

Model-Free Optimal Control of Linear Multi-Agent Systems via Decomposition and Hierarchical Approximation

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Abstract—Designing the optimal linear quadratic regulator (LQR) for a large-scale multi-agent system (MAS) is time-consuming since it involves solving a large-size matrix Riccati equation. The situation is further exasperated when the design needs to be done in a model-free way using schemes such as reinforcement learning (RL). To reduce this computational complexity, we decompose the large-scale LQR design problem into multiple smaller-size LQR design problems. We consider the objective function to be specified over an undirected graph, and cast the decomposition as a graph clustering problem. The graph is decomposed into two parts, one consisting of independent clusters of connected components, and the other containing edges that connect different clusters. Accordingly, the resulting controller has a hierarchical structure, consisting of two components. The first component optimizes the performance of each independent cluster by solving the smaller-size LQR design problem in a model-free way using an RL algorithm. The second component accounts for the objective coupling different clusters, which is achieved by solving a least squares problem in one shot. Although suboptimal, the hierarchical controller adheres to a particular structure as specified by inter-agent couplings in the objective function and by the decomposition strategy. Mathematical formulations are established to find a decomposition that minimizes the number of required communication links or reduces the optimality gap. Numerical simulations are provided to highlight the pros and cons of the proposed designs.

Index Terms—Decomposition, model-free control, reinforcement learning, linear quadratic regulator, large-scale networks

I. INTRODUCTION

The classical linear quadratic regulator (LQR) problem [1] has been solved by a variety of approaches such as Kleinman’s iterative algorithm [2], semi-definite programming [3], and policy gradient algorithms [4]. When applied to large-scale multi-agent systems (MASs), however, these methods often result in poor numerical performance due to the computation of extremely large-dimensional gain matrices. In other words, they produce encouraging results when run *offline*, but their execution becomes time-taking and numerically expensive when *real-time* control actions need to be taken. The problem becomes even more complex when the MAS model is unknown to the designer and data-driven techniques [5]-[9] such as reinforcement learning (RL) need to be used. For instance, various RL algorithms such as actor-critic methods [10], Q-learning [11], policy gradient algorithms [12], integral concurrent learning [13], and adaptive dynamic programming (ADP) [14]-[19] have been proposed for solving model-free

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LQR, but all of them involve large-size matrix inversions at every iteration of the learning phase, which drastically increases computational complexity, resulting in long learning times. The controller is usually centralized in implementation, and even if it is distributed [20]-[24] it comes at the cost of specific structural assumptions, or restrictive conditions on the cost function and the agent dynamics.

In this paper we propose a *hierarchical* RL scheme to resolve this computational bottleneck of conventional RL-based LQR. The hierarchy follows from decomposing a high-dimensional LQR problem into several smaller-sized LQR problems. We consider a MAS where each agent has its own decoupled linear dynamics. The state penalty term in the control objective (i.e., the Q matrix in the LQR cost function) couples these dynamics through a connected, undirected graph. The decomposition problem is posed as a clustering problem for this graph. The graph is decomposed into two parts, one consisting of multiple decoupled clusters of connected components, and the other containing edges that connect the different clusters. Accordingly, the proposed RL controller is composed of two components, one optimizing the performance of each decoupled cluster by solving a smaller-sized independent LQR design, and the other accounting for the performance coupling the different clusters by solving a least squares problem in one shot.

The two main benefits of this hierarchical strategy are that (i) it drastically reduces learning time, and that (ii) the resulting controller inherits a special structure from the graph embedded in the Q matrix as well from the decomposition strategy that enables it to be implemented in a cluster-wise distributed fashion, requiring far less number of communication links than conventional LQR. We formulate a mixed-integer quadratic program (MIQP) to minimize the number of communication links. Because of the decomposition the controller is suboptimal, however. We, therefore, present a minimum cut graph partitioning approach by which one can reduce the optimality gap. Compared to the conventional structure-constrained distributed LQR designs reported in the literature such as in [20], [21], our cost function and agent models are much more generic.

Hierarchical optimal control has been studied in papers such as [21], [25] (model-based) and [26] (model-free) under the assumption of time-scale separation in the plant dynamics. A related method based on spectral separation of the controllability Gramian was also recently proposed in [27]. These methods, when studied in the context of a MAS, often translate into clustering of agents [28], [29]. The clustering in our problem, in contrast, is not on the agents but rather on the graph that defines the control objective for the agents. Our design is also fundamentally different from most cooperative

control problems in the MAS literature such as [30], [31], where the objective is to *stabilize* the agents to a desired equilibrium set that describes a collective behavior. Although such stabilization objective may be characterized by an optimal control formulation [32], the cost functions in those cases are not predefined, and have to satisfy specific conditions. We illustrate the effectiveness of our designs using a classical example from formation maneuver control.

Some preliminary results on this design have been reported in our recent conference paper [33]. Compared to those results, the problem formulation as well as the control designs in this paper are significantly more extensive and in-depth. Stability proofs, solution of MIQP for optimal decomposition, and graph partitioning for minimization of the optimality gap are also completely new contributions.

The rest of the paper is organized as follows. Section II formulates the model-free optimal control problem for linear heterogeneous MAS. Section III proposes the hierarchical control framework via decomposition. Section IV analyzes the influence of the decomposition on the number of communication links and the performance gap, and presents an MIQP problem to find an optimal decomposition strategy. Section V presents numerical results on formation maneuver control. Section VI concludes the paper. Stability proofs and other theoretical results are presented in the appendix.

Notation: Throughout the paper, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denotes an undirected graph with N vertices, where $\mathcal{V} = \{1, \dots, N\}$ is the set of vertices, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges; $\mathbf{0}_{p \times q}$ denotes the p by q zero matrix; I_d is the $d \times d$ identity matrix; $e_i \in \mathbb{R}^N$ is a N -dimensional unit vector with the i -th element being 1 and other elements being 0's; \otimes denotes the kronecker product. Define $\text{diag}\{A_1, \dots, A_N\}$ as a block diagonal matrix with A_i 's on the diagonal. Given a matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_i(A)$ denotes the i -th eigenvalue of A such that $\lambda_1(A) \leq \dots \leq \lambda_n(A)$, $\lambda_{\min}(A) = \lambda_1(A)$, $\lambda_{\max}(A) = \lambda_n(A)$, $A \succ 0$ implies that A is positive definite, $A \succeq 0$ implies that A is positive semi-definite, and $\text{tr}(A)$ denotes the trace of A ; Similarly, $\sigma_1(A) \leq \dots \leq \sigma_n(A)$ are singular values of A , $\sigma_{\min}(A) = \sigma_1(A)$, $\sigma_{\max}(A) = \sigma_n(A)$. We use $\text{cond}(A) = \sigma_{\max}(A)/\sigma_{\min}(A)$ to denote the condition number of matrix A . For matrices $A, B \succeq 0$, $A \leq B$ implies that $B - A \succeq 0$, $A \geq B$ if $B \leq A$. Given a scalar x , $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x .

II. PROBLEM FORMULATION

Consider a MAS composed of N agents. Each agent i is a linear time-invariant system described as

$$\dot{x}_i = A_i x_i + B_i u_i, \quad i = 1, \dots, N \quad (1)$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ are the state and control input of agent i . Throughout the paper, we will consider A_i and B_i as unknown, but their dimensions are known. Let $x = (x_1^\top, \dots, x_N^\top)^\top$, $u = (u_1^\top, \dots, u_N^\top)^\top$, $\mathcal{A} = \text{diag}\{A_1, \dots, A_N\}$, $\mathcal{B} = \text{diag}\{B_1, \dots, B_N\}$. The overall model of the system is written in a compact form as

$$\dot{x} = \mathcal{A}x + \mathcal{B}u. \quad (2)$$

We assume that the agent dynamics are coupled through the cost function that needs to be minimized using a state-feedback LQR controller. Without loss of generality, we use an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to describe the coupling relationship among the agents in the cost function. It is important to note that the graph \mathcal{G} is not a communication or interaction graph between the agents. It is a graph that defines the coupling between the agent dynamics in the control objective. Similar settings have been considered in [20] and [21]. The cost function is given as

$$J(x(0), u) = \int_0^\infty x^\top(\tau) Q x(\tau) + u^\top(\tau) R u(\tau) d\tau, \quad (3)$$

where $Q \succeq 0$ and $R \succ 0$ are in the following forms:

$$Q = \bar{Q} + G \otimes \tilde{Q}, \quad R = \text{diag}\{R_1, \dots, R_N\}, \quad (4)$$

with $\bar{Q} = \text{diag}\{\bar{Q}_{11}, \dots, \bar{Q}_{NN}\}$ representing the subsystem-level objective, $\bar{Q}_{ii} \in \mathbb{R}^{n \times n}$ and $\bar{Q}_{ii} \succeq 0$. $R_i \in \mathbb{R}^{m \times m}$ and $R_i \succ 0$ for $i = 1, \dots, N$. $G = [G_{ij}] \in \mathbb{R}^{N \times N}$ is a Laplacian matrix corresponding to graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, i.e., $G_{ii} = \sum_{j=1}^N |G_{ij}|$ for $i = 1, \dots, N$, and $G_{ij} < 0$ if $(i, j) \in \mathcal{E}$ and $i \neq j$; $\tilde{Q} \in \mathbb{R}^{n \times n}$ and $\tilde{Q} \succeq 0$. The second component in Q represents the objective across the different subsystems. We observe that if \mathcal{G} is not connected, then the optimal control problem can be decomposed immediately according to its independent connected components. Without loss of generality, we assume \mathcal{G} is connected throughout the rest of the paper.

Given the MAS (1) with state x measurable, we would ideally like to find the controller u^* such that the performance index (3) with matrices Q and R defined in (4) is minimized despite \mathcal{A} and \mathcal{B} being unknown. We make the following assumption to guarantee existence and uniqueness of the optimal controller.

Assumption 1: The pair $(\mathcal{A}, \mathcal{B})$ is controllable and $(Q^{1/2}, \mathcal{A})$ is observable.

