m \end{aligned}\right\} \text{ Constraints}$$

$$0 \le x_j, \quad j = 1, \ldots, n \qquad \Big\} \text{ Bounds}$$

…where a_{ij}, b_i, and c_j are the model **parameters**.

Monaldo Mastrolilli
IDSIA

Linear Programming – Example

A LP problem is an optimization problem (maximization or minimization) with a linear function and linear constraints.

Example:

$$\begin{aligned} \min\ & -2x_1 + 3x_2 \\ s.t.\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{aligned}$$

Monaldo Mastrolilli
IDSIA

Formulating LP problems: Product Mix

- An engineering factory can produce 5 types of product (P1, P2, …, P5) by using 2 production processes: grinding and drilling
- Contribution to profit:

P1	P2	P3	P4	P5
$550	$600	$350	$400	$200

- Each unit requires a certain time (hours) on each process:

	P1	P2	P3	P4	P5
Grinding	12	20	-	25	15
Drilling	10	8	16	-	-

- Each product requires the final assembly: 20 hours of 1 employee: 8 employees working one shift a day
- There are 3 grinding and 2 drilling machines
- Working time: 6 day-week with 2 shifts of 8 hours each day
- **The problem is to find how much to make of each product so as to maximize the total profit**

Monaldo Mastrolilli
IDSIA

Formulating LP problems: Product Mix (2)

What are the unknowns?

The quantity for each product: x1,x2,x3,x4,x5

What is the goal?

Max 550 x1 + 600 x2 + 350 x3 + 400 x4 + 200 x5

Which are the constraints?

Grinding capacity: 12 x1 + 20 x2 + 25 x4 + 15 x5 ≤ 288

Drilling capacity: 10 x1 + 8 x2 + 16 x3 ≤ 192

Assembly capacity: 20 x1 + 20 x2 + 20 x3 + 20 x4 + 20 x5 ≤ 384

x1,x2,x3,x4,x5 ≥ 0

Solve it

One **Linear** Objective Function

Linear Constraints

LINEAR PROGRAM

Monaldo Mastrolilli
IDSIA

The Importance of Linearity

- Nowhere terms as: x^3, e^x, x_1x_2
- Linearity: severe limitation!
- Linear programming models are easier to solve than non-linear
- Non-linear expressions can sometimes be converted into a suitable linear form

Monaldo Mastrolilli
IDSIA

Integer Linear Programs (ILP)

Monaldo Mastrolilli
IDSIA

Integer Linear Programming (ILP)

- General symbolic form

$$
\begin{array}{lll}
\text{Maximize:} & c_1x_1 + c_2x_2 + \ldots c_nx_n & \} \text{ Objective} \\
\text{Subject to:} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \quad \{\leq, \geq, =\} \quad b_1 & \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \quad \{\leq, \geq, =\} \quad b_2 & \\
& \vdots & \Big\} \text{ Constraints} \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \quad \{\leq, \geq, =\} \quad b_m & \\
& 0 \leq x_j, \quad j = 1, \ldots, n & \} \text{ Bounds}
\end{array}
$$

There are some x_j that are constrained to take integral values

…where a_{ij}, b_i, and c_i are the model **parameters**.

Monaldo Mastrolilli
IDSIA

Formulating ILP problems: Product Mix

The quantity for each product:
$x1,x2,x3,x4,x5$

Max $550\,x1 + 600\,x2 + 350\,x3 + 400\,x4 + 200\,x5$

Grinding capacity: $12\,x1 + 20\,x2 + 25\,x4 + 15\,x5 \le 288$

Drilling capacity: $10\,x1 + 8\,x2 + 16\,x3 \le 192$

Assembly capacity: $20\,x1 + 20\,x2 + 20\,x3 + 20\,x4 + 20\,x5 \le 384$

$x1,x2,x3,x4,x5 \ge 0$

Integers $x1,x2,x3,x4,x5$

Solve it

Integrality

INTEGER LINEAR PROGRAM

Monaldo Mastrolilli
IDSIA