

Differential Voting: Loss Functions For Axiomatically Diverse Aggregation of Heterogeneous Preferences

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Abstract

Reinforcement learning from human feedback (RLHF) implicitly aggregates heterogeneous human preferences into a single utility function, even though the underlying utilities of the participants are in practice diverse. Hence, RLHF can be viewed as a form of voting, where the aggregation mechanism is defined by the loss function. Although Arrow’s Impossibility Theorem suggests that different mechanisms satisfy different sets of desirable axioms, most existing methods rely on a single aggregation principle, typically the Bradley–Terry–Luce (BTL) model, which corresponds to Borda count voting. This restricts the axiomatic properties of the learned reward and obscures the normative assumptions embedded in optimization. In this work, we introduce Differential Voting, a unifying framework that constructs instance-wise, differentiable loss functions whose population-level optima provably correspond to distinct classical voting rules. We develop differentiable surrogates for majority-based aggregation (BTL), Copeland, and Kemeny rules, and formally analyze their calibration properties, gradient fields, and limiting behavior as smoothing parameters vanish. For each loss, we establish consistency with the corresponding social choice rule and characterize the axioms it satisfies or violates. Our analysis shows how design choices in loss geometry—such as margin sensitivity and boundary concentration—directly translate into normative aggregation behavior. Differential Voting makes preference aggregation an explicit and controllable design choice in RLHF, enabling principled trade-offs between axiomatic guarantees and optimization stability. Code to reproduce our experiments is open-sourced.¹

1 Introduction

Reinforcement learning from human feedback (RLHF) has become a dominant paradigm for aligning machine learning

systems with human preferences, particularly in the training of large language models. In its standard form, RLHF learns a scalar reward function from pairwise human comparisons and then optimizes a policy against this learned reward. Despite its empirical success, this pipeline implicitly aggregates feedback from multiple annotators, contexts, and latent objectives into a single utility function, raising a fundamental question: *what aggregation rule is actually being implemented?*

Recent work has argued that preference learning in RLHF should be understood as a problem of *social choice*, where heterogeneous human judgments are combined into a collective decision Ge *et al.* [2024]; Siththanjan *et al.* [2024]. From this perspective, the choice of training loss is not merely a statistical detail but a normative design decision, encoding assumptions about how conflicting preferences should be reconciled. Classical impossibility results, such as Arrow’s theorem, imply that no single aggregation rule can satisfy all desirable axioms simultaneously. Nevertheless, most existing preference-learning methods rely on a narrow family of smooth pairwise losses, typically variants of the Bradley–Terry–Luce (BTL) model, whose induced aggregation behavior is rarely made explicit.

A growing theoretical literature has revealed that this choice has substantive consequences. Under heterogeneous or hidden contexts, optimizing BTL-style losses corresponds to a Borda-like scoring rule at the population level, which can violate basic axioms such as the Condorcet winner criterion and majority consistency Siththanjan *et al.* [2024]; Ge *et al.* [2024]. At the same time, alternative aggregation rules with stronger axiomatic properties—such as Copeland or Kemeny—are typically formulated as discrete or combinatorial objectives, making them difficult to integrate into gradient-based training pipelines.

In this work, we bridge this gap by introducing *Differential Voting*, a unifying framework for designing instance-wise, differentiable loss functions whose population-level optima correspond to distinct classical voting rules. Rather than treating preference aggregation as an implicit byproduct of loss minimization, Differential Voting makes the aggregation mechanism an explicit, controllable design choice.

We develop differentiable surrogates for three canonical aggregation principles: pairwise majority (via the standard BTL loss), Copeland aggregation, and Kemeny aggregation. For each loss, we characterize its gradient field, prove consistency

¹<https://github.com/ryeii/Differential-Voting>

Method	Loss / Objective	Corresp. Voting System	Axioms				Diff.
			Majority Winner	Pareto (PO)	IIA/LIIA	Condorcet (PMC)	
BTL (logistic)	$\sum_{a \neq b \in C} \log(1 + \exp(r_\theta(b) - r_\theta(a)))$	Borda-like scoring	✗	✓	✗	✗	✓
Exponential	$\sum_{a \neq b \in C} \exp(r_\theta(b) - r_\theta(a))$	Borda-like scoring	✗	✓	✗	✗	✓
Hinge	$\sum_{a \neq b \in C} \max(0, 1 + r_\theta(b) - r_\theta(a))$	Borda-like scoring	✗	✓	✗	✗	✓
Majority-based	Any rule depending only on pairwise majority directions	(Pairwise) Majority aggregation	✓	✓	✗	✓	✗
Linear Kemeny	$\sum_{a \neq b \in C} n_{a \succ b} \cdot \mathbf{1}[r_\theta(b) > r_\theta(a)]$	Kemeny (feasible-ranking variant)	✗	✓	✗	✓	✗
LCPO	Leximax Copeland subject to PO (implemented via LP feasibility checks)	Copeland (PO-constrained leximax)	✓	✓	✗	✓	✗
Soft Copeland (ours)	$-y s_{\tau, \beta}(\Delta) + \frac{\lambda}{2} \Delta^2$	Copeland (smooth surrogate)	✓	✓	✗	✓	✓
Soft Kemeny (ours)	$\sigma(-y \Delta / \tau)$	Kemeny (smooth Kendall surrogate)	✗	✓	✗	✓	✓