For heterogeneous MAS with $n \geq 2$, $Q \succ 0$ is sufficient but not necessary for observability of $(Q^{1/2}, \mathcal{A})$. However, if $n = 1$, then $(Q^{1/2}, \mathcal{A})$ is observable only if $Q \succ 0$. When \mathcal{A} and \mathcal{B} are known, the optimal controller for system (2) to minimize (3) is $u = -Kx = -R^{-1}\mathcal{B}^\top Px$ (see [1]), where P is the solution of the following algebraic Riccati equation:

$$\mathcal{P}\mathcal{A} + \mathcal{A}^\top P + Q - P\mathcal{B}R^{-1}\mathcal{B}^\top P = 0. \quad (5)$$

For MAS (1), $K \in \mathbb{R}^{mN \times nN}$ can be partitioned into N^2 blocks $K(i, j) \in \mathbb{R}^{m \times n}$. Accordingly, the control input for agent i can be written as $u_i = \sum_{j=1}^N K(i, j)x_j$. We define the communication graph associated with controller u as $\mathcal{G}_c(u) = (\mathcal{V}, \mathcal{E}_c(u))$, where

$$\mathcal{E}_c(u) = \{(i, j) \in \mathcal{V} \times \mathcal{V} : K(i, j) \neq \mathbf{0}_{m \times n}\}. \quad (6)$$

Therefore, communication between agents i and j is not required if $(i, j) \notin \mathcal{E}_c$. In the model-free case, where \mathcal{A} and \mathcal{B} are unknown, equation (5) can be transformed into an equation that is independent of \mathcal{A} and \mathcal{B} , and is based on the measurement of the states and the control inputs [15], [16], [10], [19]. The matrix P can then be obtained by implementing

an RL algorithm, which solves a least squares problem at each iteration.

Both the model-based and model-free control methods, however, are time-consuming if the MAS is of large scale. The main purpose of this paper is to provide an alternative approach for synthesizing the controller based on a decomposition of the cost function via clustering of the graph \mathcal{G} that can reduce learning time significantly. This decomposition is described in the next section.

Remark 1: The state vector of every agent is assumed to have the same dimension n for simplicity, but all of our designs and analysis in the forthcoming sections also apply to the case when agents have different dimensional states and control inputs under certain settings. More specifically, let $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$. We introduce $C_i \in \mathbb{R}^{n \times n_i}$ with $n \leq n_i$ and $\text{rank}(C_i) = n$ for $i = 1, \dots, N$. The cost function can be reformulated as $J(x(0), u) = \int_0^\infty x^\top(\tau) C^\top Q C x(\tau) + \sum_{i=1}^N u_i^\top(\tau) R_i u_i(\tau) d\tau$, where $C = \text{diag}\{C_1, \dots, C_N\}$, Q is still defined in (4).

III. HIERARCHICAL CONTROL FOR MAS

To reduce learning time, we propose a hierarchical RL design that seeks a suboptimal controller for the general case when the agents have non-identical dynamics. Throughout the section, for ease of understanding the RL design, we will assume that the decomposition strategy is given, implying that the agents have already been decomposed into multiple clusters. The actual ways of developing this decomposition will be presented in Section IV.

A. Hierarchical Control Framework

Suppose that the vertex set \mathcal{V} is decomposed into $s \leq N$ disjoint vertex sets \mathcal{V}_j , $j = 1, \dots, s$. The j -th cluster is composed of N_j nodes $\{j_1, \dots, j_{N_j}\}$. Let $n_j = n N_j$ and $m_j = m N_j$. The dynamics of the j -th cluster is written as

$$\dot{\mathbf{x}}_j = \mathcal{A}_j \mathbf{x}_j + \mathcal{B}_j \mathbf{u}_j, \quad j = 1, \dots, s, \quad (7)$$

where $\mathbf{x}_j = (x_{j_1}^\top, \dots, x_{j_{N_j}}^\top)^\top \in \mathbb{R}^{n_j}$, $\mathbf{u}_j = (u_{j_1}^\top, \dots, u_{j_{N_j}}^\top)^\top \in \mathbb{R}^{m_j}$, $\mathcal{A}_j = \text{diag}\{\mathcal{A}_{j_1}, \dots, \mathcal{A}_{j_{N_j}}\} \in \mathbb{R}^{n_j \times n_j}$, $\mathcal{B}_j = \text{diag}\{\mathcal{B}_{j_1}, \dots, \mathcal{B}_{j_{N_j}}\} \in \mathbb{R}^{n_j \times m_j}$. Without loss of generality, we assume that the agents belonging to the same cluster have contiguous indices, i.e., $j_{i+1} = j_i + 1$ and $(j+1)_1 = j_{N_j} + 1$. In the case when the indices of agents are not in such an order, one can relabel them to make this condition satisfied. The matrix G can then be decomposed as

$$G = G_1 + G_2, \quad (8)$$

where G_1 is a block-diagonal Laplacian matrix with s blocks and each block corresponds to a subgraph of \mathcal{G} , and G_2 is a Laplacian matrix where $G_2(i, i) = \sum_{j=1}^s |G_2(i, j)|$ that describes the couplings between the different clusters. Then both G_1 and G_2 are positive semi-definite.

Given the decomposition $\mathcal{V} = \mathcal{V}_1 \cup \dots \cup \mathcal{V}_s$, the graph G_2 is constructed as follows:

Step 1. Let $G_2 = G$. Replace each block on the diagonal of G_2 corresponding to each cluster \mathcal{V}_j by a zero block. The remaining off-diagonal elements remain unchanged.

$$\begin{aligned} G &= \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & \boxed{1} & 0 & 0 \\ 0 & -1 & 0 & \boxed{2} & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \longrightarrow G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} \\ G &= \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & \boxed{2} & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \longrightarrow G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Fig. 1. Two examples for the decomposition of G . In the first example, the agents are decomposed to three clusters $\{1, 2\}$, $\{3\}$ and $\{4, 5\}$. In the second example, the agents are decomposed to two clusters $\{1, 2, 3\}$ and $\{4, 5\}$.

Step 2. Replace the diagonal elements of G_2 by $G_2(i, i) = \sum_{j=1}^s |G_2(i, j)|$.

Once G_2 is constructed, G_1 will be determined accordingly. In Fig. 1, we give two examples to demonstrate two different decompositions for the same G . Similar to G , the four matrices Q , \bar{Q} , \tilde{Q} and R can also be transformed corresponding to the new indices of the agents. For simplicity of notation, we assume that Q , \bar{Q} , \tilde{Q} and R correspond directly to the relabelled agents. From (4), we have

$$Q = \bar{Q} + (G_1 + G_2) \otimes \tilde{Q}. \quad (9)$$

We define $\hat{Q} = \bar{Q} + G_1 \otimes \tilde{Q}$, and write $\hat{R}_j = \text{diag}\{\hat{R}_{j_1}, \dots, \hat{R}_{j_{N_j}}\} \in \mathbb{R}^{m_j \times m_j}$ for the j -th cluster. Then (4) can be written as

$$Q = \hat{Q} + G_2 \otimes \tilde{Q}, \quad R = \text{diag}\{\hat{R}_1, \dots, \hat{R}_s\} \quad (10)$$

where $\hat{Q} = \text{diag}\{\hat{Q}_1, \dots, \hat{Q}_s\}$. Accordingly, the global control objective in (3) can be decomposed into group-level objectives J_j , $j = 1, \dots, s$ and network-level objective J_G as

$$J(x(0), u) = \sum_{j=1}^s J_j(\mathbf{x}_j(0), \mathbf{u}_j) + J_G(x(0), u), \quad (11)$$

where

$$J_j(\mathbf{x}_j(0), \mathbf{u}_j) = \int_0^\infty \mathbf{x}_j^\top(\tau) \hat{Q}_j \mathbf{x}_j(\tau) + \mathbf{u}_j^\top(\tau) \hat{R}_j \mathbf{u}_j(\tau) d\tau, \quad (12)$$

and

$$J_G(x(0), u) = \int_0^\infty x^\top(\tau) (G_2 \otimes \tilde{Q}) x(\tau) d\tau. \quad (13)$$

The algebraic matrix Riccati equation corresponding to the group-level integral quadratic cost in (12) is

$$\mathcal{P}_j \mathcal{A}_j + \mathcal{A}_j^\top \mathcal{P}_j + \hat{Q}_j - \mathcal{P}_j \mathcal{B}_j \hat{R}_j^{-1} \mathcal{B}_j^\top \mathcal{P}_j = 0, \quad j = 1, \dots, s. \quad (14)$$

From Assumption 1, $(\mathcal{A}_j, \mathcal{B}_j)$ is controllable. However, $(\hat{Q}_j^{1/2}, \mathcal{A}_j)$ may not be observable for each $j = 1, \dots, s$. To guarantee existence and uniqueness of the solution to (14), we make the following assumption:

Assumption 2: $(\hat{Q}_j^{1/2}, \mathcal{A}_j)$ is observable for $j = 1, \dots, s$.

Remark 2: For a general matrix \mathcal{A}_j , Assumption 2 may be satisfied even when \hat{Q}_j is singular. Observability of $(Q^{1/2}, \mathcal{A})$

and $(\hat{Q}_j^{1/2}, \mathcal{A}_j)$ can both be guaranteed without knowing the dynamics of each agent if $\hat{Q}_j \succ 0$ for all $j = 1, \dots, s$.

The decomposition of the global control objective as given in (11) allows the individual groups to solve for their local optimal control gains (that minimizes (12)) in parallel. Next, we present an approximate control to account for the coupled objective given in (13). Motivated by [21], we define

$$\mathcal{R}^{-1} = R^{-1} + \tilde{R}, \quad (15)$$

where the expression for \tilde{R} will be derived shortly. Replacing R^{-1} with \mathcal{R}^{-1} in (5) yields

$$\begin{aligned} & \mathcal{P}\mathcal{A} + \mathcal{A}^\top \mathcal{P} + Q - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P} \\ &= \underbrace{\mathcal{P}\mathcal{A} + \mathcal{A}^\top \mathcal{P} + \hat{Q} - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P}}_{\text{decoupled part}} + G_2 \otimes \tilde{Q} - \mathcal{P}\mathcal{B}\tilde{R}\mathcal{B}^\top \mathcal{P} \\ &= \text{diag}\{\mathcal{P}_j \mathcal{A}_j + \mathcal{A}_j^\top \mathcal{P}_j + \hat{Q}_j - \mathcal{P}_j \mathcal{B}_j \tilde{R}_j^{-1} \mathcal{B}_j^\top \mathcal{P}_j\} \\ &\quad + G_2 \otimes \tilde{Q} - \mathcal{P}\mathcal{B}\tilde{R}\mathcal{B}^\top \mathcal{P}, \end{aligned}$$

indicating that the Riccati equation can be decomposed into multiple decoupled smaller-sized Riccati equations if

$$G_2 \otimes \tilde{Q} - \mathcal{P}\mathcal{B}\tilde{R}\mathcal{B}^\top \mathcal{P} = 0. \quad (16)$$

Therefore, we select \tilde{R} as the solution of (16), where the block diagonal matrix $\mathcal{P} = \text{diag}\{\mathcal{P}_1, \dots, \mathcal{P}_s\}$ is obtained by solving the set of s decoupled Riccati equations given in (14). The controller is designed hierarchically as

$$u_h = -\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P}x = \underbrace{-R^{-1}\mathcal{B}^\top \mathcal{P}x}_{\text{local}} - \underbrace{\tilde{R}\mathcal{B}^\top \mathcal{P}x}_{\text{global}}, \quad (17)$$

where the first term is the local controller that can be obtained by solving multiple decoupled smaller-size Riccati equations, all in parallel, and the second term is the global controller based on \tilde{R} solved from (16).

