

Common subtrees of independent random trees (and other common substructures problems)

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Acknowledgements

Based on joint work with:



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Common substructure problems

1: A TOUR OF COMMON SUBSTRUCTURES.

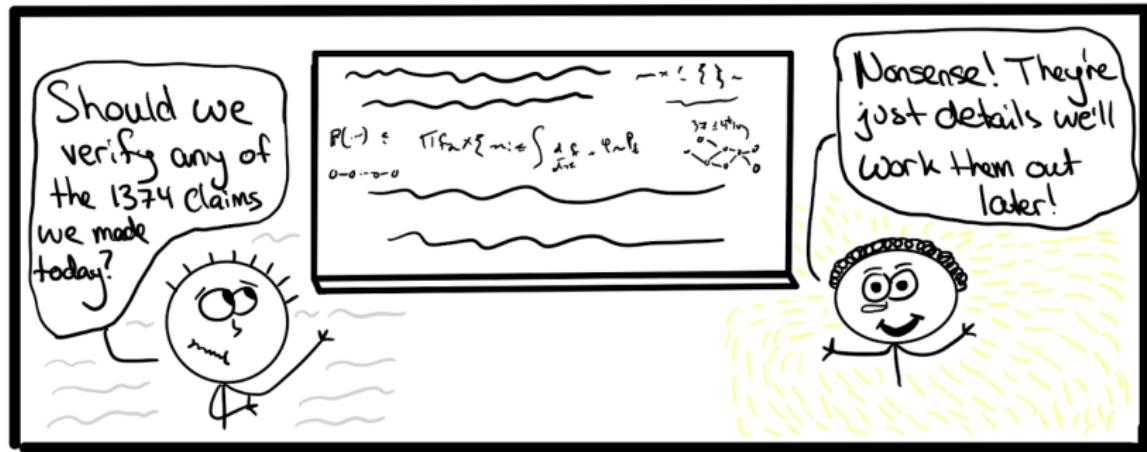
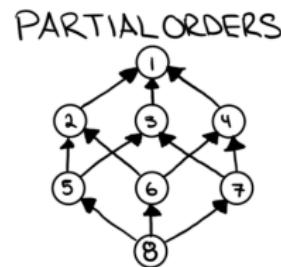
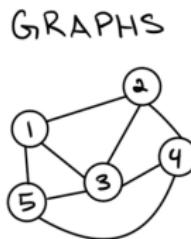
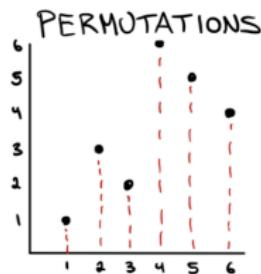


Figure: The two states of mind while doing math on the board.

Common substructure problems

Let \mathcal{S}_n be a collection of combinatorial structures built from $[n]$ with the following property:

- for any $S \in \mathcal{S}_n$ and $A \subseteq [n]$, there is some induced substructure of S on A .



Challenge: For two independent structures $S, S' \in \mathcal{S}_n$ drawn from a probability measure μ_n analyze the following quantity:

$$|\text{LCS}(S, S')| = \max \left\{ |T| : T \text{ a common substructure of } S \text{ and } S' \right\}.$$

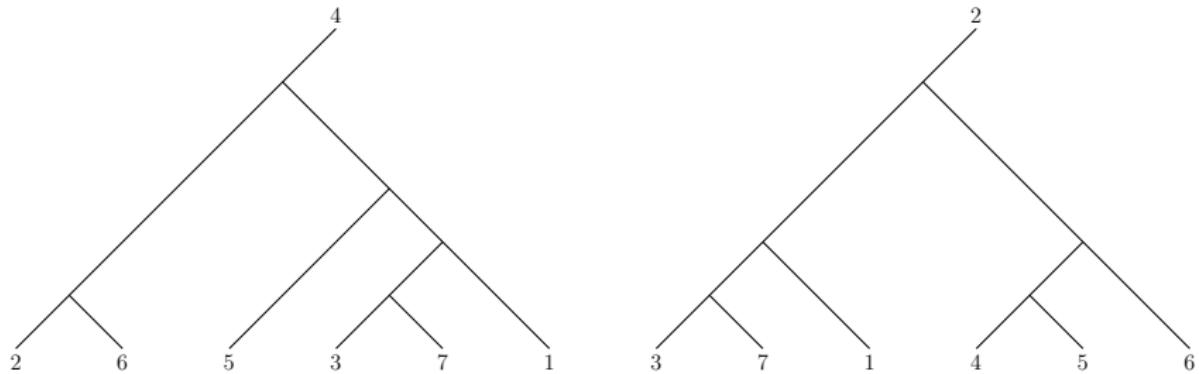
Common substructure problems

Example: The LCS of two independent strings of length n over the alphabet $\{1, \dots, k\}$, X_n and Y_n .

