SOLUZIONI - Serie 0

1. (a) $\displaystyle\int (x^3-2x+1)dx = \frac{x^4}{4} - x^2 + x + C$

   (b) $\displaystyle\int \left(\frac{1}{x} + 3x^2 - \sin x\right)\ dx = \ln|x| + x^3 + \cos x + C$

   (c) $\displaystyle\int (\cos x - 3^x)dx = \sin x - \frac{3^x}{\ln 3} + C$

   (d) $\displaystyle\int \left(e^x + \frac{1}{\cos^2(x)}\right) dx = e^x + \tan x + C$

2. (a) $\displaystyle\int \cos(2x+3)dx = \frac{1}{2}\cdot\sin(2x+3) + C$

   (b) $\displaystyle\int \frac{1}{\sin^2(3x+2)}dx = -\frac{1}{3}\cdot\cot(3x+2) + C$

3. (a) $\displaystyle\int \frac{e^x}{3+4e^x}dx = \frac{1}{4}\int\frac{4e^x}{3+4e^x}dx = \frac{1}{4}\cdot\ln|3+4e^x| + C = \frac{1}{4}\cdot\ln(3+4e^x) + C$

   (b) $\displaystyle\int \frac{1}{x\cdot\ln x}dx = \int\frac{\frac{1}{x}}{\ln x}dx = \ln|\ln x| + C$

4. (a) $\displaystyle\int x^2\cdot e^{x^3}dx = \frac{1}{3}\cdot\int 3x^2\cdot e^{x^3}dx = \frac{1}{3}\cdot e^{x^3} + C$

   (b) $\displaystyle\int \frac{1}{x\cdot\ln^3 x}dx = \int\frac{1}{x}\cdot(\ln x)^{-3}dx = -\frac{1}{2}\cdot(\ln x)^{-2} + C$

5. (a) $\displaystyle\int \underset{\downarrow}{x}\cdot\underset{\uparrow}{\cos} x dx = x\cdot\sin x - \int 1\cdot\sin x dx = x\cdot\sin x + \cos x + C$

   (b) $\displaystyle\int \underset{\downarrow}{x^2}\cdot\underset{\uparrow}{e^{3x}}dx = x^2\cdot\frac{e^{3x}}{3} - \int 2x\cdot\frac{e^{3x}}{3}dx = x^2\cdot\frac{e^{3x}}{3} - \frac{2}{3}\int\underset{\downarrow}{x}\cdot\underset{\uparrow}{e^{3x}}dx =$
   $\displaystyle= x^2\cdot\frac{e^{3x}}{3} - \frac{2}{3}\left(x\cdot\frac{e^{3x}}{3} - \int 1\cdot\frac{e^{3x}}{3}dx\right) = x^2\cdot\frac{e^{3x}}{3} - \frac{2}{9}x\cdot e^{3x} + \frac{2}{27}e^{3x} + C$

6. (a) $\displaystyle\int\frac{e^x}{\sqrt{1-e^{2x}}}\cdot dx \overset{(*)}{=} \int\frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin(e^x) + C$
   $(*)$: sostituzione $t = e^x$ quindi $dt = e^x\cdot dx$

   (b) $\displaystyle\int\frac{\arcsin x + x}{\sqrt{1-x^2}}\cdot dx = \int\frac{\arcsin x}{\sqrt{1-x^2}}\cdot dx + \int\frac{x}{\sqrt{1-x^2}}\cdot dx \overset{(*)}{=} \int t\cdot dt - \frac{1}{2}\int\frac{du}{\sqrt{u}} =$
   $\displaystyle= \frac{t^2}{2} - \sqrt{u} + C = \frac{(\arcsin x)^2}{2} - \sqrt{1-x^2} + C$
   $(*)$: sostituzioni $t = \arcsin x$, quindi $dt = \frac{dx}{\sqrt{1-x^2}}$ e $u = 1-x^2$ quindi $du = -2x\cdot dx$

7. (a) $\dfrac{2x}{1-x} = -2 + \dfrac{2}{1-x}$, quindi
$$\int \frac{2x}{1-x}\cdot dx = \int -2\cdot dx + \int \frac{2}{1-x}\cdot dx = -2x - 2\int \frac{-1}{1-x}\cdot dx =$$
$$= -2x - 2\ln|1-x| + C$$

(b) $\dfrac{x+7}{x^2-x-2} = \dfrac{x+7}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1} \quad \Rightarrow \quad x+7 = A(x+1)+B(x-2)$

$$x=-1: \quad 6 = -3B \quad \Rightarrow \quad B=-2$$
$$x=2: \quad 9 = 3A \quad \Rightarrow \quad A=3$$