However, it may not always be possible to find a \tilde{R} satisfying (16). If \mathcal{B} is square and non-singular (i.e., each agent is a fully actuated system), then \tilde{R} follows simply as $\tilde{R} = (\mathcal{P}\mathcal{B})^{-1}(G_2 \otimes \tilde{Q})(\mathcal{B}^\top \mathcal{P})^{-1}$. However, this expression does not hold when $\mathcal{B} \in \mathbb{R}^{nN \times mN}$ is rectangular and thus not invertible. In that case one may obtain \tilde{R} by solving the following least squares semidefinite program (LSSDP):

$$\begin{aligned} & \min_{\tilde{R}} \|\mathcal{P}\mathcal{B}\tilde{R}\mathcal{B}^\top \mathcal{P} - G_2 \otimes \tilde{Q}\|_F \\ & \text{s.t.} \quad \tilde{R} \succeq 0. \end{aligned} \quad (18)$$

Let $(\mathcal{P}\mathcal{B})^+$ be the Moore-Penrose inverse of $\mathcal{P}\mathcal{B}$. In [34], it was shown that the solution $(\mathcal{P}\mathcal{B})^+(G_2 \otimes \tilde{Q})(\mathcal{P}\mathcal{B})^{++}$ has minimum norm among all solutions to (18). Therefore, one may compute \tilde{R} as

$$\tilde{R} = (\mathcal{P}\mathcal{B})^+(G_2 \otimes \tilde{Q})(\mathcal{P}\mathcal{B})^{++}. \quad (19)$$

Specifically, when $\mathcal{B} \in \mathbb{R}^{nN \times mN}$ is full column rank, the least square solution is uniquely determined by

$$\tilde{R} = ((\mathcal{P}\mathcal{B})^\top \mathcal{P}\mathcal{B})^{-1} (\mathcal{P}\mathcal{B})^\top (G_2 \otimes \tilde{Q}) \mathcal{P}\mathcal{B} ((\mathcal{P}\mathcal{B})^\top \mathcal{P}\mathcal{B})^{-1}.$$

Rewrite $\mathcal{P} = \text{diag}\{p_1^\top, \dots, p_N^\top\}^\top$, where $p_i \in \mathbb{R}^{n \times nN}$. From (17), the controller for each agent i can be written as:

$$u_h^{(i)} = -R_i^{-1}B_i^\top \mathcal{P}_{j^i,i} \mathbf{x}_{j^i} - \sum_{k=1}^N \tilde{R}_{ik} B_k^\top p_k x, \quad (20)$$

where j^i specifies the cluster containing i . $\mathcal{P}_{j^i,i} \in \mathbb{R}^{n \times nN_{j^i}}$ is a submatrix consisting of the continuous n rows of \mathcal{P}_{j^i} corresponding to agent i . Since \tilde{R} inherits the structure of G_2 and \mathcal{P} is block-diagonal, there are many zero entries in \tilde{R} and p_k . As a result, agent i only requires information of agents from its own cluster and other clusters for which the corresponding entries in \tilde{R} are non-zero. Theorem 2 in Subsection IV-A characterizes what entries of \tilde{R} are zero given a G_2 . Moreover, the controller (20) can be implemented in a decentralized way, as discussed in Remark 3.

Instead of minimizing the original objective function (3), the approximate controller (17) minimizes

$$\mathcal{J}(x(0), u) = \int_0^\infty x^\top \mathcal{Q}x + u^\top \mathcal{R}u dt, \quad (21)$$

where $\mathcal{Q} = \hat{Q} + \mathcal{P}\mathcal{B}\tilde{R}\mathcal{B}^\top \mathcal{P}$, and \mathcal{R} follows from (15). The following theorem shows that the hierarchical controller (17) is stabilizing. The proof is given in Section VII.

Theorem 1: Under Assumptions 1 and 2, the hierarchical controller (17) guarantees that the MAS (1) is globally asymptotically stable. ■

B. Hierarchical RL Algorithm

We next use [19, Algorithm 3]¹ to find the decomposed LQR controllers in a model-free way. The matrix \mathcal{B} is needed for solving the least squares problem (16). For our design \mathcal{B} can be estimated by estimating \mathcal{B}_j for each cluster j at the very first step of the RL algorithm as shown in [19]. Suppose that the cost function is decomposed into s clusters, and that Assumption 2 is satisfied. Matrices G_1 and G_2 are determined accordingly. By combining the RL algorithm in [19] with our hierarchical control framework, we propose Algorithm 1 as the final RL algorithm for heterogeneous MAS.

Algorithm 1 RL Algorithm for Optimal Control of Heterogeneous MAS via Hierarchical Control

Input: \hat{Q}_j , $j = 1, \dots, s$, $G_2 \otimes \tilde{Q}$, $R = \text{diag}\{\hat{R}_1, \dots, \hat{R}_s\}$.

Output: Optimal controller u^*

1. Run [19, Algorithm 3] to solve the LQR problem (12) for each cluster j , $j = 1, \dots, s$, with group dynamics (7). Obtain \mathcal{P}_j and estimated \mathcal{B}_j for the j -th LQR problem.
2. Compute the Moore-Penrose inverse of $\mathcal{P}_j \mathcal{B}_j$ for $j = 1, \dots, s$. Then compute \tilde{R} by (19).
3. The hierarchical optimal controller is

$$u_h = -(R^{-1}\mathcal{B}^\top \mathcal{P} + \tilde{R}\mathcal{B}^\top \mathcal{P})x.$$

¹Although [19] focuses on minimum-cost variance control, the proposed algorithms can be applied to conventional LQR problems in a specific case, see [19, Remark 1].

Remark 3: Algorithm 1 is presented from the viewpoint of centralized learning. It can also be implemented in a decentralized way under specific conditions. One such condition is as follows. Consider that a *coordinator* is chosen from the agents in each cluster. The coordinator collects and transmits state information from and to all agents in that cluster. Communication between different clusters happens through these coordinators. The interaction relationship between different coordinators is determined by the structure of \tilde{R} (details are shown in Theorem 2). Through communication, the coordinator in each cluster j has access to \mathbf{x}_j and \mathbf{x}_i from any adjacent cluster i , as well as to $\mathcal{P}_j\mathcal{B}_j$ and $\mathcal{P}_i\mathcal{B}_i$. Based on these information, each coordinator can use the steps of Algorithm 1 to learn the hierarchical optimal controller for cluster j in a decentralized fashion. Once the hierarchical control gain is learned, the controller (20) can be implemented in a decentralized way as well. In this scenario, coordinators in different clusters exchange state information of agents in their clusters. Then each agent is able to obtain from its coordinator the required state information of agents in other clusters.

Remark 4: Not just model-free RL, the proposed hierarchical design is equally applicable to model-based optimal control. The strategy in that case will be to decompose the Riccati equation into multiple smaller-sized Riccati equations.

C. Numerical Comparison with Conventional RL

We close this section by citing a numerical example that compares the conventional RL algorithm of [16] with our hierarchical RL algorithm.

Example 1: Consider \mathcal{G} as a graph composed of s cliques. Each clique has c agents, and will be referred to as a cluster. The graph between cliques is considered as an unweighted path graph. Moreover, there is only one link between any two connected cliques i and j . See Fig. 2 as a demonstration of \mathcal{G} for $s = c = 3$. We set $G = L$, $\bar{Q} = 0.5I_{nN}$, where L is the Laplacian matrix, $Q = G \otimes I_n + \bar{Q}$ and $R = I_{mN}$. Then Assumption 2 always holds for arbitrary decomposition. For $n = 4$ and $m = 2$, we set $A_i = \begin{pmatrix} -I_2 & I_2 \\ \mathbf{0} & -\frac{i}{i+1}I_2 \end{pmatrix}$ and $B_i = \begin{pmatrix} \mathbf{0} \\ \frac{i}{i+1}I_2 \end{pmatrix}$, $i = 1, \dots, N$. For $n = 8$ and $m = 4$, we set $A_i = A_i \otimes I_2$ and $B_i = B_i \otimes I_2$. The component of each agent's initial state is randomly selected from $\{1, -1, 0\}$.

The computational time and the resulting performance index J for each case are shown in Table I. We see that the hierarchical RL algorithm saves a significant amount of time compared with the conventional RL algorithm. In Table I, “RL” denotes the conventional RL algorithm from [16], “HRL” denotes Algorithm 1, “***” implies that the time for data collection is too long for conventional RL, “OPT” is the optimal value by implementing the controller learned from conventional RL, “SOP” means “suboptimality”, which is defined as $\frac{J_h - J^*}{J^*}$, where J_h and J^* are the performances of the hierarchical controller and the optimal controller, respectively. We find that as the problem size Nn increases, closed-loop performance of the hierarchical controller becomes closer to optimal. This is because when there are more links in the overall graph, the

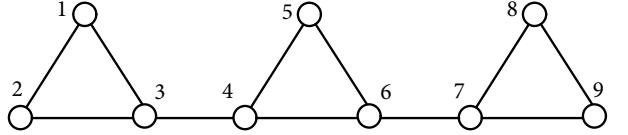


Fig. 2. Graph \mathcal{G} with 3 cliques, each clique contains 3 agents.

links contained in G_2 become less important in the computation of the total performance index. For the conventional RL algorithm, the SOP for each case is close to zero.

TABLE I
COMPARISONS BETWEEN DIFFERENT RL ALGORITHMS.

s	c	n	m	Dimension		Time(sec)		Performance		
				RL	HRL	OPT	HRL	SOP		
3	2	4	2	0.57	0.07	28.76	40.57	41.08%		
3	3	4	2	7.04	0.08	52.75	62.24	18.00%		
3	4	4	2	29.69	0.24	81.03	87.57	8.08%		
4	4	8	4	**	9.83	198.89	209.29	5.23%		

In Example 1, the agents corresponding to each clique are considered to form a cluster. There exist other ways to decompose the graph \mathcal{G} in Fig. 2. Given the number of clusters as three, the decomposition strategy used for this example is optimal in terms of the performance of the hierarchical controller. The reason will be shown in the next section. We will revisit this example in Subsection IV-D.