- Many applications in genetics and computational biology.
- By super-additivity, $E[|\text{LCS}(X_n, Y_n)|] n^{-1} \rightarrow \gamma_k$ as $n \rightarrow \infty$. [Chvatal, Sankoff 1975.]
- $\sqrt{k}\gamma_k \rightarrow 2$ as $k \rightarrow \infty$ [Kiwi, Loebl, Matoušek 2003].
- $\frac{|\text{LCS}(X_n, Y_n)| - \gamma_k}{\text{Var}(|\text{LCS}(X_n, Y_n)|)} \rightarrow N(0, 1)$ as $n \rightarrow \infty$. [Houdré, İslak 2023].

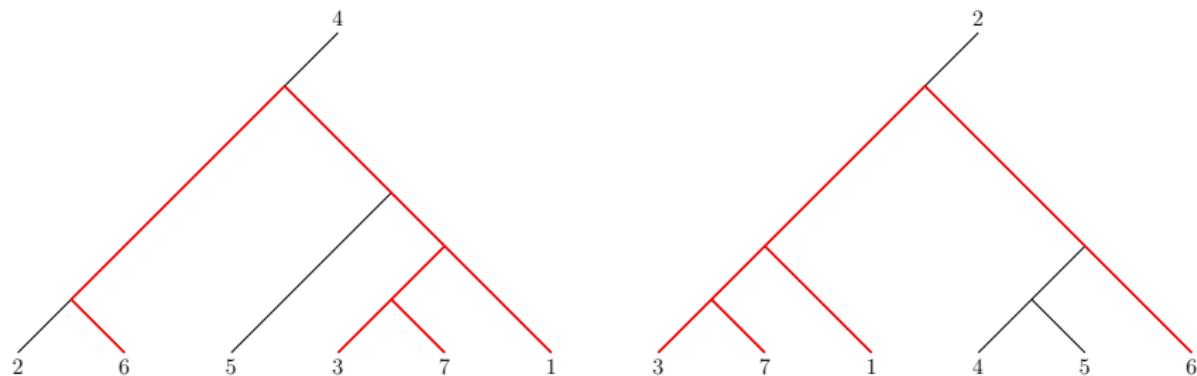
Common substructure problems

Example: The LCS (or MAST) of two independent uniform cladograms.



Common substructure problems

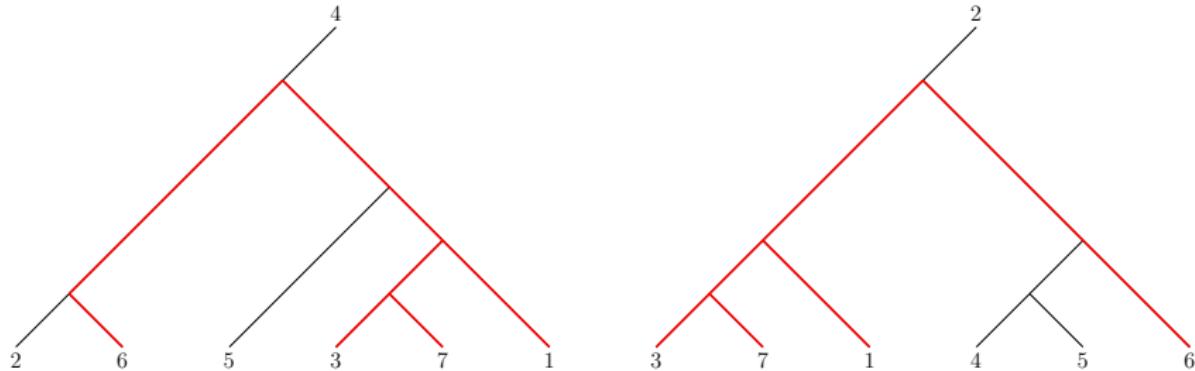
Example: The LCS (or MAST) of two independent uniform cladograms.



- The highlighted parts of the tree is the LCS. The leaves $\{1, 3, 6, 7\}$ have the same ancestral relationships in both trees.

Common substructure problems

Example: The LCS (or MAST) of two independent uniform cladograms.



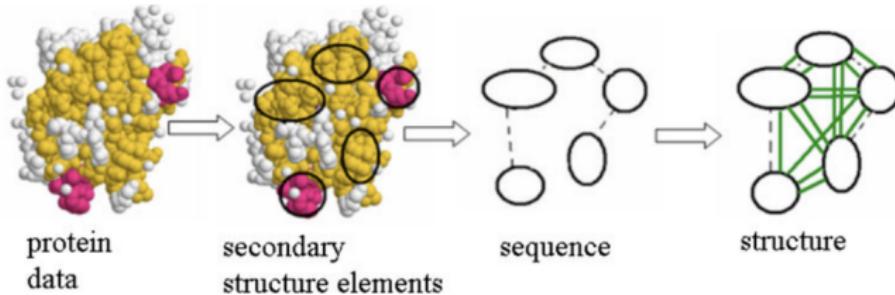
Best known bounds: $n^{0.446} \leq \text{LCS} \leq n^{1/2-\epsilon}$.

LB: [Khezeli (2022)] UB: Budzinski, Sénizergues (2023)]

Conjecture: $\text{LCS} = n^{\gamma+o(1)}$ for $\gamma < 1/2$. [Aldous (2022)]

Common substructure problems

Application (network correlation): We can use sizes of common subgraphs as a way to measure similarity in graphs! This has been studied a lot under the name of **graph matching**.