Table 1: Existing aggregation objectives in (linear) RLHF-style social choice and their axiomatic properties. "Diff." = Differentiable. Here PO denotes standard Pareto/unanimity (if all voters rank a above b , then the social outcome ranks a above b), and PMC denotes the Condorcet winner criterion (if a Condorcet winner exists, it is selected). Borda-like scoring satisfies PO but violates Majority-winner, IIA/LIIA, and PMC in general; Copeland and Kemeny satisfy PMC but violate IIA/LIIA. Checkmarks follow the setting and claims discussed in Siththaranjan *et al.* [2024]; Ge *et al.* [2024] together with standard social-choice facts.

with the corresponding social choice rule in the appropriate limit of smoothing parameters, and analyze which axioms are satisfied or violated. Our analysis reveals how geometric features of the loss—such as margin sensitivity, saturation, and boundary concentration—translate directly into normative aggregation behavior. In particular, we show that losses that are locally calibrated for pairwise classification can nonetheless induce globally distinct aggregation rules.

We complement our theoretical results with a suite of controlled synthetic experiments. These experiments demonstrate that different losses recover different classical aggregation rules at the population level, behave differently under heterogeneous preferences, and exhibit characteristic optimization geometries that explain their axiomatic properties. Together, these results show that preference optimization losses are not interchangeable surrogates but implement qualitatively different social choice mechanisms.

Contributions. Our main contributions are:

- We introduce *Differential Voting*, a framework for constructing differentiable, instance-wise losses whose population optima correspond to classical voting rules.
- We propose differentiable surrogates for Copeland and Kemeny aggregation and prove their consistency and limiting behavior under smoothing.
- We analyze the axiomatic properties and loss geometry of standard and proposed objectives, clarifying the normative assumptions encoded by loss design.
- We empirically validate our theory through controlled experiments that isolate aggregation behavior, axiomatic satisfaction, and optimization dynamics.

2 Related Work

RLHF and preference optimization. Reinforcement learning from human feedback (RLHF) aligns models by learning a reward function from pairwise human comparisons and optimizing a policy against it Stiennon *et al.* [2020]; Ouyang *et al.* [2022]. Most implementations learn a Bradley–Terry–Luce (BTL) / logistic preference model and fine-tune with a

KL-regularized RL objective. Direct Preference Optimization (DPO) shows that an equivalent objective can be optimized via a classification-style loss under a particular reward parameterization Rafailov *et al.* [2023]. These methods build on classical pairwise ranking losses such as RankNet, which optimize smooth surrogates of ranking quality using logistic link functions Burges *et al.* [2005].

Social choice and axioms for alignment. Recent work reframes preference learning in RLHF as a problem of *preference aggregation*, emphasizing evaluation through axioms from social choice theory. Ge *et al.* introduce an axiomatic framework for learning reward functions from comparisons and show that BTL and broad convex generalizations violate basic desiderata in their *linear social choice* setting Ge *et al.* [2024]. They further demonstrate that function-approximation constraints fundamentally affect which axioms are achievable, motivating aggregation rules such as LCPO that satisfy stronger guarantees than standard loss-based methods.

Heterogeneous preferences and hidden context. Feedback in deployed alignment is inherently heterogeneous, reflecting diverse annotators and latent objectives. Siththaranjan *et al.* formalize this via *hidden context* and show that standard preference learning implicitly aggregates across contexts according to a Borda-like scoring rule, which can lead to counterintuitive outcomes Siththaranjan *et al.* [2024]. This work highlights that the choice of loss encodes normative assumptions about how conflicting judgments are combined.

Pluralism and alternative aggregation targets. Complementary lines of work study alignment under pluralistic or adversarial settings. Götz *et al.* analyze *distortion* between achieved and optimal welfare, sharply distinguishing RLHF/DPO from alternatives such as Nash Learning from Human Feedback (NLHF) Götz *et al.* [2025]. Other approaches modify the aggregation target itself: maximal lotteries provide a probabilistic Condorcet extension for cyclic preferences Maura-Rivero *et al.* [2025], while NLHF frames alignment as computing equilibria of learned preference models Munos *et al.* [2024].

Positioning. Prior work shows that standard differentiable losses implicitly implement specific aggregation rules that

can violate appealing axioms, while methods with stronger normative guarantees often rely on discrete or non-smooth objectives. Differential Voting complements these approaches by making the aggregation rule an explicit *loss-level* design choice, enabling gradient-based optimization while targeting distinct voting rules via calibrated, differentiable surrogates.

3 Differential Voting: From Classical Aggregation to Differentiable Losses

We view RLHF as *implicit preference aggregation*: heterogeneous pairwise judgments are combined into a single scalar reward function, and the choice of loss determines which aggregation principle is implemented at the population optimum. We construct instance-wise, differentiable losses whose induced population objectives correspond to classical voting rules.

Let \mathcal{D} be a distribution over pairwise comparisons (x_i, x_j, y, c) , where $y \in \{-1, +1\}$ indicates whether $x_i \succ x_j$ under context c . A reward model $r_\theta(x, c) \in \mathbb{R}$ induces the margin $\Delta_{ij}(c) = r_\theta(x_i, c) - r_\theta(x_j, c)$ and win probability

$$p_\theta(x_i \succ x_j \mid c) = \sigma\left(\frac{\Delta_{ij}(c)}{\tau}\right), \quad \sigma(z) = \frac{1}{1 + e^{-z}}, \quad (1)$$

with temperature $\tau > 0$. Define the true pairwise preference probability

$$\eta(x_i, x_j, c) := \Pr(y = +1 \mid x_i, x_j, c).$$

We also use the shorthand $\Delta = \Delta_{ij}(c)$ and $p = \sigma(\Delta/\tau)$.