IV. OPTIMAL DECOMPOSITION FOR MINIMIZING COMMUNICATION AND SUBOPTIMALITY

To employ the hierarchical control framework presented in Section III, the decomposition strategy should be pre-specified. Given a MAS, there can be multiple ways to decompose \mathcal{G} that may result in different performances and different amounts of control energy required. In this section, we analyze how a decomposition affects the communication graph \mathcal{G}_c and the performance of the resulting hierarchical controller. Based on this analysis, we propose an approach to choose a decomposition minimizing the number of inter-agent communication links, or optimizing the performance of the hierarchical controller.

A. Relationship Between Decomposition and Communication Graph

Let $\tilde{R} \in \mathbb{R}^{mN \times mN}$ be the solution to (18). As $\mathcal{B}^\top \mathcal{P}$ is block-diagonal according to the clusters, the graph structure among different clusters is preserved in \tilde{R} and $\tilde{R}\mathcal{B}^\top \mathcal{P}$. We partition \tilde{R} into s^2 blocks $\tilde{R}(i, j) \in \mathbb{R}^{m_i \times m_j}$, and partition $G_2 \in \mathbb{R}^{N \times N}$ into s^2 blocks $G_2(i, j) \in \mathbb{R}^{N_i \times N_j}$, $i, j = 1, \dots, s$, respectively, according to the s clusters. We state the following theorem.

Theorem 2: Consider the MAS (1) decomposed into s clusters. For any two distinct clusters i and j , if $G_2(i, j) = \mathbf{0}_{N_i \times N_j}$, then $\tilde{R}(i, j) = \mathbf{0}_{m_i \times m_j}$, and $\mathcal{E}_c(u_h)$ in (6) contains no edges between the agents in cluster i and in cluster j , where u_h is the hierarchical controller (17). ■

This theorem indicates that our hierarchical controller (17) inherits the structure of G_2 for different clusters in the sense

that any pair of agents from two different clusters do not need to interact with each other through the feedback if there is no link between those two clusters in graph \mathcal{G} . Let $j \sim k$ and $j \not\sim k$ denote the adjacent and non-adjacent relationships between any two clusters j and k , respectively. The controller for agent i in (20) can then be rewritten as

$$u_h^{(i)} = -R_i^{-1}B_i^\top \mathcal{P}_{j^i,i}\mathbf{x}_{j^i} - \sum_{j^i \sim k} \tilde{R}_i(j^i,k)\mathcal{B}_k^\top \mathcal{P}_k \mathbf{x}_k, \quad (22)$$

where $\tilde{R}_i(j^i,k) \in \mathbb{R}^{m \times mN_j}$ is a submatrix consisting of m rows of $R(j^i,k)$ corresponding to agent i .

Consequently, the number of communication links required for our hierarchical controller is upper bounded by $N^2 - \kappa$, where

$$\kappa = \sum_{i \not\sim j} N_i N_j \quad (23)$$

is the number of pairs of agents that do not need to communicate with each other.

B. Relationship Between Decomposition and Performance

Let u^* and u_h^* be the optimal controllers corresponding to the original cost function $J(x(0), u)$ in (3) and the approximate cost function $\mathcal{J}(x(0), u)$ in (21), respectively. Here $\mathcal{J}(x(0), u)$ is a function of matrix \mathcal{Q} , which depends on the decomposition strategy. It can be verified that

$$x^\top(0)Px(0) = J(x(0), u^*),$$

$$x^\top(0)\mathcal{P}x(0) = \mathcal{J}(x(0), u_h^*),$$

where P and \mathcal{P} are the solutions to the following two Riccati equations, respectively:

$$PA + A^\top P + Q - P\mathcal{B}R^{-1}\mathcal{B}^\top P = 0, \quad (24)$$

$$\mathcal{P}A + A^\top \mathcal{P} + \mathcal{Q} - \mathcal{P}\mathcal{B}(R^{-1} + \tilde{R})\mathcal{B}^\top \mathcal{P} = 0. \quad (25)$$

The following theorem shows the relationship between the optimal performance for the approximate problem, the optimal performance of the original problem, and the performance of the original problem with our approximate hierarchical controller.

Theorem 3: Given MAS (1) with initial state $x(0)$, the optimal value of the approximate cost (21), the optimal value of the original cost (3), and the value of the original cost for our hierarchical controller have the following relationship:

$$\mathcal{J}(x(0), u_h^*) \leq J(x(0), u^*) \leq J(x(0), u_h^*). \quad (26)$$

It is desirable to design a decomposition strategy such that the corresponding hierarchical controller u_h^* makes $J(x(0), u_h^*)$ close to $J(x(0), u^*)$. To this end, we try to find the optimal decomposition such that

$$\Delta J_h = J(x(0), u_h^*) - J(x(0), u^*) \geq 0$$

is minimized. Since the performance error always depends on the initial state $x(0)$, we propose to analyze the average performance and the expectation $\mathbb{E}(\Delta J_h)$ given a random initial state vector $x(0)$. The following theorem shows an explicit form of $\mathbb{E}(\Delta J_h)$.

Theorem 4: Given MAS (1), let $K_h = \mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P}$ be the hierarchical control gain matrix, and $K = R^{-1}\mathcal{B}^\top P$ be the optimal control gain matrix. Suppose that the initial state vector $x(0)$ is a random variable with zero mean, and $\sigma^2 I_{NN}$ ($\sigma > 0$) as its covariance matrix². Then,

$$\mathbb{E}(\Delta J_h) = \sigma^2 \text{tr}(V), \quad (27)$$

where V is the solution of

$$\mathcal{A}_s^\top V + V\mathcal{A}_s + W = 0, \quad (28)$$

where $\mathcal{A}_s = \mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P}$ and $W = (K_h - K)^\top R(K_h - K)$. ■

Next, we present an upper bound for $\text{tr}(V)$ depending on \mathcal{P} and G_2 , as stated in the following lemma.

Lemma 1: For the MAS given in (1), the following statements hold for the matrices V and W defined in Theorem 4:

(i) If $\bar{Q} \succ 0$, then

$$\text{tr}(V) \leq \frac{\lambda_{\max}(\mathcal{P}) \text{cond}(\mathcal{P}) \text{tr}(W)}{\lambda_{\min}(\bar{Q})}; \quad (29)$$

(ii)

$$\text{tr}(W) \leq f_1(\text{tr}(G_2), \lambda_{\min}(\mathcal{P})) + f_2(\text{tr}(G_2), \lambda_{\min}(\mathcal{P})), \quad (30)$$

with

$$f_1 = \frac{\text{tr}^2(G_2) \text{tr}^2(\bar{Q})}{\lambda_{\min}^2(\mathcal{P}) \sigma_l^2(\mathcal{B})} \lambda_{\max}(R), \quad (31)$$

$$f_2 = [\lambda_{\max}(\mathcal{P}) - \lambda_{\min}(\mathcal{P})] \times \left[\text{tr}(\mathcal{B}R^{-1}\mathcal{B}^\top) + \frac{\sigma_{\max}^2(\mathcal{B}) \text{tr}(G_2) \text{tr}(\bar{Q})}{\lambda_{\min}^2(\mathcal{P}) \sigma_l^2(\mathcal{B})} \right], \quad (32)$$

where $\sigma_l(\mathcal{B})$ is the minimum nonzero singular value of \mathcal{B} . ■

Lemma 1 implies that if there exists a decomposition such that compared with all other possible decomposition, $\text{tr}(G_2)$ and $\lambda_{\max}(\mathcal{P})$ are minimal, and $\lambda_{\min}(\mathcal{P})$ is maximal, implying that $\text{cond}(\mathcal{P})$ is minimal as well, then the upper bound of $\mathbb{E}(\Delta J_h)$ corresponding to this decomposition is also minimal. However, there may not exist a decomposition such that all these indices are optimized simultaneously. Since minimizing $\text{cond}(\mathcal{P})$ is necessary for minimizing $\lambda_{\max}(\mathcal{P})$ and maximizing $\lambda_{\min}(\mathcal{P})$, we only consider $\text{tr}(G_2)$ and $\text{cond}(\mathcal{P})$ as the two most important indices for evaluating $\mathbb{E}(\Delta J_h)$.

In conclusion, when $\bar{Q} \succ 0$, we can view $\text{tr}(G_2)$ and $\text{cond}(\mathcal{P})$ as two of the most important factors affecting the performance of our hierarchical controller corresponding to any given decomposition.

Remark 5: From (14), \mathcal{P} depends not only on \bar{Q} , but also on \mathcal{A} , \mathcal{B} and R . For a heterogeneous MAS, if different agents have largely different dynamics and R_i differs largely for different i , then these differences will strongly influence the resulting \mathcal{P} for different decomposition strategies. Since the dynamics of the agents are unknown, we cannot utilize this information for analyzing the relationship between the decomposition and the performance of the hierarchical controller. The

²The assumption on the initial state here is to eliminate the dependence of the control performance on the initial conditions in our analysis for the relationship between decomposition and performance. The original MAS optimal control problem is still deterministic.

performance is mainly determined by $\text{tr}(G_2)$ and $\text{cond}(\mathcal{P})$ only when the differences between the agent dynamics are small, and R_i for different i are similar.

C. Approaches for Optimizing Decomposition

Subsection IV-A shows that maximizing $\kappa = \sum_{i \sim j} N_i N_j$ helps in reducing the number of required communication links in the hierarchical controller u_h . On the other hand, Subsection IV-B shows that minimizing $\text{tr}(G_2)$ and $\text{cond}(\mathcal{P})$ help optimize the performance of the closed-loop system. The index $\text{cond}(\mathcal{P})$ is always influenced by \mathcal{A} , \mathcal{B} and R , so it is impractical to optimize $\text{cond}(\mathcal{P})$ by choosing the decomposition strategy without explicit knowledge of \mathcal{A} and \mathcal{B} . However, minimizing $\text{tr}(G_2)$ is independent of the system dynamics, and, therefore, is a tractable problem. Accordingly, we seek to maximize κ for minimizing the number of communication links, and minimize $\text{tr}(G_2)$ for optimizing the performance of our hierarchical controller.

Given the desired number of clusters, minimizing $\text{tr}(G_2)$ is equivalent to minimizing the total number of edges crossing any two different clusters, which is actually a minimum s -cut problem. In [38], an algorithm is proposed to solve the minimum s -cut problem in polynomial time for a specified s . Maximizing κ , in comparison, is more complicated because the structure of the graph is important in computing κ . For example, in the first example shown in Fig. 1, $\kappa = 2$ and $\text{tr}(G_2) = 6$, and the interaction between cluster 2 and cluster 3 is not required in the hierarchical controller. If we use another decomposition, say, $\mathcal{V}_1 = \{1\}$, $\mathcal{V}_2 = \{2, 3\}$ and $\mathcal{V}_3 = \{4, 5\}$, then $\kappa = 0$ and $\text{tr}(G_2) = 6$, in which case, although the number of edges crossing the clusters remains the same, the resulting hierarchical control may require communications between two different clusters.