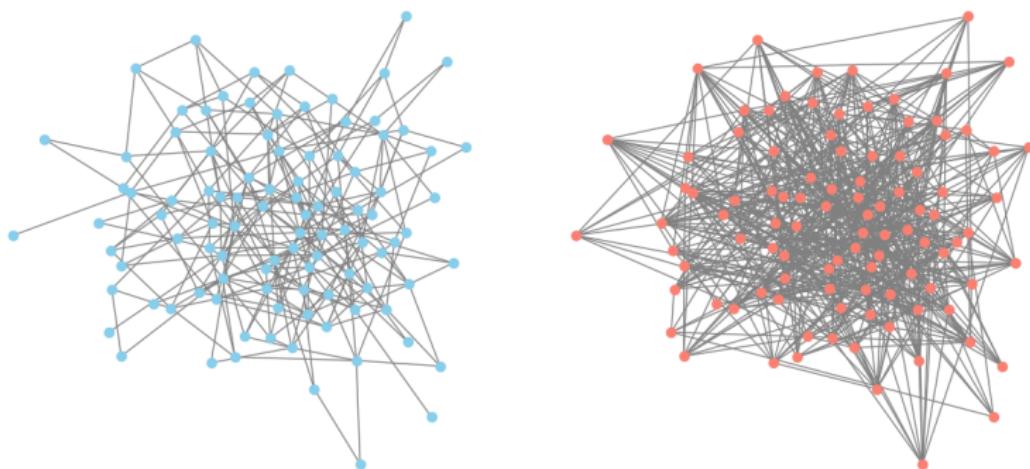


[Livi, Rizzi 2013.]

Common substructure problems

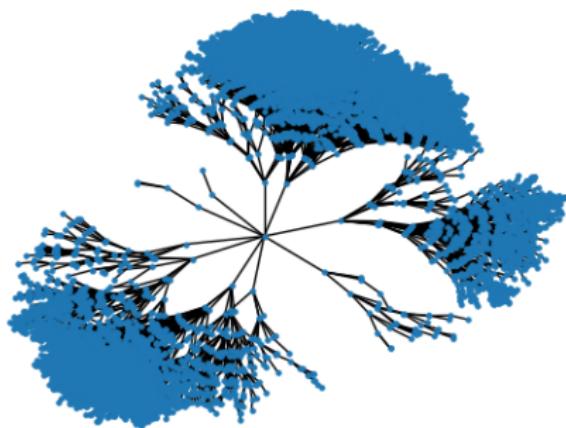
Example: The largest common induced subgraph of two Erdős-Rényi random graphs, with equivalence up to isomorphism.

- Connects to the famous graph isomorphism problem in computer science.
- The LCS in the dense case has **logarithmic size** and exhibits **two point concentration**. [Chatterjee, Diaconis 2023. Surya, Warnke, Zhu 2025.]



Common substructure problems

Example: The largest common subtree of two independent uniform random recursive trees. [Baumler, Kerriou, Martin, Lodewijks, Powierski, Rácz, Sridhar 2025+.]



Best bounds: $n^{0.83} \leq \text{LCS}(T_n, T'_n) \leq 0.99n$
Conjecture: $\text{LCS}(T_n, T'_n) = n^{1-o(1)}$.

Bienaym   trees

2: THE LCS OF CONDITIONED BIENAYM   TREES.



Figure: Its easy to forget that everyone is not thrilled by random trees.

Bienaymé trees

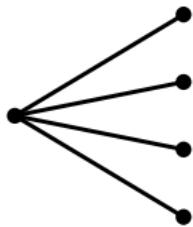
A random tree model: Given μ a measure on $\{0, 1, 2, \dots\}$ we define a random plane tree T as follows:

- Start with a root, it has a random number of children drawn from μ .
- Given the tree up to generation k , give each vertex in generation k a number of children drawn independently from μ .

Bienaymé trees

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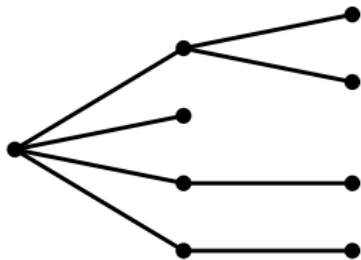
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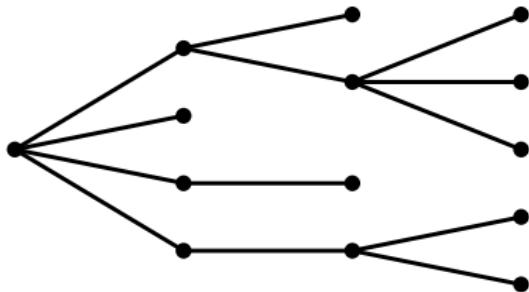
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Bienaymé trees