Given a context c (or suppressing c when fixed), the learned reward induces an ordering by sorting $r_\theta(\cdot, c)$. Our results concern the population optima of the losses and their relationship to the underlying pairwise probabilities η and classical social choice rules.

Population optima, realizability, and scope of claims. All consistency and axiomatic statements in this section concern *population* objectives and their minimizers over the hypothesis class $\{r_\theta(\cdot, c) : \theta \in \Theta\}$. When we say that a loss “implements” or “corresponds to” a classical voting rule, this means that the induced population objective coincides with the objective of that rule *restricted to the set of rankings realizable by the model class*. If the hypothesis class is rich enough to represent a ranking selected by the classical rule, then the corresponding population minimizer coincides with that rule; otherwise, the minimizer coincides with the rule *restricted to the feasible set*. We do not make claims about optimization dynamics or convergence of specific algorithms, only about population-level optima.

3.1 BTL / Logistic Loss: Local Majority Calibration

We begin with the standard Bradley–Terry–Luce (BTL) / logistic loss as a baseline. Prior work shows that, when pairwise losses are aggregated across alternatives and heterogeneous contexts, minimizing the BTL objective induces a Borda-like scoring rule at the population level Siththanjan *et al.* [2024]; Ge *et al.* [2024]. Here we record a complementary and well-known fact at a different level of analysis: *for a fixed pair*

under a fixed context, the BTL loss is classification-calibrated for the pairwise majority direction.

This result is classical, but we state it explicitly to clarify the distinction between *local statistical calibration* and *global aggregation behavior*, a theme that will recur for Copeland- and Kemeny-style losses.

Setup. Fix a comparison (x_i, x_j, c) and let

$$\eta := \Pr(y = +1 \mid x_i, x_j, c).$$

Let $\Delta = r_\theta(x_i, c) - r_\theta(x_j, c)$.

Definition 1 (BTL / logistic loss).

$$\mathcal{L}_{\text{BTL}}(\Delta, y) = \log\left(1 + \exp\left(-\frac{y\Delta}{\tau}\right)\right), \quad \tau > 0.$$

Proposition 1 (Pairwise majority calibration). *Consider the conditional risk*

$$\mathcal{R}_{\text{BTL}}(\Delta) = \mathbb{E}[\mathcal{L}_{\text{BTL}}(\Delta, Y) \mid x_i, x_j, c]$$

$$= \eta \log(1 + e^{-\Delta/\tau}) + (1 - \eta) \log(1 + e^{\Delta/\tau}).$$

- If $\eta \in (0, 1)$, \mathcal{R}_{BTL} is strictly convex and has the unique minimizer

$$\Delta^* = \tau \log \frac{\eta}{1 - \eta}, \quad \text{sign}(\Delta^*) = \text{sign}(\eta - \frac{1}{2}).$$

- If $\eta = 1$ (resp. $\eta = 0$), the risk has no finite minimizer and its infimum is approached as $\Delta \rightarrow +\infty$ (resp. $\Delta \rightarrow -\infty$).

Thus, whenever $\eta \neq \frac{1}{2}$, minimizing the BTL loss recovers the correct pairwise majority direction.

Remark (local vs. global behavior). Proposition 1 is a *local* statement: it concerns the optimal margin for a single ordered pair (x_i, x_j) under a fixed context. It does not characterize the *multi-alternative aggregation rule* induced by minimizing sums of pairwise BTL losses across datasets. As shown in prior work, when such losses are aggregated under heterogeneous contexts and standard sampling schemes, the resulting population objective implements a Borda-like scoring principle rather than a majority rule Siththanjan *et al.* [2024]; Ge *et al.* [2024].

3.2 Soft Copeland Voting

Copeland aggregation counts only whether an alternative wins or loses each head-to-head contest, ignoring margin magnitudes. A differentiable surrogate should therefore (i) be odd in the margin, (ii) saturate for confident wins/losses, and (iii) concentrate gradient mass near the win/loss boundary. At the same time, to obtain a well-posed population objective with finite minimizers, some form of regularization is required.

Soft Copeland edge score. Define the soft Copeland edge score

$$s_{\tau, \beta}(\Delta) = \tanh\left[\beta (\sigma(\Delta/\tau) - \frac{1}{2})\right], \quad (2)$$

where $\sigma(z) = (1 + e^{-z})^{-1}$, $\tau > 0$ controls smoothness of the pairwise decision boundary, and $\beta > 0$ controls saturation. The function $s_{\tau, \beta}$ is odd, bounded in $[-1, 1]$, and strictly increasing in Δ .

Definition 2 (Regularized Soft Copeland Loss). *For one instance (x_i, x_j, y, c) define*

$$\mathcal{L}_{\text{Cop}}(\theta; x_i, x_j, y, c) = -y s_{\tau, \beta}(\Delta_{ij}(c)) + \frac{\lambda}{2} \Delta_{ij}(c)^2, \quad (3)$$

where $\lambda > 0$ is a (small) margin-regularization parameter.

Remark (well-posedness). Without the quadratic term, the conditional risk is minimized only by sending $\Delta \rightarrow \pm\infty$ whenever $\eta \neq \frac{1}{2}$. The regularizer makes the objective coercive and yields finite population optima while preserving the Copeland-style saturation geometry. In practice, such regularization corresponds to explicit weight decay, KL penalties, or implicit regularization from early stopping.