Given s as the desired number of clusters, we next find the decomposition for maximizing κ . Let $\eta_i \in \mathbb{R}^N$ be a vector where each component is either 0 or 1, $i = 1, \dots, s$. We use η_i to determine the members of cluster i . Let $\eta_i(k)$ be the k th element of η_i and define

$$\eta_i(k) = \begin{cases} 1, & \text{if } k \in \mathcal{V}_i; \\ 0, & \text{otherwise,} \end{cases}$$

where \mathcal{V}_i is the set of agents in the i -th cluster. Then the number of agents in clusters i is $N_i = |\mathcal{V}_i| = \eta_i^\top \mathbf{1}_N$. Given s as the number of clusters, we use matrix $E = (\eta_1, \dots, \eta_s) \in \mathbb{R}^{N \times s}$ to denote a decomposition. Accordingly, $G_1(E)$ and $G_2(E)$ become the two decomposed matrices in (8) following this decomposition E , i.e.,

$$G = G_1(E) + G_2(E). \quad (33)$$

Note that $G_1(E)$ may not be block-diagonal but can be transformed to be so by relabelling the agents.

Next, we formulate a MIQP problem for seeking the decomposition maximizing κ . From the definition of E , we have $\kappa = \sum_{i \sim j} N_i N_j = \sum_{i \sim j} \eta_i^\top \mathbf{1}_N \mathbf{1}_N^\top \eta_j$. Let $\bar{G} = |G|$ where $\bar{G}_{ij} = |G_{ij}|$. For any two clusters i and j , the total amount of interaction weights between them is $\langle \eta_i \eta_j^\top, \bar{G} \rangle =$

$\eta_i^\top \bar{G} \eta_j$. Therefore, $i \sim j$ if and only if $\eta_i^\top \bar{G} \eta_j = 0$. Let $l_{ij} \leq 1 - \eta_i^\top \bar{G} \eta_j / T$ and $\tau_{ij} = \eta_i^\top \mathbf{1}_N \mathbf{1}_N^\top \eta_j$, where T is an integer greater than or equal to the sum of the weights of \bar{G} . Then $\eta_i^\top \bar{G} \eta_j / T = 0$ if $i \sim j$, and $\eta_i^\top \bar{G} \eta_j / T \leq 1$ otherwise. If we consider l_{ij} as a binary variable, then the maximum value of l_{ij} is 1 if $i \sim j$, and is 0 otherwise. Therefore, the decomposition that maximizes κ is given by the solution to the following MIQP problem:

$$\begin{aligned} \min_{\eta_1, \dots, \eta_s} \quad & - \sum_{i=1}^s \sum_{j=1}^s l_{ij} \tau_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^s \eta_i = \mathbf{1}_N, \quad \eta_i^\top \mathbf{1}_N \geq 1, \\ & \tau_{ij} = \eta_i^\top \mathbf{1}_N \mathbf{1}_N^\top \eta_j, \quad l_{ij} \leq 1 - \eta_i^\top \bar{G} \eta_j / T, \\ & l_{ij}, \eta_i(k) \in \{0, 1\}, \quad i, j = 1, \dots, s. \end{aligned} \quad (34)$$

The MIQP problem (34) can be solved, for example, by branch and bound algorithms. Commercial software such as Matlab, Lindo and Gurobi can be used for this purpose, as will be shown in our simulations. We next present two properties.

Theorem 5: Let $\kappa^*(s)$ be the maximum value of κ for s clusters. Given matrix $G \in \mathbb{R}^{N \times N}$ in (4), $\kappa^*(s)$ is non-decreasing in s and $\kappa^*(s) \leq z_0$, where z_0 is the number of zero elements in G . ■

Theorem 5 can be used to adjust s to reduce the number of required communication links in the hierarchical controller. For example, in the second example of Fig. 1, $\kappa(E) = 0$ for any decomposition if $s = 2$. Once we set $s = 3$, there are two communication links that can be removed, as shown in the first example of Fig. 1.

D. Numerical Verifications

As we discussed in Subsection IV-B, when $\bar{Q} \succ 0$, and $\text{cond}(\mathcal{P})$ is similar for different decompositions, $\text{tr}(G_2)$ is the key factor determining the performance of the hierarchical controller. In what follows, we will present two numerical examples where $\text{cond}(\mathcal{P})$ is similar for different decompositions.

Example 2: We reconsider the example with graph \mathcal{G} in Fig. 2. Matrices Q , R and the agent dynamics are the same as those considered in Example 1. By generating the initial states of agents 1000 times such that each time every component of the agent states is a random number chosen from a normal distribution with zero mean and variance 0.5, the average performance values for different decompositions are shown in Table II. By solving a mixed integer linear program (MILP) formulation for the minimum s -cut problem, we obtain the second decomposition, which minimizes $\text{tr}(G_2)$ and generates the hierarchical controller that has the best performance. By solving MIQP (34) using Gurobi in Matlab (total time taken is 7.8843 seconds), we obtain the third decomposition that maximizes κ , resulting in a controller that requires the fewest communication links.

Remark 6: When \bar{Q} is well-conditioned (i.e., when $\text{cond}(\bar{Q})$ is small), and $\lambda_{\max}(G)$ is not large, \hat{Q} is well-conditioned because $\text{cond}(\hat{Q}) \leq (\lambda_{\max}(G_1) + \lambda_{\max}(\bar{Q})) / \lambda_{\min}(\bar{Q})$. Moreover, when all agents have identical dynamics and equal values of R_i , \mathcal{P} will be well-conditioned

TABLE II
COMPARISONS BETWEEN DIFFERENT DECOMPOSITIONS.

Decomposition	κ	$\text{tr}(G_2)$	$\text{cond}(\mathcal{P})$	J	n_c	SOP
{1,2},{3,...,7},{8,9}	4	8	16.4	15.9	32	23.57%
{1,2,3},{4,5,6},{7,8,9}	9	4	17.1	14.3	27	10.20%
{1,...,3},{4},{5,...,9}	15	6	17.0	15.8	21	22.15%
Undecomposed	n/a	n/a	n/a	12.9	36	0

as well. In that case, $\text{tr}(G_2)$ is the most important factor determining the performance of the hierarchical controller, as can be seen clearly from the above examples. On the other hand, when \bar{Q} is ill-conditioned, different decompositions may cause largely different $\text{cond}(\hat{Q})$ and $\text{cond}(\mathcal{P})$. In that case, $\text{cond}(\hat{Q})$ becomes a more important factor determining the performance of the hierarchical controller. An example on this will be shown in Section V.

V. APPLICATION TO MULTI-AGENT FORMATION MANEUVER CONTROL

A. Problem Formulation

Consider a group of agents, denoted by the vertex set \mathcal{V} , traveling between multiple waypoints in a two-dimensional plane. The set \mathcal{V} is categorized into two subsets, $\mathcal{V} = \mathcal{L} \cup \mathcal{F}$, where \mathcal{L} is the set of leaders tracking specific positions at each waypoint, and \mathcal{F} is the set of followers adjusting their positions according to measured information from other agents.

1) *Agent Dynamics:* To make this example more realistic, we consider that the model of each agent i is perturbed by a bounded time-varying disturbance d_i , so that the dynamics can be written as

$$M_i \ddot{q}_i + C_i \dot{q}_i = u_i + d_i, \quad i = 1, \dots, N, \quad (35)$$

where $q_i \in \mathbb{R}^2$, $\dot{q}_i \in \mathbb{R}^2$, $M_i \succ 0$, $C_i \succ 0$, $u_i \in \mathbb{R}^2$ and $d_i \in \mathbb{R}^2$ are the position, the velocity, the inertial matrix, the centrifugal term, the control input, and the disturbance input of agent i , respectively. The matrices M_i and C_i are considered to be unknown. The disturbance d_i is assumed to be measurable. Similar assumptions have been made in the existing literature of model-free LQR problems with noise, e.g., [18], [19].

2) *Control Objective:* The general control objective is to design a state-feedback control law without knowing M_i and C_i for each agent i , such that the agents maintain a desired formation shape at each waypoint, achieve a flocking behavior while traveling between each of two neighboring waypoints, and take the minimum control effort for transforming from one formation shape to the other.

3) *Formation Graph and Communication Graph:* We consider the formation graph $\mathcal{G}_f = (\mathcal{E}_f, \mathcal{V})$ and the communication graph $\mathcal{G}_c = (\mathcal{E}_c, \mathcal{V})$ as two different graphs. Graph \mathcal{G}_f , together with the target position $h = (h_1^\top, \dots, h_N^\top)^\top \in \mathbb{R}^{2N}$, form a framework (\mathcal{G}_f, h) describing the desired formation. Here h_i is the target position of agent i at the next waypoint. The edges determined by \mathcal{E}_f specify the agent pairs with constrained relative positions, while the edges determined by \mathcal{E}_c specify those agent pairs who communicate with each other. Usually \mathcal{E}_c is determined by the wireless sensing range of each

agent. Therefore, staying cohesive guarantees that there are more agent pairs that can communicate with each other.

4) *LQR Problem Formulation:* Let $x_i = (q_i^\top - h_i^\top, \dot{q}_i^\top)^\top$ for $i = 1, \dots, N$. Then the i^{th} agent model can be rewritten as

$$\dot{x}_i = A_i x_i + B_i u_i + B_i d_i, \quad i = 1, \dots, N \quad (36)$$

where $A_i = \begin{pmatrix} \mathbf{0} & I_2 \\ \mathbf{0} & -M_i^{-1} C_i \end{pmatrix}$, and $B_i = \begin{pmatrix} 0 \\ M_i^{-1} \end{pmatrix}$. Let $S_1 = (I_2, \mathbf{0}_{2 \times 2})$, $S_2 = (\mathbf{0}_{2 \times 2}, I_2)$, then $q_i - h_i = S_1 x_i$, and $\dot{q}_i = S_2 x_i$. In the literature, asymptotic convergence of formation stabilization has been widely studied [30]. However, the performance of the group trajectory during transience is usually not guaranteed. To capture an optimal trajectory of the agents during travel between waypoints, we make the group minimize the following performance index:

$$\begin{aligned} J_1 &= \int_0^\infty \sum_{(i,j) \in \mathcal{E}_f} \|q_i - q_j - (h_i - h_j)\|^2 + \sum_{i \in \mathcal{L}} \|q_i - h_i\|^2 dt \\ &= \int_0^\infty x^\top (L \otimes S_1^\top S_1) x + x^\top (\Lambda \otimes I_2) x dt \end{aligned} \quad (37)$$

where $L \in \mathbb{R}^{N \times N}$ is the Laplacian matrix corresponding to the formation graph \mathcal{G}_f , $\Lambda = \text{diag}\{\Lambda_1, \dots, \Lambda_N\}$, $\Lambda_i = 1$ if $i \in \mathcal{L}$ and $\Lambda_i = 0$ otherwise.