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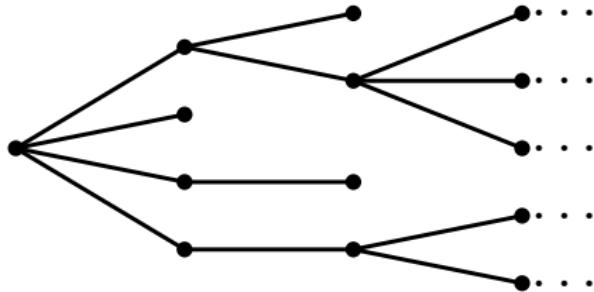
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Bienaymé trees

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Bienaymé trees

- We will focus on trees with offspring distribution μ such that $\sum_{j=1}^{\infty} j\mu(j) = 1$ and $\sigma^2 = \sum_{j=0}^{\infty} \mu(j)(j - 1)^2 < \infty$. These are right on the boundary of being finite.
- Specifically, we care about large- n asymptotics of critical Bienaymé trees conditioned to have size n .

Proposition [Kesten, Ney, and Spitzer 1966]

$$\mathbf{P}(\text{Ht}(\tau) \geq x) \sim \frac{2}{x\sigma^2}.$$

Proposition [Folklore 1900s]

$$\mathbf{P}(|\tau| = n) \sim c_1 n^{-3/2} \text{ and } \mathbf{P}(|\tau| \geq n) \sim c_2 n^{-1/2}.$$

Bienaymé trees

Your favourite tree is a Bienaymé tree: By picking μ carefully and conditioning on our trees to have size n we get many canonical trees.

- $\mu(d) = 1 - \mu(0) = 1/d$ for some $d \geq 2 \implies$ uniform d -ary tree.
- $\mu = \text{Geometric}(1/2)$ for all $k \geq 0 \implies$ uniform rooted plane tree.
- $\mu = \text{Poisson}(1)$ for all $k \geq 0$ plus a randomly labelling the vertices \implies uniform labelled tree.

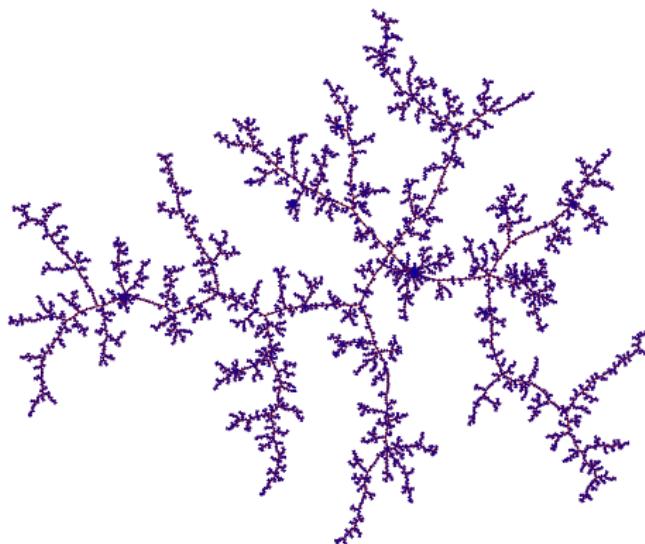
Notation: $\tau_n \leftrightarrow \tau$ conditioned to have size n .

Bienaym  trees

Conditioned Bienaym  trees are quite spiny:

- The height of τ_n is of the order \sqrt{n} .
- The distance between uniform random vertices is of order \sqrt{n} .

Are common subtrees of them similarly spiny?



[images by Igor Kortchemski!]

LCS of independent Bienaym   trees

Thm (Angel, A., Brandenberger, Donderwinkel, Khanfir 2025+)

Let τ_n and τ'_n be two independent Bienaym   trees conditioned to have size n with a mutual offspring distributions μ such that

$$\sum_{j=1}^{\infty} j^{2+\kappa} \mu(j) < \infty.$$

Then, there exists $X > 0$ such that

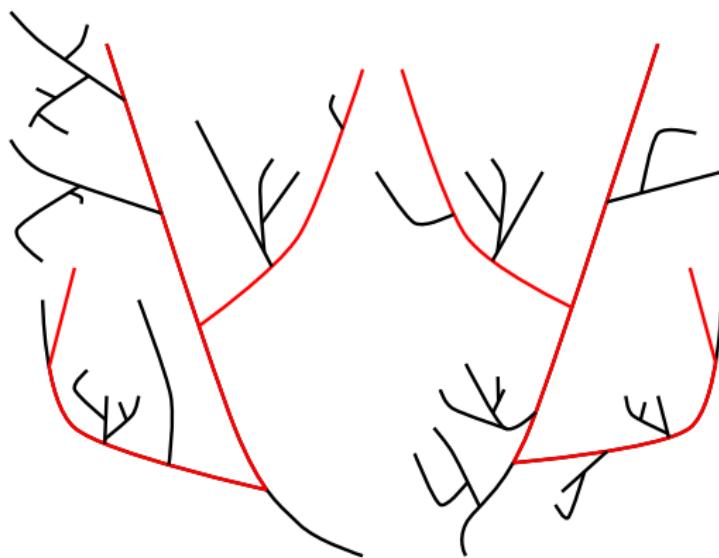
$$\frac{1}{\sqrt{n}} |\text{LCS}(\tau_n, \tau'_n)| \xrightarrow{d} X.$$

TLDR/Heuristic: Large common subtrees under a second and $(2 + \kappa)$ th moment assumption above are super thin.