Gradient field. Let $p = \sigma(\Delta/\tau)$. Since

$$\frac{d}{d\Delta} s_{\tau,\beta}(\Delta) = \frac{\beta}{\tau} p(1-p) \operatorname{sech}^2(\beta(p - \frac{1}{2})),$$

we obtain

$$\frac{\partial \mathcal{L}_{\text{Cop}}}{\partial \Delta} = -y \frac{\beta}{\tau} p(1-p) \operatorname{sech}^2(\beta(p - \frac{1}{2})) + \lambda \Delta. \quad (4)$$

Parameter gradients follow by multiplying by $\nabla_{\theta} \Delta_{ij}(c) = \nabla_{\theta} r_{\theta}(x_i, c) - \nabla_{\theta} r_{\theta}(x_j, c)$.

Pairwise Copeland consistency. We characterize the population-optimal margin for a fixed pair.

Theorem 1 (Pairwise Copeland direction consistency with finite optima). *Fix (x_i, x_j, c) and let $\eta = \Pr(y = +1 | x_i, x_j, c)$. The conditional risk*

$$\mathcal{R}_{\text{Cop}}(\Delta) = -(2\eta - 1) s_{\tau,\beta}(\Delta) + \frac{\lambda}{2} \Delta^2$$

admits at least one finite minimizer Δ^* . Moreover,

$$\operatorname{sign}(\Delta^*) = \operatorname{sign}(2\eta - 1) \quad \text{whenever } \eta \neq \frac{1}{2},$$

and if $\eta = \frac{1}{2}$ then $\Delta^* = 0$ is the unique minimizer. Finally, as $\lambda \downarrow 0$ with (τ, β) fixed, any sequence of minimizers satisfies $|\Delta^*| \rightarrow \infty$ when $\eta \neq \frac{1}{2}$.

Proof. Since $s_{\tau,\beta}(\Delta) \in [-1, 1]$ and $\frac{\lambda}{2} \Delta^2 \rightarrow \infty$ as $|\Delta| \rightarrow \infty$, \mathcal{R}_{Cop} is coercive and attains a finite minimum.

Write $\gamma := 2\eta - 1$. The function $s_{\tau,\beta}$ is odd and strictly increasing, while Δ^2 is even. Hence for any $\Delta > 0$,

$$\begin{aligned} \mathcal{R}_{\text{Cop}}(\Delta) - \mathcal{R}_{\text{Cop}}(-\Delta) &= -\gamma(s_{\tau,\beta}(\Delta) - s_{\tau,\beta}(-\Delta)) \\ &= -2\gamma s_{\tau,\beta}(\Delta). \end{aligned}$$

If $\eta > \frac{1}{2}$ (so $\gamma > 0$), then $s_{\tau,\beta}(\Delta) > 0$ for $\Delta > 0$, and therefore $\mathcal{R}_{\text{Cop}}(\Delta) < \mathcal{R}_{\text{Cop}}(-\Delta)$. Consequently, no minimizer can be negative; by symmetry, if $\eta < \frac{1}{2}$ then no minimizer can be positive.

It remains to rule out $\Delta^* = 0$ when $\eta \neq \frac{1}{2}$. Differentiate:

$$\mathcal{R}'_{\text{Cop}}(\Delta) = -\gamma s'_{\tau,\beta}(\Delta) + \lambda \Delta.$$

Since $s_{\tau,\beta}$ is strictly increasing and smooth, $s'_{\tau,\beta}(0) > 0$, and thus

$$\mathcal{R}'_{\text{Cop}}(0) = -\gamma s'_{\tau,\beta}(0).$$

If $\eta > \frac{1}{2}$ then $\gamma > 0$ and $\mathcal{R}'_{\text{Cop}}(0) < 0$, so 0 cannot be a minimizer; hence every minimizer satisfies $\Delta^* > 0$. Similarly, if $\eta < \frac{1}{2}$ then $\mathcal{R}'_{\text{Cop}}(0) > 0$ and every minimizer satisfies $\Delta^* < 0$. If $\eta = \frac{1}{2}$ then $\gamma = 0$ and the risk reduces to $\frac{\lambda}{2} \Delta^2$, uniquely minimized at $\Delta = 0$.

Finally, when $\eta \neq \frac{1}{2}$ and $\lambda \downarrow 0$ with (τ, β) fixed, the bounded term $-\gamma s_{\tau,\beta}(\Delta)$ is minimized by pushing $s_{\tau,\beta}(\Delta)$ toward its extremum, which occurs only as $|\Delta| \rightarrow \infty$. \square

From pairwise direction to Copeland score. Define the induced soft Copeland score of an alternative x by

$$C_{\tau,\beta}(x) = \mathbb{E}_{x' \sim \rho} [s_{\tau,\beta}(r_{\theta}(x, c) - r_{\theta}(x', c))], \quad (5)$$

where ρ is the comparison-opponent distribution induced by \mathcal{D} .

Theorem 2 (Limit to classical Copeland counting). *Assume $r_{\theta}(\cdot, c)$ induces strict inequalities almost surely. Then for each fixed θ ,*

$$\lim_{\tau \rightarrow 0} \lim_{\beta \rightarrow \infty} C_{\tau,\beta}(x) = \mathbb{E}_{x' \sim \rho} [\operatorname{sign}(r_{\theta}(x, c) - r_{\theta}(x', c))],$$

which is exactly the (normalized) Copeland win-minus-loss count under the induced pairwise outcomes.