Flocking behavior would require the agents to stay cohesive and have a common velocity. To model this, we additionally define the following performance index:

$$J_2 = \int_0^\infty \dot{q}^\top (L \otimes I_2) \dot{q} dt = \int_0^\infty x^\top (L \otimes S_2^\top S_2) x dt. \quad (38)$$

The overall goal then is to minimize

$$J = J_1 + J_2 = \int_0^\infty [x^\top ((L + \Lambda) \otimes I_4) x + u^\top u] dt \quad (39)$$

subject to the plant dynamics (36) for $i = 1, \dots, N$.

Remark 7: Different from formation control problems, where the steady states of all agents depend on their initial states, in the maneuver control problem each agent has a specific target position at each waypoint. In this case, it is easy for each agent to obtain its own as well as other's target steady states by either (i) using available target relative positions from its neighbors and communicating with them, or (ii) achieving the target position through a centralized task assignment.

B. Feasible Decompositions for Hierarchical Control

To use our hierarchical control design, Assumptions 1 and 2 should be satisfied. Let $\mathcal{A} = \text{diag}\{A_1, \dots, A_N\}$ and $\mathcal{B} = \text{diag}\{B_1, \dots, B_N\}$, where A_i and B_i are shown in (36). Since $M_i \succ 0$ and $C_i \succ 0$ for each agent i , $(\mathcal{A}, \mathcal{B})$ must be controllable. Moreover, when graph \mathcal{G}_f is connected and there exists at least one leader in the whole group, it holds that $Q = (L + \Lambda) \otimes I_4 \succ 0$, implying that $(Q^{1/2}, \mathcal{A})$ is observable. Then Assumption 1 is satisfied. To ensure validity of Assumption 2, the decomposition must satisfy the following result.

Theorem 6: Assumption 2 is satisfied if and only if each cluster contains at least one leader and its formation graph is connected. ■

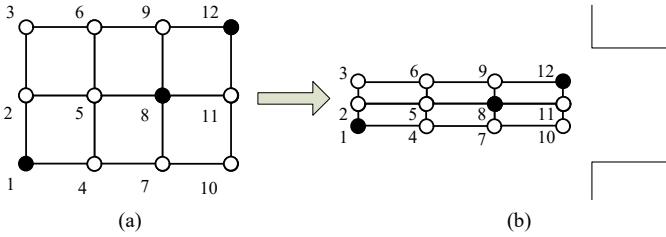


Fig. 3. (a). The initial positions of agents; (b) The target positions of agents at the next waypoint.

An immediate consequence of Theorem 6 is that the number of clusters cannot exceed the number of leaders in the MAS. To find the decomposition maximizing κ , we modify the MIQP (34) to include the following two constraints:

- (i) $\xi^\top \eta_i \geq 1$ for $i = 1, \dots, s$, where let $\xi \in \mathbb{R}^N$ with $\xi(i) = 1$ if $i \in \mathcal{L}$ and $\xi(i) = 0$ otherwise;
- (ii) $G_{kk} - \sum_{i=1}^s \eta_i^\top g_k e_k^\top \eta_i \geq \epsilon (\sum_{i=1}^s \eta_i^\top \mathbf{1}_N e_k^\top \eta_i - 1)/N$, $k = 1, \dots, N$, where $g_k \in \mathbb{R}^N$ is the k -th column of G , and ϵ is the minimum nonzero entry of \bar{G} .

The constraint (i) implies that each cluster contains at least one leader, while constraint (ii) means that each agent k either forms a cluster ($\sum_{i=1}^s \eta_i^\top \mathbf{1}_N e_k^\top \eta_i = 1$) or has a nonempty neighbor set in the cluster that it belongs to.

C. Numerical Experiments

We consider 12 agents governed by (36) with the formation graph \mathcal{G}_f as a mesh grid shown in Fig. 3. The black nodes are leaders, and the other nodes are followers. The mission of these agents is to transform their formation shape from Fig. 3 (a) to Fig. 3 (b) before entering into a narrow space as shown towards the right in the figure. The leader set is $\mathcal{L} = \{1, 8, 12\}$. The parameters in the dynamics of each agent i are set as $M_i = \text{diag}\{(i+1)/2, i/2\}$ and $C_i = \text{diag}\{i/4, i/5\}$. The disturbance is set as $d_i(t) = ((0.05i, 0.1i)^\top \times \cos t)/(t+1)$ for $i = 1, \dots, N$.

We first find the optimal LQR without considering the disturbance d_i . By implementing the optimal control law in the presence of disturbance, the value of the performance index is found as $J^* = 1247.7575$, and that for the control effort as $J_u = \int_0^\infty u^\top u dt = 428.8475$. The number of communication links is 66, which implies that the communication graph $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}_c)$ is fully connected. Under the same communication graph \mathcal{G}_c , by using the formation stabilization law in [30] we get

$$u_i = - \sum_{(i,j) \in \mathcal{E}_c} (q_i - q_j - (h_i - h_j)) - k_i(q_i - h_i) \quad (40)$$

where $k_i = 1$ if $i \in \mathcal{L}$ and $k = 0$ otherwise. The overall performance and control input performance are computed as $J = 2176.9360$ and $J_u = 983.9627$, respectively. Fig. 4 and Fig. 5 show the state trajectories of the agents corresponding to the optimal LQR and the formation stabilization law (40), respectively. From these two figures we observe that compared to the stabilization law (40), LQR provides a better closed-loop path response and thereby saves the overall control effort.

Next, we compute the model-free LQR using the conventional ADP algorithm from [16]. In the data collection

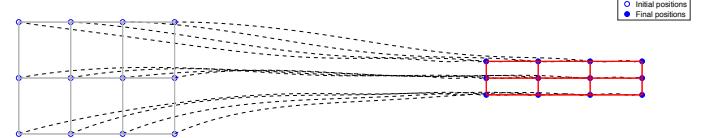


Fig. 4. The optimal trajectory for maneuver control.

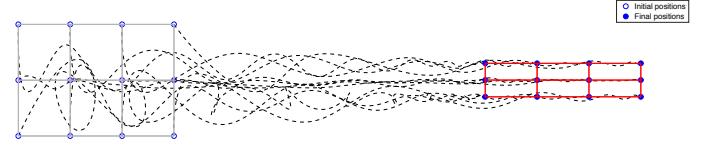


Fig. 5. State trajectories for maneuver control using the formation stabilization law from [30].

phase, $u_i + d_i$ (instead of u_i) is viewed as the control input. By implementing the learned controller, the overall performance and control input performance are computed as $J^{RL} = 1247.7601$ and $J_u^{RL} = 429.2707$, respectively. The learning time is 58.8262s.

Finally, we apply our proposed hierarchical approximation. Similar to when applying conventional RL, we regard $u_i + d_i$ as the control input during the data collection phase. For validity of Assumption 2, according to Theorem 6, we decompose the group into $s = 3$ clusters such that each cluster has a connected formation graph, and contains at least one leader. In Fig. 6, the decomposition strategies maximizing κ and minimizing $\text{tr}(G_2)$ are shown, respectively, where the nodes with the same color belong to the same cluster. The first decomposition is obtained by solving (34) with the additional constraints that were presented in Subsection V-B. The learning time is found to be 22.7734 seconds. Fig. 7 shows the evolution of the position error of the agents by comparing the hierarchical control law with the decomposition maximizing κ and the optimal LQR. The second decomposition is obtained by solving a MILP formulation for the minimum s -cut problem with additional constraints in Subsection V-B. Fig. 8 shows the evolution of the position errors by comparing the hierarchical control law with the decomposition minimizing $\text{tr}(G_2)$ and the optimal LQR. From Fig. 7 and Fig. 8, we observe that the decomposition maximizing κ has a better performance than the decomposition minimizing $\text{tr}(G_2)$.

More detailed simulation results of implementing the hierarchical RL algorithm (Algorithm 1) based on three decompositions are listed in Table III. For simplicity, each decomposition is done according to the order of the agents. As an example, for the first decomposition in Table III, the indices of agents in the three clusters are $\{1, \dots, 6\}$, $\{7, 8, 9\}$ and $\{10, 11, 12\}$. It is observed from Table III that the first decomposition maximizes κ , and induces the fewest communication links required by the hierarchical controller. The second decomposition minimizes $\text{tr}(G_2)$, but does not induce the best performance. Instead, the first decomposition, which induces the minimum condition numbers on \mathcal{P} and \bar{Q} , yields the best performance. This is because $\text{cond}(\bar{Q}) = \text{cond}(\Lambda) = \infty$, which implies that \bar{Q} is ill-conditioned. In this case, different decomposition

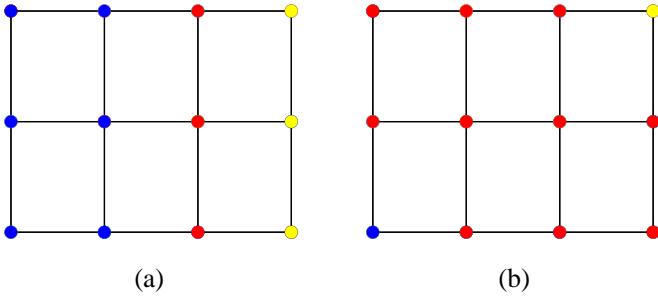


Fig. 6. (a). The decomposition maximizing κ ; (b) The decomposition minimizing $\text{tr}(G_2)$.

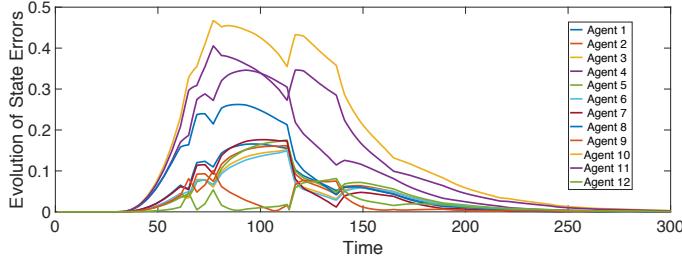


Fig. 7. The evolution of agent state errors by implementing the hierarchical controller with the decomposition maximizing κ and the optimal LQR, respectively.

strategies lead to largely different $\text{cond}(\mathcal{P})$ and $\text{cond}(\hat{Q})$, and thus $\text{tr}(G_2)$ becomes a less important factor to determine the suboptimality.