LCS of independent Bienaym   trees

Question: What is the distribution of X ?

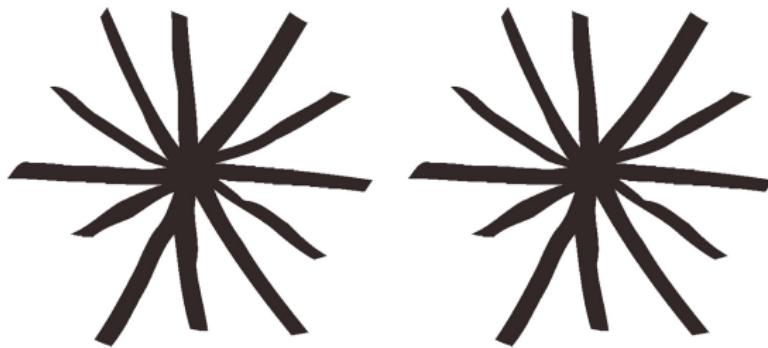
Answer: The limiting length of the longest common Y between τ_n and τ'_n up to a constant depending on μ . (A Y is a subtree with exactly one degree 3 vertex, and no degree ≥ 4 vertex.)



LCS of independent Bienaym   trees

Question: Is the $2 + \kappa$ moment assumption actually necessary?

Answer: Yes, it is used to avoid large degrees, which allow the creation of common stars. There are counterexamples when the $2 + \kappa$ condition fails.



[Vonnegut 1973]

The issue of large degrees

4: THE ISSUE OF LARGE DEGREES.

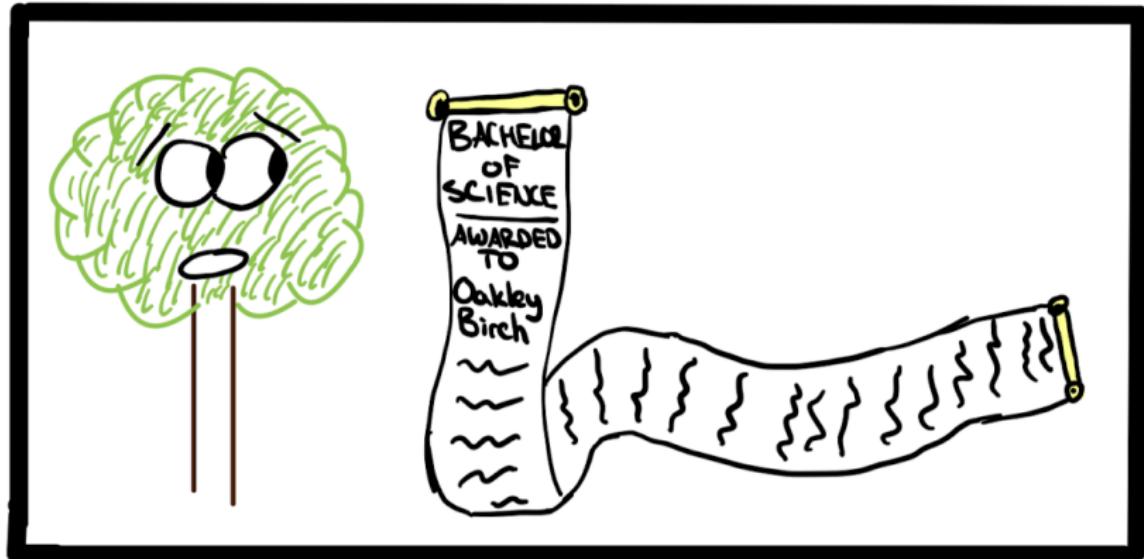
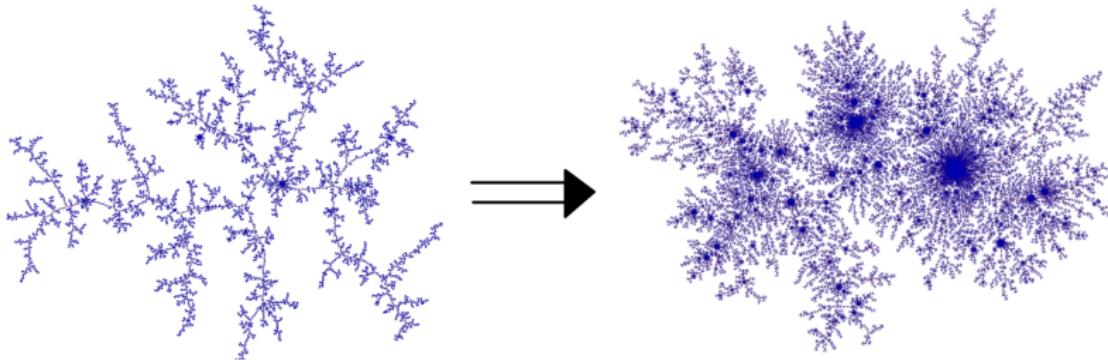


Figure: A tree with a concerningly large degree.