Proof. Fix $\tau > 0$ and write $\Delta = r_{\theta}(x, c) - r_{\theta}(x', c)$. For $\Delta \neq 0$, we have $\sigma(\Delta/\tau) - \frac{1}{2}$ has the same sign as Δ , and hence as $\beta \rightarrow \infty$,

$$\begin{aligned} s_{\tau,\beta}(\Delta) &= \tanh(\beta (\sigma(\Delta/\tau) - \frac{1}{2})) \\ &\longrightarrow \operatorname{sign}(\sigma(\Delta/\tau) - \frac{1}{2}) = \operatorname{sign}(\Delta). \end{aligned}$$

Because $|s_{\tau,\beta}(\Delta)| \leq 1$, dominated convergence yields

$$\lim_{\beta \rightarrow \infty} C_{\tau,\beta}(x) = \mathbb{E}_{x' \sim \rho} [\operatorname{sign}(\Delta)].$$

The right-hand side does not depend on τ , so taking $\tau \rightarrow 0$ leaves it unchanged, proving the claim. \square

Axiomatic properties.

Proposition 2 (Neutrality and anonymity). *The Soft Copeland loss is neutral and anonymous.*

Proof. Neutrality follows from the oddness of $s_{\tau,\beta}$ and the invariance of Δ^2 under sign flips. Anonymity holds because empirical and population risks are averages of identical persistence losses. \square

Proposition 3 (Pairwise monotonicity). *For any instance with $y = +1$ and any Δ satisfying $|\Delta| < \frac{1}{\lambda} \frac{\beta}{4\tau}$, we have $\frac{\partial \mathcal{L}_{\text{Cop}}}{\partial \Delta} < 0$. The symmetric statement holds for $y = -1$.*

Proof. From (4), the first term has sign $-y$ and magnitude at most $\frac{\beta}{4\tau}$, while the regularization term has magnitude $\lambda |\Delta|$. For $|\Delta| < \frac{1}{\lambda} \frac{\beta}{4\tau}$, the former dominates, yielding the stated sign. \square

IIA failure. As with classical Copeland, the induced aggregation violates independence of irrelevant alternatives. Since Soft Copeland converges to Copeland counting in the limit (Theorem 2), it inherits the standard IIA counterexamples.

3.3 Soft Kemeny Voting

Kemeny aggregation selects a ranking that minimizes the total number of pairwise disagreements (equivalently, the Kendall τ distance) with respect to observed comparisons. To approximate this objective using gradient-based optimization, we use a smooth surrogate that directly targets pairwise disagreement and converges to exact disagreement counting as the temperature vanishes.

Smooth disagreement surrogate. For a comparison (x_i, x_j, y, c) with margin

$$\Delta = \Delta_{ij}(c) = r_\theta(x_i, c) - r_\theta(x_j, c),$$

define the *Soft Kemeny loss*

$$\mathcal{L}_{\text{Kem}}(\theta; x_i, x_j, y, c) = \sigma\left(-\frac{y\Delta_{ij}(c)}{\tau}\right), \quad \tau > 0, \quad (6)$$

where $\sigma(z) = (1 + e^{-z})^{-1}$.

This loss is a smooth approximation to the pairwise disagreement indicator: for $\Delta \neq 0$,

$$\sigma\left(-\frac{y\Delta}{\tau}\right) \rightarrow \mathbf{1}[y\Delta < 0] \quad (\tau \rightarrow 0).$$

Gradient field. Writing $\Delta = \Delta_{ij}(c)$, we have

$$\frac{\partial \mathcal{L}_{\text{Kem}}}{\partial \Delta} = -\frac{y}{\tau} \sigma\left(-\frac{y\Delta}{\tau}\right) \left(1 - \sigma\left(-\frac{y\Delta}{\tau}\right)\right). \quad (7)$$

Parameter gradients follow by multiplying $\nabla_\theta \Delta_{ij}(c) = \nabla_\theta r_\theta(x_i, c) - \nabla_\theta r_\theta(x_j, c)$.

Global pairwise monotonicity. The loss always corrects pairwise errors in the appropriate direction.

Proposition 4 (Global pairwise monotonicity). *Fix an instance with label $y \in \{-1, +1\}$. Then for all $\Delta \in \mathbb{R}$,*

$$\text{sign}\left(\frac{\partial \mathcal{L}_{\text{Kem}}}{\partial \Delta}\right) = -y.$$

Thus, gradient descent increases Δ when $y = +1$ and decreases Δ when $y = -1$, for all margin values.

Proof. From (7), the derivative is proportional to $-y$ multiplied by $\sigma(-y\Delta/\tau)(1 - \sigma(-y\Delta/\tau)) \geq 0$, yielding the stated sign. \square

Vanishing gradients for confident correct orderings. If $y\Delta \rightarrow +\infty$, then $\sigma(-y\Delta/\tau) \rightarrow 0$ and

$$\left| \frac{\partial \mathcal{L}_{\text{Kem}}}{\partial \Delta} \right| \rightarrow 0.$$

Thus, correctly ordered pairs with large margins contribute vanishing gradient mass, matching the intuition of disagreement counting: once a pair is confidently ordered correctly, there is no incentive to further increase its margin.