Quindi $\dfrac{x+7}{x^2-x-2} = \dfrac{3}{x-2} - \dfrac{2}{x+1}$
$$\int \frac{x+7}{x^2-x-2}\cdot dx = 3\int \frac{dx}{x-2} - 2\int \frac{dx}{x+1} = 3\ln|x-2| - 2\ln|x+1| + C$$

8. (a) $\displaystyle\int \left(x^{72} - \frac{1}{x^3}\right)\, dx = \frac{x^{73}}{73} + \frac{x^{-2}}{2} + C$

(b) $\displaystyle\int x\cdot\cot(x^2+1)\, dx = \frac{1}{2}\int 2x\cdot\cot(x^2+1)\, dx = \frac{1}{2}\cdot\ln\left|\sin(x^2+1)\right| + C$

(c) $\displaystyle\int \frac{2x+3}{2x+1}\, dx = \int\left(\frac{2x+1}{2x+1} + \frac{2}{2x+1}\right)\, dx = \int 1\, dx + \int \frac{2}{2x+1}\, dx =$
$= x + \ln|2x+1| + C$

(d) $\displaystyle\int \frac{\sqrt{\tan x}}{\cos^2 x}\cdot dx \overset{(*)}{=} \int \sqrt{t}\cdot dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}(\tan x)^{\frac{3}{2}} + C$
$(*)$: sostituzione $t=\tan x$ quindi $dt = \frac{1}{\cos^2 x}\cdot dx$

(e) $\displaystyle\int \left(e^{3x+1} - x^2\right)\, dx = \frac{e^{3x+1}}{3} - \frac{x^3}{3} + C$

(f) $\dfrac{x}{(x-1)^3} = \dfrac{A}{(x-1)^3} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-1} \quad \Rightarrow \quad x = A + B(x-1) + C(x-1)^2$

$x=1: \quad 1 = A$
$x=0: \quad 0 = 1 - B + C \quad \Rightarrow \quad B = 1 + C$
$x=-1: \quad -1 = 1 - 2B + 4C \quad \Rightarrow \quad -2 = -2 + 2C + 4C \quad \Rightarrow \quad C=0 \quad \Rightarrow \quad B=1$

Dunque: $\dfrac{x}{(x-1)^3} = \dfrac{1}{(x-1)^3} + \dfrac{1}{(x-1)^2}$
$$\int \frac{x}{(x-1)^3}\cdot dx =$$
$$= \int \frac{1}{(x-1)^3}\cdot dx + \int \frac{1}{(x-1)^2}\cdot dx =$$
$$= \frac{(x-1)^{-3+1}}{-3+1} + \frac{(x-1)^{-2+1}}{-2+1} + C =$$
$$= -\frac{1}{2}\frac{1}{(x-1)^2} - \frac{1}{x-1} + C$$

(g) $\displaystyle\int \frac{e^x}{e^x+1}\, dx = \ln|e^x+1| + C$

(h) $\displaystyle \int x(x+1)(x+2)\ dx = \int (x^3+3x^2+2x)\ dx = \frac{x^4}{4} + x^3 + x^2 + C$

(i) $\displaystyle \int x \cdot e^{-(x^2+1)}\ dx = -\frac{1}{2} \cdot \int -2x \cdot e^{-x^2-1}\ dx = -\frac{1}{2} \cdot e^{-x^2-1} + C$

(j) $\displaystyle \int \frac{3}{4x-5}\ dx = \frac{3}{4} \cdot \int \frac{4}{4x-5}\ dx = \frac{3}{4} \cdot \ln|4x-5| + C$

(k) $\displaystyle \int \frac{x^2}{x^3-2}\ dx = \frac{1}{3} \cdot \int \frac{3x^2}{x^3-2}\ dx = \frac{1}{3} \cdot \ln\left|x^3-2\right| + C$

(l) $\displaystyle \int \frac{\sin 3x}{3+\cos 3x}\ dx = -\frac{1}{3} \cdot \int \frac{-\sin(3x)\cdot 3}{3+\cos(3x)}\ dx = -\frac{1}{3} \cdot \ln|3+\cos(3x)| + C$

(m) $\displaystyle \int \ln^2(x)\ dx = \int \underset{\uparrow}{1} \cdot \underset{\downarrow}{\ln^2(x)}\ dx = x \cdot \ln^2 x - \int x \cdot 2\ln x \cdot \frac{1}{x}\ dx =$

$\displaystyle = x \cdot \ln^2 x - 2\int \ln x\ dx = x \cdot \ln^2 x - 2(x \cdot \ln x - x) + C$

(n) $\displaystyle \int \left(\frac{1}{1+x^2} + \frac{1}{\sqrt{1-x^2}}\right)\ dx = \arctan x + \arcsin x + C$