The issue of large degrees

“Facts”: Degrees in conditioned Bienaymé trees behave like i.i.d. μ distributed random variables. If we assume the largest finite moment of μ is a γ th moment for $\gamma > 1$, then we should expect the maximum degree to be order $n^{1/\gamma}$.



- Once the largest degree in our two trees gets too close to \sqrt{n} , the LCS starts to change.

The issue of large degrees

Thm (Angel, A., Brandenberger, Donderwinkel, Khanfir 2025+)

Let μ be critical, satisfying $\mu(k) \sim ck^{-3} \log^{-3/2}(k)$. Then, for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\liminf_{n \rightarrow \infty} \mathbf{P}\left(|\text{LCS}(\tau_n, \tau'_n)| > \delta \log^{1/4}(n) \sqrt{n}\right) > 1 - \varepsilon.$$

In particular, there is a critical offspring distribution such that $\text{LCS}(\tau_n, \tau'_n) \geq \log^{1/4}(n) \sqrt{n}$ and

$$\sum_{k \geq 0} \mu(k) k^2 \log^{1/4}(k) < \infty.$$

The issue of large degrees

Extreme value theory: Let $(X_i)_{i=1}^n$ and $(Y_i)_{i=1}^n$ be non-negative i.i.d. random variables with tails like

$$\mathbf{P}(X_1 \geq x) = \mathbf{P}(Y_1 \geq x) \sim cx^{-1}.$$

The i th order statistic of $(X_i)_{i=1}^n$ and $(Y_i)_{i=1}^n$ (the i th largest entry of the respective vectors) are both close to $\frac{n}{i}$. Thus,

$$\sum_{i=1}^n (X_i \wedge Y_i) \approx C \sum_{i=1}^n \frac{n}{i} \wedge \frac{n}{i} \approx Cn \log(n)$$

The issue of large degrees

How to build the counter-example:

- We can find vertices of out-degree $\Theta(\sqrt{n} \log^{-3/4}(n)) = \Delta_n$ in both τ_n and τ'_n .
- the subtrees rooted above a vertex **essentially** behave like independent unconditioned Bienaymé trees.
- The heights of unconditioned Bienaymé trees satisfy $\mathbf{P}(\text{Ht}(\tau) \geq x) \sim cx^{-1}$.

Order the subtrees $\tau_n(1), \dots, \tau_n(\Delta_n)$ and $\tau'_n(1), \dots, \tau'_n(\Delta_n)$ in decreasing order of height and match the tallest subtrees.

$$|\text{LCS}(\tau_n, \tau'_n)| \geq \sum_{i=1}^{\Delta_n} \text{Ht}(\tau_n(i)) \wedge \text{Ht}(\tau'_n(i)) \approx C\Delta_n \log(\Delta_n).$$

Proof sketch (rooted trees)

3: WHY IS THE LCS THIN?

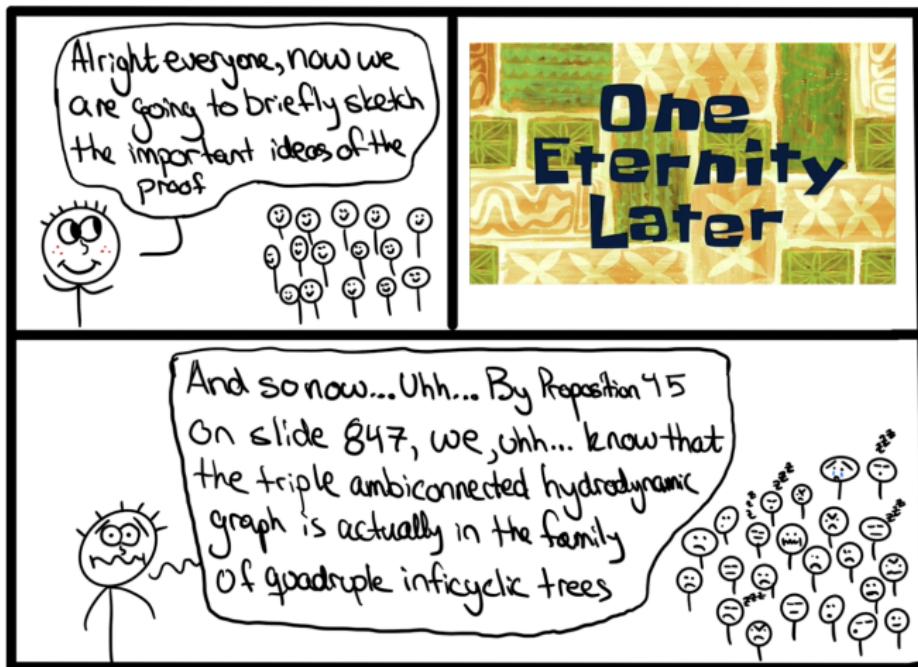


Figure: Trying to explain a proof of your favourite theorem.

Proof sketch (rooted trees)

Theorem

For any $\epsilon > 0$, $\mathbf{P}(|\text{LCS}^\bullet(\tau_n, \tau'_n)| \geq n^{1/2+\epsilon}) \rightarrow 0$.