Limit to Kendall disagreement. We recover exact pairwise disagreement counting in the low-temperature limit.

Theorem 3 (Limit to pairwise disagreement objective). *Assume $\Pr(\Delta_{ij}(c) = 0) = 0$ under \mathcal{D} . Then for each fixed θ ,*

$$\lim_{\tau \rightarrow 0} \mathbb{E}_{\mathcal{D}}[\mathcal{L}_{\text{Kem}}(\theta)] = \mathbb{E}_{\mathcal{D}}[\mathbf{1}[y\Delta_{ij}(c) < 0]].$$

Proof. Fix $\Delta \neq 0$ and $y \in \{\pm 1\}$. If $y\Delta < 0$, then $-y\Delta/\tau \rightarrow +\infty$ as $\tau \rightarrow 0$, so $\sigma(-y\Delta/\tau) \rightarrow 1$. If $y\Delta > 0$, then $-y\Delta/\tau \rightarrow -\infty$, so $\sigma(-y\Delta/\tau) \rightarrow 0$. Thus $\mathcal{L}_{\text{Kem}}(\theta) \rightarrow \mathbf{1}[y\Delta < 0]$ pointwise. Since $0 \leq \mathcal{L}_{\text{Kem}} \leq 1$, dominated convergence applies. \square

Theorem 4 (Kemeny consistency under realizability). *Let π_θ be the strict ranking induced by sorting $r_\theta(\cdot, c)$ and $\Pi_\Theta = \{\pi_\theta : \theta \in \Theta\}$. Assume the conditions of Theorem 3 hold.*

For any sequence $\tau_n \downarrow 0$, let

$$\theta_n \in \arg \min_{\theta \in \Theta} \mathbb{E}[\mathcal{L}_{\text{Kem}}^{(\tau_n)}(\theta)].$$

Then every accumulation point θ^ of $\{\theta_n\}$ induces a ranking π_{θ^*} satisfying*

$$\pi_{\theta^*} \in \arg \min_{\pi \in \Pi_\Theta} \mathbb{E}[\mathbf{1}[\pi \text{ disagrees with } y]].$$

In particular, if a Kemeny-optimal ranking is realizable by Π_Θ , then any such limit point induces a Kemeny-optimal ranking; otherwise, it induces a Kemeny-optimal ranking restricted to the feasible set Π_Θ .

Proof. For a comparison $(x_i, x_j, y, c) \sim \mathcal{D}$ write $\Delta_\theta = r_\theta(x_i, c) - r_\theta(x_j, c)$ and $T_\theta = y\Delta_\theta$. For $\tau > 0$ define the population risks

$$F_\tau(\theta) = \mathbb{E}\left[\sigma\left(-\frac{T_\theta}{\tau}\right)\right], \quad F_0(\theta) = \mathbb{E}[\mathbf{1}[T_\theta < 0]].$$

By Theorem 3, $F_\tau(\theta) \rightarrow F_0(\theta)$ pointwise for every fixed θ .

Let $\tau_n \downarrow 0$ and choose $\theta_n \in \arg \min_{\theta \in \Theta} F_{\tau_n}(\theta)$. Let θ^* be an accumulation point of $\{\theta_n\}$ (passing to a subsequence if necessary).

We first show that

$$\lim_{n \rightarrow \infty} F_{\tau_n}(\theta_n) = F_0(\theta^*). \quad (8)$$

Since $\theta_n \rightarrow \theta^*$, we have $T_{\theta_n} \rightarrow T_{\theta^*}$ almost surely. Under the assumption $\Pr(T_{\theta^*} = 0) = 0$, the sign of T_{θ_n} eventually agrees with that of T_{θ^*} . Hence

$$\sigma\left(-\frac{T_{\theta_n}}{\tau_n}\right) \rightarrow \mathbf{1}[T_{\theta^*} < 0] \quad \text{almost surely.}$$

Because the integrand is bounded in $[0, 1]$, dominated convergence yields (8).

Now fix any $\theta \in \Theta$. By optimality of θ_n for F_{τ_n} ,

$$F_{\tau_n}(\theta_n) \leq F_{\tau_n}(\theta) \quad \text{for all } n.$$

Taking limits, using (8) on the left and Theorem 3 on the right, gives

$$F_0(\theta^*) \leq F_0(\theta).$$

Since θ was arbitrary, $\theta^* \in \arg \min_{\theta \in \Theta} F_0(\theta)$.

Finally, $F_0(\theta)$ depends on θ only through the strict ranking π_θ induced by $r_\theta(\cdot, c)$. Thus minimizing F_0 over $\theta \in \Theta$ is equivalent to minimizing expected pairwise disagreement over realizable rankings Π_Θ . Consequently, π_{θ^*} is Kemeny-optimal within Π_Θ . If a classical Kemeny-optimal ranking is realizable, it is recovered; otherwise the optimum is attained within the feasible set. \square

Axiomatic properties. Neutrality and anonymity follow from invariance of the pairwise margins $\Delta_{ij}(c)$ under candidate relabeling and from the instance-wise additive structure of the empirical and population objectives.

Soft Kemeny directly targets pairwise disagreement rather than margin maximization. The logistic form concentrates gradient mass near decision boundaries, yields vanishing gradients for confidently correct orderings, and converges exactly to Kendall disagreement counting as the temperature vanishes.