(o) $\displaystyle \int \underset{\uparrow}{x^3} \cdot \underset{\downarrow}{\ln(x)}\ dx = \frac{x^4}{4} \cdot \ln x - \int \frac{x^4}{4} \cdot \frac{1}{x}\ dx = \frac{x^4}{4} \cdot \ln x - \frac{1}{16}x^4 + C$

(p) $\displaystyle \int x^2 \cdot \sqrt{1+x^3}\ dx = \frac{1}{3} \cdot \int 3x^2 \cdot \sqrt{1+x^3}\ dx = \frac{1}{3} \cdot \frac{2}{3} \cdot (1+x^3)^{\frac{3}{2}} + C = \frac{2}{6} \cdot (1+x^3)^{\frac{3}{2}} + C$

(q) $\displaystyle \int \frac{1}{\sqrt{4-x^2}}\ dx = \arcsin\left(\frac{x}{2}\right) + C$

(r) $\displaystyle \int \frac{\sqrt{x}+\ln x}{x}\ dx = \int \frac{\sqrt{x}}{x}\ dx + \int \frac{\ln x}{x}\ dx = \int \frac{1}{\sqrt{x}}\ dx + \int \frac{1}{x} \cdot (\ln x)^1\ dx =$
$\displaystyle 2\sqrt{x} + \frac{1}{2} \cdot (\ln x)^2 + C$

(s) $\displaystyle \int \frac{\sin(\ln x)}{x}\ dx = \int \frac{1}{x} \cdot \sin(\ln x)\ dx = -\cos(\ln x) + C$

(t) $\displaystyle \frac{x^2-15x+8}{x^3+3x^2-9x+5} = \frac{x^2-15x+8}{(x-1)^2(x+5)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+5}$

$\Rightarrow \quad x^2-15x+8 = A(x+5) + B(x-1)(x+5) + C(x-1)^2$

$$\begin{aligned} x=1: &\quad -6 = 6A \quad \Rightarrow \quad A=-1 \\ x=-5: &\quad 108 = 36C \quad \Rightarrow \quad C=3 \\ x=0: &\quad 8 = 5A-5B+C \quad \Rightarrow \quad 5B = -5+3-8 \quad \Rightarrow \quad B=-2 \end{aligned}$$

Dunque $\displaystyle \frac{x^2-15x+8}{x^3+3x^2-9x+5} = \frac{-1}{(x-1)^2} - \frac{2}{x-1} + \frac{3}{x+5}$

$\displaystyle \int \frac{x^2-15x+8}{x^3+3x^2-9x+5} \cdot dx =$

$\displaystyle = -\int \frac{dx}{(x-1)^2} - 2\int \frac{dx}{x-1} + 3\int \frac{dx}{x+5} =$

$\displaystyle = \frac{1}{x-1} - 2\ln|x-1| + 3\ln|x+5| + C$

(u) $\int \underset{\downarrow}{x^2} \cdot \underset{\uparrow}{\sin(x)}\ dx = x^2 \cdot (-\cos x) - \int 2x \cdot (-\cos x)\ dx =$

$= -x^2 \cdot \cos x + 2 \int \underset{\downarrow}{x} \cdot \underset{\uparrow}{\cos x}\ dx = -x^2 \cdot \cos x + 2 \cdot \left(x \cdot \sin x - \int 1 \cdot \sin x\ dx \right) =$

$-x^2 \cdot \cos x + 2x \cdot \sin x + 2 \cdot \cos x + C$

(v) $\int \left(\frac{1}{x^2} - \frac{1}{x^3} \right)\ dx = -x^{-1} + \frac{x^{-2}}{2} + C$

(w) $\int \frac{1}{x \cdot (4 - \ln^2 x)} \cdot dx \overset{(*)}{=} \int \frac{dt}{4 - t^2} = - \int \frac{dt}{t^2 - 4} = -\frac{1}{4} \ln \left| \frac{t-2}{t+2} \right| + C = \frac{1}{4} \ln \left| \frac{\ln x + 2}{\ln x - 2} \right| + C$

$(*)$: sostituzione $t = \ln x$ quindi $dt = \frac{1}{x} \cdot dx$

(x) $\int (x^2+3)^2\ dx = \int (x^4 + 6x^2 + 9)\ dx = \frac{x^5}{5} + 2x^3 + 9x + C$

(y) $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\ dx = 2 \cdot \int \frac{1}{2\sqrt{x}} \cdot \cos(\sqrt{x})\ dx = 2 \cdot \sin(\sqrt{x}) + C$

9. (a) $f(x) = 3x^2 - x^3$. Vale $f(x) = 0 \quad \Leftrightarrow \quad x = 0$ oppure $x = 3$, quindi

$$\mathcal{A} = \int_0^3 f(x)dx = x^3 - \tfrac{1}{4}x^4 \Big|_0^3 = (27 - \tfrac{81}{4}) - 0 = \tfrac{27}{4}$$