Lemma

For any $\epsilon, \gamma > 0$ there is a $C > 0$ so that

$$\mathbf{P}\left(\underbrace{\{|\text{LCS}^\bullet(\tau, \tau')| \geq h^{1+\epsilon}\} \cap \{\text{Ht}(\tau) \wedge \text{Ht}(\tau') \leq h\}}_{:= P_{\epsilon,h}}\right) \leq Ch^{-\gamma}.$$

Lemma

For any $\epsilon, \nu > 0$, $\mathbf{P}(P_{\epsilon,h}) \leq C\mathbf{P}(P_{\epsilon-\nu,h}) \frac{1}{h^{\epsilon-\nu}} + Ch^2 \exp(-h^{\nu/2})$.

Proof sketch (rooted trees)

Theorem

For any $\epsilon > 0$, $\mathbf{P}(|\text{LCS}^\bullet(\tau_n, \tau'_n)| \geq n^{1/2+\epsilon}) \rightarrow 0$.

Proving Theorem using Lemma:

- From last slide we can choose γ large enough that

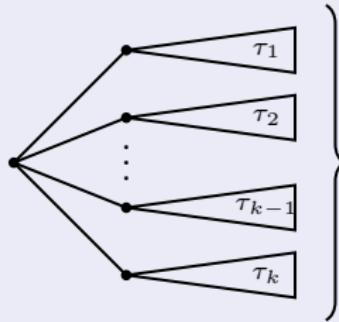
$$\mathbf{P}(P_{\epsilon/2, n^{1/2+\epsilon/2}} \mid |\tau| = |\tau'| = n) \rightarrow 0.$$

- From known results about Bienaymé tree heights we know that

$$\mathbf{P}(\text{Ht}(\tau) \wedge \text{Ht}(\tau') \geq n^{1/2+\epsilon/2} \mid |\tau| = |\tau'| = n) \rightarrow 0.$$

Proof sketch (rooted trees)

Proposition (the branching property)



The subtrees τ_i and τ_j are i.i.d. Bienaymé trees for all $1 \leq i < j \leq k$.

Proposition

There exist $c_1, c_2 > 0$ such that $\mathbf{P}(|\tau| = n) \sim c_1 n^{-3/2}$ and $\mathbf{P}(|\tau| \geq n) \sim c_2 n^{-1/2}$.

Proposition (a linear bound for $|\text{LCS}^\bullet(\tau, \tau')|$)

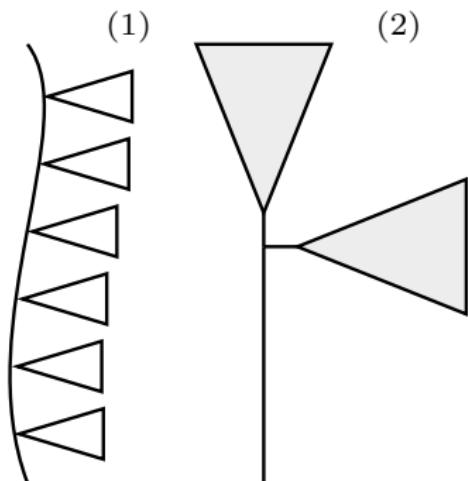
$$\mathbf{P}(|\text{LCS}^\bullet(\tau, \tau')| \geq n) \leq \mathbf{P}(|\tau| \geq n) \mathbf{P}(|\tau'| \geq n) \sim c_2^2 n^{-1}.$$

Proof sketch (rooted trees)

$$P_{\epsilon,h} = \{|\text{LCS}^\bullet(\tau, \tau')| \geq h^{1+\epsilon}\} \cap \{\text{Ht}(\tau) \wedge \text{Ht}(\tau') \leq h\}$$

Idea: Build a path \mathcal{P} in the LCS^\bullet from the root, where we always walk into the largest subtree. There are two cases:

- 1 Each subtree hanging off of \mathcal{P} is smaller than $h^{1+\epsilon-\nu}$;
- 2 There is a vertex on \mathcal{P} that has some subtree of size at least $h^{1+\epsilon-\nu}$ hanging off \mathcal{P} .

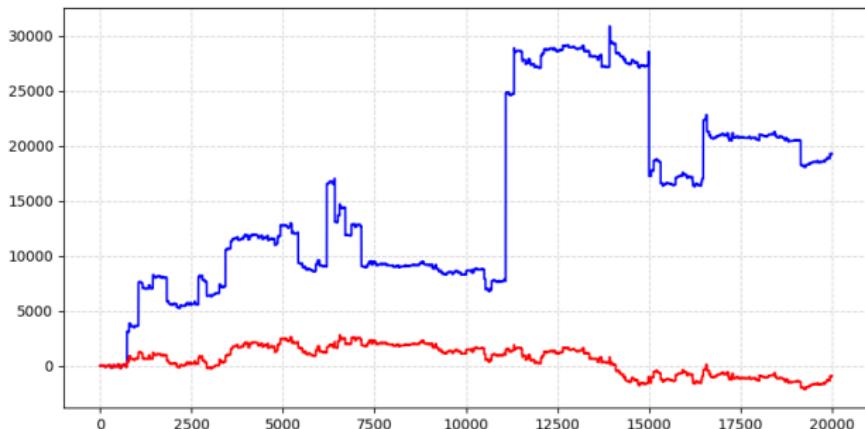


Proof sketch (rooted trees)