Remark 8: The inter-agent interactions during implementation of Algorithm 1 can be explained as follows. Suppose we choose the first decomposition strategy in Table III. During the implementation of Algorithm 2, no communication is required between agents $\{1, \dots, 6\}$ and agents $\{10, 11, 12\}$. The communication between the rest of agents can be achieved through three coordinators in the three clusters. In other words, the hierarchical RL algorithm can be implemented distributively, as described in Remark 3.

VI. CONCLUSION

We presented a model-free hierarchical RL algorithm for optimal control of linear MAS with heterogeneous agents, based on a decomposition approach that can significantly reduce learning time. The derived controller is suboptimal but has a specific structure. Two indices of this hierarchical controller, namely, the number of communication links and the closed-loop performance of the agents, are analyzed and shown to be dependent on the decomposition strategy. Optimizing these two indices is formulated as a MIQP and a minimum s -cut problem, respectively. The hierarchical controller is applied to a formation maneuver control problem. Simulation results are presented to illustrate its effectiveness in saving learning time, and corresponding trade-offs in closed-loop performance.

VII. APPENDIX

Proof of Theorem 1: Because \mathcal{P} satisfies (14) with Assumption 1 and 2, \mathcal{P} is positive definite. Consider the Lyapunov

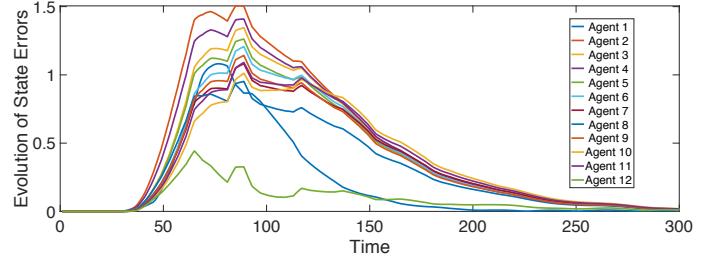


Fig. 8. The evolution of agent state errors by implementing the hierarchical controller with the decomposition minimizing $\text{tr}(G_2)$ and the optimal LQR, respectively.

function $\Phi = x^\top \mathcal{P} x$, whose derivative is given by

$$\dot{\Phi} = 2x^\top \mathcal{P}(\mathcal{A}x + \mathcal{B}u_h) \quad (41)$$

$$= 2x^\top \mathcal{P}(\mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P} - \mathcal{B}\tilde{\mathcal{R}}\mathcal{B}^\top \mathcal{P})x \quad (42)$$

$$= 2x^\top (\mathcal{P}\mathcal{A} - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P})x - 2x^\top \mathcal{P}\mathcal{B}\tilde{\mathcal{R}}\mathcal{B}^\top \mathcal{P}x \quad (43)$$

$$= x^\top (\mathcal{P}\mathcal{A} + \mathcal{A}^\top \mathcal{P} - 2\mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P})x - 2x^\top \mathcal{P}\mathcal{B}\tilde{\mathcal{R}}\mathcal{B}^\top \mathcal{P}x \quad (44)$$

$$= x^\top (-\hat{Q} - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P})x - 2x^\top \mathcal{P}\mathcal{B}\tilde{\mathcal{R}}\mathcal{B}^\top \mathcal{P}x \leq 0. \quad (45)$$

From LaSalle's Invariance Principle, we know that x converges to the largest invariant set \mathcal{E} defined as

$$\mathcal{E} = \{x \mid x^\top \hat{Q}x = x^\top \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P}x = 2x^\top \mathcal{P}\mathcal{B}\tilde{\mathcal{R}}\mathcal{B}^\top \mathcal{P}x = 0\}. \quad (46)$$

Since \hat{Q} is positive semidefinite, it follows that $\hat{Q}^{1/2}x = 0$ in \mathcal{E} . From (17), $u_h = 0$ in \mathcal{E} . Thus, we consider the dynamics in \mathcal{E} as $\dot{x} = \mathcal{A}x$, and $\hat{Q}^{1/2}\dot{x} = 0$. It follows that $\hat{Q}^{1/2}\mathcal{A}^c x = 0$ for $c \in \{0, 1, 2, \dots\}$. Since $(\hat{Q}^{1/2}, \mathcal{A})$ is observable, we have $x = 0$. Then we conclude that the only solution that stays in \mathcal{E} is the trivial solution $x = 0$ and, therefore, the closed-loop system is globally asymptotically stable. ■

Proof of Theorem 2: From the uniqueness of Moore-Penrose inverse, we can define $\Xi = \text{diag}\{\Xi_1, \dots, \Xi_s\} \in \mathbb{R}^{mN \times nN}$, where $\Xi_j = (\mathcal{P}_j \mathcal{B}_j)^+ \in \mathbb{R}^{m_j \times n_j}$. Let $\Phi = G_2 \otimes \tilde{Q}$. Next, we partition Φ into s^2 blocks $\Phi_{ij} \in \mathbb{R}^{n_i \times n_j}$, $i, j = 1, \dots, s$, according to the s clusters. For two different i and j such that $G_2(i, j) = \mathbf{0}_{N_i \times N_j}$, we have $\Phi_{ij} = \mathbf{0}_{n_i \times n_j}$. As a result,

$$\tilde{R}(i, j) = \Xi_i \Phi_{ij} \Xi_j^\top = \mathbf{0}_{m_i \times m_j}. \quad (47)$$

From (17), the hierarchical controller of the j -th cluster is in the following form

$$\mathbf{u}_j = -\hat{R}_j^{-1} \mathcal{B}_j^\top \mathcal{P}_j \mathbf{x}_j - \sum_{i=1}^s \tilde{R}(i, j) \mathcal{B}_i^\top \mathcal{P}_i \mathbf{x}_i, \quad (48)$$

which implies that communication between cluster i and cluster j is not required if $\tilde{R}(i, j) = \mathbf{0}_{m_i \times m_j}$. ■

Proof of Theorem 3: The second inequality $J(x(0), u^*) \leq J(x(0), u_h^*)$ is valid because u^* is the optimal controller corresponding to $J(x(0), u)$. Next, we prove the first inequality.

From (14), we observe that \mathcal{P} is also the solution to

$$\mathcal{P}\mathcal{A} + \mathcal{A}^\top \mathcal{P} + \hat{Q} - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \mathcal{P} = 0. \quad (49)$$

TABLE III
COMPARISONS BETWEEN DIFFERENT DECOMPOSITIONS.

Decomposition			Performance Indices								
N_1	N_2	N_3	κ	$\text{tr}(G_2)$	$\text{cond}(\mathcal{P})$	$\text{cond}(\tilde{Q})$	J	J_u	n_c	Time(sec)	SOP
6	3	3	18	12	248.7647	46.1346	1259.7985	426.9677	48	0.9248	0.82%
1	10	1	1	8	339.4430	83.7524	1347.0390	431.6374	65	13.9719	7.96%
7	2	3	0	12	285.1161	55.3510	1267.7974	442.4165	66	2.1165	1.61%

Note that $Q - \hat{Q} = G_2 \otimes \tilde{Q} \succeq 0$, i.e., $\hat{Q} \leq Q$. Using (24) and [35, Lemma 3], we obtain $\mathcal{P} \leq P$. It follows that $\mathcal{J}(x(0), u_h^*) \leq J(x(0), u^*)$. ■

Proof of Theorem 4: We first study $\mathbb{E}(J(x(0), u_h^*))$. We can write

$$\begin{aligned} J(x(0), u_h^*) &= \int_0^\infty x^\top Q x dt + \int_0^\infty u_h^{*\top} R u_h^* dt \\ &= \int_0^\infty x^\top (Q + K_h^\top R K_h) x dt \end{aligned} \quad (50)$$

where $x = e^{(\mathcal{A} - \mathcal{B}K_h)t} x(0)$. It follows that

$$\mathbb{E}(J(x(0), u_h^*)) = \text{tr}(U \mathbb{E}(x(0)x^\top(0))) = \sigma^2 \text{tr}(U), \quad (51)$$

where U is the solution of

$$\mathcal{A}_s^\top U + U \mathcal{A}_s + Q + K_h^\top R K_h = 0, \quad (52)$$

with $\mathcal{A}_s = \mathcal{A} - \mathcal{B}K_h$.

Next, we analyze $\mathbb{E}(J(x(0), u^*))$. It can be verified that

$$\mathbb{E}(J(x(0), u^*)) = \mathbb{E}(x^\top(0)Px(0)) = \sigma^2 \text{tr}(P),$$

where P is the solution of (24). Let $V = U - P$. Then $\mathbb{E}(\Delta J_h) = \mathbb{E}(J(x(0), u_h^*)) - \mathbb{E}(J(x(0), u^*)) = \sigma^2 \text{tr}(V)$. Note that (24) can be rewritten as

$$P\mathcal{A}_s + \mathcal{A}_s^\top P + Q + K_h^\top R K_h + K_h^\top R K - K^\top R K = 0. \quad (53)$$

The subtraction of (52) and (53) yields (28). ■

Proof of Lemma 1: (i) Using (14), the positive definiteness of \tilde{Q} , and [37, Corollary 4.5.11], the following holds:

$$\begin{aligned} &- \lambda_{\max}(\mathcal{P}\mathcal{A}_s\mathcal{P}^{-1} + \mathcal{A}_s^\top) \\ &= -\lambda_{\max}(\mathcal{P}\mathcal{A}\mathcal{P}^{-1} - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top + \mathcal{A}^\top - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top) \\ &= -\lambda_{\max}(-\mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top - \mathcal{Q}\mathcal{P}^{-1}) \\ &= -\lambda_{\max}(-\mathcal{P}^{1/2}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top\mathcal{P}^{1/2} - \mathcal{P}^{-1/2}\mathcal{Q}\mathcal{P}^{-1/2}) \\ &= \lambda_{\min}(\mathcal{P}^{1/2}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top\mathcal{P}^{1/2} + \mathcal{P}^{-1/2}\mathcal{Q}\mathcal{P}^{-1/2}) \\ &= \lambda_{\min}(\mathcal{P}^{1/2}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top\mathcal{P}^{1/2} + \mathcal{P}^{-1/2}\bar{Q}\mathcal{P}^{-1/2}) \\ &\geq \lambda_{\min}(\mathcal{P}^{1/2}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top\mathcal{P}^{1/2}) + \lambda_{\min}(\mathcal{P}^{-1/2}\bar{Q}\mathcal{P}^{-1/2}) \\ &\geq \lambda_{\min}(\mathcal{P})\lambda_{\min}(\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top) + \lambda_{\min}(\bar{Q})/\lambda_{\max}(\mathcal{P}) > 0. \end{aligned}$$