(b) $f(x) = -x^3 + 4x^2 - 3x$. Vale $f(x) = 0 \quad \Leftrightarrow \quad x = 0$ oppure $x = 1$ oppure $x = 3$, quindi

$$\begin{aligned} \mathcal{A} &= -\int_0^1 f(x)dx + \int_1^2 f(x)dx = \\ &= -\left(-\tfrac{1}{4}x^4 + \tfrac{4}{3}x^3 - \tfrac{3}{2}x^2\right)\Big|_0^1 + \left(-\tfrac{1}{4}x^4 + \tfrac{4}{3}x^3 - \tfrac{3}{2}x^2\right)\Big|_1^2 = \ldots = \tfrac{3}{2} \end{aligned}$$

(c) $f(x) = x^3 - 6x^2 + 9x$ e $g(x) = -\frac{1}{2}x^2 + 2x$. Vale

$$f(x) = g(x) \quad \Leftrightarrow \quad \ldots \quad \Leftrightarrow \quad x_1 = 0 \vee x_2 = 2 \vee x_3 = \tfrac{7}{2}$$

Quindi

$$\begin{aligned} \mathcal{A} &= \int_0^2 (f(x) - g(x))dx + \int_2^{\frac{7}{2}} (g(x) - f(x))dx = \\ &= \int_0^2 (x^3 - \tfrac{11}{2}x^2 + 7x)dx + \int_2^{\frac{7}{2}} (-x^3 + \tfrac{11}{2}x^2 - 7x)dx = \\ &= \left(\tfrac{x^4}{4} - \tfrac{11}{6}x^3 + \tfrac{7}{2}x^2\right)\Big|_0^2 + \left(-\tfrac{x^4}{4} + \tfrac{11}{6}x^3 - \tfrac{7}{2}x^2\right)\Big|_2^{\frac{7}{2}} = \ldots = \frac{937}{192} \cong 4.88 \end{aligned}$$

(d) Punto di intersezione:

$$\begin{cases} y = \frac{1}{2}x^2 \\ y = \frac{1}{2}x + 1 \end{cases} \quad \Leftrightarrow \quad \ldots \quad \Leftrightarrow \quad x = \left\langle \begin{array}{l} 2 \\ -1 < 0 \text{ non acc} \end{array} \right.$$

Quindi l'area colorata è

$$\begin{aligned} \mathcal{A} &= \int_0^2 \left(\tfrac{1}{2}x + 1 - \tfrac{1}{2}x^2\right) \cdot dx + \int_2^3 \left(\tfrac{1}{2}x^2 - \tfrac{1}{2}x - 1\right) \cdot dx = \\ &= \left(\tfrac{x^2}{4} + x - \tfrac{x^3}{6}\right)\Big|_0^2 + \left(\tfrac{x^3}{6} - \tfrac{x^2}{4} - x\right)\Big|_2^3 = \ldots = \tfrac{31}{12} \end{aligned}$$

10. $f(x) = ax - x^3 = 0 \quad \Leftrightarrow \quad x = 0 \vee x = \pm\sqrt{a}$. Il parametro a deve essere positivo, altrimenti $f(x) < 0$ per $x > 0$, quindi il grafico di f non si troverebbe nel primo quadrante. Dunque

$$\mathcal{A} = \int_0^{\sqrt{a}} (ax - x^3)dx = a\tfrac{x^2}{2} - \tfrac{x^4}{4}\Big|_0^{\sqrt{a}} = \tfrac{a^2}{2} - \tfrac{a^2}{4} = \tfrac{a^2}{4} = 9 \quad \Rightarrow \quad a = 6$$

11. (a) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx = \arctan x\Big|_{-\infty}^{+\infty} = \lim_{x\to+\infty} \arctan x - \lim_{x\to-\infty} \arctan x = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

(b) $\displaystyle\int_{-1}^{0} \frac{1}{\sqrt{1+x}}\, dx = 2\sqrt{1+x}\Big|_{-1}^{0} = 2 - 0 = 2$

(c) $\displaystyle\int_0^1 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_0^1 = -\left(1 - \lim_{x\to0^+} \frac{1}{x}\right) = +\infty$, quindi l'integrale diverge.

(d) $\displaystyle\int_0^{+\infty} 2^{-x}\, dx = \left(-\frac{2^{-x}}{\ln 2}\right)\Big|_0^{+\infty} = -\frac{1}{\ln 2} \cdot \left(\lim_{x\to+\infty} 2^{-x} - 2^0\right) = \frac{1}{\ln 2}$