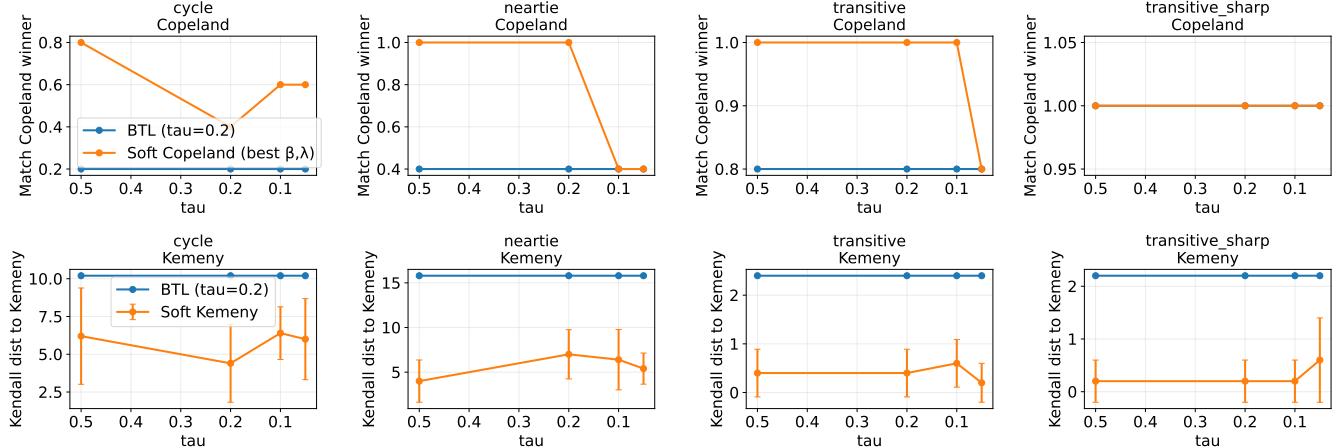


Figure 1: Population-level recovery of classical aggregation rules under smoothing. Top row: agreement with the Copeland winner; bottom row: Kendall distance to the Kemeny-optimal ranking. Soft Copeland and Soft Kemeny converge to their respective classical rules with appropriate τ , while BTL remains misaligned across cyclic, near-tie, and transitive regimes.

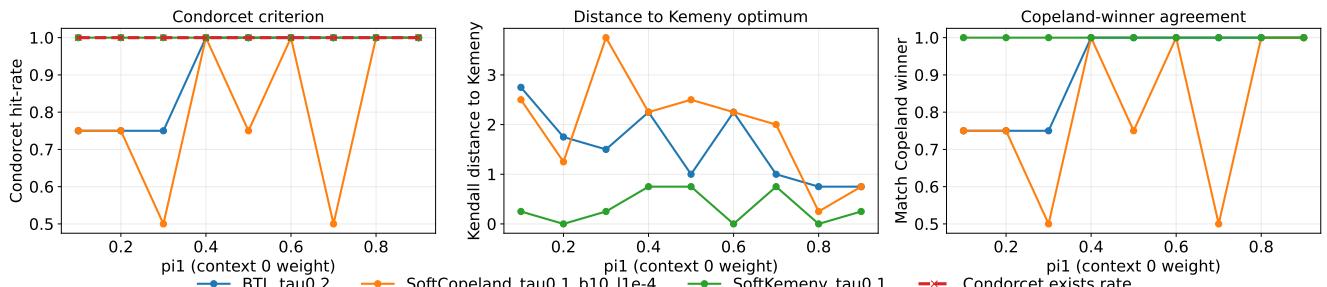


Figure 2: Aggregation under heterogeneous (hidden-context) preferences. Left: Condorcet winner criterion; middle: distance to the Kemeny optimum; right: Copeland-winner agreement. Soft Copeland and Soft Kemeny inherit the axiomatic behavior of their classical counterparts, whereas BTL systematically violates Condorcet consistency as context mixtures vary.

4 Experiments

Our theoretical analysis shows that the choice of loss function in preference optimization implicitly determines a social choice rule at the population level. We evaluate this claim empirically through a controlled suite of synthetic experiments designed to isolate aggregation behavior, axiomatic properties, and optimization geometry.

4.1 Research Questions

Our experiments are guided by the following three research questions.

RQ1: Population-level rule recovery. Do differentiable losses recover their intended classical aggregation rules (Copeland or Kemeny) at the population level as smoothing parameters vanish, and how does this behavior compare to standard BTL-style losses?

RQ2: Axiomatic behavior under heterogeneous preferences. In the presence of heterogeneous (hidden-context) preferences, do different losses satisfy or violate classical axioms such as the Condorcet winner criterion and consistency with Copeland or Kemeny outcomes?

RQ3: Loss geometry and optimization behavior. How do differences in loss geometry (e.g., gradient concentration and saturation) explain the distinct normative and optimization behaviors of the proposed losses?

4.2 Experimental Design

All experiments are implemented in a single, fully reproducible script and use minimal reward models to isolate aggregation effects.

Synthetic preference generation. We consider $m \in \{5, 7, 9\}$ alternatives and generate pairwise preferences from known population-level structures: (i) transitive utilities, (ii) near-tie utilities, (iii) cyclic (Condorcet-cycle) preferences, and (iv) sharply transitive utilities. Pairwise labels are sampled from Bernoulli distributions with probabilities η_{ab} derived from these structures.

We compare the standard BTL (logistic) loss against our proposed Soft Copeland and Soft Kemeny losses. The reward model is a vector $r \in \mathbb{R}^m$ (one scalar per alternative), ensuring that differences arise solely from the loss design.