In most applications, usually \mathcal{B} is not square, implying that $\lambda_{\min}(\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top) = 0$. Therefore, we further omit the term associated with $\lambda_{\min}(\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top)$, i.e.,

$$-\lambda_{\max}(\mathcal{P}\mathcal{A}_s\mathcal{P}^{-1} + \mathcal{A}_s^\top) \geq \lambda_{\min}(\bar{Q})/\lambda_{\max}(\mathcal{P}). \quad (54)$$

From [36, Corollary 3.2], it holds that:

$$\text{tr}(V) \leq -\frac{\lambda_{\max}(\mathcal{P}) \text{tr}(\mathcal{P}^{-1}W)}{\lambda_{\max}(\mathcal{P}\mathcal{A}_s\mathcal{P}^{-1} + \mathcal{A}_s^\top)}. \quad (55)$$

Reusing [37, Corollary 4.5.11], we have

$$\begin{aligned} \text{tr}(\mathcal{P}^{-1}W) &= \text{tr}(\mathcal{P}^{-1/2}W\mathcal{P}^{-1/2}) \\ &\leq \lambda_{\max}^2(\mathcal{P}^{-1/2}) \text{tr}(W) = \frac{\text{tr}(W)}{\lambda_{\min}(\mathcal{P})} \end{aligned} \quad (56)$$

Combining (54), (55) and (56), the bound on $\text{tr}(U)$ stated in (29) is obtained.

(ii) It can be verified that

$$W = \mathcal{P}\tilde{M}\mathcal{P} + \mathcal{P}\bar{M}\mathcal{P} + \Delta P\mathcal{M}\Delta P - P\tilde{M}P, \quad (57)$$

where $\mathcal{M} = \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top$, $\tilde{M} = \mathcal{B}\tilde{R}\mathcal{B}^\top$, $\bar{M} = \mathcal{B}\bar{R}\mathcal{R}\tilde{R}\mathcal{B}^\top$, and $\Delta P = P - \mathcal{P}$. We have shown in Theorem 3 that $\mathcal{P} \leq P$. It follows that $\text{tr}(P^2 - \mathcal{P}^2) = \text{tr}((P + \mathcal{P})^{1/2}(P - \mathcal{P})(P + \mathcal{P})^{1/2}) \geq 0$. As a result, $\text{tr}(\mathcal{P}\tilde{M}\mathcal{P} - P\tilde{M}P) = \text{tr}((\mathcal{P}^2 - P^2)\tilde{M}) = -\text{tr}(\tilde{M}^{1/2}(P - \mathcal{P})\tilde{M}^{1/2}) \leq -\lambda_{\min}(\tilde{M})\text{tr}(P^2 - \mathcal{P}^2) \leq 0$. This implies that $\text{tr}(W) \leq \text{tr}(\mathcal{P}\bar{M}\mathcal{P}) + \text{tr}(\Delta P\mathcal{M}\Delta P)$. To prove the statement (ii), we will prove $\text{tr}(\mathcal{P}\bar{M}\mathcal{P}) \leq f_1$ and $\text{tr}(\Delta P\mathcal{M}\Delta P) \leq f_2$ successively.

For the first inequality, by [37, Corollary 4.5.11], the following holds:

$$\begin{aligned} \text{tr}(\mathcal{P}\mathcal{B}\tilde{R}\mathcal{R}\tilde{R}\mathcal{B}^\top\mathcal{P}) &\leq \lambda_{\max}^2((\mathcal{P}\mathcal{B})(\mathcal{P}\mathcal{B})^+) \times \\ &\quad \text{tr}((G_2 \otimes \tilde{Q})(\mathcal{P}\mathcal{B})^{+T}R(\mathcal{P}\mathcal{B})^+(G_2 \otimes \tilde{Q})) \\ &= \text{tr}(R^{1/2}(\mathcal{P}\mathcal{B})^+(G_2 \otimes \tilde{Q})^2(\mathcal{P}\mathcal{B})^{+T}R^{1/2}) \quad (58) \\ &\leq \text{tr}^2(G_2 \otimes \tilde{Q})\lambda_{\max}[(\mathcal{P}\mathcal{B})^{+T}R(\mathcal{P}\mathcal{B})^+], \\ &= \text{tr}^2(G_2)\text{tr}^2(\tilde{Q})\lambda_{\max}(SRS), \end{aligned}$$

where $S^2 = (\mathcal{P}\mathcal{B})^+(\mathcal{P}\mathcal{B})^{+T}$, i.e., $S = T\Lambda^{1/2}T^\top$ in which T and Λ follow from the eigen-decomposition of $(\mathcal{P}\mathcal{B})^+(\mathcal{P}\mathcal{B})^{+T}$ satisfying $T\Lambda T^\top = (\mathcal{P}\mathcal{B})^+(\mathcal{P}\mathcal{B})^{+T}$. Since S is square and symmetric, we can apply [37, Corollary 4.5.11] to analyze the eigenvalues of SRS .

The singular values of S are given by the diagonal entries of $\Lambda^{1/2}$. The minimum singular value is zero while the maximum singular value is less than $\sigma_{\max}((\mathcal{P}\mathcal{B})^+)$. Therefore, we have

$$\lambda_{\max}(SRS) \leq \sigma_{\max}^2((\mathcal{P}\mathcal{B})^+)\lambda_{\max}(R). \quad (59)$$

Using the fact that $\sigma_{\max}((\mathcal{P}\mathcal{B})^+) = 1/\sigma_l(\mathcal{P}\mathcal{B})$, where $\sigma_l(\mathcal{P}\mathcal{B})$ is the minimum nonzero singular value of $\mathcal{P}\mathcal{B}$, we next look for a lower bound on $\sigma_l(\mathcal{P}\mathcal{B})$.

Note that $\sigma_l(\mathcal{P}\mathcal{B}) = \sigma_l(\mathcal{B}^\top\mathcal{P})$. From [37, Corollary 4.5.11], it holds that

$$\begin{aligned} \sigma_l(\mathcal{B}^\top\mathcal{P}) &= \lambda_l^{1/2}(\mathcal{P}\mathcal{B}\mathcal{B}^\top\mathcal{P}) \\ &\in [\lambda_{\min}(\mathcal{P})\sigma_l(\mathcal{B}), \lambda_{\max}(\mathcal{P})\sigma_{\max}(\mathcal{B})]. \end{aligned} \quad (60)$$

We then have

$$\lambda_{\max}(SRS) \leq \frac{1}{\lambda_{\min}^2(\mathcal{P})\sigma_l^2(\mathcal{B})} \lambda_{\max}(R). \quad (61)$$

Together with (58), we have $\text{tr}(\mathcal{P}\bar{M}\mathcal{P}) \leq f_1$.

In the following derivations for f_2 , some steps that are similar to the approach stated above will be omitted for brevity. We use the following two inequalities:

$$\begin{aligned} \text{tr}(\Delta PBR^{-1}\mathcal{B}^\top \Delta P) &\leq \lambda_{\max}(\Delta P) \text{tr}(\mathcal{B}\tilde{\mathcal{R}}\mathcal{B}^\top) \\ &\leq (\lambda_{\max}(P) - \lambda_{\min}(\mathcal{P})) \text{tr}(\mathcal{B}\tilde{\mathcal{R}}\mathcal{B}^\top), \end{aligned} \quad (62)$$

$$\begin{aligned} \text{tr}(\Delta P\mathcal{B}\tilde{\mathcal{R}}\mathcal{B}^\top \Delta P) &\leq \lambda_{\max}(\Delta P)\sigma_{\max}^2(\mathcal{B}) \text{tr}(\tilde{\mathcal{R}}) \\ &\leq [\lambda_{\max}(P) - \lambda_{\min}(\mathcal{P})]\sigma_{\max}^2(\mathcal{B}) \frac{\text{tr}(G_2 \otimes \tilde{Q})}{\lambda_{\min}^2(\mathcal{P})\sigma_l^2(\mathcal{B})}. \end{aligned} \quad (63)$$

Combining the two inequalities, we get $\text{tr}(\Delta P\mathcal{M}\Delta P) \leq f_2$. ■

Proof of Theorem 5: Without loss of generality, given s as the number of clusters, and the decomposition $E = (\eta_1, \dots, \eta_s)$ corresponding to the maximum $\kappa = \kappa^*(s) = \sum_{i \neq j} N_i N_j$. Next we show that for $s+1$, there exists a decomposition $\bar{E} = (\eta_1, \dots, \eta_{s+1})$ such that the corresponding $\kappa' \geq \kappa^*(s)$. For any cluster with at least two agents, if we decompose it into two clusters k and l , the resulting κ will be $\kappa' = \kappa^*(s) + N_k N_l > \kappa^*(s)$ if $k \approx l$, and $\kappa' = \kappa^*(s)$ otherwise. This shows the nondecreasing property of $\kappa^*(s)$. Since $s \leq N$, and $\kappa(N) = z_0$, we conclude that κ is bounded by z_0 . ■

Proof of Theorem 6: For each cluster j , the performance index can be written as $J = \int_0^\infty [\mathbf{x}_j^\top ((L_j + \Lambda_j) \otimes I_2) + u^\top u] dt$, where L_j is the Laplacian matrix corresponding to the subgraph \mathcal{G}_f^j of \mathcal{G}_f involving agents in \mathcal{V}_j , Λ_j is a diagonal matrix.

Sufficiency: When there is a leader in cluster j , there is at least one positive value on diagonal of Λ_j . Since \mathcal{G}_f^j is connected, using [39, Lemma 3], we have $\hat{Q}_j = (L_j + \Lambda_j) \otimes I_4 \succ 0$. As a result, $(\hat{Q}_j^{1/2}, \mathcal{A}_j)$ is observable.

Necessity: Suppose that no leaders exist in cluster j or \mathcal{G}_f^j is not connected. Then $\hat{Q}_j = L_j \otimes I_4$ and L_j has at least one zero eigenvalue. Let ζ be the eigenvector associated with eigenvalue 0 of L_j . We observe that \hat{Q}_j and \mathcal{A}_j have a common eigenvector corresponding to eigenvalue 0, which is $\zeta \otimes (1, 1, 0, 0)^\top$. Therefore, $(\hat{Q}_j^{1/2}, \mathcal{A}_j)$ can never be detectable. ■

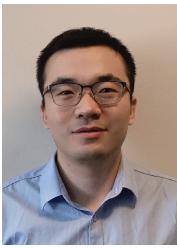
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