Proposition (tails for truncated sums)

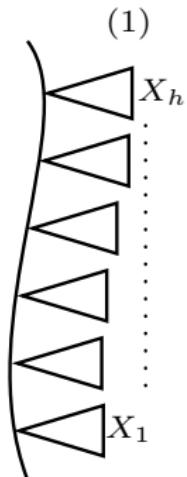
Take $(X_n)_{n=1}^{\infty}$ i.i.d. with $\mathbf{P}(X_i \geq x) \leq cx^{-1}$. For $\gamma > 0$ there exists C such that for any $t, m \geq 0, s > 1$, for $S_m = \sum_{i=1}^m (X_i \wedge sm^{1+\gamma})$,

$$\mathbf{P}(S_m \geq tm^{1+\gamma}) \leq C \exp(-t/s).$$



Proof sketch (rooted trees)

$$P_{\epsilon,h} = \{|\text{LCS}^\bullet(\tau, \tau')| \geq h^{1+\epsilon}\} \cap \{\text{Ht}(\tau) \wedge \text{Ht}(\tau') \leq h\}$$



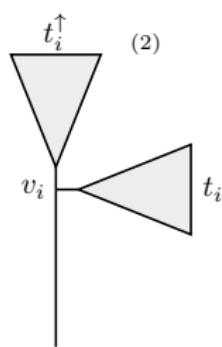
- By branching property and the linear LCS^\bullet bound, X_i 's are i.i.d. with a distribution that has tails like $\mathbf{P}(X_i \geq x) \leq Cx^{-1}$.
- $|\text{LCS}^\bullet(\tau, \tau')| \leq \sum_{i=1}^h (X_i \wedge h^{1+\epsilon-\gamma})$ by definition of $P_{\epsilon,h}$ and (1).
- We can apply the tail bounds from the last slide with $t = h^{\gamma/2}$, $s = 1$, and $\gamma = \gamma/2!$

Conclusion:

$$\mathbf{P}(P_{\epsilon,h} \cap (1)) \leq Ch^2 \exp(-h^{\gamma/2}).$$

Proof sketch (rooted trees)

$$P_{\epsilon,h} = \{|\text{LCS}^\bullet(\tau, \tau')| \geq h^{1+\epsilon}\} \cap \{\text{Ht}(\tau) \wedge \text{Ht}(\tau') \leq h\}$$



By construction of \mathcal{P} , $|t_i| \geq h^{1+\epsilon-\nu}$ and $|t_i^{\uparrow}| \geq h^{1+\epsilon-\nu}$. We can use a union bound and the linear LCS bound:

$$\begin{aligned}\mathbf{P}(P_{\epsilon,h} \cap (2)) &\leq Ch\mathbf{P}(P_{\epsilon-\nu,h})^2 \\ &\leq Ch\mathbf{P}(P_{\epsilon-\nu,h})\mathbf{P}(|t_i| \geq h^{1+\epsilon-\nu}) \\ &\leq C\mathbf{P}(P_{\epsilon-\nu,h})\frac{1}{h^{\epsilon-\nu}}.\end{aligned}$$

Conclusion [cases (1) and (2)]:

$$\mathbf{P}(P_{\epsilon,h}) \leq C\mathbf{P}(P_{\epsilon-\nu,h})\frac{1}{h^{\epsilon-\nu}} + Ch^2 \exp(-h^{\nu/2})$$

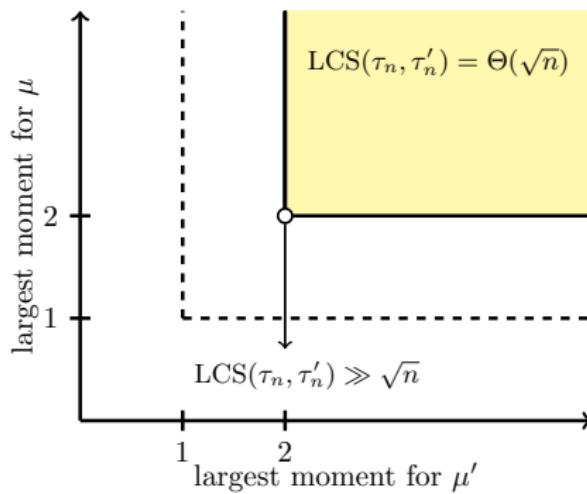
5: COOL THINGS FOR THE FUTURE.



Figure: My application to NSERC for funding (rejected).

Future directions

- What if the two trees are not the same size? For example, take τ_n and τ'_m , where $m = n^\alpha$ for some $\alpha \in (0, 1)$.
- What happens if we allow some distortion or sample the trees with dependence?
- Other moment assumptions?
- and much much more...



Future directions

- Thank you all for listening! These slides, as well as a mostly comprehensive list of common substructure references, are available on my website.

↓↓ QR code for the references :) ↓↓