Experiment 1: Population recovery under smoothing. We optimize each loss across a grid of temperature parameters τ

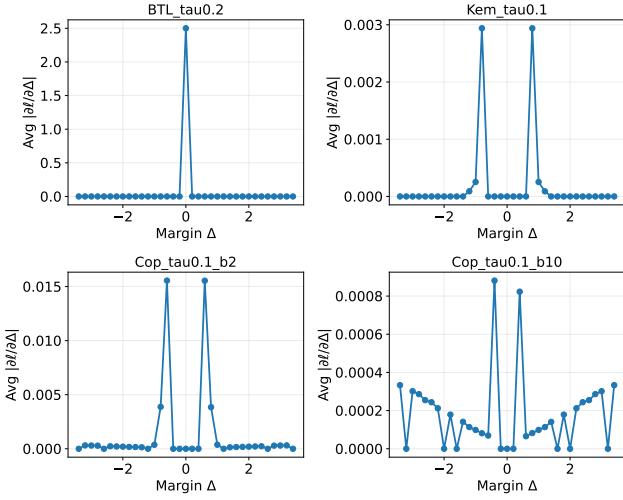


Figure 3: Loss geometry analysis. Average gradient magnitude as a function of margin Δ at convergence. BTL concentrates gradients near zero margins, Soft Kemeny targets pairwise disagreement boundaries, and Soft Copeland exhibits tunable majority-style saturation controlled by β .

(and, for Soft Copeland, saturation β and regularization λ). We measure: (i) agreement with the Copeland winner and (ii) Kendall distance to the Kemeny-optimal ranking. This experiment directly addresses RQ1.

Experiment 2: Hidden-context aggregation. To model heterogeneous preferences, each comparison is generated by first sampling a latent context, each with its own utility function. We evaluate: (i) satisfaction of the Condorcet winner criterion, (ii) distance to the Kemeny optimum, and (iii) agreement with the Copeland winner as context mixture weights vary. This experiment addresses RQ2.

Experiment 3: Loss geometry analysis. We analyze the gradient magnitude $|\partial\ell/\partial\Delta|$ as a function of the margin Δ for each loss at convergence. This experiment isolates how loss geometry concentrates optimization pressure near decision boundaries and addresses RQ3. (We omit additional noise-robustness experiments for clarity.)

4.3 Results

Experiment 1: Recovery of classical aggregation rules. Figure 1 shows population-level recovery across four preference regimes. BTL exhibits stable but systematically misaligned behavior: it fails to recover Copeland winners in cyclic and near-tie settings and maintains large Kendall distance to Kemeny optima across all regimes. In contrast, Soft Copeland reliably recovers the Copeland winner when a stable majority structure exists, while exhibiting sensitivity in near-degenerate cases, mirroring the behavior of classical Copeland aggregation. Soft Kemeny consistently achieves substantially lower Kendall distance than BTL.

Experiment 2: Axioms under heterogeneous preferences. Figure 2 evaluates aggregation under hidden contexts. BTL frequently violates the Condorcet winner criterion and diverges

from both Copeland and Kemeny outcomes as context mixtures vary. Soft Copeland closely tracks the existence of a Condorcet winner and selects it whenever it is stable, while Soft Kemeny always satisfies the Condorcet criterion and achieves the smallest distance to the Kemeny optimum. These results empirically confirm that different differentiable losses implement distinct axiomatic commitments under aggregation.

Experiment 3: Loss geometry. Figure 3 visualizes gradient magnitude as a function of margin. BTL concentrates gradients sharply at $\Delta \approx 0$ and rapidly decays, reflecting margin calibration rather than win/loss counting. Soft Kemeny exhibits symmetric peaks near decision boundaries and vanishing gradients for confidently ordered pairs, consistent with minimizing pairwise disagreement. Soft Copeland shows tunable boundary concentration controlled by β , interpolating between smooth optimization and near-discrete majority counting. These geometric differences explain the normative and optimization behavior observed in Experiments 1 and 2.

4.4 Answers to Research Questions

RQ1. Yes. Soft Copeland and Soft Kemeny recover their intended classical aggregation rules at the population level in the appropriate limits, while BTL converges to a distinct, scoring-based objective.

RQ2. Under heterogeneous preferences, loss choice determines which axioms are satisfied. Soft Copeland and Soft Kemeny inherit the Condorcet consistency of their classical counterparts, whereas BTL systematically violates it.

RQ3. Differences in loss geometry directly explain differences in aggregation behavior. Boundary concentration and saturation encode normative assumptions about whether margins, wins, or disagreements should dominate optimization.

Overall, these experiments demonstrate that preference optimization losses are not interchangeable statistical surrogates: they implement qualitatively different social choice rules. Differential Voting makes these choices explicit, controllable, and compatible with gradient-based learning.

5 Limitations

Our analysis focuses on population-level objectives and their minimizers, rather than finite-sample guarantees or full RLHF optimization dynamics; in practice, training may be affected by sampling noise, model misspecification, and interactions with policy optimization that we do not study. We consider differentiable surrogates for deterministic classical voting rules in controlled synthetic settings; extending this framework to probabilistic aggregation, multi-winner objectives, or large-scale real-world annotation pipelines remains future work. Finally, Differential Voting does not eliminate the trade-offs imposed by classical impossibility results but makes them explicit at the level of loss design.

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